The Crossing Number of $K_{7,11}$

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A drawing of $K_{7,11}$ with 225 crossings
Abstract

Researching the crossing number of $K_{7,11}$, a special case of Zarankiewicz’s conjecture (1954), we find both pre-computer and post-computer results in the literature, bringing the possible solutions back to three options: 221, 223 and 225. In our own research we use two quite old theorems by Zarankiewicz and Turán to add new constraints to a modern day semi definite programming problem in order to find a better lower bound for the crossing number of bipartite graphs, and use computers to calculate that bound. Unfortunately the results of this experiment were only positive for $cr(K_{3,n})$ and $cr(K_{4,n})$ which are both already known since at latest 1969, but we can say some things about the drawings in which we achieve that minimal number of crossings. But this result does not bring $cr(K_{7,11})$ any closer within our reach.
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Preface

Searching for a nice, discrete and combinatorial open problem with good visualisation options to attack during my research project, I found the open problem: $cr(K_{7,11}) = Z(7, 11) = 225?$. on the web. After running some background checks I decided to go for it. All literature was very interesting, and also very varied in methods. Unfortunately I was not very succesful solving the problem, but I did learn a great deal about what works in certain cases and what not, and I know much better where the dead ends are in this particular field of research. And to quote my favorite Dutch philosopher Bas Haring: "... er is niets mis met doodlopende wegen: het kunnen prachtige wegen zijn." [2]. I have had fun and intellectual challenges on my path to writing this paper, and I will hopefully also have graduated because of it, which does make it 100% worthwhile. My final wish is for you to enjoy reading it.
1 Introduction to the problem at hand

In this paper we try to find the crossing number of $K_{7,11}$. This is a special case of the more general problem of the crossing number of complete bipartite graphs, which was put forward by P. Turán in 1952 in a lecture he gave in Warsaw. It stems from a practical problem from Turán’s wartime job in a brickwork factory where there were rails from several burning-ovens to several storerooms to transport the bricks. The crossings of rails were causing problems, so it seemed smart to him to minimize the number of crossings.[1]

Schematic example of a brickwork factory

This problem can be represented by the minimum number of crossings in a complete bipartite graph $K_{m,n}$, which is called the crossing number $cr(K_{m,n})$. In 1954 Zarankiewicz [7], who attended Turán’s lecture in Warsaw, conjectured the answer to be $Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Since then the problem is referred to as Zarankiewicz’s conjecture. Actually, he claimed to have proven this conjecture, or theorem as he called it, but the proof appeared false as we will see in the next chapter.

Example of a bipartite graph: $K_{3,5}$

$K_{m,n}$ has vertex set $V$ which is partitioned into $A$ and $B$ containing respectively $m$ and $n$ vertices. There is an edge from every vertex in $A$ to every vertex in $B$, and no other edges. In principle $m$ and $n$ can be any positive integers, but often we assume $m \leq n$. For completeness: a crossing is defined as a point where two lines meet that is not a vertex.
2 Zarankiewicz’s results

Though the proof of his conjecture turned out to be false, there were some important results Zarankiewicz did prove. First of all he established an upper bound for the crossing number of $K_{m,n}$, which is equal to $Z(m,n)$. He has done this by designing a way to draw $K_{m,n}$ with $Z(m,n)$ crossings:

We draw two axes, and place the $m$ vertices from $A$ on the horizontal axis such that $\lfloor \frac{m}{2} \rfloor$ are on the left of the vertical axis, and $\lceil \frac{m}{2} \rceil$ on the right. We place the $n$ vertices from $B$ on the vertical axis divided in the same manner, and draw straight lines from all vertices in $A$ to all vertices in $B$. For an example see the figure on the front page.

Then in each quadrant every two pairs of vertices, one from $A$ and one from $B$, give rise to one crossing:

One crossing per pair of pairs

The total number of crossings in this drawing is then equal to (and the minimal number of crossings less or equal than):

$$cr(K_{m,n}) \leq \left( \left\lfloor \frac{m}{2} \right\rfloor \right) \left( \left\lceil \frac{m}{2} \right\rceil \right) \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \left( \left\lceil \frac{n}{2} \right\rceil \right)$$

$$= \left( \left\lfloor \frac{m}{2} \right\rfloor + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) \left( \left\lceil \frac{m}{2} \right\rceil \right) \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \left( \left\lceil \frac{n}{2} \right\rceil \right)\right)$$

$$= \left( \left\lfloor \frac{m}{2} \right\rfloor (\left\lfloor \frac{n}{2} \right\rfloor - 1) + \left\lfloor \frac{n}{2} \right\rfloor (\left\lfloor \frac{m}{2} \right\rfloor - 1) \right) \left( \left\lfloor \frac{m}{2} \right\rfloor (\left\lfloor \frac{n}{2} \right\rfloor - 1) + \left\lfloor \frac{n}{2} \right\rfloor (\left\lfloor \frac{m}{2} \right\rfloor - 1) \right)$$

$$= \left( \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-2}{2} \right\rfloor \right) \left( \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \right)$$

$$= Z(m,n)$$

Secondly Zarankiewicz has proven that $cr(K_{3,n}) = Z(3,n)$, which is the smallest non-trivial case. He did this by proving it for $n = 3$ and then using induction on $n$.

And lastly he has proven that induction can be used to prove his conjecture under the assumption that one can always find a pair of vertices in $A$ or in $B$ such that the edges incident with either of those vertices have no crossings between them. Because the crossings which involve this pair of vertices and any other vertex must be at least $cr(K_{m,3})$ in number, and those not involving the pair at least $cr(K_{m,n-2})$, we get:

$$cr(K_{m,n}) \geq cr(K_{m,n-2}) + (n - 2)cr(K_{m,3})$$

And with some arithmatic one can see that if $cr(K_{m,n-2}) = Z(m,n-2)$ and $cr(K_{m,3}) = Z(m,3)$, then $cr(K_{m,n}) \geq Z(m,n)$.

