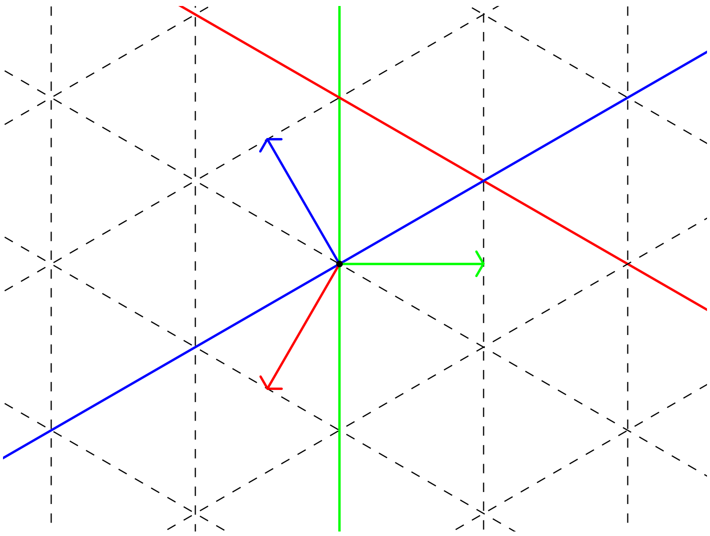
Affine Lie algebras and affine root systems

A Killing-Cartan type classification of affine Lie algebras, reduced irreducible affine root systems and affine Cartan matrices

Jan S. Nauta



MSc Thesis

under supervision of
Dr. J.V. Stokman

Universiteit van Amsterdam

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A Killing-Cartan type classification of affine Lie algebras, reduced irreducible affine root systems and affine Cartan matrices

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Author: Jan S. Nauta
Supervisor: Dr. Jasper Stokman
Second reader: Dr. Hessel Posthuma
Member of examination board: Prof. Dr. Gerard van der Geer
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Abstract

In this thesis we will construct a commutative triangle of canonical bijections between the isomorphism classes of Lie algebras isomorphic to affine Lie algebras, the similarity classes of reduced irreducible affine root systems and the affine Cartan matrices up to simultaneous permutations of rows and columns. Together with a classification of affine Cartan matrices up to simultaneous permutations of rows and columns this classifies all three types of mathematical objects. The construction of the triangle will be based on the Killing-Cartan classification of semisimple Lie algebras. After setting up the axiomatic theory of affine root systems from scratch, we will explicitly show that each irreducible affine root system gives rise to an affine Cartan matrix and that the set of real roots of an affine Lie algebra can be naturally viewed as the associated affine root system of the affine Lie algebra.

jan_nauta@hotmail.com

Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam
Science Park 904, 1098 XH Amsterdam

He exhibited the characteristic equation of an arbitrary element of the Weyl group when Weyl was 3 years old and listed the orders of the Coxeter transformation 19 years before Coxeter was born!

A.J. Coleman about W.K.J. Killing, 1989

Contents

Introduction

The classification of semisimple finite-dimensional Lie algebras over the complex numbers was settled by Killing and Cartan at the end of the nineteenth century. This celebrated classification is based on classifying objects that are directly related to semisimple Lie algebras, namely reduced root systems and Cartan matrices. It is considered a milestone in the history of mathematics. The classification can be summarized by the following commutative triangle of bijections Δ, A and $\mathfrak{g}$

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\quad\Delta\quad} & \mathscr{R} \\ & {\scriptstyle\mathfrak{g}}\nwarrow \qquad \swarrow{\scriptstyle A} & \\ & \mathscr{C} & \end{array} \tag{0.0.1}$$

together with a classification of indecomposable Cartan matrices up to simultaneous permutations of rows and columns. In (0.0.1) we have denoted by $\mathscr{L}$ the isomorphism classes of simple Lie algebras, by $\mathscr{R}$ the similarity classes of reduced irreducible root systems and by $\mathscr{C}$ the indecomposable Cartan matrices up to simultaneous permutations of rows and columns. The maps Δ, A and $\mathfrak{g}$ will be made explicit in Chapter 1.

In 1966, Serre showed that each semisimple Lie algebra is defined up to isomorphism by finitely many generators and relations that only depend on the entries of the corresponding Cartan matrix (this actually leads to the map $\mathfrak{g} : \mathscr{C} \to \mathscr{L}$). This inspired Kac and Moody in 1967 to independently start the study of Lie algebras on generators that satisfy a natural generalization of the relations of Serre with slightly weaker conditions on the corresponding Cartan matrix. The most important of such Lie algebras are now called Kac-Moody algebras, and apart from semisimple Lie algebras Kac-Moody algebras are infinite-dimensional. One particularly interesting Kac-Moody algebra is the affine Lie algebra which is a Kac-Moody algebra associated to an affine Cartan matrix. An affine Cartan matrix has all the properties of the Cartan matrix from the case of simple Lie algebras, except that is has determinant zero. This makes an affine Lie algebra an infinite-dimensional generalization of a simple Lie algebra.

In this thesis we want to establish a commutative triangle similar to (0.0.1) for affine Lie algebras. In this new triangle Cartan matrices will be replaced by affine Cartan matrices and reduced irreducible root systems by reduced irreducible affine root systems. Furthermore, we will give a classification of affine Cartan matrices up to simultaneous permutations of rows and columns analogous to the classification of indecomposable Cartan matrices up to simultaneous permutations of rows and columns. Together with the new triangle this also classifies similarity classes of reduced irreducible affine root systems and isomorphism classes of affine Lie algebras.

In 1983 Peterson and Kac solved the 'isomorphism problem' for symmetrizable Kac-Moody algebras. They showed in particular that affine Lie algebras up to isomorphism correspond bijectively to affine Cartan matrices up to simultaneous permutations of rows and columns. This takes care of the map $\mathfrak{g}$ for affine Cartan matrices and affine Lie algebras. In the seventies Macdonald has independently studied affine root systems. We will study a slightly adapted version of the theory of affine root systems as was known to Macdonald in full detail. In this manuscript we will explicitly construct the map A for reduced irreducible affine root systems and affine Cartan matrices. Finally, we establish a reduced irreducible affine root system R related to an affine Lie algebra L in a canonical way. We will show that if we take $\Delta := (\mathfrak{g} \circ A)^{-1}$, then the triangle commutes such that $\Delta(\overline{L}) = \overline{R}$. Here $\overline{L}$ is the isomorphism class of L and $\overline{R}$ is the similarity class of R. This shows that every affine Lie algebra up to isomorphism is uniquely determined by its corresponding reduced irreducible affine root system up to similarity. Finally I would like to note that since affine Lie algebras are explicitly constructed as Lie algebras on generators and relations, we will show how to generalize the structures that we define on affine Lie algebras to all Lie algebras isomorphic to affine Lie algebras. However for simplicity we will most of the time only work with affine Lie algebras.

Outline of this thesis

Chapter 1 consists of a brief summary of the classification of all semisimple finite-dimensional Lie algebras over the complex numbers up to isomorphism through the use of root systems and Cartan matrices as originated in the works of Killing and Cartan. We will establish the canonical bijective correspondence between simple Lie algebras, reduced irreducible root systems and indecomposable Cartan matrices up to appropriate isomorphism equivalences in the form of a commutative triangle (0.0.1). Classifying the indecomposable Cartan matrices then finishes the classification of semisimple Lie algebras. Along this Chapter we will establish some definitions and results about Lie algebras and root systems that will be useful in the remaining Chapters.

In Chapter 2 we will introduce Kac-Moody algebras, affine Lie algebras and the necessary tools to study root systems of Kac-Moody algebras such as the Weyl group, the invariant nondegenerate symmetric bilinear form and real and imaginary roots. We show that Kac-Moody algebras are a generalization of semisimple Lie algebras, and show that all finite-dimensional Kac-Moody algebras are semisimple Lie algebras. We give the classification of affine Cartan matrices, and show that affine Lie algebras up to isomorphism correspond bijectively to affine Cartan matrices up to simultaneous permutations of rows and columns. Finally, we will study the invariant bilinear form, the Weyl group and the real roots of affine Lie algebras. We will observe that the real roots of an affine Lie algebra satisfy axioms that generalize the axioms of a reduced irreducible finite root system. This will justify the study of reduced irreducible affine root systems in Chapter 3.

Chapter 3 consists of a detailed axiomatic study of affine root systems with the goal of classifying them. First we introduce useful notions as reducedness, irreducibility and similarity that are analogous to the case of finite root systems. Then we study the geometry of affine root systems and the affine Weyl group to obtain a special set of generators of an irreducible affine root system. Using these generators we will relate affine Cartan matrices up to simultaneous permutations of rows and columns bijectively to the similarity classes of reduced irreducible affine root systems. The classification of affine Cartan matrices then also classifies the reduced irreducible root systems up to similarity. Along the way we will explicitly realize a complete set of representatives for the similarity classes of reduced irreducible affine root systems. We end the Chapter by putting together the commutative triangle that gives the canonical bijections between isomorphism classes of Lie algebras that are isomorphic to affine Lie algebras, similarity classes of reduced irreducible affine root systems and affine Cartan matrices up to simultaneous permutations of rows and columns.

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Notations and conventions

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the natural numbers, integers, rationals, reals and complex numbers respectively.
$\mathbb{R}_{\neq 0}$ denotes the real numbers apart from zero.
$\coprod$ denotes a disjoint union of sets.
$A \ltimes B$ denotes the semidirect product of (sub)groups A and B (of a group G) with B normal.
$\delta_{ij} := \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ is the Kronecker delta.
K° is the interior and $\overline{K}$ is the closure of the subset K in a topological space.
V^* denotes the dual of a vector space V.

Chapter 1

The classification of semisimple Lie algebras

In this Chapter we want to give a brief exposition of the classification of all semisimple Lie algebras over $\mathbb{C}$ up to isomorphism through the use of root systems and Cartan matrices as originated in the works of Killing and Cartan (see [4]). After setting up the necessary theory we will establish a canonical bijective correspondence between simple Lie algebras, irreducible root systems and indecomposable Cartan matrices up to appropriate isomorphism equivalences in the form of a commutative triangle. Classifying the indecomposable Cartan matrices then finishes the classification of semisimple Lie algebras. Along this Chapter we will go through some definitions and results on Lie algebras and root systems that will be useful in the remaining of this manuscript. All results quoted in this Chapter can be found in [6] except for the results on nonreduced root systems which come from [13] and the axiomatic definition of a Cartan matrix which comes from [8].

1.1 Lie algebras

A Lie algebra arises naturally as a vector space of linear transformations with the commutator of two linear transformations as a product on it. This product is bilinear but in general neither commutative nor associative. In this Section we introduce the abstract notion of a Lie algebra, and we go through some basic algebraic tools that will be useful throughout this manuscript.

Definition 1.1.1. A vector space $\mathfrak{g}$ over $\mathbb{C}$ together with a bilinear operation $[.,.]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$ is called a *Lie algebra* if the operation $[.,.]$ satisfies the following two conditions:
(1) $[x, x] = 0$ for all $x \in \mathfrak{g}$;
(2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for all $x, y, z \in \mathfrak{g}$ (*Jacobi identity*).
The operation $[.,.]$ is called the (*Lie*) *bracket* of $\mathfrak{g}$.

Note that condition (1) of Definition 1.1.1 together with the bilinearity of the bracket implies that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$, hence the bracket of a Lie algebra is *anticommutative*. A Lie algebra $\mathfrak{g}$ is called *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. Using the anticommutativity of $[.,.]$ this is equivalent to $[x, y] = [y, x]$ for all $x, y \in \mathfrak{g}$.

Example 1.1.2. Let V be a finite-dimensional complex vector space, and let $\mathrm{End}(V)$ denote the set of linear endomorphisms of V. Then $\mathrm{End}(V)$ is a complex vector space of dimension $\dim(V)^2$ and a ring with composition of maps as multiplication. Furthermore, End(V) becomes a Lie algebra with the bracket $[x,y]=xy-yx$ and is called the *general linear algebra* which is denoted by $\mathfrak{gl}(V)$.

A linear map $\phi:\mathfrak{g}_1\to\mathfrak{g}_2$ between Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is called a (*Lie algebra*) *homomorphism* if ϕ is compatible with the Lie brackets of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, i.e. $[\phi(x),\phi(y)]=\phi([x,y])$ for all $x,y\in\mathfrak{g}_1$. In particular, if $\mathfrak{g}_1=\mathfrak{g}_2$ then ϕ is called a (*Lie algebra*) *endomorphism*. Two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are said to be *isomorphic* if there exist a linear isomorphism $\phi:\mathfrak{g}_1\to\mathfrak{g}_2$ such that ϕ is compatible with the Lie brackets of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Then the map ϕ is called a (*Lie algebra*) *isomorphism*. Write $\operatorname{ad} x:\mathfrak{g}\to\mathfrak{g}$ for the map $y\mapsto[x,y]$ for some fixed $x\in\mathfrak{g}$, then it follows from the axioms of a Lie algebra that $\operatorname{ad} x$ is actually a Lie algebra endomorphism of $\mathfrak{g}$.

A (*Lie*) *subalgebra* is a vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ such that $[x,y]\in\mathfrak{h}$ for all $x,y\in\mathfrak{h}$. A subspace $\mathfrak{h}\subset\mathfrak{g}$ such that $[x,y]\in\mathfrak{h}$ for all $x\in\mathfrak{h}$ and $y\in\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$. Notice that an ideal is also a subalgebra, but not the other way around. A nontrivial example of an ideal of a Lie algebra $\mathfrak{g}$ is the *center* $Z(\mathfrak{g}):=\{x\in\mathfrak{g}:[x,y]=0 \text{ for all } y\in\mathfrak{g}\}$. Let $\tau_1,\dots,\tau_r$ be ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is said to be the *direct sum* of $\tau_1,\dots,\tau_r$, and we write $\mathfrak{g}=\tau_1\oplus\cdots\oplus\tau_r$, if $\mathfrak{g}$ is the direct sum of $\tau_1,\dots,\tau_r$ as vector spaces. This condition forces $[x_i,x_j]=0$ for all $x_i\in\tau_i$ and $x_j\in\tau_j$ for $i\neq j$, hence the bracket of $\mathfrak{g}$ acts componentwise on the direct sum.

Analogous to other algebraic theories, we have the notion of the *quotient Lie algebra* $\mathfrak{g}/\tau$ of a Lie algebra $\mathfrak{g}$ and an ideal $\tau\subset\mathfrak{g}$. Here $\mathfrak{g}/\tau$ coincides with the quotient space together with a bilinear operation $[x+\tau,y+\tau]_\tau:=[x,y]+\tau$ for all $x+\tau,y+\tau\in\mathfrak{g}/\tau$. It is not hard to see that this operation is well defined, and that it turns $\mathfrak{g}/\tau$ into a Lie algebra with bracket $[.,.]_\tau$ in a natural way.

A Lie algebra $\mathfrak{g}$ is said to be *free* on a subset $X\subset\mathfrak{g}$ if for any given Lie algebra $\mathfrak{g}'$ together with an injection $\phi:X\hookrightarrow\mathfrak{g}'$ there exists a unique Lie algebra homomorphism $\psi:\mathfrak{g}\to\mathfrak{g}'$ such that $\psi|_X=\phi$. If $\mathfrak{g}$ is a free Lie algebra on X, then it follows straightforwardly that $\mathfrak{g}$ is the unique free Lie algebra on X up to isomorphism. For the existence of a free Lie algebra on a set X consider the complex vector space V having X as basis. Let $T^0V=\mathbb{C}$, put $T^mV=V\otimes\cdots\otimes V$ (m copies of V) for $m\in\mathbb{N}$ and define the vector space $\mathfrak{T}(V):=\bigoplus_{i=0}^{\infty}T^iV$. Further, introduce an associative product on the homogeneous generators of $\mathfrak{T}(V)$ by

$$(v_1\otimes\cdots\otimes v_k)(w_1\otimes\cdots\otimes w_m)=v_1\otimes\cdots\otimes v_k\otimes w_1\otimes\cdots\otimes w_m\in T^{k+m}V$$

and extend this bilinearly to an associative product on all of $\mathfrak{T}(V)$. Then $\mathfrak{T}(V)$ is an associative graded algebra with 1 which is generated by 1 along with X, and it is called the *tensor algebra* on V. Consider $\mathfrak{T}(V)$ as Lie algebra with the bracket $[x,y]:=x\otimes y-y\otimes x$ and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{T}(V)$ generated by X, then it turns out that $\mathfrak{g}$ is a free Lie algebra on X.

Let $\mathfrak{g}$ be a free Lie algebra on X and let τ be an ideal of $\mathfrak{g}$ generated as subalgebra of $\mathfrak{g}$ by elements k_i for i in an index set I. Consider the canonical Lie algebra homomorphism $\phi:\mathfrak{g}\to\mathfrak{g}/\tau$, then the quotient Lie algebra $\mathfrak{g}/\tau$ is said to be the Lie algebra with *generators* $\phi(x)$ for $x\in X$ and *relations* $\phi(k_i)=0$ for $i\in I$.

1.2 Root systems

In the following we will define semisimple Lie algebras, Cartan subalgebras and the Killing form. We show that a semisimple Lie algebra together with a choice of a Cartan subalgebra gives rise to a finite set of vectors in a Euclidean space that is invariant under a reflection group that they generate themselves called the Weyl group. This set of vectors is known as a root system. We will treat the theory of root systems independent of semisimple Lie algebras, and classify reduced irreducible root systems according to their Cartan matrix. Finally, we will discuss the class of nonreduced irreducible root systems.

1.2.1 Root systems of semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ over a finite-dimensional complex vector space is called *simple* if the only ideals of $\mathfrak{g}$ are $\{0\}$ and itself, and if $\mathfrak{g}$ is not abelian. A Lie algebra that can be written as a direct sum of finitely many simple Lie algebras is called *semisimple*[1].

Example 1.2.1. Consider the general linear algebra $\mathfrak{gl}(V)$ for a finite-dimensional complex vector space V. Any subalgebra of $\mathfrak{gl}(V)$ is called a *linear Lie algebra*. There are four important families of linear Lie algebras A_l $(l \geq 1)$, B_l $(l \geq 1)$, C_l $(l \geq 1)$ and D_l $(l \geq 2)$ which turn out to be simple and together they are called the *classical Lie algebras*:

A_l: Let $\dim(V) = l+1$ and let $\mathfrak{sl}(l+1,\mathbb{C})$ denote the subalgebra of $\mathfrak{gl}(V)$ of linear transformations with trace 0. This Lie algebra is called the *special linear algebra*, and is of dimension $(l+1)^2 - 1$.

B_l: Let $\dim(V) = 2l+1$ and choose a basis $\{v_1, \ldots, v_{2l+1}\}$ of V. Let f be the nondegenerate symmetric bilinear form on V defined by the matrix $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$. Define $\mathfrak{o}(2l+1,\mathbb{C})$ to be the subalgebra of $\mathfrak{gl}(V)$ of linear transformations $x \in \mathfrak{gl}(V)$ such that $f(x(v), w) = -f(v, x(w))$ for all $v, w \in V$. The Lie algebra $\mathfrak{o}(2l+1,\mathbb{C})$ is called the *orthogonal algebra* (with V of odd dimension!) and is of dimension $2l^2 + l$.

C_l: Let $\dim(V) = 2l$ and choose a basis $\{v_1, \ldots, v_{2l}\}$ of V. Let f be the nondegenerate symmetric bilinear form on V defined by the matrix $s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. Define $\mathfrak{sp}(2l,\mathbb{C})$ to be the subalgebra of $\mathfrak{gl}(V)$ of linear transformations $x \in \mathfrak{gl}(V)$ such that $f(x(v), w) = -f(v, x(w))$ for all $v, w \in V$. The Lie algebra $\mathfrak{sp}(2l,\mathbb{C})$ is called the the *symplectic algebra* and is of dimension $2l^2 + l$.

D_l: Let $\dim(V) = 2l$ and choose a basis $\{v_1, \ldots, v_{2l}\}$ of V. Let f be the nondegenerate symmetric bilinear form on V defined by the matrix $s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. Define $\mathfrak{o}(2l,\mathbb{C})$ to be the subalgebra of $\mathfrak{gl}(V)$ of linear transformations $x \in \mathfrak{gl}(V)$ such that $f(x(v), w) = -f(v, x(w))$ for all $v, w \in V$. The Lie algebra $\mathfrak{o}(2l,\mathbb{C})$ is called the *orthogonal algebra* (with V of even dimension!) and is of dimension $2l^2 - l$.

It turns out that one can define an invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)_{\mathfrak{g}}$ on a semisimple Lie algebra $\mathfrak{g}$ called the *Killing form* by $(x,y)_{\mathfrak{g}} = \mathrm{tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$ for all $x, y \in \mathfrak{g}$. Here *invariant* means that $([x,y],z)_{\mathfrak{g}} = (x,[y,z])_{\mathfrak{g}}$ for all $x,y,z \in \mathfrak{g}$, *nondegenerate* means that there exists no nonzero $x \in \mathfrak{g}$ such that $(x,y)_{\mathfrak{g}} = 0$ for all $y \in \mathfrak{g}$ and 'tr' denotes the trace of a linear endomorphism of $\mathfrak{g}$.

Consider a semisimple Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called *ad-semisimple* if ad x is a diagonalizable endomorphism of $\mathfrak{g}$ with respect to a suitable basis of $\mathfrak{g}$ as vector space. Fix a maximal subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting only of ad-semisimple elements. Such a subalgebra $\mathfrak{h}$ is said to be a *Cartan subalgebra* of $\mathfrak{g}$. One can show that $\mathfrak{h} \neq \{0\}$ and that $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$. Hence the Jacobi identity yields that $H := \{\mathrm{ad}\ h : h \in \mathfrak{h}\}$ is a commutating set of endomorphisms of $\mathfrak{g}$. By a standard result of linear algebra one observes that the endomorphisms of H are simultaneously diagonalizable. This leads to the *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, i.e. as vector

[1] This is actually not the definition of a semisimple Lie algebra from [6], but an equivalent characterization which can also be found in [6].

spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha \right),$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

is called the *root space* associated to $\alpha \in \mathfrak{h}^*$ and

$$\Delta(\mathfrak{g},\mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

is called the *root system* with respect to $\mathfrak{h}$. The $\alpha \in \Delta$ are called *roots*. When there is no ambiguity about $\mathfrak{g}$ and $\mathfrak{h}$ we will write Δ for $\Delta(\mathfrak{g},\mathfrak{h})$.

The Killing form $(.,.)_\mathfrak{g}$ turns out to be is nondegenerate when restricted to $\mathfrak{h}$, hence we can define the linear isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ depending on $(.,.)_\mathfrak{g}$ by

$$\nu(h)(h') = (h,h')_\mathfrak{g}$$

for $h, h' \in \mathfrak{h}$. This leads to a nondegenerate symmetric bilinear form $(.,.)$ on $\mathfrak{h}^*$ defined by

$$(\alpha,\beta) := (\nu^{-1}(\alpha), \nu^{-1}(\beta))_\mathfrak{g}$$

for $\alpha, \beta \in \mathfrak{h}^*$.

Proposition 1.2.2. *The root system Δ of of semisimple Lie algebra $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$ satisfies the following four conditions:*

(1) *Δ is finite, does not contain 0 and spans an $\mathbb{R}$-vector space $V \subset \mathfrak{h}^*$ such that* $\dim_\mathbb{R}(V) = \dim_\mathbb{C} \mathfrak{h}^*$ *and the form $(.,.)$ on $\mathfrak{h}^*$ restricted to V defines an inner product on V;*
(2) $\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Delta$ *for all* $\alpha, \beta \in \Delta$*;*
(3) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Delta$*;*
(4) *If $\alpha \in \Delta$, then* $\mathbb{R}\alpha \cap \Delta = \{\alpha, -\alpha\}$.

1.2.2 An axiomatic approach of root systems

To obtain a classification of semisimple Lie algebras it turns out to be useful to study root systems independently of their semisimple Lie algebra, and to classify them. We will briefly discuss the axiomatic theory of root systems in this Subsection, and do the classification in the next Subsection.

Call an $\mathbb{R}$-vector space V of dimension $l < \infty$ endowed with a positive definite symmetric bilinear form, or inner product, $(.,.)$ a *Euclidean space*.

Definition 1.2.3. A subset Δ of a Euclidean space V that is endowed with the inner product $(.,.)$ is called a *root system* if the following three conditions are satisfied:

(1) Δ is finite, does not contain 0 and spans V;
(2) $\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Delta$ for all $\alpha, \beta \in \Delta$;
(3) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.

If the following condition is also satisfied we say that Δ is a *reduced* root system, otherwise Δ is called *nonreduced*:

(4) If $\alpha \in \Delta$, then $\mathbb{R}\alpha \cap \Delta = \{\alpha, -\alpha\}$.

The dimension of V is called the *rank* of Δ.

From the Proposition 1.2.2 we observe that a root system of a semisimple Lie algebra is a reduced root system in the sense of Definition 1.2.3. Furthermore, we will see in Section 1.3 that each reduced root system Δ gives rise to a semisimple Lie algebra that has Δ as its corresponding root system. In the following we will assume root systems to satisfy conditions (1)-(3) of Definition 1.2.3, and state explicitly when a result only holds for a reduced root system.

For a nonzero vector $\alpha \in V$ the *orthogonal reflection* in the hyperplane $H_\alpha = \{v \in V : (v, \alpha) = 0\}$ orthogonal to α is the map $w_\alpha : V \to V$ defined by

$$w_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha.$$

Clearly, w_α is a linear isometry that fixes H_α and sends α to $-\alpha$. Let $W_0(\Delta)$ denote the group of linear transformations of V generated by the orthogonal reflections w_α for $\alpha \in \Delta$. We will call this group the *Weyl group* of Δ. Condition (2) of Definition 1.2.3 is then equivalent to saying that $W_0(\Delta)$ stabilizes Δ, and together with condition (3) we notice that the $\mathbb{Z}$-span of Δ is a $W_0(\Delta)$-stable lattice in V.

We call two root systems $\Delta \subset V$ and $\Delta' \subset V'$ *similar*, and write $\Delta \simeq \Delta'$, if there exists a linear isomorphism $\psi : V \xrightarrow{\sim} V'$ such that

$$2\frac{(\psi(\alpha),\psi(\beta))_{V'}}{(\psi(\alpha),\psi(\alpha))_{V'}} = 2\frac{(\alpha,\beta)_V}{(\alpha,\alpha)_V}$$

for all $\alpha, \beta \in \Delta$ which restricts to a bijection of Δ onto Δ'. Here $(.,.)_V$ (resp. $(.,.)_{V'}$) is the inner product on V (resp. V'). Similarity yields an equivalence relation on the collection of all root systems of which we call the equivalence classes *similarity classes* of root systems.

A root system Δ is said to be *irreducible* if Δ can not be written as the disjoint union of two nonempty subsets $\Delta_1, \Delta_2 \subset \Delta$ such that $(\alpha,\beta) = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, otherwise Δ is called *reducible*. For example, the root system of a simple Lie algebra is irreducible (and reduced as we already noted). Let Δ be reducible and let $\Delta = \coprod_{i=1}^r \Delta_i$ be a partition of Δ such that $(\Delta_i, \Delta_j) = 0$ for all $i \neq j$, then it turns out that each Δ_i is a root system in $\mathrm{span}_{\mathbb{R}}(\Delta_i) \subset V$. Moreover, each reducible root system Δ can be decomposed into subsets $\Delta_1, \ldots, \Delta_s$ such that $(\Delta_i, \Delta_j) = 0$ for all $i \neq j$ and each Δ_i is an irreducible root system in $\mathrm{span}_{\mathbb{R}}(\Delta_i) \subset V$. This decomposition is unique up to an ordering of the Δ_i.

Example 1.2.4. As an example we will give here for each classical Lie algebra of Example 1.2.1 an 'abstract' root system that is similar to its own root system. These root systems are called *classical root systems*. We will consider these classical root systems in various spaces $\mathbb{R}^n$ with the standard inner product and standard orthonormal unit basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$.

A_l: Let $V = \mathbb{R}^{l+1}$, then $\Delta = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ is a reduced root system in V.

B_l: Let $V = \mathbb{R}^l$, then $\Delta = \{\pm\varepsilon_i\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}$ is a reduced root system in V.

C_l: Let $V = \mathbb{R}^l$, then $\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\} \cup \{\pm 2\varepsilon_i\}$ is a reduced root system in V.

D_l: Let $V = \mathbb{R}^l$, then $\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}$ is a reduced root system in V.

It turns out that all these root systems are irreducible except D_2 which decomposes into two orthogonal root systems A_1. Furthermore, B_1 and C_1 are similar to A_1, B_2 is similar to C_2 and D_3 is similar to A_3. In Figure 1.2 we have depicted the classical root systems of rank 2 up to similarity.

A subset $\Pi \subset \Delta$ is called a *basis* of the root system Δ if Π is a basis of the vector space V, and if each root $\beta \in \Delta$ can be written as $\sum_{\alpha\in\Pi} c_\alpha \alpha$, where the coefficients c_α are all nonpositive or all nonnegative integers. Since Π is a basis of V we can write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, and we call the elements

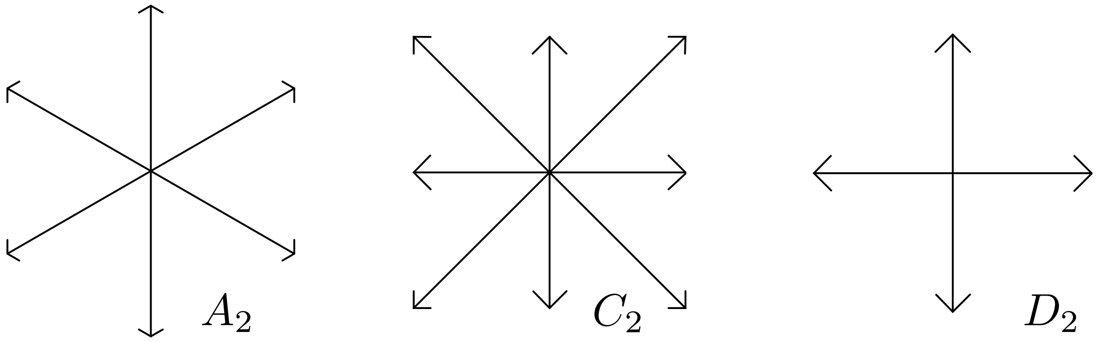


Figure 1.1: The classical root systems of rank 2 up to similarity.

of Π *simple roots.* Furthermore the expression $\sum_{\alpha\in\Pi} c_\alpha \alpha = \sum_{i=1}^{l} c_i^\beta \alpha_i$ for $\beta \in \Delta$ is unique, so we can define the number $\mathrm{ht}(\beta) = \sum_{i=1}^{l} c_i^\beta$ called the *height* of β. It turns out that every root system Δ has a unique root ϕ with respect to Π such that $\mathrm{ht}(\alpha) < \mathrm{ht}(\phi)$ for all $\alpha \in \Delta \setminus \{\phi\}$, and ϕ is called the *highest root* of Δ with respect to Π. If all $c_\alpha \geq 0$ (resp. all $c_\alpha \leq 0$), then β is said to be *positive* (resp. *negative*) with respect to Π. The positive (resp. negative) roots of Δ are denoted by Δ_+ (resp. Δ_-), and $\Delta = \Delta_+ \coprod \Delta_-$. Finally, introduce the partial ordering $\geq$ on Δ by setting $\alpha \geq \beta$ if $c_i^\alpha \geq c_i^\beta$ for all $1 \leq i \leq l$ where $\alpha, \beta \in \Delta$.

The roots of an irreducible reduced root system can have at most 2 lengths with respect to the norm induced by $(.,.)$. If there are two root lengths in an irreducible root system we will call the roots having the lowest norm *short roots* and those having the highest norm *long roots.* With respect to a chosen basis Π of Δ there exists a unique highest short (resp. long) root $\theta \in \Delta$ (resp. $\phi \in \Delta$). This means that $\theta \in \Delta$ (resp. $\phi \in \Delta$) is a short (resp. long) root such that $\mathrm{ht}(\alpha) < \mathrm{ht}(\theta)$ (resp. $\mathrm{ht}(\beta) < \mathrm{ht}(\phi)$) for all short (resp. long) roots $\alpha \in \Delta \setminus \{\theta\}$ (resp. $\beta \in \Delta \setminus \{\phi\}$). Here the highest long root coincides with the highest root of Δ with respect to Π.

Put

$$\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$$

to be the *dual root* of $\alpha \in \Delta$, then the *dual* $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$ of Δ is a root system in V having the same Weyl group as Δ. It turns out that if $\{\alpha_1, \ldots, \alpha_l\}$ is a basis for the reduced root system Δ, then $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ is a basis for the dual root system $\Delta^\vee$.

1.2.3 The classification of reduced irreducible root systems

To classify root systems it suffices to consider irreducible root systems, since each root system can be decomposed into a finite disjoint union of irreducible root systems that are mutually orthogonal. In this Subsection we consider the case that an irreducible root system Δ is reduced. We will classify all similarity classes of reduced irreducible root systems using Cartan matrices and Dynkin diagrams.

First, we want to relate a similarity invariant matrix to a root system. The notion of irreducibility will translate to a property of such a matrix called indecomposability. Let A_1 and A_2 be matrices, then the matrix of the form $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ is called the *direct sum* of A_1 and A_2. A matrix A is said to be *indecomposable* if there do not exist matrices A_1 and A_2 such that any matrix obtained from A by simultaneous permutations of its rows and columns is a direct sum of A_1 and A_2. After simultaneously permuting rows and columns any matrix A can be decomposed into a direct sum of indecomposable matrices.

Definition 1.2.5. A *Cartan matrix* is a rational integral $l \times l$-matrix $A = (a_{ij})_{1 \le i,j \le l}$ that satisfies the following four conditions:

(1) $a_{ii} = 2$ for $1 \le i \le l$;
(2) $a_{ij} \le 0$ if $i \ne j$;
(3) $a_{ij} = 0$ implies $a_{ji} = 0$;
(4) All principal minors of A are strictly positive.

Choose a basis $\Pi \subset \Delta$ and fix an ordering $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of simple roots. Then the matrix

$$A(\Delta, \Pi) := ((\alpha_i^\vee, \alpha_j))_{1 \le i,j \le l} \tag{1.2.1}$$

is called the *Cartan matrix* of Δ with respect to the ordered basis Π, and it is actually a Cartan matrix as defined in Definition 1.2.5. Since we consider Δ to be irreducible, the Cartan matrix $A(\Delta, \Pi)$ is indecomposable.

For any choice of basis Π of Δ it turns out that the Weyl group $W_0(\Delta)$ is generated by the orthogonal reflections w_α for $\alpha \in \Pi$. Furthermore $W_0(\Delta)$ acts transitively on the bases of Δ. Now the generators of $W_0(\Delta)$ are linear isometries of V, so all elements of $W_0(\Delta)$ are. This implies that the Cartan matrix of Δ does not depend on the choice of basis Π, but only on the ordering of Π. Here different orderings of the same basis Π yield Cartan matrices that coincide up to simultaneous permutations of rows and columns. Furthermore, one can show that up to simultaneous permutations of rows and columns $A(\Delta, \Pi)$ determines Δ up to similarity, and one can show that for each indecomposable Cartan matrix A there exists a corresponding reduced irreducible root system Δ (and some ordered basis Π of Δ) such that $A = A(\Delta, \Pi)$.

Write $\overline{A}$ for the equivalence class of the indecomposable Cartan matrix A under the equivalence relation of simultaneous permutations of rows and columns of matrices, and put $\mathscr{C}$ for the collection of indecomposable Cartan matrices up to simultaneous permutations of rows and columns. Further, put $\overline{\Delta}$ for the similarity class of Δ and $\mathscr{R}$ for the collection of similarity classes of reduced irreducible root systems. Then we can summarize the above as follows.

Theorem 1.2.6. *The map* $A : \mathscr{R} \to \mathscr{C}$ *defined by* $\overline{\Delta} \mapsto \overline{A(\Delta, \Pi)} =: A(\overline{\Delta})$ *is a bijection.*

We can associate a graph to each Cartan matrix up to simultaneous permutations of rows and columns. Let $A = (a_{ij})_{1 \le i,j \le l}$ be a Cartan matrix, then define the graph $S(A)$ called the *Dynkin diagram* of A as follows. The graph $S(A)$ has l nodes, and for $i \ne j$ the i-th and j-th node are joined by $a_{ij}a_{ji}$ edges. Furthermore, these edges are equipped with an arrow pointing towards the i-th node if $|a_{ij}| > 1$. Now it turns out that $a_{ij}a_{ji} \le 3$ for all $i \ne j$ for any Cartan matrix $A = (a_{ij})_{1 \le i,j \le l}$, so there is no ambiguity about the factorization of the number of edges between node i and node j of $S(A)$ to obtain $a_{ij}a_{ji}$ again. Thus given a Dynkin diagram D one can reconstruct the associated Cartan matrix $A = (a_{ij})_{i,j \in I}$ up to a permutation of the index set I of A. It must be noted that the implicit ordering of the nodes of a Dynkin diagram that we used here is not part of the definition of a Dynkin diagram. It is only used here to describe its construction.

Each indecomposable Cartan matrix up to simultaneous permutations of rows can be represented by a unique connected Dynkin diagram of which the whole list of possibilities can be found in Figure 1.2. This gives a classification of the indecomposable Cartan matrices and reduced irreducible root systems using the bijection of Theorem 1.2.6.

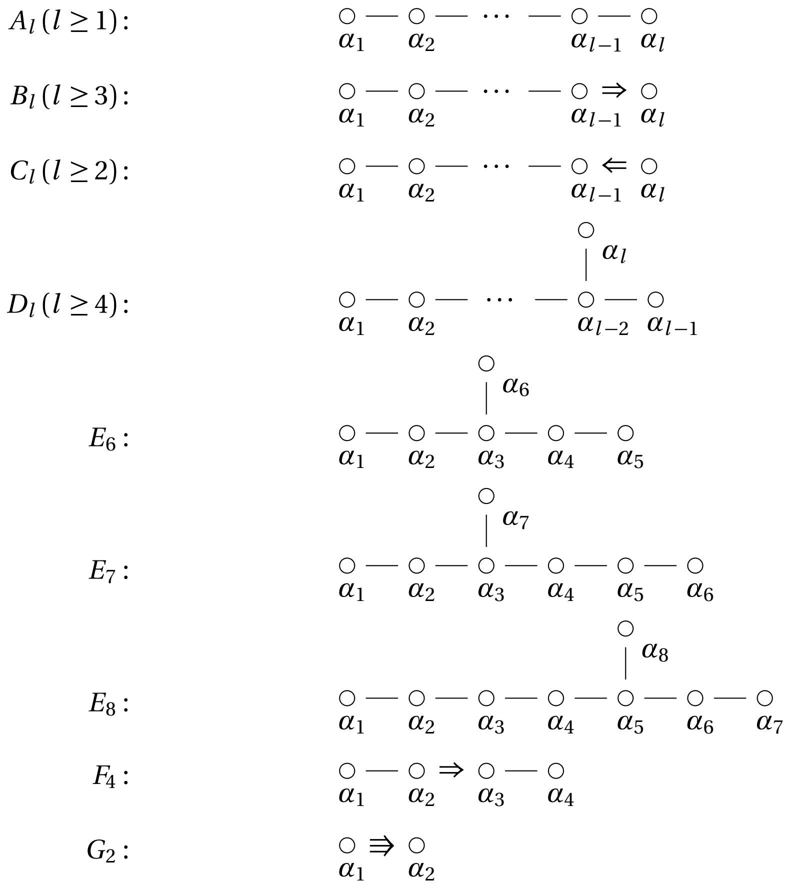


Figure 1.2: All possible Dynkin diagrams corresponding to finite Cartan matrices. The labels of the nodes solely serve the purpose of enumerating the nodes, and are not part of the definition of a Dynkin diagram.

The left column of Figure 1.2 contains the name X_l of each Dynkin diagram $S(A)$ that is depicted in the right column where l is the rank of A and X_l is called the *type* of $S(A)$. So we can define the type of a reduced irreducible root system and an indecomposable Cartan matrix as the type of the corresponding Dynkin diagram. Each classical root system X_l of Example 1.2.4 is of type X_l for $X = A, B, C, D$ up to similarity. Further, the labels of the nodes of the Dynkin diagrams in Figure 1.2 are not part of the Dynkin diagram, but only serve the purpose now of enumerating the nodes.

1.2.4 Nonreduced irreducible root systems

In this Subsection we would like to give some useful details on nonreduced irreducible root systems and classify them up to similarity. Since the root systems of semisimple Lie algebras are reduced, nonreduced root systems will not play a role in the classification of semisimple Lie algebras. However they will give rise to certain affine root systems, and therefore become an integral part of the classification of affine root systems in Chapter 3.

Let Δ be a nonreduced irreducible root system, and suppose $\alpha \in \Delta$. Then condition (3) of Definition 1.2.3 yields that $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha\}$, $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha, \pm\frac{1}{2}\alpha\}$ and $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha, \pm 2\alpha\}$ are the only possibilities for multiples of α in Δ. Hence we can define the *indivisible roots* $\Delta^{ind} := \{\alpha \in \Delta : \frac{1}{2}\alpha \notin \Delta\}$ of Δ and the *unmultipliable roots* $\Delta^{umn} := \{\alpha \in \Delta : 2\alpha \notin \Delta\}$ of Δ. So $\Delta = \Delta^{ind} \cup \Delta^{unm}$ where we note that the union is not disjoint because of the case that $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha\}$ for a root $\alpha \in \Delta$. Furthermore, Δ^{ind} and Δ^{unm} are itself reduced irreducible root systems both having the same Weyl group as Δ. Since the unmultipliable roots all follow as integral multiples of indivisible roots, it turns out that all bases for Δ are also bases for Δ^{ind}. So also Δ has the same Dynkin diagram as Δ^{ind}.

One can observe that both Δ^{ind} and Δ^{unm} have two root lengths. The long roots of Δ^{ind} are the short roots of Δ^{unm}, and the long roots of Δ^{unm} are the short root of Δ^{ind} multiplied by a factor 2. Therefore a nonreduced irreducible root system has 3 root lengths. Furthermore, the long roots of Δ^{unm} are the roots of Δ having the highest root length. After choosing a basis for Δ we have a unique highest root $\phi \in \Delta$ which is a long root of Δ^{unm}.

Finally, one can show that there is only one similarity class of nonreduced irreducible root systems for a fixed rank l. Therefore a nonreduced irreducible root system is similar to its dual root system. Moreover, $(\Delta^{ind})^\vee = (\Delta^\vee)^{unm}$ and $(\Delta^{unm})^\vee = (\Delta^\vee)^{ind}$ where the short (resp. long) roots of Δ^{ind} (resp. Δ^{unm}) become the long (resp. short) roots of $(\Delta^\vee)^{unm}$ (resp. $(\Delta^\vee)^{ind}$). A nonreduced irreducible root system Δ is said to be of type BC_l. This terminology comes from the fact that Δ^{ind} is always of type B_l and Δ^{unm} is always of type C_l (see Figure 1.2).

1.3 Serre's Theorem, a commutative triangle and the classification of simple Lie algebras

To classify semisimple Lie algebras it suffices to consider simple Lie algebras, since every semisimple Lie algebra is a direct sum of simple Lie algebras. For the classification of simple Lie algebras we will use a Theorem of Serre to construct a corresponding simple Lie algebra for each reduced irreducible root system or indecomposable Cartan matrix. This will give us a bijection between the isomorphism classes of simple Lie algebras, the similarity classes of reduced irreducible root systems and indecomposable Cartan matrices up to simultaneous permutations of rows and columns in the form of a commutative triangle. Together with the classification of the similarity classes of reduced irreducible root systems from the previous Section this finishes the classification of simple Lie algebras.

In Subsections 1.2.1 and 1.2.2 we mentioned that the choice of a Cartan subalgebra $\mathfrak{h}$ in a simple Lie algebra $\mathfrak{g}$ gives rise to a reduced irreducible root system $\Delta(\mathfrak{g},\mathfrak{h})$. It turns out that up to similarity $\Delta(\mathfrak{g},\mathfrak{h})$ does not depend on the choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Furthermore, the isomorphism class of $\mathfrak{g}$ determines $\Delta(\mathfrak{g},\mathfrak{h})$ uniquely up to similarity. Put $\overline{\mathfrak{g}}$ for the isomorphism class of the simple Lie algebra $\mathfrak{g}$ and $\mathscr{L}$ for the set of isomorphism classes of simple Lie algebras. Then we have the well defined injective map $\Delta : \mathscr{L} \to \mathscr{R}$ given by $\overline{\mathfrak{g}} \mapsto \overline{\Delta(\mathfrak{g},\mathfrak{h})} =: \Delta(\overline{\mathfrak{g}})$.

Due to Serre we have the following Theorem that shows that for each reduced root system (or indecomposable Cartan matrix) there actually exists a corresponding semisimple Lie algebra.

Theorem 1.3.1. *Let* $\mathfrak{h}$ *be a complex vector space with basis* $\{h_1,\ldots,h_l\}$*, and let* Δ *be a reduced root system in* $\mathfrak{h}^*$ *with basis* $\Pi = \{\alpha_1,\ldots,\alpha_l\}$ *and corresponding Cartan matrix* $A = (a_{ij})_{1\le i,j\le l}$*. Consider* *the Lie algebra* $\mathfrak{g}$ *generated by* $3n$ *elements* e_i, f_i, h_i $(1 \le i \le l)$ *that satisfy the following relations*

$$\begin{cases} [e_i, f_j] = \delta_{ij} h_i, \\ [h_i, h_j] = 0, \\ [h_i, e_j] = a_{ij} e_j, \\ [h_i, f_j] = -a_{ij} f_j, \\ (ad\, e_i)^{1-a_{ij}} e_j = 0 \quad i \neq j, \\ (ad\, f_i)^{1-a_{ij}} f_j = 0 \quad i \neq j. \end{cases}$$

where $i, j = 1, \ldots, l$. *Then* $\mathfrak{g}$ *is a semisimple Lie algebra with* $\mathfrak{h}$ *a Cartan subalgebra and corresponding root system* Δ.

Proof. See §18 of [6]. □

In Serre's Theorem it turns out that if Δ is irreducible, then $\mathfrak{g}$ is simple. This leads to the following canonical bijection.

Theorem 1.3.2. *The injective map* $\Delta : \mathscr{L} \to \mathscr{R}$ *defined by* $\overline{\mathfrak{g}} \mapsto \overline{\Delta(\mathfrak{g}, \mathfrak{h})} =: \Delta(\overline{\mathfrak{g}})$ *is a bijection.*

Furthermore, if A is an indecomposable Cartan matrix and $\mathfrak{g}(A)$ its corresponding simple Lie algebra from Serre's Theorem, then we obtain a map $\mathfrak{g} : \mathscr{C} \to \mathscr{L}$ defined by $\mathfrak{g}(\overline{A}) := \overline{\mathfrak{g}(A)}$. This map is well defined and bijective by Theorem 1.3.2 and 1.2.6. Hence we obtain the following classical identification (see [4]) of $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{C}$

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\quad \Delta \quad} & \mathscr{R} \\ \;\;\nwarrow{\scriptstyle \mathfrak{g}} & & \swarrow{\scriptstyle A} \\ & \mathscr{C} & \end{array} \tag{1.3.1}$$

through the bijections Δ, A and $\mathfrak{g}$ which make the diagram commute. This gives a classification of all isomorphism classes of simple Lie algebras and similarity classes of reduced irreducible root systems using the classification of indecomposable Cartan matrices up to simultaneous permutations of rows columns according to their Dynkin diagrams (see Figure 1.2).

Remark 1.3.1. In this Chapter we have defined root systems, Weyl groups and Cartan matrices. However in the remaining of this manuscript we will also encounter other types of root systems, Weyl groups and Cartan matrices. To distinguish the former from the latter we will call a root system a *finite root system*, a Weyl group a *finite Weyl group* and an indecomposable Cartan matrix a *finite Cartan matrix* from this point on. This terminology will be explained in the next Chapter.

Chapter 2

Kac-Moody algebras, Affine Lie algebras, and the set of real roots

In this Chapter we start with the construction of the Lie algebra $\mathfrak{g}(A)$ associated to any complex matrix A. This definition is modeled after a suitable generalization of Serre's Theorem from the previous Chapter. Although $\mathfrak{g}(A)$ has a root space decomposition it is does not admit an invariant nondegenerate symmetric bilinear form in general, and it does not easily generalize the notion of a Weyl group that acts on the root system. Letting A be a generalized Cartan matrix we call $\mathfrak{g}(A)$ a Kac-Moody (Lie) algebra. Kac-Moody algebras do allow a natural definition of a Weyl group. Also, Kac-Moody algebra's admit an invariant nondegenerate symmetric bilinear form if and only if their generalized Cartan matrix is symmetrizable. Furthermore, each symmetrizable Kac-moody algebra has a Serre presentation in terms of generators and relations, and therefore generalize the notion of a semisimple Lie algebra in a very nice way. Finally, each symmetrizable Kac-moody algebra up to isomorphism corresponds to a symmetrizable generalized Cartan matrix up to simultaneous permutations of rows and columns.

Indecomposable generalized Cartan matrices can be either of finite, affine or indefinite type. We will discuss the classification of generalized Cartan matrices of finite and affine type which turn out to be both symmetrizable. This will show that generalized Cartan matrices of finite type are finite Cartan matrices and that their associated Kac-Moody algebra is simple. For the remaining of this Chapter we will be considering the next interesting class of Kac-Moody algebras, namely affine Lie algebras which are Kac-Moody algebras associated to generalized Cartan matrices of affine type. Contrary to simple Lie algebras these Lie algebras are infinite dimensional. However they do contain a simple subalgebra and can be thought of as infinite-dimensional generalizations of simple Lie algebras. We will study the set of real roots of an affine Lie algebra which generalizes some of the properties of its 'underlying' finite root system. Moreover, the set of real roots will give rise to a new kind of root system called an affine root system which was defined and analyzed independently by Macdonald (see [10]). We will study affine root systems in full detail in a general setting in the next Chapter. This Chapter consists of a summary of results on Kac-Moody algebras from [8], [12] and [15].

2.1 Kac-Moody algebras

2.1.1 Realizations of matrices and their associated Lie algebras

We ended the classification of simple Lie algebras in the previous Chapter with Serre's Theorem which yields a simple Lie algebra in terms of generators and relations that only depends on a finite Cartan matrix. We will now start out with any complex $n \times n$-matrix A, and construct a Lie algebra $\mathfrak{g}(A)$ using a suitable generalization of Serre's Theorem. Such a Lie algebra will turn out to have a root space decomposition which gives rise to a new kind of root system that we will from now on call a root system. Further, up to a special type of Lie algebra isomorphism $\mathfrak{g}(A)$ only depends on A up to simultaneous permutations of rows and columns. To handle the case that $\det(A) = 0$ we first introduce the notion of a realization of a complex matrix A.

Definition 2.1.1. A *realization* of a rank l complex $n \times n$-matrix $A = (a_{ij})_{1 \le i,j \le n}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, and $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subset in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, satisfying the following three conditions:
(1) $\dim_{\mathbb{C}}(\mathfrak{h}) = 2n - l$;
(2) both Π and $\Pi^\vee$ are linearly independent sets;
(3) $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ for $1 \le i, j \le n$,
where $\langle .,. \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denotes the pairing $\langle \alpha, h \rangle = \alpha(h)$.

Example 2.1.2. Let A be a complex $n \times n$-matrix of rank l. Reordering the indices of A, if necessary, we may assume that $A = \binom{A_1}{A_2}$ with A_1 an $l \times n$ submatrix of rank l. Then take $\mathfrak{h} = \mathbb{C}^{2n-l}$, α_i the linear functional that returns the i-th coordinate on $\mathfrak{h}$ and $\alpha_i^\vee$ the i-th row of the matrix

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix}$$

for $i = 1, \dots, n$, where I_{n-l} is the $(n-l) \times (n-l)$ identity matrix. Then $(\mathfrak{h}, \Pi, \Pi^\vee)$ with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is a realization of A.

For a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of a matrix A we call Π (resp. $\Pi^\vee$) the *root basis* (resp. *coroot basis*), and elements of Π (resp. $\Pi^\vee$) are called *simple roots* (resp. *simple coroots*). Define the *root lattice* (resp. *coroot lattice*) $Q := \sum_{i=1}^n \mathbb{Z}\alpha_i$ (resp. $Q^\vee := \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee$), and also set $Q_+ := \sum_{i=1}^n \mathbb{Z}_{\ge 0}\alpha_i$. Introduce the partial ordering $\ge$ on $\mathfrak{h}^*$ by setting $\alpha \ge \beta$ if $\alpha - \beta \in Q_+$ for $\alpha, \beta \in \mathfrak{h}^*$, hence $\lambda > 0$ for all $\lambda \in Q_+ \setminus \{0\}$.

There is a natural notion of isomorphism between realizations: Two realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ (of not necessarily the same matrix) are said to be *isomorphic* if there exists a linear isomorphism $\phi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $\phi(\Pi_1^\vee) = \Pi_2^\vee$ and $\phi^*(\Pi_2) = \Pi_1$, where ϕ^* is the pull-back map $\phi^*(\alpha) = \alpha \circ \phi \in \mathfrak{h}_1^*$ for $\alpha \in \mathfrak{h}_2^*$.

