Twisted K-theory and T-duality

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Abstract

We introduce twisted K-theory groups of a space X. The twist H is an integral cohomology class of X of degree three. We study the twisted K-theory of principal circle bundles. We introduce pairs (E, H), with E a principal circle bundle and H a twist on E. Under certain circumstances we call two pairs (E, H) and $(\hat{E}, \hat{H})$ T-dual. We study the T-duality isomorphism, an isomorphism of the twisted K-theory of T-dual pairs. Finally we discuss examples.

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Cheers mates!

[1] Pacific Rim Mathematical Association

Contents

Chapter 1

Introduction

The aim of this thesis is to discuss T-duality as formulated in [7] and [5]. T-duality originates from physics and in particular from string theory. This thesis will have a mathematical flavour and we will mainly focus on a small part of physical T-duality, namely a certain isomorphism in twisted K-theory. We will define and discuss some properties of twisted K-theory later on. Whenever possible we will introduce the physical motivation and ideas behind the theory. Let us now discuss what physicists mean by T-duality and how this gives rise to an isomorphism in twisted K-theory.
In general, a duality in physics is *'a transformation between different looking physical theories that have the same observable physics'* [19]. So we should not be confused with the mathematical concept of 'dualizing', where all arrows get turned around. A concrete example, which clarifies this idea, is given by Maxwell's equations.

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0$$
$$\frac{\partial E}{\partial t} = c\nabla \times B \qquad \frac{\partial B}{\partial t} = -c\nabla \times E$$

These equations define the theory of electromagnetism and satisfy the following duality. When we apply the transformation

$$E \mapsto \text{-}B \qquad B \mapsto E$$

we obtain

$$\begin{aligned} \nabla \cdot (\text{-}B) = 0 \qquad & \qquad \nabla \cdot E = 0 \\ \frac{\partial(\text{-}B)}{\partial t} = c\nabla \times E \qquad & \qquad \frac{\partial E}{\partial t} = \text{-}c\nabla \times \text{-}B \end{aligned}$$

and after rearranging the signs we see that these are the exact same equations as before the transformation.

Similarly T-duality is a set of transformation rules of some variables that produces equivalent physical theories. To be more precise we need to say some general things about string theory. The basic idea of string theory is to replace the classical point particle by strings. String theory is defined on a certain spacetime E this is the 'universe' where all events take place. As a mathematical object it is a manifold with a Lorentzian metric. There are several different types of string theory. Currently there is no complete definition of string theory, and a question that arises is if the different partial definitions are all equivalent through dualities. T-duality gives a relation between 'type IIa' and 'type IIb'. Some of the data that goes into a type IIa string theory is a closed 3-form H with integral periods on spacetime E, called the H-flux. Another piece of data is an RR-field $G \in \Omega^{\text{even}}(E)$. For a type IIb string theory we need the same data except in this case the RR-field G is an odd-dimensional form, i.e. $G \in \Omega^{\text{odd}}(E)$, [11], [16].

The T-duality transformation then tells how to obtain a T-dual spacetime $\hat{E}$, H-flux $\hat{H}$ and RR-field $\hat{G}$. If G is an even form then $\hat{G}$ is odd, and vice versa. We thus move from type IIa to IIb or the other way around. These two T-dual string theories predict the same observable physics, in other words no experiment would be able to distinguish between the theories. This is at least remarkable since in general the spacetimes E and $\hat{E}$ are different. So string theory does not uniquely decide on the 'shape' of our universe.

It was argued by Minasian-Moore [14], Hořava [11] and Moore-Witten [16] that in type IIa string theory in the absence of H-flux the RR-fields are classified by $K^0(E)$ and the charges are classified by $K^1(E)$. In type IIb string theory, however, the exact opposite holds, the RR-fields are classified by $K^1(E)$ and the charges by $K^0(E)$. Notice how $\Omega^{\text{even}}(E)$ corresponds to $K^0(X)$ and $\Omega^{\text{odd}}(E) \leftrightarrow K^1(X)$. Indeed when we calculate K-theory using the Atiyah-

Hirzebruch spectral sequence, we notice that $K^0(X)$ is a subgroup of the even dimensional cohomology of E and K^1 is a subgroup of the odd dimensional cohomology. When there is no H-flux on E and $\hat{E}$ the rules for T-duality tell us that $E = \hat{E} = M \times S^1$, for a certain manifold M. We will explain this in detail later. Thus the equivalence of the two string theories predicts an isomorphism of K-groups.

$$K^{\bullet}(M \times S^1) \to K^{\bullet+1}(M \times S^1)$$

In example 4.2.1 we show that this is indeed the case.

When the H-flux is non-trivial it has been proposed by Witten, Kapustin, Bouwknegt-Mathai and Atiyah-Segal that the RR-fields are classified by twisted K-theory, $K^0(E, H)$, [1] and the charges by $K^1(E, H)$ in case of type IIa and again vice versa for type IIb. So this time we expect the following isomorphism.

$$K^{\bullet}(E, H) \to K^{\bullet+1}(\hat{E}, \hat{H})$$

This is what we will call the 'topological T-duality isomorphism'. The mathematics involved in the above statement will be the main subject of this thesis. The physical T-duality incorporates a lot more information like a Riemannian metric and a complex valued dilaton and axion. We will set aside all this extra information and just focus on the mathematics behind topological T-duality.

In the mathematical setting the main objects are pairs (E, H) where $\pi : E \to M$ is a principal circle or $U(1)$ bundle[2]. We discuss principal bundles in chapter 2. We let $H \in H^3(E; \mathbb{Z})$ be a cohomology class.

Again we can construct a T-dual pair. The dual circle bundle, $\hat{\pi} : \hat{E} \to M$ is determined by requiring its first Chern class to be $\pi_! H$. Here $\pi_!$ is integration over the fibre of the bundle, in this case the circle. We can then find $\hat{H}$ such that $\hat{\pi}_! \hat{H} = c_1(E)$, so the situation is entirely symmetric. This $\hat{H}$ is not uniquely determined by the above conditions, this issue will be addressed later. We will prove the topological T-duality isomorphism for such pairs. Now let us have a look at an example.

Example 1.0.1. *If we let E be the three dimensional sphere S^3 and equip it with trivial H-flux, then the T-dual space will be the product $S^2 \times S^1$ with*

[1] $K^0(E, H)$ is the twisted K-theory of E twisted by H

[2] We can identify $U(1) \cong S^1$ so this is the same as a principal $U(1)$-bundle.

$\hat{H} = 1 \in \mathbb{Z} \cong H^3(S^2 \times S^1, \mathbb{Z})$. Topological T-duality tells us that the twisted K-theory of $S^2 \times S^1$ can be calculated as

$$K^\bullet(S^2 \times S^1, 1) \cong K^{\bullet+1}(S^3, 0)$$

One of the properties of twisted K-theory is that $K^\bullet(E, 0) = K^\bullet(E)$. So in this case the twisted K-theory of $S^2 \times S^1$ can be realized as a geometric twist of $S^2 \times S^1$ namely S^3.

These K-theory groups can be calculated using spectral sequences and we can check that the isomorphism holds.

(E, H)	$(S^3, 0)$	$(S^2 \times S^1, 1)$
$K^0(E, H)$	$\mathbb{Z}$	$\mathbb{Z}$
$K^1(E, H)$	$\mathbb{Z}$	$\mathbb{Z}$

This example can be extended to an example in string theory as is done in [8] where type IIb string theory on $AdS^3 \times T^4 \times S^3$ without H-flux is seen to be dual to $AdS^3 \times T^4 \times S^2 \times S^1$ with one unit H-flux. AdS^3 is the three dimensional anti de Sitter space, a maximally symmetric Lorentzian manifold with constant negative scalar curvature.

This thesis is organized as follows, in chapter 2 we introduce the mathematical tools needed to introduce twisted K-theory. We will briefly review normal K-theory in order to generalize the definition to the twisted case. We will also need to discuss principal G-bundles and some of their properties.

In chapter 3 we will define twisted K-theory and discuss some of its properties. We will introduce the Atiyah-Hirzebruch spectral sequence in order to compute some twisted K-theory groups.

In the final chapter we define what it means for two pairs (E, H) and $(\hat{E}, \hat{H})$ to be T-dual and prove that the topological T-duality isomorphism exists. Finally we will introduce some examples of T-dual pairs and do the explicit calculations of the twisted K-groups.

Chapter 2

Tools

When defining twisted K-theory, the twist H is an element of the third cohomology of the base space. To this twist we associate a principal G bundle, with G a certain group of Fredholm operators. In the first section of this chapter we discuss principal G-bundles and show that there is a 1-1 correspondence between isomorphism classes of these and $[X, BG]$. With our choice of G, this will allow us to obtain a principal G-bundle from the twist.
To calculate twisted K-theory groups we need spectral sequences, they are the subject of the second section of this chapter. We also introduce the Gysin sequence in this section, another important calculational tool. It is used for the calculation of cohomology of vector bundles.
In the third section we discuss the Fredholm operators and in particular the subgroup that we need in the definition of twisted K-theory. We end this chapter with a section on Clifford algebras. Again we are mainly interested in a certain subgroup, the spin$^{\mathbb{C}}$ group.

2.1 Principal G-bundles

In this section we give a precise definition of principal G-bundles. We will discuss the associated bundle construction and the classification of isomorphism classes of principal G bundles. Since a principal G-bundle is a special type of fiber bundle we start out by recalling the definition of a fiber bundle.

Definition 2.1.1. *A fiber bundle is a surjective map of topological spaces $p : E \to B$ such that $p^{-1}(x) = F$ for every $x \in B$. F is called the fiber of the bundle. Also, there is an open cover $\{X_\alpha\}$ of B such that over each X_α the bundle is trivial, i.e. with the property that*

$$\begin{array}{ccc} p^{-1}(X_\alpha) & \xrightarrow{\cong} & X_\alpha \times F \\ \downarrow & \swarrow & \\ X_\alpha & & \end{array}$$

commutes. The homeomorphisms $h_\alpha : p^{-1}(X_\alpha) \to X_\alpha \times F$ are called local trivialization.

