Perfect Matchings and Pfaffian Orientation

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Bachelor Thesis

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Abstract
A graph can be covered with tiles consisting of two vertices and one edge connecting the vertices. If it is possible to cover a graph by those tiles such that each site of the graph is occupied by exactly one tile, the obtained configuration is called a perfect matching. Not all graphs have perfect matchings, but when a perfect matching exists on a graph, the question is how many such configurations does this graph have? For a certain class of graphs, a method for finding the number of perfect matchings is used involving so-called Pfaffians. Some graphs can be oriented in a way that, when representing the graph by its adjacency matrix, the Pfaffian of the matrix enumerates the perfect matchings of the graph. The orientation is called a Pfaffian orientation. The physicist Kasteleyn has shown that every planar graph has a Pfaffian orientation. The method of constructing such orientation is applied to a rectangular lattice of dimension $m \times n$ and a similar approach has been used to find the number of perfect matching of a rectangular lattice with periodic boundary conditions. Coverings of lattice graphs have applications in the field of condensed matter physics and statistical physics.

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My search for a subject for my bachelor thesis started at the Institute for Theoretical Physics, since I was very interested in physics that time and my wish was to study a mathematical subject with applications in the physics world. However, string theory and quantum field theory, maybe a more common or fashionable combination of mathematics and physics, did not appeal to me. Luckily, the coordinator for Mathematical Physics mentioned the Dimer Problem and advised me to visit the department of Statistical Physics. Here, I came across the dimer problem and it turned out to be a graph theoretical problem. Since I haven’t taken any course on discrete mathematics and graph theory I needed to start from the very beginning and must become familiar with graphs and matchings.

My supervisor provided me with the article of Kasteleyn on dimers and Pfaffians. Under his guidance I studied the theory of perfect matchings and Pfaffian orientations. And now, after a long time and the usual writer’s block, the result is this thesis. Even more, my interest in graph theory has become larger with time.

I would like to express my gratitude to Marianne Hoogeveen and Timo Kluck, whom I forced to read (parts of) this thesis, their suggestions and help have been useful and sometimes even motivated me, when I just did not feel like writing anymore. They assured me that this is usual behaviour for students who are writing reports.

I would like to thank my supervisor, Dion Gijswijt, for his guidance and instructions to improve my understanding in the theory and my style of writing. His enthusiastic and patient way of explaining has been most helpful.

I do not think this will be the last time that I encounter Graph Theory.

Jeanette Nguyen
Amsterdam, May 15, 2008
Chapter 1

Introduction

The problem of enumerating perfect matchings has applications in both physics and chemistry. Perfect matchings in physics are known as dimer configurations and in chemistry the term Kekule structures is used. The components of matter, the molecules and atoms, are arranged on a crystal, a regular lattice. The arrangement of the particles on a lattice, in particular the number of ways the particles can be arranged, is of great interest in physics, prompting physicists such as Kasteleyn to explore the theory of graphs and matchings.

A dimer is a molecule consisting of two atoms linked by a bond. A dimer covering of a lattice is a collection of dimers that covers all the vertices of the lattice exactly once.

The most occurring regular lattices are the rectangular lattice, the triangular lattice and the hexagonal lattice, also known as the honeycomb lattice. Kasteleyn \[4\] and Fisher\[2\] have found the number of perfect matchings on a planar rectangular lattice, using tools from linear algebra such as determinants and Pfaffians of a matrix. They have also determined the behaviour of the number of perfect matchings when the dimensions of the lattice go to infinity.

In this thesis the results of Kasteleyn \[3\] of perfect matchings on some graphs, especially a rectangular lattice graph, are treated.

First, in Chapter 2 the concept of a graph is introduced and examples of some kinds of graphs are given. Then matchings and perfect matchings of a graph are defined.

In Chapter 3 the tools to enumerate perfect matchings in graphs will be explored. A graph will be represented by its adjacency matrix and the Pfaffian of a matrix is defined. We find out that some graphs have a Pfaffian orientation. When such an orientation is found, enumerating the perfect matchings can be done by computing the determinant.
Chapter 4 treats perfect matchings on a planar rectangular lattice. The method of the previous chapter will be used to find an expression for the number of perfect matchings of an $m \times n$ lattice graph. Also, the behaviour of this expression will be examined when letting the dimension of the graph go to infinity. Then periodic boundary conditions are introduced. It appears that a slightly different approach is required for counting the matchings. Finally, the triangular lattice and the hexagonal lattice are briefly discussed and the known results [3], [10] of these lattices are compared with the results of the rectangular lattice.
Chapter 2
Graphs and Matchings

To treat the theory of perfect matchings on various lattices, an introduction into graph theory is required. In this chapter a definition of a graph is given and examples of various graphs are shown. Furthermore, matchings and especially perfect matchings are defined. Hall’s Matching Theorem and Tutte’s 1-factor Theorem are briefly discussed.

2.1 What is a graph?

Definition 2.1. A graph $G$ consists of a set of vertices $V$ and a set of edges $E$ connecting pairs of vertices.

The order of a graph, $n(G)$, is the number of vertices of a graph $G$. The size of a graph, $e(G)$, is the number of edges of a graph $G$.

Two edges are adjacent if they have exactly one common vertex.

There are different types of graphs and they can be categorised according to their properties. Here, some kinds of graphs are briefly treated.

![Graphs](image)

Figure 2.1: The complete graph $K_5$, a bipartite graph, a planar graph and a tree.
Example 2.2. A complete graph $K_n$ is a graph whose vertices are pairwise connected by an edge.

Example 2.3. A tree is a connected graph without cycles.

Definition 2.4. A graph $G$ is bipartite if $V(G)$ is the union of two disjoint sets of $G$. Thus, a vertex belonging to one of the sets can only have edges connecting to vertices of the other set of vertices.

Definition 2.5. A graph $G$ is planar if it can be drawn without crossings of edges on a plane surface. Such drawing of a graph $G$ is called a planar embedding of $G$. A plane graph is defined by a mapping of the planar graph from every vertex into a position on $\mathbb{R}^2$ and from every edge into a curve with two extreme points, coinciding with the end vertices. All curves are disjoint except on common endpoints of the graph.

The edges of a plane graph divides the plane into regions called faces of the plane graph. A finite plane graph has one unbounded face, called the infinite face.

The complete graphs $K_{3,3}$ and $K_5$ are examples of non-planar graphs.

For plane graphs the formula of Euler holds.

**Theorem 2.6 (Euler).** If $G$ is a connected plane graph with $e$ the number of edges of $G$, $n$ the number of vertices and $f$ the number of faces, then the following equation holds:

$$n - e + f = 2.$$  \hspace{1cm} (2.1)

**Proof.** For the most simple plane graph consisting of just one vertex the equation holds, since $e = 0$, $n = 1$ and $f = 1$ (the infinite face).

![Figure 2.2](image-url)

Figure 2.2: A graph with a vertex with degree 1, a connected graph $G$ without a vertex of degree 1 and edge $u$, graph $G$ without edge $u$.

Consider now a connected plane graph $G$, an edge $u$ and use induction on $u$. When subtracting edge $u$ from $G$ a plane graph $G'$ is obtained with $n'$ vertices, $e'$ edges and $f'$ faces. Two cases can be distinguished: Either $u$ saturates a vertex $v$ with degree 1 or $u$ does not saturate a vertex with
degree 1. In the latter case, \( u \) is an edge contained in a cycle \( C \).

The first case when removing \( u \) and vertex \( v \) will give \( n' - e' + f' = n - 1 - (e - 1) + f = n - e + f = 2 \), since both sides of edge \( u \) is the same face as illustrated in Figure 2.2.