Proposition 2.1.3. *Up to isomorphism there exists a unique realization for every complex $n \times n$-matrix A. Moreover, realizations of matrices A and A' are isomorphic if and only if A' can be obtained from A by simultaneous permutation of the rows and columns of A.*

Proof. The existence follows from Example 2.1.2. For the rest of the proof we refer to Proposition 1.1 of [8]. □

Let $A = (a_{ij})_{1 \le i,j \le n}$ be a complex matrix of rank l together with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$. Further, choose $h_{n+1}, \dots, h_{2n-l} \in \mathfrak{h}$ such that $\{h_1 := \alpha_1^\vee, \dots, h_n := \alpha_n^\vee, h_{n+1} \dots, h_{2n-l}\}$ is a basis of $\mathfrak{h}$. Introduce the Lie algebra $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ with the generators e_i, f_i $(1\le i\le n)$ and h_j $(1\le j\le 2n-l)$ that satisfy the following relations:

$$\begin{cases} [e_i,f_j]=\delta_{ij}h_i & (i,j=1,\dots,n),\\ [h_i,h_j]=0 & (i,j=1,\dots,2n-l),\\ [h_i,e_j]=\langle\alpha_j,h_i\rangle e_j & (i=1,\dots,2n-l; j=1,\dots,n),\\ [h_i,f_j]=-\langle\alpha_j,h_i\rangle f_j & (i=1,\dots,2n-l; j=1,\dots,n).\end{cases} \tag{2.1.1}$$

Then the construction of $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ does not depend on the choice of $h_{n+1},\dots,h_{2n-l}$, and $\mathfrak{h}\subset\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ is a commutative subalgebra.

Denote by $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ generated by $e_1,\dots,e_n$ (resp. $f_1,\dots,f_n$), then we have the following results on $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$.

Theorem 2.1.4. (i) $\tilde{\mathfrak{n}}_+$ *(resp.* $\tilde{\mathfrak{n}}_-$*) is freely generated by* $e_1,\dots,e_n$ *(resp.* $f_1,\dots,f_n$*).*

(ii) *The Lie algebra* $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *is* $Q-graded$*. In particular, one has the root space decomposition*

$$\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)=\left(\bigoplus_{\alpha\in Q_+\setminus\{0\}}\tilde{\mathfrak{g}}_{-\alpha}\right)\oplus\tilde{\mathfrak{g}}_0\oplus\left(\bigoplus_{\alpha\in Q_+\setminus\{0\}}\tilde{\mathfrak{g}}_{\alpha}\right) \tag{2.1.2}$$

with respect to $\mathfrak{h}$*, where* $\tilde{\mathfrak{g}}_\alpha=\{x\in\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee):[h,x]=\langle\alpha,h\rangle x$ *for all* $h\in\mathfrak{h}\}$ *for all* $\alpha\in Q$ *and* $[\tilde{\mathfrak{g}}_\alpha,\tilde{\mathfrak{g}}_\beta]\subseteq\tilde{\mathfrak{g}}_{\alpha+\beta}$ *for all* $\alpha,\beta\in Q$*. Furthermore,* $\dim(\tilde{\mathfrak{g}}_{\pm\alpha})<\infty$ *for all* $\alpha\in Q_+\setminus\{0\}$ *and* $\tilde{\mathfrak{g}}_0=\mathfrak{h}$.

(iii) *If* $\alpha>0$ *(resp.* $\alpha<0$*) then* $\mathfrak{g}_\alpha$ *is the linear span of elements of the form* $[\dots[[e_{i_1},e_{i_2}],e_{i_3}]\dots,e_{i_r}]$ *(resp.* $[\dots[[f_{i_1},f_{i_2}],f_{i_3}]\dots,f_{i_r}]$*) such that* $\alpha_{i_1}+\dots+\alpha_{i_r}=\alpha$ *(resp.* $-\alpha$*).*

(iv) *In* (2.1.2) *we have*

$$\bigoplus_{\alpha\in Q_+\setminus\{0\}}\tilde{\mathfrak{g}}_{\pm\alpha}=\tilde{\mathfrak{n}}_\pm,$$

so that $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *decomposes as a vector space as* $\tilde{\mathfrak{n}}_-\oplus\mathfrak{h}\oplus\tilde{\mathfrak{n}}_+$.

(v) *There exists a unique maximal ideal* τ *of* $\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *that has a trivial intersection with* $\mathfrak{h}$*. Furthermore,* τ *is* Q*-graded and*

$$\tau=(\tau\cap\tilde{\mathfrak{n}}_-)\oplus(\tau\cap\tilde{\mathfrak{n}}_+). \tag{2.1.3}$$

Proof. This follows from Theorem 1.2 of [8] and the same Theorem of [15]. □

Now define the quotient Lie algebra

$$\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee):=\tilde{\mathfrak{g}}(A,\mathfrak{h},\Pi,\Pi^\vee)/\tau$$

which is called the *Lie algebra associated to the matrix A*. We keep the same notation for the images of e_i, f_i $(1\le i\le n)$ and $\mathfrak{h}$ in $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ under the canonical map. Notice that $\mathfrak{h}\subset\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ is a commutative subalgebra by (v) of Theorem 2.1.4. This is usually called the *Cartan subalgebra* of $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$, and as we will see in Corollary 2.2.7 it generalizes the notion of a Cartan subalgebra as we introduced it in the previous Chapter for semisimple Lie algebras.

Denote by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ generated by $e_1,\dots,e_n$ (resp. $f_1,\dots,f_n$). The following result then follows from Theorem 2.1.4.

Theorem 2.1.5. (i) *The Lie algebra* $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *is* $Q-graded$*. In particular, one has the root space decomposition*

$$\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)=\bigoplus_{\alpha\in Q}\mathfrak{g}_\alpha$$

with respect to $\mathfrak{h}$ *where* $\mathfrak{g}_\alpha = \tilde{\mathfrak{g}}_\alpha/(\tau \cap \tilde{\mathfrak{g}}_\alpha)$, *and* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ *for all* $\alpha, \beta \in Q$.
Moreover, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee) : [h,x] = \langle \alpha, h\rangle x \text{ for all } h \in \mathfrak{h}\}$ *is finite-dimensional and* $\mathfrak{g}_0 = \mathfrak{h}$.

(ii) *If* $\mathfrak{g}_\alpha \neq \{0\}$, *then* $\alpha = 0$, $\alpha > 0$ *or* $\alpha < 0$. *Furthermore, if* $\alpha > 0$ *(resp.* $\alpha < 0$*) then* $\mathfrak{g}_\alpha$ *is the linear span of elements of the form* $[\dots[[e_{i_1}, e_{i_2}], e_{i_3}]\dots, e_{i_r}]$ *(resp.* $[\dots[[f_{i_1}, f_{i_2}], f_{i_3}]\dots, f_{i_r}]$*) such that* $\alpha_{i_1} + \cdots + \alpha_{i_r} = \alpha$ *(resp.* $-\alpha$*). In particular,* $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ *and* $\mathfrak{g}_{c\alpha_i} = 0$ *for* $|c| > 1$ *(*$i = 1,\dots,n$*).*

(iii) *The Lie algebra* $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *decomposes as* $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ *as a vector space, where*

$$\mathfrak{n}_- = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \mathfrak{g}_{-\alpha} = \tilde{\mathfrak{n}}_-/(\tau \cap \tilde{\mathfrak{n}}_-), \qquad \mathfrak{n}_+ = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \mathfrak{g}_{\alpha} = \tilde{\mathfrak{n}}_+/(\tau \cap \tilde{\mathfrak{n}}_+).$$

The subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ is called the *root space* attached to α. Define the number $\mathrm{mult}(\alpha) := \dim(\mathfrak{g}_\alpha)$ which shall be called the *multiplicity* of α. An element $\alpha \in Q$ is called a *root* if $\alpha \neq 0$ and $\mathrm{mult}(\alpha) \neq 0$. We let Δ denote the set of all roots of $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ and call it the *root system* of $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$. Corollary 2.2.7 will show that Δ generalizes the notion of a finite root system.

Let $\mathfrak{g}(A_1,\mathfrak{h}_1,\Pi_1,\Pi_1^\vee)$ and $\mathfrak{g}(A_2,\mathfrak{h}_2,\Pi_2,\Pi_2^\vee)$ be Lie algebras associated to matrices A_1 and A_2 respectively. A Lie algebra isomorphism $\phi : \mathfrak{g}(A_1,\mathfrak{h}_1,\Pi_1,\Pi_1^\vee) \to \mathfrak{g}(A_2,\mathfrak{h}_2,\Pi_2,\Pi_2^\vee)$ is called a *realization preserving isomorphism* if $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$, $\phi(\Pi_1^\vee) = \Pi_2^\vee$, and $\phi^*(\Pi_2) = \Pi_1$, where ϕ^* sends $\alpha \in \mathfrak{h}_2^*$ to its pull-back $\phi^*(\alpha) = \alpha \circ \phi \in \mathfrak{h}_1^*$. Notice that in this situation ϕ induces an isomorphism of realizations $(\mathfrak{h}_1,\Pi_1,\Pi_1^\vee)$ of A_1 and $(\mathfrak{h}_2,\Pi_2,\Pi_2^\vee)$ of A_2.

We have the following characterization of the Lie algebra $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$.

Proposition 2.1.6. *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{h} \subset \mathfrak{g}$ *be a commutative subalgebra,* $e_1,\dots,e_n$, $f_1,\dots,f_n$ *elements of* $\mathfrak{g}$, *and let* $\Pi^\vee = \{\alpha_1^\vee,\dots,\alpha_n^\vee\} \subset \mathfrak{h}$ *and* $\Pi = \{\alpha_1,\dots,\alpha_n\} \subset \mathfrak{h}^*$ *be linearly independent sets such that the relations* (2.1.1) *are satisfied. Suppose that* $\{h_1 := \alpha_1^\vee,\dots,h_n := \alpha_n^\vee, h_{n+1},\dots,h_{2n-l}\}$ *is a basis of* $\mathfrak{h}$, *that* e_i, f_i *(*$1 \le i \le n$*) and* h_j *(*$1 \le j \le 2n-l$*) generate* $\mathfrak{g}$ *as a Lie algebra, and that* $\mathfrak{g}$ *has no nonzero ideals which intersect* $\mathfrak{h}$ *trivially. Finally, put* $A = (\langle \alpha_j, \alpha_i^\vee\rangle)_{1\le i,j\le n}$, *and assume that* $\dim(\mathfrak{h}) = 2n - \mathrm{rank}(A)$. *Then there exists a Lie algebra isomorphism* $\phi : \mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee) \to \mathfrak{g}$ *such that* $\phi(\mathfrak{h}) = \mathfrak{h}$, $\phi(\Pi^\vee) = \Pi^\vee$ *and* $\phi^*(\Pi) = \Pi$.

Proof. For the proof we refer to Proposition 1.4 of [15]. □

Furthermore, we have that A up to simultaneous permutation of rows and columns determines its $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ uniquely up to realization preserving isomorphism.

Theorem 2.1.7. *Let* A *(resp.* A'*) be a complex* $n \times n$*-matrix together with a realization* $(\mathfrak{h},\Pi,\Pi^\vee)$ *(resp.* $(\mathfrak{h}',\Pi',\Pi'^\vee)$*). There exists a realization preserving isomorphism between* $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ *and* $\mathfrak{g}(A',\mathfrak{h}',\Pi',\Pi'^\vee)$ *if and only if* A' *can be obtained from* A *by simultaneous permutation of the rows and columns of* A.

Proof. For the proof we refer to Proposition 1.4 of [15] and Proposition 2.1.3. □

From Theorem 2.1.7 we observe that the Lie algebra $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ does not depend on any chosen realization for A up to realization preserving isomorphism and is uniquely determined by A up to simultaneous permutation of rows and columns. From here on we will write $\mathfrak{g}(A)$ for $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ if there is no ambiguity about the chosen realization $(\mathfrak{h},\Pi,\Pi^\vee)$ for the construction of $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$. In Corollary 2.1.15 we will see that Theorem 2.1.7 still holds if the condition of a realization preserving isomorphism is be weakened to any Lie algebra isomorphism for a special choice of the matrix A.

Let $(\mathfrak{h}_1,\Pi_1,\Pi_1^\vee)$ (resp. $(\mathfrak{h}_2,\Pi_2,\Pi_2^\vee)$) be a realization of the matrix A_1 (resp. A_2), then clearly the *direct sum of realizations* $(\mathfrak{h},\Pi,\Pi^\vee) := (\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2, \Pi_1^\vee \times \{0\} \cup \{0\} \times \Pi_2^\vee)$ is a realization of the direct sum of matrices $A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. In this case Proposition 2.1.6 tells us that the Lie algebra $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ is isomorphic to the direct sum of the ideals $\mathfrak{g}(A_1,\mathfrak{h}_1,\Pi_1,\Pi_1^\vee)$ and $\mathfrak{g}(A_2,\mathfrak{h}_2,\Pi_2,\Pi_2^\vee)$.

A matrix A together with a corresponding realization is said to be *indecomposable* if there do not exist matrices A_1 and A_2 such that any matrix obtained from A by simultaneous permutations of its rows and columns is of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. After simultaneously permuting rows and columns a matrix A can be decomposed into a direct sum of indecomposable matrices, and the corresponding realization into a direct sum of indecomposable realizations (see Proposition 1.9C of [15]). This implies that we can restrict our study to Lie algebras $\mathfrak{g}(A)$ associated to an indecomposable complex matrix A.

2.1.2 The invariant nondegenerate symmetric bilinear form

Similar to the Killing form on semisimple Lie algebras we would like to define an invariant nondegenerate symmetric bilinear form on the Lie algebra $\mathfrak{g}(A)$. We will observe that we need an extra symmetry condition on the matrix A to achieve this. Then the desired form turns out to restrict to a nondegenerate symmetric bilinear form on $\mathfrak{h}$. We will use this fact to obtain a canonical linear isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$, and to induce a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$.

Definition 2.1.8. A complex $n\times n$-matrix $A=(a_{ij})_{1\le i\le n}$ is said to be *symmetrizable* if there exists an invertible diagonal matrix D and a symmetric matrix B such that $A=DB$. In this case the Lie algebra $\mathfrak{g}(A)$ is called a *symmetrizable* Lie algebra and the matrix B is called a *symmetrization* of A.

Next we will see that an invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ exists if and only if A is symmetrizable. Furthermore, the restriction of $(.,.)$ on $\mathfrak{h}$ determines $(.,.)$ uniquely on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$, and different forms $(.,.)$ coincide on the subspace $\bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee \subset \mathfrak{h}$ up to a factor in $\mathbb{C}$ if A is indecomposable.

Theorem 2.1.9. *Let A be a complex $n\times n$-matrix with realization $(\mathfrak{h},\Pi,\Pi^\vee)$, and put $\mathfrak{h}' := \bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee$.*

(i) *If A is symmetrizable with symmetrization B such that $A = diag(\varepsilon_1,\dots,\varepsilon_n)B$, then any choice of subspace $\mathfrak{h}''\subset\mathfrak{h}$ such that $\mathfrak{h}=\mathfrak{h}'\oplus\mathfrak{h}''$ gives rise to an invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$) such that $(.,.)|_{\mathfrak{h}}$ is nondegenerate and defined by*

$$\begin{cases} (\alpha_i^\vee,h)=\langle\alpha_i,h\rangle\varepsilon_i & (h\in\mathfrak{h}, i=1,\dots,n),\\ (h',h'')=0 & (h,h'\in\mathfrak{h}''); \end{cases} \tag{2.1.4}$$

(ii) *If there exists an invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$, then A is symmetrizable and there exists a symmetrization B of A and a subspace $\mathfrak{h}''\subset\mathfrak{h}$ such that $A=diag(\varepsilon_1,\dots,\varepsilon_n)B$, $\mathfrak{h}=\mathfrak{h}'\oplus\mathfrak{h}''$ and $(.,.)|_{\mathfrak{h}}$ is nondegenerate and defined by* (2.1.4).

Moreover, if (1) or (2) is satisfied then

(iii) *if $(.,.)'$ is a another invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ and A is indecomposable, then there exists a nonzero $\mu\in\mathbb{C}$ such that $(x,y)'=\mu(x,y)$ for all $x,y\in\mathfrak{h}'$;*

(iv) *if $(.,.)'$ is a nondegenerate symmetric bilinear $\mathbb{C}$-valued form on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ such that $(.,.)'|_{\mathfrak{h}} = (.,.)|_{\mathfrak{h}}$, then $(.,.)'=(.,.)$ on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$.*

Proof. (1) follows from Theorem 2.2 of [8], (2) follows from Theorem 3.1 of [15] and (4) follows from Theorem 3.2 of [15]. To prove (3) let $(.,.)$ be the invariant nondegenerate symmetric bilinear form on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ of (2) induced by $A=DB$ with D an invertible diagonal matrix and B a symmetric matrix. If $(.,.)'$ is another invariant nondegenerate symmetric bilinear form on $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$, then by (2) this form is induced by $A = D'B'$ with D' an invertible diagonal matrix and B' a symmetric matrix. Thus $B = EB'$ for some invertible diagonal matrix E. Since A is indecomposable, it also follows from $A = DB = D'B'$ that B and B' are indecomposable. But for $B = EB'$ to hold under the condition that both B and B' are symmetric and indecomposable with E an invertible diagonal matrix, it must be true that $E = \mu I$ where $\mu \in \mathbb{C}$ is nonzero and I is the $n \times n$ identity matrix. This implies that $B = \mu B'$, so $\mu D = D'$. Then (2.1.4) and bilinearity of $(.,.)$ and $(.,.)'$ show that $(x, y)' = \mu(x, y)$ for all $x, y \in \mathfrak{h}'$. □

Remark 2.1.1. At first sight it is not clear from the first equation of (2.1.4) why $(\alpha_i^\vee, \alpha_j^\vee) = (\alpha_j^\vee, \alpha_i^\vee)$ for $1 \le i, j \le n$, even though $(.,.)$ is a symmetric form. Consider the decomposition $A = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n)B$ that induces $(.,.)$ in (i) of Theorem 2.1.9. Put $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$, then $a_{ij} = \varepsilon_i b_{ij}$. Further $a_{ji} = \varepsilon_j b_{ji} = \varepsilon_j b_{ij}$, since B is a symmetric matrix. Recall that $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, then we finally obtain

$$\begin{aligned}(\alpha_i^\vee, \alpha_j^\vee) &= \langle \alpha_i, \alpha_j^\vee \rangle \varepsilon_i = a_{ji}\varepsilon_i \\ &= \varepsilon_j b_{ij} \varepsilon_i = \varepsilon_j a_{ij} = (\alpha_j^\vee, \alpha_i^\vee)\end{aligned}$$

for $1 \le i, j \le n$.

Since the bilinear form $(.,.)$ from ((i) or (ii) of) Theorem 2.1.9 is nondegenerate on $\mathfrak{h}$ we can define the linear isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ depending on $(.,.)$ by

$$\langle \nu(h), h' \rangle = (h, h')$$

for $h, h' \in \mathfrak{h}$. This leads to a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ defined by

$$(\alpha, \beta) := (\nu^{-1}(\alpha), \nu^{-1}(\beta))$$

for $\alpha, \beta \in \mathfrak{h}^*$. Furthermore, we obtain

$$\langle \alpha, h \rangle = \langle \nu(\nu^{-1}(\alpha)), h \rangle = (\nu^{-1}(\alpha), h) = (\alpha, \nu(h)) \tag{2.1.5}$$

for $\alpha \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$.

Write $A = (a_{ij})_{1\le i,j\le n}$, then from (2.1.4) we obtain $(\alpha_i^\vee, \alpha_j^\vee) = \varepsilon_j a_{ij}$ for $1 \le i, j \le n$ and

$$\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i \tag{2.1.6}$$

for $1 \le i \le n$. This leads to

$$(\alpha_i, \alpha_j) = a_{ij}/\varepsilon_i \tag{2.1.7}$$

for $1 \le i, j \le n$. By an argument similar to that in Remark 2.1.1 one observes that indeed $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$.

Finally, if $(.,.)'$ is another invariant nondegenerate symmetric bilinear form on $\mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)$, then there exists a nonzero $\mu \in \mathbb{C}$ such that $(x, y)' = \mu(x, y)$ for all $x, y \in \mathfrak{h}' := \bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee$ by (iii) of Theorem 2.1.9. Notice that $\nu(\mathfrak{h}') = \bigoplus_{i=1}^n \mathbb{C}\alpha_i$ by (2.1.6). Putting ν' for the linear isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$ induced by $(.,.)'$ we observe that

$$\nu'|_{\mathfrak{h}'} = \mu\nu|_{\mathfrak{h}'}. \tag{2.1.8}$$

Then it follows that

$$(\alpha, \beta) = \mu(\alpha, \beta)' \tag{2.1.9}$$

for all $\alpha, \beta \in \bigoplus_{i=1}^n \mathbb{C}\alpha_i$.

2.1.3 Kac-Moody algebras

In the following we introduce the notion of a generalized Cartan matrix. This is a matrix A that satisfies the properties of a finite Cartan matrix, except that A does not necessarily have all its principal minors positive. A Lie algebra associated to such a matrix is called a Kac-Moody algebra. As we will see in this Subsection, the generators of a Kac-Moody algebra $\mathfrak{g}(A)$ satisfy the natural generalization of the relations of Serre's Theorem. Furthermore, if A is also symmetrizable then $\mathfrak{g}(A)$ is defined by those relations, and up to isomorphism only depends on A up to simultaneous permutations of rows and columns. Finally, if $\mathfrak{g}(A)$ is a symmetrizable Kac-Moody algebra then A can be expressed using the invariant bilinear form $(.,.)$ in a similar way as a Cartan matrix of a finite root system is defined using the Killing form.

Definition 2.1.10. A *generalized Cartan matrix* is a rational integral $n \times n$-matrix $A = (a_{ij})_{1\leq i,j\leq n}$ satisfying the following three conditions

(1) $a_{ii} = 2$ for $1 \leq i \leq n$;
(2) $a_{ij} \leq 0$ if $i \neq j$;
(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Example 2.1.11. (i) A finite Cartan matrix is a generalized Cartan matrix.

(ii) If A is a generalized Cartan matrix, then the transposed A^T of A is also a generalized Cartan matrix.

(iii) If $A = (a_{ij})_{i,j\in I}$ is a generalized Cartan matrix with I a finite index set, then every permutation of I gives another generalized Cartan matrix.

The Lie algebra $\mathfrak{g}(A)$ associated to a generalized Cartan matrix A is called a *Kac-Moody algebra*. Kac-Moody algebras are especially interesting because they admit Serre type relations on the generators $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $\mathfrak{g}(A)$.

Proposition 2.1.12. *Let $\mathfrak{g}(A)$ be a Kac-Moody algebra associated to the $n \times n$ generalized Cartan matrix A, then*

$$(ad\, e_i)^{1-a_{ij}} e_j = 0, \qquad\qquad (ad\, f_i)^{1-a_{ij}} f_j = 0$$

for $1 \leq i, j \leq n$ and $i \neq j$.

Proof. See §3.3 of [8]. □

Recall that by (i) of Theorem 2.1.4 the subalgebra $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) of $\tilde{\mathfrak{g}}(A)$ is freely generated by $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_n$). Furthermore, by (v) of Theorem 2.1.4 the Lie algebra $\tilde{\mathfrak{g}}(A)$ has a unique maximal ideal τ that intersects $\mathfrak{h}$ trivially. Since $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$, Proposition 2.1.12 shows that

$$(\text{ad}\, e_i)^{1-a_{ij}} e_j, (\text{ad}\, f_i)^{1-a_{ij}} \in \tau$$

for $1 \leq i, j \leq n$ and $i \neq j$.

Together with (2.1.1) Proposition 2.1.12 shows that the generators e_i, f_i $(1 \leq i \leq n)$ and h_i $(1 \leq i \leq 2n - l)$ of a Kac-Moody algebra $\mathfrak{g}(A)$ satisfy a natural generalization of the relations of Serre's Theorem for semisimple Lie algebras (see Theorem 1.3.1). Using representation theory of Kac-Moody algebras it is possible to obtain a much stronger result, namely that every symmetrizable Kac-Moody algebra is defined by these relations.

Theorem 2.1.13. *Let $A=(a_{ij})_{1\le i,j\le n}$ be a symmetrizable generalized Cartan matrix, then $\mathfrak{g}(A)$ is the Lie algebra generated by e_i, f_i $(1\le i\le n)$ and h_i $(1\le i\le 2n-l)$ with relations*

$$\begin{cases} [e_i,f_j]=\delta_{ij}\alpha_i^\vee & (i,j=1,\dots,n),\\ [h_i,h_j]=0 & (i,j=1,\dots,2n-l),\\ [h_i,e_j]=\langle\alpha_j,h_i\rangle e_j & (i=1,\dots,2n-l; j=1,\dots,n),\\ [h_i,f_j]=-\langle\alpha_j,h_i\rangle f_j & (i=1,\dots,2n-l; j=1,\dots,n),\\ (ad\, e_i)^{1-a_{ij}}e_j=0 & (i,j=1,\dots,n; i\neq j),\\ (ad\, f_i)^{1-a_{ij}}f_j=0 & (i,j=1,\dots,n; i\neq j).\end{cases}$$

Proof. Follows from Theorem 9.11 of [8]. □

Using representation theory of Kac-Moody algebras it is also possible to generalize the statement of Theorem 2.1.7 to arbitrary Lie algebra isomorphisms when considering symmetrizable Kac-Moody algebras, and thereby establishing a criterion for isomorphism of two symmetrizable Kac-Moody algebras.

Theorem 2.1.14. *Let A and A' be symmetrizable generalized Cartan matrices such that $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{g}(A')$, then A' can be obtained from A by simultaneous permutation of the rows and columns of A.*

Proof. This follows from Theorem 2 (b) of [12]. □

Corollary 2.1.15. *Let A and A' be symmetrizable generalized Cartan matrices, then $\mathfrak{g}(A)$ and $\mathfrak{g}(A')$ are isomorphic as Lie algebras if and only if A' can be obtained from A by simultaneous permutation of the rows and columns of A.*

Proof. Follows from Theorem 2.1.7 and 2.1.14. □

This shows that up to isomorphism a symmetrizable Kac-Moody algebra only depends on its associated symmetrizable generalized Cartan matrix up to simultaneous permutation of the rows and columns.

Finally, if A is a symmetrizable generalized Cartan matrix then one can show that

$$A=\mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)B$$

with $\varepsilon_1,\dots,\varepsilon_n\in\mathbb{Q}_{>0}$ and B a symmetrization with rational coordinates. Fix an invariant nondegenerate symmetric bilinear form $(.,.)$ on $\mathfrak{g}(A)$ associated to this decomposition of A using Theorem 2.1.9. Then from (2.1.7) we obtain

$$(\alpha_i,\alpha_i)>0 \tag{2.1.10}$$

for $i=1,\dots,n$. The form $(.,.)$ on the symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$ provided by Theorem 2.1.9 and satisfying (2.1.10) is called a *standard invariant form*. In this setting (2.1.6) and (2.1.7) imply

$$A=\left(2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{1\le i,j\le n}. \tag{2.1.11}$$

This expression coincides with the expression of the Cartan matrix of a root system of a semisimple Lie algebra (see (1.2.1)).

2.1.4 The Weyl group and the set of real roots of a Kac-Moody algebra

In this Subsection we want to get a hint that Kac-Moody algebras also carry a nice structure on their root systems. We will observe that the root basis Π of a root system Δ of a Kac-Moody algebra $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ gives rise to a reflection group on $\mathfrak{h}^*$ that we will call the Weyl group W of the Kac-Moody algebra. It turns out that W leaves Δ invariant and that it induces an action on Δ. This will lead us to defining the real roots Δ^{re} of the root system of a Kac-Moody algebra as the roots that are W-equivalent to Π under the action of W on Δ. These are the roots that interest us the most in the remaining of this Chapter, since turn out to generalize finite root systems nicely for a certain classes of infinite-dimensional Kac-Moody algebras.

Let A be a generalized Cartan matrix with associated Kac-Moody algebra $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$. For $i = 1,\dots,n$ define the *fundamental reflection* r_i of the space $\mathfrak{h}^*$ by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \tag{2.1.12}$$

for all $\lambda \in \mathfrak{h}^*$. Clearly r_i fixes the subspace $\{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle = 0\} \subset \mathfrak{h}^*$ and $r_i(\alpha_i) = -\alpha_i$, so r_i is a reflection of $\mathfrak{h}^*$. Define the *Weyl group* W of $\mathfrak{g}(A)$ as the subgroup of the group of linear automorphisms $\mathrm{GL}(\mathfrak{h}^*)$ of $\mathfrak{h}^*$ generated by all fundamental reflections. We will see in Corollary 2.2.7 that the Weyl group generalizes the finite Weyl group of a finite root system.

Proposition 2.1.16. (i) *The root system Δ of $\mathfrak{g}(A)$ is W-invariant, and W acts faithfully on Δ;*

(ii) *If A is symmetrizable, then a standard invariant form $(.,.)$ considered on $\mathfrak{h}^*$ is W-invariant for $\mathfrak{g}(A)$ (i.e. $(\lambda,\mu) = (w(\lambda), w(\mu))$ for all $\lambda,\mu \in \mathfrak{h}^*$ and $w \in W$).*

Proof. (i) follows from Proposition 3.7 b) of [8] and (3.12.1) in the proof of Proposition 3.12 of [8], and (ii) follows from Proposition 3.9 of [8]. □

Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with root system Δ and Weyl group W. A root $\alpha \in \Delta$ is said to be *real* if there exists $w \in W$ such that $w(\alpha)$ is a simple root, otherwise α is called *imaginary*. Write Δ^{re} (resp. Δ^{im}) for the subset of Δ of real roots (resp. imaginary roots), then $\Delta = \Delta^{re} \amalg \Delta^{im}$. Clearly Δ^{re} and Δ^{im} are W-invariant.

Next, we assume that $\mathfrak{g}(A)$ is symmetrizable and consider a standard invariant form $(.,.)$ on $\mathfrak{g}(A)$. Then we have the following insightful Lemma on the terminology of 'real' and 'imaginary' root.

Lemma 2.1.17. *Let $\alpha \in \Delta^{re}$ and $\beta \in \Delta^{im}$, then*

$$(\alpha,\alpha) > 0, \qquad \textit{and} \qquad (\beta,\beta) \le 0.$$

Proof. This follows from Proposition 5.1 and 5.2 of [8]. □

Lemma 2.1.17 shows that real roots have a positive 'squared length' with respect to any standard invariant form $(.,.)$, while imaginary roots have a vanishing or negative 'squared length' with respect to $(.,.)$. This makes Δ^{re} more useful for a geometric description than Δ^{im}. It will actually turn out that Δ^{re} generalizes the notion of a finite root system in a very nice way for a certain classes of infinite-dimensional Kac-Moody algebras. Describing such a root system will be the main focus of the Chapter 3.

To end this Section we will show the useful fact that each real root induces a reflection in $\mathfrak{h}^*$ that is contained in W. From (2.1.6) and (2.1.7) we obtain $\alpha_i^\vee = 2\frac{\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathfrak{h}$. We generalize this formula as follows. For $\alpha \in \Delta^{re}$ define the *dual root*

$$\alpha^\vee := 2\frac{\nu^{-1}(\alpha)}{(\alpha,\alpha)} \in \mathfrak{h}.$$

It follows from (2.1.8) and (2.1.9) that this definition is independent of the choice of standard invariant form. Further, for $\alpha \in \Delta^{re}$ define the *reflection* r_α by

$$r_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \tag{2.1.13}$$

for all $\lambda \in \mathfrak{h}^*$. Since $\alpha \in \Delta^{re}$ there exists $w \in W$ and $i \in \{1, \ldots, n\}$ such that $\alpha = w(\alpha_i)$. Then one obtains from Proposition 2.1.16 (ii) and (2.1.13)

$$r_\alpha = w \circ r_i \circ w^{-1} \in W. \tag{2.1.14}$$

Hence W contains the reflection of $\mathfrak{h}^*$ induced by all real roots.

2.2 A classification of generalized Cartan matrices and some Kac-Moody algebras

We proceed to giving a classification of generalized Cartan matrices. First we will observe that an indecomposable generalized Cartan matrix is of finite, affine or indefinite type. For this thesis we will only be interested in the first two types which turn out to be symmetrizable. It turns out that a generalized Cartan matrix of finite type is actually a finite Cartan matrix. We will observe that the corresponding Kac-Moody algebras are simple Lie algebras. Furthermore, every semisimple Lie algebra is isomorphic to a finite-dimensional Kac-Moody algebra, and a Kac-Moody algebras is finite-dimensional if and only if it is a semisimple Lie algebra. Finite Cartan matrices and simple Lie algebras have been fully classified in Chapter 1.

Affine Cartan matrices are closely related to finite Cartan matrices, and give rise to a class of infinite-dimensional Kac-Moody algebras called affine Lie algebras. Similar to the case of simple Lie algebras in Chapter 1 we will be using Dynkin diagrams to classify generalized Cartan matrix of affine type up to simultaneous permutations of rows and columns. Since affine Cartan matrices up to simultaneous permutations of rows and columns are in bijective correspondence with affine Lie algebras up to isomorphism, this will immediately classify the latter.

2.2.1 Three types of generalized Cartan matrices

Recall that any matrix can be decomposed into a direct sum of indecomposable matrices. It therefore suffices to study only the indecomposable generalized Cartan matrices. Further, for a vector v in $\mathbb{R}^n$ we will write $v > 0$ if all coordinates of v (with respect to the stadard basis of $\mathbb{R}^n$) are strictly positive. Let A be an indecomposable generalized Cartan matrix, then one and only one of the following three possibilities holds for A by Theorem 4.3 of [8]:

(Fin) there exists $v > 0$ such that $Av > 0$;
(Aff) there exists $v > 0$ such that $Av = 0$;
(Ind) there exists $v > 0$ such that $Av < 0$.

Definition 2.2.1. We will say that an indecomposable generalized Cartan matrix A is of *finite type* if A satisfies (Fin), of *affine type* if A satisfies (Aff), and of *indefinite type* if A satisfies (Ind).

Remark 2.2.1. Kac-Moody algebras associated to indecomposable generalized Catran matrices of indefinite type are not very well understood. For example, it is an open problem in general what the root multiplicities are of these Kac-Moody algebras, although some special cases have been solved (see e.g. [5], [9]). Since these Kac-Moody algebras are not in the scope of this manuscript we will not consider the case of A being an indecomposable generalized Cartan matrix of indefinite type anymore.

It turns out that an indecomposable generalized Cartan matrix A of finite or affine type can be characterized by its principal minors.

Proposition 2.2.2. *Let A be an indecomposable generalized Cartan matrix.*
(i) *A is of finite type if and only if all principal minors of A are positive.*
(ii) *A is of affine type if and only if all proper principal minors of A are positive and* $\det(A) = 0$.
(iii) *A is of affine type if and only if there exists a real-valued vector $\delta > 0$ that is unique up to a constant factor such that $A\delta = 0$.*

Proof. This follows from Proposition 4.7 of [8]. □

We introduce the following Definition.

Definition 2.2.3. A rank l *affine Cartan matrix* is an rational integral square $(l+1)\times(l+1)$-matrix $A = (a_{ij})_{0\le i,j\le l}$ satisfying the following five conditions

(1) $a_{ii} = 2$ for $0 \le i \le l$;
(2) $a_{ij} \le 0$ if $i \ne j$;
(3) $a_{ij} = 0$ implies $a_{ji} = 0$;
(4) $\det(A) = 0$ and all the proper principal minors of A are strictly positive;
(5) A is indecomposable.

Now from Proposition 2.2.2 and the Definitions of a finite and affine Cartan matrix we observe the following.

Corollary 2.2.4. *The indecomposable generalized Cartan matrices of finite (resp. affine) type coincide with the finite (resp. affine) Cartan matrices.*

Notice that if we change $\det(A) = 0$ to $\det(A) \ne 0$ in Definition 2.2.3 we obtain the definition of a finite Cartan matrix. In that sense affine Cartan matrices can be considered within the generalized Cartan matrices as the matrices that are the closest related to finite Cartan matrices. In mathematical terms we have the following.

Corollary 2.2.5. *Every proper principal submatrix of a finite (resp. affine) Cartan matrix is a finite Cartan matrix.*

2.2.2 Kac-Moody algebras associated to finite Cartan matrices

Next we want to characterize Kac-Moody algebras corresponding to finite Cartan matrices.

Let $\mathfrak{h}_{\mathbb{R}}$ denote a real vector space such that $\Pi^\vee \subset \mathfrak{h}_{\mathbb{R}}$, $\Pi \subset \mathfrak{h}_{\mathbb{R}}^*$ and $(\mathbb{C}\otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$ is a realization of a generalized Cartan matrix A. Then we have the following equivalences.

Proposition 2.2.6. *Let A be an indecomposable generalized Cartan matrix. Then the following conditions are equivalent:*

(1) *A is of finite type;*
(2) *A is symmetrizable and any standard invariant form $(.,.)$ restricted to $\mathfrak{h}_{\mathbb{R}}$ is positive definite;*
(3) *The Kac-Moody algebra $\mathfrak{g}(A)$ is a simple Lie algebra;*

(4) *The Weyl group W of the Kac-Moody algebra $\mathfrak{g}(A)$ is a finite group;*

(5) *The root system Δ of the Kac-Moody algebra $\mathfrak{g}(A)$ is a finite set;*

(6) *The root system Δ of the Kac-Moody algebra $\mathfrak{g}(A)$ does not contain any imaginary roots, i.e. $\Delta = \Delta^{re}$ and $\Delta^{im} = \emptyset$.*

Proof. This follows from Proposition 4.9 and Theorem 5.6 of [8]. □

Notice that the term 'finite' in our terminology of generalized Cartan matrix of finite type, finite root system and finite Weyl group is now justified by characterizations (3), (4) and (5) of Proposition 2.2.6.

Corollary 2.2.7. *If A is a finite Cartan matrix, then $\mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)$ is a simple Lie algebra with $\mathfrak{h}$ a Cartan subalgebra, Δ a reduced irreducible finite root system with finite Weyl group W and Π is a basis for Δ with corresponding finite Cartan matrix $A(\Delta, \Pi) = A$.*

Proof. This follows from combining the results of Proposition 2.2.6, Theorem 1.3.1 and 2.1.13 together with the appropriate definitions of Chapter 1. □

From the classification (1.3.1) of simple Lie algebras it follows that each simple Lie algebra is isomorphic to a Kac-Moody algebra associated to a finite Cartan matrix. So indeed the Lie algebras $\mathfrak{g}(A)$ and in particular (symmetrizable) Kac-Moody algebras generalize semisimple Lie algebras. However the only other Kac-Moody algebras are infinite-dimensional.

Corollary 2.2.8. *A Kac-Moody algebra $\mathfrak{g}(A)$ is finite-dimensional if and only if $\mathfrak{g}(A)$ is a semisimple Lie algebra.*

Proof. This follow from (3) and (5) of Proposition 2.2.6. □

2.2.3 A classification of affine Cartan matrices and affine Lie algebras

In the previous Subsection we saw that Kac-Moody algebras associated to finite Cartan matrices are just simple Lie algebras which we have completely classified in Chapter 1. Now we want to shift our focus to classifying affine Cartan matrices, and their corresponding Kac-Moody algebras which are known as *affine Lie algebras*. From Corollary 2.2.8 it follows that they are infinite-dimensional Lie algebras. First we will establish a canonical bijection between affine Cartan matrices up to simultaneous permutations of rows and columns and affine Lie algebras up to isomorphism. Then we give a classification of affine Cartan matrices up to simultaneous permutations of rows and columns in a similar way as the classification of finite Cartan matrices in Subsection 1.2.3 using Dynkin diagrams.

In Section 2.1 we saw that symmetrizable generalized Cartan matrices bring a lot of structure with them on their corresponding Kac-Moody algebras. It turns out that affine Cartan matrices lie in this rich class of matrices.

Proposition 2.2.9. *Let A be an affine Cartan matrix, then A is symmetrizable.*

Proof. This follows from Lemma 4.6 of [8]. □

Write $\overline{A}$ for the equivalence class of the affine Cartan matrix A under the equivalence relation of simultaneous permutations of rows and columns of matrices. Further, let $\mathscr{C}_a$ denote the collection of affine Cartan matrices up to simultaneous permutations of rows and columns. Next, consider the set $\widehat{\mathscr{L}_a}$ of all Lie algebras $\mathfrak{g}$ that are isomorphic to an affine Lie algebra $\mathfrak{g}(A)$. Write $\overline{\mathfrak{g}}$ for the isomorphism class of the $\mathfrak{g}$ in $\widehat{\mathscr{L}_a}$, and put $\mathscr{L}_a$ for the collection of isomorphism classes of $\widehat{\mathscr{L}_a}$. Using Proposition 2.2.9 we deduce from Corollary 2.1.15 the following canonical bijection.

Theorem 2.2.10. *The map* $\mathfrak{g} : \mathscr{C}_a \to \mathscr{L}_a$ *defined by* $\overline{A} \mapsto \overline{\mathfrak{g}(A)} := \mathfrak{g}(\overline{A})$ *is a bijection.*

Remark 2.2.2. For completeness reasons we included all Lie algebras isomorphic to affine Lie algebras in the definition of $\mathscr{L}_a$. In the remaining of this thesis however we want to work only with affine Lie algebras $\mathfrak{g}(A)$ (apart from Subsection 2.3.5), and leave arbitrary isomorphic Lie algebras out of the picture to simplify things. Therefore we will summarize here how the most important structures of $\mathfrak{g}(A)$ are inherited by an isomorphic Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g} \in \overline{\mathfrak{g}(A)}$ for an affine Cartan matrix A, then there exists an isomorphism $\phi : \mathfrak{g}(A) \to \mathfrak{g}$. Now $\mathfrak{g}$ inherits some structures of $\mathfrak{g}(A)$ through ϕ. Assume $\mathfrak{g}(A) = \mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)$, then $(\phi(\mathfrak{h}), {\phi^*}^{-1}(\Pi), \phi(\Pi^\vee))$ is a realization of A with $\phi(\mathfrak{h}) \subset \mathfrak{g}$ a commuting subalgebra that leads to a root space decomposition of $\mathfrak{g}$. The corresponding root system of $\mathfrak{g}$ is then given by ${\phi^*}^{-1}(\Delta) \subset \phi(\mathfrak{h})^*$. Furthermore, each $\alpha \in {\phi^*}^{-1}(\Delta)$ can be expressed as a linear combination of elements in ${\phi^*}^{-1}(\Pi)$ with coefficients all positive or all negative integers. If $(.,.)$ is a standard invariant form on $\mathfrak{g}(A)$, then $(x, y) := (\phi^{-1}(x), \phi^{-1}(y))$ for all $x, y \in \mathfrak{g}$ defines an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ such that $(\alpha, \alpha) > 0$ for all $\alpha \in {\phi^*}^{-1}(\Pi)$. Finally, we can define the Weyl group of $\mathfrak{g}$ as $\{{\phi^*}^{-1} \circ w \circ \phi^* : w \in W\}$ and the real roots as ${\phi^*}^{-1}(\Delta^{re})$ with respect to $\mathfrak{g}(A)$ and ϕ.

In (i) of Example 3.2.3 we will consider Δ^{re} as an 'affine root system'. In a similar fashion one can show that ${\phi^*}^{-1}(\Delta^{re})$ is an affine root system. Then analogous to Subsection 3.5.4 it turns out that ${\phi^*}^{-1}(\Delta^{re})$ as an affine root system does not depend on the isomorphism ϕ up to an appropriate equivalence relation called 'similarity'. Furthermore, Δ^{re} and ${\phi^*}^{-1}(\Delta^{re})$ are similar affine root systems (compare with (3.5.6)).

Now let us classify the affine Cartan matrices up to simultaneous permutations of rows and columns. Let A be an affine Cartan matrix, then the Dynkin diagram $S(A)$ still makes sense (see Subsection 1.2.3). To distinguish between Dynkin diagrams of the different generalized Cartan matrices we will call a Dynkin diagram $S(A)$ *finite* (resp. *affine*) if A is a finite (resp. affine) Cartan matrix. It turns out that $a_{ij}a_{ji} \leq 4$ holds for any affine Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, so there is no ambiguity about the factorization of the number of edges between node i and node j of $S(A)$ to obtain $a_{ij}a_{ji}$ again. Thus given an affine Dynkin diagram D one can reconstruct the associated affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ up to a permutation of the index set I of A.

To classify the affine Cartan matrices up to simultaneous permutation of rows and columns, it suffices to classify all possible affine Dynkin diagrams. This has been done in [8], and we state the result below.

Theorem 2.2.11. (i) *All possible affine Dynkin diagrams are listed in Figure 2.1 (where the labels at the nodes are not part of the definition of a Dynkin diagram).*

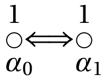
(ii) *Let D be a Dynkin diagram with all the labels as depicted in Figure 2.1. Then there exists an affine Cartan matrix A such that the following two conditions hold*

(1) $D = S(A)$

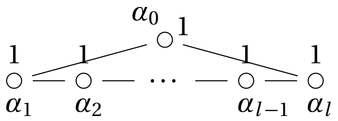
(2) *the numerical label* a_i *of the node* α_i *of the Dynkin diagram D in Figure 2.1 is the i-th coordinate of the unique column vector* $\delta > 0$ *such that* $A\delta = 0$ *where the* a_i *are positive relatively prime integers.*

Proof. See Theorem 4.8 in [8]. □

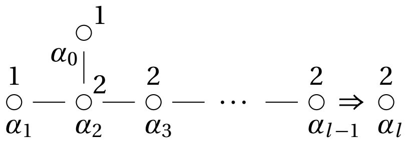
$A_1^u = A_1^t$:



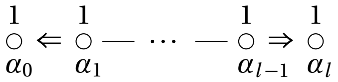
$A_l^u = A_l^t \, (l \geq 2)$:



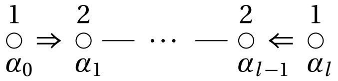
$B_l^u \, (l \geq 3)$:



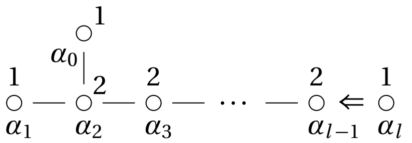
$B_l^t \, (l \geq 2)$:



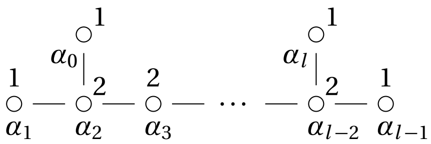
$C_l^u \, (l \geq 2)$:



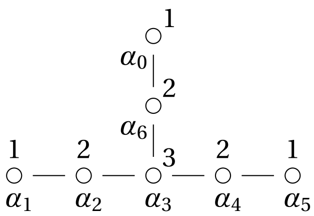
$C_l^t \, (l \geq 3)$:



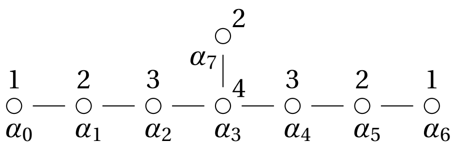
$D_l^u = D_l^t \, (l \geq 4)$:



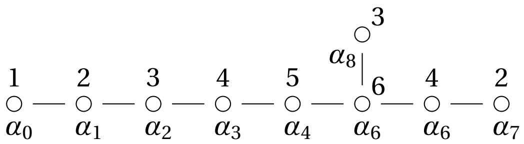
$E_6^u = E_6^t$:



$E_7^u = E_7^t$:



$E_8^u = E_8^t$:



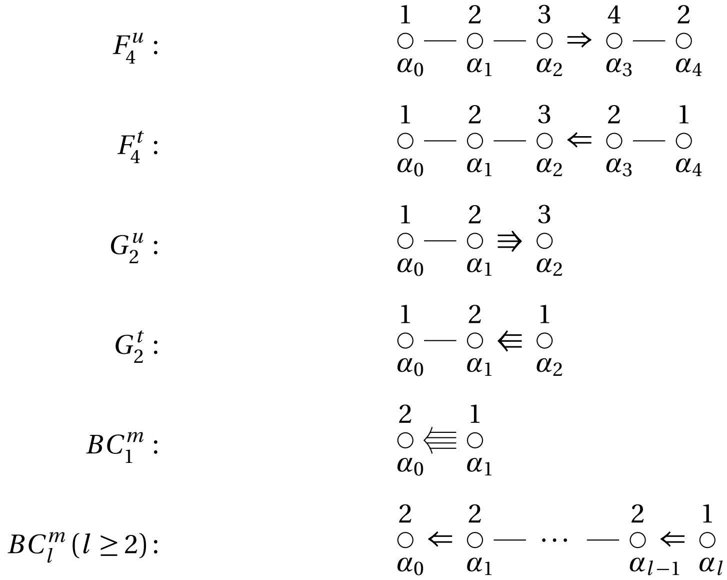


Figure 2.1: All possible affine Dynkin diagrams.

The left column of Figure 2.1 contains the name X_l^j of each affine Dynkin diagram $S(A)$ in the right column where l is the rank of A and X_l^j is called the *type* of A and $S(A)$. The naming of types of affine Dynkin diagrams as listed in Figure 2.1 differs substantially from [8]. This is because Kac classifies the affine Dynkin diagrams according to the construction of their corresponding Kac-Moody algebra, while we will do this according to their corresponding affine root system. The latter will be explained in detail in Section 3.5. For now remember that in our naming it is possible for an affine Dynkin diagrams to be of two types at the same time. For example, in Figure 2.1 there is a Dynkin diagram that is of both type A_2^u and type A_2^t, although type G_2^u and G_2^t correspond to two different affine Dynkin diagrams.

For later purposes we have fixed an enumeration of the nodes of the affine Dynkin diagrams in Figure 2.1 as follows. In Figure 2.1 the nodes of $S(A)$ are enumerated by $\alpha_0, \ldots, \alpha_l$ for a rank l affine Cartan matrix A. If $S(A)$ is of *type* X_l^j where $j \in u, t$ and $X \in \{A, \ldots, G\}$, then the Dynkin subdiagram with nodes $\alpha_1, \ldots, \alpha_l$ is the finite Dynkin diagram of the type X_l. If $j = m$, then $X_l^j = BC_l^m$ and the Dynkin subdiagram with nodes $\alpha_1, \ldots, \alpha_l$ is the finite Dynkin diagram of the type C_l which represents the finite root system of unmultipliable roots of BC_l.

2.3 Affine Lie algebras, the set of real roots and affine root systems

In the previous Section we distinguished two important classes of symmetrizable generalized Cartan matrices, namely finite and affine Cartan matrices. We saw that Kac-Moody algebras corresponding to finite Cartan matrices are just simple finite-dimensional Lie algebras which we treated and classified in Chapter 1. Further, we ended the previous Section with a classification of affine Lie algebras by classifying all affine Cartan matrices up to simultaneous permutation of rows and columns. In this Section we want to discuss affine Lie algebras in more detail. First we will explicitly choose a standard invariant form on a specifically chosen affine Lie algebra within its isomorphism class. Then it will turn out that affine Lie algebras contain a simple Lie algebra as subalgebra and in that sense are infinite-dimensional generalizations of simple Lie algebras. Using our knowledge of simple Lie algebras we will describe the real roots of an affine Lie algebra explicitly, and study its Weyl group. This will lead us to the notion of an affine root system which we will discuss in full detail in the next Chapter. As an example we end this Chapter with an explicit construction of an untwisted affine Lie algebra.

2.3.1 The normalized invariant form

In this Subsection we will fix a standard invariant form for an affine Lie algebra $\mathfrak{g}(A)$.

Let $A=(a_{ij})_{0\le ij\le l}$ be a rank l affine Cartan matrix, and consider its Dynkin diagram $S(A)$ as depicted in Figure 2.1. Assume that the index set of A is ordered such that A satisfies (ii) of Theorem 2.2.11. Put a_i for the numerical label of the node with label α_i of $S(A)$ in Figure 2.1 for $i=0,\ldots,l$. Notice that the transposed A^T of A is also a rank l affine Cartan matrix. Then $S(A)$ coincides with $S(A^T)$, except that the directions of possible arrows of $S(A^T)$ are reversed in comparison with $S(A)$. Consider the Dynkin diagram $S(A^T)$ in Figure 2.1 with the same enumeration of nodes as $S(A)$. Put $a_i^\vee$ for the numerical label of the node of $S(A^T)$ with label α_i for $i=0,\ldots,l$. From Figure 2.1 it follows that

$$a_0^\vee=1.$$

The matrix A is symmetrizable by Proposition 2.2.9, hence there exist $\varepsilon_0,\ldots,\varepsilon_l\in\mathbb{C}$ and a symmetrization B such that $A=\mathrm{diag}(\varepsilon_0,\ldots,\varepsilon_l)B$. Moreover, one can show that there exists a symmetrization B such that

$$A=\mathrm{diag}(a_0a_0^{\vee-1},\ldots,a_la_l^{\vee-1})B.$$

Consider the affine Lie algebra $\mathfrak{g}:=\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ associated to A with $\Pi:=\{\alpha_0,\ldots,\alpha_l\}$ and $\Pi^\vee:=\{\alpha_0^\vee,\ldots,\alpha_l^\vee\}$. Fix an element $d\in\mathfrak{h}$ such that $\langle\alpha_i,d\rangle=\delta_{0i}$ for $i=0,\ldots,l$, then $\{\alpha_0^\vee,\ldots,\alpha_l^\vee,d\}$ is a basis of $\mathfrak{h}$. By Theorem 2.1.9 there exists an invariant nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{g}$ such that $(.,.)$ is nondegenerate on $\mathfrak{h}$ and uniquely defined by

$$\begin{cases}(\alpha_i^\vee,\alpha_j^\vee)=\langle\alpha_i,\alpha_j^\vee\rangle\varepsilon_i=a_ja_j^{\vee-1}a_{ji} & (i,j=0,\ldots,l);\\ (\alpha_i^\vee,d)=0 & (i=1,\ldots,l);\\ (\alpha_0^\vee,d)=a_0;\\ (d,d)=0.\end{cases}$$

Remark 2.1.1 shows that indeed $(\alpha_i^\vee,\alpha_j^\vee)=(\alpha_j^\vee,\alpha_i^\vee)=a_ia_i^{\vee-1}a_{ij}$ for $0\le i,j\le l$.