Definition 2.1.2. *A principal G-bundle is a fiber bundle with fiber G, a topological group, satisfying the following properties;*

1. *E has a free fiberwise right G action so $\mu : E \times G \to E$ has $\mu : p^{-1}(x) \times G \mapsto p^{-1}(x)$ and $\mu(e, g) = e \Leftrightarrow g = 1$.*
2. *the induced action on fibers is transitive.*
3. *there exist local trivialization $\psi : p^{-1}(U) \to U \times G$ such that $\psi = (\psi_1, \psi_2)$ has the property that $\psi_1(e.g) = \psi_1(e)$ and $\psi_2(e.g) = \psi_2(e).g$ (equivariance).*

Now let E/G denote the quotient space of E under the equivalence relation $e \sim e' \Leftrightarrow \exists g \in G$ such that $e.g = e'$. Notice that E/G is homeomorphic to B since G acts fiberwise and transitivitely on fibers.

A morphism of principal G-bundles over the same base space B is an equivariant continuous map $f : E \to E'$. Equivariant means that $f(e.g) = f(e).g$. It is an isomorphism if there is an inverse morphism f' such that both compositions are the identity.

Lemma 2.1.3. *Every morphism $f : E \to E'$ of principal G-bundles is an isomorphism.*

Proof: first assume that both bundles are trivial, so $E = E' = B \times G$. In this case we have $f(b, g) = (b, \tilde{f}(b).g)$. Here $\tilde{f} : B \to G$ is a continuous map since f

is continuous. So $\tilde{g} : B \to G$ defined by $b \mapsto (\tilde{f}(b))^{-1}$ is still continuous since the inverse map on a topological group is a continuous map. And we find that $g(b, g) = (b, \tilde{g}(b)g)$ is the inverse of f.
Now let E and E' be non-trivial bundles and $f : E \to E'$ a morphism. By the arguments for trivial bundles above we see that $f|_b : E|_b \to E'|_b$ is an isomorphism for every $b \in B$. This means that f is a bijection. It remains to check that f^{-1} is continuous. This is a local question and for any point $b \in B$ we can find a neighborhood U such that both E and E' are trivial over U, but in this case f^{-1} is continuous by the above. □

Just as with regular fiber bundles we can take the pullback of a principal G-bundle over a map into the base space

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & B \end{array}$$

We claim that $f^*E = \{(x, e) \in X \times E | f(x) = \pi(e)\}$ is again a principal G-bundle over X. By definition we find $f^*E|_x = E|_{f(x)}$. So the fiber of $f^*\pi$ is G. We let G act on f^*E by $(x, e).g = (x, e.g)$. This is obviously free and transitive since the action on E is fiberwise and has these properties. Now we are ready to define the associated bundle construction.

Definition 2.1.4. *Let X be a topological space with continuous right G action and $p : E \to B$ a G-bundle. From these we can construct the associated fiber bundle.*

$$E \times_G X \to E/G \sim B$$

Here $E \times_G X$ is defined to be the quotient space of $E \times X$ under the equivalence relation $(e, x) \sim (e', x') \Leftrightarrow e' = e.g$ and $x' = x.g$. The map is given by $[(e, x)] \mapsto [e]$ and has fiber $p^{-1}([e]) = \{[(e, x)] |\, x \in X\} \cong X$.

We want to use the pullback construction to classify the isomorphism classes of principal G-bundles over a space X. It turns out that we can obtain any principal G-bundle as a pullback of the universal bundle $EG \to BG$. For the reader unfamiliar with universal spaces and the universal bundle we refer to appendix B.

Theorem 2.1.5. *The map $\psi : [X, BG] \to \{iso.\ classes\ of\ prin.\ G\text{-}bundles\ over\ X\}$ given by $f \mapsto f^*(EG)$ is a bijection.*

First we want to proof the following lemma.

Lemma 2.1.6. *Let $p : E \to B$ be a principal G-bundle and X a topological space with right G action. Write $Hom_G(E, X)$ for the set of equivariant maps from E to X and let $\pi : E \times_G X \to B$ be the associated bundle. There is a bijection*

$$Hom_G(E, X) \leftrightarrow \Gamma(B, E \times_G X).$$

Proof: Let us first take $E = B \times G$ the trivial bundle. In this case we have $Hom_G(B \times G, X) = Hom(B, X)$ and $\Gamma(B, B \times G \times_G X) = \Gamma(B, B \times X) = Hom(B, X)$ so the bijection hold.

Now let $f : E \to X$ be a G-equivariant map. We define $\tilde{f} : E \to E \times X$ by $\tilde{f}(e) = (e, f(e))$. This map is G-equivariant as well, so we get an induced map $\hat{f} : B \to E \times_G X$. This is a section since $p \circ \hat{f}([e]) = p([e, f(e)]) = [e]$.

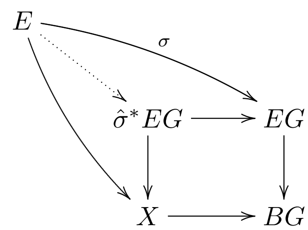
To show surjectivity we start with an arbitrary section $h : B \to E \times_G X$. We define $\tilde{h} : E \to X$ as follows, let $h(b) = [(e, x)]$, then for every element $(e', x') \in [(e, x)]$ we let $\tilde{h}(e') = x'$. Since G acts freely and transitively on E we get a unique image for every point in e. The map is equivariant since for $(e', x') \in [(e, x)]$ we have $e' = e.g$ and $x' = x.g$ so $\tilde{h}(e.g) = \tilde{h}(e') = x' = x.g = \tilde{(h)}(e).g$. And obviously $\hat{\tilde{h}} = h$. We still need to show that $\tilde{h}$ is continuous, this is a local question so we can take $U \subset B$ with $E|_U = U \times G$. We have seen that there is a bijection in that case.

To see that the map is injective, let $f, g : E \to X$ be G-equivariant maps and assume that $\hat{f} = \hat{g}$. Now by construction we have $\hat{f}([e]) = [(e, f(e))] = [(e, g(e))] = \hat{g}([e])$. □

Proof of theorem 2.1.5: First we show surjectivity. Let $E \to X$ be a principal G bundle. Associate to it $E \times_G EG \to X$, this bundle has EG as its fibre which is a contractible space. This implies that the bundle admits a section[1] , $\sigma : X \to E \times_G EG$. Now by the previous lemma we find a G equivariant map

[1] See [15]

$\tilde{\sigma} : E \to EG$. Since this is an equivariant map it induces a map on the bases $\hat{\sigma} : X \to BG$. We can construct an equivariant map from E to $\hat{\sigma}^* EG$ from the following pull-back diagram.



And thus these bundles are isomorphic. So we have shown surjectivity. Injectivity is obtained by a standard argument that for homotopic maps f, g the pullback bundles are isomorphic. For details see [6].

2.2 Spectral sequences and the Gysin sequence

In chapter three we calculate some twisted K-theory groups using spectral sequences. Spectral sequences are hard to understand without seeing them used in an example. For the reader unfamiliar with them we recommend looking up the examples in section 3.4 after having read theorem 2.2.4.

Definition 2.2.1. *A differential bigraded module over a ring R, is a collection of R-modules, $\{E^{p,q}\}$, where p and q are integers, together with an R-linear mapping, $d : E^{*,*} \to E^{*,*}$, the differential, of bidegree $(s, 1-s)$ or $(s, s-1)$, for some integer s, and satisfying $d \circ d = 0$.*

We will only be interested in the case where the bidegree is $(s, 1-s)$, this will correspond to cohomology theories, whereas bidegree $(-s, s-1)$ corresponds to homology theories.

Definition 2.2.2. *A spectral sequence is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where $r = 1, 2, ...$; the differentials are all of bidegree $(r, 1-r)$ and for all p, q, r, $E_{r+1}^{p,q}$ is isomorphic to $H(E_r^{p,q}, d_r)$, the homology with respect to the d_r map.*

Notice how $E_r^{*,*}$ together with d_r determines $E_{r+1}^{*,*}$ but they do not determine d_{r+1}. Determining the differentials is usually a hard task.

When the differentials d_r are zero the E_{r+1} page is exactly E_r. Most spectral sequences encountered in practice have the property that $d_r = 0$ for all r bigger than a certain value, say N. This means that $E_\infty^{p,q} = E_N^{p,q}$ and we say that the sequence collapses at N.

Definition 2.2.3. *A spectral sequence $\{E^{*,*}, d_r\}$ is said to converge to H^*, a graded R-module, if there is a filtration F on H^*,*

$$\{0\} \subset F^p H^p \subset \cdots \subset F^0 H^p = H^p$$

such that

$$E_\infty^{p,q} = F^p H^{p+q} / F^{p+1} H^{p+q}$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

In general it is difficult to determine the cohomology ring, H^*.

Theorem 2.2.4 (the cohomology Leray-Serre spectral sequence)**.** *Let R be a commutative ring with unit. Suppose $F \to E \to B$ is a fibration, where B is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E^{*,*}, d_r\}$, converging to $H^*(E; R)$ as an algebra, with*

$$E_2^{p,q} = H^p(B, \mathcal{H}^q(F))$$

the cohomology of the space B with local coefficients in the cohomology of the fibre. This spectral sequence is natural with respect to fibre-preserving maps. Furthermore, the cup product $\smile$ on cohomology with local coefficients and the product $\cdot_2$ on $E_2^{*,*}$ are related by $u \cdot_2 v = (l)^{p'q} u \smile v$ when $u \in E^{p,q}$ and $v \in E^{p',q'}$

Proposition 2.2.5. *If on top of the requirements in the previous theorem $\pi_1(B)$ acts trivially on $H^*(F)$ then there is a first quadrant spectral sequence $\{E_r^{p,q}, d_r\}$, with;*

$$E_2^{p,q} = H^p(X, H^q(F))$$

converging to $H^*(E)$.