The second case when removing \( u \) yields \( n' - e' + f' = n - (e - 1) + (f - 1) = n - e + f = 2 \). Hence, Euler’s formula is proved to be true.

\[ \square \]

![Figure 2.3: A lattice graph and a directed graph](image)

**Example 2.7.** A lattice graph is a graph where the points of a lattice are the vertices and the edges of the graph are the line segments of the lattice.

The previous graphs are all examples of graphs consisting of unordered pairs of vertices. However, an orientation can be adopted on a graph. This means that the edges of the graph have a direction assigned to them. A graph with an orientation is called a directed graph and consists of a set of vertices \( V \), and a set \( E \) of ordered pairs of these vertices, called edges.

### 2.2 Matchings and Perfect Matchings

**Definition 2.8.** A matching \( M \) in a graph \( G \) is a set of non-loop edges of the graph such that each vertex of the graph is contained in at most one edge. The vertices incident to the edges in \( M \) are called saturated by \( M \). A perfect matching in a graph is a matching saturating all vertices.

**Definition 2.9.** A maximum matching is a matching of maximum size among all matchings in the graph.

A maximal matching is a matching that cannot be enlarged by adding an edge.

A perfect matching is a largest possible matching in a graph. While not all graphs have a perfect matching, all graphs do have maximal matchings. It
Figure 2.4: A graph $G$ and a perfect matching $M$ of $G$.

is obvious that a graph $G$ of odd order does not have a perfect matching.

An $M$-alternating path is a path that alternates between the edges in a matching $M$ and edges not in $M$.

An $M$-augmenting path is an $M$-alternating path that begins and ends with a vertex not saturated by matching $M$.

Figure 2.5: An $M$-alternating path and an $M$-augmenting path.

Recall the symmetric difference of two sets $M \triangle N$. This can also be written as $M \triangle N = (M - N) \cup (N - M)$. When applying the symmetric difference to graphs the following definition is in order.

**Definition 2.10.** The symmetric difference $G \triangle H$ of the graphs $G$ and $H$ is the subgraph of $G \cup H$ whose edges are the edges of $G \cup H$ appearing in exactly one of $G$ and $H$.

Figure 2.6: For example, the symmetric difference of matchings $M$ and $N$, where the black lines are the edges of matching $M$ and the green lines are the edges belonging to matching $N$, consists of a path of length 3 and a cycle of length 6. The common edge $e$ is not included in the symmetric difference.

**Lemma 2.11.** The components of the graph induced by the symmetric difference of two matchings are either paths or even cycles.
Proof. Consider matchings $M$ and $N$ of a graph $G$ with vertex set $V$ and their symmetric difference $M \triangle N$. For a vertex there are four cases: either the vertex has no incident edge or an incident edge from the matching $M$ or $N$ or both, since $M$ and $N$ are matchings. A vertex in $V(M \triangle N)$ has at most two incident edges. Thus, every component of $(V(M \triangle N), M \triangle N)$ are paths and cycles with edges belonging alternating to $M - N$ and $N - M$, which means that the cycles are even. \hfill \qedsymbol

The symmetric difference can be a useful tool for investigating matchings in a graph.

**Example 2.12.** A tree $T(V, E)$ has at most one perfect matching.

*Proof.* Consider a tree with even order and assume the tree has two perfect matchings $M$ and $N$. Since perfect matchings $M$ and $N$ saturate all vertices of the graph, $(V, M \triangle N)$ consists of even cycles and isolated vertices. However, a tree does not contain a cycle, therefore neither does $(V, M \triangle N)$. This means that $M \triangle N = \emptyset$, which means $M = N$. \hfill \qedsymbol

**Definition 2.13.** A $k$-factor of a graph $G$ is a spanning $k$-regular subgraph of $G$. A subgraph $H$ of graph $G$ is a spanning subgraph if $V(H) = V(G)$.

For instance, a 1-factor of a graph is a subgraph consisting of disjoint edges. This is similar to a perfect matching, the only difference being, that a perfect matching is a set of edges, whereas a 1-factor is a subgraph.

An *odd component* of a graph $G$ is a component of odd order. The number of odd components of $G$ is denoted by $o(G)$.

![Figure 2.7](image-url)

*Figure 2.7:* The square is a subset of the graph $G$ called $U$, the circles are the odd components of $G - U$ and the triangles are the even components of $G - U$.

There is a way to check whether a graph contains a perfect matching or a 1-factor. Suppose a graph $G$ with vertex set $V$ has a perfect matching and let $U \subseteq V$. Having a perfect matching implies that every odd component of $G - U$ must have a matching edge connecting to $U$ (Figure 2.7). On the
other side if this is the case, every vertex in $U$ is connected to at most one of these edges. Therefore, $G - U$ can have at most $|U|$ odd components. Tutte proved that also the converse holds.

**Theorem 2.14 (Tutte’s 1-factor Theorem).** A graph $G$ has a perfect matching if and only if $o(G - U) \leq |U|$ for every $U \subseteq V$.

A proof of Tutte’s 1-factor Theorem is provided by Tutte[1947] and Lovasz[1975].

Now consider a bipartite graph $G$ with bipartition $V = X \cup Y$. If a matching saturates $X$, then for every $S \subseteq X$, there must be at least $|S|$ vertices in $Y$ that have a neighbour in $S$. This is necessary since the neighbours of the vertices in $S$ must be chosen from $Y$. Define $N(S)$ to be the set of vertices in $Y$ that have a neighbour in $S$. Hall’s Matching Theorem can be formally stated as:

**Theorem 2.15 (Hall’s Matching Theorem).** A bipartite graph $G$ with bipartition $X, Y$ has a matching that saturates $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

![Figure 2.8: A bipartite graph with bipartition $|X| = |Y|$ without a perfect matching.](image)

The proof for Hall’s Matching Theorem is omitted.

When the sets of bipartition have the same size, $|X| = |Y|$, Hall’s Matching Theorem is also known as the Marriage Theorem. Back in the old days when considering a group of men and an equal number of women, one wanted to know if there is a possibility to match the men with the women with regard to compatibility. If, for some $k \in \mathbb{N}$, every man is compatible with exactly $k$ women, and every woman is compatible with $k$ men, a perfect matching must exist.

It was Frobenius[1917] who originally proved the Marriage Theorem.
Chapter 3

Pfaffians

One way to count the perfect matchings of a graph is to manually count all the possible configurations. However, this task is complicated if the size of the graph becomes larger. In fact, determining the number of perfect matchings of a graph is an NP-hard problem [9]. Hence, there is probably no efficient algorithm to compute the number of perfect matchings in general graphs.

However, for planar graphs an efficient algorithm involving so-called Pfaffians can be used to enumerate the perfect matchings of the graph. First an orientation is adopted on the graph and the graph is represented by its adjacency matrix. Then the Pfaffian of a matrix is introduced. A graph is directed in such a way to insure that the Pfaffian of the adjacency matrix counts the perfect matchings of the graph. If the latter is the case, the orientation is called Pfaffian. When a Pfaffian orientation is known, counting the perfect matchings can be reduced to just computing the determinant of the adjacency matrix. Thus a polynomial algorithm is found to enumerate the perfect matchings of a special class of graphs.

3.1 The adjacency Matrix and the Pfaffian

A graph $G$ with vertex set $V = \{v_1, \ldots, v_n\}$ can be represented by a matrix in a way that the entry $a_{ij}$ of the matrix is the number of edges in $G$ with endpoints $\{v_i, v_j\}$. The matrix is called the adjacency matrix and is denoted by $A(G)$. Note that the adjacency matrix is symmetric since $a_{ij} = a_{ji}$. For a graph $H$ without multiple edges between pairs of vertices the adjacency matrix $A(H)$ has the form:

$$A(H)_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E(H) \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not-adjacent} \end{cases}. \quad (3.1)$$
Figure 3.1: A graph $G$ with labelled vertices.

**Example 3.1.** Let graph $G$ be the graph as illustrated in Figure 3.1. Using the labelling of the vertices the adjacency matrix has the form:

$$A(G) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$  \hspace{1cm} (3.2)

A perfect matching will have exactly one entry 1 in each row and column.