The fallacy of his "proof" lies in this assumption, which was noticed separately in 1965 (resp. 1966) by P. Kainen and G. Ringel. They both tried to repair the proof but failed as described by Guy [1]. To this day the proof of the conjecture has not essentially come any closer, although more partial results were found over the years.
3 Results by Guy and Kleitman

A very interesting result is mentioned by Guy [1] in 1969 and by Kleitman [3] in 1971, referred to by Guy as one-legged induction, which we have stated below in the form that Kleitman used. They clearly worked closely together in this field, where Kleitman seems to have done most of the proving, and Guy most of the writing.

First we need to introduce the notion of a good drawing. In a good drawing of $K_{m,n}$ no two arcs have more than one point in common, and that point is either one of the vertices at their ends, or a crossing. If a drawing of $K_{m,n}$ is not good, it clearly has more than the minimal number of crossings.

Secondly we make a distinction between the crossing number (eg. minimal number of crossing) of a graph $G$, denoted by $cr(G)$, and, for $D$ a drawing of a graph, $cr(D)$, which is the number of crossings in that particular drawing. Then there is also $cr(a,b)$ where $a, b \in A$ (resp. $B$) are vertices, which is the number of crossings in the subgraph induced by $a, b$ and $B$ (resp. $A$).

$$cr(D) = 3 > cr(K_{3,3}) = 1 \quad cr(a, b) = 1$$

**One-legged induction:** any good drawing $D_{m,n}$ of $K_{m,n}$ contains $n$ drawings of $K_{m,n-1}$, each obtained by suppressing a vertex $b \in B$. Each crossing between a line ending at $b$ and another line ending at $b' \in B$ occurs in $n - 2$ of these drawings, namely the ones in which neither $b$ or $b'$ are suppressed. Therefore $cr(D_{m,n}) \geq \frac{n}{n-2} cr(K_{m,n-1})$ (Counting argument).

This means that if $n$ is odd, say $n = 2s - 1$, and $cr(K_{m,n}) = Z(m, n)$, we have:

$$cr(D_{m,n+1}) \geq \frac{n+1}{n} Z(m, n)$$

$$= \frac{n+1}{n} Z(m, n + 1) \left( \frac{n-1}{2s} \right)$$

$$= \frac{2s}{2s-2} Z(m, 2s) \left( \frac{2s-2}{2s} \right)$$

$$= \frac{s}{s-1} Z(m, 2s) \left( \frac{s-1}{2s} \right)$$

$$= Z(m, n + 1)$$

Of course we can also use $a, a' \in A$ and interchange $m$ and $n$ in the argument. This implies, together with Zarankiewicz’s proof that $cr(K_{3,n}) = Z(3, n)$ for all $n$, that $K_{4,n} = Z(4, n)$ for all $n$. That means we only have to consider odd values of $m$ and $n$ when researching Zarankiewicz’s conjecture.
Another important result by Kleitman [3] is his proof of a special case of Zarankiewicz’s conjecture: \( cr(K_{5,n}) = Z(5,n) \) for all \( n \). He proves this by showing that no \( n \in \mathbb{N} \) can be the smallest \( n \) for which \( cr(K_{5,n}) < Z(5,n) \) using several ingenious arguments. Some of these arguments generalize to \( m \geq 5 \), such as the parity argument which states that every two good drawings of \( K_{2r+1,2s+1} \) \((r,s \in \mathbb{N})\) have the same parity. Since Zarankiewicz’s drawing is good, this means that good drawings of \( K_{2r+1,2s+1} \) have the same parity as \( Z(2r + 1, 2s + 1) \). Also interesting is the permutation argument, which considers the clockwise order in which the arcs leave a vertex \( a \in A \), labeling them with \( 1, 2, \ldots, n \) representing an ordering of the elements of \( B \). This concept is used in most of the following research of Zarankiewicz’s conjecture.

Another important and general result from Kleitman’s work, which he derived from his counting argument and the result \( cr(K_{5,n}) = Z(5,n) \), is a lower bound for \( cr(K_{m,n}) \) for \( m \geq 5 \):

\[
    cr(K_{m,n}) \geq \frac{m}{m-2} \frac{m-1}{m-4} \ldots \frac{7}{5} Z(5,n) \\
    \geq \frac{m(m-1)}{5} 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \\
    \geq \frac{1}{2} m(m-1) \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor
\]

Kleitman’s work implies that the crossing number of the now smallest unknown case, \( K_{7,7} \), can only be 77, 79 or 81.
4 Results by Woodall

In 1993 Woodall [6] has proven, extending Kleitmans work on permutations and using several computer programs, that \( \text{cr}(K_{7,n}) = Z(7,n) \) for \( n \leq 9 \).

First Woodall constructs cyclic-order graphs \( CO_m \) as follows:
Number the \( m \) vertices of \( A \subset V \) (\( V \) the vertex set of \( K_{m,n} \)) \( 1, 2, \ldots, m \). The vertices of \( CO_m \) are then the \( (m-1)! \) different cyclic orderings \( \pi_i \) of the set \( A = \{1, 2, \ldots, m\} \). Two such orderings are adjacent in \( CO_m \) if and only if one can be obtained from the other by transposing two adjacent elements of the ordering.

Example of a cyclic-order graph: \( CO_4 \)

**Theorem:** In a drawing \( D \) of \( K_{m,n} \), where \( \pi(a) = \pi_i \) for some \( i \in \{1, \ldots, (m-1)\!\} \) is the clockwise order in which the \( m \) arcs leave a vertex \( a \in B \), the minimal number of crossings between two vertices \( a \) and \( b \) is equal to the minimal distance \( d \) between the vertices \( \pi(a) \) and \( \pi(b)^{-1} \) in \( CO_m \).

**Proof:** We start from the observation that if \( \text{cr}(a,b) = 0 \), then \( \pi(a) = \pi(b)^{-1} \). Note also that in a drawing of \( K_{m,2} \) with vertices \( a, b \) and \( A \) we can move the vertices \( i \in A \) along the arcs \( (a, i, b) \) they are on, without changing \( \pi(a) \) or \( \pi(b) \). Therefore we can assume them to be so close to \( b \) that there are no crossings on the edges between them and \( b \).