Proposition 2.2.6. *If the base space B of a fibration $F \to E \to B$ is a finite-dimensional CW-complex then we can extend the Serre spectral sequence to* *any generalized cohomology theory h^*. That is, we have a spectral sequence $\{E_r^{*,*}, d_r\}$ with $E_2^{p,q} = H^p(B, h^q(F))$ converging to $h^*(E)$. We will call this spectral sequence the Atiyah-Hirzebruch-Serre spectral sequence.*

We can now derive the Gysin sequence for orientable vector bundles $\nu \to X$ with X path-connected. Let $S(\nu)$ be the sphere bundle of ν, defined by taking $S(\nu)|_x = \{v \in \nu_x | |v| = 1\}$. The action of $\pi_1(X)$ on $H^n(S^n) = \mathbb{Z}$ can be given by the identity or the multiplication by -1. If it would be given by the later map, there would be a continuous path in E that changes orientation. But this is impossible for an oriented bundle, thus $\pi_1(X)$ acts trivially on $H^*(S^n)$.

Proposition 2.2.7. *Let $\nu \to X$ be an orientable vector bundle of rank $n+1$. There is a long exact sequence in homology.*

$$\cdots \to H^p(X) \to H^{p+n+1}(X) \to H^{p+n+1}(S(\nu)) \to H^{p+1}(X) \to \cdots$$

This sequence is called the Gysin sequence.

Proof: the E_2 page of the Leray-Serre spectral sequence converging to $H^*(S(\nu))$ is given by $E_2^{p,q} = H^p(X, H^q(S^n))$ so it is

n	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$H^4(X,\mathbb{Z})$
$n-1$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	0	0	0	0
0	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$H^4(X,\mathbb{Z})$
E_2	0	1	2	3	4

The differentials $d_{n+1} : E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}$ are the only ones that could possibly be nonzero. The E_∞ page is given by

n	$\ker(d_{n+1})$	$\ker(d_{n+1})$	$\cdots$	$\ker(d_{n+1})$	$\ker(d_{n+1})$
$n-1$	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
1	0	0	$\cdots$	0	0
0	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$\cdots$	$H^{n+1}(X,\mathbb{Z})/\mathrm{im}(d_{n+1})$	$H^{n+2}(X,\mathbb{Z})/\mathrm{im}(d_{n+1})$
E_∞	0	1	$\cdots$	$n+1$	$n+2$

We have exact sequences

$$0 \to \ker(d_{n+1}) \to H^p(X,\mathbb{Z}) \stackrel{d_{n+1}}{\to} H^{p+n+1}(X,\mathbb{Z}) \to H^{p+n+1}(X,\mathbb{Z})/\mathrm{im}(d_{n+1}) \to 0$$

which correspond to

$$0 \to E_\infty^{p,n} \to H^p(X,\mathbb{Z}) \stackrel{d_{n+1}}{\to} H^{p+n+1}(X,\mathbb{Z}) \to E_\infty^{p+n+1,0} \to 0$$

We also see that the filtration of $H^*(S(\nu))$ given by the E_∞ page is,

$$\{0\} \subset F^{p+n}H^{p+n} = \cdots = F^{p+1}H^{p+n} \subset F^pH^{p+n} = \cdots = F^0H^{p+n} = H^{p+n}(S(\nu))$$

and $E_\infty^{p+n,0} = F^{p+n}H^{p+n}$ and $E_\infty^{p,n} = F^pH^{p+n}/F^{p+1}H^{p+n}$ thus we have another exact sequence

$$0 \to E_\infty^{p+n,0} \to H^{p+n}(S(\nu)) \to E_\infty^{p,n} \to 0$$

Combining the exact sequences we obtain the Gysin sequence. The map d_{n+1} is the cup product with $c_1(\nu)$, the first Chern class of ν **??**.

We are mainly interested in the two dimensional case, where $S(\nu)$ is a circle bundle. Next we calculate cohomology groups needed in the computations of twisted K-theory in chapter three.

Example 2.2.8. *Cohomology groups of circle bundles over M^g, the Riemannian surface of genus g. The vector bundles over M^g are classified by their first Chern class which is an element of $H^2(M^g) = \mathbb{Z}$. We will write E_k for the bundle corresponding to the integer k. To calculate the cohomology of E_k using the Gysin sequence, we need the cohomology of M^g. This can be calculated explicitly using a CW-complex structure, see for example [9], and is given by*

$$H^i(M^g,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

We now use the Gysin sequence with $n = dim(S^1) = 1$ to calculate $H^0(E_k)$ and $H^3(E_k)$,

$$0 \longrightarrow H^0(M^g) \longrightarrow H^0(E_k) \longrightarrow 0$$
$$0 \longrightarrow H^3(E_k) \longrightarrow H^2(M^g) \longrightarrow 0$$

So $H^0(E_k) = H^3(E_k) = \mathbb{Z}$*. It is clear that all cohomology groups of dimension four or higher vanish. To calculate the other two cohomology groups we use the following parts of the Gysin sequence.*

$$0 \to H^1(M^g) \to H^1(E_k) \to H^0(M^g) \to H^2(M^g) \to H^2(E_k) \to H^1(M^g) \to 0$$

$$0 \to \mathbb{Z}^{2g} \to H^1(E_k) \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to H^2(E_k) \to \mathbb{Z}^{2g} \to 0$$

When $k = 0$ *we find* $H^1(E_0) = H^2(E_0) = \mathbb{Z}^{2g+1}$ *and for* $k \neq 0$ *we obtain* $H^1(E_k) = \mathbb{Z}^{2g}$ *and* $H^2(E_k) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_k$*.*

The next example calculates the cohomology of vector bundles over the real projective spaces $\mathbb{R}P^n$ for $n > 1$.

Example 2.2.9. *In this example we are interested in the circle bundles over* $\mathbb{R}P^n$*; to be able to use the Gysin sequence we need the cohomology of* $\mathbb{R}P^n$*. The homology of* $\mathbb{R}P^n$ *is given by*

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i = 0, i = n \text{ odd} \\ \mathbb{Z}_2 & 0 < i < n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

see for example [9], so using the universal coefficient theorem we find that the cohomology is given by

$$H^i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i = 0, i = n \text{ odd} \\ \mathbb{Z}_2 & 0 < i \leq n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

The line bundles over $\mathbb{R}P^n$ *are classified by their first Chern class in* $H^2(\mathbb{R}P^n)$*. When* $n > 1$ *this groups is* $\mathbb{Z}_2$*, so there are two bundles. The trivial bundle* $\mathbb{R}P^n \times S^1$ *and* E *the non-trivial bundle. We will write* E_n *for the nontrivial bundle over* $\mathbb{R}P^n$*. To calculate the cohomology groups of* E_n *we will first look at the beginning of the sequence, next the middle and then distinguish the odd and even case for the end of the sequence. Let us start by calculating the low cohomology groups. The begin of the sequence, starting at* $H^{-1}(\mathbb{R}P^n)$*, is*

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^0(E) \longrightarrow 0$$

so $H^0(E) = \mathbb{Z}$*, then the sequence becomes*

$$0 \longrightarrow H^1(E) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(E) \longrightarrow 0$$

where the middle map is a surjection. This tells us that $H^1(E) = \mathbb{Z}$ *and* $H^2(E) = 0$.

The middle of the Gysin sequence starting at an odd dimensional number i *is given by*

$$0 \longrightarrow H^i(E) \longrightarrow \mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2 \longrightarrow H^{i+1}(E) \longrightarrow 0$$

So all middle terms vanish. Now when n *is even dimensional, the end of the Gysin sequence is given by*

$$0 \longrightarrow H^{n-1}(E) \longrightarrow \mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2 \longrightarrow H^n(E) \longrightarrow 0$$

$$0 \longrightarrow H^{n+1}(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

So $H^i(E_2n) = 0$ *for* $1 < i < 2n+1$ *and* $H^{2n+1}(E_2n) = \mathbb{Z}_2$*. Now for the odd case the Gysin sequence ends with*

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^n(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

and

$$0 \longrightarrow H^{n+1}(E) \longrightarrow \mathbb{Z} \longrightarrow 0$$

So we find $H^{2n+2}(E_{2n+1}) = \mathbb{Z}$ *and* $H^{2n+1}(E_{2n+1}) = \mathbb{Z}$ *or* $\mathbb{Z} \oplus \mathbb{Z}_2$ *depending on an extension problem. T-duality will decide which of the options this should be.*

To summarize we have found

$$H^0(E_n) = H^1(E_n) = \mathbb{Z}, \qquad H^{2n+1}(E_{2n}) = \mathbb{Z}_2,$$
$$H^{2n+1}(E_{2n+1}) = \mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad H^{2n+2}(E_{2n+1}) = \mathbb{Z}$$

2.3 Fredholm operators

The Brown representability theorem tells us that every generalized cohomology theory can be represented (or classified) by an Ω-spectrum. To make sense of this we will first give the definition of an Ω-spectrum.

Definition 2.3.1. *An Ω-spectrum is a sequence of CW-complexes E_n, such that there are morphisms $\Sigma E_n \to E_{n+1}$ and the corresponding (adjoint) morphisms $E_n \to \Omega E_{n+1}$ are weak equivalences.*

An Ω-spectrum is a special type of spectrum. A spectrum does not require the morphisms $E_n \to \Omega E_{n+1}$ to be weak equivalences. The easiest example of a spectrum is the so-called suspension spectrum of a space. In this case $E_n = \Sigma^n X$ the morphisms are the identities $\Sigma\Sigma^n X \to \Sigma^{n+1} X$. This is not an Ω-spectrum, but $E_n = \lim_{n\to\infty} \Omega^K \Sigma^{K+n} X$ is.