Consider the vector space isomorphism $\nu:\mathfrak{h}\to\mathfrak{h}^*$ induced by $(.,.)$, and put $\Lambda_0:=a_0^{-1}\nu(d)$. Then $\langle\Lambda_0,\alpha_i^\vee\rangle=\delta_{0i}$ for $i=0,\ldots,l$ and $\langle\Lambda_0,d\rangle=0$, and $\{\alpha_0,\ldots,\alpha_l,\Lambda_0\}$ is a basis of $\mathfrak{h}^*$. The induced bilinear form $(.,.)$ on $\mathfrak{h}^*$ is given by

$$\begin{cases}(\alpha_i,\alpha_j)=a_i^\vee a_i^{-1}a_{ij} & (i,j=0,\ldots,l);\\ (\alpha_i,\Lambda_0)=0 & (i=1,\ldots,l);\\ (\alpha_0,\Lambda_0)=a_0^{-1};\\ (\Lambda_0,\Lambda_0)=0.\end{cases}\tag{2.3.1}$$

Put $\delta=\sum_{i=0}^l a_i\alpha_i$, then we obtain from (ii) of Theorem 2.2.11 and (2.3.1)

$$(\alpha_i,\delta)=0\ \ (i=0,\ldots,l),\qquad (\delta,\delta)=0,\qquad (\Lambda_0,\delta)=1.\tag{2.3.2}$$

Since a_i and $a_i^{\vee-1}$ are positive integers (see Figure 2.1), and $a_{ii} = 2$ we observe from (2.3.1) that $(\alpha_i, \alpha_i) > 0$ for $i = 0, \ldots, l$. Thus $(.,.)$ is a standard invariant form on $\mathfrak{g}$, and this specific choice of $(.,.)$ is called the *normalized invariant form* on $\mathfrak{g}$. It is normalized in the sense that we fixed one specific standard invariant form, while others coincide on $\bigoplus_{i=0}^{l} \mathbb{C}\alpha_i$ up to multiplication with a complex number by (2.1.9). Also notice that restricted to $\bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$ the normalized invariant form is $\mathbb{R}$-valued.

2.3.2 The real roots of an affine Lie algebra

As we will see in this Subsection the affine Lie algebra $\mathfrak{g}$ that was defined in the previous Subsection contains a simple Lie algebra $\mathring{\mathfrak{g}}$ as subalgebra, and the root system Δ of $\mathfrak{g}$ contains the finite root system $\mathring{\Delta}$ of $\mathring{\mathfrak{g}}$. We will explicitly describe the set of real roots Δ^{re} of $\mathfrak{g}$ in terms of δ and $\mathring{\Delta}$. Further, we introduce an element $\hat{\theta}$ related to $\mathring{\Delta}$ to describe the root basis Π in terms of $\mathring{\Delta}$.

First, let us introduce some notation. Denote by $\mathring{\mathfrak{h}}$ (resp. $\mathring{\mathfrak{h}}_{\mathbb{R}}$) the linear span over $\mathbb{C}$ (resp. $\mathbb{R}$) of $\alpha_1^\vee, \ldots, \alpha_l^\vee$. Further denote by $\mathring{\mathfrak{h}}^*$ (resp. $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$) the linear span over $\mathbb{C}$ (resp. $\mathbb{R}$) of $\alpha_1, \ldots, \alpha_l$. Then we have the direct sum of subspaces

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0.$$

By (2.3.1), (2.3.2), (2.1.5) and (2.1.6) one observes that δ and Λ_0 vanish on $\mathring{\mathfrak{h}}$. Hence the set $\mathring{\mathfrak{h}}^* \subset \mathfrak{h}^*$ (resp. $\mathring{\mathfrak{h}}^*_{\mathbb{R}} \subset \mathfrak{h}^*$) can actually be identified with the dual of $\mathring{\mathfrak{h}}$ (resp. $\mathring{\mathfrak{h}}_{\mathbb{R}}$) by restriction to $\mathring{\mathfrak{h}}$ (resp. $\mathring{\mathfrak{h}}_{\mathbb{R}}$).

Consider the matrix $\mathring{A} = (a_{ij})_{1 \le ij \le l}$ which is obtained from A by removing the row and column with index 0. By Corollary 2.2.5 this is a finite Cartan matrix. Let $\mathring{\mathfrak{g}}$ denote the subalgebra of the affine Lie algebra $\mathfrak{g}$ generated by $e_1, \ldots, e_l, f_1, \ldots, f_l$, then by Theorem 1.3.1 $\mathring{\mathfrak{g}}$ is a simple Lie algebra corresponding to the finite Cartan matrix $\mathring{A}$. Now $\mathring{\mathfrak{g}} \cap \mathfrak{h} = \mathring{\mathfrak{h}}$ and $(\mathring{\mathfrak{h}}, \mathring{\Pi}, \mathring{\Pi}^\vee) := (\mathring{\mathfrak{h}}, \{\alpha_1, \ldots, \alpha_l\}, \{\alpha_1^\vee, \ldots, \alpha_l^\vee\})$ is a realization of the matrix $\mathring{A}$, so we observe that $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A}, \mathring{\mathfrak{h}}, \mathring{\Pi}, \mathring{\Pi}^\vee)$. Notice that the finite Dynkin diagram $S(\mathring{A})$ is obtained from $S(A)$ by removing the node α_0 and the vertices attached to it.

Put $\mathring{Q} := \mathrm{span}_{\mathbb{Z}} \mathring{\Pi}$, then the finite set $\mathring{\Delta} := \{\alpha \in \Delta : \alpha \in \mathring{Q}\}$ is the root system of $\mathring{\mathfrak{g}}$. Hence by Corollary 2.2.7 $\mathring{\Delta}$ is a finite root system with $\mathring{\Pi}$ a basis for $\mathring{\Delta}$. Further, put $\mathring{\Delta}_l$ (resp. $\mathring{\Delta}_s$) for the long (resp. short) roots of $\mathring{\Delta}$ with respect to $(.,.)$, and let $\mathring{W}$ be the finite Weyl group of $\mathring{\Delta}$.

Consider the real roots Δ^{re} of $\mathfrak{g} = \mathfrak{g}(A)$ and its corresponding Dynkin diagram $S(A)$ as depicted in Figure 2.1. Then $S(A)$ is of type X_l^j with $j \in \{u, t, m\}$, $X \in \{A, \ldots, G\}$, and $l \in \mathbb{N}$. If $j = u$ we will say that Δ^{re} is of *untwisted type*, if $j = t$ we will say that Δ^{re} is of *twisted type* and if $j = m$ then we will say that Δ^{re} is of *mixed type*. Notice that it is possible for Δ^{re} to be of untwisted and twisted type. Using this terminology we want to describe the real and imaginary roots of the affine Lie algebra $\mathfrak{g}$ explicitly. The set of imaginary roots Δ^{im} turns out to be generated by δ which, together with the root system $\mathring{\Delta}$ of the simple Lie algebra $\mathring{\mathfrak{g}}$, will be used to describe Δ^{re}.

Proposition 2.3.1. (i) $\Delta^{im} = \{n\delta : n \in \mathbb{Z}\}$;

(ii) $\Delta^{re} = \{\alpha + n\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$ *if* Δ^{re} *is of untwisted type;*

(iii) $\Delta^{re} = \{\alpha + n\frac{(\alpha,\alpha)}{2}\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$ *if* Δ^{re} *is of twisted type but not of untwisted type;*

(iv) $\Delta^{re} = \{\alpha + n\frac{(\alpha,\alpha)}{2}\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\} \cup \{\frac{2}{(\alpha,\alpha)}\alpha + (2n+1)\frac{2}{(\alpha,\alpha)}\delta) : \alpha \in \mathring{\Delta}_l, n \in \mathbb{Z}\}$ *if* Δ^{re} *is of mixed type.*

Proof. This follows from Theorem 5.6 and Proposition 6.3 of [8]. □

Remark 2.3.1. One can compute explicitly the *squared root length* (α,α) for each $\alpha\in\Delta^{re}$ as follows. Choose $w \in W$ such that $w(\alpha) = \alpha_i$ for some $\alpha_i \in \Pi$. Then by Proposition 2.1.16 (ii) we have $(\alpha,\alpha)=(\alpha_i,\alpha_i)$. Now the first equation of (2.3.1) we obtain

$$(\alpha_i,\alpha_i)=a_i^\vee a_i^{-1}a_{ii}=2a_i^\vee a_i^{-1}.$$

For Δ^{re} that is only of untwisted type put $r=1$, for Δ^{re} of twisted type (and possibly untwisted) but not type G_2^t put $r=2$ and for Δ^{re} of type G_2^t put $r=3$. Checking all possibilities for a_i and $a_i^\vee$ in Figure 2.1 we observe the following squared root lengths occurring in Δ^{re}.

Δ^{re} type	squared root lengths	(α_0,α_0)
untwisted	$\frac{2}{r},2$	2
twisted, not untwisted type	$2,2r$	2
mixed	$\begin{cases}1,4 & (l=1)\\ 1,2,4 & (l>1)\end{cases}$	1

Recall that if Δ^{re} is of mixed type then $\mathring{\Delta}$ is a finite root system of type C_l. It follows that if $\alpha\in\mathring{\Delta}_l$ (resp. $\beta\in\mathring{\Delta}_s$), then $\alpha\in\Delta^{re}$ and $(\alpha,\alpha)=4$ (resp. $(\beta,\beta)=2$). Then for all $\alpha\in\mathring{\Delta}_l$ we have $\frac{2}{(\alpha,\alpha)}\alpha=\frac{1}{2}\alpha$ and for all $\alpha\in\mathring{\Delta}_s$ we have $\frac{2}{(\alpha,\alpha)}\alpha=\alpha$. In other words, $\mathring{\Delta}\cup\frac{1}{2}\mathring{\Delta}_l$ is a nonreduced irreducible finite root system with $\mathring{\Delta}$ the unmultipliable roots and $\frac{1}{2}\mathring{\Delta}_l$ the indivisible short roots.

Introduce the following element

$$\hat{\theta}:=\delta-a_0\alpha_0=\sum_{i=1}^{l}a_i\alpha_i\in\mathring{Q}, \tag{2.3.3}$$

then by (2.3.1) and (2.3.2)

$$(\hat{\theta},\hat{\theta})=2a_0. \tag{2.3.4}$$

Proposition 2.3.2. (i) *Let Δ^{re} of untwisted type, and let ϕ be the highest root of $\mathring{\Delta}$ with respect to the basis $\mathring{\Pi}=\{\alpha_1,\dots,\alpha_l\}$. Then $\hat{\theta}=\phi$ and $\Pi=\{\alpha_0,\alpha_1,\dots,\alpha_l\}$ with $\alpha_0=\delta-\phi$.*

(ii) *Let Δ^{re} of twisted but not untwisted type, and let θ be the highest short root of $\mathring{\Delta}$ with respect to the basis $\mathring{\Pi}=\{\alpha_1,\dots,\alpha_l\}$. Then $\hat{\theta}=\theta$ and $\Pi=\{\alpha_0,\alpha_1,\dots,\alpha_l\}$ with $\alpha_0=\delta-\theta$.*

(iii) *Let Δ^{re} of mixed type, and let ϕ be the highest root of $\mathring{\Delta}$ with respect to the basis $\mathring{\Pi}=\{\alpha_1,\dots,\alpha_l\}$. Then $\hat{\theta}=\phi$ and $\Pi=\{\alpha_0,\alpha_1,\dots,\alpha_l\}$ with $\alpha_0=\frac{1}{2}(\delta-\phi)$.*

Proof. This follows from Proposition 6.4 of [8]. □

2.3.3 The Weyl group of an affine Lie algebra

In this Subsection we study the Weyl group W of $\mathfrak{g}$ in more detail. It turns out that W is the semidirect product of the finite Weyl group $\mathring{W}$ of $\mathring{\Delta}$ and a translation group over a lattice. Furthermore, we can interpret the action of W on certain real subspaces of $\mathfrak{h}^*$ modulo δ as an affine Weyl group action.

First, notice that $(\delta,\alpha_i)=0$ for $i=0,\dots,l$ by (2.3.2). Since W is generated by the fundamental reflections r_i for $i=0,\dots,l$, we observe from (2.1.13) that $w(\delta)=\delta$ for all $w\in W$. Now let $W'\subset W$ be the subgroup generated by the fundamental reflections r_i for $i=1,\dots,l$. Then $w(\Lambda_0)=\Lambda_0$ for all $w\in W'$, hence W' acts trivially on $\mathbb{C}\delta+\mathbb{C}\Lambda_0$. Also, W' leaves $\mathring{\mathfrak{h}}^*=\bigoplus_{i=1}^l\mathbb{C}\alpha_i$ invariant and W' acts faithfully on $\mathring{\mathfrak{h}}^*$ since W acts faithfully on $\mathfrak{h}^*$ ((i) of Proposition 2.1.16). Hence we can identify W' with $\mathring{W}$ as groups, and we will write $\mathring{W}\subset W$.

After a direct (but perhaps tedious) computation using (2.3.3) and (2.3.4) one can observe that for $\lambda\in\mathfrak{h}^*$ we have

$$r_0 r_{\hat\theta}(\lambda)=\lambda+(\lambda,\delta)a_0^{-1}\hat\theta-((\lambda,a_0^{-1}\hat\theta)+\frac{1}{2}(a_0^{-1}\hat\theta,a_0^{-1}\hat\theta)(\lambda,\delta))\delta. \tag{2.3.5}$$

In general it is interesting to consider for $\alpha\in\mathring{\mathfrak{h}}^*$ the linear endomorphism t_α of $\mathfrak{h}^*$ defined by

$$t_\alpha(\lambda)=\lambda+(\lambda,\delta)\alpha-((\lambda,\alpha)+\frac{1}{2}(\alpha,\alpha)(\lambda,\delta))\delta. \tag{2.3.6}$$

Then $(\Lambda_0,\delta)=1$ and $(\Lambda_0,\mathring{\mathfrak{h}}^*)=0$ imply

$$t_\alpha(\Lambda_0)=\Lambda_0+\alpha-\frac{1}{2}(\alpha,\alpha)\delta \tag{2.3.7}$$

and for $\lambda\in\mathfrak{h}^*$ such that $(\lambda,\delta)=0$ we have

$$t_\alpha(\lambda)=\lambda-(\lambda,\alpha)\delta. \tag{2.3.8}$$

This describes t_α completely on $\mathfrak{h}^*$ by (2.3.2). It follows now from a straightforward calculation that

$$t_\alpha\circ t_\beta=t_{\alpha+\beta},$$

and

$$w\circ t_\alpha\circ w^{-1}=t_{w(\alpha)}$$

for $w\in\mathring{W}$, and from (2.3.5) and (2.3.6) we have

$$r_0=t_{a_0^{-1}\hat\theta}r_{\hat\theta}$$

for the fundamental reflection r_0. These identities inspire the idea that the Weyl group W is the semidirect product of the finite Weyl group $\mathring{W}$ and a lattice that is generated by the $\mathring{W}$-orbit of $a_0^{-1}\hat\theta$. It turns out that this lattice can also be described in terms of the coroot lattice $\mathring{Q}^\vee$ or the root lattice $\mathring{Q}$ of $\mathring{\mathfrak{g}}$.

Proposition 2.3.3. *$W=\mathring{W}\ltimes T$ with $T=\{t_\alpha:\alpha\in M\}$ an abelian group and*

$$M=\mathbb{Z}(\mathring{W}(a_0^{-1}\hat\theta))=\begin{cases}\nu(\mathring{Q}^\vee) & \text{if } \Delta^{re} \text{ is of untwisted type,}\\ \mathring{Q} & \text{if } \Delta^{re} \text{ is of twisted or mixed type,}\end{cases}$$

a lattice that spans $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. In particular, $\nu(\mathring{Q}^\vee)=\mathring{Q}$ if Δ^{re} is of both untwisted and twisted type.

Proof. See Proposition 6.5 of [8]. □

Let us introduce some extra notation so that we can describe the action of W on real subspaces of $\mathfrak{g}^*$. Recall that $\mathring{\mathfrak{h}}^*_{\mathbb{R}} = \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i$ and put

$$\mathfrak{h}^*_{\mathbb{R}} := \mathring{\mathfrak{h}}^*_{\mathbb{R}} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0,$$

then $\Delta \subset \mathfrak{h}^*_{\mathbb{R}}$ and $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*_{\mathbb{R}}$ since $\Delta \subset Q = \sum_{i=0}^{l} \mathbb{Z}\alpha_i$. Notice that the restriction of the normalized invariant form $(.,.)$ on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ (resp. $\mathring{\mathfrak{h}}^*_{\mathbb{R}} \oplus \mathbb{R}\delta$) is $\mathbb{R}$-valued and even positive definite (resp. positive semidefinite and vanishing on $\mathbb{R}\delta$) by (2.1.10) and (2.3.2).

It turns out that W acts on certain hyperplanes of $\mathfrak{h}^*_{\mathbb{R}}$ relative to δ in a very special way. For $s \in \{0,1\}$ consider the following subsets of $\mathfrak{h}^*_{\mathbb{R}}$

$$\mathfrak{h}^*_s := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} : (\lambda, \delta) = s\}.$$

Then $\mathfrak{h}^*_0 = \mathring{\mathfrak{h}}^*_{\mathbb{R}} \oplus \mathbb{R}\delta = \sum_{i=0}^{l} \mathbb{R}\alpha_i$, $\mathfrak{h}^*_1 = \mathfrak{h}^*_0 + \Lambda_0$ and both $\mathfrak{h}^*_0$ and $\mathfrak{h}^*_1$ are W-invariant. By (i) of Proposition 2.1.16 we also observe that W acts faithfully on $\mathfrak{h}^*_0$.

Consider the $\mathbb{R}$-vector space $\mathfrak{h}^*_{\mathbb{R}}$ and its subspace $\mathbb{R}\delta$, then the relation $x \sim y$ if and only if $x - y \in \mathbb{R}\delta$ on $\mathfrak{h}^*_{\mathbb{R}}$ yields the quotient space $\mathfrak{h}^*_{\mathbb{R}}/\mathbb{R}\delta$. Next, consider the subset $\mathfrak{h}^*_1 = \mathring{\mathfrak{h}}^*_{\mathbb{R}} \oplus \mathbb{R}\delta + \Lambda_0$ of $\mathfrak{h}^*_{\mathbb{R}}$. Then for $x, y \in \mathfrak{h}^*_1$ there exist unique $\mu, \nu \in \mathring{\mathfrak{h}}^*_{\mathbb{R}}$ and $\xi, \xi' \in \mathbb{R}$ such that $x = \mu + \xi\delta + \Lambda_0$ and $y = \nu + \xi'\delta + \Lambda_0$. Hence the equivalence relation $\sim$ restricted to $\mathfrak{h}^*_1$ becomes $\mu + \xi\delta + \Lambda_0 \sim \nu + \xi'\delta + \Lambda_0$ if and only if $\mu = \nu$. Write $\overline{\mu + \Lambda_0}$ for the equivalence class of $\mu + \xi\delta + \Lambda_0 \in \mathfrak{h}^*_1$, and put $\mathfrak{h}^*_1/\mathbb{R}\delta$ for the set of equivalence classes of $\mathfrak{h}^*_1$ under $\sim$.

Consider $W = \mathring{W} \ltimes T$ with $T = \{t_\alpha : \alpha \in M\}$ as in Proposition 2.3.3, then we obtain an induced action of W on $\mathfrak{h}^*_1/\mathbb{R}\delta$ as follows. Since $\mathring{W}$ fixes δ and Λ_0 we get

$$w(\overline{\mu + \Lambda_0}) = \overline{w(\mu) + \Lambda_0} \tag{2.3.9}$$

for $w \in \mathring{W}$, and $\mathring{W}$ acts faithfully on $\mathfrak{h}^*_1/\mathbb{R}\delta$ since $\mathring{W}$ acts faithfully on $\mathring{\mathfrak{h}}^*$. Further, (2.3.7) and (2.3.8) imply

$$t_\alpha(\overline{\mu + \Lambda_0}) = \overline{\mu + \alpha + \Lambda_0} \tag{2.3.10}$$

for $\alpha \in M \subset \mathring{\mathfrak{h}}^*_{\mathbb{R}}$, and it immediately follows that T acts faithfully on $\mathfrak{h}^*_1/\mathbb{R}\delta$. Thus we conclude by Proposition 2.3.3 that W acts faithfully on $\mathfrak{h}^*_1/\mathbb{R}\delta$.

Notice that $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ considered as an abelian group acts faithfully and transitively on $\mathfrak{h}^*_1/\mathbb{R}\delta$ by $\mathring{\mathfrak{h}}^*_{\mathbb{R}} \times \mathfrak{h}^*_1/\mathbb{R}\delta \to \mathfrak{h}^*_1/\mathbb{R}\delta, (\mu, \overline{\nu + \Lambda_0}) \mapsto \overline{\mu + \nu + \Lambda_0}$. In the language of Chapter 3 we call $\mathfrak{h}^*_1/\mathbb{R}\delta$ an *affine space* with *space of translations* $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. Further, we can identify $\mathfrak{h}^*_1/\mathbb{R}\delta$ with $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ by the projection $\pi(\overline{\nu + \Lambda_0}) = \nu$. This gives a bijective correspondence $\pi : \mathfrak{h}^*_1/\mathbb{R}\delta \to \mathring{\mathfrak{h}}^*_{\mathbb{R}}$. Forcing the vector space structure of $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ onto $\mathfrak{h}^*_1/\mathbb{R}\delta$ through π we can view $\mathfrak{h}^*_1/\mathbb{R}\delta$ as a vector space with origin Λ_0. Since we identify $\mathfrak{h}^*_1/\mathbb{R}\delta$ with $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ by the means of projection it is clear that this induces a symmetric bilinear form on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ that is just the restriction of $(.,.)$ onto $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. This form on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ is positive definite as we noted earlier.

The action of W on $\mathfrak{h}^*_1/\mathbb{R}\delta$ translates to a (nonlinear) action of W on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ which we denote by 'af' through the group isomorphism $\mathrm{af} : W \to \mathrm{af}(W)$ defined by $\mathrm{af}(w) = \pi \circ w \circ \pi^{-1}$. This leads to

$$\mathrm{af}(w)(\mu) = w(\mu)$$

for $w \in \overset{\circ}{W}$ and $\mu \in \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ by (2.3.9), and

$$\mathrm{af}(t_\alpha)(\mu) = \mu + \alpha$$

for $\alpha \in M$ and $\mu \in \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ by (2.3.10). Hence $\mathrm{af}(T)$ acts as a group on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ by translation over the lattice $M \subset \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$. This shows that if Δ^{re} is of untwisted type, then $\mathrm{af}(W) = \overset{\circ}{W} \ltimes \mathrm{af}(T)$ is the *affine Weyl group* of the finite root system $\overset{\circ}{\Delta}$ in the language of [2] and [7] which explains the word 'affine' in our terminology.

2.3.4 The set of real roots as an affine root system

In this Subsection we will compare some important properties of $\overset{\circ}{\Delta}$ and Δ^{re}. We want to observe that Δ^{re} satisfies axioms analogous to the axioms of a reduced irreducible finite root system. This will give rise to a new root system called an affine root system which we will rigorously introduce in Chapter 3.

A root $\beta \in \overset{\circ}{\Delta}$ defines a linear functional on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ by the mapping $\mu \mapsto (\beta, \mu)$. Further r_β is a reflection of $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ in the kernel this linear functional which is the hyperplane orthogonal to β. As we will see in Example 3.2.3, we can interpret Δ^{re} as a set of so called 'affine linear functions' on the 'affine Euclidean space' $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$. This means that real roots are linear functionals on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ composed with a translation of $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$, and that $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ is a vector space where we have 'forgotten the origin'. Then $\mathrm{af}(r_{\alpha+n\delta})$ for $\alpha + n\delta \in \Delta^{re}$ can be thought of as the reflection of $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ in the kernel of $\alpha + n\delta$. In the following calculations we want to show that this generalizes the action of r_β for $\beta \in \overset{\circ}{\Delta}$ on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$.

Let $\alpha \in \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ and $n \in \mathbb{R}$ such that $\alpha + n\delta \in \Delta^{re}$ and let $\lambda \in \mathfrak{h}^*_1$, then by (2.1.13), (2.3.1) and (2.3.2) we have

$$\begin{aligned} r_{\alpha+n\delta}(\lambda) &= \lambda - 2\frac{(\lambda, \alpha+n\delta)}{(\alpha+n\delta, \alpha+n\delta)}(\alpha+n\delta) \\ &= \lambda - 2\frac{(\lambda,\alpha)+n}{(\alpha,\alpha)}(\alpha+n\delta) \end{aligned}$$

Since $\lambda \in \mathfrak{h}^*_1$ there exist $\mu \in \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ and $\xi \in \mathbb{R}$ such that $\lambda = \mu + \xi\delta + \Lambda_0$. So on the class of λ in $\mathfrak{h}^*_1/\mathbb{R}\delta$ we have

$$r_{\alpha+n\delta}(\overline{\lambda}) := \overline{r_{\alpha+n\delta}(\lambda)} = \overline{\mu - 2\frac{(\mu,\alpha)+n}{(\alpha,\alpha)}\alpha + \Lambda_0}.$$

This shows that

$$\mathrm{af}(r_{\alpha+n\delta})(\mu) = \mu - 2\frac{(\mu,\alpha)+n}{(\alpha,\alpha)}\alpha \tag{2.3.11}$$

defines a reflection of $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ in the translated hyperplane $\{\mu \in \overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}} : (\mu,\alpha) = -n\}$. In other words, the action of $\mathrm{af}(r_{\alpha+n\delta})$ on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ coincides with a translation of the action of r_α on $\overset{\circ}{\mathfrak{h}}^*_{\mathbb{R}}$ that is uniquely defined by n and α.

Although Δ^{re} is an infinite set, which one directly observes from Proposition 2.3.1, we will see in the following Theorem that Δ^{re} satisfies some axioms that are analogous to the axioms of a reduced irreducible finite root system.

Theorem 2.3.4. (i) Δ^{re} *spans* $\mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$*, and* $(\alpha,\alpha) > 0$ *for all* $\alpha \in \Delta^{re}$*;*

(ii) $r_\alpha(\beta) \in \Delta^{re}$ *for all* $\alpha, \beta \in \Delta^{re}$*;*

(iii) $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Delta^{re}$*;*

(iv) af(W) *considered as a discrete group acts properly on* $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$*;*

(v) *Let* $p : \mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \to \mathring{\mathfrak{h}}_{\mathbb{R}}^*$ *be the projection along the direct sum. For each* $\alpha \in \Delta^{re}$ *the fiber* $p(\alpha)^{-1}$ *contains at least 2 distinct elements;*

(vi) *Let* $\alpha \in \Delta^{re}$*, then* $\mathbb{R}\alpha \cap \Delta^{re} = \{\pm\alpha\}$*;*

(vii) *There do not exist nonempty subsets* $\Delta_1, \Delta_2 \subset \Delta^{re}$ *such that* $\Delta_1 \amalg \Delta_2 = \Delta^{re}$ *and* $(\alpha,\beta) = 0$ *for all* $\alpha \in \Delta_1$*,* $\beta \in \Delta_2$*.*

Proof. (i) Trivially, $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ spans $Q = \sum_{i=0}^{l} \mathbb{Z}\alpha_i$ over $\mathbb{Z}$. Further $\Pi \subset \Delta^{re} \subset Q$, thus Δ^{re} spans $\mathfrak{h}_0^* = \sum_{i=0}^{l} \mathbb{R}\alpha_i$ over $\mathbb{R}$. Further by Lemma 2.1.17 (or Proposition 2.3.1) we observe that $(\alpha,\alpha) > 0$ for all $\alpha \in \Delta^{re}$.

(ii) Let $\alpha \in \Delta^{re}$, then $r_\alpha \in W$ by (2.1.14). So for $\beta \in \Delta^{re}$ we have $r_\alpha(\beta) \in \Delta^{re}$ since Δ^{re} is W-invariant.

(iii) Let $\alpha, \beta \in \Delta^{re}$, then there exists $w \in W$ and $\alpha_i \in \Pi$ such that $w(\alpha) = \alpha_i$. Hence by (ii) of Proposition 2.1.16 we have

$$2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{(w(\beta),\alpha_i)}{(\alpha_i,\alpha_i)}.$$

Since $w(\beta) \in \Delta \subset Q$ there exist $c_0, \ldots, c_l \in \mathbb{Z}$ such that $w(\beta) = \sum_{j=0}^{l} c_j \alpha_j$, hence

$$2\frac{(w(\beta),\alpha_i)}{(\alpha_i,\alpha_i)} = \sum_{j=0}^{l} c_j \cdot 2\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$$

But $2\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} = a_{ij} \in \mathbb{Z}$ by (2.1.11), so clearly $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(iv) We have af$(W) = \mathring{W} \ltimes$ af(T) by Proposition 2.3.3 and the definition of af. Here $\mathring{W}$ is a finite group by Proposition 2.2.6, and af(T) considered as a discrete group acts properly on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ since the lattice M is a discrete set in $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$. This implies that af$(W)$ considered as a discrete group acts properly on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$.

(v) This follows directly from the explicit descriptions of Δ^{re} in Proposition 2.3.1.

(vi) First, the root basis Π is contained in Δ^{re} by definition of Δ^{re}. Second, $r_i(\alpha_i) = -\alpha_i \in \Delta^{re}$ for $i = 0, \ldots, l$ since Δ^{re} is W-invariant. Then by (ii) of Theorem 2.1.5 we observe that $\mathbb{R}\alpha_i \cap \Delta^{re} = \{\pm\alpha_i\}$ for $i = 0, \ldots, l$. Now let α be any real root and assume that $c\alpha \in \Delta^{re}$ for some $c \in \mathbb{R}$, then $c \neq 0$ and there exist $w \in W$ and $\alpha_i \in \Pi$ such that $w(c\alpha) = \alpha_i$. Hence by linearity of w on $\mathfrak{h}^*$ we observe that $w(\alpha) = c^{-1}\alpha_i \in \Delta^{re}$. Thus we observe that $c = \pm 1$ and $\mathbb{R}\alpha \cap \Delta^{re} = \{\pm\alpha\}$.

(vii) This follows pretty straightforwardly from the fact that $A = (a_{ij})_{0 \leq i,j \leq l}$ is indecomposable: Assume that there exist nonempty subsets $\Delta_1, \Delta_2 \subset \Delta^{re}$ such that $\Delta_1 \amalg \Delta_2 = \Delta^{re}$ and $(\alpha,\beta) = 0$ for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$. Also assume that $\Pi \subset \Delta_1$. Let $\beta \in \Delta_2$ then there exists $\alpha_i \in \Pi$ and $w \in W$ such that $w(\beta) = \alpha_i$. Since w is a composition of finitely many fundamental reflections, there exists $\gamma \in \Delta_2$ and $\alpha_j \in \Pi$ such that $r_j(\gamma) = \gamma - 2\frac{(\gamma,\alpha_j)}{(\alpha_j,\alpha_j)}\alpha_j \in \Delta_1$. But $(\gamma,\alpha_j) = 0$ since $\gamma \in \Delta_2$ and $\alpha_j \in \Delta_1$, so $r_j(\gamma) = \gamma \in \Delta_2$. This is a contradiction with the fact that $r_j(\gamma) \in \Delta_1$. Hence we obtain that $\Pi \nsubseteq \Delta_1$, and similarly $\Pi \nsubseteq \Delta_2$.

Recall that by (2.1.11) we have $a_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Now choose $\alpha_i \in \Delta_1 \cap \Pi$ and $\alpha_j \in \Delta_2 \cap \Pi$, then $(\alpha_i,\alpha_j) = 0$ so also $a_{ij} = 0$. This shows that A is decomposable which contradicts the assumption that it is not. Hence there do not exist nonempty subsets $\Delta_1, \Delta_2 \subset \Delta^{re}$ such that $\Delta_1 \amalg \Delta_2 = \Delta^{re}$ and $(\alpha, \beta) = 0$ for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$. $\square$

Let us briefly analyze the statements of Theorem 2.3.4 with regards to the definition of a reduced irreducible finite root system. Statement (i) says that Δ^{re} spans $\mathfrak{h}_0^*$, and it implies that Δ^{re} does not contain 0 or any element of $\mathfrak{h}_0^*$ with vanishing squared length with respect to $(.,.)$. Although Δ^{re} is an infinite set, statement (iv) makes sure that Δ^{re} is not too 'large'. Now (ii) and (iii) are straightforward generalizations of (2) and (3) of Definition 1.2.3. Notice that (2.1.9) implies that the integer $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ does not depend on the choice of $(.,.)$. Further, (v) is an extra axiom of which its utility will be explained in Remark 3.2.3. Finally, (vi) captures the 'reducedness' and (vii) the 'irreducibility' of Δ^{re}.

In the language of Chapter 3, Theorem 2.3.4 asserts that Δ^{re} is an *reduced irreducible affine root system* on the *affine Euclidean space* $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ together with the form $(.,.)$. In that Chapter we will define these notions rigorously, and analyze them in a general setting. Our main goal will be to show that reduced irreducible affine root systems are in bijective correspondence with affine Cartan matrices. Together with the results of this Chapter, this will give us a commutative triangle like (1.3.1) but then for affine Lie algebras, affine root systems and affine Cartan matrices.

2.3.5 An explicit construction of untwisted affine Lie algebras

Kac realizes a complete set of representatives for the isomorphism classes of affine Lie algebras (see [8]). He distinguishes between *untwisted* and *twisted affine Lie algebras*. This terminology coincides with our terminology of untwisted and twisted type sets of real roots in the sense that each (un)twisted set of real roots corresponds to an (un)twisted affine Lie algebra. What we call mixed type, Kac also considers to be of twisted type. To give an example of Lie algebras isomorphic to affine Lie algebras we will now briefly show how to construct the untwisted affine Lie algebra $\hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ associated to the simple Lie algebra $\mathring{\mathfrak{g}}$. Here we still consider $\mathring{\mathfrak{g}}$ as subalgebra of the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)$ with all the associated notation as introduced earlier in this Section such that Δ^{re} is untwisted.

Let $\mathscr{L} := \mathbb{C}[t, t^{-1}]$ be the algebra of *Laurent polynomials* $P = \sum_{k\in\mathbb{Z}} c_k t^k$ (where only finitely many $c_k \in \mathbb{C}$ are nonzero). Consider the *loop algebra*

$$\mathscr{L}(\mathring{\mathfrak{g}}) := \mathscr{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}$$

which is an infinite-dimensional Lie algebra with bracket $[.,.]_0$ defined by

$$[t^m \otimes x, t^n \otimes y]_0 := t^{m+n} \otimes [x, y]$$

for all $m, n \in \mathbb{Z}$ and $x, y \in \mathring{\mathfrak{g}}$. Introduce an element K, and consider the vector space

$$\tilde{\mathscr{L}}(\mathring{\mathfrak{g}}) := \mathscr{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}K.$$

Defining for all $m, n \in \mathbb{Z}$, $x, y \in \mathring{\mathfrak{g}}$ and $\lambda, \mu \in \mathbb{C}$ the bracket

$$[t^m \otimes x + \lambda K, t^n \otimes y + \mu K]_1 := [t^m \otimes x, t^n \otimes y]_0 + m\delta_{m,-n}(x, y)K$$

turns $\tilde{\mathscr{L}}(\mathring{\mathfrak{g}})$ into a Lie algebra with center $\mathbb{C}K$. Finally, introduce the element d and consider

$$\hat{\mathscr{L}}(\mathring{\mathfrak{g}}) := \tilde{\mathscr{L}}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}d$$

with bracket

$$[(t^m \otimes x)+\lambda K+\mu d,(t^n \otimes y)+\lambda_1 K+\mu_1 d]_2 := [t^m \otimes x+\lambda K, t^n \otimes y+\lambda_1 K]_1+\mu n t^n \otimes y-\mu_1 m t^m \otimes x$$

for all $m,n \in \mathbb{Z}$, $x,y \in \mathring{\mathfrak{g}}$ and $\lambda,\lambda_1,\mu,\mu_1 \in \mathbb{C}$. Then $\hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ is a Lie algebra with center $\mathbb{C}K$ such that d acts on $\mathscr{L}$ as $t\frac{d}{dt}$ in the bracket $[.,.]_2$.

We can identify $\mathring{\mathfrak{g}}$ with the subalgebra $1\otimes\mathring{\mathfrak{g}} \subset \hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ by the map $x \mapsto 1\otimes x$. Furthermore,

$$\mathfrak{h}' := \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

is an $(l+2)$-dimensional commutative subalgebra of $\hat{\mathscr{L}}(\mathring{\mathfrak{g}})$. Let $\delta' \in \mathfrak{h}'^*$ such that $\delta'|_{\mathring{\mathfrak{h}}\oplus\mathbb{C}K} = 0$, $\langle \delta', d\rangle = 1$. Further, extend $\lambda \in \mathring{\mathfrak{h}}^*$ to a linear function on $\mathfrak{h}'$ by putting $\langle \lambda, K\rangle = \langle \lambda, d\rangle = 0$. In this way we can identify $\mathring{\mathfrak{h}}^*$ with a subspace of $\mathfrak{h}'^*$.

Put

$$\Delta' := \{j\delta'+\gamma : j\in\mathbb{Z}, \gamma\in\mathring{\Delta}\} \cup \{j\delta' : j\in\mathbb{Z}\} \subset \mathfrak{h}'^*,$$

then $\hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ has a root space decomposition with respect to $\mathfrak{h}'$ namely

$$\hat{\mathscr{L}}(\mathring{\mathfrak{g}}) = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha\in\Delta'} \mathscr{L}(\mathring{\mathfrak{g}})_\alpha \right)$$

where $\mathscr{L}(\mathring{\mathfrak{g}})_{j\delta'+\gamma} = t^j \otimes \mathring{\mathfrak{g}}_\gamma$ and $\mathscr{L}(\mathring{\mathfrak{g}})_{j\delta'} = t^j \otimes \mathring{\mathfrak{h}}$ for all $j\in\mathbb{Z}$ and $\gamma\in\mathring{\Delta}$.

Finally, let ϕ denote the highest root of $\mathring{\Delta}$ with respect to the basis $\mathring{\Pi} = \{\alpha_1,\ldots,\alpha_l\}$. Put

$$\Pi' := \{\alpha'_0 := \delta'-\phi, \alpha'_1 := \alpha_1, \ldots, \alpha'_l := \alpha_l\}$$

and

$$\Pi'^\vee := \{\alpha'^\vee_0 := \frac{2}{(\phi,\phi)}K - 1\otimes\phi^\vee, \alpha'^\vee_1 := 1\otimes\alpha_1^\vee, \ldots, \alpha'^\vee_l := 1\otimes\alpha_l^\vee\}.$$

Using Proposition 2.1.6 one can now show that there exists an isomorphism $\phi : \mathfrak{g}(A,\mathfrak{h}',\Pi',\Pi'^\vee) \to \hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ such that $\phi(\mathfrak{h}') = \mathfrak{h}'$, $\phi(\Pi'^\vee) = \Pi'^\vee$ and $\phi^*(\Pi') = \Pi'$. By Theorem 2.1.7 there exists a realization preserving isomorphism between $\mathfrak{g}(A,\mathfrak{h}',\Pi',\Pi'^\vee)$ and $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$. Hence there exists an isomorphism $\psi : \mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee) \to \hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$, $\psi(\Pi^\vee) = \Pi'^\vee$ and $\psi^*(\Pi') = \Pi$. Using Remark 2.2.2 and Proposition 2.3.1 we observe that we can consider $\{j\delta'+\gamma : j\in\mathbb{Z}, \gamma\in\mathring{\Delta}\} = \psi^{*-1}(\Delta^{re})$ as the real roots of $\hat{\mathscr{L}}(\mathring{\mathfrak{g}})$ with respect to $\mathfrak{g}(A,\mathfrak{h},\Pi,\Pi^\vee)$ and ψ.

Chapter 3

Affine Root Systems

In this chapter we will exhibit the abstract theory of affine root systems based on the axioms we found in Theorem 2.3.4 for Δ^{re}. Similarly to the abstract theory of finite root systems, no knowledge of Lie algebras is required. Although affine root systems have already been studied some time ago (see [10]), this thesis contains a slightly adapted and more detailed version of the theory obtained from these first explorations. First we describe the landscape of affine root systems, namely affine linear functions on affine Euclidean space and reflections of affine Euclidean space. Then we define affine root systems and notions as their duals, affine root subsystems, irreducibility, reducedness and similarity. We will use the geometry of affine root systems to define a special set of generators called the basis of an affine root system, which differs substantially from the definition of a basis of a finite root system. We classify all reduced irreducible affine root systems up to similarity by obtaining a bijective correspondence between reduced irreducible affine root systems up to similarity and affine Cartan matrices up to simultaneous permutations of rows and columns. This leads to an explicit construction of a reduced irreducible affine root system for each similarity class, which allows us to explain the naming of the classification of affine Dynkin diagrams. We finish the Chapter by relating the obtained correspondence to the bijective correspondence between affine Cartan matrices up to simultaneous permutations of rows and columns and isomorphism classes of Lie algebras that are isomorphic to affine Lie algebras to finally obtain an analogue of (1.3.1).

3.1 Affine linear functions and orthogonal reflections in affine Euclidean space

In this section we define Euclidean spaces without a specified origin, so called affine Euclidean spaces. Natural maps between such spaces arise by composing linear maps with translations. Maps that have special interest to us are functions to the reals $\mathbb{R}$, and so called orthogonal reflections in affine hyperplanes which generalize orthogonal reflections in hyperplanes in vector spaces. This Section serves to set up all the formalities to define and study affine root systems rigorously in the next Sections.

3.1.1 Affine space and affine linear maps

Definition 3.1.1. Let E be a set and let V be a finite-dimensional vector space (over a field K) acting as abelian group faithfully and transitively on E. We call E an *affine space* (*over* K), V its *space of translations* and the elements of V *translations of E*. The dimension of V over K is also said to be the *dimension* of E over K .

Example 3.1.2. Let V be a finite-dimensional vector space over a field K, and consider the action of V as an abelian group on itself by addition of vectors in V. Then V is an affine space over K in a natural way with itself as its space of translations.

Since V is an abelian group, we have $v(w(x)) = (v+w)(x) = (w+v)(x) = w(v(x))$ for all $v, w \in V$ and $x \in E$. Now assume $v(x) = x$ for some $v \in V$ and $x \in E$, then $v(w(x)) = w(v(x)) = w(x)$ for any $w \in V$. Transitivity of the group action of V on E then implies $v(y) = y$ for all $y \in E$, hence by faithfulness of the group action v must be the zero vector of V. In other words, the group action of V on E is simply transitive.

Fix a point $x \in E$, then the simply transitive group action of V on E gives a bijective correspondence between the translations of E and the points of E by sending $v \in V$ to $v(x) \in E$. Write the action of v on x as $x + v$, and if $y = x + v$ for $y \in E$ we write $v = y - x$ which is well defined because the group action is simply transitivity. We can now let E inherit the K-vector space structure of V by defining addition in E by $(x+v)+(x+w) = x+(v+w)$ and scalar multiplication by $\lambda(x+v) = x+\lambda v$ for all $v, w \in V$ and $\lambda \in K$. Let E_x denote the vector space E with x fixed, then we have a natural linear isomorphism $x + v \mapsto v$ between E_x and V. Notice that x plays the role of the origin in E_x, so in some sense an affine space E is a vector space where we forgot what the origin was. In the remaining of this Chapter we will identify an affine space E with its space of translations V as the vector space E_x in the way just described, when an origin $x \in E$ is chosen.

Remark 3.1.1. The simply transitive group action of V on E guarantees that each translation $v \in V$ of E generates a unique *translation map* $t_v : E \to E$ defined by $t_v(y) = y + v$. Choosing an origin $x \in E$ and identifying E as E_x with the vector space V we obtain the unique *translation map* $t'_v : V \to V$ by $t_v(w) = w + v$ for all $w \in V$. Since V is a vector space the map t_v is a bijection of V. Furthermore, t_v is linear if and only if $v = 0$. In the remaining of this chapter we will make no distinction in the notation of the maps t_v and t'_v, and we will write t_v for both of them specifying the domain clearly.

Definition 3.1.3. A nonempty subset L of an affine space E with space of translations V is said to be an *affine subspace* of E if there exists $x \in L$ and a subspace $U \subset V$ such that L is the U-orbit of x under the action of V on E. The subspace U is called the *space of translations* of L, and in particular if $U \subset V$ is of codimension 1 (resp. dimension 1), or a *hyperplane* (resp. *line*) in V, then L is said to be an *affine hyperplane* (resp. *affine line*) in E. Two distinct affine hyperplanes are said to be *parallel* if their spaces of translations coincide, and *nonparallel* otherwise.

Note that in the above Definition the subspace $U \subset V$ does not depend on the choice of $x \in L$. Further, V acts simply transitively on E, hence U acts simply transitively on L. This shows that an affine subspace L of an affine space E, as in the above definition, is an affine space of itself with space of translations U.

Proposition 3.1.4. *Two distinct affine hyperplanes H resp. H' in an affine space E with spaces of translations U resp. U' in V have a nonempty intersection if and only if H and H' are nonparallel affine hyperplanes in E. Furthermore, this nonempty intersection $H \cap H'$ is an affine subspace in E and an affine hyperplane in H and H' with space of translations $U \cap U'$.*

Proof. If $\dim(V) = 1$, then $\{0\} \subset V$ is the only subspace of codimension 1. Hence all affine hyperplanes in E are one-point sets, and all pairs of distinct affine hyperplanes in E are parallel and have an empty intersection. So the Proposition holds in this case and we may assume in the following that $\dim(V) > 1$.

Let H resp. H' be two distinct affine hyperplanes in E with spaces of translations U resp. U' in V, and choose $x \in H$ and $y \in H'$. Let H and H' be parallel affine hyperplanes , then $U = U'$, $H = x + U$ and $H' = y + U$. Assume now that $H \cap H' \neq \emptyset$ and let $z \in H \cap H'$, then by transitivity of the group action of U on H and H' we get $H = x + U = z + U = y + U = H'$. This contradicts the fact that H and H' are distinct. Hence we conclude that $H \cap H' = \emptyset$ and that two distinct affine hyperplanes with nonempty intersection are nonparallel.

On the other hand, if H and H' are nonparallel, then U and U' are two distinct hyperplanes in V. Hence $U \cap U'$ is of codimension 2 in V, and $U \cap U' \neq \emptyset$ since $\dim(V) > 1$. Let $x \in H$ and $y \in H'$, then by transitivity of the action of U (resp. U') on H (resp. H') there exist $v \in U$ and $v' \in U'$ such that $x + v = y + v' \in H \cap H'$. Hence $H \cap H' \neq \emptyset$, so the first statement of the Proposition is proved.

Define $L := H \cap H'$ and $W := U \cap U'$, then for $x \in L$ it is clear by simple transitivity that $L = x + W$. Now the second statement of the Proposition follows. □

Next, we proceed with the description of natural maps between affine spaces. On the one hand these maps should have a linear character on the underlying space of translation. On the other hand, as an affine space does not have a specified origin these maps may also carry a translation.

Definition 3.1.5. A map $a : E \to E'$ between affine spaces E resp. E' over the same field K with spaces of translations V resp. V' is called *affine linear* is there exists a K-linear map $Da : V \to V'$ such that for all $x \in E$ and $v \in V$

$$a(x + v) = a(x) + Da(v).$$

Let $\mathrm{Hom}(E, E')$ denote the set of affine linear maps from E to E'.

Example 3.1.6. (i) For $v \in V$ the translation map t_v of E by v (see Remark 3.1.1) is an affine linear map of E to itself such that Dt_v is the identity map id_V on V. On the other hand, if t is an affine linear map of E to itself such that $Dt = \mathrm{id}_V$, then $t = t_v$ for some $v \in V$. In particular, the identity map id_E on E coincides with t_0.

(ii) Let E, E' and E'' be affine spaces with spaces of translations V, V' and V'' respectively, all over the same field K. Consider $a \in \mathrm{Hom}(E, E')$, $b \in \mathrm{Hom}(E', E'')$, $x \in E$ and $v \in V$, then we observe from affine linearity of both a and b that

$$\begin{aligned}(b \circ a)(x + v) &= b(a(x + v)) \\ &= b(a(x) + Da(v)) \\ &= b(a(x)) + Db(Da(v)) \\ &= (b \circ a)(x) + (Db \circ Da)(v).\end{aligned}$$

Since $Da \in \mathrm{Hom}_K(V, V')$ and $Db \in \mathrm{Hom}_K(V', V'')$ we have $Db \circ Da \in \mathrm{Hom}_K(V, V'')$, so $b \circ a \in \mathrm{Hom}(E, E'')$ with

$$D(b \circ a) = Db \circ Da. \tag{3.1.1}$$

(iii) Let $a : E \to E'$ be an affine linear map such that $a(x) = y$ for some $x \in E$ and $y \in E'$, then $a : E_x \to E'_y$ is a linear map sending $x + v$ to $y + Da(v)$. Conversely, let $a : E_x \to E'_y$ be a linear map, and let $\phi : E_x \to V$ (resp. $\phi' : E'_y \to V'$) be the canonical linear isomorphism determined by $x + v \mapsto v$ (resp. $y + w \mapsto w$). Then $\alpha := \phi' \circ a \circ \phi^{-1} : V \to V'$ is the linear map such that

$a(x+v)=y+\alpha(v)$ for all $v\in V$. Now $a(x)=y$, and for $z\in E$ and $v\in V$ there exists a unique $w\in V$ such that $z=x+w$. Therefore

$$\begin{aligned} a(z+v) &= a(x+w+v) \\ &= a(x)+\alpha(w+v) \\ &= a(x)+\alpha(w)+\alpha(v) \\ &= a(x+w)+\alpha(v) \\ &= a(z)+\alpha(v). \end{aligned} \tag{3.1.2}$$

which implies that $a: E\to E'$ is an affine linear map satisfying $a(x)=y$.

Fix $x\in E$ and $y\in E'$, and let $\mathrm{Hom}_K(E_x,E'_y)$ denote the K-vector space of K-linear maps from E_x to E'_y, here considered as the set of affine linear maps from E to E' sending x to y. For $a\in \mathrm{Hom}_K(E_x,E'_y)$ it is clear from (ii) that $t_{v'}\circ a\in \mathrm{Hom}(E,E')$ for all $v'\in V'$. On the other hand, if $a\in\mathrm{Hom}(E,E')$ such that $a(x)=z$, then there exists a unique $v'\in V'$ such that $t_{v'}(z)=z+v'=y$. This implies that $(t_{v'}\circ a)(x)=y$, so $t_{v'}\circ a\in\mathrm{Hom}_K(E_x,E'_y)$ by (i). We conclude that

$$\mathrm{Hom}(E,E')=\coprod_{v'\in V'}(t_{v'}\,\mathrm{Hom}_K(E_x,E'_y)) \tag{3.1.3}$$

where $t_{v'}$ acts on each $a\in\mathrm{Hom}_K(E_x,E'_y)$ by $t_{v'}a=t_{v'}\circ a$. Also, we observe that $a:E\to E'$ is an affine linear bijection if and only if $a:E_x\to E'_y$ is a linear isomorphism. Moreover, its inverse $a^{-1}:E'_y\to E_x$ is also a linear isomorphism with $D(a^{-1})=(Da)^{-1}$, so $a^{-1}\in\mathrm{Hom}(E',E)$. We call such affine linear bijections *affine linear isomorphisms*. The translation maps t_v of E by $v\in V$ are examples of affine linear isomorphisms.

An affine linear map $a:E\to E'$ in the context of Definition 3.1.5 respects affine subspaces in the sense that an affine subspace $L\subset E$ gets mapped onto an affine subspace $L'\subset E'$. Indeed, write $L=x+U$ for some $x\in L$ with $U\subset V$ the space of translations of L, then $a(L)=a(x+U)=a(x)+Da(U)$ where $U':=Da(U)\subset V'$ is a linear subspace. By Definition 3.1.3, $L':=a(L)\subset E'$ is an affine linear subspace with space of translations U', and $\dim(U')\le\dim(U)$. In particular, if $a:E\to E'$ is an affine linear isomorphism, then $Da:V\to V'$ is a linear isomorphism. Hence in this case affine hyperplanes get mapped onto affine hyperplanes. Furthermore, no two distinct affine hyperplanes in E get mapped to the same affine hyperplane in E' and we exhaust all affine hyperplanes in E' this way.

3.1.2 Affine Euclidean space and affine linear functions

From now on we will fix K to be the field of real numbers $\mathbb{R}$, so that the space of translations V becomes an l-dimensional real vector space. Further, we turn V into a Euclidean space by equipping it with a real positive definite symmetric bilinear form $(.,.)_V$. Let $|v|_V:=\sqrt{(v,v)_V}$ for all $v\in V$, then the affine space E becomes a metric space with the distance function $|x-y|_E:=|v|_V$ for $x,y\in E$ and $v\in V$ such that $y=x+v$. We call a real affine space E endowed with the metric $|.|_E$ in this way an *affine Euclidean space*. If we choose an origin x in an affine Euclidean space E and identify E with the vector space V as E_x, then we can let E_x inherit the inner product space structure of V. Define a real positive definite symmetric bilinear form $(.,.)_{E_x}$ on E_x by $(x+v,x+w)_{E_x}:=(v,w)_V$ for all $v,w\in V$. Then the canonical identification of E_x with V defined by $x+v\mapsto v$ preserves $(.,.)_{E_x}$ and $(.,.)_V$ respectively as defined in the following. In the remaining of this Chapter we will consider

E_x to be equipped with the inner product $(.,.)_{E_x}$. Notice that the metric space E with metric $|.|_E$ coincides with E_x considered as a metric space with metric induced by $(.,.)_{E_x}$. This implies that E can be considered as a topological space having the Euclidean topology induced by $|.|_E$.

Remark 3.1.2. In the remaining of this manuscript we will view an affine Euclidean space E as topological space always as being equipped with the Euclidean topology induced by $|.|_E$.