Now assume that we have a drawing of \( K_{m,2} \) with the minimal number of crossings for certain fixed \( \pi(a) \) and \( \pi(b) \), and the vertices of \( A \) are all so close to \( b \) that there are no crossings on the edges \( (b, i), i \in A \). Then there is at most one crossing between two arcs \( (a, i, b) \) and \( (a, j, b) \) and no loops. Therefore we can find a crossing \( p \) of some arcs \( (a, i, b) \) and \( (a, j, b) \) such that there are no crossings on either segment \( (a, p) \) of these arcs, which implies \( i \) and \( j \) are adjacent elements of \( \pi(a) \). We can now reduce the number of crossings by one crossing if we open up \( p \) such that the new cyclic ordering of \( a \) is \( \pi(a)' \) where \( i \) and \( j \) are interchanged.
Opening up crossing $p$

If we keep repeating this process until there are no crossings left, we have a chain of interchanges that transforms $\pi(a)$ into $\pi(b)^{-1}$. This implies that $d(\pi(a), \pi(b)^{-1}) \leq c_{D}(a, b)$ for all drawings $D$ of $K_{m,2}$ (with orderings $\pi(a)$ and $\pi(b)$ that is). But if we have a path in $CO_{m}$ from $\pi(a)$ to $\pi(b)^{-1}$ of length $k$, we can use these interchanges to find a way to draw $K_{m,2}$ with $k$ crossings, which means that $d(\pi(a), \pi(b)^{-1}) \geq c_{r}(a, b)$.

Overall we conclude that the minimum number of crossings between $a$ and $b$ is equal to $d(\pi(a), \pi(b)^{-1})$. Q.E.D.

Let $\sigma(\pi(a), \pi(b))$ denote the distance in $CO_{m}$ between $\pi(a)$ and $\pi(b)^{-1}$. Then we have a lower bound for the number of crossings in a drawing $D$ of $K_{m,n}$ with fixed cyclic orderings $\pi(a)$ for each $a \in B$:

$$cr(D_{m,n}) = \sum_{a \in B} \sum_{b \in B, b \neq a} cr(a, b) \geq \sum_{a \in B} \sum_{b \in B, b \neq a} \sigma(\pi(a), \pi(b))$$

Woodall’s programmes are based on the idea that if a drawing of $K_{m,n}$ with $K$ or less crossings exists, there must be a set of $n$ elements $\pi_{1}, \ldots, \pi_{n}$, with the following properties ($\sigma_{i,j} = \sigma(\pi_{i}, \pi_{j})$):

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sigma_{i,j} \leq K$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \sigma_{i,j} \leq \left(\frac{n-2}{n}\right)K$$

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-2} \sigma_{i,j} \leq \left(\frac{n-3}{n-1}\right)\left(\frac{n-2}{n}\right)K$$

$$\vdots$$

$$\sum_{i=1}^{2} \sum_{j=i+1}^{2} \sigma_{i,j} \leq \left(\frac{1}{3}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-2}{n}\right)K$$

Here of course two of the $\pi_{i}$ can represent the same ordering, and should be processed with their multiplicity.

He then uses computer programs to check whether there are such sets for a number of different $m$ and $n$, and the results prove that there can be no drawings with $cr(K_{7,7}) < Z(7,7)$ or $cr(K_{7,9}) < Z(7,9)$.

This made $K_{7,11}$ and $K_{9,9}$ the smallest unknown cases instead of $K_{7,7}$, and they still are today.
5 Results using optimization

In 2006 De Klerk, Maharry, Pasechnik, Richter and Salazar [4] improved the lower bound found by Kleitman using the same counting argument but a new lower bound on $K_{7,n}$. They calculated that new lower bound by using optimization techniques and computers.

They use prior results on permutations to construct a $(m-1)!x(m-1)!$-matrix $C$ where $C_{ij} = \sigma_{i,j}$. Besides they define a $(m-1)!$-vector $x$ where $x_i$ represents the number of times the cyclic ordering $\pi_i$ occurs in $D$ scaled such that $1^T x = 1$. Then the inequality from the last chapter:

$$cr(D_{m,n}) \geq \sum_{a \in B} \sum_{b \in B, b \neq a} \sigma(\pi(a), \pi(b))$$

translates to:

$$cr(D_{m,n}) \geq \frac{1}{2} \left( (m-1)! \right) \left( (m-1)! \right) C_{ij}(x_i)(x_j) + \sum_{i=1}^{(m-1)!} C_{ii}x_i$$

$$= \frac{1}{2} n^2 \left( (m-1)! \right) \left( (m-1)! \right) C_{ii}x_i - \frac{1}{2} n \sum_{i=1}^{(m-1)!} \frac{1}{4}(m-1)^2 x_i$$

We can rewrite $C_{ii}$ as $\frac{1}{4}(m-1)^2$ since the distance in $CO_n$ between $\pi_i$ and $\pi_i^{-1}$ can be found as follows:

If we divide $\pi_i$ in two parts of length $\left\lfloor \frac{m}{2} \right\rfloor$ and $\left\lceil \frac{m}{2} \right\rceil$ we can reverse these separately in $\left( \left\lfloor \frac{m}{2} \right\rfloor \right)$ plus $\left( \left\lceil \frac{m}{2} \right\rceil \right)$ steps. As we have seen in chapter 2 the sum of those two numbers is equal to $\left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m-1}{2} \right\rceil$.

Note that $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor = \frac{1}{4}(m-1)^2$, which gives $Z_{m,n} = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil = \frac{1}{4}(m-1)^2 \frac{1}{4}(n-1)^2$.

These notations are both used regularly.

Since $D$ is arbitrary, we now have:

$$cr(K_{m,n}) \geq \frac{1}{2} n^2 \min \left\{ x^T Cx \mid x \in \mathbb{R}^{(m-1)!}, 1^T x = 1 \right\} - \frac{1}{2} n \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor$$

Finding $\alpha_m = \min \left\{ x^T Cx \mid x \in \mathbb{R}^{(m-1)!}, 1^T x = 1 \right\}$ is a standard quadratic optimization problem, which cannot be solved in this form for the values of $m$ we need. De Klerk ea. [4] relax the problem to a convex optimization problem, using $x^T Cx = tr(CX)$ where $X = xx^T$ is a PSD matrix, of the same dimensions as $C$, and $tr$ is the trace.