To say that such an object represents a generalized cohomology theory is to say that the cohomology groups $h^n(X)$ of a CW-complex X are given by the homotopy classes of maps from X to E_n.

$$h^n(X) \cong [X, E_n]$$

For example, singular cohomology is classified by the Eilenberg-Maclane spectrum of $\mathbb{Z}$, thus the spectrum with $E_n = K(\mathbb{Z}, n)$.

A generalized cohomology theory is a contravariant functor from the category of CW-complexes to the category of abelian groups satisfying the Eilenberg-Steenrod axioms (except for the dimension axiom). It is a well known fact that K-theory is a generalized cohomology theory, so there should be an Ω spectrum classifying it.

In appendix A we discuss such a classification; $K^n(X) = [X, \Omega^{n+1} U]$ and we prove that K-theory is two-periodic. In this section we introduce the space of Fredholm operators on the infinite dimensional Hilbert space $\mathcal{H}$. This space will be denoted by $\mathrm{Fred}(\mathcal{H})$. We show that the spaces $\mathrm{Fred}(\mathcal{H})$ and $\Omega\mathrm{Fred}(\mathcal{H})$ give another representation of K-theory.

As in the finite case, we say *the* infinite dimensional Hilbert space, when we actually mean the isomorphism class of infinite Hilbert spaces. In other words we do not specify a basis. The following is an easy lemma from linear algebra.

Lemma 2.3.2. *Let $f : V \to W$ be a linear map between two n-dimensional complex Hilbert spaces, the following are equivalent;*

1. *The map f is an isomorphism.*

2. *The kernel of f is zero.*

3. *The cokernel of f is zero.*

Proof: Injectivity is the same as having an empty kernel and surjectivity is the same as the cokernel being zero. So it is obvious that (1) implies (2) and (3). To see that (2) implies (1) we note that the image must be n-dimensional thus f is surjective. Similarly if the cokernel is empty, the image is n-dimensional so the kernel is zero dimensional and we are done. □

Definition 2.3.3. *A linear map $f : \mathcal{H} \to \mathcal{H}$ is called bounded linear if there is a number $M > 0$ such that for every vector $v \in \mathcal{H}$, $\|f(v)\| < M\|v\|$.*

Definition 2.3.4. *A Fredholm operator is a bounded linear map*

$$f : \mathcal{H} \to \mathcal{H}$$

with finite kernel and cokernel.

In this infinite dimensional setting, the Fredholm operators are nearly isomorphisms. The set of Fredholm operators can be given the norm topology. The norm of a bounded, and in particular Fredholm, operator is given by $\|f\| = \min\{c \mid \|f(v)\| < c\|v\| \ \forall \ v \in \mathcal{H}\}$. The set of Fredholm operators is closed under multiplication since

$$\dim(\ker(TS)) \leq \dim(\ker(T)) + \dim(\ker(S))$$

and

$$\dim(\operatorname{coker}(TS)) \leq \dim(\operatorname{coker}(T)) + \dim(\operatorname{coker}(S)).$$

We can use Fredholm operators to classify complex K-theory, i.e. $K(X) = [X, \mathrm{Fred}(\mathcal{H})]$. $[X, \mathrm{Fred}(\mathcal{H})]$ inherits a product from $\mathrm{Fred}(\mathcal{H})$ by pointwise multiplication and thus is a semi-group.

Definition 2.3.5. *We define the index map by,*

$$\begin{aligned} \mathrm{index} : \mathrm{Fred}(\mathcal{H}) &\to \mathbb{Z} \\ f &\mapsto \dim(\ker(f)) - \dim(\operatorname{coker}(f)) \end{aligned}$$

For a morphism between finite dimensional vector spaces, the following lemma holds.

Lemma 2.3.6. *Let $f : V \to W$ be a linear map between vector spaces (over the same field), then* $\text{index}(f) = \dim(V) - \dim(W)$.

Proof: This follows from the rank-nullity theorem in linear algebra, which states that for a linear map between finite dimensional vector spaces such as f the following equality holds;

$$\dim(V) = \dim(\text{im}(f)) + \dim(\ker(f))$$

Now the cokernel of f is $W/\text{im}(f)$ so $\dim(\text{coker}(f)) = \dim(W) - \dim(\text{im}(f))$, so indeed,

$$\begin{aligned}\dim(\ker(f)) - \dim(\text{coker}(f)) &= \dim(V) - \dim(\text{im}(f)) - \dim(W) + \dim(\text{im}(f)) \\ &= \dim(V) - \dim(W).\end{aligned}$$

So in the finite case the index only depends on the dimensions of the vector spaces. □

We would like to define a similar map, which by abuse of notation we will also call 'index', in the following setting. Let X be a compact space.

$$\text{index} : [X, \text{Fred}(\mathcal{H})] \to K(X)$$

In fact, we will prove that this map is an isomorphism of semi-groups. First we describe how the index map is constructed. For a more elaborate discussion see [2]. We want to construct vector bundles that resemble the kernels and cokernels of a set of operators. The K-theory of vector bundles over a point is the group $\mathbb{Z}$. All positive integers correspond to a vector space of that dimension (these are very trivial vector bundle, but there is not much one can do over a point). In general K-theory measures more than just the dimension of a bundle. However it is nice to keep the idea of K-theory measuring the dimension in mind, since we want to generalize a map that takes the difference of the dimension of two spaces.

Let $f : X \to \text{Fred}(\mathcal{H})$, we write f_x for the Fredholm operator $f(x)$. It turns out that one can find a closed subspace V of finite codimension in $\mathcal{H}$ such that

$V \cap \ker(f_x) = 0$ for all $x \in X$. Furthermore, for any space V with this property, the space $\cup_{x\in X}\mathcal{H}/f_x(V) =: \mathcal{H}/f(V)$ topologized as the quotient space of $X \times \mathcal{H}$ is a vector bundle over X. For a proof of this statement we refer to the appendix of [2].

This bundle will be our candidate for the vector bundle that resembles the cokernel. For the 'kernel' bundle we take the trivial bundle $X \times \mathcal{H}/V$. Note that $\dim(\mathcal{H}/V)$ is a bit bigger than $\dim(\ker f_x)$, in particular let $\tilde{f}_x := f_x|_{\mathcal{H}/V}$ then the difference is $\dim(\mathrm{im}(\tilde{f}_x))$. We compensate by the 'cokernel' bundle. The dimension, $\dim(\mathcal{H}/f_x(V))$ is a bit bigger than $\dim(\mathrm{coker}(f_x)) = \dim(\mathcal{H}/f_x(\mathcal{H}))$ and again the difference is $\dim(\mathrm{im}(\tilde{f}_x))$. This motivates the construction of the index map as

$$\begin{aligned} \mathrm{index} : [X, \mathrm{Fred}(\mathcal{H})] &\rightarrow K(X) \\ [f] &\mapsto [\mathcal{H}/V] - [\mathcal{H}/f(V)]\,. \end{aligned}$$

We need to show that this map is independent of the choice of V and that it does not depend on the choice of representative of the homotopy class.

Let us assume that W is another closed subspace of $\mathcal{H}$ of finite codimension, with $\ker(f_x) \cap W = 0$ for every $x \in X$. Then $V \cap W$ is a closed subspace of $\mathcal{H}$ of finite codimension and also $\ker(f_x) \cap (V \cap W) = 0\ \forall x \in X$. So we may assume that $W \subset V$. Now we need to show that

$$[\mathcal{H}/V] - [\mathcal{H}/f(V)] = [\mathcal{H}/W] - [\mathcal{H}/f(W)]$$

To do so look at the following exact sequences of vector bundles

$$0 \rightarrow V/W \rightarrow \mathcal{H}/W \rightarrow \mathcal{H}/V \rightarrow 0$$
$$0 \rightarrow V/W \rightarrow \mathcal{H}/f(W) \rightarrow \mathcal{H}/f(V) \rightarrow 0$$

The first sequence of vector bundles is obviously exact. To see that the second sequence is exact note that (1) $f_x(W) \subset f_x(V)$ so fiberwise the second map is just a quotient map and thus surjective. (2) Fiberwise the first map is given by $f_x|_{V/W}$. This map is well defined, when two vectors $v, v' \in V/W$ are equivalent then $v = v' + w$ for some $w \in W$. But then $f_x(v) = f_x(v') + f_x(w)$ so $[f_x(v)] =$

$[f_x(v')]$. To see that this map is injective we use a similar argument, if $[f_x(v)] = [f_x(v')]$ then $f_x(v) = f_x(v') + f_x(w)$ for some $w \in W$, but then by linearity $f_x(v) = f_x(v' + w)$. f_x is injective on V thus $v = v' + w$ or $[v] = [v']$.
Now we find

$$[\mathcal{H}/W] - [\mathcal{H}/V] = [V/W] = [\mathcal{H}/f(W)] - [\mathcal{H}/f(V)]$$

thus rearranging the terms we obtain the wanted equality. □
Next we need to show that homotopic maps $f_0, f_1 : X \to \text{Fred}(\mathcal{H})$ have the same index. To do so we first prove the following lemma.