Consider a graph $G$ with vertex set $V(G) = \{v_1, \ldots, v_{2n}\}$. Now assign to each edge $e$ a variable $x_e$ and let an edge have a direction, i.e. apply an orientation to the graph. Define the matrix $B$ of the oriented graph $G$ as follows:

$$B(x) = (b_{ij})_{2n \times 2n},$$

where $b_{ij} = \begin{cases}
  x_e, & \text{if } e = (v_i, v_j) \\
  -x_e, & \text{if } e = (v_j, v_i) \\
  0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent}
\end{cases}$. \hspace{1cm} (3.3)

Matrix $B$ is antisymmetric:

$$B(x) = \begin{pmatrix}
  0 & b_{1,2} & \cdots & b_{1,2n} \\
-b_{1,2} & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & b_{2n-1,2n} \\
-b_{1,2n} & \cdots & -b_{2n-1,2n} & 0
\end{pmatrix}.$$

From now on we will only consider graphs with an even number of vertices, since an odd number of vertices excludes the existence of a perfect matching. The vertices of the graph are labelled by $1, 2, \cdots, 2n$. A partition of the set
Define \( \Pi \) to be the set of all those partitions. Thus, a perfect matching of the graph is an element of \( \Pi \).

The **Pfaffian** \( \text{Pf}(B) \) of matrix \( B(x) \) is defined by:

\[
\text{Pf}(B) = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{e \in \Pi} b_e,
\]

where \( b_e = b_{ij} \) if \( e = \{i, j\} \) with \( i < j \) and the sign \( \text{sgn}(\pi) \) of a partition \( \pi = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\} \) is defined by:

\[
\text{sgn}(\pi) := \text{sgn}
\begin{pmatrix}
1 & 2 & 3 & \cdots & 2n \\
i_1 & j_1 & i_2 & \cdots & j_n
\end{pmatrix}
\prod_{k=1}^{n} \sigma(i_k, j_k),
\]

where \( \sigma(i, j) = \begin{cases} 
1, & \text{if } i < j \\
-1, & \text{if } i > j \\
0 & \text{otherwise}
\end{cases} \).

The sign of partition \( \pi \) is well-defined. The order of two numbers forming a pair is arbitrary, as is the order of the pairs. Hence, when interchanging two partners, say \( i_3j_3 \to j_3i_3 \), the sign of the permutation will flip but one matrix element will also gain an opposite sign. Nett, the sign of the perfect matching will not change. Switching two blocks of partners will leave the sign of the permutation unchanged since the only consequence is the interchange of two matrix elements.

Define the sign of a perfect matching to be the sign corresponding to the term in the Pfaffian.

**Lemma 3.2.** If \( B \) is an antisymmetric matrix, then \( \det(B) = (\text{Pf}(B))^2 \)

**Proof.** An antisymmetric matrix \( B \) of dimension \( n \) satisfies \( B = -B^T \), where \( B^T \) is the transposed matrix of \( B \). Taking the determinant of the matrix leads to:

\[
\det(B) = \det(B^T) = \det(-B) = (-1)^n \det(B).
\]

If \( B \) is a matrix of odd order, it says \( \det(B) = -\det(B) \). Hence, the determinant vanishes.

Consider the case where \( B \) is of order \( 2n \):

Use the definition of the determinant of a matrix:

\[
\det(B) = \sum_{\pi \in S_{2n}} \text{sgn}(\pi) \prod_{i=1}^{2n} b_{i,\pi(i)} \cdot \sigma(i, \pi(i)),
\]
where \( b_{i,\pi(i)} \) is the variable assigned to edge \( \{i, j\} \) and \( \sigma(i, j) = \begin{cases} 1, & \text{if } i < j \\ -1, & \text{if } i > j \\ 0, & \text{otherwise} \end{cases} \).

The Pfaffian squared has the form:

\[
\text{Pf}(B)^2 = \left( \sum_{M \in \Pi} \text{sgn}(M) \prod_{e \in M} b_e \right) \left( \sum_{N \in \Pi} \text{sgn}(N) \prod_{f \in N} b_f \right) = \sum_{M,N \in \Pi} \text{sgn}(M) \cdot \text{sgn}(N) \prod_{e \in M} b_e \prod_{f \in N} b_f. \tag{3.9}
\]

Now, to prove the lemma one must show that a term in the Pfaffian squared has a one to one correspondence with a term in the determinant. The remaining terms of the determinant will have to cancel each other.

Consider the expression for the determinant of \( B \). The permutations \( \pi \in S_{2n} \) can be decomposed into even and odd cycles. Suppose the decomposition of a term \( \pi \) contains an odd cycle \( (i_1 i_2 \cdots i_{2k+1}) \), where \( k \geq 1 \). Consider the odd cycle in the decomposition containing the smallest element. For each such term there is another term in the determinant containing this cycle in opposite direction \( (i_2 i_3 \cdots i_1) \) and the rest of the decomposition remains the same, say \( \pi' \). Then \( \prod_{e=1}^{2n} b_{i,\pi(e)} = -\prod_{e=1}^{2n} b_{i,\pi'(e)} \) and since \( \pi \) and \( \pi' \) consists of cycles of the same length \( \text{sgn}(\pi) = \text{sgn}(\pi') \). Hence, all terms that contain at least one odd cycle in the decomposition cancel each other in pairs.

Now one must show that the remaining terms of \( \text{det}(B) \), the permutations whose decomposition consists only of even cycles, have a one to one correspondence with the terms of \( \text{Pf}(B)^2 \). Each term of the Pfaffian squared has the form:

\[
\text{sgn}(M) \cdot \text{sgn}(N) \prod_{e \in M} b_e \prod_{f \in N} b_f, \tag{3.10}
\]

where \( M = \{\{i_1 j_1\} \cdots \{i_n j_n\}\} \) and \( N = \{\{i'_1 j'_1\} \cdots \{i'_n j'_n\}\} \). Choose \( i_1 \) to be any number in the set \( \{1, 2, \cdots, 2n\} \), now \( j_1 \) is fixed since it must be the partner of \( i_1 \). Let this line be an element of matching \( M \). The number \( j_1 \) occurs also in \( N \), so choose \( j_1 = j'_1 \). This time \( j'_1 \) is fixed, it can be either \( i_1 \) again or another number and setting this number \( j'_1 = i_2 \). This procedure can be carried on until \( j'_r \) is reached which is equal to \( i_1 \).

An even cycle is obtained with edges alternating between matching \( M \) and \( N \) as illustrated in Figure 3.2. So, \( M \cup N \) corresponds to evenly oriented cycles, these, in their turn, correspond to permutations with only even cycles.
The sign of $M$ is given by:

$$\text{sgn}(M) = \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \cdots & 2n \\ i_1 & j_1 & \cdots & j_n \end{array} \right) \prod_{k=1}^{n} \sigma(i_k, j_k),$$

(3.11)

and the sign of $N$ is:

$$\text{sgn}(N) = \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \cdots & 2n \\ j_1 & j'_1 & \cdots & j'_n \end{array} \right) \prod_{l=1}^{n} \sigma(j_l, j'_l).$$

(3.12)

The claim is that the permutation $\pi$ in the term of the determinant is equal to $\rho_{\pi} = \rho_{N}\rho_{M}^{-1}$.

$$\rho_{N}\rho_{M}^{-1} = \left( \begin{array}{cccc} 1 & 2 & \cdots & 2n \\ j_1 & j'_1 & \cdots & j'_n \end{array} \right) \left( \begin{array}{cccc} i_1 & j_1 & \cdots & j_n \\ 1 & 2 & \cdots & 2n \end{array} \right) = \left( \begin{array}{cccc} i_1 & j_1 & \cdots & j_n \\ j_1 & j'_1 & \cdots & j'_n \end{array} \right) = \rho_{\pi}.$$  