Proposition 3.1.7. *If $a : E \to E'$ is an affine linear map between affine Euclidean spaces E and E', then $a : E \to E'$ is also a continuous map.*

Proof. Choose an origin $x \in E$ and $y \in E'$, then $a : E_x \to E'_y$ is a linear map between finite-dimensional vector spaces by (iii) of Example 3.1.6. Hence $a : E_x \to E_y$ is a continuous map. Since E_x (resp. E'_y) and E (resp. E') coincide as topological spaces we conclude that $a : E \to E'$ is a continuous map. □

Corollary 3.1.8. *If $a : E \to E'$ is an affine linear isomorphism between affine Euclidean spaces E and E', then $a : E \to E'$ is also a homeomorphism.*

Proof. This follows directly from Proposition 3.1.7. □

In affine Euclidean space we have the notion of convexness from the structure of the space of translations over the totally ordered field $\mathbb{R}$.

Definition 3.1.9. A subset U of an affine Euclidean space E with space of translation V is called *convex* if for every $x, y \in U$ and for all $0 \leq \lambda \leq 1$ we have $x + \lambda v \in U$ where $v \in V$ is the unique vector such that $y = x + v$. The convex subset $\{x + \lambda v : 0 \leq \lambda \leq 1\}$ (resp. $\{x + \lambda v : 0 < \lambda < 1\}$) of E is said the be the *line segment* (resp. *open line segment*) between points $x, y \in E$.

Example 3.1.10. Any affine subspace H of an affine Euclidean space E is a convex subset by linearity of the space of translations of H. In particular, this holds for affine hyperplanes and affine lines. Also notice that any convex subset of an affine Euclidean space is arc-connected, hence also connected.

Recall from Example 3.1.2 that a vector space has the natural structure of an affine space, hence we can consider the reals $\mathbb{R}$ as an affine Euclidean space. Let $\widehat{E} := \mathrm{Hom}(E, \mathbb{R})$ denote the set of all *affine linear functions* $a : E \to \mathbb{R}$, and choose $0 \in \mathbb{R}$ as origin to turn $\mathbb{R}$ into an $\mathbb{R}$-vector space. For $a, b \in \widehat{E}$ and $\lambda \in \mathbb{R}$ define $a + b$ as a function on E by $(a + b)(x) = a(x) + b(x)$ for all $x \in E$, and λa by $(\lambda a)(x) = \lambda \cdot a(x)$. Clearly both $a + b$ and λa are affine linear functions on E with $D(a + b) = Da + Db$ and $D(\lambda a) = \lambda Da$ respectively. Then the vector space properties of $\mathbb{R}$ turn $\widehat{E}$ into an $\mathbb{R}$-vector space.

Example 3.1.11. The constant functions on E with values in $\mathbb{R}$ are contained in $\widehat{E}$ and have all gradient 0. On the other hand, an affine linear function on E is constant if it has has gradient 0. Further, choose an origin $x \in E$ and consider the set of $\mathbb{R}$-linear functionals on E_x denoted by E_x^* (where 0 is the origin of $\mathbb{R}$). From (iii) of Example 3.1.6 we observe that $E_x^* \subset \widehat{E}$.

Remark 3.1.3. It will happen more than once that we want to say something about the constant function $c \in \widehat{E}$ that is identically one on the affine Euclidean space E. For the simplicity of the statements we will refer to this function as the *constant one function* on E.

Now, identify V with its dual space V^* by means of the scalar product $(.,.)_V$. The *gradient* of an affine linear function $a \in \widehat{E}$ is the unique vector $Da \in V$ such that $a(x+v) = a(x) + (Da, v)_V$ for all $x \in E$ and $v \in V$. We define a real positive semidefinite symmetric bilinear form on $\widehat{E}$ by

$$(a, b)_{\widehat{E}} := (Da, Db)_V$$

for all $a, b \in \widehat{E}$. Two affine linear functions $a, b \in \widehat{E}$ are called *orthogonal* if $(a, b)_{\widehat{E}} = 0$. Further, a nonzero $a \in \widehat{E}$ is called *isotropic* if $(a, a)_{\widehat{E}} = 0$, and *nonisotropic* otherwise. Note that $a \in \widehat{E}$ is isotropic if and only if a is a nonzero constant function on E.

Proposition 3.1.12. *The vector space $\widehat{E}$ of affine linear functions on E is of dimension* $\dim_{\mathbb{R}}(E) + 1$. *Moreover, if we let c denote the constant one function on E, then $\widehat{E}$ can be identified with $V \oplus \mathbb{R}c$ by the linear isomorphism $\phi_x : a \mapsto (Da, a(x))$ after fixing an origin $x \in E$.*

Proof. Fix a point $x \in E$ and let $a \in \widehat{E}$, then $a(x+v) = a(x) + (Da, v)_V$ for each $v \in V$ with Da the gradient of a. Hence every $a \in \widehat{E}$ is the sum of a unique constant function with value $a(x)$ on E and a unique linear functional $(x + Da, .)_{E_x}$ on E_x sending $x + v$ to $(Da, v)_V$ for all $v \in V$. Thus $\phi'_x : \widehat{E} \to E_x^* \oplus \mathbb{R}c$, $a \mapsto ((x + Da, .)_{E_x}, a(x))$ is a well defined injective linear map. Moreover ϕ'_x is surjective, because the constant functions $\mathbb{R}c$ and the linear functionals E_x^* on E_x are contained in $\widehat{E}$ as we have seen in Example 3.1.11. Identify E_x^* with V by the linear isomorphism $(x + Da, .)_{E_x} \mapsto Da$, where the inverse map sends a vector $\alpha \in V$ to the linear functional $(x + \alpha, .)_{E_x}$ on E_x such that $x + v \mapsto (\alpha, v)_V$ for all $v \in V$. This yields a linear isomorphism $\phi_x : \widehat{E} \to V \oplus \mathbb{R}c$, $a \mapsto (Da, a(x))$ with inverse map sending $\alpha + \mu c \in V \oplus \mathbb{R}c$ to the affine linear function a with gradient α defined by $a(x+v) = \mu + (\alpha, v)_V$ for all $v \in V$ (affine linearity follows analogous to the computation done in (3.1.2)). □

Corollary 3.1.13. *Choose an origin $x \in E$ and identify $\widehat{E}$ with $V \oplus \mathbb{R}c$, then the map $D : \widehat{E} \to V$ that sends $a \in \widehat{E}$ to its gradient Da is a projection onto V along the direct sum $\widehat{E} = V \oplus \mathbb{R}c$.*

Define the real positive semidefinite symmetric bilinear form

$$(\alpha + \lambda c, \beta + \mu c)_{V \oplus \mathbb{R}c} := (\alpha, \beta)_V \tag{3.1.4}$$

on $V \oplus \mathbb{R}c$ for $\alpha, \beta \in V$ and $\lambda, \mu \in \mathbb{R}$. Then we immediately obtain the following Corollary.

Corollary 3.1.14. *The linear isomorphism ϕ_x preserves the forms on $\widehat{E}$ and $V \oplus \mathbb{R}c$ respectively.*

3.1.3 Orthogonal reflections in affine Euclidean space

Next, we want to consider orthogonal reflections in affine hyperplanes in the affine Euclidean space E. We first recall orthogonal reflections in hyperplanes in Euclidean spaces (see for example [6] Chapter 3, §9). For each nonzero $v \in V$ define

$$v^\vee := \frac{2v}{(v, v)_V},$$

then the orthogonal reflection $w_v : V \to V$ in the hyperplane orthogonal to v is given by

$$w_v(u) = u - (v, u)_V \, v^\vee \tag{3.1.5}$$

for all $u \in V$. Indeed, w_v is a linear isometry of V fixing the hyperplane orthogonal to v and mapping v to $-v$.

Let H be an affine hyperplane in E with space of translations $U \subset V$, then $E = H + U^\perp$ where $U^\perp = \{z \in V : (z, u)_V = 0 \text{ for all } u \in U\}$ is the orthoplement of U in V.

Definition 3.1.15. The *orthogonal reflection* in the affine hyperplane H (with space of translations U) is the map $w_H : E \to E$ defined by $w_H(h+w) = h - w$ for all $h \in H$ and $w \in U^\perp$.

Clearly, w_H is a bijection of E with itself by simply transitivity of the group action of V on E, and w_H fixes the affine hyperplane $H \subset E$. Moreover, we have the following observation.

Proposition 3.1.16. *The orthogonal reflection $w_H : E \to E$ in the affine hyperplane $H \subset E$ is a metric preserving affine linear isomorphism.*

Proof. Let $h \in H$ be an origin of E. We will proof that $w_H : E_h \to E_h$ is an orthogonal reflection in the hyperplane orthogonal to $h + v$ for some nonzero $v \in U^\perp$ as described by (3.1.5). This implies that w_H preserves the metric on E, and by (iii) of Example 3.1.6 we obtain that $w_H : E \to E$ is an affine linear isomorphism.

Let $v' \in V$ then there are unique $u \in U$ and $u' \in U^\perp$ such that $v' = u + u'$. Hence

$$w_H(h+v') = w_H(h+u+u') = h + u - u' = h + v' - 2u'. \tag{3.1.6}$$

If $u' = 0$ we are done, else notice that for $v \in U^\perp \setminus \{0\}$ there exists $\lambda \in \mathbb{R}_{\neq 0}$ such that $v = \lambda u'$. Then $(v, u')_V = \operatorname{sgn}(\lambda)|v|_V|u'|_V$ where $\operatorname{sgn}(\lambda) \in \{\pm 1\}$ denotes the sign of λ. Since $(v, u)_V = 0$ we have

$$(v, v')_V\, v^\vee = (v, u')_V\, v^\vee = 2\frac{(v, u')_V}{(v, v)_V} v = 2\operatorname{sgn}(\lambda) v \frac{|u'|_V}{|v|_V} = 2u'. \tag{3.1.7}$$

Substituting the left-hand side of the equations of (3.1.7) into the right-hand side of (3.1.6) shows that $w_H : E_h \to E_h$ is an orthogonal reflection in the hyperplane orthogonal to $h + v$ as described by (3.1.5). □

Now for each nonisotropic $a \in \widehat{E}$ the gradient $Da \in V$ is nonvanishing. This implies that $\ker((Da, .)_V) = \{v \in V : (Da, v)_V = 0\}$ has codimension 1 in V, hence $\ker((Da, .)_V)$ is a hyperplane in V. So for $x \in E$ such that $a(x) = 0$, we have $\{y \in E : a(y) = 0\} = x + \ker((Da, .)_V)$. Defining

$$H_a := \{y \in E : a(y) = 0\}, \tag{3.1.8}$$

we conclude that H_a is an affine hyperplane in E with space of translations $\ker((Da, .)_V) \subset V$.

Definition 3.1.17. Let E be an affine Euclidean space with space of translation V, then we call a nonzero vector $v \in V$ a *normal vector* to an affine hyperplane $H \subset E$ with space of translations $U \subset V$ if $(v, w)_V = 0$ for all $w \in U$.

Example 3.1.18. The normal vectors of H_a are λDa for $\lambda \in \mathbb{R}_{\neq 0}$.

Further, for each nonisotropic $a \in \widehat{E}$ define

$$a^\vee := \frac{2a}{(a, a)_{\widehat{E}}} = \frac{2a}{(Da, Da)_V},$$

and the map $w_a : E \to E$,

$$w_a(x) = x - a^\vee(x) Da = x - a(x) Da^\vee. \tag{3.1.9}$$

Proposition 3.1.19. *The map $w_a : E \to E$ coincides with orthogonal reflection w_{H_a}. Furthermore, we have $Dw_a = w_{Da}$.*

Proof. For $x \in E$ we have the unique expression $x = h + w$ for some $h \in H_a$ and $w \in U^{\perp}$, where U is the space of translations of the affine hyperplane $H_a \subset E$. Thus

$$\begin{aligned} w_a(x) &\overset{(3.1.9)}{=} x - a(x)Da^{\vee} \\ &= h + w - a(h+w)Da^{\vee} \\ &= h + w - (Da, w)_V Da^{\vee}. \end{aligned}$$

But $Da \in U^{\perp}$, so $(Da, w)_V Da^{\vee} = 2w$ by a similar argument as (3.1.7). Thus w_a coincides with the orthogonal reflection w_{H_a} in the affine hyperplane H_a by definition of w_{H_a}.

From affine linearity of a and w_a we obtain for $x \in E$ and $v \in V$

$$\begin{aligned} w_a(x) + Dw_a(v) &= w_a(x+v) \\ &\overset{(3.1.9)}{=} x + v - (a(x+v))Da^{\vee} \\ &= x + v - (a(x) + (Da, v)_V)Da^{\vee} \\ &= x - a(x)Da^{\vee} + w_{Da}(v) \qquad (3.1.10) \\ &\overset{(3.1.9)}{=} w_a(x) + w_{Da}(v), \end{aligned}$$

so we immediately observe

$$Dw_a = w_{Da}. \qquad (3.1.11)$$

□

Propositions 3.1.16 and now imply the following Corollary.

Corollary 3.1.20. *The map $w_a : E \to E$ for $a \in \widehat{E}$ is a metric preserving affine linear isomorphism.*

3.1.4 Orthogonal reflections and affine linear automorphisms

By (iii) of Example 3.1.6, $\mathrm{End}(E) := \mathrm{Hom}(E, E) = \coprod_{v \in V}(t_v\ \mathrm{End}_{\mathbb{R}}(E_x))$ for a fixed $x \in E$ where $\mathrm{End}_{\mathbb{R}}(E_x)$ is the $\mathbb{R}$-vector space of $\mathbb{R}$-linear maps from E_x to itself, here considered as the set affine linear maps from E to itself fixing x. Let $\mathrm{GL}(E) \subset \mathrm{End}(E)$ be the group of affine linear isomorphisms of E to itself, or *affine linear automorphisms* of E, with id_E as identity element and the composition of maps as group operation. Composition of maps is a well defined associative operation on $\mathrm{GL}(E)$ by (ii) of Example 3.1.6, and the existence of inverse elements follows from (ii) of that same Example. Further, let $t(V) \subset \mathrm{GL}(E)$ denote the subgroup of translation maps of E, then $t(V)$ is canonically isomorphic to the group of translation maps of V which we will also denote by $t(V)$ (see Remark 3.1.1). Finally, let $\mathrm{GL}_{\mathbb{R}}(V)$ (resp. $\mathrm{GL}_{\mathbb{R}}(E_x)$) denote the group of linear automorphisms of V (resp. E_x) with the composition of maps as group operation. Notice that

$$\mathrm{GL}_{\mathbb{R}}(E_x) \cong \mathrm{GL}_{\mathbb{R}}(V) \qquad (3.1.12)$$

by conjugation with the linear isomorphism $E_x \xrightarrow{\sim} V$, $x + v \to v$.

Proposition 3.1.21. *The group $GL(E)$ is isomorphic to the semidirect product $t(V) \rtimes GL_{\mathbb{R}}(V)$.*

Proof. Let $x \in E$, let $t(V) \subset \mathrm{GL}(E)$ be the subgroup of translation maps of E, and consider the subgroup $\mathrm{GL}(E)_x \subset \mathrm{GL}(E)$ of affine linear automorphisms of E that keep x fixed. Then $t(V) \cap \mathrm{GL}(E)_x = \{\mathrm{id}_E\}$ since $\mathrm{id}_E = t_0 \in t(V)$ is the only translation map that leaves x fixed. Now, let $v \in V$ and let $y \in E$, then there exists a unique $w \in V$ such that $y = x + w$. Further, let $g \in \mathrm{GL}(E)_x$, then by

(iii) of Example 3.1.6 we have that $Dg \in \mathrm{GL}_{\mathbb{R}}(V)$ with $(Dg)^{-1} = D(g^{-1})$. Furthermore, we observe that

$$\begin{aligned}(g \circ t_v \circ g^{-1})(y) &= (g \circ t_v \circ g^{-1})(x+w)\\ &= g(x + (Dg)^{-1}(w) + v)\\ &= x + w + Dg(v)\\ &= t_{Dg(v)}(x+w)\\ &= t_{Dg(v)}(y),\end{aligned}$$

so $g \circ t_v \circ g^{-1} = t_{Dg(v)} \in t(V)$. Because $\mathrm{GL}(E) \subset \mathrm{End}(E) = \coprod_{v\in V}(t_v\ \mathrm{End}_{\mathbb{R}}(E_x))$ we have $\mathrm{GL}(E) = t(V)\mathrm{GL}(E)_x := \{t \circ g : t \in t(V) \text{ and } g \in \mathrm{GL}(E)_x\}$, so we conclude that $t(V) \subset \mathrm{GL}(E)$ is a normal subgroup, and that $\mathrm{GL}(E) = t(V) \rtimes \mathrm{GL}(E)_x$.

Now consider $t(V)$ canonically as the group of translation maps of V. By (ii) and (iii) of Example 3.1.6 in combination with the isomorphism of (3.1.12) we can also identify $\mathrm{GL}(E)_x$ canonically with the group $\mathrm{GL}_{\mathbb{R}}(V)$ by the group isomorphism $a \mapsto Da$. This implies that that $\mathrm{GL}(E) \cong t(V) \rtimes \mathrm{GL}_{\mathbb{R}}(V)$. □

For $a \in \widehat{E}$ the orthogonal reflection w_a in the affine hyperplane H_a is a metric preserving affine linear automorphism of E by Corollary 3.1.20, hence $w_a \in \mathrm{GL}(E)$. Then we observe from (3.1.10) together with Proposition 3.1.12 that for $x \in E$ we have

$$w_a = t_{-a(x)Da^\vee} \circ w_{x+Da}, \tag{3.1.13}$$

where $w_{x+Da} \in \mathrm{GL}_{\mathbb{R}}(E_x)$. Furthermore, we observe by the proof of Proposition 3.1.12 that

$$w_{x+C_1} \circ \cdots \circ w_{x+C_k} \circ t_v \circ w_{x+C_k}^{-1} \circ \cdots \circ w_{x+C_1}^{-1} = t_{w_{C_1} \circ \cdots \circ w_{C_k}(v)} \tag{3.1.14}$$

for all $v \in V$ and nonzero $C_1, \ldots, C_k \in V$.

Now $\mathrm{GL}(E)$ acts canonically on the affine Euclidean space E as a group of affine linear automorphisms of E. Furthermore, this action implies a linear action of $\mathrm{GL}(E)$ on the vector space $\widehat{E}$. For $g \in \mathrm{GL}(E)$ and $a \in \widehat{E}$ put $g(a) := a \circ g^{-1}$, then $a \circ g^{-1} \in \widehat{E}$ follows from the fact $a \in \widehat{E}$ and $g^{-1} \in \mathrm{End}(E)$ together with (ii) of Example 3.1.6. It follows easily by evaluation in E that this action is linear. Also, because $g \in \mathrm{GL}(E)$ we observe that $a \neq b$ in $\widehat{E}$ implies $a \circ g^{-1} \neq b \circ g^{-1}$, and if $a \in \widehat{E}$ is a constant function then $a \circ g^{-1} = a$. Thus for $g \in \mathrm{GL}(E)$ and $a \in \widehat{E}$ the mapping $a \mapsto a \circ g^{-1}$ is a linear automorphism of $\widehat{E}$ that fixes the constant functions.

In particular, for $a, b \in \widehat{E}$ the orthogonal reflection w_a acts on $\widehat{E}$ by $w_a(b) = b \circ w_a^{-1} = b \circ w_a$, or explicitly

$$w_a(b) = b - (a^\vee, b)_{\widehat{E}}\, a \tag{3.1.15}$$

which follows from (3.1.9). Also, for $v \in V$ it follows easily that the action of the translation map t_v on $b \in \widehat{E}$ becomes

$$t_v(b) = b - (Db, v)_V\, c, \tag{3.1.16}$$

where $c \in \widehat{E}$ is the constant one function on E.

Let $\mathrm{GL}_{\mathbb{R},c}(\widehat{E})$ denote the group of linear automorphisms of $\widehat{E}$ that fix the constant functions with composition of maps as group operation, then we observe the following.

Proposition 3.1.22. *For $g \in \mathrm{GL}(E)$ and $a \in \widehat{E}$, the mapping $g \mapsto \{a \mapsto a \circ g^{-1}\}$ defines a group isomorphism between* $\mathrm{GL}(E)$ *and* $\mathrm{GL}_{\mathbb{R},c}(\widehat{E})$.

Proof. We already saw that the mapping $\phi : g \mapsto \{a \mapsto a \circ g^{-1}\}$ is well defined, and it is follows in a straightforward way that ϕ is a group homomorphism. Let $f \neq g$ in $\mathrm{GL}(E)$, then there exists $x \in E$ such that $f^{-1}(x) \neq g^{-1}(x)$, say $y = f^{-1}(x)$ and $z = g^{-1}(x)$. Choose y as origin of E and identify E with V as the vector space E_y. By this identification there exists a unique vector $v \in V$ such that $z = y + v$. Consider an $\mathbb{R}$-linear functional a on E_y that sends $z = y + v$ to $1 \in \mathbb{R}$. By (iii) of Example 3.1.6, a can be viewed as an affine linear function on E, and we note $0 = a(y) \neq a(z) = 1$. Thus $a \circ f^{-1} \neq a \circ g^{-1}$, and we observe that ϕ is injective.

Next, we proof surjectivity of ϕ. Choose an origin $x \in E$ to identify E as E_x with the vector space V. Also idenfity $\widehat{E}$ with $E_x^* \oplus \mathbb{R}c$ as in the proof of Proposition 3.1.12, where c is the constant one function on E and E_x^* is the vector space of $\mathbb{R}$-linear functionals on E_x. Let $T \in \mathrm{GL}_{\mathbb{R},c}(\widehat{E})$, then T can be viewed as a linear automorphism of $E_x^* \oplus \mathbb{R}c$ such that T restricted to $\mathbb{R}c$ is the identity map on $\mathbb{R}c$. Consider the restriction $T' := T|_{E_x^*} : E_x^* \to E_x^* \oplus \mathbb{R}c$, and let $p_x : E_x^* \oplus \mathbb{R}c \to E_x^*$ (resp. $p_r : E_x^* \oplus \mathbb{R}c \to \mathbb{R}c$) be the linear projection onto E_x^* (resp. $\mathbb{R}c$) along the direct sum. Since T is an isomorphism that fixes $\mathbb{R}c$ the image $T'(E_x^*)$ has trivial intersection with $\mathbb{R}c$, and T' must be of rank $\dim(E_x^*) = \dim(E_x) = \dim(V) = l$. Hence $S := p_x \circ T' : E_x^* \xrightarrow{\sim} E_x^*$ is a linear automorphism and $\gamma := p_r \circ T' : E_x^* \to \mathbb{R}$ is a linear map such that $T'(\alpha) = S(\alpha) + \gamma(\alpha)c$ for $\alpha \in E_x^*$. Thus there exists a linear automorphism $\sigma : E_x \xrightarrow{\sim} E_x$ such that $S(\alpha) = \alpha \circ \sigma^{-1}$ for $\alpha \in E_x^*$. By (iii) of Example 3.1.6, σ is an affine linear automorphism of E, so $\sigma \in \mathrm{GL}(E)$. Further, there exists a vector $\eta \in V$ such that $\gamma(\alpha) = (D\alpha, \eta)_V$ for all $\alpha \in E_x^* \subset \widehat{E}$, where $D\alpha \in V$ is the gradient of α. Let $a \in \widehat{E}$, then $a = \alpha + \lambda c$ for some $\lambda \in \mathbb{R}$ and $\alpha \in E_x^*$. This leads to

$$\begin{aligned} T(a) &= T(\alpha + \lambda c) \\ &= T'(\alpha) + \lambda c \\ &= S(\alpha) + \gamma(\alpha)c + \lambda c \\ &= \alpha \circ \sigma^{-1} + (D\alpha, \eta)_V\, c + \lambda c \end{aligned}$$

Since $\sigma \in \mathrm{GL}(E)$ we have that $c \circ \sigma^{-1} = c$, so

$$\begin{aligned} \alpha \circ \sigma^{-1} + (D\alpha, \eta)_V\, c + \lambda c &= \alpha \circ \sigma^{-1} + (D\alpha, \eta)_V\, c \circ \sigma^{-1} + \lambda c \circ \sigma^{-1} \\ &= (\alpha + \lambda c + (D\alpha, \eta)_V\, c) \circ \sigma^{-1}. \end{aligned}$$

Then (3.1.16) implies that

$$\begin{aligned} (\alpha + \lambda c + (D\alpha, \eta)_V\, c) \circ \sigma^{-1} &= (\alpha + \lambda c) \circ t_{-\eta} \circ \sigma^{-1} \\ &= a \circ t_{-\eta} \circ \sigma^{-1}, \end{aligned}$$

where $t_{-\eta} \circ \sigma^{-1} \in \mathrm{GL}(E)$. Hence $\phi^{-1}(T) = (t_{-\eta} \circ \sigma^{-1})^{-1} = \sigma \circ t_\eta$ which proves the surjectivity of ϕ. □

We end this Section with a general result on metric preserving affine linear automorphisms of E. This will prove to be useful in the next Section when we want to consider compositions of orthogonal reflections of affine Euclidean space.

Proposition 3.1.23. *Let $w \in \mathrm{GL}(E)$ be metric preserving on E and let $a \in \widehat{E}$, then $Dw : V \to V$ is a linear isometry and we have the gradient $D(w(a)) = (Dw)(Da)$. Furthermore, for nonisotropic $a, b \in \widehat{E}$ we have $(w(a)^\vee, w(b))_{\widehat{E}} = (a^\vee, b)_{\widehat{E}}$.*

Proof. Let $w \in \mathrm{GL}(E)$ be a metric preserving affine linear isomorphism, then by (iii) of Example 3.1.6 we observe that $Dw : V \to V$ is linear isometry. By Proposition 3.1.22 w acts on $\widehat{E}$ by $w(a) = a \circ w^{-1}$, so we obtain for $x \in E$ and $v \in V$

$$\begin{aligned}
(D(w(a)), v)_V &= w(a)(x+v) - w(a)(x) \\
&= a(w^{-1}(x+v)) - a(w^{-1}(x)) \\
&= a(w^{-1}(x) + (Dw)^{-1}(v)) - a(w^{-1}(x))) \\
&= (Da, (Dw)^{-1}(v))_V \\
&= ((Dw)(Da), v)_V.
\end{aligned}$$

In other words,

$$D(w(a)) = (Dw)(Da). \tag{3.1.17}$$

Further, the action of w on $\widehat{E}$ fixes the constant functions, hence it sends nonisotropic vectors to nonisotropic vectors. This implies that for nonisotropic $a, b \in \widehat{E}$ the expression $(w(a)^\vee, w(b))_{\widehat{E}}$ is well defined. Now, by definition of $(.,.)_{\widehat{E}}$ we have

$$(w(a)^\vee, w(b))_{\widehat{E}} = 2\frac{(Dw(a), Dw(b))_V}{(Dw(a), Dw(a))_V},$$

so by (3.1.17)

$$2\frac{(Dw(a), Dw(b))_V}{(Dw(a), Dw(a))_V} = 2\frac{((Dw)(Da), (Dw)(Db))_V}{((Dw)(Da), (Dw)(Da))_V}.$$

Finally, since Dw is a linear isometry of V, hence

$$(w(a)^\vee, w(b))_{\widehat{E}} = 2\frac{((Dw)(Da), (Dw)(Db))_V}{((Dw)(Da), (Dw)(Da))_V} = 2\frac{(Da, Db)_V}{(Da, Da)_V} = (a^\vee, b)_{\widehat{E}}.$$

□

3.2 Affine root systems

In this section we will define the main objects that we would like to investigate in this Chapter, namely affine root systems. Analogously to the theory of finite root systems we will introduce concepts like the dual affine root system, reducedness, similarity, and irreducibility. The latter concept is very subtle in the theory of affine root systems, therefore we will use two Subsections to fully study it. We let E be an affine Euclidean space with space of translations V of dimension $l > 0$, and $\widehat{E}$ the vector space of affine linear functions on E. The Euclidean metric on E induces the Euclidean topology on E, making it into a locally compact space.

3.2.1 Affine root systems

We define our main object of interest.

Definition 3.2.1. A subset R of nonisotropic vectors in $\widehat{E}$ (i.e. nonconstant functions on E) is called an *affine root system* on E if the following five conditions are satisfied

(1) R spans $\widehat{E}$;
(2) $w_a(b) \in R$ for all $a, b \in R$;

(3) $(a^\vee, b)_{\widehat{E}} \in \mathbb{Z}$ for all $a, b \in R$;

(4) the subgroup $W(R) \subset \mathrm{GL}(E)$ generated by the orthogonal reflections w_a for $a \in R$ considered as a discrete group acts properly on E;

(5) for each gradient $C \in \{Da : a \in R\}$ there are at least two distinct $b, b' \in R$ such that $C = Db = Db'$.

The elements of R are called *affine roots* and the dimension of V is said to be the *rank* of R.

Remark 3.2.1. Condition (4) of the above Definition is equivalent to: for any two compact subsets K and K' of E there are only finitely many $w \in W(R)$ such that $w(K) \cap K' \neq \emptyset$ (see (D'2) in Chapter V §3 on p.77 of [2] and the Remark of Chapter III §4.5 of [3]).

Definition 3.2.2. For an affine root system $R \subset \widehat{E}$ the group $W(R)$ is called the *affine Weyl group* of R, and its generators w_a for $a \in R$ are called *reflections*.

By condition (2) of Definition 3.2.1 we observe that that the affine Weyl group $W(R)$ acts on R as a group. Then condition (2) of Definition 3.2.1 can also be read as: $W(R)$ stabilizes R.

Example 3.2.3. (i) Let A be an affine Cartan matrix, and consider the set of real roots $\Delta^{re} \subset \mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ corresponding to the affine Lie algebra $\mathfrak{g}(A)$ in the context of Section 2.3. Then we can consider the vector space $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ with positive definite normalized invariant form $(.,.)$ as an affine Euclidean space by Example 3.1.2. Now we want to identify $\mathfrak{h}_0^*$ with the space of affine linear functions $\widehat{\mathring{\mathfrak{h}}_{\mathbb{R}}^*}$ on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ in a natural way preserving the forms on both spaces.

First, $\widehat{\mathring{\mathfrak{h}}_{\mathbb{R}}^*}$ can be identified with $\mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c$ by Proposition 3.1.12 and Corollary 3.1.14, where c is the constant one function on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ after choosing the origin $0 \in \mathring{\mathfrak{h}}_{\mathbb{R}}^*$. Then $(\alpha + nc, \beta + mc)_{\mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c} = (\alpha, \beta)$ by Corollary 3.1.14. Hence from the proof of Proposition 3.1.12 it follows that it is natural to consider each $\alpha \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* \subset \mathfrak{h}_0^*$ as a linear functional on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ defined by $\mu \mapsto (\alpha, \mu)$. Further, notice that it is natural to identify δ with c because in Section 2.3 we have chosen to work in the affine hyperplane $\mathfrak{h}_1^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : (\lambda, \delta) = 1\}$ of $\mathfrak{h}_0^*$. If we would have started out with $\mathfrak{h}_s^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : (\lambda, \delta) = s\}$ for any other nonzero $s \in \mathbb{R}$, then it would be natural to identify δ with sc. This would turn Δ^{re} in an affine root system that is 'similar' to the one that we have discussed here starting out with $\mathfrak{h}_1^*$ (see (ii) of Example 3.2.9).

We conclude that it is natural to identify $\widehat{\mathring{\mathfrak{h}}_{\mathbb{R}}^*} = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c$ with $\mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ by the linear isomorphism that is the identity map on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ and sends c to δ. This means that $\alpha + n\delta \in \mathfrak{h}_0^*$ acts on $\mu \in \mathring{\mathfrak{h}}_{\mathbb{R}}^*$ by

$$(\alpha + n\delta)(\mu) = (\alpha, \mu) + n.$$

Clearly the form $(.,.)_{\mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c}$ and the normalized invariant form $(.,.)$ on $\mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ are preserved by this identification. Now by (3.1.9) we have

$$w_{\alpha+n\delta}(\mu) = \mu - (\alpha + n\delta)(\mu) \cdot 2\frac{\alpha}{(\alpha,\alpha)} = \mu - 2\frac{(\mu,\alpha) + n}{(\alpha,\alpha)}\alpha$$

for the reflection of $\mu \in \mathring{\mathfrak{h}}_{\mathbb{R}}^*$ in the affine hyperplane $H_{\alpha+n\delta}$. For real roots this expression coincides by (2.3.11) with

$$\mathrm{af}(r_{\alpha+n\delta})(\mu).$$

Further af(W) is generated by af(r_{α_i}) for $\alpha_i \in \Pi$, so surely by af($r_{\alpha+n\delta}$) for $\alpha+n\delta \in \Delta^{re}$. Also, $W(\Delta^{re})$ is generated by $w_{\alpha+n\delta}$ for $\alpha+n\delta \in \Delta^{re}$. Hence af($W$) and $W(\Delta^{re})$ are isomorphic by the obvious mapping af($r_{\alpha+n\delta}$) $\mapsto w_{\alpha+n\delta}$. Then it follows from Theorem 2.3.4 and Corollary 3.1.13 that Δ^{re} is an affine root system on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ with affine Weyl group af(W) $= \mathring{W} \ltimes$ af(T).

(ii) Let R_0 be a reduced irreducible finite root system in V with finite Weyl group $W_0(R_0) \subset \mathrm{GL}_{\mathbb{R}}(V)$. Let $\{\alpha_1,\dots,\alpha_l\}$ be a basis of R_0, then $\{\alpha_1^\vee,\dots,\alpha_l^\vee\}$ is a basis of $R_0^\vee$. Furthermore, $R_0^\vee$ is contained in the *coroot lattice* $Q^\vee := \sum_{i=1}^l \mathbb{Z}\alpha_i^\vee \subset V$. Write $t(Q^\vee) := \{t_v : v \in Q^\vee\}$ for subgroup of translations of $t(V)$ over the lattice $Q^\vee$. Consider the vector space V as affine Euclidean space E with V as space of translations (see Example 3.1.2). Let c be the constant one function on E, and identify $\widehat{E}$ with $V \oplus \mathbb{R}c$ after choosing the origin $0 \in E$ (see Prop. 3.1.12 and Cor. 3.1.14). Then the subset

$$R^u_{R_0} := \{mc + \alpha\}_{m\in\mathbb{Z},\alpha\in R_0}$$

of $V \oplus \mathbb{R}c$ turns out to be an affine root system on E with affine Weyl group $W(R^u_{R_0}) = t(Q^\vee) \rtimes W_0(R_0)$, where $t(Q^\vee), W_0(R_0) \subset W(R^u_{R_0})$ are subgroups. In Section 3.5 we will give an extensive proof of this result (see Proposition 3.5.5).

Analogous to the theory of finite root systems as described in Chapter 1 we have the notions of indivisible and unmultipliable affine roots in an affine root system R. Condition (3) of Definition 3.2.1 yields that $\mathbb{R}a \cap R = \{\pm a\}$, $\mathbb{R}a \cap R = \{\pm a, \pm\frac{1}{2}a\}$ and $\mathbb{R}a \cap R = \{\pm a, \pm 2a\}$ are the only possibilities for multiples of an affine root $a \in R$. Define the *indivisible affine roots* $R^{ind} := \{a \in R : \frac{1}{2}a \notin R\}$ of R and the *unmultipliable affine roots* $R^{unm} := \{a \in R : 2a \notin R\}$ of R, then $R = R^{ind} \cup R^{unm}$ (union not disjoint). From a straightforward check of the five conditions of Definition 3.2.1 it follows that both R^{ind} and R^{unm} are affine root systems on E with affine Weyl group $W(R)$.

Definition 3.2.4. An affine root system R is said to be *reduced* if each $a \in R$ is indivisible. It is called *nonreduced* otherwise.

Example 3.2.5. (i) From the possibilities of multiples of affine roots in an affine root system R on E as noted in the previous paragraph, we observe that both R^{ind} and R^{unm} are reduced affine root systems on E.

(ii) Example 3.2.3 (i) made clear that the set of real roots Δ^{re} corresponding to the affine Lie algebra $\mathfrak{g}(A)$ is an affine root system. By (vi) of Theorem 2.3.4 it then follows that Δ^{re} is a reduced affine root system.

(iii) The affine root system $R^u_{R_0}$ of Example 3.2.3 is reduced since R_0 is reduced.

For our purposes we will only be interested in reduced affine root systems, but not until the last Section of this Chapter will we make this distinction.

Analogous to the theory of finite root systems, we also have the notion of the dual of an affine root system.

Definition 3.2.6. If R is an affine root system, then we define the *dual of* R to be the set $R^\vee := \{a^\vee : a \in R\}$.

Proposition 3.2.7. *The dual $R^\vee$ of an affine root system R on E is an affine root system on E with the same affine Weyl group as R.*

Proof. We will check all conditions of Definition 3.2.1 for $R^\vee$. Since $R^\vee$ is obtained from R by positively scaling each affine root in R, the dual $R^\vee \subset \widehat{E}$ is a subset of nonisotropic vectors satisfying condition (1) and (5). Further, $(a^\vee)^\vee = a$ for all $a \in R$, so $R^\vee$ also satisfies condition (3). Next, we observe from (3.1.15) that $w_{a^\vee} = w_a$ in $\mathrm{GL}_{\mathbb{R},c}(\widehat{E})$ for all $a \in R$. This implies that $W(R^\vee) = W(R)$, so surely condition (4) is satisfied. Thus to proof condition (2), it is enough to show that $w_a(R^\vee) \subset R^\vee$ for all $a \in R$. Let $a \in R$ and $b' \in R^\vee$, then there exists $b \in R$ such that $b' = b^\vee$, so $w_a(b') = w_a(b^\vee) = b^\vee - (a^\vee, b^\vee)_{\widehat{E}}\, a = w_a(b) \cdot \frac{2}{(b,b)_{\widehat{E}}}$. Now by 3.1.17, $D(w_a(b)) = (Dw_a)(Db)$, where $Dw_a : V \to V$ is a linear isometry. But then $(b,b)_{\widehat{E}} = (Db, Db)_V = (Dw_a(Db), Dw_a(Db))_V = (Dw_a(b), Dw_a(b))_V = (w_a(b), w_a(b))_{\widehat{E}}$. Hence $w_a(b') = w_a(b) \cdot \frac{2}{(w_a(b), w_a(b))_{\widehat{E}}} = w_a(b)^\vee \in R^\vee$. □

3.2.2 Similar affine root systems

Next, we define an appropriate equivalence relation called 'similarity' on the set of affine root systems analogous to the notion of similarity for finite root systems. We will show that similarity preserves important structures on affine root systems like the affine Weyl group, the integers $(a^\vee, b)_{\widehat{E}}$ of Definition 3.2.1 and the collections of affine hyperplanes induced by the affine roots.

Definition 3.2.8. Let $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ be two affine root systems. We call R and R' *similar*, and write $R \simeq R'$, if there exists a linear isomorphism $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ such that

$$(T(a)^\vee, T(b))_{\widehat{E}} = (a^\vee, b)_{\widehat{E}}$$

for all $a, b \in R$, which restricts to a bijection of R onto R'.

The equivalence classes under the equivalence relation similarity on the collection of affine root systems are called *similarity classes*.

Example 3.2.9. (i) Every $w \in W(R)$ realizes a similarity of R with itself. Indeed, $w \in W(R)$ acts on $\widehat{E}$ as linear automorphism that fixes the constant functions by Proposition 3.1.22. Then (ii) of Definition 3.2.1 implies that w maps R bijectively onto itself, and Proposition 3.1.23 tells us that $(w(a)^\vee, w(b))_{\widehat{E}} = (a^\vee, b)_{\widehat{E}}$ for all $a, b \in R$.

(ii) Recall from (i) of Example 3.2.3 that we identified $\delta \in \mathfrak{h}_0^*$ with the constant one function c on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$. However if we interpret δ as nc for some $n \in \mathbb{R}$, then it follows easily that the map $\phi : \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c \to \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}c$ that is the identity on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ and sends c to nc realizes a similarity of Δ^{re} with itself.

Proposition 3.2.10. *If the linear isomorphism $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ realizes a similarity between affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$, then*

(i) *R is (non)reduced implies R' is (non)reduced*

(ii) *T sends constant functions to constant functions*

(iii) $W(R) \cong W(R')$.

Proof. (i) Clearly, similarity respects reducedness since T is linear.

(ii) Since R is nonempty, we can choose distinct affine roots b and b' such that $Db = Db' =: C$ by (5) of Definition 3.2.1. By affine linearity of b and b' we have $b(x+v) - b'(x+v) = b(x) - b'(x)$ for all $x \in E$ and $v \in V$, hence $b - b' \in \widehat{E}$ is a constant function on E. Then for all $a \in R$ we have

$$\begin{aligned}(T(a)^\vee, T(b-b'))_{\widehat{E}} &= (T(a)^\vee, T(b) - T(b'))_{\widehat{E}} \\ &= (T(a)^\vee, T(b))_{\widehat{E}} - (T(a)^\vee, T(b'))_{\widehat{E}} \\ &= (a^\vee, b)_{\widehat{E}} - (a^\vee, b')_{\widehat{E}} \\ &= (Da^\vee, C)_V - (Da^\vee, C)_V = 0.\end{aligned}$$

Since R spans $\widehat{E}$, and R' spans $\widehat{E'}$, and $T|_R : R \to R'$ is a bijection, we observe that $(a', T(b-b'))_{\widehat{E}} = 0$ for all $a' \in \widehat{E'}$. This implies that $T(b-b') \in \widehat{E'}$ is a contant function on E'. Hence by linearity T maps constant function to constant functions.

(iii) Now if $R \simeq R'$, realized by the linear isomorphism T, then it is easy to check that

$$T \circ w_a \circ T^{-1} = w_{T(a)} \tag{3.2.1}$$

on $\widehat{E'}$ for all $a \in R$ by evaluating both sides of the equation at $T(b)$ for $b \in R$. Since $T \circ w_a \circ w_b \circ T^{-1} = T \circ w_a \circ T \circ T^{-1} \circ w_b \circ T^{-1} = w_{T(a)} \circ w_{T(b)}$ and $T|_R : R \to R'$ is a bijection, we have the surjective group homomorphism $\widetilde{T} : W(R) \to W(R')$, $w \mapsto T \circ w \circ T^{-1}$. On the other hand, the surjective group homomorphism $W(R') \to W(R)$, $w \mapsto T^{-1} \circ w \circ T$ is clearly the inverse of $\widetilde{T}$, hence $W(R) \cong W(R')$. □

Let $c \in \widehat{E}$ and $c' \in \widehat{E'}$ be constant one functions, then by (ii) of Proposition 3.2.10 $T(c) \in \mathbb{R}_{\neq 0} c'$.

Definition 3.2.11. A linear isomorphism $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ that realizes a similarity between affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ such that $T(c) \in \mathbb{R}_{>0} c'$ is said to be a *similarity transformation* between R and R'. Furthermore, if $T(c) = c'$ then we will call T a *normalized similarity transformation.*

Example 3.2.12. (i) From (i) of Example 3.2.9 we observe that $w \in W(R)$ is a normalized similarity transformation of R with itself.

(ii) For an affine root system R and $\lambda \in \mathbb{R}_{\neq 0}$ define $\lambda R := \{\lambda a\}_{a \in R}$, then $R \simeq \lambda R$ realized by the similarity transformation $T : \widehat{E} \xrightarrow{\sim} \widehat{E}$, $a \mapsto |\lambda| a$. The affine root system λR is called a *rescaling* of R.

(iii) If the linear isomorphism $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ realizes a similarity between affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$, then $-T$ also realizes a similarity between R and R'. Furthermore, either T or $-T$ must be a similarity transformation. We conclude that if $R \simeq R'$, then there exists a similarity transformation realizing a similarity between R and R'.

Example 3.2.12 shows that we may always assume without loss of generality that a similarity between two affine root systems is realized by a similarity transformation. Furthermore, it turns out that each similarity transformation $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ can be given in terms of a unique affine linear isomorphism $\psi : E' \to E$.

Proposition 3.2.13. *Let $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ be a similarity transformation realizing a similarity between the affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$, then there exists $\lambda \in \mathbb{R}_{>0}$ and an affine linear isomorphism $\psi : E \to E'$ such that $T(a) = \lambda(a \circ \psi^{-1})$ for all $a \in \widehat{E}$.*

Proof. Consider two affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ such that $R \simeq R'$ realized by the similarity transformation T. Let c (resp. c') denote the constant one function on E (resp. E'), then $T(c) = \lambda c'$ for some $\lambda \in \mathbb{R}_{>0}$. Write $T = \lambda \widetilde{T}$ where $\widetilde{T} : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ is the linear isomorphism defined by $\widetilde{T}(a) = \lambda^{-1} T(a)$ that sends c to c'. Choose and origin $x \in E$ (resp. $y \in E'$) and identify E (resp. E') with V (resp. V') as E_x (resp. E'_y). Further, identify $\widehat{E}$ (resp. $\widehat{E'}$) with $E_x^* \oplus \mathbb{R}c$ (resp. ${E'_y}^* \oplus \mathbb{R}c'$) as in the proof of Proposition 3.1.12. Next, we consider the restriction $T' := \widetilde{T}|_{E_x^*} : E_x^* \to {E'_y}^* \oplus \mathbb{R}c'$, and let $p_y : {E'_y}^* \oplus \mathbb{R}c' \to {E'_y}^*$ (resp. $p_r : {E'_y}^* \oplus \mathbb{R}c' \to \mathbb{R}c'$) be the linear projection onto ${E'_y}^*$ (resp. $\mathbb{R}c'$) along the direct sum. Similarly to the proof of surjectivity in Proposition 3.1.22, we obtain that $S := p_y \circ T' : E_x^* \xrightarrow{\sim} {E'_y}^*$ is a linear isomorphism and $\gamma := p_r \circ T' : E_x^* \to \mathbb{R}$ is a linear map such that $T'(\alpha) = S(\alpha) + \gamma(\alpha)c'$ for all $\alpha \in E_x^*$. Hence there exists a linear automorphism $\sigma : E_x \xrightarrow{\sim} E'_y$ such that $S(\alpha) = \alpha \circ \sigma^{-1}$ for $\alpha \in E_x^*$. By (iii) of Example 3.1.6, σ is an affine linear automorphism of E.

Further, there exists a vector $\eta \in V$ such that $\gamma(\alpha) = (D\alpha, \eta)_V$ for all $\alpha \in E_x^* \subset \widehat{E}$. Analogously to the proof of surjectivity in Proposition 3.1.22, we find that $\widetilde{T}(a) = a \circ \psi^{-1}$ for $a \in \widehat{E}$, where $\psi : E \to E'$ is defined to be the affine linear isomorphism $\sigma \circ t_\eta$. Thus $T(a) = \lambda \widetilde{T}(a) = \lambda(a \circ \psi^{-1})$ for all $a \in \widehat{E}$ which concludes the proof. □

Let $\mathscr{H} := \{H_a : a \in R\}$ (resp. $\mathscr{H}' := \{H_{a'} : a' \in R'\}$) be the collection of affine hyperplanes in E (resp. E') induced by $R \subset \widehat{E}$ (resp. $R' \subset \widehat{E'}$). Then a similarity transformation realizing $R \simeq R'$ induces a bijection between $\mathscr{H}$ and $\mathscr{H}'$.

Corollary 3.2.14. *Let $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$ be a similarity transformation realizing a similarity between the affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$, and let $\lambda > 0$ and $\psi : E \to E'$ the affine linear isomorphism such that $T(a) = \lambda(a \circ \psi^{-1})$ for all $a \in \widehat{E}$. Then T induces a bijection from $\mathscr{H}$ onto $\mathscr{H}'$ given by the mapping $H_a \mapsto \psi(H_a) = H_{T(a)}$.*

Proof. Let $H \in \mathscr{H}$, then there exists $a \in R$ (not unique) such that $H = H_a$. For $a \in R$ there exists a unique affine root $a' \in R'$ such that $a' = T(a)$, because $T|_R : R \to R'$ is a bijection. We observe that the mapping $H_a \mapsto H_{T(a)}$ from $\mathscr{H}$ to is well defined and yields a bijection from $\mathscr{H}$ onto $\mathscr{H}'$. Finally, let $\lambda > 0$ and $\psi : E \to E'$ the affine linear isomorphism such that $T(a) = \lambda(a \circ \psi^{-1})$ for all $a \in \widehat{E}$ as in Proposition 3.2.13, then

$$\begin{aligned}\psi(H_a) &= \psi(\{x \in E : a(x) = 0\}) \\ &= \{y \in E' : \lambda(a \circ \psi^{-1}) = 0\} \\ &= H_{T(a)}.\end{aligned}$$

□

3.2.3 Affine root subsystems and irreducibility

Next we will introduce the notion of an affine root subsystem of an affine root system, and we will show that each affine root subsystem can be viewed as affine root system in a natural way. Further, we show how affine root systems decompose uniquely into elementary affine root subsystems called irreducible affine root subsystems.

Definition 3.2.15. A nonempty subset R' of an affine root system R on an affine Euclidean space E is called an *affine root subsystem* of R if $w_a(R') \subset R'$ for all $a \in R'$ and if for each gradient $C \in \{Db : b \in R'\}$ there are at least two distinct affine roots in R' with C as gradient.

Example 3.2.16. Trivially R is an affine root subsystem of itself. Further, it is not difficult to see that both R^{ind} and R^{unm} are reduced affine root subsystems of the affine root system R.

Remark 3.2.2. Let $T : \widehat{E} \to \widehat{E'}$ be a linear isomorphism realizing a similarity between the affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$. For $\widetilde{R} \subset R$ an affine root subsystem we define $\widetilde{R}' := T(\widetilde{R}) \subset R'$. Let $a', b' \in \widetilde{R}'$, then there exist unique $a, b \in \widetilde{R}$ such that $a' = T(a)$ and $b' = T(b)$. Thus by (3.2.1) we have

$$w_{a'}(b') = w_{T(a)}(T(b)) = (T \circ w_a \circ T^{-1})(T(b)) = T(w_a(b)).$$

Since $w_a(b) \in \widetilde{R}$ by Definition 3.2.15, we observe that $w_{a'}(b') = T(w_a(b)) \in \widetilde{R}'$.

Let $C \in \{Da' : a' \in \widetilde{R}'\}$, then there exists $b' \in \widetilde{R}'$ such that $Db' = C$. Furthermore, there exists a unique $b \in \widetilde{R}$ such that $b' = T(b)$. By Definition 3.2.15 we observe that there exists $a \in \widetilde{R}$ distinct of b such that $Da = Db$, therefore $b - a$ is a constant function on E. This implies that $T(b-a) =$

$T(b) - T(a)$ is a constant function on $\widetilde{E'}$ because T sends constant function to constant functions. We conclude that $DT(a) = DT(b) = C$, hence $\widetilde{R}' \subset R'$ is an affine root subsystem. In this sense similarity respects affine root subsystems.

In the following we want to show that we can identify an affine root subsystem R' of an affine root system R on E with an affine root system $\widetilde{R}'$ on an affine Euclidean space E' in a natural way. If $R' \subset R$ is an affine root subsystem such that R' spans $\widehat{E}$, then the gradients $\{Db : b \in R'\}$ of R' span the space of translation V of E by Proposition 3.1.12. Furthermore, it follows straightforwardly from Definition 3.2.15 that R' itself is an affine root system on E. Now assume that R' does not span $\widehat{E}$, then the gradients $\{Db : b \in R'\}$ of R' do not span V. Moreover, because $R' \subset R$ is a nonempty set of nonisotropic vector contained in $\widehat{E}$, the gradients of R' span a nontrivial proper subspace V' of V.

Let $V' \subset V$ be a nontrivial proper subspace, and let $V'^{\perp} = \{v \in V : (v, w) = 0 \text{ for all } w \in V'\}$ be the orthogonal complement of the subspace V' in V. Then V decomposes as the direct sum of subspaces $V' \oplus V'^{\perp}$. The group action of V on E induces an action of the subgroup $V'^{\perp}$ (resp. V') of V on E. Define $E' := \{x + V'^{\perp} : x \in E\}$ (resp. $E'' := \{x + V' : x \in E\}$) to be the set of orbits of E under the action of the subgroup $V'^{\perp} \subset V$ (resp. $V' \subset V$) on E.

Proposition 3.2.17. *The set E' (resp. E'') can be viewed as affine Euclidean space with space of translation V' (resp. $V'^{\perp}$) in a canonical way such that the map $p' : E \to E'$, $x \mapsto x + V'^{\perp}$ (resp. $p'_{\perp} : E \to E''$, $x \mapsto x + V'$) is affine linear with $Dp' : V \to V'$ (resp. $Dp'_{\perp} : V \to V'^{\perp}$) the orthogonal projection onto V' (resp. $V'^{\perp}$).*

Proof. Let $x, y \in E$ such that x and y lie in the same $V'^{\perp}$-orbit, hence there exists $v' \in V'^{\perp}$ such that $x + v' = y$. Then also $(x+v)+v' = (x+v')+v = y+v$ for any $v \in V$, hence $x+v$ and $y+v$ lie in the same $V'^{\perp}$-orbit. Thus the action of V on E implies an action of V on E', namely for an orbit $O \in E'$ with representative $x \in E$ and any $v \in V$ we have $O + v = (x + V'^{\perp}) + v := (x+v) + V'^{\perp}$. In a similar fashion, one observes that V acts on E'', namely for an orbit $O' \in E''$ with representative $y \in E$ and any $v' \in V$ we have $O' + v' = (y + V') + v' := (y + v') + V'$.