Lemma 2.3.7. *Let $g : Y \to X$ be a continuous map, then for a map $f : X \to$ Fred$(\mathcal{H})$ we have*

$$\text{index}(fg) = g^*\text{index}(f).$$

Proof: When $V \cap \ker(f_x) = 0$ for all x then $V \cap \ker((fg)_y) = V \cap \ker f_{g(y)} = 0$ for all $y \in Y$. Now

$$g^*(X \times \mathcal{H}/V) = \{(y, x, \bar{h}) \in Y \times X \times \mathcal{H}/V \mid g(y) = x\} = Y \times \mathcal{H}/V$$

and

$$\begin{aligned} g^*(\cup_{x\in X}\mathcal{H}/f_x(V)) &= \{(y, x, \bar{h}) \in Y \times \cup_{x\in X}\mathcal{H}/f_x(V) \mid g(y) = x\} \\ &= \cup_{y\in Y}\mathcal{H}/fg_y(V) \end{aligned}$$

This proves the lemma. □

Next we show that *index* is a homomorphism, we can do this since we have a multiplication structure on both sides. Let $f, g : X \to \text{Fred}(\mathcal{H})$ be two maps. Let $V, W \subset \mathcal{H}$ both be of finite codimension and such that $ker(f_x) \cap V = ker(g_x) \cap W = 0\ \forall x \in X$. Let π_W be the projection onto W. The maps f and $\pi_W \circ f$ are homotopic, they differ on $\mathcal{H}/W$ only. Since W is of finite codimension we can construct the homotopy $h_t : \mathcal{H}/W \to \mathcal{H}/W$ by $f_t(v) = t.Id$, where Id is the identity matrix and we choose the standard basis for $\mathcal{H}/W$. Thus we may assume $f(\mathcal{H}) \subset W$.
So $ker(g_x \circ f_x) = ker(f_x) \forall x \in X$ since $ker(g_x) \cap W = 0$. So we have a vector

bundle $\mathcal{H}/gf(V)$. As before, we find that the following sequence of vector bundles is short exact.

$$0 \longrightarrow W/f(V) \xrightarrow{g} \mathcal{H}/gf(V) \longrightarrow \mathcal{H}/g(W)$$

We also have the short exact sequence,

$$0 \longrightarrow W/f(V) \longrightarrow \mathcal{H}/f(V) \longrightarrow \mathcal{H}/W.$$

Thus we find that the index of gf, the pointwise multiplication of Fredholm operators f_x, g_x, is equal to the sum $\text{index}(f) + \text{index}(g)$.

$$\begin{aligned} \text{index}(gf) &= [\mathcal{H}/V] - [\mathcal{H}/gf(V)] \\ &= [\mathcal{H}/V] - [W/f(V)] - [\mathcal{H}/g(W)] \\ &= [\mathcal{H}/V] - [\mathcal{H}/f(V)] + [\mathcal{H}/W] - [\mathcal{H}/g(W)] \\ &= \text{index}(f) + \text{index}(g) \end{aligned}$$

To prove the final proposition we introduce the algebra $\mathcal{A}^*$ of bounded automorphisms of $\mathcal{H}$.

Proposition 2.3.8. *We have an exact sequence of semi-groups*

$$[X, \mathcal{A}^*] \to [X, \mathit{Fred}(\mathcal{H})] \overset{index}{\to} K(X) \to 0$$

We will only prove surjectivity of the last map. For a proof of exactness in the middle, see [2]. Let E be a vector bundle over X. To construct an operator with index $[E]$, we need to introduce the following standard operators.
Let $(e_1, e_2, e_3, ...)$ be an orthonormal basis of $\mathcal{H}$ then define $f_k \in \text{Fred}(\mathcal{H})$ by

$$f_k(e_i) = \begin{cases} e_{i-k} & \text{if } i > k \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.3.9. *The index of f_k is k.*

Proof: We distinguish between the following two cases
(i) $k > 0$
(ii) $k \leq 0$

In the first case the map sends every basis vector e_i with $i \leq k$ to zero, so $\ker(f_x) = \text{span}\{e_1, ..., e_k\}$ and we may pick $V = \mathcal{H}/\text{span}\{e_1, ..., e_k\}$ then indeed

$$\text{index}(f_k) = [X \times \text{span}\{e_1, ..., e_k\}] - [\mathcal{H}/\mathcal{H}] = k$$

In case (ii) all basis vectors get shifted $|k|$ places by f_k. The kernel of f_k is empty and we may take $V = \mathcal{H}$, then

$$\text{index}(f_k) = [\mathcal{H}/\mathcal{H}] - [\mathcal{H}/f_k(\mathcal{H})] = -[\text{span}\{e_1, ..., e_k\}] = -|k| = k$$

Proof of proposition 2.3.8: Let F be a vector bundle over X such that $E \oplus F \equiv X \times V$. Let $\pi_x : V \to V$ be the projection onto E_x. Define the map

$$f : X \to \text{Fred}(\mathcal{H} \otimes V) \cong \text{Fred}(\mathcal{H})$$

by $f_x = f_{-1} \otimes \pi_x + f_0 \otimes (1 - \pi_x)$. Now $\ker(f_x) = 0$ for all x, and $H \otimes V / f(\mathcal{H} \otimes V) \cong E$ thus

$$\text{index}(f) = -[E]$$

Also

$$\text{index}(f_k f) = k - [E]$$

so the index is surjective since every element of $K(X)$ is of the form $k - [E]$.

Finally, to prove that the index map is injective and thus an isomorphism, we use Kuiper's theorem.

Theorem 2.3.10 (Kuiper's theorem)**.** $[X, \mathcal{A}^*] = 1$.

For a proof we refer to [13]. This indeed implies injectivity, by exactness.

We end this section by showing that $PU(\mathcal{H})$ acts on $\text{Fred}(\mathcal{H})$. Here $PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{C}^\times$ with $U(\mathcal{H})$ the unitary operators on $\mathcal{H}$. Let $U(\mathcal{H})$ act on $\text{Fred}(\mathcal{H})$ by

$$\begin{aligned} U(\mathcal{H}) \times \text{Fred}(\mathcal{H}) &\to \text{Fred}(\mathcal{H}) \\ (g, f) &\mapsto g^{-1} \circ f \circ g \end{aligned}$$

Since g is an isomorphism this is well defined. Now if $\alpha \in \mathbb{C}^\times$ then

$$(\alpha g).f = (\alpha g)^{-1} \circ f \circ \alpha g = g^{-1} \circ f \circ g$$

so indeed $PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{C}^\times$ acts on $\text{Fred}(\mathcal{H})$.

2.4 Clifford algebras

In this section we discuss the Clifford algebra and the group spin$^{\mathbb{C}}$. We will need this group and some of its properties in order to be able to define the Thom morphisms which in turn is needed for the T-duality isomorphism.

Clifford algebras where invented to generalize the complex numbers and quaternions. In particular, we are interested in groups that act on $\mathbb{R}^n$ and produce the rotations. Both the unit complex numbers and the unit quaternions are such groups. The spin(n) groups will be the higher dimensional analogs.

These groups are defined as subgroups of the units in the Clifford algebras Cl_n. $\mathbb{R}^n$ is a subspace of Cl_n so the spin group acts naturally on $\mathbb{R}^n$. The spin$^{\mathbb{C}}$ group will be defined similarly in the complex case. We do not wish to go into full detail in defining these concepts and will leave a lot unproven, for an elaborate introduction we refer to [4].

Clifford algebras are closely related to exterior algebras (also known as Grassman algebras). Recall that the exterior algebra of a vector space V is defined to be $T(V)/\langle x \otimes x\rangle$, the tensor algebra on V with the ideal generated by all elements of the form $x \otimes x$ quotiented out.

Now let V be a vector space over a field k and let $\Phi : V \to k$ be a quadratic form on V.

Definition 2.4.1. *We define the Clifford algebra with respect to this quadratic form, $Cl(\Phi)$ to be*

$$Cl(\Phi) := T(V)/\langle x \otimes x - \Phi(x)\rangle$$

We will be interested in the special case were $V = \mathbb{R}^n$ and $\Phi(x_1, x_2, \cdots, x_n) = -(x_1^2 + \cdots x_n^2)$. We will denote this Clifford algebra by Cl_n.

Lemma 2.4.2. *Let V be an n-dimensional vector space. If Φ is a quadratic form we can define the bilinear form $B(u, v) = \frac{1}{2}(\Phi(u + v) - \Phi(u) - \Phi(v))$ and there is an orthonormal basis of V with respect to B. Let $(e_1, \cdots, e_n)$ denote this basis. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_k} | i_1 < \cdots < i_k, 0 \leq k \leq n\}$ is a basis for $Cl(\Phi)$.*

The first statements should be familiar from linear algebra. Now an element in $Cl(\Phi)$ is given by $x_1 \otimes \cdots \otimes x_r$ with $x_i = \sum_{j=0}^{n} x_{i_j} e_j \in V$. So it is a sum of

elements of the form $a\, e_{j_1} \otimes \cdots \otimes e_{j_r}$.
The following relations hold.

$$e_i \otimes e_j = \begin{cases} 1 & \text{when } i = j \\ -e_j \otimes e_i & \text{when } i \neq j \end{cases}$$

Indeed $1 = B(e_i, e_i) = \frac{1}{2}(\Phi(2e_i) - 2\Phi(e_i)) = \frac{1}{2}(4\Phi(e_i) - 2\Phi(e_i)$ so $e_i \otimes e_i = \Phi(e_i) = 1$. And furthermore $0 = B(e_i, e_j) = \frac{1}{2}(\Phi(e_i + e_j) - \Phi(e_i) - \Phi(e_j))$ so by the previous calculation we find $\Phi(e_i+e_j)-2 = 0$ and thus $(e_i+e_j)\otimes(e_i+e_j) = \Phi(e_i + e_j) = 2$. Now

$$\begin{aligned} 2 &= (e_i + e_j) \otimes (e_i + e_j) \\ &= e_i \otimes e_i + e_i \otimes e_j + e_j \otimes e_i + e_j \otimes e_j = 2 + e_i \otimes e_j + e_j \otimes e_i \end{aligned}$$

and thus $e_i \otimes e_j = -e_j \otimes e_i$. These two rules gives us a way to reduce $e_{j_1} \otimes \cdots \otimes e_{j_r}$ to an element of the suggested basis.