The problem then becomes (with $J$ the all 1 matrix):

$$\alpha_m = \min \left\{ tr(CX) \mid X \in \mathbb{R}^{(m-1)!x(m-1)!}, X \text{ PSD}, tr(JX) = 1 \right\}$$

Then they apply different techniques using the symmetry of $C$ to reduce the calculating time, and ultimately calculate with the help of computers that $\alpha_7 = 4.3593$. This gives us the following lower bound: $cr(K_{7,n}) \geq 2.1796n^2 - 4.5n$. This is an improvement for all $n \geq 23$. 
The former lower bound for all $m \geq 7$ by Kleitman [3] from chapter 3 can be rewritten thus:

$$cr(K_{m,n}) \geq \frac{1}{5} m(m - 1) \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil$$

$$\frac{cr(K_{m,n})}{Z(m,n)} \geq \frac{1}{5} \left\lceil \frac{m(m-1)}{4(m-1)^2} \right\rceil$$

$$\geq \frac{4}{5} \frac{m}{m-1}$$

But by using $cr(K_{7,n}) \geq 2.1796n^2 - 4.5n$ instead of $cr(K_{5,n}) = Z(m,n)$ the new lower bound for all $m \geq 7$, if we take the limit for $n \rightarrow \infty$, becomes:

$$cr(K_{m,n}) \geq \frac{m(m-1)}{4 \cdot 6} (2.1796n^2 - 4.5n)$$

$$\lim_{n \rightarrow \infty} \frac{cr(K_{m,n})}{Z(m,n)} \geq \frac{m(m-1)}{42 \left\lceil \frac{1}{4}(m-1)^2 \right\rceil} \cdot \frac{2.1796n^2 - 4.5n}{\left\lceil \frac{1}{4}(n-1)^2 \right\rceil}$$

$$\geq 0.83 \frac{m}{m-1}$$

In 2007 De Klerk, Pasechnik and Schrijver [5] use a technique based on matrix representations to reduce the dimensions to such an extent that the optimization problem can be solved for $m = 8$ and $m = 9$. They find that $\alpha_8 = 5.859985644 \ldots$ and $\alpha_9 = 7.7352126 \ldots$, implying:

$$cr(K_{8,n}) \geq 2.92999n^2 - 6n$$

$$cr(K_{9,n}) \geq 3.8676063n^2 - 8n$$

This last result improves the lower bound for $m \geq 9$ to:

$$\lim_{n \rightarrow \infty} \frac{cr(K_{m,n})}{Z(m,n)} \geq 0.8594 \frac{m}{m-1}$$

Summarizing the results so far using optimization are an improved lower bound for $cr(K_{7,n})$, $cr(K_{8,n})$ and $cr(K_{9,n})$ specifically (but only for higher numbers of $n$, for the smallest unknown cases we already had a better lower bound), and more generally an improved ratio between $cr(K_{m,n})$ and $Z(m,n)$ for $n \rightarrow \infty$ and $m \geq 9$ (and a little less improved ratios for $m \geq 7$ and $m \geq 8$).
6 Ideas for improving the lower bound of \( cr(K_{7,11}) \)

After studying the above thoroughly we came up with several ideas to improve the lower bound of \( cr(K_{7,11}) \). The best lower bound known so far was 220 (calculated using \( cr(K_{7,10}) = Z(7, 10) \) \[6\] and \( cr(D_{m,n}) \geq \frac{n}{n-2} cr(K_{m,n-1}) \) \[3\]), whereas \( Z(7, 11) = 225 \). But as mentioned in chapter 3 we know the parity of the crossing number is the same as that of \( Z(7, 11) \), which means it must be 221, 223 or 225.

We first look at Kleitman’s method of proving \( cr(K_{5,n}) = Z(5,n) \), to see which of his arguments would work for \( m = 7 \) too. Unfortunately some of the ones that reduced the feasible options substantially for \( m = 5 \) did not do so for \( m = 7 \), as was to be expected.

We also thought about looking at tight and pseudo-tight groups like Woodall suggested, or perhaps try his method of calculating \( cr(K_{7,7}) \) and \( cr(K_{7,9}) \) with modern day computers. This was not the course of action we chose to follow in the end.

Since we know that \( cr(K_{7,11}) \) is either 221, 223 or 225, we thought of constructing all graphs with seven vertices of degree 11, eleven of degree 7 and 221 (resp. 223) of degree 4, and check whether they are planar and run some other tests to see if they are counterexamples of the hypothesis \( cr(K_{7,11}) = Z(7, 11) = 225 \). It might be possible, but it is complicated to automatically construct all these graphs.

Then we looked at the optimization problem. Since it is based entirely on the minimum number of crossings between a pair of vertices of certain orientations, we thought we could improve the result by finding a lower bound for the number of crossings for triplets instead of pairs. Unfortunately it appeared very complicated due to the fact that the choice of drawings rises dramatically when we go from two to three vertices.

We finally decided on trying to use optimization (semidefinite programming: SDP) to calculate the lower bound of \( cr(K_{7,11}) \), as was done by De Klerk, Maharry, Pasechnik, Richter and Salazar and later De Klerk, Pasechnik and Schrijver, and add constraints to try and increase the minimum.
7 Finding new constraints for the optimization problem

To achieve a better lower bound for $K_{7,11}$ we experiment with new constraints implemented in the SDP method. We hope to find a higher $\alpha_7$ using the following result by Zarankiewicz:

If, in a given drawing $D_{m,n}$ of $K_{m,n}$, there is a pair of vertices without crossings between them, we can conclude that

$$cr(D_{m,n}) \geq cr(K_{m,n-2}) + (n-2)cr(K_{m,3}).$$

To find out whether this is a better lower bound than the one found with the SDP methods of Chapter 4, we need to know if the following inequality holds:

$$cr(K_{m,n-2}) + (n-2)cr(K_{m,3}) > \frac{1}{2} n^2 \alpha_m - \frac{1}{2} n\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor.$$