Now let $O, O' \in E'$ be distinct orbits, then we can choose representatives $x, y \in E$ such that $O = x + V'^{\perp}$ and $O' = y + V'^{\perp}$ with $y - x = v'$ for some $v' \in V'$. Hence $O + v' = O'$ which shows that the action of V on E' is transitive, moreover the restricted action of the subgroup $V' \subset V$ on E' is transitive. Further, for $v \in V$ we have $v = v_1 + v_2$ for unique $v_1 \in V'$ and $v_2 \in V'^{\perp}$, so

$$\begin{aligned} O + v &= (x + V'^{\perp}) + v \\ &= ((x + V'^{\perp}) + v_2) + v_1 \\ &= ((x + v_2) + V'^{\perp}) + v_1 \\ &= (x + V'^{\perp}) + v_1 \\ &= O + v_1. \end{aligned} \tag{3.2.2}$$

Since $V' \subset V$ is a nontrivial proper subspace this implies that the group action of V on E' is not faithful. Now let $w, w' \in V'$ such that $O + w = (x + V'^{\perp}) + w = (x + V'^{\perp}) + w' = O + w'$, then $(x + w) + V'^{\perp} = (x + w') + V'^{\perp}$. Thus there exists a vector $z \in V'^{\perp}$ such that $x + w = x + w' + z$. Since the action of V on E is simply transitive we have $w = w' + z$, hence $z = 0$. We conclude that the action of V' on E' is simply transitive. Analogously one observes that the action of $V'^{\perp}$ on E'' induced by the action of V on E'' is simply transitive. Also for $x \in E$ and $v \in V$ such that $v = v_1 + v_2$

for $v_1 \in V'$ and $v_2 \in V'^{\perp}$ we have

$$\begin{aligned}(x+V')+v &= ((x+V')+v_1)+v_2 \\ &= ((x+v_1)+V')+v_2 \\ &= (x+V')+v_2.\end{aligned} \tag{3.2.3}$$

Let $x, y \in E$, then there exists a unique $v \in V$ such that $y - x = v$. Now $v = v_1 + v_2$ for unique $v_1 \in V'$ and $v_2 \in V'^{\perp}$. We turn E' into a metric space with distance function $d((x+V'^{\perp}),(y+V'^{\perp}))_{E'} := \sqrt{(v_1, v_1)_{V'}}$, where $(.,.)_{V'}$ is the positive definite symmetric bilinear form on the vector space V' induced by the by the scalar product on $V' \subset V$. Notice that the metric $d(.,.)_{E'}$ is well defined on the pairs orbits of E', because different representatives of the same orbit only differ by vectors in $V'^{\perp}$. Similarly, we turn E'' into a metric space with distance function $d((x+V'),(y+V'))_{E''} := \sqrt{(v_2, v_2)_{V'^{\perp}}}$, where $(.,.)_{V'^{\perp}}$ is the positive definite symmetric bilinear form on the vector space $V'^{\perp}$ induced by the by the scalar product on $V'^{\perp} \subset V$. In this setting E' (resp. E'') is an affine Euclidean space with space of translations V' (resp. $V'^{\perp}$). Furthermore, from (3.2.2) (resp. (3.2.3)) we observe that the mapping $p' : E \to E'$, $x \mapsto x + V'^{\perp}$ (resp. $p'_{\perp} : E \to E''$, $x \mapsto x + V'$) is affine linear with $Dp' \in \mathrm{Hom}_{\mathbb{R}}(V, V')$ (resp. $Dp'_{\perp} \in \mathrm{Hom}_{\mathbb{R}}(V, V'^{\perp})$) the orthogonal projection of $V = V' \oplus V'^{\perp}$ onto V' (resp. $V'^{\perp}$) along the direct sum. □

Next we turn the set $E' \times E''$ into an affine Euclidean space with space of translations $V' \oplus V'^{\perp}$ by defining the action of $V' \oplus V'^{\perp}$ on $E' \times E''$ componentwise by

$$(x_1, x_2) + (v_1, v_2) = (x_1 + v_1, x_n + v_n) \tag{3.2.4}$$

where $x_1 \in E'$, $x_2 \in E''$, $v_1 \in V'$ and $v_2 \in V'^{\perp}$. Since the componentwise actions are simply transitive the action of $V' \oplus V'^{\perp}$ on $E' \times E''$ is simply transitive. Further, define the inner product $(.,.)_{V' \oplus V'^{\perp}}$ on $V' \oplus V'^{\perp}$ by

$$((u_1, u_2), (v_1, v_2))_{V' \oplus V'^{\perp}} = (u_1, v_1)_{V'} + (u_2, v_2)_{V'^{\perp}} \tag{3.2.5}$$

where $u_1, v_1 \in V'$ and $u_2, v_2 \in V'^{\perp}$. This turns $E' \times E''$ into an affine Euclidean space with $V' \oplus V'^{\perp}$ as space of translations.

Lemma 3.2.18. *The map $p : E \to E' \times E''$ sending $x \in E$ to its orbits $(p'(x), p'_{\perp}(x)) = (x + V'^{\perp}, x + V')$ is an affine linear isomorphism.*

Proof. Since both $p' : E \to E'$ and $p'_{\perp} : E \to E''$ are affine linear maps, it follows straightforwardly that p is an affine linear map with $Dp : V \xrightarrow{\sim} V' \oplus V'^{\perp}$, $v \mapsto (Dp'(v), Dp'_{\perp}(v)) = v$ a linear isomorphism. Because Dp is a linear isomorphism (iii) of Example 3.1.6 shows that p is an affine linear isomorphism. □

Let $\widehat{E'}$ be the space of affine linear functions on E' endowed with the bilinear form $(.,.)_{\widehat{E'}}$ defined by $(a', b')_{\widehat{E'}} := (Da', Db')_{V'}$ for all $a', b' \in \widehat{E'}$. Then for $a' \in \widehat{E'}$, $x \in E$ and $v = v_1 + v_2 \in V' \oplus V'^{\perp} = V$ we obtain using (3.2.2) that

$$\begin{aligned}(a' \circ p')(x+v) &= a'((x+v)+V'^{\perp}) \\ &= a'((x+V'^{\perp})+v) \\ &= a'((x+V'^{\perp})+v_1) \\ &= a'(x+V'^{\perp}) + (Da', v_1)_{V'}\end{aligned} \tag{3.2.6}$$

where the last equation follows from affine linearity of a' on E'. Then $a'(x+V'^{\perp})+(Da',v_1)_{V'} = (a'\circ p')(x)+(Da',v)_V$. Hence $(a'\circ p')$ is an affine linear function on E with gradient $D(a'\circ p') = Da'$. Furthermore $(a'\circ p')\neq(b'\circ p')$ for distinct $a',b'\in\widehat{E'}$. So we can define the injective linear homomorphism $\pi:\widehat{E'}\hookrightarrow\widehat{E}$ by $\pi(a') = a'\circ p'$. Notice that the subspace $\pi(\widehat{E'})$ of $\widehat{E}$ contains all constant functions on $\widehat{E}$. Indeed, the constant one function $c':E'\to\mathbb{R}$ gets sent to the constant one function $\pi(c') = c'\circ p':E\to\mathbb{R}$ on E which we will denote by c. As usual we write $(.,.)_{\widehat{E}}$ for the bilinear form on $\widehat{E}$, and $(.,.)_V$ for the inner product on V. Then for $a',b'\in\widehat{E'}$ we have

$$\begin{aligned}(\pi(a'),\pi(b'))_{\widehat{E}} &= (D\pi(a'),D\pi(b'))_V\\ &= (Da',Db')_{V'}\\ &= (a',b')_{\widehat{E'}}.\end{aligned}\tag{3.2.7}$$

Hence the map π preserves the forms on $\widehat{E'}$ and $\widehat{E}$ respectively.

We return to the specific situation that $V'\subset V$ is subspace spanned by the gradients of the affine roots contained in the affine root subsystem $R'\subset R$. By Proposition 3.1.12, $\widehat{E'}$ is a vector space of dimension $\dim(V')+1$, hence the injectivity of the linear map π guarantees that $\pi(\widehat{E'})\subset\widehat{E}$ is a subspace of the same dimension. Furthermore one observes that $\{a\in\widehat{E}:Da\in V'\}\subset\widehat{E}$ is a subspace, and by the isomorphism of Proposition 3.1.12 it is clear that $\{a\in\widehat{E}:Da\in V'\}$ is of dimension $\dim(V')+1$. We observe from (3.2.6) that $\pi(\widehat{E'})\subset\{a\in\widehat{E}:Da\in V'\}$, thus we may even conclude that

$$\pi(\widehat{E'}) = \{a\in\widehat{E}:Da\in V'\}.\tag{3.2.8}$$

Finally, $R'\subset\{a\in\widehat{E}:Da\in V'\}$ by definition of V', so $R'\subset\pi(\widehat{E'})$.

Proposition 3.2.19. *The set $\widetilde{R}' := \pi^{-1}(R')\subset\widehat{E'}$ is an affine root system on E'.*

Proof. Since π preserves the forms on $\widehat{E'}$ and $\widehat{E}$ it is clear that $\widetilde{R}'$ is a nonempty subset of non-isotropic vectors of $\widehat{E'}$. Now we will check the five conditions of Definition 3.2.1.

(1) To show that $\widetilde{R}'$ spans $\widehat{E'}$ it suffices to show that R' spans $\pi(\widehat{E'}) = \{a\in\widehat{E}:Da\in V'\}$. For $a'\in R'$ there exists $a\in R'$ distinct from a' such that $Da = Da'$ by Definition 3.2.15. Choose $x\in E$ and identify $\{a\in\widehat{E}:Da\in V'\}$ with $V'\oplus\mathbb{R}c$ using Proposition 3.1.12. Then clearly a and a' span $\mathbb{R}Da\oplus\mathbb{R}c$, hence R' spans $V'\oplus\mathbb{R}c$. We observe that $\widetilde{R}'$ satisfies criterion (1) of Definition 3.2.1.

(2) Let $a',b'\in\widetilde{R}'$, then there exist $a,b\in R'$ such that $a=\pi(a')$ and $b=\pi(b')$. Hence looking at orthogonal reflections in $\widehat{E'}$ we observe

$$\begin{aligned}w_{a'}(b') &\overset{(3.1.15)}{=} b'-(a'^{\vee},b')_{\widehat{E'}}\,a'\\ &= \pi^{-1}(b)-(\pi^{-1}(a)^{\vee},\pi^{-1}(b))_{\widehat{E}}\,\pi^{-1}(a)\\ &= \pi^{-1}(b-(\pi^{-1}(a)^{\vee},\pi^{-1}(b))_{\widehat{E}}\,a)\\ &\overset{(3.2.7)}{=} \pi^{-1}(b-(a^{\vee},b)_{\widehat{E}}\,a)\\ &\overset{(3.1.15)}{=} \pi^{-1}(w_a(b)).\end{aligned}\tag{3.2.9}$$

Since $w_a(b)\in R'$ by Definition 3.2.15, we observe that $w_{a'}(b') = \pi^{-1}(w_a(b))\in\widehat{R}'$. Therefore criterion (2) of Definition 3.2.1 is satisfied

(3) Since π preserves the forms on $\widehat{E'}$ and $\widehat{E}$ it is immediate that criterion (3) of Definition 3.2.1 is satisfied.

(4) Let $W'\subset W(R)$ be the subgroup generated by the reflections w_a such that $a\in R'$, and let $W(\widetilde{R}')$ be the group generated by the orthogonal reflections $w_{a'}$ of E' for $a'\in\widetilde{R}'$. Let $w_a\in W'$ for

some $a \in R'$ and consider $w_a \in \mathrm{GL}(E)$ (see Proposition 3.1.22), then by (3.1.11) we have $Dw_a = w_{Da}$. Now $Da \in V'$ since $a \in R'$, thus by (3.1.5) $Dw_a|_{V'} = w_{Da}|_{V'} : V' \to V'$ and $Dw_a|_{V'^\perp} = w_{Da}|_{V'^\perp} = \mathrm{id}_{V'^\perp}$. Let $w' \in W'$, then $w' = w_1 \circ \cdots \circ w_k$ with reflections $w_1, \dots w_k \in W'$. Hence (ii) of Example 3.1.6 guarantees that $Dw' = Dw_1 \circ \cdots \circ Dw_k$ which implies that $Dw'|_{V'} : V' \to V'$ and $Dw'|_{V'^\perp} = \mathrm{id}_{V'^\perp}$.

First we want to establish an isomorphism between W' and $W(\widetilde{R}')$, so that we can later exploit the fact that $W(R)$, and hence W', acts properly on E to prove that $W(\widetilde{R}')$ acts properly on E'. Using Proposition 3.1.22 we can consider W' (resp. $W(\widetilde{R}')$) as subgroup of $\mathrm{GL}_{\mathbb{R},c}(\widehat{E})$ (resp. $\mathrm{GL}_{\mathbb{R},c'}(\widehat{E'})$). Let $w' \in W'$, then $w' \circ \pi : \widehat{E'} \to \widehat{E}$ is a linear map that sends c' to c. Now $\pi(\widehat{E'}) = \{a \in \widehat{E} : Da \in V'\}$ by (3.2.8), and by Proposition 3.1.23 we have the gradient $D(w'(a)) = (Dw')(Da)$ for all $a \in \widehat{E}$. But we know that $Dw' \in \mathrm{GL}(V)$ satisfies $Dw'|_{V'} : V' \to V'$ and $Dw'|_{V'^\perp} = \mathrm{id}_{V'^\perp}$, so the image $(w' \circ \pi)(\widehat{E'})$ is contained in $\{a \in \widehat{E} : Da \in V'\} = \pi(\widehat{E'})$. Let $w \in \mathrm{GL}_{\mathbb{R},c'}(\widehat{E'})$ be the unique linear automorphism such that $\pi \circ w = w' \circ \pi$, where we consider $\pi : \widehat{E'} \xrightarrow{\sim} \pi(\widehat{E'})$ as linear isomorphism onto its image. Then it follows straightforwardly from the uniqueness of w that $w' \mapsto w =: \phi(w')$ defines a group homomorphism $\phi : W' \to W(\widetilde{R}')$. Furthermore, (3.2.9) implies that ϕ satisfies $\phi(w_a) = w_{\pi^{-1}(a)}$ for all $a \in R'$. Thus we observe from the definition of $\widetilde{R}'$ and $W(\widetilde{R}')$ that the homomorphism ϕ is surjective.

Next we want to prove injectivity of ϕ. Suppose $w' \in W'$ such that $\phi(w') = \mathrm{id}_{\widehat{E'}}$, then for all $a' \in \widehat{E'}$ we have $\pi(a) = \pi(\phi(w')(a)) = w'(\pi(a))$. This implies that $w'|_{\pi(\widehat{E'})} = \mathrm{id}_{\pi(\widehat{E'})}$. Choosing an origin $x \in E$ and using Proposition 3.1.12, we can identify $\widehat{E}$ with $V \oplus \mathbb{R}c = V' \oplus V'^\perp \oplus \mathbb{R}c$, where the subspace $\pi(\widehat{E'}) = \{a \in \widehat{E} : Da \in V'\}$ of $\widehat{E}$ is identified with $V' \oplus \mathbb{R}c$. Further, we identify E as the vector space E_x with V through the linear isomorphism $E_x \to V, x + v \mapsto v$. Then for $a \in \widehat{E}$ there exist $\lambda \in \mathbb{R}$, $\alpha \in V'$ and $\beta \in V'^\perp$ such that $a = \alpha + \beta + \lambda c$ and $a(x + v_1 + v_2) = (\alpha, v_1)_{V'} + (\beta, v_2)_{V'^\perp} + \lambda$ for all $v_1 \in V'$ and $v_2 \in V'^\perp$. We already know that $w'|_{\pi(\widehat{E'})} = \mathrm{id}_{\pi(\widehat{E'})} = \mathrm{id}_{V' \oplus \mathbb{R}c}$, so by linearity of w' we only need to show that $w'|_{V'^\perp} = \mathrm{id}_{V'^\perp}$ to prove injectivity of ϕ.

By Proposition 3.1.21 we have the identification of $\mathrm{GL}(E)$ with $t(V) \rtimes \mathrm{GL}_{\mathbb{R}}(V)$ using $x \in E$ as origin. Hence there exists a unique $u \in V$ such that $w'^{-1} = t_u \circ Dw'^{-1}$. For the proof that $w'|_{V'^\perp} = \mathrm{id}_{V'^\perp}$ we need to establish that $u \in V'$. Since $w'^{-1} \in W'$ we can write $w'^{-1} = w_{a_1} \circ \cdots \circ w_{a_r}$ with $a_i \in R'$ for $1 \le i \le r$. Then by (3.1.13) we have $w_{a_i} = t_{-a_i(x)Da_i^\vee} \circ w_{Da_i}$ for each i. Notice that $-a_i(x)Da_i^\vee \in V'$ because $a_i \in R'$. Put $u_i := -a_i(x)Da_i^\vee \in V'$, then

$$w'^{-1} = t_{u_1} \circ w_{Da_1} \circ t_{u_2} \circ w_{Da_2} \circ \cdots \circ t_{u_{r-1}} \circ w_{Da_{r-1}} \circ t_{u_r} \circ w_{Da_r}.$$

We can rewrite this as

$$\begin{aligned} w'^{-1} = t_{u_1} \circ (w_{Da_1} \circ t_{u_2} \circ w_{Da_1}^{-1}) \circ (w_{Da_1} \circ w_{Da_2} \circ t_{u_3} \circ w_{Da_2}^{-1} \circ w_{Da_1}^{-1}) \circ \dots \\ \circ (w_{Da_1} \circ \cdots \circ w_{Da_{r-1}} \circ t_{u_r} \circ w_{Da_{r-1}}^{-1} \circ \cdots \circ w_{Da_1}^{-1}) \circ (w_{Da_1} \circ w_{Da_2} \circ \cdots \circ w_{Da_r}) \end{aligned}$$

Using (3.1.14) and the fact that $Dw'^{-1} = w_{Da_1} \circ w_{Da_2} \circ \cdots \circ w_{Da_r}$ we can write

$$\begin{aligned} w'^{-1} &= t_{u_1} \circ t_{w_{Da_1}(u_2)} \circ \cdots \circ t_{w_{Da_1}(w_{Da_2}(\dots w_{Da_{r-1}}(u_r)\dots))} \circ Dw'^{-1} \\ &= t_{u_1 + w_{Da_1}(u_2) + \cdots + w_{Da_1}(w_{Da_2}(\dots w_{Da_{r-1}}(u_r)\dots))} \circ Dw'^{-1}. \end{aligned}$$

Since $u \in V$ such that $w'^{-1} = t_u \circ Dw'^{-1}$ is unique, we must have $u = u_1 + w_{Da_1}(u_2) + \cdots + w_{Da_1}(w_{Da_2}(\dots w_{Da_{r-1}}(u_r)\dots))$. As we have already seen $a_i \in R'$ implies that $w_{Da_i} : V' \to V'$, so because $u_1, \dots, u_r \in V'$ we observe that

$$u = u_1 + w_{Da_1}(u_2) + \cdots + w_{Da_1}(w_{Da_2}(\dots w_{Da_{r-1}}(u_r)\dots)) \in V'.$$

Now let us show that $w'(\beta) = \beta$ for $\beta \in V'^{\perp}$, then by linearity of w' we will have $w' = \mathrm{id}_{V' \oplus V'^{\perp} \oplus \mathbb{R}c} = \mathrm{id}_{\widehat{E}}$. Notice that $Dw'|_{V'} : V' \to V'$ and $Dw'|_{V'^{\perp}} = \mathrm{id}_{V'^{\perp}}$ implies that $Dw'^{-1}|_{V'} : V' \to V'$ and $Dw'^{-1}|_{V'^{\perp}} = \mathrm{id}_{V'^{\perp}}$. For $v \in V$ write $v = v_1 + v_2$ for unique $v_1 \in V'$ and $v_2 \in V'^{\perp}$, then we have $w'^{-1}(v) = (t_u \circ Dw'^{-1})(v) = Dw'^{-1}(v_1) + v_2 + u$. So

$$\begin{aligned} w'(\beta)(v) &= (\beta \circ w'^{-1})(v) \\ &= (\beta, Dw'^{-1}(v_1) + v_2 + u)_V. \end{aligned}$$

Further, $\beta \in V'^{\perp}$ and $Dw'^{-1}(v_1), u \in V'$, so $w'(\beta)(v) = (\beta, v_2)_V = (\beta, v)_V = \beta(v)$. This shows that $w' = \mathrm{id}_{V' \oplus V'^{\perp} \oplus \mathbb{R}c} = \mathrm{id}_{\widehat{E}}$ which implies that ϕ is injective. Thus ϕ is a group isomorphism.

Before we can conclude the proof we want to observe how the characterizing formula $w' \circ \pi = \pi \circ \phi(w')$ of ϕ for $w' \in \mathrm{GL}_{\mathbb{R},c}(\widehat{E})$ translates to a formula for w' considered in $\mathrm{GL}(E)$. Let $a' \in \widetilde{R}'$ then there exists a unique $a \in R'$ such that $a = \pi(a') = a' \circ p'$. By (3.1.9) and the properties of the affine linear map $p' : E \to E'$ we have for $x \in E$

$$\begin{aligned} p'(w_a(x)) &= p'(x - a(x)Da^{\vee}) \\ &= p'(x) - a(x)Dp'(Da^{\vee}) \\ &= p'(x) - a(x)Da'^{\vee} \\ &= p'(x) - a'(p(x))Da'^{\vee} \\ &= x + V'^{\perp} - a'(x + V'^{\perp})Da'^{\vee} \\ &= w_{a'}(x + V'^{\perp}) = w_{a'}(p'(x)), \end{aligned}$$

where we used that $Da \in V'$ since $a \in R'$. Then for $p'_{\perp}$ we obtain

$$\begin{aligned} p'_{\perp}(w_a(x)) &= p'_{\perp}(x - a(x)Da^{\vee}) \\ &= p'_{\perp}(x) - a(x)Dp'_{\perp}(Da^{\vee}) \\ &= p'_{\perp}(x). \end{aligned}$$

Thus $p' \circ w_a = w_{a'} \circ p'$ and $p'_{\perp} \circ w_a = p'_{\perp}$. Moreover, one can directly compute in a similar fashion that $p' \circ \phi(w)^{-1} = w \circ p'$ and $p'_{\perp} \circ \phi(w)^{-1} = p'_{\perp}$ for all $w \in W(\widetilde{R}')$.

Finally, we can deduce condition (4): Let $K'_1, K'_2 \subset E'$ and $K'' \subset E''$ be compact. Suppose $w \in W(\widetilde{R}')$ such that $w(K'_1) \cap K'_2 \neq \emptyset$, then also $(w(K'_1) \times K'') \cap (K'_2 \times K'') \neq \emptyset$. The map $p : E \to E' \times E''$ is an affine linear isomorphism, hence Corollary 3.1.8 implies that p is a homeomorphism. Put $K_1 := p^{-1}(K'_1 \times K'') \subset E$ and $K_2 := p^{-1}(K'_2 \times K'') \subset E$, then both K_1 and K_2 are compact in E since p is an affine linear isomorphism. Further

$$p'(\phi^{-1}(w)(K_1)) = w(p'(K_1)) = w(K'_1)$$

and

$$p'_{\perp}(\phi^{-1}(w)(K_1)) = p'_{\perp}(K_1) = K'',$$

hence $\phi^{-1}(w)(K_1) \cap K_2 \neq \emptyset$. This only holds for only finitely many $w \in W(\widetilde{R}')$ since $W(R)$ acts properly on E and ϕ is an isomorphism. Thus there exist only finitely many $w \in W(\widetilde{R}')$ such that $w(K'_1) \cap K'_2 \neq \emptyset$. We conclude that $W(\widetilde{R}')$ acts properly on E', hence criterion (4) of Definition 3.2.1 is holds for $\widetilde{R}'$.

(5) Let $a' \in \widetilde{R}'$, then $a := \pi(a') \in R'$ so by (5) of Definition 3.2.15 there exists $b \in R'$ such that $Da = Db$. This implies that $a - b$ is a constant function on E. Put $b' := \pi^{-1}(b)$, then $a' \neq b'$ in $\widetilde{R}'$ by

injectivity of π. Further $\pi^{-1}(a-b)=\pi^{-1}(a)-\pi^{-1}(b)=a'-b'$ must also be a constant function since π sends constant functions to constant functions. Thus $Da'=Db'$ which proves that criterion (5) of Definition 3.2.1 holds for $\widetilde{R}'$.

This shows that $\widetilde{R}'$ is an affine root system on E' with affine Weyl group isomorphic to the subgroup $W'\subset W(R)$. □

Definition 3.2.20. We will call the affine root system $\widetilde{R}'$ on E' as constructed in this Subsection the affine root system *associated to* the affine root subsystem R'. The form preserving injective linear map $\pi:\widehat{E'}\hookrightarrow\widehat{E}$ that sends $\widetilde{R}'$ bijectively onto R' as in the previous is called the *realizing map* of $\widetilde{R}'$. If $R'\subset R\subset\widehat{E}$ is already an affine root system on E, then R' is its own associated affine root system.

Lemma 3.2.21. *If there exists a nonempty subset R' of an affine root system R on E such that $(a,b)_{\widehat{E}}=0$ for all $a\in R'$ and $b\in R\setminus R'$, then both R' and $R\setminus R'$ are affine root subsystems of R.*

Proof. Let $a',b'\in R'$, then $(a',a)_{\widehat{E}}=(b',a)_{\widehat{E}}=0$ for all $a\in R\setminus R'$. Since $R\subset\widehat{E}$ is a nonempty subset of nonisotropic vectors which span $\widehat{E}$ there must exist an affine root $b\in R$ such that $(w_{a'}(b'),b)_{\widehat{E}}\neq 0$, else $w_{a'}(b')\in R$ would be isotropic. Now $w_{a'}(b')=b'-(a'^{\vee},b')_{\widehat{E}}\,a'$, so we obtain that $(w_{a'}(b'),a)_{\widehat{E}}=(b',a)_{\widehat{E}}-(a'^{\vee},b')_{\widehat{E}}(a',a)_{\widehat{E}}=0$ for all $a\in R\setminus R'$. This implies that there exists $b\in R'$ such that $(w_{a'}(b'),b)_{\widehat{E}}\neq 0$, hence $w_{a'}(b')\in R'$ for all $a',b'\in R'$.

Further, let $a'\in R'$, then there exists an affine root $a\in R$ distinct from a' such that $Da=Da'$ by (5) of Definition 3.2.1. Because R consists only of nonisotropic vectors contained in $\widehat{E}$ we obtain $(a,a')_{\widehat{E}}=(Da,Da')_V=(Da,Da)_V=(a,a)_{\widehat{E}}\neq 0$. The definition of R' now implies that $a\in R'$, so for each gradient of an affine root of R' there exist at least two distinct affine roots in R' with the same gradient. This shows that $R'\subset R$ is an affine root subsystem. Furthermore, since this argument is symmetric in R' and $R\setminus R'$ we observe that $R\setminus R'$ is also an affine root subsystem of R. □

Lemma 3.2.21 invokes the following Definition.

Definition 3.2.22. An affine root system R on E is said to be *reducible* if there exist nonempty subsets $R',R''\subset R$ such that $R=R'\amalg R''$, and $(a,b)_{\widehat{E}}=0$ for all $a\in R'$ and $b\in R''$. It is called *irreducible* otherwise. An affine root subsystem $R'\subset R$ is said to be *(ir)reducible* if the affine root system $\widetilde{R}'$ associated to R' is (ir)reducible.

Example 3.2.23. (i) Let R be an affine root system on E of rank 1 and assume that R is reducible. Hence there exist nonempty subsets $R',R''\subset R$ such that $R=R'\amalg R''$ and $(a,b)_{\widehat{E}}=0$ for all $a\in R'$ and $b\in R''$. Let $a\in R'$ and $b\in R''$ then $Da,Db\neq 0$, because R consists of nonisotropic vectors contained in $\widehat{E}$. Since R is of rank 1 the space of translations V of E is of dimension 1. This implies that $(a,b)_{\widehat{E}}=(Da,Db)_V\neq 0$ which contradicts the assumption that R is reducible. We conclude that an affine root system R on E of rank 1 is always irreducible.

(ii) Let R (resp. R') be an affine root system on E (resp. E'), and let $T:\widehat{E}\to\widehat{E'}$ be a linear isomorphism realizing a similarity between R and R'. Assume that R is irreducible, and R' is reducible. Then $R'=R'_1\amalg R'_2$ such that $(a',b')_{\widehat{E'}}=0$ for all $a'\in R'_1$ and $b'\in R'_2$. Now $R=T^{-1}(R'_1)\amalg T^{-1}(R'_2)$, and for $a'\in R'_1$ and $b'\in R'_2$ we have $(T^{-1}(a')^{\vee},T^{-1}(b'))_{\widehat{E}}=(a'^{\vee},b')_{\widehat{E'}}=0$. This implies that $(T^{-1}(a'),T^{-1}(b'))_{\widehat{E}}=0$, or $(a,b)_{\widehat{E}}=0$ for all $a\in T^{-1}(R'_1)$ and $b\in T^{-1}(R'_2)$. This implies that R is reducible which contradicts our assumptions. We conclude that both R and R' must be either reducible or irreducible, so similarity respects the notion (ir)reducibility.

(iii) Example 3.2.5 (ii) made clear that the set of real roots Δ^{re} corresponding to the affine Lie algebra $\mathfrak{g}(A)$ is a reduced affine root system. By (vii) of Theorem 2.3.4 it follows that Δ^{re} is also irreducible.

(iv) The reduced affine root system $R^u_{R_0}$ of Example 3.2.5 (iii) is irreducible since R_0 is irreducible as we will see in Proposition 3.5.5 in slightly more detail.

Proposition 3.2.24. *For an affine root system R on E there exist irreducible affine root subsystems $R_1, \dots, R_m \subset R$ such that R can be written uniquely as the disjoint union $\coprod_{i=1}^m R_i$ with $(a,b)_{\widehat{E}} = 0$ for each $a \in R_i$, $b \in R_j$ and $i \neq j$ up to a reordering of the indices.*

Proof. We proceed with induction on the rank l of R. Affine root systems of rank one must be irreducible by (i) of Example 3.2.23. Now assume we are done up to some $l > 1$. If R is reducible write $R = R' \amalg R''$ such that $(a,b)_{\widehat{E}} = 0$ for all $a \in R'$ and $b \in R''$. By construction of the affine root system $\widetilde{R'}$ on E' (resp. $\widetilde{R''}$ on E'') associated to R' (resp. R'') we observe that the rank of $\widetilde{R'}$ (resp. $\widetilde{R''}$) is strictly smaller than l. Hence the induction hypothesis holds for both $\widetilde{R'}$ and $\widetilde{R''}$. This means that we can write $\widetilde{R'} = R'_1 \amalg \cdots \amalg R'_k$ (resp. $\widetilde{R''} = R''_1 \amalg \cdots \amalg R''_n$) for irreducible affine root subsystems $R'_1, \dots, R'_k \subset \widetilde{R'}$ (resp. $R''_1, \dots, R''_n \subset \widetilde{R''}$) such that $(a,b)_{\widehat{E'}} = 0$ (resp. $(a,b)_{\widehat{E''}} = 0$) for each $a \in R'_i$, $b \in R'_j$ (resp. $a \in R''_i$, $b \in R''_j$) and $i \neq j$.

Consider the injective linear map $\pi' : \widehat{E'} \hookrightarrow \widehat{E}$ (resp. $\pi'' : \widehat{E''} \hookrightarrow \widehat{E}$) that realizes the affine root system $\widetilde{R'}$ (resp. $\widetilde{R''}$) associated to R' (resp. R''). We want to show that the decomposition we are looking for is $R = \coprod_{i=1}^k \pi'(R'_i) \amalg \coprod_{j=1}^n \pi''(R''_j)$, where $R' = \pi'(\widetilde{R'}) = \coprod_{i=1}^k \pi'(R'_i)$ and $R'' = \pi''(\widetilde{R''}) = \coprod_{j=1}^n \pi''(R''_j)$. Notice that the injectivity of both π' and π'' justify the disjoint unions. Furthermore, since the realizing maps π' and π'' are form preserving we observe that $(a,b)_{\widehat{E}} = 0$ for all $a \in \pi'(R'_i)$, $b \in \pi'(R'_j)$ (resp. $a \in \pi''(R''_i)$, $b \in \pi''(R''_j)$) and $i \neq j$. Because $R = R' \amalg R''$ such that $(a',b')_{\widehat{E}} = 0$ for $a' \in R'$ and $b' \in R''$ we also have $(a,b)_{\widehat{E}} = 0$ for all $a \in \pi'(R'_i)$ and $b \in \pi''(R''_j)$.

Since both π' and π'' are bijections onto their image with similar arguments (but with reversed realizing maps) as in the proofs of (2) and (5) from the proof of Proposition 3.2.19 that $\pi'(R'_i)$ for $i = 1, \dots, k$ and $\pi''(R''_j)$ for $j = 1, \dots, n$ are affine root subsystems of R. Consider the affine root system $\widetilde{R'_i}$ on E'_i associated to R'_i realized by the injective linear map $\pi'_i : \widehat{E'_i} \hookrightarrow \widehat{E'}$ for $i \in \{1, \dots, k\}$. Also, consider the affine root system $\widetilde{\pi'(R'_i)}$ on $E^{\pi'}_i$ associated to $\pi'(R'_i) \subset R$ realized by $\pi_i : \widehat{E^{\pi'}_i} \hookrightarrow \widehat{E}$. Then $\widetilde{R'_i}$ gets bijectively mapped onto $\pi'(R'_i) \subset R$ by $\pi' \circ \pi'_i$, and $\widetilde{\pi'(R'_i)}$ gets bijectively mapped onto $\pi'(R'_i) \subset R$ by π_i (see the following diagram).

$$\begin{array}{ccc}
\widehat{E'_i} \supset \widetilde{R'_i} & \xrightarrow{\pi'_i} & R'_i \subset \widetilde{R'} \subset \widehat{E'} \\
 & & \big\downarrow \pi' \\
\widehat{E^{\pi'}_i} \supset \widetilde{\pi'(R'_i)} & \xrightarrow{\pi_i} & \pi'(R'_i) \subset R' \subset \widehat{E}
\end{array}$$

Because $\widetilde{R'_i} \subset \widehat{E'_i}$ (resp. $\widetilde{\pi'(R'_i)} \subset \widehat{E^{\pi'}_i}$) is an affine root system, its $\mathbb{R}$-span is all of $\widehat{E'_i}$ (resp. $\widehat{E^{\pi'}_i}$). Then by linearity we notice that $\pi'(\pi'_i(\widehat{E'_i})) = \pi_i(\widehat{E^{\pi'}_i})$. So considering π_i as a linear isomorphism on its image, we obtain the linear isomorphism $\pi_i^{-1} \circ \pi' \circ \pi'_i : \widehat{E'_i} \xrightarrow{\sim} \widehat{E^{\pi'}_i}$ that maps $\widetilde{R'_i}$ bijectively onto $\widetilde{\pi'(R'_i)}$. Furthermore, π_i^{-1}, π' and π'_i are form preserving, so we observe that $\pi_i^{-1} \circ \pi' \circ \pi'_i$ realizes a similarity between $\widetilde{R'_i}$ and $\widetilde{\pi'(R'_i)}$. Since $\widetilde{R'_i}$ is irreducible by assumption, we observe from (ii) of Example 3.2.23 that $\widetilde{\pi'(R'_i)}$ is also irreducible for $i \in \{1, \dots, k\}$. In a similar fashion one obtains that $\widetilde{\pi''(R''_j)}$ is also irreducible for $j \in \{1, \dots, n\}$. We conclude that R has a decomposition into irreducible root subsystems as stated in the Proposition.

Next, we want to prove the uniqueness of this decomposition. Assume we can write $R = \coprod_{i=1}^m R_i$ and $R = \coprod_{j=1}^n R'_j$ for irreducible affine root subsystems $R_i, R'_j \subset R$ ($i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$)

such that $(a,b)_{\widehat{E}} = 0$ for all $a \in R_i, b \in R_k$ (resp. $a \in R'_i, b \in R'_k$) and $i \neq k$. Further, assume that after reordering the indices there exists $q > 1$ such that $R_1 \subset R'_1 \amalg \cdots \amalg R'_q$ and $R_1 \cap R'_j \neq \emptyset$ for $j = 1,\ldots,q$. Consider the affine root system $\widetilde{R_1}$ on E_1 associated to R_1 with realizing map $\pi : \widehat{E_1} \hookrightarrow \widehat{E}$. Since π preserves the bilinear forms on $\widehat{E_1}$ and $\widehat{E}$ respectively, we observe that $\widetilde{R_1} = \pi^{-1}(R'_1 \cap R_1) \amalg \cdots \amalg \pi^{-1}(R'_q \cap R_1)$ with $\pi^{-1}(R'_k \cap R_1) \neq \emptyset$ for $k = 1,\ldots,q$ and $(a',b')_{\widehat{E_1}} = 0$ for all $a' \in \pi^{-1}(R'_i \cap R_1), b' \in \pi^{-1}(R'_j \cap R_1)$ and $i \neq j$. This implies that $\widetilde{R_1}$ is reducible which contradicts the assumption that $\widetilde{R_1}$ is irreducible. We conclude that for each i there exists a unique j such that $R_i \subset R'_j$. Since the argument is symmetric in R_i and R'_j, we also have that for every j there exists a unique k such that $R'_j \subset R_k$. But then for each i there exist unique j and k such that $R_i \subset R'_j \subset R_k$ which implies that $i = k$ and $R_i = R'_j$. Furthermore we obtain $m = n$, and after reordering the indices it becomes clear that $R_i = R'_i$ for all $i = 1,\ldots,n$. □

The unique decomposition of R into $\coprod_{i=1}^m R_i$ as in Proposition 3.2.24 is called the *orthogonal decomposition* of R.

3.2.4 The direct sum of affine root systems

In this Subsection we construct new affine root systems by joining affine root systems together in a natural way called a direct sum. Further, we will show that each affine root system is similar to a direct sum of irreducible affine root systems, and that one only needs to study a complete set of representatives of the similarity classes of irreducible affine root systems to understand affine root systems up to similarity.

Let E_1 (resp. E_2) be an affine Euclidean space with a finite-dimensional space of translation V_1 (resp. V_2) that is endowed with the inner product $(.,.)_{V_1}$ (resp. $(.,.)_{V_2}$). Also for $i = 1,2$ let $\widehat{E_i}$ be the space of affine linear functions on E_i, endowed with the bilinear form $(.,.)_{\widehat{E_i}}$ such that $(a_i,b_i)_{\widehat{E_i}} = (Da_i,Db_i)_{V_i}$ for all $a_i,b_i \in \widehat{E_i}$. Put $E := E_1 \times E_2$ and $V := V_1 \oplus V_2$, then E turns into an affine Euclidean space with V as space of translations as follows: The action of V on E is defined componentwise as

$$(x_1,x_2)+(v_1,v_2) = (x_1+v_1,x_2+v_2)$$

where $x_i \in E_i$ and $v_i \in V_i$ for $i = 1,2$, and the inner product $(.,.)_V$ on V is defined componentwise as

$$((u_1,u_2),(v_1,v_2))_V = (u_1,v_1)_{V_1} + (u_2,v_2)_{V_2}$$

where $x_i \in E_i$ and $u_i, v_i \in V_i$ for $i = 1,2$. Thus E has a Euclidean metric again.

For each $i \in \{1,2\}$ let $p_i : E \to E_i$ be the projection map that sends (x_1,x_2) to x_i. It follows in a straightforward way that $p_i \in \mathrm{Hom}(E,E_i)$ for $i = 1,2$. Furthermore, $Dp_i \in \mathrm{Hom}_{\mathbb{R}}(V,V_i)$ is the orthogonal projection on the i-th coordinate of V defined by $Dp_i(v_1,v_2) = v_i$, where each $v_j \in V_j$. For $x \in E$ and $v \in V$ there exist $x_i \in E_i$ and $v_i \in V_i$ for $i = 1,2$ such that $x = (x_1,x_2)$ and $v = (v_1,v_2)$. Hence for $a_i \in \widehat{E_i}$ we have

$$\begin{aligned}(a_i \circ p_i)(x+v) &= (a_i \circ p_i)((x_1,x_2)+(v_1,v_2)) \\ &= a_i(p_i((x_1+v_1,x_2+v_2))) \\ &= a_i(x_i+v_i) \\ &= a_i(x_i) + (Da_i,v_i)_{V_i},\end{aligned}$$

where the last equation follows from affine linearity of a_i on E_i. For $i = 1,2$, define $\iota_i : V_i \hookrightarrow V$ to be the injective linear homomorphism that embeds V_i canonically into $V = V_1 \oplus V_2$. Then

$a_i(x_i)+(Da_i, v_i)_{V_i} = (a_i \circ \pi_i)(x)+(\iota_i(Da_i), v)_V$, hence $(a_i \circ \pi_i)$ is an affine linear function on E with gradient $D(a_i \circ \pi_i) = \iota_i(Da_i)$. Furthermore, $(a_i \circ \pi_i) \neq (b_i \circ \pi_i)$ for distinct $a_i, b_i \in \widehat{E_i}$. So if we let $\widehat{E}$ denote the space of affine linear functions on E, we can define the injective linear homomorphism $\pi_i : \widehat{E_i} \hookrightarrow \widehat{E}$ by $\pi(a_i) = a_i \circ p_i$ for $i = 1, 2$.

For $a_i \in \widehat{E_i}$ and $a'_j \in \widehat{E_j}$, we have

$$
\begin{aligned}
(\pi_i(a_i), \pi_j(a'_j))_{\widehat{E}} &= (D\pi_i(a_i), D\pi_j(a'_j))_V \\
&= (\iota_i(Da_i), \iota_j(Da'_j))_V \\
&= \begin{cases} (a_i, a'_j)_{\widehat{E_i}} & (i = j), \\ 0 & (i \neq j). \end{cases}
\end{aligned}
\tag{3.2.10}
$$

Hence the maps π_i preserve the forms on $\widehat{E_i}$ and $\widehat{E}$ respectively, and $(\pi_1(\widehat{E_1}), \pi_2(\widehat{E_2}))_{\widehat{E}} = 0$. Notice that both $\pi_1(\widehat{E_1})$ and $\pi_2(\widehat{E_2})$ contain all constant functions on $\widehat{E}$, because the constant one function $c_i : E_i \to \mathbb{R}$ gets sent to the constant one function $c := \pi(c_i) = c_i \circ p_i : E \to \mathbb{R}$ for $i = 1, 2$. On the other hand, if $a \in \pi_1(\widehat{E_1}) \cap \pi_2(\widehat{E_2})$, then $(a, a)_{\widehat{E}}$ must vanish by (3.2.10) which implies that a is a constant function on E. Thus $\pi_1(\widehat{E_1}) \cap \pi_2(\widehat{E_2}) = \mathbb{R}c$. Finally, let $a \in \widehat{E}$, $x_i \in E_i$ and $v_i \in V_i$ for $i = 1, 2$, then we have

$$
\begin{aligned}
a((x_1, x_2) + (v_1, v_2)) &= a((x_1, x_2)) + (Da, (v_1, v_2))_V \\
&= a((x_1, x_2)) + (Da, \iota_1(v_1))_V + (Da, \iota_2(v_2))_V \\
&= a((x_1, x_2)) + (Dp_1(Da), v_1)_{V_1} + (Dp_2(Da), v_2)_{V_2}
\end{aligned}
$$

By Proposition 3.1.12, we can choose $a_i \in \widehat{E_i}$ such that $Da_i = Dp_i(Da)$ for $i = 1, 2$ and $a_1(x_1) + a_2(x_2) = a((x_1, x_2))$. Hence $a = \pi_1(a_1) + \pi_2(a_2)$, and we observe that $\pi_1(\widehat{E_1})$ and $\pi_2(\widehat{E_2})$ generate $\widehat{E}$.

Now let R_i be an affine root system on E_i for $i = 1, 2$.

Proposition 3.2.25. *The set $R := \pi_1(R_1) \cup \pi_2(R_2)$ is an affine root system on E.*

Proof. As we have seen, the injective linear maps $\pi_1 : \widehat{E_1} \hookrightarrow \widehat{E}$ and $\pi_2 : \widehat{E_2} \hookrightarrow \widehat{E}$ send constant functions to constant functions. Since $R_1 \subset \widehat{E_1}$ and $R_2 \subset \widehat{E_2}$ are affine root systems we observe that $R = \pi_1(R_1) \cup \pi_2(R_2) \subset \widehat{E}$ is a subset of nonisotropic vectors. Now we will check the five conditions of Definition 3.2.1.

(1) Since $R_1 \subset \widehat{E_1}$ and $R_2 \subset \widehat{E_2}$ are affine root systems we know that R_1 (resp. R_2) spans $\widehat{E_1}$ (resp. $\widehat{E_2}$). Recall that the subspaces $\pi_1(\widehat{E_1})$ and $\pi_2(\widehat{E_2})$ of $\widehat{E}$ generate $\widehat{E}$. We observe that $R = \pi_1(R_1) \cup \pi_2(R_2)$ spans $\widehat{E}$.

(2) For $a_i \in \widehat{E_i}$ and $a'_j \in \widehat{E_j}$ we have the orthogonal reflection in $\widehat{E}$

$$
\begin{aligned}
w_{\pi_i(a_i)}(\pi_j(a'_j)) &= \pi_j(a'_j) - (\pi_i(a_i)^\vee, \pi_j(a'_j))_{\widehat{E}}\, \pi_i(a_i) \\
&= \pi_j(a'_j) - (D\pi_i(a_i)^\vee, D\pi_j(a'_j))_V\, \pi_i(a_i) \\
&= \pi_j(a'_j) - (\iota_i(Da_i)^\vee, \iota_j(Da'_j))_V\, \pi_i(a_i) \\
&= \begin{cases} \pi_j(a'_j) - (Da_i^\vee, Da'_j)_{V_i}\, \pi_i(a_i) & (i = j), \\ \pi_j(a'_j) & (i \neq j), \end{cases} \\
&= \begin{cases} \pi_j(w_{a_i}(a'_j)) & (i = j), \\ \pi_j(a'_j) & (i \neq j), \end{cases}
\end{aligned}
$$

where w_{a_i} is a reflection in $W(R_i)$. Using the fact that both R_1 and R_2 satisfy condition (2) of Definition 3.2.1 we observe that $w_a(b) \in R$ for all $a, b \in R = \pi_1(R_1) \cup \pi_2(R_2)$.

(3) From (3.2.10) it follows that $(a^\vee, b)_{\widehat{E}} \in \mathbb{Z}$ for all $a \in \pi_1(R_1)$ and $b \in \pi_2(R_2)$, because both $R_1 \subset \widehat{E_1}$ and $R_2 \subset \widehat{E_2}$ satisfy condition (3) of Definition 3.2.1.

(4) We want to exploit the fact that $W(R_1)$ (resp. $W(R_2)$) acts properly on E_1 (resp. E_2) to prove that $W(R)$ acts properly on $E = E_1 \times E_2$. Therefore we first establish a group isomorphism between $W(R_1) \times W(R_2)$ and $W(R)$. Consider $w_a \in W(R)$ for some $a \in R$, then $a = \pi_i(a_i)$ for some $a_i \in R_i$ and $i \in \{1,2\}$, hence $Da = D\pi_i(a_i) = \iota_i(Da_i)$. Let $x = (x_1, x_2) \in E_1 \times E_2 = E$, then by (3.1.9) we have

$$\begin{aligned} w_a(x) &= x - a(x) Da^\vee \\ &= (x_1, x_2) - a_i(p_i((x_1, x_2)))\iota_i(Da_i)^\vee \\ &= (x_1, x_2) - a_i(x_i)\iota_i(Da_i)^\vee. \end{aligned}$$

We observe that $w_a|_{E_j}$ coincides with the identity function id_{E_j} on $E_j \subset E$ for $j \neq i$, and that $w_a|_{E_i}$ coincides with $w_{a_i} \in W(R_i)$ on E_i. In other words, either $w_a = (w_{a_1}, \mathrm{id}_{E_2})$ or $w_a = (\mathrm{id}_{E_1}, w_{a_2})$ where these expressions acts componentwise on $E = E_1 \times E_2$. Consider the injective group homomorphisms $\partial_1 : \mathrm{GL}(E_1) \hookrightarrow \mathrm{GL}(E)$ (resp. $\partial_2 : \mathrm{GL}(E_2) \hookrightarrow \mathrm{GL}(E)$) given by $\partial_1(w_1) = (w_1, \mathrm{id}_{E_2})$ (resp. $\partial_2(w_2) = (\mathrm{id}_{E_1}, w_2)$ where the last expression acts componentwise on $E = E_1 \times E_2$. Then ∂_i restricts to an injective group homomorphism $W(R_i) \hookrightarrow W(R)$ for $i = 1,2$ such that $\partial_i(w_{a_i}) = w_{\pi_i(a_i)}$ for $a_i \in R_i$. Thus we obtain the group isomorphism $\partial : W(R_1) \times W(R_2) \xrightarrow{\sim} W(R)$ sending (w_1, w_2) to $\partial_1(w_1) \circ \partial_2(w_2)$.

Assume that $W(R)$ does not act properly on E, then there exist compact subsets K_1 and K_2 of E such that $w(K_1) \cap K_2 \neq \emptyset$ for infinitely many $w \in W(R)$. Notice that E with its Euclidean topology is homeomorphic to $E_1 \times E_2$ with the to the product topology of the Euclidean topologies on the E_i. For $i = 1,2$, this turns $p_i : E \to E_i$ into a continuous function by definition of the product topology of $E = E_1 \times E_2$, hence the projected sets $p_i(K_1)$ and $p_i(K_2)$ in E_i are compact. Let $w \in W(R)$ such that $w(K_1) \cap K_2 \neq \emptyset$, then by ∂ we have the unique expression $w = \partial_1(w_1) \circ \partial_2(w_2)$ with $w_i \in W(R_i)$ such that $w_i(p_i(K_1)) \cap p_i(K_2) \neq \emptyset$ for $i = 1,2$. Since $W(R_i)$ acts properly on E_i, there only exist finitely many $w_i \in W(R_i)$ such that $w_i(p_i(K_1)) \cap p_i(K_2) \neq \emptyset$ for $i = 1,2$. Using the isomorphism ∂ we observe that this can only yield finitely many $w \in W(R)$ such that $w(K_1) \cap K_2 \neq \emptyset$. This contradiction tells us that $W(R)$ acts properly on E.

(5) Let $a \in R$, then there exists $i \in \{1,2\}$ and $a' \in R_i$ such that $a = \pi_i(a')$. By condition (5) Definition 3.2.1 for R_i there exists $b' \in R_i$ distinct of a' such that $Da' = Db'$. This implies that $a' - b'$ is a constant function on E_i. Put $b := \pi_i(b')$, then $a \neq b$ in R by injectivity of π_i. Further $\pi_i(a' - b') = \pi_i(a') - \pi_i(b') = a - b$ must also be a constant function since π_i sends constant functions to constant functions. We conclude that $Da = Db$. □

The constructions of this subsection inspire the following definition of a direct sum of affine root systems.

Definition 3.2.26. The affine root system $R = \pi_1(R_1) \cup \pi_2(R_2)$ on $E = E_1 \times E_2$ as defined in Proposition 3.2.25 is called the *direct sum* of the affine root systems R_1 (resp. R_2) on E_1 (resp. E_2), and we denote it by $R = R_1 \oplus R_2$.

Corollary 3.2.27. *If the affine root system R on $E_1 \times E_2$ is the direct sum of R_1 (resp. R_2) on E_1 (resp. E_2), then $\pi_i(R_i) \subset R$ is an affine root subsystem for $i = 1,2$ and $(a, b) = 0$ for all $a \in \pi_1(R_1)$ and $b \in \pi_2(R_2)$.*

Proof. By definition of R_1 and R_2, we observe that both $\pi_1(R_1)$ and $\pi_2(R_2)$ are nonempty subsets of R. From the proof of condition (2) and (5) of Proposition 3.2.25 we conclude the first statement. The latter statement follows from (3.2.10). □

Proposition 3.2.28. *Let R be a reducible affine root system, and write $R' \amalg R''$ such that $(a,b) = 0$ for all $a \in R'$ and $b \in R''$. Let $\widetilde{R'}$ (resp. $\widetilde{R''}$) be the affine root system asociated to R' (resp. R''), then $R \simeq \widetilde{R'} \oplus \widetilde{R''}$ realized by a normalized similarity transformation.*

Proof. By Lemma 3.2.21 both R' and R'' are affine root subsystems of R, so the affine root system $\widetilde{R'}$ on E' (resp. $\widetilde{R''}$ on E'') asociated to R' (resp. R'') is well defined. Consider the realizing map $\pi' : \widehat{E'} \hookrightarrow \widehat{E}$ (resp. $\pi' : \widehat{E''} \hookrightarrow \widehat{E}$) of $\widetilde{R'}$ (resp. $\widetilde{R''}$) defined by $\pi'(a) = a \circ p'$ (resp. $\pi''(b) = b \circ p'_\perp$) with $p' : E \to E'$ (resp. $p'_\perp : E \to E''$) the canonical map that sends E to the space of orbits E' as in Proposition 3.2.17. Then $p : E \to E' \times E''$, $x \mapsto (p'(x), p'_\perp(x))$ is an affine linear isomorphism by Lemma 3.2.18. Further, the map $\pi : \widehat{E' \times E''} \to \widehat{E}$, $a \mapsto a \circ p$ is clearly injective since p is bijective, and it follows from a direct computation that π is also linear. The dimensions of that spaces of translations of $E' \times E''$ and E are the same, hence by Proposition 3.1.12 the dimensions of $\widehat{E' \times E''}$ and $\widehat{E}$ are the same. This implies that π is a linear isomorphism.