Example 2.4.3. *$Cl_1 = \mathbb{C}$, the basis of Cl_1 is $(1, e_1)$ and $e_1 \otimes e_1 = \Phi(e_1) = -1$.*

Multiplication in $Cl(\Phi)$ is done by tensoring elements. The following two morphisms on $Cl(\Phi)$ are defined on basis elements by;

$$\begin{aligned} \alpha : Cl(\Phi) &\to Cl(\Phi) \\ e_{i_1} \otimes \cdots \otimes e_{i_k} &\mapsto (-1)^k e_{i_1} \otimes \cdots \otimes e_{i_k} \\ t : Cl(\Phi) &\to Cl(\Phi) \\ e_{i_1} \otimes \cdots \otimes e_{i_k} &\mapsto e_{i_k} \otimes \cdots \otimes e_{i_1} \end{aligned}$$

we then extend linearly.
Also let $\bar{x} := \alpha(x^t)$ and define the norm of an element to be $N(x) = x\bar{x}$. This corresponds exactly to the regular complex conjugate and norm in the example above. Note that for $x \in \mathbb{R}^n$ we have $N(x) = x \cdot \alpha(x^t) = x \cdot -x = -x^2$.
We are mainly interested in the spin$^{\mathbb{C}}$ groups, which are subgroups of the *complex* Clifford algebras. These are defined as $Cl_n^c := Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ and there are morphisms similar to the ones above.

$$\begin{aligned} \alpha(x \otimes z) &= \alpha(x) \otimes z \\ (x \otimes z)^t &= x^t \otimes \bar{z} \end{aligned}$$

Again $\overline{x \otimes z} = \alpha((x \otimes z)^t)$ and $N(x \otimes z) = (x \otimes z)(\overline{x \otimes z})$.

Definition 2.4.4. *We define the Clifford group Γ_n^c to be the elements $x \in (Cl_n^c)^\times$ with the property that for every $y \in \mathbb{R}^n$, $\alpha(x)yx^{-1}$ is an element of $\mathbb{R}^n$ as well.*

We can define the map $\rho : \Gamma_n^c \to Aut(\mathbb{R}^n)$ by $\rho(x) : y \mapsto \alpha(x)yx^{-1}$.

Fact 2.4.5. *The norm map N maps from Γ_n^c to $\mathbb{C}^*$ and is a homomorphism of groups. Also the kernel of ρ is precisely $\mathbb{C}^*$.*

For a proof of these facts see [4]. Next we define the $Pin^{\mathbb{C}}(n)$ group.

Definition 2.4.6. *The group $Pin^{\mathbb{C}}(n)$ consists of the elements $x \in \Gamma_n^c$ with $N(x) = 1$.*

Lemma 2.4.7. *The restriction of ρ to $Pin^{\mathbb{C}}(n)$,*

$$\rho|_{Pin^{\mathbb{C}}(n)} : Pin^{\mathbb{C}}(n) \to Aut(\mathbb{R}^n)$$

surjects onto $O(n)$ with kernel $U(1) = \{1 \otimes z \in Cl_n^c \,||z| = 1\}$, so we have a short exact sequence

$$0 \to U(1) \to Pin^{\mathbb{C}}(n) \xrightarrow{\rho} O(n) \to 0$$

To see that $\rho(x)$ is an element of the orthonormal group we need to show that for every $y \in \mathbb{R}^n$ the equality $|y| = |\rho(x)y|$ holds. We know that for $y \in \mathbb{R}^n$ we have $|y| = \sqrt{N(y \otimes 1)}$ so it is enough to show that $N(y \otimes 1) = N(\rho(x)(y \otimes 1))$. Now

$$N(\rho(x)y\otimes 1) = N(\alpha(x)y\otimes 1x^{-1}) = N(\alpha(x))N(y\otimes 1)N(x^{-1}) = N(\alpha(x))N(x^{-1})N(y)$$

So if we show that $N(\alpha(x)) = N(x)$ we are done.

$$N(\alpha(x)) = \alpha(x)\alpha(\alpha(x)^t) = \alpha(x)x^t = x\alpha(x^t) = N(x)$$

We still need to show that ρ surjects onto $O(n)$ and that the kernel is $U(1)$. Let e_1 be a basis vector in $\mathbb{R}^n$, then $N(e_1 \otimes 1) = N(e_1) = -e_1e_1 = 1$ and

$$\alpha(e_1)e_ie_1^{-1} = \begin{cases} -e_1 & i = 1 \\ e_i & i \neq 1 \end{cases}$$

So $e_1 \in \mathrm{Pin}^{\mathbb{C}}(n)$, by the same reasoning all vectors of length one lie in $\mathrm{Pin}^{\mathbb{C}}(n)$. For such a unit vector $v \in \mathbb{R}^n$, $\rho(v)$ is the reflection in the hyperplane perpendicular to v. These reflections generate $O(n)$ so ρ is indeed surjective.
The kernel of ρ restricted to $\mathrm{Pin}^{\mathbb{C}}(n)$ is clearly $\ker(\rho) \cap \mathrm{Pin}^{\mathbb{C}}(n)$, so all elements $x \otimes z$ in $(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^*$ with $N(x) = 1$. This is exactly $U(1)$.

Definition 2.4.8. *The group* $\mathrm{spin}^{\mathbb{C}}(n)$ *is defined to be the subgroup of* $\mathrm{Pin}^{\mathbb{C}}(n)$ *that maps to* $SO(n)$ *under* ρ.

Chapter 3

Twisted K-theory

In this chapter we define twisted K-theory and prove some of its properties. The existence of the Umkehr map for certain fibre bundles will be of great importance for T-duality. Twisted K-theory is, as the name suggests, a twisted version of K-theory. The T-duality isomorphism that we will discuss gives isomorphisms between the twisted K-theory of the trivial bundle and the normal K-theory of a non-trivial bundle which depends on the twisting. So in that case we can think of the twisted K-theory as a geometric twist in the vector bundle.

3.1 Definition and first properties

The twist in twisted K-theory is an element of $H^3(X, \mathbb{Z})$, where X is the base space of the vector bundles. To this element we can associate a principal $PU(\mathcal{H})$ bundle. Where $\mathcal{H}$ is an infinite dimensional Hilbert space. We obtain this principal bundle by using the following isomorphisms.

$$\begin{aligned} H^3(X, \mathbb{Z}) &\cong [X, K(\mathbb{Z}, 3)] \cong \left[X, B^2K(\mathbb{Z}, 1)\right] \cong [X, BPU(\mathcal{H})] \\ &\cong \{\text{isomorphism classes of principal } PU(\mathcal{H})\text{-bundles over } X\} \end{aligned}$$

The first isomorphism is a well-known fact from algebraic topology, a proof can be found in [9]. For a proof of the second isomorphism, see lemma B.8. The third isomorphism is treated in B.9. And finally the last isomorphism was discussed in 2.1.5.

In the previous chapter we saw that the group $PU(\mathcal{H})$ acts on $\mathrm{Fred}(\mathcal{H})$, the group of Fredholm operators. This allows us to define an action on $\Omega\mathrm{Fred}(\mathcal{H})$ as well. Now we can construct two new bundles with the associated bundle construction, let $H \in H^3(X;\mathbb{Z})$ be our twist and E_H the principal $PU(\mathcal{H})$-bundle obtained by the above isomorphisms. We define;

$$\begin{aligned} Y_H^0 &= E_H \times_{PU(\mathcal{H})} \mathrm{Fred}(\mathcal{H}) \to X \\ Y_H^{-1} &= E_H \times_{PU(\mathcal{H})} \Omega\mathrm{Fred}(\mathcal{H}) \to X \end{aligned}$$

We define the twisted K-theory is as the homotopy classes of sections of these bundles.

Definition 3.1.1. *The twisted K-theory of X with twist $H \in H^3(X,\mathbb{Z})$ is given by*

$$\begin{aligned} K_H^0(X) &= \Gamma\left[X, Y_H^0\right] \\ K_H^{-1}(X) &= \Gamma\left[X, Y_H^{-1}\right] \end{aligned}$$

The higher twisted K-theory groups are defined by Bott periodicity, i.e. $K^n(X,H) = \Gamma\left[X, E_H \times_{PU(\mathcal{H})} \Omega^n \mathit{Fred}(\mathcal{H})\right] \cong \Gamma\left[X, E_H \times_{PU(\mathcal{H})} \Omega^k \mathit{Fred}(\mathcal{H})\right]$, *where $k = 0$ if n is even and 1 otherwise.*

Twisted K-theory with zero twist reduces to regular K-theory.

Theorem 3.1.2. *Twisted K-theory satisfies the following properties.*

1. *If $H = 0$ then $K^\bullet(X,H) \cong K^\bullet(X)$.*

2. *$K^\bullet(X,H)$ is a module over $K^\bullet(X)$.*

3. *There is a cup product homomorphism*

$$K^p(X,H) \otimes K^q(X,H') \to K^{p+q}(X,H+H')$$

4. *If $f : X \to Y$ is a continuous map, then there is a homomorphism*

$$f^* : K^\bullet(Y,H) \to K^\bullet(X,f^*H)$$

5. *Let $f : X \to Y$ be a smooth, K-oriented map, then there is a homomorphism*

$$f_! : K^\bullet(X) \to K^{\bullet+d}(Y)$$

where $d = \dim(M) - \dim(N)$.