Substituting $cr(K_{m,n-2})$ with its lower bound $\frac{1}{2} (n-2)^2 \alpha_m - \frac{1}{2} (n-2)\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$, and $cr(K_{m,3})$ with $\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$ we find that this is the case if:

$$\frac{1}{2} (n-2)^2 \alpha_m - \frac{1}{2} (n-2)\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor + (n-2)\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor > \frac{1}{2} n^2 \alpha_m - \frac{1}{2} n\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$$

$$\frac{1}{2} (n-2)^2 \alpha_m - \frac{1}{2} n^2 \alpha_m > -(n-1)\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$$

$$(2n-2)\alpha_m < (n-1)\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$$

$$2\alpha_m < \left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$$

This is true for $5 \leq m \leq 9$, see the table below:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\alpha_m$</th>
<th>$\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1.9472</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2.9519</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4.3593</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>5.8599</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>7.7352</td>
<td>16</td>
</tr>
</tbody>
</table>

For $m = 3$ and $m = 4$ we have equality, which means that the inductive lower bound is still not worse than the combinatoric one. But also some calculation shows that if $\alpha_m \geq \frac{1}{2} \left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$, then when $n$ is even we get $cr(K_{m,n}) \geq Z(m,n)$ and when $n$ is odd $cr(K_{m,n}) \geq Z(m,n) - \frac{1}{4} \left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor$.

Overall we can conclude that, for $5 \leq m \leq 9$:

$$cr(K_{m,n-2}) + (n-2)cr(K_{m,3}) > \frac{1}{2} n^2 \alpha_m - \frac{1}{2} n\left\lfloor \frac{1}{4} (m-1)^2 \right\rfloor.$$
If we now calculate an $\alpha_m'$ with the extra constraint that there is no pair of vertices $(a, b) \in B$ with $cr(a, b) = 0$ in the drawing $D_{m,n}$ which achieves the minimum, we have a possibly better lower bound:

$$cr(K_{m,n}) \geq \min \left\{ cr(K_{m,n-2}) + (n - 2)cr(K_{m,3}), \frac{1}{2}n^2\alpha_m' - \frac{1}{2}n\left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor \right\}$$

To calculate $\alpha_m'$ we need to translate the constraint "there is no pair of vertices $(a, b) \in B$ with $cr(a, b) = 0"$ to a statement about the cyclic orientations of pairs of vertices $(a, b) \in B$. One way to do this is to conclude that there are two options: either all pairs of vertices have at least one crossing, and therefore we can write $C_{i,j} = 1$ if $i = j^{-1}$ (instead of zero) to calculate $\alpha_m'$, or there is a pair of vertices without crossings between them and $cr(K_{m,n-2}) + (n - 2)cr(K_{m,3})$ is the lower bound.

A stricter way to do this is by using Turán's theorem to prove that there are no pairs of vertices with opposite orientations at all in drawings with $cr(D_{m,n}) < cr(K_{m,n-2}) + (n - 2)cr(K_{m,3})$. Then we can demand that $X_{i,j} = 0$ if $i = j^{-1}$, and calculate $\alpha''_m$. Please note that $cr(K_{m,3}) = \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor$.

**Theorem:** If there is a pair of vertices $(a, b) \in B$ with $\pi(a) = \pi(b)^{-1}$, then $cr(D_{m,n}) \geq cr(K_{m,n-2}) + (n - 2)cr(K_{m,3})$.

**Proof:** If we have a drawing $D_{m,n}$ where there is a pair of vertices $a, b \in B$ with opposite orientations ($A = \{1, 2, \ldots, m\}$, $B = \{a, b, c_1, c_2, \ldots, c_{n-2}\}$) we can write:

$$cr(D_{m,n}) = cr(c_1, c_2, \ldots, c_{n-2}) + \sum_{i=1}^{n-2} (cr(c_i, a, b) - cr(a, b)) + cr(a, b)$$

$$\geq cr(K_{m,n-2}) + \sum_{i=1}^{n-2} (cr(c_i, a, b) - cr(a, b)) + cr(a, b)$$

Here $cr(a, b, c)$ means the number of crossings in the subgraph of $D_{m,n}$ induced by the vertices $a, b, c$ and $\{1, 2, \ldots, m\}$ as in Chapter 3.

Assume that $cr(D_{m,n}) < cr(K_{m,n-2}) + (n - 2)cr(K_{m,3})$. Then the following inequality must hold:

$$\sum_{i=1}^{n-2} (cr(c_i, a, b) - cr(a, b)) + cr(a, b) < (n - 2)\left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor.$$ 

To prove this can not be the case we construct the graphs $G_i$ with vertices $j \in \{1, 2, \ldots, m\}$ and edges $(j, k)$ if and only if there is a crossing either between the edge $(c_i, j)$ and the path $(a, k), b)$, or between the edge $(c_i, k)$ and the path $(a, j, b)$ of $D_{m,n}$ for $i \in \{1, 2, \ldots, n-2\}$. Then $cr(c_i, a, b) - cr(a, b) \geq |EG_{G'}|$, the number of edges of $G_i$.

**Turán:** If $G$ is a graph with $m$ vertices without triangles, the number of edges is less or equal than $\frac{m^2}{2} - \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{1}{4}(m-1)^2 \right\rceil$ edges in the complementary graph.

As a consequence there are at least $\binom{n}{2} - \frac{m^2}{2} \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{1}{4}(m-1)^2 \right\rceil$ edges in the complementary graph.

If we assume that $|EG_{G'}| < \left\lceil \frac{1}{4}(m-1)^2 \right\rceil$ we know from Turán that there is a triangle in the complementary graph of $G_i$, which means $\exists j, k, l \in \{1, 2, \ldots, m\}$ such that the subgraph induced by $(c_i, j, k, l)$ has no crossings with the subgraph induced by $(a, b, j, k, l)$. But since there are only two orientations for $m = 3$, and $a$ and $b$ have opposite orientations, the orientation of $c_i$ regarding $(j, k, l)$ must be the same as one of theirs. But in that case we know from Woodall that there must be at least one crossing.
Therefore we know that $|EG_i| \geq \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor$ for all $i$ and that:

$$\sum_{i=1}^{n-2} (cr(c_i, a, b) - cr(a, b)) + cr(a, b) \geq \sum_{i=1}^{n-2} |EG_i| + cr(a, b) \geq (n-2)\left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor$$

Q.E.D.