(3.13)

Since $\text{sgn}(\rho_{M}) = \text{sgn}(\rho_{M}^{-1})$ and $\sigma(i, j) \cdot \sigma(j, j') = \sigma(j, j')$, because $i \rightarrow j \rightarrow j'$, is $\text{sgn}(\pi) = \text{sgn}(M) \cdot \text{sgn}(N)$. Thus, proving the lemma.

$$\square$$

### 3.2 Pfaffian Orientation

The Pfaffian of a matrix can be used to count the number of perfect matchings of a graph. Let $G$ be a graph with $2n$ vertices and given an arbitrary orientation $\vec{G}$ of $G$. The adjacency matrix of $G$ is defined as follows:

$$A_s(\vec{G}) = (a_{ij})_{2n \times 2n},$$

where $a_{ij} = \begin{cases} 1, & \text{if } (u_i, u_j) \in E(\vec{G}), \\ -1, & \text{if } (u_j, u_i) \in E(\vec{G}), \\ 0, & \text{if otherwise.} \end{cases}$

(3.14)
This matrix is antisymmetric. Now recall the definition of the Pfaffian of a matrix. The terms of the Pfaffian of this adjacency matrix are equal to $-1, 1$ or $0$. Note that the nonzero terms of the Pfaffian have a one to one correspondence with the perfect matchings of the graph. To insure the Pfaffian of the matrix enumerates all perfect matchings with the right sign, an orientation must be adopted to the graph such that the corresponding terms of the Pfaffian are counted with the same sign. When this is the case, the orientation is called a Pfaffian orientation.

The question now is: when does a graph have a Pfaffian orientation and how does one construct such orientation? The remainder of this section is dedicated to Kasteleyn’s result that every planar graph has a Pfaffian orientation.

Consider a directed graph $G$ and let $C$ be any even undirected cycle in $G$. For any given routing around $C$, if $C$ has an even number of edges following the routing then $C$ has also an even number of edges with an orientation opposite of the routing. Now the following can be defined:

**Definition 3.3.** If $C$ is an even undirected cycle in $G$, then $C$ is **evenly oriented** if it has an even number of edges following the routing, otherwise $C$ is **oddly oriented**.

For a perfect matching $M$ of a graph $G$ the sign of $M$, $\text{sgn}(M)$ is defined as:

$$\text{sgn}(M) := \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \ldots & 2n \\ i_1 & j_1 & \ldots & j_n \end{array} \right) \prod_{k=1}^{n} \sigma(i_k, j_k),$$

for $M = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ where $\sigma(i, j) = \begin{cases} 1, & \text{if } i \rightarrow j \\ -1, & \text{if } j \rightarrow i \end{cases}$.

**Lemma 3.4.** Let $\vec{G}$ be an arbitrary orientation of a graph $G$ and let $M$ and $N$ be any two perfect matchings of $G$. Let $k$ denote the number of evenly oriented alternating cycles formed in $M \cup N$. Then $\text{sgn}(M) \cdot \text{sgn}(N) = (-1)^k$.

**Proof.** First we note that if the lemma holds for an orientation, then the lemma holds for all orientations. To see this consider an edge $e$. This edge $e$ can be situated in:
1. $e \notin (M \cup N)$
2. $e \in (M \cap N)$
3. $e \in E(M) \oplus E(N)$, $e$ is an edge in an alternating cycle.

When reversing the direction of the edge $e$ the following will occur:

- In the case of (1), $e$ is not part of either $M$ nor of $N$. There is no effect on the equation of the lemma.
For an edge $e$ to be in both $M$ and $N$, the sign of each matching will change, but the value of $k$ remains the same.

If, as in case (3), $e$ is a component of an alternating cycle, then the sign of matching $M$ will change, when reversing the direction of $e$, but $k$ changes by one.

Since an orientation can be chosen at will, orient the graph as follows: for a line $e$ in the case of (1) and (2): orient the line arbitrarily. If the line is a component of an alternating cycle, orient $e$ such that the cycle becomes directed, evenly or oddly either way.

Relabelling the vertices of $G$ affects its adjacency matrix $A(\vec{G})$ such that the rows and the columns of the matrix are permuted by a permutation $\pi$. In the expression of the Pfaffian of the matrix, the perfect matchings are multiplied by $\pi$. Therefore, this lemma remains invariant under relabelling of the vertices of the graph $G$.

Now label the vertices as follows:

For an edge $e \in (M \cap N)$ label such that the head of the line equals the tail plus one. For an edge $e \in E(M) \oplus E(N)$, choose any line belonging to $M$ and label it with the next unused consecutive numbers with its head equals its tail plus one. Continue this process until all alternating cycles are labelled.

Now, the term of the Pfaffian corresponding to the perfect matching $M$ contains the sign of the identity permutation, which is +1 and each term $a_{mn}$ in the product is also positive. The Pfaffian of the perfect matching $N$ contains only positive $a_{mn}$, due to the chosen labelling. Therefore, the sign of $N$ is the sign of the corresponding permutation $\rho$.

Since each alternating even cycle of $G$ correspond to an even cycle of the permutation $\rho$ and the cycles are disjoint, it follows that $\text{sgn}(N) = (-1)^k$, thus proving the lemma.

**Definition 3.5.** A cycle $C$ in a graph $G$ is nice if the subgraph $G - C$ contains a perfect matching.

**Theorem 3.6.** Let $G$ be an even graph and $\vec{G}$ an orientation of $G$. Then the following are equivalent:

1. $\vec{G}$ is a Pfaffian orientation.
2. Every perfect matching of $G$ has the same sign relative to $\vec{G}$.
3. Every nice cycle in $G$ is oddly oriented relative to $\vec{G}$.
4. If $G$ has a perfect matching, then for some perfect matching $M$, every $M$-alternating cycle is oddly oriented relative to $\vec{G}$.

**Proof.** By definition statement (1) implies (2).

To see that statement (2) implies (3) assume that $\vec{G}$ is a Pfaffian orientation and let $C$ be a nice cycle of $G$. Let $F$ be a perfect matching of $G - C$ and
$C_1$ and $C_2$ perfect matchings of $G$. Define two perfect matchings $M$ and $N$ of $G$ as follows: $M := F \cup C_1$ and $N := F \cup C_2$. By statement (2) $M$ and $N$ have the same sign and $C$ is a $M, N$-alternating cycle in $M \cup N$. Thus by Lemma 3.4 $C$ is oddly oriented.

Now assume that every nice cycle in $G$ is oddly oriented as is stated in (3) and let $M$ be a perfect matching of $G$. Let $C$ be a $M$-alternating cycle. Then $M - C$ is a perfect matching of $G - C$, thus arriving at statement (4): $C$ is nice and is oddly oriented.

Now let $M$ be a perfect matching of $G$ and assume that every $M$-alternating cycle is oddly oriented. Let $N$ be another perfect matching of $G$. Every $M, N$-alternating cycle is oddly oriented. Hence, by Lemma 3.4 $M$ and $N$ have the same sign. Therefore, $\vec{G}$ is Pfaffian.

---

**Example 3.7.** Consider the complete graph $K_{3,3}$ illustrated in Figure 3.2. As can be seen the graph has 6 perfect matchings. However, the graph does not have a Pfaffian orientation. To see this start with an orientation of $K_{3,3}$ as illustrated in Figure 3.7: all lines are directed downwards.

All nice cycles consist of 4 or 6 edges. With the chosen direction the cycles are evenly oriented. For the cycles to be oddly oriented the direction of an odd number of edges of each cycle must be reversed.

There are 9 cycles of length 4. Each edge of $K_{3,3}$ appears 4 times in these 9 cycles. If $F \subseteq E$, where $E$ is the set of edges of $K_{3,3}$, are the reversed edges,
then $4 \cdot |F| = 9 \pmod{2}$ to have an odd number of clockwise oriented edges. This is not possible.