Proof:

1. When $H = 0$, the corresponding principal $PU(\mathcal{H})$ bundle is the trivial bundle. $E_H = X \times PU(\mathcal{H})$ and so $Y_H^n = X \times \Omega^n \mathrm{Fred}(\mathcal{H})$. Now homotopy classes of sections of this bundle correspond to homotopy classes of maps $X \to \Omega^n \mathrm{Fred}(\mathcal{H})$. In section 2.3 we proved that $\mathrm{Fred}(\mathcal{H})$ is a classifying space for K-theory. So indeed

$$K^\bullet(X, 0) = [X, \Omega^\bullet \mathrm{Fred}(\mathcal{H})] = K^\bullet(X)$$

2. This property follows from property 1 and 3. If we have a cup product

$$K^p(X, H) \otimes K^q(X, H') \to K^{p+q}(X, H + H')$$

then for $H = 0$ we obtain

$$K^p(X) \otimes K^q(X, H') \to K^{p+q}(X, H')$$

this gives a module structure.

3. We want to construct a cup product structure. In normal K-theory the cup product comes from a product $\mu : BU \times BU \to BU$ which classifies the tensor product. This means that

$$\begin{aligned} \tilde{K}^0(X) \times \tilde{K}^0(X) &\to \tilde{K}^0(X) \\ (\xi, \eta) &\mapsto \xi \otimes \eta \end{aligned}$$

corresponds to

$$[X, BU] \times [X, BU] \to [X, BU] \tag{3.1}$$

$$(f_\xi, f_\eta) \mapsto \mu(f_\xi, f_\eta) \tag{3.2}$$

under the isomorphism $\tilde{K}^0(X) \cong [X, BU]$.
On $\text{Fred}(\mathcal{H})$ there is an equivalent map

$$\mu : \text{Fred}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \to \text{Fred}(\mathcal{H})$$

given as follows. Since $\mathcal{H}$ is an infinite dimensional Hilbert space, there exists an isomorphism; $i : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ Now define $\mu(f, g) = i \circ f \otimes g \circ i^{-1}$. Now we can define maps

$$\mu_{m,n} : \Omega^m \text{Fred}(\mathcal{H}) \times \Omega^n \text{Fred}(\mathcal{H}) \to \Omega^{m+n} \text{Fred}(\mathcal{H})$$

by

$$\mu_{m,n}(f, g) : S^{m+n} \cong S^m \wedge S^n \stackrel{f \wedge g}{\to} \text{Fred}(\mathcal{H}) \wedge \text{Fred}(\mathcal{H}) \stackrel{\mu}{\to} \text{Fred}(\mathcal{H})$$

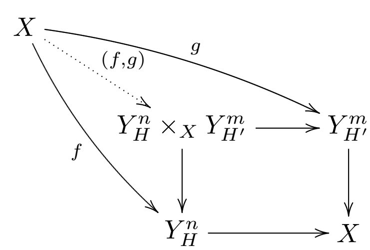
This defines the product in untwisted K-theory, if $x, y \in K^*(X)$ and we represent these by

$$f_x : X \to \Omega^m \text{Fred}(\mathcal{H}) \quad \text{and} \quad f_y : X \to \Omega^n \text{Fred}(\mathcal{H})$$

then $f_x \smile f_y$ is represented by the composition

$$X \stackrel{\Delta}{\to} X \times X \stackrel{(f_x, f_y)}{\to} \Omega^m \text{Fred}(\mathcal{H}) \times \Omega^n \text{Fred}(\mathcal{H}) \stackrel{\mu_{m,n}}{\to} \Omega^{m+n} \text{Fred}(\mathcal{H})$$

Now in the case of twisted K-theory we start with two sections $f : X \to E_H \times_{PU(\mathcal{H})} \Omega^n \text{Fred}(\mathcal{H})$ and $g : X \to E_{H'} \times_{PU(\mathcal{H})} \Omega^m \text{Fred}(\mathcal{H})$ and would like to obtain a section of $E_{H+H'} \times_{PU(\mathcal{H})} \Omega^{n+m} \text{Fred}(\mathcal{H})$. It is not possible to just use the same composition. We will prove that $Y^n_H \times_X Y^m_{H'} \cong E_{H+H'} \times_{PU} (\Omega^n \text{Fred} \times \Omega^m \text{Fred})$ in lemma 3.1.3. Using this lemma we obtain a section h of $E_{H+H'} \times_{PU(\mathcal{H})} \Omega^{n+m} \text{Fred}(\mathcal{H})$ as follows. Since $Y^n_H \times_X Y^m_{H'}$ is a fibered product, the maps f and g give a map $(f, g) : X \to Y^n_H \times_X Y^m_{H'}$.



Now let $h : X \to E_{H+H'} \times_{PU(\mathcal{H})} \Omega^{n+m}\text{Fred}(\mathcal{H})$ be given by the composition;

$$X \overset{(f,g)}{\to} Y_H^n \times_X Y_{H'}^m \cong E_{H+H'} \times_{PU} (\Omega^n\text{Fred} \times \Omega^m\text{Fred}) \overset{\text{id}\times\mu_{m,n}}{\to} Y_{H+H'}^{m+n}.$$

We now prove the lemma.

Lemma 3.1.3. *There is an isomorphism*

$$\begin{aligned} E_{H+H'} \times_{PU} (\Omega^n \mathit{Fred} \times \Omega^m \mathit{Fred}) &\cong (E_H \times_{PU} \Omega^n \mathit{Fred}) \times_X (E_{H'} \times_{PU} \Omega^m \mathit{Fred}) \\ &= Y_H^n \times_X Y_{H'}^m \end{aligned}$$

This is a bit messy. Let us first discuss the elements on the right hand side of the isomorphism.
$E_H = f_H^* EPU$, $f_H : X \to K(\mathbb{Z}, 3) = BPU$. So we have

$$E_H = \{(x, \alpha) \in X \times EPU | f_H(x) = \pi(\alpha)\}$$

or in other words the following diagram commutes

$$\begin{array}{ccc} PU & = & PU \\ \downarrow & & \downarrow \\ f_H^* EPU & \longrightarrow & EPU \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f_H} & BPU \end{array}$$

The action of PU on E_H comes from the action on the universal bundles and is given by

$$\begin{array}{rcl} PU \times E_H & \to & E_H \\ ([g], (x, \alpha)) & \mapsto & (x, [g].\alpha) \end{array}$$

The action of PU on Fred is given by

$$\begin{array}{rcl} PU \times \text{Fred} & \to & \text{Fred} \\ ([g], f) & \mapsto & (x, g^{-1} \circ f \circ g) \end{array}$$

And on $\Omega^n\text{Fred}$ we have

$$\begin{aligned} PU \times \Omega^n\text{Fred} &\rightarrow \Omega^n\text{Fred} \\ ([g], \phi) &\mapsto \phi_g \\ & \phi_g : S^n \rightarrow \text{Fred} \\ & \phi_g(v) = g^{-1} \circ \phi(v) \circ g \end{aligned}$$

So the right-hand side is given by

$$\begin{aligned} Y_H^n \times_X Y_{H'}^m &= \{([(x,\alpha),\psi],[(y,\beta),\phi],z) \in Y_H^n \times Y_{H'}^m \times X | ... \\ &\quad ...x = y = z, \pi(\alpha) = f_H(x), \pi(\beta) = f_{H'}(y)\} \\ &= \{[(x,\alpha),\psi],[(x,\beta),\phi] | \pi(\alpha) = f_H(x), \pi(\beta) = f_{H'}(y)\} \end{aligned}$$

And where $[(x,\alpha),\psi] \sim [(x, g.\alpha), \psi_g]$ and similarly for $[(x,\beta),\phi]$. The left hand side is given by

$$E_{H+H'} \times_{PU} (\Omega^n\text{Fred} \times \Omega^n\text{Fred}) = \{[(y,\gamma),\chi_1,\chi_2] | f_{H+H'}(y) = \pi(\gamma)\}$$

and where $[(y,\gamma),\chi_1,\chi_2] \sim [(y, g.\gamma), g.\chi_1, g.\chi_2]$.

1. The universal bundle $EPU \rightarrow BPU$ has as its fibre the group PU, thus for all $p \in PU$ we have $\pi^{-1}(p) \cong PU$. Fix an element $g \in PU$ and these isomorphisms.

2. The elements of $E_{H+H'} \times_{PU} (\Omega^n\text{Fred} \times \Omega^n\text{Fred})$ are of the form $[(y,\gamma),\chi_1,\chi_2]$ where $\gamma \in \pi^{-1}(f_{H+H'}(y)) \cong PU$. We can use the action of PU to obtain an element of the following form, where $g \in PU \cong \pi^{-1}(y)$.

$$((y,g), \chi_{1g\gamma^{-1}}, \chi_{2g\gamma^{-1}})$$

3. Similarly, we can represent a class $([(x,\alpha),\phi][(x,\beta)],\psi)$ in $Y_H^n \times_X Y_{H'}^m$ by an element

$$\big(\big((x,g),\phi_{g\alpha^{-1}}\big), \big((x,g),\psi_{g\beta^{-1}}\big)\big)$$

4. We now define

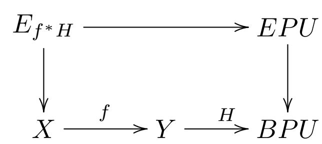
$$\begin{aligned} \Psi : E_{H+H'} \times_{PU} (\Omega^n\text{Fred} \times \Omega^n\text{Fred}) &\rightarrow Y_H^n \times_X Y_{H'}^m \\ ((y,g), \chi_{1g\gamma^{-1}}, \chi_{2g\gamma^{-1}}) &\mapsto (((y,g),\chi_{1g\gamma^{-1}}), ((y,g),\chi_{2g\gamma^{-1}})) \\ \Phi : Y_H^n \times_X Y_{H'}^m &\rightarrow E_{H+H'} \times_{PU} (\Omega^n\text{Fred} \times \Omega^n\text{Fred}) \\ \big(\big((x,g),\phi_{g\alpha^{-1}}\big), \big((x,g),\psi_{g\beta^{-1}}\big)\big) &\mapsto ((x,g), \phi_{g\alpha^{-1}}, \psi_{g\beta^{-1}}) \end{aligned}$$

These maps give the isomorphism and conclude the proof of property 3.