Now we have proven we can add the constraint $X_{i,j} = 0$ if $i = j-1$ when calculating $\alpha''_m$.

We used mathematica to calculate $C$ for $m = 3$ to $m = 7$, some of the notebooks are appendices. Then we wrote the SDP problem in a certain format which was then transformed by a perl programme, sdp2neos.pl, written by A. Schrijver to the sparse SDP format. This result we then submitted to the Neos server online, which solved the problem and calculated $\alpha_m$, some outputs are also appendices. Of course we also calculated $\alpha'_m$ and $\alpha''_m$ for $m = 3$ to $m = 6$. It was not necessary to calculate the case $m = 7$ because of predictive results, and which also would have required a more complicated approach involving regular $*$-representations.
8 Results

We have calculated $\alpha'_3$ to $\alpha'_6$ and $\alpha''_3$ to $\alpha''_6$ with the following results:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\alpha_m$</th>
<th>$\alpha'_m$</th>
<th>$\alpha''_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.666</td>
<td>1.3333</td>
</tr>
<tr>
<td>5</td>
<td>1.9472</td>
<td>1.9472</td>
<td>1.9472</td>
</tr>
<tr>
<td>6</td>
<td>2.9519</td>
<td>2.9519</td>
<td>2.9519</td>
</tr>
</tbody>
</table>

The results for $m = 3$ and $m = 4$ were promising, but unfortunately there was no improvement from $m \geq 5$ where it starts to get interesting.

The results for $m = 3$ and $m = 4$ are still interesting though, since there are some conclusions that can be deduced:

If there is no pair $(a, b) \in B$ such that $\pi(a) = \pi(b) - 1$, $\alpha''_3$ gives us the following lower bound, $n \geq 2$:

$$\frac{1}{2}n^2\alpha''_3 - \frac{1}{2}n\left\lfloor\frac{1}{4}(3 - 1)^2\right\rfloor = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$> \frac{1}{4}n^2 - \frac{1}{2}n + \frac{1}{4}$$

$$\geq \left\lfloor\frac{1}{4}(n - 1)^2\right\rfloor$$

$$= Z(3, n)$$

If there is no pair $(a, b) \in B$ such that $\pi(a) = \pi(b) - 1$, $\alpha''_4$ gives us the following lower bound, $n \geq 3$:

$$\frac{1}{2}n^2\alpha''_4 - \frac{1}{2}n\left\lfloor\frac{1}{4}(4 - 1)^2\right\rfloor = \frac{1}{2}n^2\frac{4}{3} - \frac{1}{2}n \cdot 2$$

$$= \frac{2}{3}n^2 - n$$

$$> \frac{1}{2}n^2 - n + \frac{1}{2}$$

$$\geq 2\left\lfloor\frac{1}{4}(n - 1)^2\right\rfloor$$

$$= Z(4, n)$$

This means that for $m = 3$ (resp. $m = 4$) a drawing with $Z(3, n)$ (resp. $Z(4, n)$) crossings must always have a pair $(a, b) \in B$ such that $\pi(a) = \pi(b)^{-1}$ for all $n \geq 2$ (resp. $n \geq 3$).

In fact such a drawing consists solely of pairs with opposite orientations. We can see this by removing the pair $(a, b)$ as in Chapter 7 and assuming $cr(c_1, c_2, \ldots, c_{n-2}) > Z(m, n - 2)$, which gives us the inequality:

$$Z(m, n) = cr(D_{m,n})$$

$$= cr(c_1, c_2, \ldots, c_{n-2}) + \sum_{i=1}^{n-2} (cr(c_i, a) - cr(a, b)) + cr(a, b)$$

$$\geq cr(c_1, c_2, \ldots, c_{n-2}) + (n - 2)\left\lfloor\frac{1}{4}(m - 1)^2\right\rfloor$$

$$> Z(m, n - 2) + (n - 2)\left\lfloor\frac{1}{4}(m - 1)^2\right\rfloor$$

$$= Z(m, n)$$
which is a contradiction. Therefore $cr(c_1, c_2, \ldots, c_{n-2}) \leq Z(m, n - 2)$. For $m \leq 6$ this implies equality, and again there is a pair $c_i, c_j$ with opposite orientations and no crossings in the smaller drawing. The equation above also shows that $cr(a, b) = 0$ when $cr(D_{m,n}) = Z(m, n)$.

For $n = 3$, since there are only two orientations possible, we can conclude that there Zarankiewicz’s way of drawing it such that $cr(K_{3,n}) = Z(3, n)$ is unique.

For $n = 4$, with six possible orientations, it does not work that way. But here too we can conclude that any drawing $D_{4,n}$ with $cr(D_{4,n}) = Z(4, n)$ consists of pairs of vertices with opposite orientations and no internal crossings, except of course that when $n$ is odd we have one vertex left over.

In the figure above we show that there is a drawing of $D_{4,4}$ with $cr(D_{4,4}) = Z(4, 4) = 4$ with two different pairs of opposite orientations. Therefore Zarankiewicz’s way of drawing $K_{4,n}$ is not unique.
9 Conclusions

We must conclude we have not made any progress finding $cr(K_{7, 11})$ with our research, and have not increased the lower bound for any other unknown case of Zarankiewicz’s conjecture either. Apparently the new constraint "no pair of vertices with opposite orientations" is not strong enough.

We can also conclude however, that Zararankiewicz’s drawing of $K_{4, n}$ is unique to a great extent, and that a drawing of $K_{4, n}$ with $Z(4, n)$ crossings consists of $\lceil \frac{n}{2} \rceil$ pairs of vertices with opposite orientations (and, if $n$ is odd, one extra vertex).