From now on when considering a planar graph the chosen routing will be clockwise.

**Lemma 3.8.** If $\vec{G}$ is a connected plane directed graph such that every boundary face, except for the infinite face, has an odd number of edges oriented clockwise, then in every cycle the number of edges oriented clockwise is of opposite parity to the number of points of $G$ inside the cycle. Therefore, $\vec{G}$ is Pfaffian.

**Proof.** Let $C$ be any cycle in $\vec{G}$ and define the following:
- $f$ is the number of faces inside $C$, $f_1, f_2, \ldots$ are the faces of $C$
- $c_i$ is the number of edges on the boundary of face $f_i$ oriented clockwise.
- $e$ is the number of edges inside $C$
- $v$ is the number of vertices inside $C$
- $k$ is the number of edges on $C$, this is equal to the number of vertices on $C$.
- $c$ is the number of edges on $C$ oriented clockwise.

Now we want to show that $c$ is of opposite parity to $v$. Let $G'$ be the graph obtained by deleting all vertices in the exterior of $C$. Recall Euler’s formula for a connected plane graph:

$$\text{number of vertices} - \text{number of edges} + \text{number of face} = 2.$$  \hspace{1cm} (3.16)

That means that when applying Euler’s formula to $G'$:

$$(v + k) - (e + k) + (f + 1) = 2,$$

$$v - e + f = 1.$$  \hspace{1cm} (3.17)

If $c_i$ is the number of edges oriented clockwise on the boundary of face $f_i$, then it is by hypothesis odd, since faces do not have vertices inside its interior. Thus, $c_i \equiv 1 \pmod{2}$ and $f \equiv \sum_{i=1}^{f} c_i \pmod{2}$. Because any edge inside of $C$ is counted as clockwise exactly once $\sum_{i=1}^{f} c_i = c + e$. Implementing Euler’s formula leads to:

$$f = c + e = c - 1 + f + v \pmod{2},$$

$$c - 1 + v = 0 \pmod{2}.$$  \hspace{1cm} (3.18)

This means that $c + v = 1 \pmod{2}$, thus proving the lemma.

In particular, if $C$ is a $M, N$ alternating with $M$ and $N$ two perfect matchings
of $G$, then the number of vertices inside $C$ must be even, otherwise it will contradict planarity, therefore, the number of edges on $C$ oriented clockwise is odd. Thus, $G$ is Pfaffian.

\[ \square \]

**Theorem 3.9 (Kasteleyn).** Every planar graph $G$ has a Pfaffian orientation.

**Proof.** We may assume that $G$ is connected. If $G$ is a tree, then any orientation on this graph is Pfaffian.

Let $e$ be an edge in a cycle $C$ in the border of the infinite face. The graph $G - e$ has a Pfaffian orientation by induction. Adding the edge $e$ to $G - e$ and orient it in a way that the face had an odd number of clockwise oriented edges, results in a Pfaffian orientation of $G$ by Lemma 3.8. Every planar graph has a Pfaffian orientation by induction.

\[ \square \]

Lemma 3.8 can be used to construct a Pfaffian orientation on a planar graph: Consider a planar graph $G$ and remove a line $e$ belonging to a cycle bordering the infinite face.

![Figure 3.5](image.png)

Figure 3.5: To construct a Pfaffian orientation on a graph $G$, first remove a line bordering the infinite face.

Now apply an orientation to the remaining lines in a way such that all boundary faces have an odd number of clockwise oriented edges. Now reinstall line $e$ and give it an orientation such that the faces remain oddly clockwise oriented.

The graph $G$ now has a Pfaffian orientation.

When labelling the vertices as illustrated in Figure 3.6, the adjacency matrix
corresponding to this directed graph is given by:

\[ A(G) = \begin{pmatrix}
0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & -1 & 0
\end{pmatrix}. \quad (3.19)

The determinant of this matrix is 81 and using Lemma 3.2 this implies that the number of perfect matchings is 9. Of course, the graph is of such form that it can be easily checked that the number of perfect matchings is indeed 9 by constructing the configurations.
Chapter 4

The Rectangular Lattice

An application of matchings in physics is the dimer problem. A dimer is a molecule consisting of two atoms linked by a bond. The dimers are arranged on lattices and one wants to find the number of possibilities of the dimer coverings on the lattice. In this chapter a rectangular lattice graph of dimension \( m \times n \) is treated. Using the method of the previous chapter, an expression is derived for the number of configurations of the dimers on the lattice, i.e. the number of perfect matchings of the lattice graph.

By winding the lattice on a torus, the lattice does not have a Pfaffian orientation anymore. However, counting the perfect matchings on a lattice graph on a torus does involve Pfaffians. It appears that there is a relation between the genus of the surface on which the lattice graph is embedded and the number of Pfaffians needed to enumerate the perfect matchings of the graph.

4.1 The Planar Rectangular Lattice

Consider a rectangular lattice graph \( L(m, n) \) of dimension \( m \times n \), where \( m \) and \( n \) are even. A Pfaffian orientation on the graph can be constructed as illustrated in Figure 4.1.

Considering only the horizontal bonds each row of the graph can be represented by an \( m \times m \) matrix \( A \):

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 0
\end{pmatrix}.
\]
Label the vertices of the first row by 1, 2, \ldots, m and the vertices of the next row by \( m + 1, m + 2, \ldots, 2m \) and proceed in this way until all vertices are labelled. The adjacency matrix \( A_s(G) \) belonging to the orientation is:

\[
A_s(G) = \begin{pmatrix}
A & I & 0 & \cdots & 0 \\
-I & -A & I & \ddots & \vdots \\
0 & -I & A & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & I \\
0 & \cdots & 0 & -I & (-1)^{n-1}A
\end{pmatrix}, \tag{4.2}
\]

where \( I \) is the identity matrix representing the vertical bonds connecting each row.

When multiplying the first row of the adjacency matrix by \(-1\), then the third and fourth row, the fourth and fifth column, then the seventh and eighth row and so on, the absolute value of the matrix is not changed and the following matrix \( M \) is obtained:

\[
M = \begin{pmatrix}
-A & I & 0 & \cdots & 0 \\
I & -A & I & \ddots & \vdots \\
0 & I & -A & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & I \\
0 & \cdots & 0 & I & -A
\end{pmatrix}. \tag{4.3}
\]
If $B$ is the $n \times n$ matrix given by:

$$B = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \ddots & \vdots \\
0 & 1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}, \quad (4.4)$$

then $M$ can be written as a sum of two tensor products:

$$M = -I \otimes A + B \otimes I. \quad (4.5)$$

The number of perfect matchings of this lattice can be found by means of the obtained result $\det(M) = (\text{Pf}(M))^2$. The determinant of $M$ can be determined by its eigenvalues. Let $A$ have the eigenvalues $\lambda_1, \ldots, \lambda_m$ with eigenvectors $\vec{a}_1, \ldots, \vec{a}_m$ and let $B$ have the eigenvalues $\mu_1, \ldots, \mu_n$ with eigenvectors $\vec{b}_1, \ldots, \vec{b}_n$. Because

$$M(\vec{b}_i \otimes \vec{a}_j) = (-I \otimes A + B \otimes I)(\vec{b}_i \otimes \vec{a}_j)$$
$$= (-I \otimes A)(\vec{b}_i \otimes \vec{a}_j) + (B \otimes I)(\vec{b}_i \otimes \vec{a}_j)$$
$$= (-I\vec{b}_i \otimes A\vec{a}_j) + (B\vec{b}_i \otimes I\vec{a}_j)$$
$$= -\lambda_j(\vec{b}_i \otimes \vec{a}_j) + \mu_i(\vec{b}_i \otimes \vec{a}_j)$$
$$= (\mu_i - \lambda_j)(\vec{b}_i \otimes \vec{a}_j) \quad (4.6)$$

matrix $M$ has eigenvalues $\mu_i - \lambda_j$ with eigenvectors $\vec{b}_i \otimes \vec{a}_j$.