Consider the direct sum $\widetilde{R'} \oplus \widetilde{R''}$ on $E' \times E''$ with canonical embeddings $\pi_1 : \widehat{E'} \hookrightarrow \widehat{E' \times E''}$ and $\pi_2 : \widehat{E''} \hookrightarrow \widehat{E' \times E''}$. Here $\pi_1(a) = a \circ p_1$ (resp. $\pi_2(b) = b \circ p_2$) where $p_1 : E' \times E'' \to E'$ (resp. $p_2 : E' \times E'' \to E''$) is the affine linear map that projects to the first (resp. second) coordinate of $E' \times E''$. Let $a \in \widetilde{R'} \oplus \widetilde{R''}$, say $a \in \pi_1(\widetilde{R'})$, then there exists a unique $a' \in \widetilde{R'}$ such that $a = \pi_1(a') = a' \circ p_1$. We observe that $\pi(a) = a \circ p = a' \circ p_1 \circ p = a' \circ p' = \pi'(a') \in R'$. This yields a bijection of $\pi_1(\widetilde{R'})$ onto R', and in a similar manner one obtains a bijection of $\pi_2(\widetilde{R''})$ onto R''. Thus $\pi|_{\widetilde{R'} \oplus \widetilde{R''}} : \widetilde{R'} \oplus \widetilde{R''} \to R$ is a bijection. Finally, π_1 and π_2 are form preserving maps, so it follows that also π is form preserving. This shows that π realizes a similarity between R and $\widetilde{R'} \oplus \widetilde{R''}$. Furthermore, notice that π sends the constant one function on $E' \times E''$ to the constant one function on E which makes it a normalized similarity transformation. □

Next, we would like to generalize the notion of the direct sum of two affine root system to the direct sum of any finite amount of affine root systems. Therefore we first show associativity of the direct sum of affine root systems, i.e. $(R_1 \oplus R_2) \oplus R_3 = R_1 \oplus (R_2 \oplus R_3)$ with $R_i \subset \widehat{E_i}$ affine root systems for $i = 1, 2, 3$.

Let R_i be an affine root system on E_i with space of translations V_i for $i = 1, 2, 3$. Consider the direct sum $R_1 \oplus R_2$ (resp. $R_2 \oplus R_3$) on $E_{12} = E_1 \times E_2$ (resp. $E_{23} = E_2 \times E_3$) with canonical embeddings $\pi_1 : \widehat{E_1} \hookrightarrow \widehat{E_{12}}$ and $\pi_2 : \widehat{E_2} \hookrightarrow \widehat{E_{12}}$ (resp. $\pi'_2 : \widehat{E_2} \hookrightarrow \widehat{E_{23}}$ and $\pi'_3 : \widehat{E_3} \hookrightarrow \widehat{E_{23}}$). Here $\pi_1(a) = a \circ p_1$ (resp. $\pi_2(b) = b \circ p_2$) where $p_1 : E_{12} \to E_1$ (resp. $p_2 : E_{12} \to E_2$) is the affine linear map that projects to the first (resp. second) coordinate of E_{12}. Also $\pi'_2(a) = a \circ p'_2$ (resp. $\pi'_3(b) = b \circ p'_3$) where $p'_2 : E_{23} \to E_2$ (resp. $p'_3 : E_{23} \to E_3$) is the affine linear map that projects to the first (resp. second) coordinate of E_{23}. Further consider the direct sum $R := (R_1 \oplus R_2) \oplus R_3$ (resp. $R' := R_1 \oplus (R_2 \oplus R_3)$) on $E = E_{12} \times E_3$ (resp. $E' = E_1 \times E_{23}$) with canonical embeddings $\pi_{12} : \widehat{E_{12}} \hookrightarrow \widehat{E}$ and $\pi_3 : \widehat{E_3} \hookrightarrow \widehat{E}$ (resp. $\pi'_1 : \widehat{E_1} \hookrightarrow \widehat{E'}$ and $\pi'_{23} : \widehat{E_{23}} \hookrightarrow \widehat{E'}$). Here $\pi_{12}(a) = a \circ p_{12}$ (resp. $\pi_3(b) = b \circ p_3$) where $p_{12} : E \to E_{12}$ (resp. $p_3 : E \to E_3$) is the affine linear map that projects to the first (resp. second) coordinate of E. Also $\pi'_1(a) = a \circ p'_1$ (resp. $\pi'_{23}(b) = b \circ p'_{23}$) where $p'_1 : E' \to E_1$ (resp. $p'_{23} : E' \to E_{23}$) is the affine linear map that projects to the first (resp. second) coordinate of E'. Then $R = \pi_{12}(\pi_1(R_1)) \amalg \pi_{12}(\pi_2(R_2)) \amalg \pi_3(R_3)$ and $R' = \pi'_1(R_1) \amalg \pi'_{23}(\pi'_2(R_2)) \amalg \pi'_{23}(\pi'_3(R_3))$.

As affine Euclidean spaces we have $E = E' = E_1 \times E_2 \times E_3$ with space of translations $V = V_1 \oplus$

$V_2 \oplus V_3$ acting on E by

$$(x_1, x_2, x_3) + (v_1, v_2, v_3) = (x_1 + v_1, x_2 + v_2, x_3 + v_3)$$

where $x_i \in E_i$ and $v_i \in V_i$ for $i = 1,2,3$, and the inner product $(.,.)_V$ on V is defined as

$$((u_1, u_2, u_3), (v_1, v_2, v_3)_V = (u_1, v_1)_{V_1} + (u_2, v_2)_{V_2} + (u_3, v_3)_{V_3}$$

where $x_i \in E_i$ and $u_i, v_i \in V_i$ for $i = 1,2,3$. This implies that $\widehat{E} = \widehat{E'}$.

Let $a \in R$, say $a \in \pi_{12}(\pi_1(R_1))$, then there exists a unique $b \in R_1$ such that $a = \pi_{12}(\pi_1(b)) = b \circ p_1 \circ p_{12}$. Thus there also exists a unique $a' \in \pi_1'(R_1) \subset R'$ such that $a' = \pi_1'(b) = b \circ p_1'$. Clearly, $p_1 \circ p_{12} = p_1'$, so $a = a'$. In this manner we observe that $\pi_{12}(\pi_1(R_1)) = \pi_1'(R_1)$. Similarly on obtains $\pi_{12}(\pi_2(R_2)) = \pi_{23}'(\pi_2'(R_2))$ and $\pi_3(R_3) = \pi_{23}'(\pi_3'(R_3))$, so we find that $R = R'$. Hence we may write $R_1 \oplus R_2 \oplus R_3 := (R_1 \oplus R_2) \oplus R_3 = R_1 \oplus (R_2 \oplus R_3)$. Inductively we can now define the *direct sum of affine root systems R_i* for $i = 1, \ldots, n$ to be $\bigoplus_{i=1}^n R_i$ for $n > 1$. For $n = 1$ we put $\bigoplus_{i=1}^n R_i = R_1$.

Lemma 3.2.29. *Let R_i and R_i' be affine root systems such that $R_i \simeq R_i'$ for $i = 1,2$, then there exist $\lambda_1, \lambda_2 \in \mathbb{R}_{\neq 0}$ such that $R_1 \oplus R_2 \simeq \lambda_1 R_1' \oplus \lambda_2 R_2'$.*

Proof. Let $R_i \subset \widehat{E_i}$ (resp. $R_i' \subset \widehat{E_i'}$) be affine root systems for $i = 1,2$. Consider the direct sum $R_1 \oplus R_2 \subset \widehat{E_1 \times E_2}$ (resp. $R_1' \oplus R_2' \subset \widehat{E_1' \times E_2'}$) with canonical embeddings $\pi_i : \widehat{E_i} \hookrightarrow \widehat{E_1 \times E_2}$ (resp. $\pi_i' : \widehat{E_i'} \hookrightarrow \widehat{E_1' \times E_2'}$) for $i = 1,2$. Here $\pi_i(a) = a \circ p_i$ (resp. $\pi_i'(b) = b \circ p_i'$) where $p_i : E_1 \times E_2 \to E_i$ (resp. $p_i' : E_1' \times E_2' \to E_i'$) is the affine linear that projects to the i-th coordinate of $E_1 \times E_2$ (resp. $E_1' \times E_2'$) for $i = 1,2$. Assume that $T_i : \widehat{E_i} \xrightarrow{\sim} \widehat{E_i'}$ realizes a similarity between R_i and R_i' for $i = 1,2$. By (ii) of Proposition 3.2.10 the linear isomorphism T_i sends constant functions to constant functions for $i = 1,2$. Therefore there exists $\lambda_i \in \mathbb{R}_{\neq 0}$ such that $\lambda_i T_i$ sends the constant one function on E_i to the constant one function on E_i' for $i = 1,2$. Notice that by (ii) of Example 3.2.12 the linear isomorphism $\lambda_i T_i$ is a normalized similarity transformation realizing a similarity between R_i and the rescaling $\lambda_i R_i'$ of R_i' for $i = 1,2$.

We will now construct a linear isomorphism $T : \widehat{E_1 \times E_2} \to \widehat{E_1' \times E_2'}$ that realizes $R_1 \oplus R_2 \simeq \lambda_1 R_1' \oplus \lambda_2 R_2'$. By Proposition 3.2.13 there exists an affine linear isomorphism $\psi_i : E_i \to E_i'$ such that $\lambda_i T_i(a) = a \circ \psi_i^{-1}$ for $i = 1,2$. Consider the map $p : E_1' \times E_2' \to E_1 \times E_2, (x_1, x_2) \mapsto (\psi_1^{-1}(x_1), \psi_2^{-1}(x_2))$. Since both ψ_1 and ψ_2 are affine linear isomorphisms it follows from a direct computation that also p is an affine linear isomorphism. Define the map $T : \widehat{E_1 \times E_2} \to \widehat{E_1' \times E_2'}$ by $T(a) = a \circ p$. Now T is well defined and injective, because p is an affine linear isomorphism. Furthermore, it follows straightforwardly that T is a linear map. Using Proposition 3.1.12 and the linear isomorphism T_1 (resp. T_2) we observe that the spaces of translations of E_1 and E_1' (resp. E_2 and E_2') have the same dimension. Using Proposition 3.1.12 again this implies that $\widehat{E_1 \times E_2}$ and $\widehat{E_1' \times E_2'}$ have the same dimension. We conclude that π is a linear isomorphism.

Let $a \in R_1 \oplus R_2$, say $a \in \pi_i(R_i)$, then there exists a unique $a_i \in R_i$ such that $a = \pi_i(a_i) = a_i \circ p_i$. We observe that

$$T(a) = a \circ p = a_i \circ p_i \circ p = a_i \circ \psi_i^{-1} \circ p_i' = \pi_i'(\lambda_i T_i(a_i)) \in \pi_i'(\lambda_i R_i') \tag{3.2.11}$$

for $i = 1,2$. This yields a bijection $T|_{R_1 \oplus R_2} : R_1 \oplus R_2 \to \lambda_1 R_1' \oplus \lambda_2 R_2'$. Finally, because π_i and π_i' are form preserving maps, and $\lambda_i T_i$ is a linear isomorphism realizing a similarity for $i = 1,2$ it follows from (3.2.11) that $(T(a)^\vee, T(b))_{\widehat{E_1' \times E_2'}} = (a^\vee, b)_{\widehat{E_1 \times E_2}}$ for all $a, b \in R_1 \oplus R_2$. This shows that T realizes a similarity between $R_1 \oplus R_2$ and $\lambda_1 R_1' \oplus \lambda_2 R_2'$. □

Now we generalize Lemma 3.2.29.

Proposition 3.2.30. *Let R_i and R_i' be affine root systems such that $R_i \simeq R_i'$ for $i = 1, \dots, m$, then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}_{\neq 0}$ such that $\bigoplus_{i=1}^m R_i \simeq \bigoplus_{i=1}^m \lambda_i R_i'$.*

Proof. Let R_i and R_i' be affine root systems such that $R_i \simeq R_i'$ for $i = 1, \dots, m$. We proceed with induction to m. For $m = 1$ there is nothing to prove, so assume that we are done up to $m > 1$. By the induction hypothesis there exist $\lambda_1, \dots, \lambda_{m-1} \in \mathbb{R}_{\neq 0}$ such that $\bigoplus_{i=1}^{m-1} R_i \simeq \bigoplus_{i=1}^{m-1} \lambda_i R_i'$. Put $R := \bigoplus_{i=1}^{m-1} R_i$ (resp. $R' := \bigoplus_{i=1}^{m-1} R_i'$), then by Lemma 3.2.29 there exists $\mu, \lambda \in \mathbb{R} \setminus \{0\}$ such that $R \oplus R_m \simeq \mu R' \oplus \lambda R_m'$. This leads to $\bigoplus_{i=1}^m R_i \simeq (\bigoplus_{i=1}^{m-1} \mu \lambda_i R_i') \oplus \lambda R_m'$ which finishes the proof. □

This leads to the following special case.

Corollary 3.2.31. *Let R_i and R_i' be affine root systems such that $R_i \simeq R_i'$ realized by a normalized similarity transformation T_i for $i = 1, \dots, m$. Then $\bigoplus_{i=1}^m R_i \simeq \bigoplus_{i=1}^m R_i'$ realized by a normalized similarity transformation.*

Proof. Since T_i sends the constant one function to the constant one function we have $\lambda_i = 1$ for $i = 1, \dots, n$ in the proof of Lemma 3.2.29. The Corollary now follows from Proposition 3.2.30. □

Finally, we want to observe how affine root systems in general are built up as direct sums of irreducible affine root systems associated to the affine root subsystems of the orthogonal decomposition.

Proposition 3.2.32. *Let R be an affine root system with orthogonal decomposition $R_1 \amalg \cdots \amalg R_m$, and let $\widetilde{R_i}$ be the affine root system associated to R_i for $i = 1, \dots, m$. Then $R \simeq \bigoplus_{i=1}^m \widetilde{R_i}$ realized by a normalized similarity transformation.*

Proof. We proceed with induction on the rank l of R. Affine root systems of rank one must be irreducible by (i) of Example 3.2.23, hence R is its own associated affine root system which concludes this case. Now assume we are done up to some $l > 1$. If R is irreducible we are done again, so suppose that R is reducible. Write $R = R' \amalg R''$ such that $(a, b) = 0$ for all $a \in R'$ and $b \in R''$, and let $\widetilde{R'}$ on E' (resp. $\widetilde{R''}$ on E'') be the affine root system associated to R' (resp. R''). By Proposition 3.2.28 we have

$$R \simeq \widetilde{R'} \oplus \widetilde{R''}$$

realized by a normalized similarity transformation.

By Proposition 3.2.24 we have the orthogonal decomposition $\widetilde{R'} = R_1' \amalg \cdots \amalg R_k'$ (resp. $\widetilde{R''} = R_1'' \amalg \cdots \amalg R_n''$). Now let $\widetilde{R_i'}$ (resp. $\widetilde{R_j''}$) be the affine root system associated to R_i' (resp. R_j'') realized by the injective linear map $\pi_i' : \widehat{E_i'} \hookrightarrow \widehat{E'}$ (resp. $\pi_j' : \widehat{E_j''} \hookrightarrow \widehat{E''}$) for $1 \leq i \leq k$ (resp. $1 \leq j \leq n$). By construction of the affine root system $\widetilde{R'}$ on E' (resp. $\widetilde{R''}$ on E'') we observe that the rank of $\widetilde{R'}$ (resp. $\widetilde{R''}$) is strictly smaller than l. Therefore the induction hypothesis holds for both $\widetilde{R'}$ and $\widetilde{R''}$. Thus $\widetilde{R'} \simeq \bigoplus_{i=1}^k \widetilde{R_i'}$ (resp. $\widetilde{R''} \simeq \bigoplus_{j=1}^n \widetilde{R_j''}$) realized by a normalized similarity transformation. So by Corollary 3.2.31 we obtain

$$\widetilde{R'} \oplus \widetilde{R''} \simeq \bigoplus_{i=1}^k \widetilde{R_i'} \oplus \bigoplus_{j=1}^n \widetilde{R_j''}$$

realized by a normalized similarity transformation.

Next consider the injective linear map $\pi' : \widehat{E'} \hookrightarrow \widehat{E}$ (resp. $\pi'' : \widehat{E''} \hookrightarrow \widehat{E}$) that realizes the affine root system $\widetilde{R'}$ (resp. $\widetilde{R''}$) associated to R' (resp. R''). Then we find from the proof of Proposition

3.2.24 that the orthogonal decomposition of R is $\coprod_{i=1}^{k}\pi'(R'_i)\amalg\coprod_{j=1}^{n}\pi''(R''_j)$. Let $\widetilde{\pi'(R'_i)}\subset\widehat{E_i^{\pi'}}$ (resp. $\widetilde{\pi''(R''_j)}\subset\widehat{E_j^{\pi''}}$) be the affine root system associated to $\pi'(R'_i)\subset R$ (resp. $\pi''(R''_j)\subset R$) realized by the injective linear map $\pi_i:\widehat{E_i^{\pi'}}\hookrightarrow\widehat{E}$ (resp. $\pi_j:\widehat{E_j^{\pi''}}\hookrightarrow\widehat{E}$) for $i=1,\dots,k$ (resp. $j=1,\dots,n$). Then we have to show $R\simeq\bigoplus_{i=1}^{k}\widetilde{\pi'(R'_i)}\oplus\bigoplus_{j=1}^{n}\widetilde{\pi''(R''_j)}$ realized by a normalized similarity transformation to prove this Proposition. By the proof of Proposition 3.2.24 we also have that $\widetilde{R'_i}\simeq\widetilde{\pi'(R'_i)}$ (resp. $\widetilde{R''_j}\simeq\widetilde{\pi''(R''_j)}$) realized by a normalized similarity transformation for $i=1,\dots,k$ (resp. $j=1,\dots,n$). So by Corollary 3.2.31 we obtain

$$\bigoplus_{i=1}^{k}\widetilde{R'_i}\oplus\bigoplus_{j=1}^{n}\widetilde{R''_j}\simeq\bigoplus_{i=1}^{k}\widetilde{\pi'(R'_i)}\oplus\bigoplus_{j=1}^{n}\widetilde{\pi''(R''_j)}$$

realized by a normalized similarity transformation. We conclude that

$$R\simeq\bigoplus_{i=1}^{k}\widetilde{\pi'(R'_i)}\oplus\bigoplus_{j=1}^{n}\widetilde{\pi''(R''_j)}$$

realized by a normalized similarity transformation, as we had to show. □

Proposition 3.2.32 leads to the following statement.

Corollary 3.2.33. *Every affine root system R is similar to a direct sum $\bigoplus_{i=1}^{n}R_i$ of irreducible affine root systems R_i.*

We arrive at the main result of this Subsection, namely that it suffices to study a complete set of representatives of the similarity classes of the irreducible affine root systems to understand the similarity classes of all affine root systems.

Theorem 3.2.34. *Let $\{R_i\}_{i\in I}$ be a complete set of representatives of similarity classes of the irreducible affine root systems with index set I, then for each affine root system R there exist $\lambda_1,\dots,\lambda_m\in\mathbb{R}_{\neq 0}$ such that $R\simeq\bigoplus_{j=1}^{m}\lambda_j R_j$.*

Proof. Let R be an affine root system with orthogonal decomposition $R_1\amalg\cdots\amalg R_m$ for $m>1$ (see Proposition 3.2.24). Then by Proposition 3.2.32 we have $R\simeq\bigoplus_{j=1}^{m}\widetilde{R_j}$ realized by a normalized similarity transformation, where $\widetilde{R_j}$ is the affine root system associated to R_j for $j=1,\dots,m$. By definition of the orthogonal decomposition, $\widetilde{R_j}$ is an irreducible affine root system for $j=1,\dots,m$. This implies that for each $1\leq j\leq m$ there exists a unique $i_j\in I$ such that $\widetilde{R_j}\simeq R_{i_j}$. Then by Proposition 3.2.30 there exist $\lambda_1,\dots,\lambda_m\in\mathbb{R}_{\neq 0}$ such that $\bigoplus_{j=1}^{m}\widetilde{R_j}\simeq\bigoplus_{j=1}^{m}\lambda_i R_{i_j}$. □

This result shows that the irreducible affine root systems are the building blocks of all affine root systems. Therefore we will work mostly with irreducible affine root systems in the rest of this chapter, unless stated explicitly. Furthermore, we will classify the similarity classes of the reduced irreducible affine root systems in the last Section of this Chapter.

Remark 3.2.3. In the work of Macdonald (see [10]) the definition of an affine root system coincides with ours, except that condition (5) of Definition 3.2.1 is omitted. It turns out that if an affine root system in Macdonald's sense is irreducible, then condition (5) is satisfied and we have an irreducible affine root system in our sense. However, a reducible affine root system in Macdonald's sense that does not satisfy condition (5) decomposes as a mixture of both irreducible affine root systems and irreducible finite root systems. It has been suggested by Mark Reeder to add condition (5) to the definition of an affine root system in order to avoid such decompositions. Stokman (see [14]) proposes an equivalent condition: The additive subgroup $V_R := \{v \in V : t_v \in W(R)\} \subset V$ spans V. One proves this equivalence as follows.

Assume $R \subset \widehat{E}$ satisfies Definition 3.2.1 including (5) and consider distinct $a, b \in R$ such that $Da = Db =: C$. Then $b(x+v) - a(x+v) = b(x) + (C, v)_V - a(x) - (C, v)_V = b(x) - a(x)$ for all $x \in E$ and all v in the space of translations V of E. Thus $b(x) - a(x) = \mu$ for all $x \in E$ and some fixed $\mu \in \mathbb{R}_{\neq 0}$. Then for any $x \in E$ (3.1.13) and (3.1.14) reveal that

$$\begin{aligned} w_a \circ w_b &= t_{-a(x)C^\vee} \circ w_{x+C} \circ t_{-b(x)C^\vee} \circ w_{x+C} \\ &= t_{-a(x)C^\vee} \circ t_{b(x)C^\vee} \\ &= t_{(b(x)-a(x))C^\vee} \\ &= t_{\mu C^\vee}, \end{aligned} \tag{3.2.12}$$

hence we observe that $w_a \circ w_b = t_{\mu C^\vee} \in W(R)$ for some $\mu \in \mathbb{R}_{\neq 0}$. In other words $\mu C^\vee \in V_R$. Since the gradients $\{Da : a \in R\}$ of R span V by (1) of Definition 3.2.1 together with Proposition 3.1.12 and $\mu C^\vee$ is a nonvanishing scalar multiple of C, we find one implication of the equivalence.

For the other implication assume that R is an affine root system in Macdonald's sense and that V_R spans V. Choose $C \in \{Da : a \in R\}$, then $C \neq 0$ and there exists $a \in R$ such that $C = Da$. Because $\dim(V) > 0$ and V_R spans V we can choose $v \in V_R$ such that $(v, C)_V \neq 0$. Then $t_v \in W(R)$, $t_v(a) \in R$ and $t_v(a) = a - (C, v)_V\, c \neq a$. Then we obtain $Dt_v(a) = Da \circ Dt_{-v} = Da$ from (i) and (ii) of Example 3.1.6 and Proposition 3.1.23 which shows that (5) of Definition 3.2.1 holds. Thus R is an affine root system.

3.3 The geometry of affine root systems

In this section let R be an affine root system on an affine Euclidean space E of rank l, and let V denote the space of translations of E. In the following we are interested in the collection $\mathscr{H} := \{H_a : a \in R\}$ of affine hyperplanes in E induced by R and its configuration in E (see (3.1.8)). The affine hyperplanes $\mathscr{H}$ define a partition of the affine Euclidean space E and give rise to special subsets of E called alcoves. Since these alcoves turn out to be closely related to the structure of affine root systems, we will give a precise geometrical description of them. Along the way we will obtain an important result on the generators of $W(R)$ and define the gradient root system $D(R)$ of R.

Because of the geometric flavor of this Section we will sometimes clarify a proof with a sketch of the situation. Furthermore, many statements in this Section are very intuitive when they are considered with $\mathscr{H}$ explicitly depicted (see for example Figure 3.1). It must be noted that most proofs of in Section are adapted versions from statements that can be found in [2].

3.3.1 Affine hyperplanes and alcoves in affine Euclidean space

Recall that we consider an affine Euclidean space E as being equipped with the Euclidean topology which makes E a locally compact space. Now a collection of subsets $\mathscr{P}$ of a locally compact space X is called *locally finite* if all compact subsets K of X meet with only finitely many distinct elements of $\mathscr{P}$ ([11], Chapter 3, §26, §29 and §39).

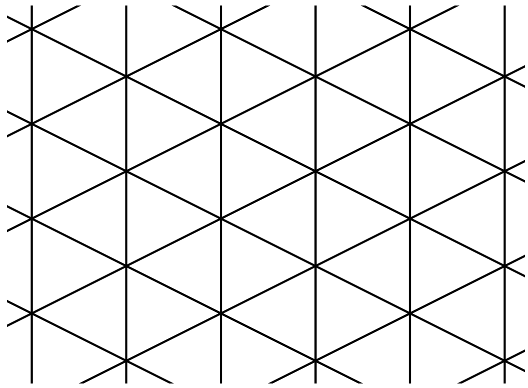


Figure 3.1: Example of a hyperplane configuration $\mathscr{H}$ (of a reduced irreducible affine root system of type A_2^u).

Proposition 3.3.1. *$\mathscr{H}$ is a locally finite collection of subsets of E.*

Proof. Let $K \subset E$ be a compact subset and assume that $H \in \mathscr{H}$ meets K, then there exists $a \in R$ such that $H = H_a$. Now the orthogonal reflection $w_a \in W(R)$ in the affine hyperplane H_a is a homeomorphism of E by Corollary 3.1.8, hence $w_a(K) \subset E$ is also compact. Further w_a fixes H_a in E, so $K \cap H_a = w_a(K \cap H_a) = w_a(K) \cap H_a \neq \emptyset$ which implies $w_a(K) \cap K \neq \emptyset$. This only holds for finitely many $a \in R$ by Remark 3.2.1, so only finitely many $H \in \mathscr{H}$ meet K. We conclude that $\mathscr{H}$ is locally finite. □

Until further notice we forget the special nature of $\mathscr{H}$, and let $\mathscr{H}$ denote any locally finite collection of affine hyperplanes in the affine Euclidean space E. Further, we develop some notation to study such more general collections. Recall that a nonempty subset U of a topological space X is a connected component of X if and only if for every connected subset $U' \subset X$, we have $U' \cap U = \emptyset$ or $U' \subset U$ ([11], Chapter 3, §25). Let H be an affine hyperplane in E, and A be a nonempty connected subset of E not meeting H. Let $D_H(A)$ denote the connected component of $E \setminus H$ containing A, or in other words, $D_H(A)$ is the unique open half-space in E bounded by H containing A. So $D_H(A) = D_H(B)$ means that the connected subsets A and B of E are contained in the same connected component of $E \setminus H$, or that A and B *lie on the same side* of the affine hyperplane H. For a collection of affine hyperplanes $\mathscr{N}$ of E which do not meet A, write $D_{\mathscr{N}}(A) := \bigcap_{H \in \mathscr{N}} D_H(A)$. If A is a singleton, say $A = \{x\}$, we put $D_H(x) := D_H(A)$. Finally write $E_{reg} := E \setminus \bigcup_{H \in \mathscr{H}} H$ for the *regular points* of E relative to $\mathscr{H}$ which are the points of E that do not lie on an affine hyperplane of $\mathscr{H}$. In the remaining of this Subsection we will develop some theory about $\mathscr{H}$ in this general setting. In the next Subsection we will apply this theory to (irreducible) affine root systems.

Lemma 3.3.2. *E_{reg} is open in E. Moreover, the connected components of E_{reg} are open in E.*

Proof. We will proof the latter, then the former is implied. Let C be a connected component of E_{reg}, and let $x \in C$ be a point. Since E is locally compact, there exists a compact subset K of E such that x lies in the interior K° of K. The collection $\mathscr{H}$ is locally finite by Proposition 3.3.1, so K only meets finitely many affine hyperplanes of $\mathscr{H}$. This means that there exists a finite subset $\mathscr{N} \subset \mathscr{H}$ consisting of affine hyperplanes that meet K. Then $K^\circ \setminus H$ is an open neighborhood of x in E for all $H \in \mathscr{N}$, so $A := \bigcap_{H \in \mathscr{N}} K^\circ \setminus H \subset E_{reg}$ is also an open neighborhood of x. Because E is equipped with the Euclidean topology E is locally connected. So there exists a connected open neighborhood B of x in E such that $B \subset A$. But x lies in the connected component C, so we must have $B \subset C$. This shows that C is open in E. □

Lemma 3.3.3. *For every $x \in E$ there exist at most finitely many affine hyperplanes passing through x that belong to $\mathscr{H}$. Also, there exists a connected open neighborhood of x that does not meet any affine hyperplane belonging to $\mathscr{H}$, except for those passing through x.*

Proof. The first statement is clear since $\mathscr{H}$ is locally finite. So consider the collection $\mathscr{N}$ of affine hyperplanes contained in $\mathscr{H}$ that do not pass through x. Since $\mathscr{H}$ is locally finite, so is $\mathscr{N}$. Hence Lemma 3.3.2 applies to E_{reg} relative to $\mathscr{N}$, in which x is contained. Thus x lies in one of its connected components, which is open in E and clearly only meets affine hyperplanes of $\mathscr{H}$ passing through x. □

Lemma 3.3.4. *Let $L \in \mathscr{H}$ be an affine hyperplane, then there exists a point $x \in L$ that does not belong to any other affine hyperplane $H \neq L$ of $\mathscr{H}$.*

Proof. Since $\mathscr{H}$ is a locally finite collection of affine hyperplanes in E, the collection $\mathscr{L} := \{L \cap H : H \in \mathscr{H} \setminus \{L\}\}$ of affine hyperplanes in the affine Euclidean space L (see Prop. 3.1.4) is locally finite. If $y \in L$ does not belong to any affine hyperplane in $\mathscr{H} \setminus \{L\}$ we are done, so suppose the opposite. Since $\mathscr{L}$ is locally finite in L, Lemma 3.3.3 yields that there are only finitely many affine hyperplanes of $\mathscr{L}$ passing through y, say $H_1, \ldots, H_m$, and y has a connected open neighborhood $U \subset L$ that does not meet any affine hyperplane of $\mathscr{L}$ other than $H_1, \ldots, H_m$. The affine Euclidean space L is in bijective correspondence with it space of translations, hence finitely many affine hyperplanes do not exhaust the space L. This means that we can consider an affine line S in L through y that is not contained in (but does intersect) the affine hyperplanes $H_1, \ldots, H_m$ (for L of dimension 1 we let $S = L$). Then $S \cap U \setminus \{y\} \subset L$ is nonempty and contains a point x does not meet any affine hyperplane of $\mathscr{L}$ other than $H_1, \ldots, H_m$ since $x \in U \setminus \{y\}$. But also $x \notin H_i$ for $1 \leq i \leq m$ because $x \in S \setminus \{y\}$. This shows that there exists a point $x \in L$ that does not belong to any other affine hyperplane $H \neq L$ of $\mathscr{H}$. □

Introduce a relation $\sim$ on E by saying that $x \sim y$ if for any affine hyperplane $H \in \mathscr{H}$ either $x \in H$ and $y \in H$, or $D_H(x) = D_H(y)$. It is a straightforward check that $\sim$ is an equivalence relation on E.

Definition 3.3.5. An equivalence class F under the equivalence relation $\sim$ on E is called a *facet* of E relative to $\mathscr{H}$.

Let x be a point in a facet F of E relative to $\mathscr{H}$, then an affine hyperplane $H \in \mathscr{H}$ contains F if and only if $x \in H$. By Lemma 3.3.3 there are only finitely many of such affine hyperplanes, hence their intersection defines an affine subspace L of E by Proposition 3.1.4 called the *support* of F. Let $\mathscr{N}$ be the collection of affine hyperplanes in $\mathscr{H}$ that do not contain the facet F, then by definition of $\sim$ we observe that

$$F = L \cap D_{\mathscr{N}}(x). \tag{3.3.1}$$

In other words, $F \subset L$ and $F \subset D_H(x)$ for all $H \in \mathscr{N}$. So if we write $\overline{F}$, $\overline{L}$ and $\overline{D_H(x)}$ for the closures in E of F, L and $D_H(x)$ respectively, then $\overline{F} \subset \overline{L}$ and $\overline{F} \subset \overline{D_H(x)}$ for every $H \in \mathscr{H}$. Now $\overline{L} = L$, because L is an affine subspace which must be closed in E. This implies that $\overline{F} \subset L \cap (\bigcap_{H \in \mathscr{N}} \overline{D_H(x)})$. On the other hand consider $y \in L \cap (\bigcap_{H \in \mathscr{N}} \overline{D_H(x)})$, then we have $y \in L$. Since $\overline{D_H(x)} = D_H(x) \cup H$ we also observe that for each $H \in \mathscr{N}$ either $y \in D_H(x)$ or $y \in H$. Since $x \in F$ we have by (3.3.1) that $x \in L$, and for each $H \in \mathscr{N}$ we have $x \in D_H(x)$ and $x \notin H$. So the open line segment with extremities x

and y is contained in L and in $D_H(x)$ for each $H \in \mathcal{N}$, hence in F by (3.3.1). Thus certainly y is contained in $\overline{F}$, and we obtain

$$\overline{F} = L \cap \bigcap_{H \in \mathcal{N}} \overline{D_H(x)}. \tag{3.3.2}$$

Definition 3.3.6. An *alcove* of E relative to $\mathcal{H}$ is a facet of E relative to $\mathcal{H}$ that is not contained in any affine hyperplane belonging to $\mathcal{H}$.

Hence the alcoves in Figure 3.1 are the white triangles. It is not a coincidence that they are connected.

Proposition 3.3.7. *The alcoves of E relative to $\mathcal{H}$ are the connected components of E_{reg}.*

Proof. First we show that an alcove $C \subset E_{reg}$ is connected. Let $x, y \in C$, then $D_H(x) = D_H(y) = D_H(C)$ for all $H \in \mathcal{H}$ by Definition 3.3.5 and 3.3.6. Consider the line segment S between x and y, then by linearity $D_H(z) = D_H(x) = D_H(C)$ for all $z \in S$ and $H \in \mathcal{H}$. This implies $S \subset C$, so C is arc-connected and in particular connected.

Now, assume that there exists a connected subset $U \subset E_{reg}$ such that $C \subseteq U$. Then for $x \in C$ and $y \in U$ we can not have that $D_H(x) \neq D_H(y)$ for some $H \in \mathcal{H}$, else $U = (D_H(x) \cap U) \cup (D_H(y) \cap U)$ is a partition in disjoint open sets of U in E_{reg}. Thus $D_H(x) = D_H(y)$ for all $x \in C$, $y \in U$ and $H \in \mathcal{H}$. But this means that y is contained in the same facet as x, which is C. We conclude that $C = U$ and that C is a connected component of E_{reg}.

On the other hand, let U be a connected component of E_{reg}. For $x, y \in U$ and $H \in \mathcal{H}$ we obtain $D_H(x) = D_H(y)$ by the argument of the last paragraph. By Definition 3.3.6 and the definition of the relation $\sim$, we observe that U is contained in an alcove C. But since all alcoves are connected components of E_{reg}, U and C coincide. □

Remark 3.3.1. Lemma 3.3.2 combined with the last Proposition reveals that the alcoves of E relative to $\mathcal{H}$ are open subsets of E. This will also be proven in Theorem 3.3.23 when we describe the alcoves explicitly.

Next, we generalize the conclusion of Remark 3.3.1 from alcoves to arbitrary facets.

Lemma 3.3.8. *Every facet F of E is an open subset in its support L relative to $\mathcal{H}$.*

Proof. From Remark 3.3.1 we observe that alcoves of E relative to $\mathcal{H}$ are open in E. Now consider an arbitrary facet $F \subset E$ that is not an alcove relative to $\mathcal{H}$. Then by (3.3.1) we have $F = L \cap D_{\mathcal{N}}(x)$ for some $x \in F$, where $L = H_1 \cap \cdots \cap H_m$ and $\mathcal{N} = \mathcal{H} \setminus \{H_1, \ldots, H_m\}$ for some $H_1, \ldots, H_m \in \mathcal{H}$. Notice that $\mathcal{N} \subset \mathcal{H}$ is locally finite, so the collection of affine hyperplanes $\{H \cap L : H \in \mathcal{N}\}$ is a locally finite collection of affine hyperplanes in the affine subspace $L \subset E$. This implies that in L we can write $F = \bigcap_{H \in \mathcal{N}} D_{H \cap L}(x)$ for some $x \in F$. We observe from (3.3.1) that F is an alcove of L relative to $\{H \cap L : H \in \mathcal{N}\}$, thus Remark 3.3.1 ensures us that F is an open subset in L. □

Lemma 3.3.9. *Let F and F' be facets relative to $\mathcal{H}$ such that $F' \cap \overline{F} \neq \emptyset$, then $F' \subset \overline{F}$.*

Proof. If F' meets $F \subset \overline{F}$ then Definition 3.3.5 implies that $F = F'$. So assume that F' does not meet F but does meet $\overline{F}$, and let $y \in F' \cap (\overline{F} \setminus F)$. Further let $x \in F$, then as in (3.3.1) we have $F = L \cap D_{\mathcal{N}}(x)$ where L is the support of F and $\mathcal{N} \subset \mathcal{H}$ the affine hyperplanes that do not contain F. Write $\mathcal{N}'$ for subset of affine hyperplanes of $\mathcal{N}$ that meet y and put $\mathcal{N}'' := \mathcal{N} \setminus \mathcal{N}'$. Then for any $H \in \mathcal{N}''$ we have $y \notin H$ and $y \in \overline{D_H(x)}$ by (3.3.2), so $y \in D_H(x)$ and $D_H(x) = D_H(y)$. By (3.3.2) $\overline{F} \subset L$, so in particular $y \in L$. Hence $y \in L \cap \bigcap_{H \in \mathcal{N}'} H \cap \bigcap_{H \in \mathcal{N}''} D_H(x)$, so since y is contained in the facet F' we obtain from (3.3.1)

$$F' = L \cap \bigcap_{H \in \mathcal{N}'} H \cap \bigcap_{H \in \mathcal{N}''} D_H(x).$$

On the other hand, rewriting (3.3.2) for $\overline{F}$ we obtain

$$\overline{F} = L \cap \bigcap_{H \in \mathcal{N}'} \overline{D_H(x)} \cap \bigcap_{H \in \mathcal{N}''} \overline{D_H(x)}.$$

This leads indeed to $F' \subset \overline{F}$. □

Lemma 3.3.10. *Every $x \in E$ is in the closure of at least one alcove of E.*

Proof. If E is an affine Euclidean space of dimension 0, then E is a singleton and does not have affine hyperplanes so the Lemma is clear. Otherwise let $x \in E$, then by Lemma 3.3.3 there are only finitely many affine hyperplanes of $\mathcal{H}$ passing through x, say $H_1, \dots, H_m$, and x has an open neighborhood $U \subset E$ that does not meet any affine hyperplane of $\mathcal{H}$ other than $H_1, \dots, H_m$. Consider the affine line S through x that is not contained in any of the affine hyperplanes $H_1, \dots, H_m$ (for L of dimension 1 we let $S = L$), and let $y \neq x$ be a point on S close enough to x such that the open line segment S' with extremities x and y lies in U. Since S' also does not meet any affine hyperplane H_i for $i = 1, \dots, m$ it is a connected subset of E_{reg}, and hence contained in an alcove C by Proposition 3.3.7. But x lies in the closure of S', so clearly x is contained in the closure of C. □

Definition 3.3.11. A *wall* of an alcove C is an affine hyperplane $L \in \mathcal{H}$ such that L is the support of a facet that is contained in the closure $\overline{C}$.

As one can see, each alcove in Figure 3.1 has three walls. Next, we obtain a criterion for $H \in \mathcal{H}$ to be a wall.

Lemma 3.3.12. *Let C be an alcove of E relative to $\mathcal{H}$ and let $L \in \mathcal{H}$ be an affine hyperplane in E. Assume that there exists a point $x \in L \cap \overline{C}$ that does not belong to any of the affine hyperplanes in $\mathcal{N} := \mathcal{H} \setminus \{L\}$, then $L \cap D_{\mathcal{N}}(C)$ is the unique facet with support L contained in $\overline{C}$, hence L is a wall of C.*

Proof. By Lemma 3.3.3 there exists a connected open neighborhood U of x in E that does not meet any of the affine hyperplanes of $\mathcal{N}$. But since $x \in \overline{C}$, the open set U does meet C. Then for all $H \in \mathcal{N}$, we have

$$D_H(x) = D_H(U) = D_H(U \cap C) = D_H(C) = D_H(y) \tag{3.3.3}$$

for some $y \in C$. By the relation "$\sim$", the facet F containing x has support L. Now by (3.3.1) and (3.3.3), $F = L \cap D_{\mathcal{N}}(x) = L \cap D_{\mathcal{N}}(C)$. Further, (3.3.2) and (3.3.3) tell us,

$$\overline{C} = \bigcap_{H \in \mathcal{H}} \overline{D_H(y)} = \overline{D_L(y)} \cap \bigcap_{H \in \mathcal{N}} \overline{D_H(y)} = \overline{D_L(y)} \cap \bigcap_{H \in \mathcal{N}} \overline{D_H(C)}. \tag{3.3.4}$$

So since $L \subset \overline{D_L(y)}$ and $D_H(C) \subset \overline{D_H(C)}$ for all $H \in \mathcal{N}$, we observe that $F \subset \overline{C}$. Also by (3.3.4), any facet contained in $\overline{C}$ with support L must lie on the same side of all affine hyperplanes in $\mathcal{N}$ as C. By definition, different facets with support L can not be on the same side for all affine hyperplanes in $\mathcal{N}$, thus only one such facet exists, namely $F = L \cap D_{\mathcal{N}}(C)$. □

Proposition 3.3.13. *Every $L \in \mathcal{H}$ is the wall of at least one alcove of E relative to $\mathcal{H}$.*

Proof. Let $L \in \mathscr{H}$ be an affine hyperplane, then by Lemma 3.3.4 there exists a point $x \in L$ that does not belong to any other affine hyperplane $H \neq L$ of $\mathscr{H}$. By Lemma 3.3.10 there exists an alcove C such that x is contained in its closure. Finally, Lemma 3.3.12 implies that L is a wall of C. □

The next Lemma shows that an alcove C and the facets that are contained in its closure relative to $\mathscr{H}$ only depend on the walls of the alcove C.

Lemma 3.3.14. *Let C be an alcove of E relative to $\mathscr{H}$ and let $\mathscr{M} \subset \mathscr{H}$ be the set of walls of C.*
(i) *$C = D_{\mathscr{M}}(C)$;*
(ii) *$F \subset \overline{C}$ is a facet of E relative to $\mathscr{H}$ if and only if then F is also a facet of E relative to $\mathscr{M}$.*

Proof. (i) Fix $x \in C$, then $D_H(C) = D_H(x)$ for all $H \in \mathscr{H}$ since C is an alcove relative to $\mathscr{H}$. Thus to prove $C = D_{\mathscr{M}}(C)$ with $\mathscr{M} \subset \mathscr{H}$ the set of walls of C relative to $\mathscr{H}$, it is enough to show that $C = D_{\mathscr{M}}(x)$. Now because $\mathscr{M} \subset \mathscr{H}$ and $C = D_{\mathscr{H}}(x)$ by (3.3.1), we observe that $C \subset D_{\mathscr{M}}(x)$. On the other hand let $y \in D_{\mathscr{M}}(x)$, then by linearity the line segment S with extremities x and y is also contained in $D_{\mathscr{M}}(x)$. This implies that S does not meet any wall of C. Writing $\mathscr{N}$ for the subcollection of affine hyperplanes of $\mathscr{H}$ meeting S we get $\mathscr{M} \cap \mathscr{N} = \emptyset$. Furthermore S is compact in E, so the locally finiteness of $\mathscr{H}$ leads to $\mathscr{N}$ being finite. Now $x \in C \cap S = D_{\mathscr{H}}(C) \cap S$, S is connected and S does not meet any affine hyperplane of $\mathscr{H} \setminus \mathscr{N}$, thus $S \subset D_{\mathscr{H}\setminus\mathscr{N}}(C)$. So if we can show that $D_{\mathscr{H}\setminus\mathscr{N}}(C) = C$, then $y \in C$ which implies $D_{\mathscr{M}}(x) \subset C$ so we are done. In fact we will show in the following that for each finite subset $\mathscr{N} \subset \mathscr{H}$ such that $\mathscr{N} \cap \mathscr{M} = \emptyset$ it holds that $C = D_{\mathscr{H}\setminus\mathscr{N}}(C)$. To do that we proceed with induction to the cardinal $n \in \mathbb{Z}_{\geq 0}$ of $\mathscr{N}$.

If $n = 0$ we have $\mathscr{H} \setminus \mathscr{N} = \mathscr{H}$ for which we already know that $C = D_{\mathscr{H}}(C)$. So we assume that we are done up to $n > 0$ including n. Put $\mathscr{L} := \mathscr{H} \setminus \mathscr{N}$, then $\mathscr{M} \subset \mathscr{L}$ and by the induction hypothesis we have $C = D_{\mathscr{L}}(C)$. Notice that $\mathscr{L}$ is a locally finite collection of affine hyperplanes in E. Since C is an alcove relative to $\mathscr{H}$ we have $D_H(C) = D_H(x)$ for all $H \in \mathscr{H}$. In particular $D_H(C) = D_H(x)$ for all $H \in \mathscr{L}$, so $C = D_{\mathscr{L}}(C) = D_{\mathscr{L}}(x)$ which is an alcove relative to $\mathscr{L}$ by (3.3.1).

Next we want to show that $\mathscr{M}$ is also the set of walls of C relative to $\mathscr{L}$. By Lemma 3.3.12, $\mathscr{M}$ is contained in the set of walls of C relative to $\mathscr{L}$. On the other hand, any wall H of C relative to $\mathscr{L}$ must contain a point $x \in \overline{C}$ that does not meet any affine hyperplane belonging to $\mathscr{L} \setminus \{H\}$ by Definition 3.3.11. Hence by Lemma 3.3.3 there exists an open neighborhood $U_x \subset E$ of x such that U_x does not meet any affine hyperplane contained in $\mathscr{L}$ apart from H. Moreover using Lemma 3.3.3 we can choose U_x small enough such that U_x only meets hyperplanes of $\mathscr{H}$ passing through x, and by choosing U_x bounded we make sure that the closure $\overline{U_x}$ is compact. Then the locally finiteness of $\mathscr{H}$ guarantees that U_x only meets finitely many affine hyperplanes contained in $\mathscr{H}$. If x is not contained in any other affine hyperplane of $\mathscr{H}$ than H, then H is also a wall of C relative to $\mathscr{H}$ by definition of x. Otherwise, suppose that there exists an affine hyperplane $H' \in \mathscr{H}$ different from H that contains x. Then H' induces a partition of H as $H = H_1 \amalg H_2 \amalg (H \cap H')$ such that $D_{H'}(H_1) \neq D_{H'}(H_2)$ (see Figure 3.2). Since the only hyperplane of $\mathscr{L}$ that meets U_x is H, we observe that all points of $U_x \cap H$ are in the same facet as x relative to $\mathscr{L}$. This in turn implies that $U_x \cap H \subset \overline{C}$ because H is a wall of C relative to $\mathscr{L}$. Now choose $y \in D_{H'}(H_1)$ and $z \in D_{H'}(H_2)$ such that $y, z \in U_x \cap H$. Then there exist open neighborhoods U_y of y and U_z of z respectively such that $U_y \subset D_{H'}(H_1)$ and $U_z \subset D_{H'}(H_2)$ (see Figure 3.2). But $U_y \cap C \neq \emptyset$ and $U_z \cap C \neq \emptyset$ since $y, z \in \overline{C}$. This implies that $C \cap D_{H'}(H_1) \neq \emptyset$ and $C \cap D_{H'}(H_2) \neq \emptyset$ which contradicts C being an alcove relative to $\mathscr{H}$. This leads to H being the only affine hyperplane of $\mathscr{H}$ passing through x, so H is also a wall of C relative to $\mathscr{H}$. We conclude that $\mathscr{M}$ is also the set of walls of C relative to $\mathscr{L}$.

Let $L \in \mathscr{L}$ such that L is not a wall of C relative to $\mathscr{L}$, then $L \notin \mathscr{M}$ as we have just seen. Because $C = D_{\mathscr{L}}(C)$ by the induction hypothesis we want to show that $D_{\mathscr{L}\setminus\{L\}}(C) = D_{\mathscr{L}}(C)$ in the following

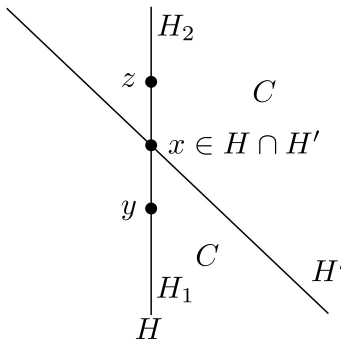


Figure 3.2: Assuming that $x \in \overline{C}$ is contained in a wall H of the alcove C relative to $\mathscr{L} \subsetneq \mathscr{M}$ but not relative to $\mathscr{M}$ leads to a contradiction. Namely, these assumptions imply that x must also lie on $H' \in \mathscr{M}$, which leads to C not being alcove relative to $\mathscr{M}$.

to complete the induction. We immediately observe that $D_{\mathscr{L}}(C) \subset D_{\mathscr{L}\setminus\{L\}}(C)$. On the other hand assume that L meets $D_{\mathscr{L}\setminus\{L\}}(C)$ and choose z in their intersection, then $z \notin H$ for any $H \in \mathscr{L} \setminus \{L\}$. Hence $z \notin L \cap \overline{C}$, because otherwise L would be a wall of C relative to $\mathscr{L}$ by Lemma 3.3.12. Further by Lemma 3.3.3 there exists a connected open neighborhood U of z not meeting any affine hyperplane $H \in \mathscr{L} \setminus \{L\}$, so $U \subset D_{\mathscr{L}\setminus\{L\}}(C)$. Notice that $\emptyset \neq U \cap D_L(C)$, so since $C = D_{\mathscr{L}}(C)$ and $U \subset D_{\mathscr{L}\setminus\{L\}}(C)$ we observe that $\emptyset \neq U \cap D_L(C) = U \cap C \subset C$. But any open ball with center z meets $U \cap D_L(C)$. In other words $z \in \overline{C}$ which contradicts the fact that $z \notin L \cap \overline{C}$ since $z \in L$. Hence our assumption that L meets $D_{\mathscr{L}\setminus\{L\}}(C)$ was wrong. Now every open halfspace $D_H(C)$ for $H \in \mathscr{L}$ is convex, hence the intersection of convex sets $D_{\mathscr{L}\setminus\{L\}}(C)$ is also convex. It follows directly from arc-connectedness of a convex set that $D_{\mathscr{L}\setminus\{L\}}(C)$ is connected in E. Then $C = D_{\mathscr{L}}(C) \subset D_{\mathscr{L}\setminus\{L\}}(C)$ together with $L \cap D_{\mathscr{L}\setminus\{L\}}(C) = \emptyset$ implies that $D_{\mathscr{L}\setminus\{L\}}(C) \subset D_L(C)$, so we obtain that $D_{\mathscr{L}\setminus\{L\}}(C) \subset D_L(C)$. We conclude that $D_{\mathscr{L}\setminus\{L\}}(C) = D_{\mathscr{L}}(C)$, which coincides with C by the induction hypothesis. Hence for each finite subset $\mathscr{N} \subset \mathscr{H}$ such that $\mathscr{N} \cap \mathscr{M} = \emptyset$ it holds that $C = D_{\mathscr{H}\setminus\mathscr{N}}(C)$. As the first paragraph of this proof indicates, we obtain now $C = D_{\mathscr{M}}(C)$.

(ii) Let F be a subset of the closure $\overline{C}$ of C and suppose that F is a facet of E relative to $\mathscr{M}$ with support L. Then F is open in L by Lemma 3.3.8. Now let H' be an affine hyperplane that meets F but does not contain F. Since $F \subset L$ is open this implies that $F \not\subset \overline{D_{H'}(x)}$ for all $x \in E \setminus H'$. Since $F \subset \overline{C}$ we observe that also $\overline{C} \not\subset \overline{D_{H'}(x)}$ for all $x \in E \setminus H'$. But by (3.3.2) we have $\overline{C} = \bigcap_{H\in\mathscr{H}} \overline{D_H(y)}$ for some $y \in C$, so the only possibility is that $H' \notin \mathscr{H}$. This shows that all $H \in \mathscr{H}$ that meet F also contain F, hence F is also a facet of E relative to $\mathscr{H}$.

On the other hand, suppose $F \subset \overline{C}$ is a facet of E relative to $\mathscr{H}$. Since the collection of walls $\mathscr{M}$ of C is part of the locally finite collection of affine hyperplanes $\mathscr{H}$, $\mathscr{M}$ is also a locally finite collection of affine hyperplanes in E. As we have seen in the previous part of this Lemma it holds that $C = D_{\mathscr{M}}(C) = \bigcap_{H\in\mathscr{M}} D_H(x)$ for some $x \in C$. Hence (3.3.1) shows that C is also an alcove relative to $\mathscr{M}$. So let us assume that $F \neq C$. By (3.3.2) we obtain $\overline{C} = \bigcap_{H\in\mathscr{M}} \overline{D_H(x)}$ for some $x \in C$, so $F \subset \overline{C} \setminus C$ is contained in $\bigcup_{H\in\mathscr{M}} H$. Hence the support of F relative to $\mathscr{H}$ is the intersection of finitely many walls of C, say $H_1, \dots, H_m \in \mathscr{M}$. By (3.3.1) we obtain $F = H_1 \cap \cdots \cap H_m \bigcap_{H\in\mathscr{H}\setminus\{H_1,\dots,H_m\}} D_H(x)$ for some $x \in F$. Then clearly $F \subset F' := H_1 \cap \cdots \cap H_m \bigcap_{H\in\mathscr{M}\setminus\{H_1,\dots,H_m\}} D_H(x)$ for some $x \in F$ where we note that F' is a facet relative to $\mathscr{M}$ by (3.3.1). Further it follows that $F' \subset \bigcap_{H\in\mathscr{M}} \overline{D_H(y)} = \overline{C}$ for some $y \in C$. Now by Lemma 3.3.8 both F and F' are open subsets of their support $H_1 \cap \cdots \cap H_m$. Suppose that $F \subsetneq F'$ then there exists $z \in F' \setminus F$ and $H' \in \mathscr{H}$ such that $D_{H'}(z) \neq D_{H'}(F)$. Hence we observe that similarly to the previous paragraph that $F' \not\subset \overline{D_{H'}(x)}$ for all $x \in E \setminus H'$, hence $\overline{C} \not\subset \overline{D_{H'}(x)}$

for all $x \in E \setminus H'$. This gives a contradiction with (3.3.2) which yields $\overline{C} = \bigcap_{H \in \mathscr{H}} \overline{D_H(y)}$ for some $y \in C$. Thus we conclude that $F = F'$, so F is also a facet relative to $\mathscr{M}$. □

3.3.2 Alcoves of irreducible affine root systems

We return to studying the the collection of affine hyperplanes $\mathscr{H} = \{H_a : a \in R\}$ generated by an affine root system R on an affine Euclidean space E with space of translations V. Our main goal is to explicitly describe the alcoves relative to $\mathscr{H}$. Before we can do that we need some results on the affine Weyl group $W(R)$.