Lastly we did show that Zarankiewicz’s inductive lower bound

$$cr(D_{m, n}) \geq cr(K_{m, n-2}) + (n - 2)cr(K_{m, 3})$$

also holds under the condition: there is no pair of vertices in $B$ "with opposite orientations" in $D_{m, n}$, instead of "without crossings".
A Mathematica notebook computing input for SDP $K_{4,n}$

<< Combinatorica"

f[x_] := Prepend[x, 0];

i = 1;
g[x_] := Append[x, i++];

A = Sort[Map[f, Permutations[Range[3]]]]

\{(0, 1, 2, 3), (0, 1, 3, 2), (0, 2, 1, 3), (0, 2, 3, 1), (0, 3, 1, 2), (0, 3, 2, 1)\}

aa = Map[g, Sort[Map[f, Permutations[Range[3]]]]]

\{(0, 1, 2, 3, 1), (0, 1, 3, 2, 2), (0, 2, 1, 3, 3), (0, 2, 3, 1, 4), (0, 3, 1, 2, 5), (0, 3, 2, 1, 6)\}

i = 1;

\(\text{a = Table[Permute[aa[[j]], \{1, 4, 3, 2, 5\}], \{j, 1, 6\}]\)}

\{(0, 3, 2, 1, 1), (0, 2, 3, 3, 2), (0, 3, 1, 2, 3), (0, 1, 3, 2, 4), (0, 2, 1, 3, 5), (0, 1, 2, 3, 6)\}

b = Sort[a]

\{(0, 1, 2, 3, 6), (0, 1, 3, 2, 4), (0, 2, 1, 3, 5), (0, 2, 3, 1, 2), (0, 3, 1, 2, 3), (0, 3, 2, 1, 1)\}

d = Table[b[[j]][[5]], \{j, 6\}]

\{6, 4, 5, 2, 3, 1\}

\(\text{h[i_] := d[[1]]}\)

\(\text{Table[i, h[i]], \{i, 6\}\)}

\{(1, 6), (2, 4), (3, 5), (4, 2), (5, 3), (6, 1)\}

P[1] = \{1, 3, 2, 4\};
P[2] = \{1, 2, 4, 3\};
P[3] = \{1, 4, 2, 3\};
P[4] = \{1, 3, 4, 2\};

B = Table[Permute[A[[k]], P[[j]]], \{k, 6\}, \{j, 4\}]

\{(0, 2, 1, 3), (0, 2, 1, 3), (0, 2, 1, 3), (0, 2, 1, 3), (0, 2, 1, 3), (0, 2, 1, 3)\}

Clear[k, j, i];

DionQ[k_, l_] :=

Catch[q; For[j = 1, j < 5, j++, If[B[[k]][[j]] == A[[l]], Throw[0]]]]

c = Table[DionQ[k, g], \{k, 6\}, \{g, 6\}]

\{(0, 1, 1, 1, 1, 0), (1, 0, 1, 0, 1, 1), (1, 1, 0, 1, 0, 1), (1, 0, 1, 1, 0, 1), (0, 1, 1, 1, 1, 0)\}

Needs["GraphUtilities"]

merlijn = FromAdjacencyMatrix[c]

- Graph:<12,6,Undirected>
ShowGraph[merlijn]

cce = GraphDistanceMatrix[c]

{0., 1., 1., 1., 1., 2.}, {1., 0., 1., 2., 1., 1.}, {1., 1., 0., 1., 2., 1.}, 
{1., 2., 1., 0., 1., 1.}, {1., 1., 2., 0., 1.}, {2., 1., 1., 1., 1., 0.}]

Sum[Sum[cce[[ij]][[hi]] x[i][j], {i, 1, 6}], {j, 1, 6}]

4 x[1][1] + 2 x[1][2] + 2 x[1][3] + 2 x[1][4] + 2 x[1][5] + 0 x[1][6] + 

Table[x[i][j], {i, 1, 6}, {j, 1, 6}]

{(x[1][1], x[1][2], x[1][3], x[1][4], x[1][5], x[1][6]), 
(x[2][1], x[2][2], x[2][3], x[2][4], x[2][5], x[2][6]), 
(x[3][1], x[3][2], x[3][3], x[3][4], x[3][5], x[3][6]), 
(x[4][1], x[4][2], x[4][3], x[4][4], x[4][5], x[4][6]), 
(x[5][1], x[5][2], x[5][3], x[5][4], x[5][5], x[5][6]), 
(x[6][1], x[6][2], x[6][3], x[6][4], x[6][5], x[6][6])}

Sum[2 x[i][j], {i, 1, 6}, {j, 1, 6}]

2 x[1][1] + 2 x[1][2] + 2 x[1][3] + 2 x[1][4] + 2 x[1][5] + 2 x[1][6] + 
B  Sparse SDP input for calculating $\alpha_4$

Minimize

\begin{align*}
& 2 \ x_1_1 + 2 \ x_1_2 + 2 \ x_1_3 + 2 \ x_1_4 + 2 \ x_1_5 + \\
& 0 \ x_1_6 + 2 \ x_2_2 + 2 \ x_2_3 + 0 \ x_2_4 + 2 \ x_2_5 + \\
& 2 \ x_2_6 + 2 \ x_3_3 + 2 \ x_3_4 + 0 \ x_3_5 + 2 \ x_3_6 + \\
& 2 \ x_4_4 + 2 \ x_4_5 + 2 \ x_4_6 + 2 \ x_5_5 + 2 \ x_5_6 + \\
& 2 \ x_6_6
\end{align*}

Subject to

\begin{align*}
x_1_1, x_1_2, x_1_3, x_1_4, x_1_5, x_1_6; \\
x_2_2, x_2_3, x_2_4, x_2_5, x_2_6; \\
x_3_3, x_3_4, x_3_5, x_3_6; \\
x_4_4, x_4_5, x_4_6; \\
x_5_5, x_5_6; \\
x_6_6 \geq 0
\end{align*}

\begin{align*}
x_1_1 &= 0 \\
x_1_2 &= 0 \\
x_1_3 &= 0 \\
x_1_4 &= 0 \\
x_1_5 &= 0 \\
x_1_6 &= 0 \\
x_2_2 &= 0 \\
x_2_3 &= 0 \\
x_2_4 &= 0 \\
x_2_5 &= 0 \\
x_2_6 &= 0 \\
x_3_3 &= 0 \\
x_3_4 &= 0 \\
x_3_5 &= 0 \\
x_3_6 &= 0 \\
x_4_4 &= 0 \\
x_4_5 &= 0 \\
x_4_6 &= 0 \\
x_5_5 &= 0 \\
x_5_6 &= 0 \\
x_6_6 &= 0
\end{align*}