To find the eigenvalues of matrix $B$ we have the characteristic polynomial $p_n(\mu)$:
\[ p_n(\mu) = \begin{vmatrix} -\mu & 1 & 0 & \ldots & 0 \\ 1 & -\mu & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 1 & -\mu \end{vmatrix} \] 

\[ = -\mu \begin{vmatrix} -\mu & 1 & 0 & \ldots & 0 \\ 1 & -\mu & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 1 & -\mu \end{vmatrix} \]

So, \( p_n(\mu) = -\mu p_{n-1}(\mu) - p_{n-2} \). This is valid for \( n \geq 1 \) when setting \( p_0(\mu) = -1 \) and \( p_{-1}(\mu) = 0 \). We look for a solution of \( p_n \) of the form \( p_n = s^n \). Then \( S \) must satisfy:

\[ s^n = -\mu s^{n-1} - s^{n-2}, \]

and hence

\[ \mu^2 + \mu s + 1 = 0. \]

The roots of the obtained expression are

\[ s = \frac{-\mu + \sqrt{\mu^2 - 4}}{2}, \quad t = \frac{-\mu - \sqrt{\mu^2 - 4}}{2}, \]

then:

\[ p_n = c_1 s^{n+1} + c_2 t^{n+1}, \]  

for some \( c_1, c_2 \in \mathbb{C} \).  

(4.9)

Taking \( n = 0 \) and \( n = -1 \), we find for \( c_1 \) and \( c_2 \):

\[ 0 = c_1 + c_2, \]

\[ 1 = c_1 s + c_2 t. \]

(4.10)

Thus,

\[ c_1 = \frac{1}{\sqrt{\mu^2 - 4}}, \quad c_2 = \frac{-1}{\sqrt{\mu^2 - 4}}. \]

(4.11)
and:

\[ p_n(\mu) = \frac{1}{\sqrt{\mu^2 - 4}} (s^{n+1} - t^{n+1}). \]  

(4.12)

Now, to find the eigenvalues \( \mu_i \), set this equation equal to zero. Then

\[ s^{n+1} = t^{n+1}, \]

\[ s = e^2 t, \quad \text{with} \quad \epsilon = e^{\frac{k \pi}{n+1}}, \quad 0 \leq k \leq n. \]  

(4.13)

Solving \( s = e^2 t \) for \( \mu \):

\[ \mu = \pm \left( \epsilon + \frac{1}{\epsilon} \right) = \pm 2 \cos \left( \frac{k \pi}{n+1} \right). \]  

(4.14)

Since \( -\cos \left( \frac{k \pi}{n+1} \right) = \cos \left( \frac{\pi + k}{n+1} \right) \), the minus sign can be omitted and hence, the eigenvalues of \( B \) are \( 2 \cos \left( \frac{\pi k}{n+1} \right) \) where \( k = 1, \ldots, n \).

The eigenvalues of \( A \) can be found in a similar way and are \( 2i \cos \left( \frac{\pi k}{m+1} \right) \) where \( k = 1, \ldots, m \). Therefore the eigenvalues of \( M \) are \( 2 \left( \cos \left( \frac{\pi k}{m+1} \right) - i \cos \left( \frac{\pi l}{n+1} \right) \right) \) with \( k = 1, \ldots, m \) and \( l = 1, \ldots, n \). The determinant of matrix \( M \) is the product of these numbers. The absolute value of the determinant is given by:

\[ |\det M| = 2^m n \prod_{k=1}^{m} \prod_{l=1}^{n} \left( \cos^2 \left( \frac{\pi k}{m+1} \right) + \cos^2 \left( \frac{\pi l}{n+1} \right) \right)^{\frac{1}{2}}. \]  

(4.15)

Hence, the number of perfect matchings is:

\[ n_{PM} = PfM = (\det M)^{\frac{1}{2}} \]

\[ = 2^{mn/2} \prod_{k=1}^{m} \prod_{l=1}^{n} \left( \cos^2 \left( \frac{\pi k}{m+1} \right) + \cos^2 \left( \frac{\pi l}{n+1} \right) \right)^{\frac{1}{2}}. \]  

(4.16)

**Example 4.1.** Consider a chessboard. Its dimensions are \( 8 \times 8 \) and a domino tile covers 2 sites. Using the obtained equation (4.16) of the number of perfect matchings, it is found that the domino tiles can cover the chess board in 12,988,816 ways.

To determine the behaviour of expression (4.16) in the limit as \( m, n \to \infty \) take the logarithm of the equation and transform the summations into
integrals \[7\[8\]:

\[
\frac{1}{mn} \ln(n_{\text{PM}}) = \frac{1}{2} \ln 2 + \frac{1}{4mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \ln \left( \cos^2 \left( \frac{\pi k}{m+1} \right) + \cos^2 \left( \frac{\pi l}{n+1} \right) \right) - \frac{1}{2} \ln 2 + \frac{1}{4 \pi^2} \int_0^\pi \int_0^\pi \ln \left\{ \cos^2 \left( \frac{\pi k}{m+1} \right) + \cos^2 \left( \frac{\pi l}{n+1} \right) \right\} \quad (4.17)
\]

The integral can be written as:

\[
\frac{1}{4 \pi^2} \int_0^\pi \int_0^\pi \ln \{ \cos^2 x + \cos^2 y \} \, dxdy = \frac{1}{4 \pi^2} \int_0^\pi \int_0^\pi \ln \left\{ \frac{1}{2} + \frac{1}{2} \cos 2x + \frac{1}{2} \cos 2y \right\} \, dxdy. \quad (4.18)
\]

To evaluate the integral consider the following:

\[
F(X, Y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \ln \{ X + Y + X \cos u + Y \cos v \} \, dudv. \quad (4.19)
\]

Taking the \( X \)-derivative of \( F \) leads to:

\[
\frac{\partial F}{\partial X} = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{(1 + \cos u)du}{X + Y + X \cos u + Y \cos v}. \quad (4.20)
\]

Evaluating the integral over \( v \) in the expression for \( \frac{\partial F}{\partial X} \) gives:

\[
\frac{\partial F}{\partial X} = \frac{1}{\pi} \int_0^\pi \frac{(1 + \cos u)du}{\{(X + Y + X \cos u)^2 - Y^2\}^{\frac{3}{2}}}. \quad (4.21)
\]

When substituting \( A = 1 + \cos u \) and automatically \( \frac{dA}{du} = -\sin u \), \( \sin^2 u = 1 - \cos^2 u = (1 + \cos u)(1 - \cos u) \), \( 1 - \cos u = -A + 2 \), the expression can be written like:

\[
\frac{\partial F}{\partial X} = \frac{1}{\pi} \int_0^2 \frac{dA}{\{X(2 - A)(XA + 2Y)\}^{\frac{3}{2}}}. \quad (4.22)
\]

Using the identity:

\[
\frac{d}{dx} \tan^{-1} p = \frac{p'}{1 + p^2}, \quad (4.23)
\]
the following can be found:

\[
\{X(2 - A)(XA + 2Y)\}^{-\frac{1}{2}} = -\frac{2}{X} \frac{d}{dA} \tan^{-1} \left\{ \frac{X(2 - A)}{XA + 2Y} \right\}^{\frac{1}{2}}. \tag{4.24}
\]

With the expansion of the arctangent it follows that:

\[
\frac{\partial F}{\partial X} = \frac{2}{\pi X} \tan^{-1} \left( \frac{X}{Y} \right)^{\frac{1}{2}}
= \frac{2}{\pi X} \left\{ \left( \frac{X}{Y} \right)^{\frac{1}{2}} - \frac{1}{3} \left( \frac{X}{Y} \right)^{\frac{3}{2}} + \frac{1}{5} \left( \frac{X}{Y} \right)^{\frac{5}{2}} - \ldots \right\}. \tag{4.25}
\]