In this Subsection we will consider $W(R)$ as subgroup of $\mathrm{GL}(E)$ since we will work mostly in E (see Proposition 3.1.22). Recall that $H_a = \{y \in E : a(y) = 0\}$ for $a \in R$, so for $b \in R$ the orthogonal reflection $w_b \in W(R)$ acts on H_a by $w_b(H_a) = H_{w_b(a)}$ by (i) of Example 3.2.12 and Corollary 3.2.14. Hence $W(R)$ acts on the collection of affine hyperplanes $\mathscr{H}$. Now every $w \in W(R)$ is an affine linear automorphism of E, so also a homeomorphism of E by Corollary 3.1.8. In particular w permutes the connected components of E_{reg}, which are the alcoves of E relative to $\mathscr{H}$ by Proposition 3.3.7. Clearly this is a group action of $W(R)$ on the collection of alcoves of E.

Lemma 3.3.15. *Let C be an alcove relative to $\mathscr{H}$, let H be a wall of C and let $w \in W(R)$, then $w(H)$ is a wall of $w(C)$.*

Proof. Because H is a wall of C, H is the support of a facet F that is contained in $\overline{C}$. This implies that there exists $x \in F$ such that x does not meet any affine hyperplane of $\mathscr{H}$ other than H. Since the homeomorphism $w \in W(R)$ of E permutes $\mathscr{H}$ and the collection of alcoves relative to $\mathscr{H}$ we observe that $w(H) \in \mathscr{H}$ and that $w(C)$ is an alcove relative to $\mathscr{H}$. Furthermore, $w(x)$ does not meet any affine hyperplane of $\mathscr{H}$ other than $w(H)$. Let U be an open neighborhood of $w(x)$, then $w^{-1}(U)$ is an open neighborhood of x. Since $x \in \overline{C}$ we have $w^{-1}(U) \cap C \neq \emptyset$. This implies that $U \cap w(C) \neq \emptyset$, hence $w(x) \in \overline{w(C)}$. By Lemma 3.3.12 we conclude that $w(H)$ is a wall of the alcove $w(C)$. □

In the following Theorem we will see that the action of $W(R)$ on the collection of alcoves is transitive, and that the affine Weyl group is generated by the reflections in the walls of any fixed alcove.

Theorem 3.3.16. *Fix an alcove C of E relative to $\mathscr{H}$.*

(i) *For any $x \in E$ there exists $w \in W(R)$ such that $w(x) \in \overline{C}$;*

(ii) *$W(R)$ acts transitively on the collection of all alcoves;*

(iii) *$W(R)$ is generated by the orthogonal reflections w_a for $a \in R$ such that H_a is a wall of C.*

Proof. Consider the subgroup W' of $W(R)$ generated by the orthogonal reflections of E in the walls of C. We will first proof (i) and (ii) for W', and then show that $W' = W(R)$ in complete the proof of the whole Theorem.

(i) Let $x \in E$ and write J for the W'-orbit of x. Let $z \in C$, then there is a closed ball A with center z meeting J. Since A is a compact set in the locally compact space E, Remark 3.2.1 ensures us that A has finite intersection with J. Hence there exists a point y in J with minimum distance to z.

We shall prove that $y \in \overline{C}$. Recall from (i) of Lemma 3.3.14 that $C = D_{\mathscr{M}}(C)$ where $\mathscr{M} \subset \mathscr{H}$ is the set of walls of C, then also $C = D_{\mathscr{M}}(x')$ for all $x' \in C$. We observe from (3.3.1) that C is an alcove relative to $\mathscr{M}$, hence by (3.3.2) we have $\overline{C} = \overline{D_{\mathscr{M}}(x')} = \overline{D_{\mathscr{M}}(C)}$. Therefore we will show that $y \in \overline{D_H(C)}$ for all $H \in \mathscr{M}$.

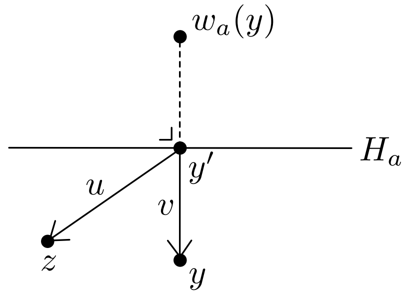


Figure 3.3: The only option for y is to lie on the same side of H_a as z, else $w_a(y)$ lies closer to z than y itself.

Let $H \in \mathscr{M}$, then $H = H_a$ for some $a \in R$. Further $w_a \in W'$ so $w_a(y) \in J$ which implies that $|y - z|_E \leq |w_a(y) - z|_E$ by definition of y. Let $y' \in H$ be the point such that there exists $v \in V$ with $y = y' + v$ and $w_a(y) = y' - v$, then y' lies exactly in between y and $w_a(y)$, and v is orthogonal to H. Also consider the vector $u \in V$ such that $z = y' + u$. Then $|y - z|_E \leq |w_a(y) - z|_E$ implies $(v - u, v - u)_V \leq (v + u, v + u)_V$, or $(v, u)_V \geq 0$ (see Figure 3.3). Because v is orthogonal to H this inequality show that $y \in \overline{D_H(z)} = \overline{D_H(C)}$ for $H \in \mathscr{M}$. Thus $y \in \overline{C}$, and since $y \in J$ there exists $w \in W'$ such that $y = w(x)$.

(ii) Let C' be an alcove of E relative to $\mathscr{H}$, let x be a point in C', and write J for the W'-orbit of x. Let $z \in C$, then similarly to (i) there exists a point y in J with minimum distance to z. Suppose that $y \notin C$, then there exists a wall H_a of C for some $a \in R$ such that $D_{H_a}(z) \neq D_{H_a}(y)$. Clearly, $y, w_a(y)$ and z do not lie on the same line else $|y - z|_E > |w_a(y) - z|_E$ which contradicts the choice of $y \in J$.

Consider the trapezoid with vertices $z, w_a(z), y, w_a(y)$, then z (resp. y) is a vertex of the long (resp. short) side or of the short (resp. long) side of the pair of parallel sides of the trapezoid (see Figure 3.4). Choose γ in the first case as the angle of $w_a(y)$ and γ' in the second case as the angle of z, then $\frac{\pi}{2} \leq \gamma, \gamma' < \pi$. This implies that $-1 < \cos(\gamma), \cos(\gamma') \leq 0$. Although it is also possible that

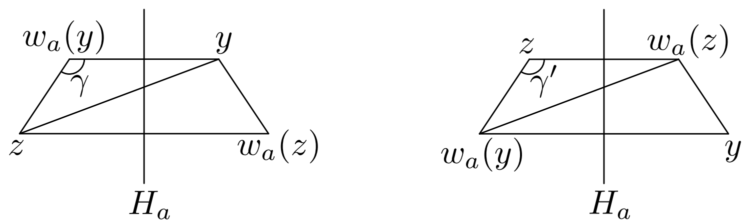


Figure 3.4: The two possible trapezoids with vertices $z, w_a(z), y, w_a(y)$.

both parallel sides have the same length, both cases we are considering here cover that situation since γ and γ' possibly equal $\frac{\pi}{2}$. In the first case the law of cosines yields

$$\begin{aligned}|y - z|_E^2 &= |w_a(y) - y|_E^2 + |w_a(y) - z|_E^2 - 2|w_a(y) - y|_E |w_a(y) - z|_E \cos(\gamma)\\ &> |w_a(y) - z|_E^2,\end{aligned}$$

and in the second case it yields

$$\begin{aligned}|w_a(y) - w_a(z)|_E^2 &= |w_a(y) - z|_E^2 + |w_a(z) - z|_E^2 - 2|w_a(y) - z|_E |w_a(z) - z|_E \cos(\gamma')\\ &> |w_a(y) - z|_E^2.\end{aligned}$$

But w_a preserves the metric on E, so $|w_a(y)-w_a(z)|_E = |y-z|_E$. In both cases we obtain $|w_a(y)-z|_E < |y-z|_E$, and because $w_a \in W'$ we obtain that $w_a(y) \in J$ has strictly smaller distance to z than y. This contradicts the choice of y as a point in J with minimum distance to z. We conclude that $y \in C$, hence there exists $w \in W'$ such that $w(x) = y \in C$. Since w permutes the alcoves of E, we obtain $w(C') = C$.

(iii) Since $W' \subset W(R)$, we only have to prove that $w_a \in W'$ for all $a \in R$ to show that $W' = W(R)$. By Proposition 3.3.13, for every $a \in R$ there exists an alcove C' of E such that H_a is a wall of C'. By the previous part of this Theorem, there exists an orthogonal reflection $w \in W'$ such that $C' = w(C)$. Furthermore by Lemma 3.3.15 we observe that $w^{-1}(H_a)$ is a wall of C, hence there exists an affine root $b \in R$ such that $w^{-1}(H_a) = H_b$. Consequently $H_a = w(H_b) = H_{w(b)}$, so $a = \lambda w(b)$ for some $\lambda \in \mathbb{R}_{\neq 0}$. By (3.2.1) and (i) of Example 3.2.9 we have $w \circ w_b \circ w^{-1} = w_{w(b)}$. But $w_{w(b)}$ and w_a are orthogonal reflections in the same affine hyperplane, so $w_{w(b)} = w_a$. We conclude that $w_a = w \circ w_b \circ w^{-1} \in W'$. □

The set of walls of a fixed alcove was an important ingredient of Theorem 3.3.16. We want to understand this set a little better by looking at vectors normal to these walls. For a fixed alcove C relative to $\mathscr{H}$, let $\mathscr{M}$ be the set of walls of C. For each wall $H \in \mathscr{M}$, let $t_H \in V$ be the unit vector normal to H such that $h + t_H$ lies on the same side of H as C for all $h \in H$. Write $N := \{t_H : H \in \mathscr{M}\}$, then it turns out that the angles between the vectors of N are not acute.

Lemma 3.3.17. *For all $t, t' \in N$ it holds that $(t, t')_V \leq 0$.*

Proof. Assume that $H_a \neq H_b$ are parallel walls of C for some $a, b \in R$, then $t_{H_a} = -t_{H_b}$ or $t_{H_a} = t_{H_b}$ by Definition 3.1.3. In the former case we clearly have $(t_{H_a}, t_{H_b})_V \leq 0$, so assume the latter then $\overline{D_{H_a}(C)} \cap \overline{D_{H_b}(C)} = \overline{D_{H_b}(C)}$ or $\overline{D_{H_a}(C)}$. By symmetry of variables we may assume the outcome is $\overline{D_{H_a}(C)}$, hence $H_b \cap \overline{D_{H_a}(C)} = \emptyset$. By (3.3.2) $\overline{C} \subset \overline{D_{H_a}(C)}$, so $\overline{C} \cap H_b = \emptyset$. This contradicts H_b being a wall of C so we can discard this case.

Now assume $H_a \neq H_b$ are nonparallel walls of C, then $H_a \cap H_b \neq \emptyset$ by Proposition 3.1.4. Choose an origin $x \in H_a \cap H_b$ of E and identify E with V as the vector space E_x, then H_a and H_b are hyperplanes in E_x. Moreover $H_a \cap H_b \subset E_x$ is of codimension 2, so we can let X be the plane perpendicular to $H_a \cap H_b$ passing through x. Further, let $\Gamma := X \cap D_{H_a}(C) \cap D_{H_b}(C)$ be the intersection of two open half planes in X and let $S_a := X \cap H_a \cap \overline{\Gamma}$ (resp. $S_b := X \cap H_b \cap \overline{\Gamma}$) be a closed halfline in X contained in $X \cap H_a$ (resp. $X \cap H_b$) and the closure of Γ. Clearly the boundary $\partial\Gamma$ of Γ is given by $S_a \cup S_b$ (see Figure 3.5).

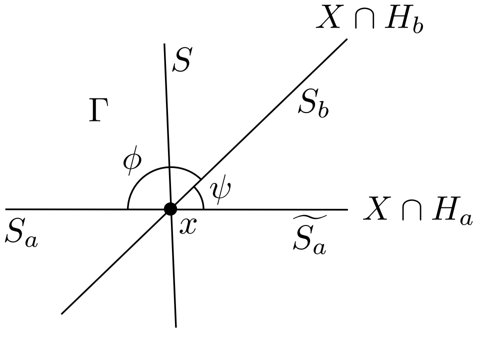


Figure 3.5: The plane X.

Now assume the contrary to what we would like to prove, namely that $(t_{H_a}, t_{H_b})_V > 0$. Then the angle ϕ between S_a and S_b in $\overline{\Gamma}$ lies strictly between $\frac{\pi}{2}$ and π. Consider $\widetilde{S_a} := (X \cap H_a) \setminus (S_a \setminus \{x\})$, the closure of complement of S_a in the line $H_a \cap X$, then the angle $\psi = \pi - \phi$ between $\widetilde{S_a}$ and S_b in $X \cap \overline{D_{H_a}(C)}$ lies strictly between 0 and $\frac{\pi}{2}$. Also note that $\widetilde{S_a} \cap X \cap D_{H_b}(C) = \emptyset$. Consider $L := w_b(H_a) = H_{w_b(a)} \in \mathscr{H}$ which contains $H_a \cap H_b$, then $S := w_b(\widetilde{S_a}) \subset L \cap X$ is a closed halfline in $X \cap \overline{D_{H_b}(C)}$ making an angle $\psi' = \psi$ with S_b between 0 and $\frac{\pi}{2}$. Hence $S \subset \overline{\Gamma}$ and $S \cap S_a = S \cap S_b = \{x\}$. This means that in X we have $D_{L \cap X}(S_a \setminus \{x\}) \neq D_{L \cap X}(S_b \setminus \{x\})$. Then because $E_x = X \oplus (H_a \cap H_b)$ we notice that $D_L(S_a \setminus \{x\}) \neq D_L(S_b \setminus \{x\})$ in E. Since $L \in \mathscr{H}$ we have $C \subset D_L(S_a \setminus \{x\})$ or $C \subset D_L(S_b \setminus \{x\})$. Without loss of generality we can assume that

$$C \subset D_L(S_a \setminus \{x\}) = D_L(C). \tag{3.3.5}$$

We want to show that $\overline{C} \cap H_b \subset H_a \cap H_b$, because then there is no facet contained in $\overline{C}$ with support H_b which is equivalent to H_b not being a wall of C. This contradiction then yields $(t_{H_a}, t_{H_b})_V \leq 0$. Indeed, let $q \in \overline{C} \cap H_b$ then $q \in \overline{D_L(C)}$ by (3.3.5), $q \in \overline{D_{H_a}(C)}$ by (3.3.2) for $\overline{C}$ and certainly $q \in H_b$. This leads to

$$q \in \overline{D_L(C)} \cap \overline{D_{H_a}(C)} \cap H_b. \tag{3.3.6}$$

Now choose x and X such that $q \in X$. From the structure of $\overline{\Gamma}$ (see also Figure 3.5) we observe that $S_b = X \cap \overline{D_{H_a}(C)} \cap H_b$ which contains q by (3.3.6) and the fact that $q \in X$. Also $q \in \overline{D_L(C)}$ by (3.3.6), so let us assume that $q \in D_L(C)$. Since $D_L(C) = D_L(S_a \setminus \{x\}) \neq D_L(S_b \setminus \{x\})$, we note that $q \notin S_b \setminus \{x\}$. Further we have $x \in H_a \cap H_b \subset L$, so $x \notin D_L(C)$ which leads to $q \neq x$. This leads to $q \notin S_b$ which contradicts our observation that $q \in S_b$. Thus we conclude $q \in \overline{D_L(C)} \setminus D_L(C) = L$. In particular, $q \in L \cap S_b = \{x\} \subset H_a \cap H_b$. This show that $\overline{C} \cap H_b \subset H_a \cap H_b$ as we wanted. □

This Lemma leads to the first result on the description of the alcoves relative to $\mathscr{H}$ induced by R.

Proposition 3.3.18. *The set of walls $\mathscr{M} \subset \mathscr{H}$ of an alcove C of E relative to $\mathscr{H}$ is finite.*

Proof. Let N be the set of normal unit vectors of the walls of a fixed alcove C relative to $\mathscr{H}$ lying on the same side as C as in Lemma 3.3.17. Consider t_H and $t_{H'}$ with $H \neq H'$ in $\mathscr{M}$. Since $(t_H, t_{H'})_V \leq 0$ by Lemma 3.3.17 we have

$$|t_H - t_{H'}|_V^2 = 2 - 2(t_H, t_{H'})_V \geq 2. \tag{3.3.7}$$

Next, consider the unit sphere $P \subset V$ together with a cover $\mathscr{P}$ of open balls with radius $\sqrt{2}$. Then $N \subset P$ and each open ball of $\mathscr{P}$ contains at most one vector of N by (3.3.7). Since P is compact there exists a finite subcover $\mathscr{Q} \subset \mathscr{P}$ of P. This shows that that N is finite, hence $\mathscr{M}$ is finite. □

Moreover, we obtain a finiteness result on the directions of normal vectors to the affine hyperplanes of $\mathscr{H}$.

Proposition 3.3.19. *There are only finitely many normal unit vectors possible to the affine hyperplanes of $\mathscr{H} = \{H_a : a \in R\}$.*

Proof. Let C be an alcove of E relative to $\mathscr{H}$ and let $\mathscr{M} \subset \mathscr{H}$ its set of walls. By Lemma 3.3.9 every facet relative to $\mathscr{H}$ (resp. $\mathscr{M}$) that meets $\overline{C}$ is contained in $\overline{C}$. Furthermore, the facets contained in $\overline{C}$ relative to $\mathscr{H}$ are the same as the facets contained in $\overline{C}$ relative to $\mathscr{M}$ by (ii) of Lemma 3.3.14. By Proposition 3.3.18 the collection of affine hyperplanes $\mathscr{M}$ is a finite. Hence $\overline{C}$ is the union of finitely many facets relative to $\mathscr{M}$ (or to $\mathscr{H}$) by definition of a facet and (3.3.2). But every facet only meets finitely many affine hyperplanes of $\mathscr{H}$, so there are only finitely many hyperplanes of $\mathscr{H}$ meeting $\overline{C}$. Then the set $N(C)$ of unit vectors normal to the affine hyperplanes in $\mathscr{H}$ that meet $\overline{C}$ is finite. Furthermore, there exists $\lambda < 1$ such that $(t,t')_V \leq \lambda$ for all distinct $t,t' \in N(C)$.

Define $N(\mathscr{H})$ to be the set of unit vectors such that each vector is normal to an affine hyperplanes $H \in \mathscr{H}$. Let $t,t' \in N(\mathscr{H})$ be distinct, then t and t' are parallel vectors if and only if $t = -t'$. Otherwise, let t (resp. t') be a unit vector normal to the affine hyperplane H (resp. H') in $\mathscr{H}$. Since t and t' are not parallel we have $H \cap H' \neq \emptyset$. Let $x \in H \cap H'$, then there exists $w \in W(R)$ such that $w(x) \in \overline{C}$ by (i) of Theorem 3.3.16. Since $w(H), w(H') \in \mathscr{H}$ and $w(x) \in w(H) \cap w(H')$ we observe that the unit vectors normal to $w(H)$ and $w(H')$ are contained in $N(C)$. By (iii) of Example 3.1.6 the affine linear map $w : E \to E$ can be considered as a linear map $w : E_x \to E_{w(x)}$ defined by $x + v \mapsto w(x) + Dw(v)$ for all v in the space of translations V of E. Now $w \in W(R)$, hence $w = w_{a_1} \circ \cdots \circ w_{a_m}$ for $a_1, \ldots a_m \in R$. Then by (ii) of Example 3.1.6 and (3.1.11) we have $Dw = w_{Da_1} \circ \cdots \circ w_{Da_m}$ which is a linear isometry because each w_{Da_i} is an orthogonal reflection in V. This shows that $Dw(t)$ (resp. $Dw(t')$) is a unit vector normal to $w(H)$ (resp. $w(H')$), hence $Dw(t), Dw(t') \in N(C)$ and $(t,t')_V = (Dw(t), Dw(t'))_V \leq \lambda$. Then $|t-t'|_V = 2 - 2(t,t')_V \geq 2 - 2\lambda$. Hence covering the unit sphere P in V with open balls of diameter $\sqrt{2-2\lambda}$ and using the compactness of P we observe similarly to the proof of Proposition 3.3.18 that $N(\mathscr{H})$ is a finite set. □

Next we need to study the gradients of an affine root system R. Let $D(R) := \{Da : a \in R\} \subset V$ be the set of gradients of affine roots in R. Then $D(R)$ is a finite root system in a natural way called the *gradient root system* of R.

Proposition 3.3.20. *$D(R)$ is a finite root system in V, it is irreducible if R is, and the mapping D : $W(R) \to W_0(D(R))$, $w \mapsto Dw$ is a surjective group homomorphism with kernel the subgroup $t(R) \subset W(R)$ of translation maps of E that are contained in $W(R)$.*

Proof. Since R spans $\widehat{E}$ and only contains nonisotropic vectors of $\widehat{E}$ one observes from Proposition 3.1.12 that $D(R)$ spans V and does not contain 0. Also, because of condition (3) of Definition 3.2.1 we have that $\mathbb{R}Da \cap R = \{\pm Da\}$, $\mathbb{R}Da \cap R = \{\pm Da, \pm\frac{1}{2}Da\}$ and $\mathbb{R}Da \cap R = \{\pm Da, \pm 2Da\}$ are the only possibilities for multiples of the gradient Da of an affine root $a \in R$. Further, there are only finitely many normal unit vectors possible to the affine hyperplanes of $\mathscr{H}$ by Proposition 3.3.19. Since each gradient Da is a normal vector to $H_a \in \mathscr{H}$ we observe that $D(R)$ must be finite. Now, let $a,b \in R$ then $w_a(b) \in R$, so $D(w_a(b)) \in D(R)$. But by (3.1.17) and (3.1.11) we have $D(w_a(b)) = (Dw_a)(Db) = w_{Da}(Db) \in D(R)$. Finally, for all $\alpha, \beta \in D(R)$ we have $(\alpha^\vee, \beta)_V = (Da^\vee, Db)_V = (a^\vee, b)_{\widehat{E}} \in \mathbb{Z}$ for some $a,b \in R$. Thus we conclude that $D(R)$ is a finite root system in V. It follows directly from the previous argument that $D(R)$ is irreducible if R is.

By (3.1.1), $D : W(R) \to W_0(D(R))$ is a group homomorphism. Further, we have $Dw_a = w_{Da}$ for all $a \in R$ from (3.1.11). Since $W_0(D(R))$ is generated by the reflections w_{Da} for $a \in R$ it follows that D is surjective. Finally, $Dw = 1$ if and only if $w(x+v) = w(x)+v$ for all $x \in E$ and $v \in V$. Since the kernel of a group homomorphism is a subgroup, we observe that it must be the subgroup $t(R) \subset W(R)$ of translations. □

We will see later on in Remark 3.5.1 that the gradient root system $D(R)$ need not be reduced even if R is reduced.

From here we will assume that R is an irreducible affine root system for the sake of simplicity of the statements of our results. To study the alcoves relative to $\mathscr{H}$ in more detail, we need to know more about the unit vectors normal to the walls of an alcove. We continue with a general Lemma about a finite set of vectors.

Lemma 3.3.21. *Let $\{t_0, \dots, t_n\}$ be a linearly dependent set of vectors spanning V such that*

(1) $(t_i, t_j)_V \leq 0$ *for $i \neq j$;*

(2) *there is no partition of $\{0, \dots, n\}$ into two nonempty subsets I and J such that $(t_i, t_j)_V = 0$ for $i \in I$ and $j \in J$.*

Then there exist coefficients $c_0, \dots, c_n \in \mathbb{R}_{>0}$ such that $\sum_{i=0}^n c_i t_i = 0$, and if $c'_0, \dots, c'_n \in \mathbb{R}$ are such that $\sum_{i=0}^n c'_i t_i = 0$, then there exists a constant $\xi \in \mathbb{R}$ such that $c_i = \xi c'_i$ for all $0 \leq i \leq n$, i.e. $n = \dim(V)$.

Proof. Let $\{e_0, \dots, e_n\}$ be a standard basis of $\mathbb{R}^{n+1}$. Put $a_{ij} := (t_i, t_j)_V$ for $0 \leq i, j \leq n$, then the matrix $A = (a_{ij})_{0 \leq i,j \leq n}$ is a real symmetric square matrix, so we can define a symmetric bilinear form q on $\mathbb{R}^{n+1}$ by

$$q(\sum_{i=0}^n x_i e_i, \sum_{j=0}^n y_j e_j) := \sum_{i,j=0}^n x_i y_j a_{ij}$$

for $x_0, \dots, x_n, y_0, \dots, y_n \in \mathbb{R}$. Also we can define its related quadratic form $q(v) := q(v, v)$ for $v \in \mathbb{R}^{n+1}$, so

$$q(\sum_{i=0}^n x_i e_i) = \sum_{i,j=0}^n x_i x_j a_{ij} = |\sum_{i=0}^n x_i t_i|_V^2 \geq 0 \tag{3.3.8}$$

for $x_0, \dots, x_n \in \mathbb{R}$. By the Corollary of Proposition 2 §7.1 of [1] the kernel of q, which is defined as $N := \{v \in \mathbb{R}^{n+1} : q(v, w) = 0 \text{ for all } w \in \mathbb{R}^{n+1}\}$, equals the subspace of isotropic vectors with respect to the quadratic form q. Thus (3.3.8) yields $N = \{(c_0, \dots, c_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n c_i t_i = 0\}$, and since $\{t_0, \dots, t_n\}$ is a linearly dependent set we conclude that $\dim(N) \geq 1$. To study N further fix $\sum_{i=0}^n c_i e_i \in N$, then

$$\begin{aligned} q(\sum_{i=0}^n |c_i| e_i) &= \sum_{i,j=0}^n |c_i||c_j| a_{ij} = \sum_{0 \leq i \neq j \leq n} |c_i c_j| (t_i, t_j)_V + \sum_{i=0}^n c_i^2 |t_i|_V^2 \\ &\leq \sum_{0 \leq i \neq j \leq n} c_i c_j (t_i, t_j)_V + \sum_{i=0}^n c_i^2 |t_i|_V^2 = \sum_{i,j=0}^n c_i c_j a_{ij} = q(\sum_{i=0}^n c_i e_i) = 0 \end{aligned}$$

since $(t_i, t_j)_V \leq 0$ for $i \neq j$. By (3.3.8) we conclude that $q(\sum_{i=0}^n |c_i| e_i) = 0$, so $\sum_{i=0}^n |c_i| e_i \in N$. This in turn shows that $0 = q(\sum_{i=0}^n |c_i| e_i, e_j) = \sum_{i=0}^n |c_i| a_{ij}$ for all j.

Let $I = \{i : c_i \neq 0\}$ and let $j \notin I$, then $|c_i| a_{ij} \leq 0$ for $i \in I$ and $|c_i| a_{ij} = 0$ for $i \notin I$. Since $\sum_{i=0}^n |c_i| a_{ij} = 0$ for all j, it is immediate that $a_{ij} = 0$ for $i \in I$ and $j \notin I$. Assumption (2) of this Lemma implies then that $I = \emptyset$ or $I = \{0, \dots, n\}$, hence every nonzero vector of N has nonvanishing coordinates. If the dimension of N would be greater than 1, then the intersection of N with the hyperplane $x_i = 0$ for some i would be of dimension at least 1. This would contradict that every nonzero vector of N has nonvanishing coordinates, hence $\dim(N) = 1$. Further, for $\sum_{i=0}^n c_i e_i \in N$ also $\sum_{i=0}^n |c_i| e_i \in N$, so N contains a vector with only positive coordinates. Since $\dim(N) = 1$, this vector generates $N = \{(c_0, \dots, c_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n c_i t_i = 0\}$, and we can draw all conclusions stated in the Proposition. □

Fix an alcove C relative to $\mathscr{H}$ and let $\mathscr{M}$ be its set of walls. Let $N = \{t_H \in V : H \in \mathscr{M}\}$ be as in Lemma 3.3.17. Hence $t_H \in N \subset V$ is the unit vector normal to $H \in \mathscr{M}$ such that $h + t_H$ is on the same side of H as C for all $h \in H$.

Lemma 3.3.22. *The set $N \subset V$ satisfies the conditions of Lemma 3.3.21.*

Proof. By Proposition 3.3.20 the gradient root system $D(R) = \{Da : a \in R\}$ spans V. Hence the only fixed point of V under the action of the finite Weyl group $W_0(D(R))$ is 0. Now the affine Weyl group $W(R)$ is generated by the reflections w_a for $H_a \in \mathscr{M}$ by Theorem 3.3.16. Thus the surjective homomorphism $D : W(R) \to W_0(D(R))$ of Proposition 3.3.20 implies that $W_D := \{Dw_a : H_a \in \mathscr{M}\} = \{w_{Da} : H_a \in \mathscr{M}\}$ generates $W_0(D(R))$ as a group. This means that W_D only leaves 0 as a fixed point in V. Then the normal vectors $\{Da : H_a \in \mathscr{M}\}$ of the affine hyperplanes $\mathscr{M}$ span V. Since each vector $t_{H_a} \in N$ coincides with Da up to a nonzero scaling for $H_a \in \mathscr{M}$ we conclude that N spans V.

Now assume that N is a linearly independent set, then the walls of C intersect in exactly one point, say $x \in E$. Since the reflections $w_a \in W(R)$ for $H_a \in \mathscr{M}$ generate $W(R)$, we observe that x is a fixed point for the action of $W(R)$ on E. So if $a \in R$ then $w_a(x) = x - a(x)Da^\vee = x$, which implies $a(x) = 0$. Consequently for $v \in V$ it holds that $a(x + v) = (Da, v)_V$, so after choosing the origin $x \in E$ we have that $a \in E_x^*$ for every $a \in R$. By Proposition 3.1.12 this is in contradiction with the fact that R spans $\widehat{E}$, hence N is a linearly dependent set.

Finally, N is finite by Proposition 3.3.18, criterion (1) of Lemma 3.3.21 holds by Lemma 3.3.17 and criterion (2) of the Lemma is satisfied since R is irreducible. □

This leads us to the main Theorem of this Section which gives a precise geometric description of an alcove relative to $\mathscr{H}$ induced by an irreducible affine root system R. This description is important for the remaining of this Chapter since it will give us a way to define the analogue of a basis a finite root system.

Theorem 3.3.23. *The alcoves of E relative to $\mathscr{H}$ are open l-simplices with $l+1$ walls.*

Proof. Let $\mathscr{M}$ be the set of walls of a fixed alcove C and N their unit normals as in Lemma 3.3.22. Since $\dim(V) = l$ Lemma 3.3.21 and lemma:alc.7 yield that N has exactly $l+1$ elements, say $N = \{t_0, \dots, t_l\}$. By the proof of the Lemma 3.3.17 there can not exist distinct parallel walls of C with the same unit normal in N, hence $\mathscr{M}$ contains exactly $l+1$ affine hyperplanes of $\mathscr{H}$. Put $\mathscr{M} = \{H_0, \dots, H_l\}$ such that $t_i \in N$ is a normal vector of $H_i \in \mathscr{M}$ for each $i \in \{0, \dots, l\}$. Assume that $\{t_1, \dots, t_l\}$ is a linearly dependent set, then there exist $d_1, \dots, d_l \in \mathbb{R}$, not all vanishing, such that $\sum_{i=1}^l d_i t_i = 0$. On the other hand, $\sum_{i=0}^l c_i t_i = 0$ for some $c_0, \dots, c_l \in \mathbb{R}_{>0}$ by Lemma 3.3.21. So subtracting a small or large enough multiple of $\sum_{i=1}^l d_i t_i$ to $\sum_{i=0}^l c_i t_i$ yields a new vanishing relation between $t_0, \dots, t_l$ with coefficients not all positive or all negative. This contradicts the second statement of Lemma 3.3.21. Hence $\{t_1, \dots, t_l\}$ forms a basis of V, which implies that the hyperplanes $H_1, \dots, H_l$ have a unique intersection point x_0.

Let x_0 be the origin of E and write E_{x_0} for this vector space. Choose a basis $\{t'_1, \dots, t'_l\}$ in V dual to $\{t_1, \dots, t_l\}$, i.e. $(t_i, t'_j)_V = \delta_{ij}$ for all $1 \le i, j \le l$. By Lemma 3.3.22, there exist $q_1, \dots, q_l \in \mathbb{R}_{>0}$ such that $t_0 = -(q_1 t_1 + \dots + q_l t_l)$. Since t_0 is orthogonal to H_0, there exists $q \in \mathbb{R}$ such that $H_0 = \{x_0 + t : t \in V \text{ and } (t, t_0)_V = -q\}$. Also every $x \in E$ can be written uniquely in the form $x = x_0 + t$ with $t = \xi_1 t'_1 + \dots \xi_l t'_l$ for some real coefficients $\xi_1, \dots, \xi_l$, since $\{t'_1, \dots, t'_l\}$ is a basis of V. A point x written in this form belongs to $C = D_{\mathscr{M}}(C)$ (see (i) of Lemma 3.3.14) if and only if $D_{H_i}(x) = D_{H_i}(x_0 + t) = D_{H_i}(C)$ for $0 \le i \le l$. In other words, $x = x_0 + t$ is on the same side of H_i as t_i for $0 \le i \le l$, or $(t, t_i)_V > 0$ for $1 \le i \le l$ and $(t, t_0)_V > -q$. Equivalently there exist $\xi_1 \dots, \xi_l \in \mathbb{R}_{>0}$ such that $q_1\xi_1 + \dots + q_l \xi_l < q$. Note that $q > 0$ since $C \ne \emptyset$. Write $\xi_i = \frac{\lambda_i}{q_i} q$ for $1 \le i \le l$, then $x \in C$ if and only if $\lambda_i \in \mathbb{R}_{>0}$ for $1 \le i \le l$ and $\sum_{i=1}^l \lambda_i < 1$. Putting $x_i = x_0 + \frac{q}{q_i} t'_i$ for $1 \le i \le l$, we observe that $x = x_0 + \sum_{i=1}^l \lambda_i (x_i - x_0) = (1 - \sum_{i=1}^l \lambda_i) x_0 + \sum_{i=1}^l \lambda_i x_i$, so C is the open l-simplex with vertices $x_0, \dots, x_l$. □

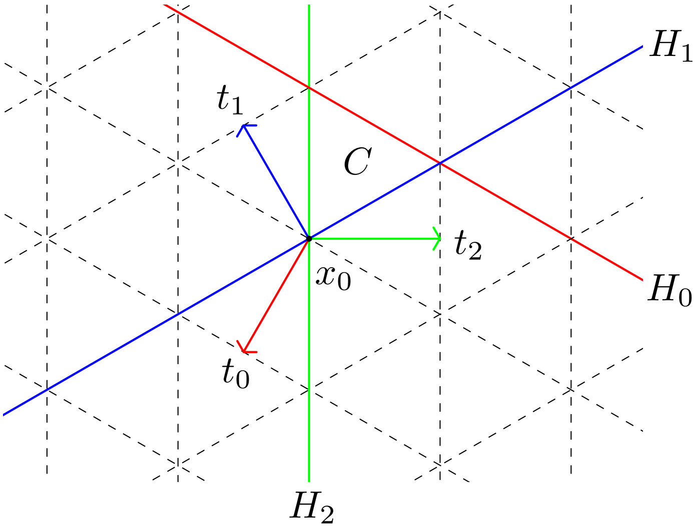


Figure 3.6: An alcove C with walls H_0, H_1, H_2 and normal vectors t_0, t_1, t_2 as in Theorem 3.3.23. The hyperplane configuration corresponds to a reduced irreducible affine root system of type A_2^u.

3.4 Bases of irreducible affine root systems

Let R be an irreducible affine root system of rank l on an affine Euclidean space E. Let $\mathscr{H} = \{H_a : a \in R\}$ be the set of affine hyperplanes in E generated by R (see (3.1.8)), and write $E_{reg} = E \setminus \bigcup_{H \in \mathscr{H}} H$ for the regular points of E relative to $\mathscr{H}$. Fix an alcove C of E relative to $\mathscr{H}$ and let $\mathscr{M} \subset \mathscr{H}$ be its set of walls. Let $B(C)$, or just B if there is no ambiguity about the fixed alcove, be the set of indivisible $a \in R$ such that $H_a \in \mathscr{M}$ and $a(x) > 0$ for all $x \in C$. The set B is said to be a *basis* of R (relative to the alcove C), its elements are called *simple affine roots* and the reflections $w_a \in W(R)$ with $a \in B$ are called *simple reflections*. In this Section we will show that B generates R^{ind} through the action of $W(R)$. Since $W(R)$ is generated by the simple reflections (Theorem 3.3.16), this shows that B contains all the information necessary to construct R^{ind}. Therefore B will play a central role in the next Section when we classify all reduced irreducible affine root systems. We will also show here that B satisfies a natural generalization of the criteria of a basis for a finite root system.

Proposition 3.4.1. *B is a basis of the $l+1$-dimensional $\mathbb{R}$-vector space $\widehat{E}$.*

Proof. From the proof of Theorem 3.3.23 we have $\mathscr{M} = \{H_0, \dots, H_l\}$, and the normal vectors $t_1, \dots, t_l$ to $H_1, \dots, H_l$ respectively form a basis for V. Hence $H_1, \dots, H_l$ have the unique intersection point $x_0 \in E$. Identify E with V as a vector space by choosing the origin x_0. Put $B = \{a_0, \dots, a_l\}$ such that a_i vanishes on H_i for all i. Then $a_1, \dots, a_l$ can be considered as linear functionals on V by (iii) of Example 3.1.6, so for each $i \in \{1, \dots, l\}$ there exists a unique vector $Da_i \in V$ such that $a_i = (Da_i, .)_V$. Then each Da_i is a normal vector to H_i. Thus the linear functionals $a_1, \dots, a_l$ must be linearly independent, else we get a contradiction with the normal vectors $t_1, \dots, t_l$ being a basis of V. This implies that the linear functionals $a_1, \dots, a_l$ form a basis of V^*. Now, there exists $Da_0 \in V$ such that $a_0(x_0 + v) = a(x_0) + (Da_0, v)_V$ for all $v \in V$. Furthermore, C is an l-simplex by Theorem 3.3.23, so H_0 does not contain x_0 and $a(x_0) \neq 0$. Hence there exist $c_1, \dots, c_l \in \mathbb{R}$ such that $a_0 + \sum_{i=1}^{l} c_i a_i$ is the nonzero constant function on E that is identically $a_0(x_0)$. By Proposition 3.1.12 B is now a basis of the $(l+1)$-dimensional space $\widehat{E}$. □

In the following we will use transitive action of $W(R)$ on the collection of alcoves of E relative to $\mathscr{H}$ to show that B generates R^{ind} by letting the $W(R)$ act on B.

Proposition 3.4.2. *For each indivisible $a \in R$ there exists $w \in W(R)$ such that $a = w(b)$ for some $b \in B$.*

Proof. Let $a \in R$, then the affine hyperplane H_a is a wall of some alcove C' by Proposition 3.3.13. Since $H_a = \{y \in E : a(y) = 0\}$ and since $C' \subset E_{reg}$ is connected, we have $a(x) > 0$ or $a(x) < 0$ for all $x \in C'$. Assume that a is positive on C'. By Theorem 3.3.16 there exists an orthogonal reflection $w \in W(R)$ such that $C' = w(C)$, hence $w^{-1}(a)$ is positive on C. We observe by Lemma 3.3.15 that $H_{w^{-1}(a)} = w^{-1}(H_a)$ is a wall of C, so if we let $a \in R$ be indivisible then $w^{-1}(a) \in B$.

If a is negative on C', then $D_{H_a}(C') \neq D_{H_a}(w_a(C'))$ are the connected components of $E \setminus H_a = E \setminus \{y \in E : a(y) = 0\}$ so a is positive on the alcove $w_a(C')$. By Lemma 3.3.15 we have that H_a is a wall of $w_a(C')$. We can now repeat the argument of the last paragraph with a and the alcove $w_a(C')$. □

Moreover, we observe the following.

Proposition 3.4.3. *B and R generate the same $\mathbb{Z}$-lattice in $\widehat{E}$ of rank $l+1$.*

Proof. Write $L(B)$ (resp. $L(R)$) for the lattice generated by B (resp. R). Clearly $L(B) \subset L(R)$, and $L(R)$ is generated by the indivisible affine roots of R. By Proposition 3.4.2 it is enough to prove that $L(B)$ is stable under $W(R)$, moreover by Theorem 3.3.16 it is enough to show that $w_a(L(B)) \subset L(B)$ for all $a \in B$. For $a, b \in B$ we have $w_a(b) = b - (a^\vee, b)_{\widehat{E}}\, a$ by (3.1.15) with $(a^\vee, b)_{\widehat{E}} \in \mathbb{Z}$ by Definition 3.2.1, which implies that $w_a(b) \in L(B)$. Therefore $w_a(L(B)) \subset L(B)$ for all $a \in B$, which completes the proof. □

Definition 3.4.4. An affine root $a \in R$ is called *positive* (resp. *negative*) (*relative to* C) if $a(x) > 0$ (resp. $a(x) < 0$) for all $x \in C$.

Since no affine root vanishes on E_{reg}, every affine root in R is either positive or negative. Let R^+ denote the subset of positive roots of R and R^- the negative roots, then we have the disjoint union $R = R^+ \cup R^-$. Moreover, $R^- = -R^+$ since $w_a(a) = -a$ for all $a \in R$ by (3.1.15). This also shows that $\mathscr{H} = \{H_a : a \in R\} = \{H_a : a \in R^+\}$.

By Theorem 3.3.23 we have that an alcove C is an open l-simplex in E. Let $x_0, \dots, x_l$ be the vertices of C, and turn E into a vector space by choosing an origin. Then C is given by the points $x \in E$ of the form $x = \sum_{i=0}^{l} \lambda_i x_i$ with $\sum_{i=0}^{l} \lambda_i = 1$ and each $\lambda_i > 0$. Put $B = B(C) = \{a_0, \dots, a_l\}$ such that $a_i(x_j) = 0$ for $i \neq j$. In other words, $x_j \in H_{a_i}$ for all $i \neq j$, so since C is an l-simplex $x_i \notin H_{a_i}$. But it does hold that $x_i \in \overline{C}$ and by definition of B we have that a_i is positive on C, hence $a_i(x_i) > 0$ for each i. Also by definition of B, we observe that $B \subset R^+$. Moreover, analogous to the theory of finite root systems we have the following decomposition of positive (resp. negative) affine roots into positive (resp. negative) sums of simple affine roots.

Proposition 3.4.5. *For every positive (resp. negative) affine root $a \in R$ there exist positive (resp. negative) $\lambda_0, \dots, \lambda_l \in \mathbb{Z}$ such that $a = \sum_{i=0}^{l} \lambda_i a_i$.*

Proof. By Proposition 3.4.3, we can write $a \in R$ as $a = \sum_{i=0}^{l} \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$ for $0 \leq i \leq l$. Evaluating both sides of this equation at x_i yields $\lambda_i = a(x_i)/a_i(x_i)$ for $0 \leq i \leq l$. Assume $a \in R^+$, then $a(x) \geq 0$ for all $a \in \overline{C}$ and in particular $a(x_i) \geq 0$. Because $a_i(x_i) > 0$, we obtain $\lambda_i \geq 0$ for $0 \leq i \leq l$. Likewise, if a is negative, we observe that $\lambda_i \leq 0$ for $0 \leq i \leq l$. □

In the next Proposition we will see that a similarity transformation between two affine root systems is compatible with the notion of a basis of an affine root system.

Proposition 3.4.6. *Let $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ be irreducible affine root system such that $R \simeq R'$ realized by the similarity transformation $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$. If B is a basis of R, then $B' := T(B)$ is a basis of R'.*

Proof. First, consider two irreducible affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ such that $R \simeq R'$ realized by a similarity transformation $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$. Then by Proposition 3.2.13 there exists $\lambda \in \mathbb{R}_{>0}$ and an affine linear isomorphism $\psi : E \to E'$ such that $T(a) = \lambda(a \circ \psi^{-1})$ for all $a \in \widehat{E}$.

Next, let $\mathscr{H} = \{H_a : a \in R^+\}$ (resp. $\mathscr{H}' := \{H_{a'} : a' \in R'^+\}$) be the collection of affine hyperplanes in E (resp. E') induced by R (resp. R'), and let $B = \{a_0, \dots, a_l\}$ be a basis of R with respect to an alcove $C \subset E$ relative to $\mathscr{H}$. First we show that $\psi(C) \subset E'$ is an alcove relative to $\mathscr{H}'$, and then we show that the set $B' := T(B) = \{\lambda(a_0 \circ \psi^{-1}), \dots, \lambda(a_l \circ \psi^{-1})\} =: \{a'_0, \dots, a'_l\} \subset R'$ is a basis of R' with respect to $\psi(C)$.

By Corollary 3.2.14, ψ induces a bijection from $\mathscr{H}$ onto $\mathscr{H}'$ given by the mapping $H_a \mapsto \psi(H_a) = H_{T(a)}$. Since $\psi : E \to E'$ is an affine linear isomorphism, ψ is also a homeomorphism relative to the Euclidean topologies on E and E' respectively by Corollary 3.1.8. Hence $\psi' := \psi|_{E_{reg}} : E_{reg} \to E'_{reg}$ is a homeomorphism relative the induced topologies on E_{reg} and E'_{reg} respectively. Now C is an alcove in E, so Proposition 3.3.7 together with the fact that ψ is a homeomorphism yields that $\psi(C)$ is an alcove in E' relative to $\mathscr{H}'$.

Now B' consists of indivisible roots because T is linear and $T|_R : R \to R'$ is a bijection. Further, using Theorem 3.3.23 we can write $\mathscr{M} := \{H_{a_0}, \dots, H_{a_l}\} \subset \mathscr{H}$ for the set of walls of the alcove $C \subset E$. Then by a similar argument as in Lemma 3.3.15 one observes that $\mathscr{M}' := \{\psi(H_{a_0}), \dots, \psi(H_{a_l})\} \subset \mathscr{H}'$ is the set of walls of $\psi(C)$. Finally, let $y \in \psi(C)$, then for $1 \le i \le l$ we have $a'_i(y) = \lambda a_i(\psi^{-1}(y)) = \lambda a_i(x)$ for some $x \in C$. But $\lambda > 0$, and a_i is positive on C because B is a basis relative to C. We conclude that all elements in B' are positive on the alcove $\psi(C)$, thus B' is a basis for R' with respect to the alcove $\psi(C)$. □

Remark 3.4.1. Notice that being a similarity transformation is a necessary condition on T in Proposition 3.4.6. Otherwise we obtain $\lambda < 0$ from the proof of Proposition 3.2.13, and in that case one can not conclude that the elements in B' are positive on the alcove $\psi(C)$.

We end this section with a result on finite root systems contained in an irreducible affine root system R. For each i let $\widehat{E}_i$ be the subspace of affine linear functions of $\widehat{E}$ that vanish on the vertex x_i of the alcove C, and write $R_i := R \cap \widehat{E}_i$. Since $\widehat{E} = \mathrm{Hom}(E, \mathbb{R})$ (where 0 is the origin in $\mathbb{R}$), we observe by (3.1.3) that $\widehat{E}_i$ coincides with the vector space $\mathrm{Hom}_{\mathbb{R}}(E_{x_i}, \mathbb{R})$. Since $\widehat{E}_i$ does not contain any nonzero constant functions the symmetric bilinear form $(.,.)_{\widehat{E}}$ is positive definite on $\widehat{E}_i$, hence $\widehat{E}_i$ is an inner product space with respect to this form.

Proposition 3.4.7. *R_i is a finite root system in $\widehat{E}_i$, and it is reduced if R is. A basis of R_i is given by $B \setminus \{a_i\}$.*

Proof. By Lemma 3.3.3 there exist only finitely many affine hyperplanes in $\mathscr{H}$ passing through x_i. Furthermore, we know from nonreduced affine root systems that maximally four affine roots induce the same affine hyperplane of $\mathscr{H}$. Hence R_i is finite, and it does not contain 0 since R does not contain any constant functions. Further, $a_j(x_i) = 0$ for $i \neq j$, so $B \setminus \{a_i\} \subset R_i \subset \widehat{E}_i$. Since B is a basis of the $(l+1)$-dimensional space $\widehat{E}$, $B \setminus \{a_i\}$ is a linearly independent set of l vectors in $\widehat{E}_i$. As $\widehat{E}_i$ is a strict subspace of $\widehat{E}$, we conclude that $B \setminus \{a_i\}$ and hence R_i spans $\widehat{E}_i$. Also for $a, b \in R_i$, it is clear from the definition of R that $(a^\vee, b)_{\widehat{E}} \in \mathbb{Z}$, and that $w_a(b) = b - (a^\vee, b)_{\widehat{E}}\, a \in R$. In particular $w_a(b) \in R_i$ since a and b vanish at x_i. Finally, it is immediate that R_i is reduced if R is.

In the last paragraph we already saw that $B \setminus \{a_i\}$ is a basis of the vector space $\widehat{E}_i$. Now let $a \in R_i$, then by Proposition 3.4.5 there exist either all positive or all negative $\lambda_0, \ldots, \lambda_l \in \mathbb{Z}$ such that $a = \sum_{j=0}^{l} \lambda_j a_j$. Evaluating both sides of this equation at x_i yields $\lambda_i = 0$, hence $a = \sum_{i \neq j} \lambda_j a_j$ with either all positive or all negative $\lambda_j \in \mathbb{Z}$ for each $j \neq i$. Thus $B \setminus \{a_i\}$ is a basis of the finite root system R_i. □

3.5 A classification of reduced irreducible affine root systems

In this Section we will proceed with the classification of reduced irreducible affine root systems up to similarity. First, we will relate affine Cartan matrices to irreducible affine root systems using the notion of a basis. Then we will construct a bijective correspondence between reduced irreducible affine root systems up to similarity class, and affine Cartan matrices up to simultaneous permutations of rows and columns. Using the earlier developed classification of affine Cartan matrices, this will finish the classification. Along the way we will also explicitly construct a complete set of representatives for the similarity classes of reduced irreducible affine root systems, which we will use to explain the naming of the classification of affine Dynkin diagrams. Finally, we will relate the obtained correspondence to the bijective correspondence between affine Cartan matrices up to simultaneous permutations of rows and columns and isomorphism classes of Lie algebras that are isomorphic to affine Lie algebras to obtain an analogue of (1.3.1).

3.5.1 Affine Cartan matrices of irreducible affine root systems

Fix an ordered basis $B = (a_0, \ldots, a_n)$ of an irreducible affine root system $R \subset \widehat{E}$, and define the matrix $A(R, B) = (a_{ij})_{0 \le i,j \le l}$ where $a_{ij} := (a_i^\vee, a_j)_{\widehat{E}}$. If there is no ambiguity about R and B, we will just write A for $A(R, B)$. The coefficients a_{ij} for $0 \le i, j \le l$ are called the *affine Cartan integers* of R, and the matrix A is said to be the *affine Cartan matrix* of R. The following result justifies this definition.

Proposition 3.5.1. *For each basis B of R, the $(l+1) \times (l+1)$-matrix $A = A(R, B)$ is a rank l affine Cartan matrix.*

Proof. We will check all conditions of Definition 2.2.3 explicitly for the $(l+1) \times (l+1)$-matrix $A = A(R, B)$. Firstly, $A = (a_{ij})_{0 \le i,j \le l}$ is a rational integral matrix such that $a_{ii} = 2$ for all i by criterion (3) of Definition 3.2.1. Secondly, $a_{ij} = 0$ implies $a_{ji} = 0$ by the definition of the affine Cartan integers. Further, recall that each vector Da_k for $0 \le k \le l$ is a nonzero multiple of the vector $e_{H_{a_k}}$ as defined before Lemma 3.3.21. Since a_i and a_j are positive on the fixed alcove C, we even have that Da_k is a positive multiple of $e_{H_{a_k}}$ for $0 \le k \le l$. Now $a_{ij} = (a_i^\vee, a_j)_{\widehat{E}} = (Da_i^\vee, Da_j)_V$ with $Da_i^\vee = \frac{2}{(a_i, a_i)_{\widehat{E}}} Da_i$, so Lemma 3.3.21 and 3.3.22 give $a_{ij} \le 0$ if $i \neq j$.

Next, $\{Da_0, \ldots, Da_l\}$ is a linearly dependent set in V since $\dim(V) = l$. Since $a_{ij} = (Da_i^\vee, Da_j)_V$ for $0 \le i, j \le l$, we observe that $\det(A) = 0$. Further, by Proposition 3.4.7, $B \setminus \{a_k\}$ is a basis for a finite root system. Hence the proper principal submatrix $A' := (a_{ij})_{i,j \in \{0,\ldots,l\} \setminus \{k\}}$ of A is a Cartan matrix of a finite root system. This implies that A' is decomposable into a direct sum of finite Cartan matrices. From linear algebra we have that $A' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ implies $\det(A') = \det(A_1)\det(A_2)$, hence every principal minor of A' is the product of principal minors finite Cartan matrices. It follows from Proposition 2.2.2 that every principal minor of A' is strictly positive. This implies that all proper principal minors of A are strictly positive. Finally, from the irreducibility of R it follows directly that A is an indecomposable matrix. □

Next, we are interested in the dependency of the affine Cartan matrix of an irreducible affine root system on the different choices of representatives of its similarity class and on the choices of bases.