\begin{align*}
x_1_1 + 2 \ x_1_2 + 2 \ x_1_3 + 2 \ x_1_4 + 2 \ x_1_5 + \\
2 \ x_1_6 + x_2_2 + 2 \ x_2_3 + 2 \ x_2_4 + 2 \ x_2_5 + \\
2 \ x_2_6 + x_3_3 + 2 \ x_3_4 + 2 \ x_3_5 + 2 \ x_3_6 + \\
x_4_4 + 2 \ x_4_5 + 2 \ x_4_6 + x_5_5 + 2 \ x_5_6 + x_6_6 - 1 \geq 0
\end{align*}
C  Sparse SDP input for calculating $\alpha'_4$

\begin{align*}
\text{Minimize} \\
2 \ x_{1.1} + 2 \ x_{1.2} + 2 \ x_{1.3} + 2 \ x_{1.4} + 2 \ x_{1.5} + \\
2 \ x_{1.6} + 2 \ x_{2.2} + 2 \ x_{2.3} + 2 \ x_{2.4} + 2 \ x_{2.5} + \\
2 \ x_{2.6} + 2 \ x_{3.3} + 2 \ x_{3.4} + 2 \ x_{3.5} + 2 \ x_{3.6} + \\
2 \ x_{4.4} + 2 \ x_{4.5} + 2 \ x_{4.6} + 2 \ x_{5.5} + 2 \ x_{5.6} + \\
2 \ x_{6.6} \\
\text{Subject to} \\
x_{1.1}, x_{1.2}, x_{1.3}, x_{1.4}, x_{1.5}, x_{1.6}; \\
x_{2.2}, x_{2.3}, x_{2.4}, x_{2.5}, x_{2.6}; \\
x_{3.3}, x_{3.4}, x_{3.5}, x_{3.6}; \\
x_{4.4}, x_{4.5}, x_{4.6}; \\
x_{5.5}, x_{5.6}; \\
x_{6.6} \geq 0
\end{align*}

\begin{align*}
x_{1.1} &\geq 0 \\
x_{1.2} &\geq 0 \\
x_{1.3} &\geq 0 \\
x_{1.4} &\geq 0 \\
x_{1.5} &\geq 0 \\
x_{1.6} &\geq 0 \\
x_{2.2} &\geq 0 \\
x_{2.3} &\geq 0 \\
x_{2.4} &\geq 0 \\
x_{2.5} &\geq 0 \\
x_{2.6} &\geq 0 \\
x_{3.3} &\geq 0 \\
x_{3.4} &\geq 0 \\
x_{3.5} &\geq 0 \\
x_{3.6} &\geq 0 \\
x_{4.4} &\geq 0 \\
x_{4.5} &\geq 0 \\
x_{4.6} &\geq 0 \\
x_{5.5} &\geq 0 \\
x_{5.6} &\geq 0 \\
x_{6.6} &\geq 0
\end{align*}

\begin{align*}
x_{1.1} + 2 \ x_{1.2} + 2 \ x_{1.3} + 2 \ x_{1.4} + 2 \ x_{1.5} + \\
2 \ x_{1.6} + x_{2.2} + 2 \ x_{2.3} + 2 \ x_{2.4} + 2 \ x_{2.5} + \\
2 \ x_{2.6} + x_{3.3} + 2 \ x_{3.4} + 2 \ x_{3.5} + 2 \ x_{3.6} + \\
x_{4.4} + 2 \ x_{4.5} + 2 \ x_{4.6} + x_{5.5} + 2 \ x_{5.6} + x_{6.6} - 1 \geq 0
\end{align*}
D Sparse SDP input for calculating $\alpha_4''$

Minimize
\[
2 \ x_1 \_1 + 2 \ x_1 \_2 + 2 \ x_1 \_3 + 2 \ x_1 \_4 + 2 \ x_1 \_5 + \\
2 \ x_2 \_2 + 2 \ x_2 \_3 + 2 \ x_2 \_5 + 2 \ x_2 \_6 + 2 \ x_3 \_3 + \\
2 \ x_3 \_4 + 2 \ x_3 \_6 + 2 \ x_4 \_4 + 2 \ x_4 \_5 + 2 \ x_4 \_6 + \\
2 \ x_5 \_5 + 2 \ x_5 \_6 + 2 \ x_6 \_6
\]

Subject to
\[
x_1 \_1, x_1 \_2, x_1 \_3, x_1 \_4, x_1 \_5, ; \\
, x_2 \_2, x_2 \_3, , x_2 \_5, x_2 \_6; \\
, , x_3 \_3, x_3 \_4, , x_3 \_6; \\
, , , x_4 \_4, x_4 \_5, x_4 \_6; \\
, , , , x_5 \_5, x_5 \_6; \\
, , , , , x_6 \_6 \geq 0
\]

\[
x_1 \_1 \geq 0 \\
x_1 \_2 \geq 0 \\
x_1 \_3 \geq 0 \\
x_1 \_4 \geq 0 \\
x_1 \_5 \geq 0 \\
x_2 \_2 \geq 0 \\
x_2 \_3 \geq 0 \\
x_2 \_5 \geq 0 \\
x_2 \_6 \geq 0 \\
x_3 \_3 \geq 0 \\
x_3 \_4 \geq 0 \\
x_3 \_6 \geq 0 \\
x_4 \_4 \geq 0 \\
x_4 \_5 \geq 0 \\
x_4 \_6 \geq 0 \\
x_5 \_5 \geq 0 \\
x_5 \_6 \geq 0 \\
x_6 \_6 \geq 0
\]
### Output NEOS server for $\alpha_5$

<table>
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<th>part of output NEOS server</th>
<th>approximation</th>
</tr>
</thead>
<tbody>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
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<td>1 1 1 4 1.009629413221890468e-03</td>
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</tr>
<tr>
<td>1 1 1 5 1.009629309627346283e-03</td>
<td>0.001</td>
</tr>
<tr>
<td>1 1 1 6 4.16666352718987124e-03</td>
<td>0.004</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>0</td>
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<tr>
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</tr>
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F  Output NEOS server for $\alpha''_5$

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References