Integrating each term we obtain:

\[
F(X, Y) = \frac{1}{2} \ln(X) + \frac{4}{\pi} \left\{ \left( \frac{X}{Y} \right)^{\frac{1}{2}} - \frac{1}{3} \left( \frac{X}{Y} \right)^{\frac{3}{2}} + \frac{1}{5} \left( \frac{X}{Y} \right)^{\frac{5}{2}} - \ldots \right\}. \tag{4.26}
\]

Now, returning to equation (4.17), evaluating \( F(X, Y) \) where \( X = Y = \frac{1}{2} \), the result is:

\[
\frac{1}{mn} \ln(n_{PM}) \rightarrow \frac{1}{2} \ln 2 + \frac{1}{4} \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \ln \left\{ \cos^2 \left( \frac{\pi k}{m + 1} \right) + \cos^2 \left( \frac{\pi l}{n + 1} \right) \right\} \\
= \frac{1}{2} \ln 2 + \frac{1}{4} F \left( \frac{1}{2}, \frac{1}{2} \right) \\
= \frac{1}{\pi} \left\{ 1 + \frac{1}{3^2} - \frac{1}{5^2} + \ldots \right\} \\
= \frac{1}{\pi} C \approx 0.29. \tag{4.27}
\]

where \( C = (1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots) \), is the Catalan constant.

Thus, the number of perfect matchings of a planar rectangular lattice is asymptotically:

\[
n_{PM} \sim e^{\frac{mnC}{\pi}}. \tag{4.28}
\]
4.2 The Rectangular Lattice on a Torus

Consider now a rectangular lattice on a torus. One might expect to find the number of perfect matchings in the previous way of constructing a Pfaffian orientation. However, it appears that wrapped around a torus a rectangular lattice does not have a Pfaffian orientation.

**Theorem 4.2.** A rectangular lattice \( L \) with even dimensions on a torus does not have a Pfaffian orientation.

**Proof.** Consider a lattice with even dimensions on a torus.

![Figure 4.2: A directed rectangular lattice graph of dimension 4 x 4 on a torus](image)

If the same orientation is applied as on the planar rectangular lattice with the extra directions of the boundaries to be given by:

\[
(m, j; 1, j) = (-1)^{j-1},
\]

\[
(i, n; i, 1) = -1.
\]

(4.29)

where \( i, 1 \leq i \leq m \), denotes the vertex in the horizontal direction and \( j, 1 \leq j \leq n \), the vertex in the vertical direction, the cycles formed by going straight in the horizontal and in the vertical direction are evenly oriented nice cycles as can be seen in Figure 4.2. For an orientation to be Pfaffian, every nice cycle must have an odd orientation.

The chosen routing for the squares is clockwise and for cycles looping in the horizontal direction the routing is to the left.

Suppose the graph has a Pfaffian orientation, thus the vertical and the horizontal bonds are oddly oriented and the unit cells, given by the square surrounded by the dashed lines in Figure 4.3 as well. Consider now two unit cells bordering each other as illustrated in Figure 4.3. Define the following: \( H \) is the cycle looping in the horizontal direction.

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Figure 4.3: Two bordering unit cells.

$P$ and $Q$ are the left and right unit cells.

$a_1$ is the number of oddly oriented edges of $P \cap H$.

$a_2$ is the number of oddly oriented edges of $Q \cap H$.

$e$ is the common edge of $P$ and $Q$.

$b_1$ is the number of oddly oriented edges of $P - a_1 - e$.

$b_2$ is the number of oddly oriented edges of $Q - a_2 - e$.

The number of oddly oriented edges of $H$ is given by $c + a_1 + a_2$. The bond $e$ is counted as oddly oriented for one cell and evenly oriented for the other cell. Hence, the cycle formed by these cells is of odd parity. Choose $e$ to be directed upwards. By hypothesis $c + a_1 + a_2 \equiv 1 \pmod{2}$. We want to show that the cycle denoted by the green line in Figure 4.4 is evenly oriented. So, $c + b_1 + b_2 \equiv 0 \pmod{2}$.

Figure 4.4: The cycle denoted by green is an evenly oriented nice cycle.

Consider the cycle formed by $c + b_2 + a_1 + e$. The parity of this cycle is equal to $(1 - a_1) + b_2 + e \equiv 1 \pmod{2}$ since a cycle formed by two oddly oriented cycles with one common border is also oddly oriented. The cycle $c + e + b_1 + a_2$ is equal to $(1 - a_1) + b_1 + 1 \equiv 1 \pmod{2}$. This all leads to $b_1 + b_2 = a_1 + a_2 + 1$.

So:

$$c + b_1 + b_2 = c + a_1 + a_2 + 1 \equiv 0 \pmod{2} \quad (4.30)$$

This obtained evenly oriented cycle $G$ is also nice; a perfect matching of $L - G$ can be found by pairing the 'left over' vertex with the vertex situated above it. On this horizontal cycle all vertices can be paired with its horizontal neighbour except for one vertex. This one, in its turn, can be matched to
the vertex above it and continue in this way until a situation like illustrated in Figure 4.5 is reached.

Thus, even if graph $L$ is oriented such that the horizontal, vertical and unit cells are oddly oriented nice cycles, there is always within this configuration another cycle, denoted by green in Figure 4.4 that is evenly oriented and nice. Therefore, a rectangular lattice on a torus does not have a Pfaffian orientation.

Since a rectangular lattice on a torus does not have a Pfaffian orientation the number of perfect matchings of this lattice must be obtained in an other way than used in the previous section. Recall the planar rectangular lattice with its Pfaffian orientation as can be seen in Figure 4.6. The green lines denote the perfect matching $M$.

This will be the standard configuration. By winding this lattice on a torus, periodical boundary conditions are introduced. There can be four classes of configurations derived from the standard configuration as illustrated in Figure 4.7 where the standard configuration $M$ is denoted by green and black denotes the other configuration $N$:
\begin{itemize}
  \item \((e,e)\): \(M \cup N\) forming even numbers of loops in the horizontal as well as in the vertical direction.
  \item \((e,o)\): \(M \cup N\) forming even numbers of loops in the horizontal direction and odd numbers of loops in the vertical direction.
  \item \((o,e)\): \(M \cup N\) forming odd numbers of loops in the horizontal direction and even numbers of loops in the vertical direction.
  \item \((o,o)\): \(M \cup N\) forming odd numbers of loops in the horizontal as well as in the vertical direction.
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{configurations.png}
\caption{The four configurations for a rectangular lattice graph derived from the standard configuration.}
\end{figure}

The added bonds to wrap the lattice on a torus do not have an orientation yet. Define for \(1 \leq i \leq m\), where \(i\) denotes the site in the horizontal direction, and for \(1 \leq j \leq n\), where \(j\) is the site in the vertical direction:

\[ A(m, j; 1, j) = 1, \quad A(i, n; i, 1) = (-1)^i. \quad (4.31) \]

The matrix \(A_1\) is defined corresponding to this orientation. According to Lemma 3.4, \(\text{sgn}(M) \cdot \text{sgn}(N) = (-1)^k\), where \(M\) is the matching in the standard configuration and \(N\) another perfect matching and \(k\) is the number of evenly oriented cycles with alternating edges belonging to \(M\) and \(N\). With this information one can see that the Pfaffian of matrix \(A_1\) counts only the \((e,e)\) configurations correctly since there are even numbers of loops in the horizontal and vertical directions. The loops are even oriented; non-looping polygons in this configuration are oddly oriented. The other configurations will be counted with a negative sign.
To count the other configurations correctly, three additional matrices are defined:

\[ A_2 \text{ with } A(m, j; 1, j) = 1; \ A(i, n; i, 1) = (-1)^{i+1}, \]  
\[ A_3 \text{ with } A(m, j; 1, j) = -1; \ A(i, n; i, 1) = (-1)^i, \]  
\[ A_4 \text{ with } A(m, j; 1, j) = -1; \ A(i, n; i, 1) = (-1)^{i+1}. \]  

Again, with aid of Lemma 3.4 one can see that the matrices do not count all configurations correctly. The counting rules are summarised as follows:

<table>
<thead>
<tr>
<th>Class of configurations</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e,e)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(e,o)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>(o,e)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(o,o)</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.1: Summary of the counting rules of the Pfaffians.