Proposition 3.5.2. *Let $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ be irreducible affine root systems such that $R \simeq R'$ realized by the similarity transformation $T : R \xrightarrow{\sim} R'$, and let $B = \{a_0, \ldots, a_l\}$ be an ordered basis of R.*

(i) *Let $B' := T(B) = \{T(a_0), \ldots, T(a_l)\} =: \{a'_0, \ldots, a'_l\}$, then $A(R, B) = A(R', B')$.*

(ii)*Let $\widetilde{B}$ be another ordered basis of R, then $A(R, B)$ coincides with $A(R, \widetilde{B})$ up to simultaneous permutation of rows and columns.*

Proof. (i) Consider two irreducible affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ such that $R \simeq R'$ realized by a similarity transformation $T : \widehat{E} \xrightarrow{\sim} \widehat{E'}$. Further, let $B = \{a_0, \ldots, a_l\}$ be an ordered basis of R with respect to the alcove $C \subset E$. Then by Proposition 3.4.6 we obtain that $B' := T(B) = \{T(a_0), \ldots, T(a_l)\} =: \{a'_0, \ldots, a'_l\}$ is an ordered basis of R' with respect to the alcove $\psi(C) \subset E'$. Here $\psi : E \to E'$ is the affine linear isomorphism such that $T(a) = \lambda(a \circ \psi^{-1})$ for some fixed $\lambda > 0$ from Proposition 3.2.13. Now it follows from the definition of similarity that $(a_i'^{\vee}, a'_j)_{\widehat{E'}} = (T(a_i)^{\vee}, T(a_j))_{\widehat{E'}} = (a_i^{\vee}, a_j)_{\widehat{E}}$ for all $0 \le i, j \le l$, hence the matrices $A(R, B)$ and $A(R', B')$ coincide.

(ii) Next, let $\widetilde{B}$ be an ordered basis of R with respect to the alcove $\widetilde{C} \subset E$. By Theorem 3.3.16 there exists $w \in W(R)$ such that $w(\widetilde{C}) = C$ where w is considered as an element of $\mathrm{GL}(E)$. Then w considered as an element of $\mathrm{GL}_{\mathbb{R},c}(\widehat{E})$ is a normalized similarity transformation of R with itself by Example 3.2.12. Hence we observe from the last paragraph that $B' := \{w(a_0), \ldots, w(a_l)\} \subset R$ is an ordered basis of R with respect to the alcove $w^{-1}(C) = \widetilde{C}$.

Now for each $a \in \widetilde{B}$ there exists a unique $b \in B'$ such that both a and b vanish on a wall of C. Since both a and b are indivisible roots, they must coincide. Consequently $\widetilde{B} = B'$ up to an ordering, hence $\widetilde{B}$ is the w-image of B up to an ordering. Since w acting on $\widehat{E}$ is a similarity transformation of R with itself we have $(w(a_i)^{\vee}, w(a_j))_{\widehat{E}} = (a_i^{\vee}, a_j)_{\widehat{E}}$ for all $0 \le i, j \le l$. Hence we observe that the matrices $A(R, B)$ and $A(R, B')$ coincide up to simultaneous permutations of rows and columns. □

We conclude the matrix $A(R, B)$ up to simultaneous permutations of rows and columns does not depend on the choice of basis B of R nor on the choice of representative of the similarity class of R.

Recall that we write $\overline{A}$ for the equivalence class of the affine Cartan matrix A under the equivalence relation of simultaneous permutations of rows and columns of matrices. Further, recall that $\mathscr{C}_a$ denotes the collection of indecomposable affine Cartan matrices up to simultaneous permutations of rows and columns. Now, write $\overline{R}$ for the similarity class of the irreducible affine root system R, and put $\mathscr{R}_a$ for the similarity classes of reduced irreducible affine root systems. Then Proposition 3.5.2 immediately implies the following result.

Corollary 3.5.3. *There exists a well defined map $A : \mathscr{R}_a \to \mathscr{C}_a$ given by $\overline{R} \mapsto \overline{A(R, B)} =: A(\overline{R})$, where B is any choice of ordered basis of the representative R of the similarity class $\overline{R}$.*

Moreover, the affine Cartan matrix up to simultaneous permutations of rows and columns of a reduced irreducible affine root systems R determines R uniquely up to similarity.

Proposition 3.5.4. *The map $A : \mathscr{R}_a \to \mathscr{C}_a$ is injective.*

Proof. Let $\overline{R}, \overline{R'} \in \mathscr{R}_a$, and assume that we have representatives $R \in \overline{R}$ (resp. $R' \in \overline{R'}$) together with an ordered basis $B \subset \widehat{E}$ (resp. $B' \subset \widehat{E'}$) such that $A(\overline{R}) = \overline{A(R, B)} = \overline{A(R', B')} = A(\overline{R'})$. Then we can fix an new ordering on $B = (a_0, \dots, a_l)$ and $B' = (a'_0, \dots, a'_l)$ such that $A(R,B) = A(R',B')$. In the following we will construct a linear isomorphism $T : \widehat{E} \to \widehat{E'}$ that realizes a similarity between R and R'. This will imply that the similarity classes $\overline{R}$ and $\overline{R'}$ coincide which shows the injectivity of A.

Since B (resp. B') is a basis of $\widehat{E}$ (resp. $\widehat{E'}$), there exists a unique linear isomorphism $T : \widehat{E} \to \widehat{E'}$ defined by $T(a_i) = a'_i$ for $0 \le i \le l$. For $a_i, a_j \in B$ we obtain

$$\begin{aligned} w_{T(a_i)}(T(a_j)) &= T(a_j) - (T(a_i)^\vee, T(a_j))_{\widehat{E'}}\, T(a_i) \\ &= T(a_j) - (a_i^\vee, a_j)_{\widehat{E}}\, T(a_i) \\ &= T(a_j - (a_i^\vee, a_j)_{\widehat{E}}\, a_i) \\ &= T(w_{a_i}(a_j)). \end{aligned}$$

Since B is a basis of $\widehat{E}$ we have $T \circ w_{a_i} \circ T^{-1} = w_{T(a_i)}$ on $\widehat{E'}$ for all $a_i \in B$. Further, $T \circ w_{a_i} \circ w_{a_j} \circ T^{-1} = T \circ w_{a_i} \circ T \circ T^{-1} \circ w_{a_j} \circ T^{-1} = w_{T(a_i)} \circ w_{T(a_j)}$ for $a_i, a_j \in B$, and the simple reflections w_{a_i} for $a_i \in B'$ generate $W(R')$. Hence we have a group isomorphism $\widetilde{T} : W(R) \xrightarrow{\sim} W(R')$, $w \mapsto T \circ w \circ T^{-1}$ with inverse $w' \mapsto T^{-1} \circ w' \circ T$.

Next, we want to observe that $T|_R : R \to R'$ is a bijection. By Proposition 3.4.2 we have that each $a \in R$ is $W(R)$-conjugate to some $a_i \in B$, say $a = w(a_i)$ for some $w \in W(R)$. Consequently, $T(a) = T(w(a_i)) = T(w(T^{-1}(T(a_i)))) = (T \circ w \circ T^{-1})(a'_i) \in R'$ since $T \circ w \circ T^{-1} \in W(R')$ by $\widetilde{T}$, thus $T|_R : R \to R'$ is a well-defined map. But T is a linear isomorphism, so since each affine root in R is a unique sum of simple affine roots in B, $T|_R$ is also injective. This argument is symmetric in R and R', so we easily observe that also $T^{-1}|_{R'} : R' \to R$ is well defined and injective. Consequently we obtain that $T|_R : R \to R'$ is a bijection.

Finally, for all $a, b \in R$ we have $w_{T(a)}(T(b)) = T(b) - (T(a)^\vee, T(b))_{\widehat{E'}}\, T(a)$ and $(T \circ w_a \circ T^{-1})(T(b)) = T(b) - (a^\vee, b)_{\widehat{E}}\, T(a)$. If we would know that $w_{T(a)} = T \circ w_a \circ T^{-1}$ for all $a \in R$, then we obtain $(a^\vee, b)_{\widehat{E}} = (T(a)^\vee, T(b))_{\widehat{E'}}$ for all $a \in R$. To show that $w_{T(a)} = T \circ w_a \circ T^{-1}$ for $a \in R$, write $a = w(a_i)$ for some $w \in W(R)$ and $a_i \in B$. Further, we can write $w = w_{i_1} \circ \dots \circ w_{i_r}$ for some $r \in \mathbb{N}$, where each $w_{i_j} = w_{a_{i_j}}$ for some $i_j \in \{0, \dots, l\}$. In the following we leave out the $\circ$-symbol in the formulas to improve readability. Firstly, we already know that $T w_{a_i} T^{-1} = w_{T(a_i)} = w_{a'_i}$, so put $w_{i'_j} := w_{a'_{i_j}} = w_{T(a_{i_j})} = T w_{a_{i_j}} T^{-1} = T w_{i_j} T^{-1}$. Secondly, by (i) of Example 3.2.9 and (3.2.1) we obtain that $w w_{a_i} w^{-1} = w_{w(a_i)}$ and $w_{i'_1} \dots w_{i'_r} w_{a'_i} (w_{i'_1} \dots w_{i'_r})^{-1} = w_{(w_{i'_1} \dots w_{i'_r})(a'_i)}$. This leads directly to

$$\begin{aligned} T w_a T^{-1} &= T w_{w(a_i)} T^{-1} \\ &= T w w_{a_i} w^{-1} T^{-1} \\ &= T w_{i_1} \dots w_{i_r} w_{a_i} w_{i_r} \dots w_{i_1} T^{-1} \\ &= T w_{i_1} T^{-1} \dots T w_{i_r} T^{-1} T w_{a_i} T^{-1} T w_{i_r} T^{-1} \dots T w_{i_1} T^{-1} \\ &= w_{i'_1} \dots w_{i'_r} w_{a'_i} w_{i'_r} \dots w_{i'_1} \\ &= w_{i'_1} \dots w_{i'_r} w_{a'_i} (w_{i'_1} \dots w_{i'_r})^{-1} \\ &= w_{(w_{i'_1} \dots w_{i'_r})(a'_i)} \\ &= w_{(T w_{i_1} T^{-1} \dots T w_{i_r} T^{-1})(T(a_i))} \\ &= w_{T w T^{-1}(T(a_i))} = w_{T(w(a_i))} = w_{T(a)}. \end{aligned}$$

Hence we conclude that $(a^\vee, b)_{\widehat{E}} = (T(a)^\vee, T(b))_{\widehat{E'}}$ for all $a \in R$.

Thus we have shown that T realizes a similarity between R and R', but then $\overline{R}$ and $\overline{R'}$ must coincide. Since $A(\overline{R}) = A(\overline{R'})$ we conclude that $A : \mathscr{R}_a \to \mathscr{C}_a$ is injective. $\square$

3.5.2 Explicit constructions of reduced irreducible affine root systems

In Proposition 3.5.4 of the previous Subsection we saw that the map $A : \mathscr{R}_a \to \mathscr{C}_a$ from Corollary 3.5.3 is injective. Our goal in this Subsection is to show that A is also a surjective map. First we will construct some concrete reduced irreducible affine root systems based on the explicit realizations of Δ^{re} in Proposition 2.3.1. Then we will pick a specific set of these realizations and show that the corresponding affine Cartan matrices form a complete set of representatives for the classes in $\mathscr{C}_a$. This leads to the bijectivity of $A : \mathscr{R}_a \to \mathscr{C}_a$ which implies that we have constructed a complete set of representatives for the similarity classes in $\mathscr{R}_a$.

Let R_0 be an irreducible finite root system (possibly nonreduced) in V with finite Weyl group $W_0(R_0) \subset \mathrm{GL}_{\mathbb{R}}(V)$. Let $\{\alpha_1, \ldots, \alpha_l\}$ be a basis of R_0, then $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ is a basis of $R_0^\vee$. Furthermore, R_0 (resp. $R_0^\vee$) is contained in the *root lattice* $Q := \sum_{i=1}^l \mathbb{Z}\alpha_i \subset V$ (resp. *coroot lattice* $Q^\vee := \sum_{i=1}^l \mathbb{Z}\alpha_i^\vee \subset V$). Write $t(L) := \{t_v : v \in L\}$ for subgroup of translations of $t(V)$ over the lattice $L \subset V$. Next, consider the vector space V as affine Euclidean space E with V as space of translations (see Example 3.1.2). Let c be the constant one function on E, and identify $\widehat{E}$ with $V \oplus \mathbb{R}c$ after choosing the origin $0 \in E$ (see Prop. 3.1.12 and Cor. 3.1.14). By Proposition 3.1.21 we have the identification of GL(E) with $t(V) \rtimes \mathrm{GL}_{\mathbb{R}}(V)$ using the same choice of origin. Hence for each $a := \lambda c + \alpha \in \widehat{E}$ with $\alpha \in V \setminus \{0\}$ and $\lambda \in \mathbb{R}$ we have

$$w_a = t_{-\lambda\alpha^\vee} \circ w_\alpha \tag{3.5.1}$$

by (3.1.13) and (3.1.12), where $w_\alpha \in \mathrm{GL}_{\mathbb{R}}(V)$. Here $t_v(a) = a - (Da, v)_V\, c$ for $v \in V$ and $a \in \widehat{E}$ by (3.1.16).

Consider the subsets of $V \oplus \mathbb{R}c$

$$R^u_{R_0} := \{mc + \alpha\}_{m\in\mathbb{Z}, \alpha \in R_0^{ind}} \cup \{(2m+1)c + \beta\}_{m \in \mathbb{Z}, \beta \in R_0 \setminus R_0^{ind}}$$

and

$$R^t_{R_0} := \{m\frac{|\alpha|_V^2}{2}c + \alpha\}_{m\in\mathbb{Z}, \alpha\in R_0^{unm}} \cup \{(2m+1)\frac{|\beta|_V^2}{2}c + \beta\}_{m\in\mathbb{Z}, \beta \in R_0\setminus R_0^{unm}}$$

where R_0^{ind} (resp. R_0^{unm}) are the indivisible (resp. unmultipliable) roots of R_0.

We will now see that $R^u_{R_0}$ and $R^t_{R_0}$ are reduced irreducible affine root systems that are related by their dual.

Proposition 3.5.5. *$R^u_{R_0}$ is a reduced irreducible affine root system on E with gradient root system $D(R^u_{R_0}) = R_0$ and affine Weyl group $W(R^u_{R_0}) = t(Q^\vee) \rtimes W_0(R_0)$, where $t(Q^\vee), W_0(R_0) \subset W(R^u_{R_0})$ are subgroups.*

Proof. First we will check all conditions of Definition 3.2.1 for $R^u_{R_0}$ as introduced before this Proposition. From its definition it is clear that $R^u_{R_0}$ considered in $V \oplus \mathbb{R}c$ is a nonempty subset of nonisotropic elements with respect to the form (3.1.4) on $V \oplus \mathbb{R}c$.

(1) Since R_0 spans V it is clear that $R^u_{R_0}$ spans $V \oplus \mathbb{R}c$.

(2) It follows from a straightforward computation that the reflections generated by $R^u_{R_0}$ stabilize $R^u_{R_0}$.

(3) By Corollary 3.1.14 using the form (3.1.4) on $V \oplus \mathbb{R}c$ we obtain $((mc+\alpha)^\vee, (nc+\beta))_{V\oplus\mathbb{R}c} = (\alpha^\vee, \beta)_V \in \mathbb{Z}$ for $m, n \in \mathbb{Z}$ and $\alpha, \beta \in R_0$.

(4) Consider the group $W(R^u_{R_0})$ generated by the reflections w_a for $a \in R^u_{R_0}$. Then by (3.5.1) we have

$$w_a = w_{mc+\alpha} = t_{-m\alpha^\vee} w_\alpha \tag{3.5.2}$$

if we write $a = mc + \alpha$ for some $m \in \mathbb{Z}$ and $\alpha \in R_0$. Letting $m = 0$, we observe that $w_\alpha \in W(R)$ for all $\alpha \in R_0$, hence $W_0(R_0) \subset W(R)$ is a subgroup. Also, letting $m = -1$ we observe that $t_{\alpha^\vee} = w_a w_\alpha^{-1} = w_a w_\alpha \in W(R^u_{R_0})$ for all $\alpha \in R_0$. Hence we observe that $t(Q^\vee) \subset W(R^u_{R_0})$ is a subgroup. Furthermore, $t(Q^\vee) \cap W_0(R_0) = \{\mathrm{id}_V\}$, because $t_0 = \mathrm{id}_V$ is the only translation map that fixes 0. By (3.1.14) we also have for $w \in W_0(R_0)$ and $\gamma \in Q^\vee$ that $w t_\gamma w^{-1} = t_{w(\gamma)}$. Now R_0 is $W_0(R_0)$-invariant so $Q^\vee$ as well, hence we obtain that $t_{w(\gamma)} \in t(Q^\vee)$. Further, (3.5.2) implies that $W(R^u_{R_0}) = t(Q^\vee) W_0(R_0) := \{tw : t \in t(Q^\vee) \text{ and } w \in W_0(R_0)\}$. So we conclude that $t(Q^\vee) \subset W(R^u_{R_0})$ is a normal subgroup and that $W(R^u_{R_0}) = t(Q^\vee) \rtimes W_0(R_0)$. Then $t(Q^\vee)$ acts properly on E as a discrete group, because $Q^\vee \subset V$ is a discrete set. Furthermore, $W_0(R_0) \subset W(R^u_{R_0})$ is a finite group, so $W(R^u_{R_0}) = t(Q^\vee) \rtimes W_0(R_0)$ acts properly on E.

(5) It follows from the definition of $R^u_{R_0}$ that for each $\alpha \in R_0$ there are at least two distinct $a, b \in R^u_{R_0}$ such that $Da = Db = \alpha$.

We conclude that $R^u_{R_0}$ is an affine root system on E. Further, by definition of $R^u_{R_0}$ all its affine roots are indivisible, hence $R^u_{R_0}$ is reduced. Also, since R_0 is irreducible and $((mc+\alpha)^\vee, (nc+\beta))_{V\oplus\mathbb{R}c} = (\alpha^\vee, \beta)_V$ for $m, n \in \mathbb{Z}$ and $\alpha, \beta \in R_0$, the affine root system $R^u_{R_0}$ is irreducible as well. Finally, it is clear from Corollary 3.1.13 that $D(R^u_{R_0}) = R_0$. □

Remark 3.5.1. The construction of $R^u_{R_0}$ gives rise to an example of a reduced irreducible affine root system with nonreduced irreducible gradient root system. Let R_0 be a nonreduced irreducible finite root system, then the affine root system $R^u_{R_0}$ is reduced with gradient root system R_0.

Proposition 3.5.6. *$R^t_{R_0^\vee}$ coincides with the dual of $R^u_{R_0}$.*

Proof. Let $m \in \mathbb{Z}$ and $\alpha \in R_0$ such that $mc + \alpha \in R^u_{R_0}$, then

$$(mc+\alpha)^\vee = \frac{2}{(mc+\alpha, mc+\alpha)_V}(mc+\alpha) = \frac{2}{|\alpha|_V^2}(mc+\alpha) = m\frac{2}{|\alpha|_V^2}c + \alpha^\vee = m\frac{|\alpha^\vee|_V^2}{2}c + \alpha^\vee. \tag{3.5.3}$$

This leads to

$$\begin{aligned}
(R^u_{R_0})^\vee &= \{(mc+\alpha)^\vee\}_{m\in\mathbb{Z},\alpha\in R_0^{ind}} \cup \{((2m+1)c+\beta)^\vee\}_{m\in\mathbb{Z},\beta\in R_0\setminus R_0^{ind}} \\
&\overset{(3.5.3)}{=} \{m\frac{|\alpha^\vee|_V^2}{2}c + \alpha^\vee\}_{m\in\mathbb{Z},\alpha\in R_0^{ind}} \cup \{(2m+1)\frac{|\beta^\vee|_V^2}{2}c + \beta^\vee\}_{m\in\mathbb{Z},\beta\in R_0\setminus R_0^{ind}} \\
&= \{m\frac{|\alpha|_V^2}{2}c + \alpha\}_{m\in\mathbb{Z},\alpha\in (R_0^{ind})^\vee} \cup \{(2m+1)\frac{|\beta|_V^2}{2}c + \beta\}_{m\in\mathbb{Z},\beta\in R_0^\vee\setminus (R_0^{ind})^\vee} \\
&= \{m\frac{|\alpha|_V^2}{2}c + \alpha\}_{m\in\mathbb{Z},\alpha\in (R_0^\vee)^{unm}} \cup \{(2m+1)\frac{|\beta|_V^2}{2}c + \beta\}_{m\in\mathbb{Z},\beta\in R_0^\vee\setminus (R_0^\vee)^{unm}} \\
&= R^t_{R_0^\vee},
\end{aligned}$$

where we used that $(R_0^{ind})^\vee = (R_0^\vee)^{unm}$. □

Corollary 3.5.7. *$R^t_{R_0}$ is a reduced irreducible affine root system on E with gradient root system $D(R^t_{R_0}) = R_0$ and affine Weyl group $W(R^t_{R_0}) = t(Q) \rtimes W_0(R_0)$, where $t(Q), W_0(R_0) \subset W(R^t_{R_0})$ are subgroups.*

Proof. By Proposition 3.5.6, $R^t_{R_0^\vee}$ coincides with the dual of $R^u_{R_0}$. Since $R_0^{\vee\vee} = R_0$, we observe that $R^t_{R_0}$ coincides with the dual of $R^u_{R_0^\vee}$. Therefore by Proposition 3.5.5 together with Proposition 3.2.7, $R^t_{R_0}$ is an affine root system with affine Weyl group $W(R^t_{R_0}) = t(Q) \rtimes W_0(R_0^\vee)$, where $t(Q), W_0(R_0^\vee) \subset W(R^t_{R_0})$ are subgroups. But $W_0(R_0) = W_0(R_0^\vee)$, so we obtain $W(R^t_{R_0}) = t(Q) \rtimes W_0(R_0)$. Furthermore, similarly to the case of $R^u_{R_0}$ we have that $R^t_{R_0}$ is irreducible (see proof of Prop. 3.5.5). Also, $R^t_{R_0}$ is the dual of the reduced affine root system $R^u_{R_0}$, and it follows straightforwardly from the definition of the dual of an affine root system that $R^t_{R_0}$ needs to be reduced too. Finally, it is clear from Proposition 3.1.12 that $D(R^t_{R_0}) = R_0$. □

It turns out that we have already come across reduced irreducible affine root systems of the form $R^u_{R_0}$ and $R^t_{R_0}$.

Example 3.5.8. Let A be an affine Cartan matrix, and consider the set of real roots $\Delta^{re} \subset \mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ corresponding to the affine Lie algebra $\mathfrak{g}(A)$ in the context of Section 2.3. By Example 3.2.3 (i), 3.2.5 (ii), and 3.2.23 (iii) we have that Δ^{re} is a reduced irreducible affine root system on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$. By Proposition 2.3.1 we have

(1) $\Delta^{re} = \{\alpha + n\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$ if Δ^{re} is of untwisted type;

(2) $\Delta^{re} = \{\alpha + n\frac{(\alpha,\alpha)}{2}\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$ if Δ^{re} is of twisted type but not of untwisted type;

(3) $\Delta^{re} = \{\alpha + n\frac{(\alpha,\alpha)}{2}\delta : \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\} \cup \{\frac{(\alpha,\alpha)}{2}\alpha + (2n+1)\frac{(\alpha,\alpha)}{2}\delta) : \alpha \in \mathring{\Delta}_l, n \in \mathbb{Z}\}$ if Δ^{re} is of mixed type.

Letting $V := \mathring{\mathfrak{h}}_{\mathbb{R}}^*, (.,.)_V = (.,.), R_0 := \mathring{\Delta}$ and $c := \delta$ we observe that $R^u_{R_0} = \Delta^{re}$ if Δ^{re} is of untwisted type and $R^t_{R_0} = \Delta^{re}$ if Δ^{re} is of twisted but not untwisted type. If Δ^{re} is of mixed type then we observe from Remark 2.3.1 that $\mathring{\Delta} \cup \frac{1}{2}\mathring{\Delta}_l$ is a nonreduced irreducible finite root system with $\mathring{\Delta}$ the unmultipliable roots and $\frac{1}{2}\mathring{\Delta}_l$ the indivisible short roots. So if we put $R_0 := \mathring{\Delta} \cup \frac{1}{2}\mathring{\Delta}_l$, then $R^t_{R_0} = \Delta^{re}$.

Analogue to the case of reduced (irreducible) finite root systems we have the following duality of bases.

Lemma 3.5.9. *If $(a_0, \ldots, a_l)$ is a basis of the reduced irreducible affine root system R, then $(a_0^\vee, \ldots, a_l^\vee)$ is a basis of the dual $R^\vee$.*

Proof. Let $\{a_0, \ldots, a_l\}$ be a basis of R with respect to some alcove C. Now $a^\vee = \frac{2}{(a,a)_{\hat{E}}} a = 0$ if and only if $a = 0$, so $R^\vee$ and R generate the same collection of affine hyperplanes $\mathscr{H}$ in E. This means that they also generate the same alcoves. Since $a^\vee$ is a positive multiple of $a \in R$ and R is reduced we observe that $\{a_0^\vee, \ldots, a_l^\vee\}$ is a basis of the reduced irreducible affine root system $R^\vee$ with respect to the alcove C. □

Finally, we obtain the following important Proposition.

Proposition 3.5.10. (i) *The injective map $A : \mathscr{R}_a \to \mathscr{C}_a$ defined by the mapping $\overline{R} \mapsto \overline{A(R,B)} =: A(\overline{R})$ is also surjective.*

(ii) *The following reduced irreducible affine root systems form a complete set of representatives of the similarity classes in $\mathscr{R}_a$:*

(1) *$R^u_{R_0}$ with R_0 running through the similarity classes of reduced irreducible finite root systems (i.e. R_0 of type A_l $(l \geq 1)$, B_l $(l \geq 3)$, C_l $(l \geq 2)$, D_l $(l \geq 4)$, E_6, E_7, E_8, F_4, G_2);*

(2) $R^t_{R_0}$ *with* R_0 *running through the similarity classes of reduced irreducible finite root systems having two root lengths (i.e.* R_0 *of type* B_l $(l \geq 3), C_l$ $(l \geq 2), F_4, G_2$*);*
(3) $R^u_{R_0}$ *with* R_0 *a nonreduced irreducible finite root systems (i.e.* R_0 *of type* BC_l $(l \geq 1)$*).*

Proof. First we will construct bases for the reduced irreducible affine root systems in each of the three cases as stated in (ii).

(1) Let $B' := \{\alpha_1, \dots, \alpha_l\}$ be a basis of R_0. Further, let ϕ be the unique highest root of R_0 with respect to B'. We will show that $B := \{\alpha_0 := c - \phi, \alpha_1, \dots, \alpha_l\}$ is a basis of $R^u_{R_0}$ with respect to a certain alcove C. Note that B contains only indivisible affine roots since $R^u_{R_0}$ is reduced. Now consider the set $C := \{x \in V : (\alpha_i, x)_V > 0 \text{ for } 1 \leq i \leq l \text{ and } (\phi, x)_V < 1\}$, then by definition of C all elements of B are positive on C. We claim that $C = \{x \in V : 0 < (\alpha, x)_V < 1 \text{ for all } \alpha \in R_0^+\}$. Clearly, $\{x \in V : 0 < (\alpha, x)_V < 1 \text{ for all } \alpha \in R_0^+\} \subseteq C$ holds. On the other hand if $x \in C$, then $(\alpha_i, x)_V > 0$ for $1 \leq i \leq l$, so also $(\alpha, x)_V > 0$ for all $\alpha \in R_0^+$ since B' is a basis of R_0. Further, ϕ is the highest root of R_0, so $\phi - \alpha \geq 0$ for all $\alpha \in R_0^+$ in the dominance partial order on R_0 with respect to B'. Hence for $\alpha \in R_0^+$ we can write $\phi - \alpha = \sum_{i=1}^l c_i \alpha_i$ where $c_i \geq 0$ for all i. For $x \in C$ we have $(\alpha_i, x)_V > 0$ for all i, which leads to $(\phi - \alpha, x)_V = (\sum_{i=1}^l c_i \alpha_i, x)_V \geq 0$. Thus $(\alpha, x)_V \leq (\phi, x)_V < 1$ for all $\alpha \in R_0^+$ which shows that $C = \{x \in V : 0 < (\alpha, x)_V < 1 \text{ for all } \alpha \in R_0^+\}$. So every $x \in C$ lies in between the affine hyperplanes H_α and $H_{1-\alpha}$ for all $\alpha \in R_0^+$. Further, by definition of $R^u_{R_0}$ there are no affine roots yielding parallel affine hyperplanes to H_α lying in between H_α and $H_{1-\alpha}$ for $\alpha \in R_0^+$. This implies $C \subset E_{reg}$ relative to $\mathscr{H} = \{H_a : a \in R^u_{R_0}\}$, and C is connected in E_{reg}. Moreover if $x \in E_{reg}$ but $x \notin C$, then there exists $\alpha \in R_0^+$ such that $(\alpha, x)_V < 0$ or $(\alpha, x)_V > 1$. Thus C is a connected component of E_{reg}, and an alcove of E relative to $\mathscr{H}$ by Lemma 3.3.7. Then by Theorem 3.3.23, C is an open simplex with $l + 1$ walls, and it is clear that $B = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ generates the walls of $C = \{x \in V : (\alpha_i, x)_V > 0 \text{ for } 1 \leq i \leq l \text{ and } (\phi, x)_V < 1\}$. Thus B is a basis of $R^u_{R_0}$ with respect to the alcove C.

(2) Since $R^t_{R_0} = (R^u_{R_0^\vee})^\vee$ we deduce from case (1) and Lemma 3.5.9 that $B := \{\alpha_0 := \frac{|\theta|_V^2}{2} c - \theta, \alpha_1, \dots, \alpha_l\}$ is a basis of $R^t_{R_0}$ with respect to the alcove C. Here $B' := \{\alpha_1, \dots, \alpha_l\}$ is a basis of R_0, and $\theta \in R_0$ is the highest short root of R_0 with respect to B'.

(3) Let $B' = \{\alpha_1, \dots, \alpha_l\}$ be basis of R_0, then B' is also a basis of the reduced irreducible finite root system R_0^{ind}. Further, let ϕ be the highest root of R_0, hence $\phi = 2\alpha$ for some $\alpha \in R_0^{ind}$. Then $B := \{\alpha_0 := c - \phi, \alpha_1, \dots, \alpha_l\}$ is a set of indivisible affine roots since $R^u_{R_0}$ is reduced. Define $C = \{x \in V : (\alpha_i, x)_V > 0 \text{ for } 1 \leq i \leq l \text{ and } (\phi, x)_V < 1\}$, then the affine roots in B are positive on C. By similar arguments as in (1), $C = \{x \in V : 0 < (\alpha, x)_V < 1 \text{ for all } \alpha \in R_0^{unm+}\}$. So every $x \in C$ lies between H_α and $H_{1-\alpha}$ for $\alpha \in R_0^{unm+}$, and by definition of $R^u_{R_0}$ there are no affine roots yielding parallel affine hyperplanes to H_α lying in between H_α and $H_{1-\alpha}$, so $C \subset E_{reg}$ and is connected. Then analogous to (1) one observes that B is a basis of $R^u_{R_0}$ with respect to the alcove C.

It is an elementary exercise to compute the affine Cartan matrix and draw the affine Dynkin diagram for each reduced irreducible affine root system with the constructed basis B as described in this proof. Using the classification of affine Dykin diagrams from Figure 2.1 we get the following correspondence. If $R^u_{R_0}$ is as in (1) with R_0 of type X_l, then the affine Dynkin diagram $S(R^u_{R_0})$ is of type X^u_l. If $R^t_{R_0}$ is as in (2) with R_0 of type X_l and not of type C_2, then the affine Dynkin diagram $S(R^t_{R_0})$ is of type X^t_l. If $R^t_{R_0}$ is as in (2) with R_0 of type C_2, then the affine Dynkin diagram $S(R^t_{R_0})$ is of type B^t_2. If $R^u_{R_0}$ is as in (3) with R_0 of type BC_l, then the affine Dynkin diagram $S(R^u_{R_0})$ is of type BC^m_l.

We conclude that the reduced irreducible affine root systems as stated in (ii) are in bijective correspondence with all affine Dynkin diagrams. It follows that the injective map A of Proposition 3.5.4 is surjective, and that the reduced irreducible affine root systems of (ii) form a complete set of representatives of $\mathscr{R}_a$. □

Remark 3.5.2. (i) Consider case (1) (resp. (2)) of the proof of Proposition 3.5.10. If we equip the nodes of the affine Dynkin diagram $S(R^u_{R_0})$ (resp. $S(R^t_{R_0})$) with the 'α'-labeling of Figure 2.1, then removing node α_0 and the edges connected to it yields the finite Dynkin diagram corresponding to R_0 (forgetting the labeling of the nodes again). In case (3) we need to remove node α_l in $S(R^u_{R_0})$ and the edges connected to it to obtain the finite Dynkin diagram for R_0 (which is of type B_l).

(ii) The construction of a basis for $R^t_{R_0}$ in case (2) of Proposition 3.5.10 (ii) is valid for any reduced irreducible finite root system R_0. In the situation that R_0 has only one root length, the highest short root coincides with the highest root with respect to a chosen basis for R_0. Then it is immediate that the corresponding affine Cartan matrix coincides with the affine Cartan matrix of $R^u_{R_0}$, hence $R^u_{R_0} \simeq R^t_{R_0}$ by Proposition 3.5.4.

In view of Proposition 3.5.10 and (ii) of Remark 3.5.2 we call an affine root system R of *untwisted type* if $R \simeq R^u_{R_0}$ with R_0 reduced, of *twisted type* if $R \simeq R^t_{R_0}$ with R_0 reduced and of *mixed type* if $R \simeq R^u_{R_0}$ with R_0 nonreduced. Notice from (ii) of Remark 3.5.2 that affine roots systems R such that $R \simeq R^u_{R_0}$ with R_0 of type $A_l\,(l \geq 1), D_l\,(l \geq 4), E_6, E_7$ and E_8 are both of untwisted and twisted type.

3.5.3 The naming of affine Dynkin diagrams explained

In this Subsection we want to give the rationale behind the naming of the types of affine Dynkin diagrams as can be found in Figure 2.1.

First, this naming is directly linked to the classification of similarity classes of reduced irreducible affine root systems. Second, it relies on the classification of similarity classes of reduced irreducible finite root systems. This is possible since the similarity class (or type if you want) of the gradient root system $D(R)$ is an invariant for the similarity class of the affine root system R.

Lemma 3.5.11. *If $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ are similar affine root systems, then $D(R)$ and $D(R')$ are similar gradient root systems.*

Proof. Let V (resp. V') be the space of translations of the affine Euclidean space E (resp. E'), and put c (resp. c') for the constant one function on E (resp. E'). Choose an origin $x \in E$ (resp. $y \in E'$), then by Proposition 3.1.12 we can identify $\widehat{E}$ (resp. $\widehat{E'}$) with $V \oplus \mathbb{R}c$ (resp. $V' \oplus \mathbb{R}c'$) with respect to the choice of origin. Assume that $T : V \oplus \mathbb{R}c \to V' \oplus \mathbb{R}c'$ realizes the similarity between R and R', and let $p_{V'} : V' \oplus \mathbb{R}c' \to V'$ be the projection onto V' along the direct sum. Then $t := p_{V'} \circ T|_V : V \to V'$ is a linear isomorphism that maps $D(R)$ onto $D(R')$ by Corollary 3.1.13. Furthermore, it follows directly from the properties of T that $(t(\alpha)^\vee, t(\beta))_{V'} = (\alpha^\vee, \beta)_V$ for all $\alpha, \beta \in D(R)$. □

If the reduced irreducible affine root system R is of type X^j_l, then R is similar to exactly one of the reduced irreducible affine root systems that is stated in (ii) of Proposition 3.5.10. Then Lemma 3.5.11, Proposition 3.5.5, Corollary 3.5.7 and the proof of Proposition 3.5.10 show that $D(R)$ is of type X_l, except if R is of type B^t_2 then $D(R)$ is of type C_2. The latter exception has to do with the fact that the finite Dynkin diagram of type C_2 can also be considered as being of type B_2, since it also naturally fits into the family B_l as the rank 2 Dynkin diagram. However the affine Dynkin diagram of type C^u_2 (resp. B^t_2) fits naturally into the family C^u_l (resp. B^t_l) as the rank 2 Dynkin diagram. For the sake of the classification of finite Dynkin diagrams we made the choice of considering the finite Dynkin diagram of type C_2 as part of the C family.

The final ingredient of the naming of a type of affine Dynkin diagram is the untwised, twisted or mixed type of the affine root system. By definition these types are similarity invariants of reduced irreducible affine root systems. If R is of type X_l^j, then $j \in \{u, t, m\}$. If $j = u$, then R is of untwisted type. If $j = t$, then R is of twisted type. If $j = m$, then R is of mixed type. On the other hand, if R is a reduced irreducible affine root system of untwisted, twisted or mixed type respectively with $D(R)$ of type X_l but not C_2, then R is of type X_l^j with j equal to u, t or m respectively. If $D(R)$ is of type C_2 and R is of untwisted (resp. twisted) type, then R is of type C_2^u (resp. B_2^t).

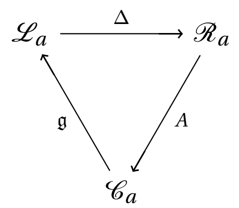
3.5.4 A new commutative triangle

We end this Chapter relating reduced irreducible affine root systems back to affine Lie algebras and real roots. This will gives us a canonical bijection between affine Lie algebras, reduced irreducible affine root systems and affine Cartan matrices up to the appropriate equivalent relations in the form of a commuting triangle. Furthermore, this will prove that every affine Lie algebra up to isomorphism is uniquely determined by its set of real roots up to similarity.

Proposition 3.5.10 (i) together with the classification of affine Cartan matrices from Figure 2.1 gives us a classification of reduced irreducible affine root systems. Together with Theorem 2.2.10 we obtain the following canonical bijections between the reduced irreducible affine root systems up to similarity $\mathscr{R}_a$, the affine Cartan matrices up to simultaneous permutations of rows and columns $\mathscr{C}_a$ and the isomorphism classes $\mathscr{L}_a$ of Lie algebras isomorphic to an affine Lie algebra

$$\mathscr{R}_a \xrightarrow{A} \mathscr{C}_a \xrightarrow{\mathfrak{g}} \mathscr{L}_a.$$

Define $\Delta := (\mathfrak{g} \circ A)^{-1} : \mathscr{L}_a \to \mathscr{R}_a$, then we obtain the commutative diagram



(3.5.4)

Finally, using the map Δ we want to show that for an affine Lie algebra its set of real roots is its naturally associated affine root system. So consider $\overline{\mathfrak{g}'} \in \mathscr{L}_a$, then there exists an affine Cartan matrix A such that $\overline{\mathfrak{g}'} = \mathfrak{g}(\overline{A})$. Let the affine Lie algebra $\mathfrak{g} := \mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)$ be a representative where of $\overline{\mathfrak{g}'}$. We consider $\mathfrak{g}$ as in the context of Section 2.3. So let W denote the Weyl group of $\mathfrak{g}$, let $(.,.)$ denote the normalized invariant form that is defined on $\mathfrak{h}^*$ by 2.3.1, let $\Pi = \{\alpha_0, \dots, \alpha_l\}$ and put $\mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ with $\mathring{\mathfrak{h}}_{\mathbb{R}}^* = \bigoplus_{i=1}^l \mathbb{R}\alpha_i$ and $\delta = \sum_{i=0}^l a_i \alpha_i$ such that the vector $(a_0, \dots, a_l)$ is in the kernel of A and has positive relatively prime integer coordinates. Recall that $(.,.)$ is positive definite on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ and vanishes on $\mathbb{R}\delta$.

By (2.1.11) we observe that

$$A = \left(2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{0 \le i,j \le l}.$$

Hence by definition of $\Delta : \mathscr{L}_a \to \mathscr{R}_a$ we must have that Π corresponds to the basis of a reduced irreducible affine root system in $\mathfrak{h}_0^* = \mathring{\mathfrak{h}}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ with bilinear form $(.,.)$. Consider $\mathfrak{h}_0^*$ as the space of affine linear function on the affine space $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ as in (i) of Example 3.2.3. Since $\Pi \subset \mathfrak{h}_0^*$ is a basis of a reduced irreducible affine root system R, (iii) of Theorem 3.3.16 tells us that the affine Weyl group $W(R)$ is generated by the simple reflections w_{α_i} in $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$. Consider $W(R)$ in $\mathrm{GL}_{\mathbb{R},c}(\mathfrak{h}_0^*)$ using

Proposition 3.1.22, then for $\beta \in \mathfrak{h}_0^*$ we have

$$w_{\alpha_i}(\beta) = \beta - 2\frac{(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i = r_i(\beta) \tag{3.5.5}$$

where $r_i \in W$ is the fundamental reflection of $\mathfrak{h}^*$ generated by $\alpha_i \in \Pi$. Now W is generated by the fundamental reflections r_i for $i = 0, \ldots, l$ and the action of W on $\mathfrak{h}_0^*$ is faithful, so we can identify W with $W(R)$ using the identification of their generators in (3.5.5). By Proposition 3.4.2 we have that for each $a \in R$ there exists $\alpha_i \in \Pi$ and $w \in W(R)$ such that $a = w(\alpha_i)$ and R is $W(R)$-invariant. But Δ^{re} is a W-invariant subset of $\mathfrak{h}_0^*$ and for each $\alpha \in \Delta^{re}$ there exists $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$, so we must have $R = \Delta^{re}$. In other words,

$$\Delta(\overline{\mathfrak{g}(A, \mathfrak{h}, \Pi, \Pi^\vee)}) = \overline{\Delta^{re}}. \tag{3.5.6}$$

In particular, every affine Lie algebra up to isomorphism is uniquely determined by its set of real roots up similarity. We conclude that we can classify affine Lie algebras not only according to their affine Cartan matrix, but also according to their *associated affine root system* which corresponds with the set of real roots of the affine Lie algebra.

Summary (in Dutch)

Het is een bekend wiskundig feit dat simpele Lie-algebras, gereduceerde irreduciebele wortelsystemen en eindige Cartan-matrices op de juiste isomorfie-equivalenties na in bijectieve correspondentie met elkaar zijn. In deze scriptie trachten we op een analoge wijze zo een bijectieve correspondentie te beschrijven tussen affiene Lie-algebras, gereduceerde irreduciebele affiene wortelsystemen en affiene Cartan-matrices. Aangezien de bijectie tussen affiene Lie-algebras en affiene Cartan-matrices al duidelijk is, beslaat het merendeel van dit manuscript het bestuderen van affiene wortelsystemen met het doel ze te relateren aan affiene Cartan-matrices. Om enig inzicht te geven in affiene wortelsystemen, zullen we hier de iets eenvoudigere 'gewone' wortelsystemen bespreken. Aangezien wortelsystemen veel te maken hebben met spiegelingen en spiegelsymmetrieën zullen we daar eerst iets over vertellen.

Beschouw een eindig-dimensionale reële vectorruimte V met een inproduct $(.,.)$. Het inproduct $(.,.)$ geeft ons de mogelijkheid om te praten over lengtes van en hoeken tussen vectoren in V. We brengen in herinnering dat we met een *hypervlak* in V een lineaire deelruimte van V bedoelen van één dimensie lager dan V zelf.

We kunnen de ruimte V orthogonaal spiegelen in een hypervlak H in V. Dit leidt tot een lineaire transformatie van V. Een (*orthogonale*) *spiegeling* in V is een lineair automorfisme van V dat de punten van een hypervlak H in V op zijn plaats houdt en een vector α die loodrecht op H staat naar $-\alpha$ stuurt. Aangezien V wordt opgespannen door het hypervlak H en de vector α, legt dit een spiegeling compleet vast vanwege lineariteit (zie Figuur 4.1). We noteren deze spiegeling met w_α.

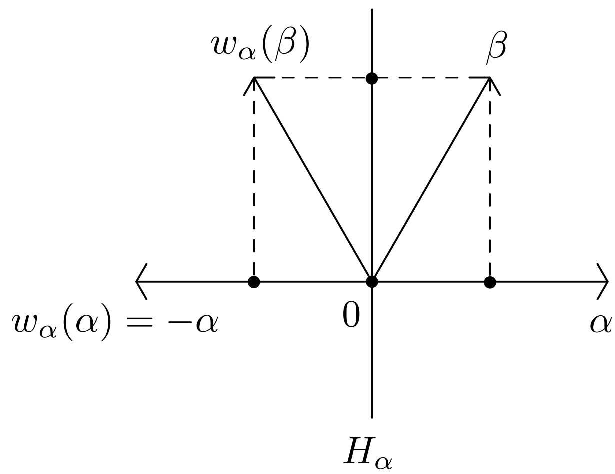


Figure 4.1: Een voorstelling van een spiegeling w_α in een 2-dimensionale ruimte.

Er bestaat een eenvoudige formule voor een spiegeling w_α, namelijk

$$w_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$$

voor alle $v \in V$. Het is gemakkelijk in te zien dat in de bovenstaande forumule inderdaad de punten van het hypervlak

$$H_\alpha = \{w \in V : (w,\alpha) = 0\}$$

loodrecht op α op zijn plaats worden gehouden, en dat α naar $-\alpha$ wordt gestuurd. Intuïtief kan H_α gezien worden als een 'dubbelzijdige spiegel' in V, waarbij w_α het spiegelbeeld geeft van elke punt in V ten opzichte van H_α (zie Figuur 4.1). Verder kan worden opgemerkt dat als je dezelfde spiegeling twee keer achter elkaar uitvoert dan worden alle punten van V op zijn plaats gelaten (vergelijk dit met het feit dat jij zelf het spiegelbeeld van jouw eigen spiegelbeeld bent).

We noemen een eindige ondergroep van de groep van lineaire automorfismen van V die wordt voortgebracht door spiegelingen een *eindige reflectiegroep*. In deze scriptie beschouwen we eindige refectiegroepen die aan een extra symmetrie-eis voldoen. Een *rooster* L in V is een abelse ondergroep van V van de vorm

$$\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_l$$

met $\{v_1,\ldots,v_l\}$ een basis van V. In Figuur 4.2 staat een voorbeeld van een rooster in een 2-dimensionale ruimte afgebeeld. We zeggen dat een eindige reflectiegroep W *kristallografisch* is als W een rooster invariant laat in V. Deze naamgeving komt uit de kristallografie waar symmetriegroepen van kristalroosters een belangrijke rol spelen.

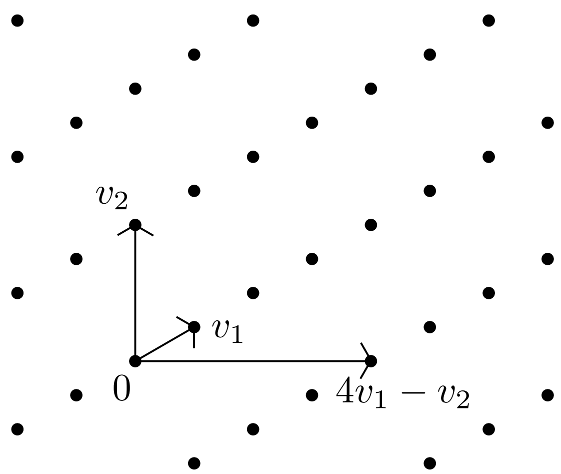


Figure 4.2: Een rooster $\mathbb{Z}v_1 + \mathbb{Z}v_2$.

Een *wortelsysteem* Δ is een verzameling vectoren in V die aan de volgende vier condities voldoet

(1) Δ is eindig, bevat 0 niet en spant V op;
(2) $\Delta \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ voor alle $\alpha \in \Delta$;
(3) $w_\alpha(\beta) \in \Delta$ voor alle $\beta \in \Delta$;
(4) $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ voor alle $\alpha,\beta \in \Delta$.

De groep W_0 voortgebracht door de spiegelingen w_α voor $\alpha \in \Delta$ is een kristallografische eindige reflectiegroep genaamd de *Weyl-groep* van Δ. Het blijkt dat elke kristallografische eindige reflectiegroep een Weyl-groep van een wortelsysteem is. Echter er bestaan structureel verschillende wortelsystemen met dezelfde Weyl-groep.

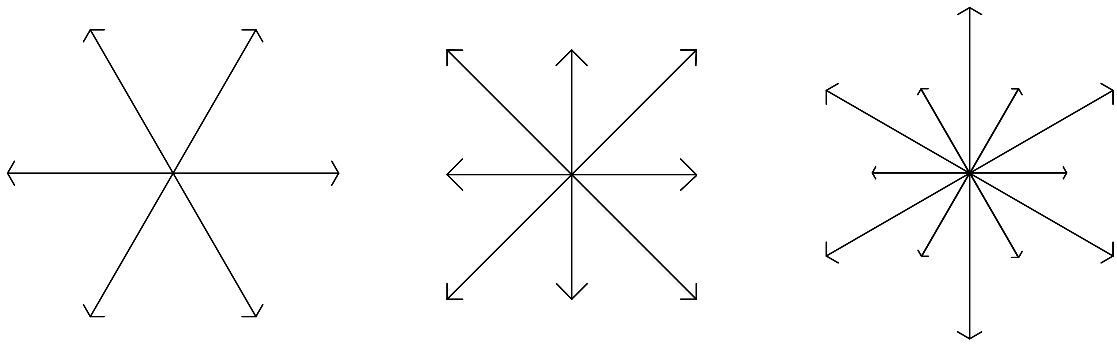


Figure 4.3: Drie wortelsystemen in een 2-dimensionale ruimte.

Een wortelsysteem Δ kan gezien worden als de kleinste verzameling vectoren in V die V opspant, zodanig dat Δ invariant is onder de kristallografische eindige reflectiegroep W_0, en dat de spiegelingen w_α voor $\alpha \in \Delta$ de groep W_0 voortbrengen. Dit impliceert dat Δ een aantal spiegelsymmetrieën heeft. In Figuur 4.3 staan drie wortelsystemen afgebeeld in een 2-dimensionale ruimte V, waarbij elke pijl een vector $\alpha \in \Delta$ voorstelt. Het blijkt dat Δ te partitioneren is in twee verzamelingen Δ^+ en Δ^-, zodat zowel Δ^+ als Δ^- precies de normalen bevat van de hypervlakken waarin gespiegeld wordt binnen W_0. Bovendien geldt dan $\Delta^+ = -\Delta^-$.

Indien we een hypervlak in V transleren met een vector krijgen we een deelverzameling van V die mogelijk niet de oorsprong bevat en geen lineaire deelruimte van V is. We noemen zo een verschoven hypervlak een *affien hypervlak*. Zo kan het hypervlak

$$H_\alpha = \{w \in V : (w, \alpha) = 0\}$$

loodrecht op $\alpha \in V$ na een translatie met een vector in V beschreven worden als het affiene hypervlak

$$H_{\alpha,k} := \{w \in V : (w, \alpha) = k\}$$

voor een zekere $k \in \mathbb{R}$. Het is nog steeds mogelijk om orthogonaal te spiegelen in een *affien hypervlak $H_{\alpha,k}$*. Bovendien wordt de formule voor zo een *affiene spiegeling $w_{\alpha,k}$* gegeven door slechts een kleine aanpassing te maken in de formule voor de spiegeling w_α:

$$w_{\alpha,k}(v) = v - 2\frac{(v,\alpha) - k}{(\alpha,\alpha)}\alpha$$

voor $v \in V$. Dit is geen lineaire transformatie van V maar een zogeheten *affiene transformatie*. Affiene transformaties bestaan in het algemeen uit een lineaire transformatie samengesteld met een translatie.

Beschouw nu een wortelsysteem Δ. De groep W_a van affiene transformaties van V die wordt voortgebracht door de affiene spiegelingen $w_{\alpha,k}$ met $\alpha \in \Delta$ en $k \in \mathbb{Z}$ heet de *affiene Weyl-groep* van Δ. Een affiene Weyl-groep is groep van oneindige orde. Voor het linker wortelsysteem Δ in Figuur 4.3 hebben we in Figuur 4.4 de hypervlakken $H_{\alpha,k}$ met $\alpha \in \Delta$ en $k \in \mathbb{Z}$ afgebeeld. De affiene Weyl-groep van Δ permuteert deze hypervlakken.

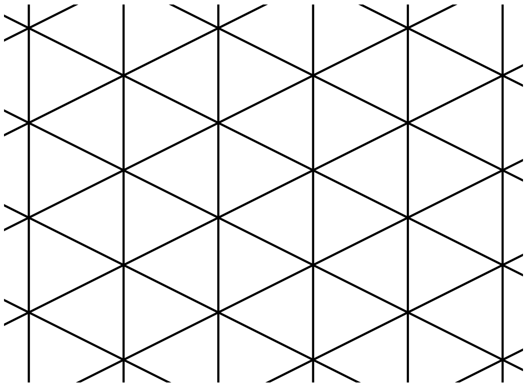


Figure 4.4: De affiene hypervlakken gerelateerd aan een affiene Weyl-groep.

In deze scriptie bestuderen we onder andere Macdonald's generalisatie van de axiomatische definitie van een wortelsysteem met een aantal noodzakelijke technische modificaties. Dit noemen we een *affien wortelsysteem.* Analoog aan de situatie bij wortelsystemen heeft elk affiene wortelsysteem R een gerelateerde affiene Weyl-groep W_a. Een affien wortelsysteem R kan dan ook, analoog aan het geval van een wortelsysteem, gezien worden als een W_a-invariante verzameling normalen van de affiene hypervlakken waarin gespiegeld wordt binnen de gerelateerde affiene Weyl-groep W_a. Om affiene wortelsystemen te bestuderen blijkt het makkelijk te zijn om te werken in zogenaamde *affiene ruimtes.* Dit zijn vectorruimtes waarbij we 'vergeten' zijn waar de oorsprong zich bevindt. Uiteindelijk relateren we affiene wortelsystemen aan de hand van speciale generatoren aan affiene Cartan-matrices en laten we zien hoe een affiene Lie-algebra op natuurlijke wijze een geassocieerd affien wortelsysteem heeft.

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