Thus, the number of perfect matchings of a lattice on a torus can be found by means of four Pfaffians and is given by:

\[ n_{PM} = \frac{1}{2} \left( -\text{Pf}(A_1) + \text{Pf}(A_2) + \text{Pf}(A_3) + \text{Pf}(A_4) \right). \]  

(4.35)

In comparison with the planar rectangular lattice, the occurrence of matrix \( A \) is replaced by matrices \( P \) and \( Q \):

\[
P = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & -1 \\
-1 & 0 & 1 & \ldots & \ldots & 0 \\
0 & -1 & \ldots & \ldots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & -1 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 1 \\
-1 & 0 & 1 & \ldots & \ldots & 0 \\
0 & -1 & \ldots & \ldots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & 1 \\
-1 & 0 & \ldots & 0 & -1 & 0
\end{pmatrix}
\]  

(4.36)

due to the periodic boundary conditions. The four Pfaffians can be found
analogously as in the planar case and this leads to:

\[ n_{PM(torus)} = -\frac{1}{2} \prod_{k=1}^{m} \prod_{l=1}^{n} 2 \left\{ \sin^2 \left( \frac{2k\pi}{m} \right) + \sin^2 \left( \frac{2l\pi}{n} \right) \right\}^{\frac{1}{2}} + \frac{1}{2} \prod_{k=1}^{m} \prod_{l=1}^{n} 2 \left\{ \sin^2 \left( \frac{2k\pi}{m} \right) + \sin^2 \left( \frac{(2l-1)\pi}{n} \right) \right\}^{\frac{1}{2}} + \frac{1}{2} \prod_{k=1}^{m} \prod_{l=1}^{n} 2 \left\{ \sin^2 \left( \frac{(2k-1)\pi}{m} \right) + \sin^2 \left( \frac{2l\pi}{n} \right) \right\}^{\frac{1}{2}} + \frac{1}{2} \prod_{k=1}^{m} \prod_{l=1}^{n} 2 \left\{ \sin^2 \left( \frac{(2k-1)\pi}{m} \right) + \sin^2 \left( \frac{(2l-1)\pi}{n} \right) \right\}^{\frac{1}{2}}. \] (4.37)

When \( m \) and \( n \) are pushed to infinity, the logarithm of the expression turns out to be the same as for the planar case (4.17):

\[ \frac{1}{mn} \ln(n_{PM(torus)}) \rightarrow \frac{G}{\pi}. \] (4.38)

**Example 4.3.** Consider a very peculiar chessboard, namely a chessboard wrapped around a torus. So the dimension is \( 8 \times 8 \), then the number of ways to cover the chessboard with domino tiles is 311, 853, 312.

We have seen that although a rectangular lattice on a torus does not permit a Pfaffian orientation, the perfect matchings of this graph can be enumerated with a linear combination of four Pfaffians corresponding to four matrices that are slightly different from the standard configuration. This can be generalised to graphs that can be embedded in a surface of genus \( g \). Then the enumeration of perfect matchings of the graph requires a linear combination of \( 2^g \) Pfaffians corresponding to a slightly modification of the adjacency matrix(Kasteleyn \[6\]).
Chapter 5
Discussion and Conclusion

The previous chapter treats the enumeration of perfect matchings of rectangular lattices. However, there are more types of regular lattices, for instance the triangular lattice and the hexagonal lattice. In this chapter we will give a review of some results of these lattices. What is the expression for the number of perfect matchings and what is the behaviour of this expression in the limit $m, n \to \infty$, where $m$ and $n$ are the dimensions of the lattices.

A Pfaffian orientation on the triangular lattice and the hexagonal lattice is illustrated in Figure 5.1.

![Figure 5.1: The triangular lattice and the hexagonal lattice, both with Pfaffian orientation.](image)

A calculation using Pfaffians of the number of perfect matchings on the triangular lattice is given by Fendley, Moessner and Sondhi [1]. Taking the logarithm of the expression for the number of perfect matchings in the limit as $m, n \to \infty$, the result is $S_{\text{Triangular}} = 0.4286$. For the hexagonal lattice the number of perfect matchings depends on the boundary conditions. For the lattices with dimensions $m$ and $n$ as illustrated in Figure 5.2 the number of...
configurations of tilings is given by Hosoya [3]:

\[ n_P = \binom{m+n}{n} \frac{(m+n)!}{m!n!}, \]

\[ n_H = \frac{1}{2^{m-1}} \prod_{k=m}^{2m-1} \binom{n+k}{n} \frac{(n+k)!}{m-1} \prod_{k=1}^{m-1} \binom{n+k}{n}. \quad (5.1) \]

![Figure 5.2: The hexagonal lattices P and H with different boundaries.](image)

In the limit as the dimensions of the hexagonal lattice are infinitely large, the logarithm of the number of perfect matchings is \( S_{\text{Hexagonal}} = 0.1615 \). The results for the three types of lattices are summarised in Table 5.1:

<table>
<thead>
<tr>
<th>Type of lattice</th>
<th>( S_{\text{Lattice type}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>0.29</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.43</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 5.1: Summary of three types of regular lattices.

For a review on other lattices and the enumeration of coverings of the lattices the reader is referred to Wu [10].

We have seen that an algorithm exists for counting the perfect matchings of planar lattices and this has been applied explicitly to a rectangular lattice and some results are given for several other regular lattices. However, when considering a three dimensional regular lattice, an exact result has not yet been found.
Imagine a long and rainy Sunday afternoon and you are locked up in a room containing just a chess board. A nice game of chess would be a pleasant occupation, even if the opponent is yourself. But alas, the chess pieces are nowhere to be found. Luckily, hidden in a small cabinet you find 32 domino tiles. In what way can these two games be combined, you ask yourself. One way to spend the dreary afternoon is to place the tiles on the squares of the board such that each tile covers two squares without overlapping each other and count the ways in which this can be accomplished.

Figure 5.3: A plain chessboard

Looking at the chess board and its 64 squares, you know for sure that this activity will cost definitely this afternoon, maybe even all the afternoons from now on until some time next month! However, full with patience you start with the most simple configurations and vary a little. After a while you are fed up with feeling like a robot in a huge factory or an accountant doing the same thing over and over again without a prospect of ever finishing in the near future. So, you decide to try a more thoughtful approach, you turn to mathematics.
Let’s transform the chess board into a mathematical object, say a graph. A graph consists of a collection of dots and a collection of lines pairwise connecting the dots. The chess board in graph form looks like Figure 5.5 where the dots are the points in the middle of the squares and the lines the possible positions of the domino tiles. When assigning to each dot a label, this graph, in its turn, can be transformed into a matrix. The matrix consists of rows and columns with zero’s and ones.

This is a good thing, you can do calculations with matrices, for instance, the number of ways of covering a chess board with domino tiles! There are 12988816 ways to arrange the domino tiles on the board. And this number can be found in less than an afternoon. Feeling quite pleased with yourself, you find out that you are not the first one to discover this ingenious way. The physicist Kasteleyn has beaten you in this point.

Maybe a little bit disappointed you throw the board and the tiles away and grumble a while. But wait! What if the chess board is three dimensional?
Bibliography