4. Let $f : X \to Y$ be a continuous map, we want to show that there is a homomorphism

$$f^* : K^\bullet(Y, H) \to K^\bullet(X, f^*H)$$

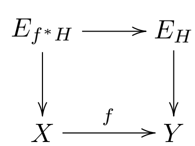
Notice that E_{f^*H} is defined as the pull-back of the following diagram



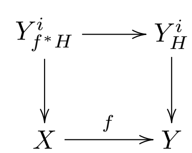
On the other hand E_H is the pullback of the universal PU bundle over H. For any two maps $X \xrightarrow{g} Y \xrightarrow{h} Z$, the pullback construction satisfies $(h \circ g)^* \cong g^* \circ h^*$ since

$$\begin{aligned} (h \circ g)^*(E) &= \{(x, e) \in X \times E | h(g(e)) = \pi(e)\} \\ g^* \circ h^*(E) &= \{(x, y, e) \in X \times Y \times E | g(x) = y, h(y) = \pi(e)\} \end{aligned}$$

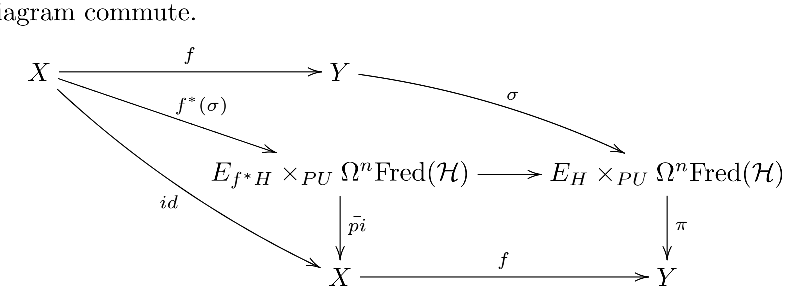
are homeomorphic spaces. So we find that $E_{f*H} = f^*E_H$ and the following diagram is a pull-back diagram;



Applying $\times\Omega^i\text{Fred}(\mathcal{H})$ to the top row we obtain commutative diagrams



Now we define $f^*(\sigma)(x) = (x, \sigma(g(x))$ as the map that makes the following diagram commute.



We see that $f^*(\sigma)$ is a section since $\bar{pi}(f^*(\sigma)) - id_X$.

5. We leave the proof of property five to section 3.3.

3.2 The Thom isomorphism

In this section we will discuss the construction of the Thom morphism in normal and twisted K-theory. An important notion in the construction of the Thom isomorphism is the Thom space.

Definition 3.2.1. *Let $\nu \to X$ be a vector bundle. Let $D(\nu)$ be the disk bundle, $D(\nu)|_x = \{v \in \nu|_x$ and $|v| \leq 1\}$ and let $S(\nu)$ be the analogously defined sphere bundle of ν. We define the Thom space to be*

$$X^\nu = D(\nu)/S(\nu)$$

Lemma 3.2.2. *If X is compact then $X^\nu = \nu \cup \{\infty\}$*

Proof: There is an open cover $\{U_\alpha\}$ of X such that $\nu|_{U_\alpha} = U_\alpha \times \mathbb{C}^m$, since X is compact we can find a subcover, $\{U_i\}_{i=0}^n$.
The Thom space of $\nu|_{U_i}$ is

$$U_i \times D^{2m}/U_i \times S^{2m-1} \simeq U_i \times S^{2m} \simeq U_i \times \mathbb{C}^m \cup \{\infty\}$$

The Thom space of the whole bundle is then given by

$$\left(\cup_{i=0}^n U_i \times D^{2m}/\sim_\nu\right) / \left(\cup_{i=0}^n U_i \times S^{2m-1}/\sim_\nu\right)$$

$$\simeq$$

$$\cup_{i=0}^n U_i \times \mathbb{C}^m \times \{\infty\} / \sim_\nu, \sim_\infty$$

$$\simeq$$

$$(\cup_{i=0}^n U_i \times \mathbb{C}^m/\sim_\nu) \cup \{\infty\}$$

$$=$$

$$\nu \cup \{\infty\}$$

A vector bundle $\nu \to X$ is called h-oriented, where h is a generalized cohomology theory, if there are isomorphisms of $h^\bullet(X)$ modules;

$$h^\bullet(X) \to h^{\bullet+d}(X^\nu)$$

where $d = \dim(\nu)$. These are called Thom isomorphisms.

It turns out that a vector-bundle is K-oriented if it is a spin$^{\mathbb{C}}$-bundle. A complex vector bundle is always a spin$^{\mathbb{C}}$-bundle and we will first construct the Thom morphism in untwisted K-theory for such a complex vector bundle. Notice that in this case d is even, so there is no dimension shift by Bott periodicity.

To construct the Thom isomorphism we have to introduce compactly supported K-theory, $K_c^\bullet(X)$. We will establish an isomorphims between $K^\bullet(X^\nu)$ and $K_c^\bullet(\nu)$. This allows us to define the Thom morphism as a map $K^\bullet(X) \to K_c^\bullet(\nu)$ which we will then generalize to a morphism in the twisted K-theory. Finally we will construct the Thom morphism for spin$^{\mathbb{C}}$-bundles in a similar way.

Definition 3.2.3. *A compactly supported vector bundle $\nu \to X$ is a vector bundle with the property that there is a compact set $K \subset X$, such that $\nu|_{X\setminus K}$ is trivial.*

Definition 3.2.4. *Compactly supported K-theory $K_c^0(X)$ is the Groethendieck group of isomorphism classes of compactly supported vector bundles on X.*

Note that when X is compact we just get the normal K-theory, we can take K to be X since every vector bundle is trivial over the empty set.

Theorem 3.2.5. *If X is a compact space and $\nu \to X$ is a vector bundle over X then there is an isomorphism*

$$\tilde{K}^*(X^\nu) \cong K_c^*(\nu)$$

Proof: Since X is compact we have $X^\nu \simeq \nu \cup \{\infty\}$. We define the following maps.

$$\begin{array}{rcl} \mathrm{Vect}(\nu \cup \{\infty\}) & \to & \mathrm{Vect}_c(\nu) \\ \mu & \mapsto & \mu|_\nu \\ \mathrm{Vect}(\nu \cup \{\infty\}) & \leftarrow & \mathrm{Vect}_c(\nu) \\ \text{add a copy of } \mathbb{C}^{\,n} \text{ at } \{\infty\} & \leftarrow & \rho \end{array}$$

We have to check that this is well defined. First we prove that $\mu|_\nu$ is compactly supported. We know that X is a compact space, so $\nu \cup \{\infty\}$ is the one-point-compactification of ν and thus compact as well. Let $\{W_j\}$ be an open cover of

$\nu \cup \{\infty\}$ such that $\mu|_{W_j} \cong W_j \times \mathbb{C}^m$. Let W_l be such that $\{\infty\} \in W_l$. Now $K := (\nu \cup \infty)\backslash W_l$ is a closed subspace of a compact space, so it is compact too. Furthermore $\mu|_{\nu\backslash K}$ is trivial.
The second map can be defined since we know that there is a compact set K such that $\rho|_{\nu/K}$ is trivial. ν/K is a neighborhood of ∞ so we can just paste in a trivial part.

Now let $p : \nu \to X$ be a vector bundle. There are three steps in the construction of the Thom homomorphism.

1. Define a $K^*(X)$ module structure on $K_c^*(\nu)$.
2. Find a Thom class μ_ν in $K_c^*(\nu)$.
3. Define the Thom morphism to be $w \mapsto w.\mu_\nu$

We define the module structure as follows.

$$\begin{array}{rcl} K^*(X) \times K_c^*(\nu) & \to & K_c^*(\nu) \\ (\eta, \xi) & \mapsto & p^*\eta \otimes \xi \end{array}$$

Next we define the Thom class to be the following alternating sum in $K_c(\nu)$.

$$\mu_\nu = \sum_{i=0}^{n} (-1)^i \wedge^i (p^*\nu)$$

To see that this is indeed an element in $K_c(\nu)$ we need to prove the following lemma.

Lemma 3.2.6. *Let ν be an n-dimensional complex vector bundle, $\mu_\nu|_{\nu\backslash\{0\}}$ is trivial in $K(\nu\backslash\{0\})$.*

Proof: We will show that there is a long exact sequence.

$$0 \to \wedge^n p^*\nu|_{\nu\backslash\{0\}} \xrightarrow{d^n} \wedge^{n-1} p^*\nu|_{\nu\backslash\{0\}} \xrightarrow{d^n-1} \cdots$$
$$\cdots \xrightarrow{d^3} \wedge^2 p^*\nu|_{\nu\backslash\{0\}} \xrightarrow{d^2} p^*\nu|_{\nu\backslash\{0\}} \xrightarrow{d^1} \mathbb{C} \times \nu\backslash\{0\} \to 0$$

For a short exact sequence of vector bundles $0 \to A \to B \to C \to 0$ over X we have $[B] = [A] + [C]$ in $K(X)$, i.e. $[A] - [B] + [C] = 0$. So when we have a long exact sequence of the above form, we can split it into the short exact sequences;

$$
\begin{aligned}
0 \to \wedge^n p^*\nu|_{\nu\backslash\{0\}} \to \wedge^{n-1} p^*\nu|_{\nu\backslash\{0\}} \to Im(d^{n-1}) \to 0 \\
0 \to Ker(d^i) \to \wedge^i p^*\nu|_{\nu\backslash\{0\}} \to Im(d^i) \to 0 \\
0 \to Ker(d^1) \to p^*\nu|_{\nu\backslash\{0\}} \to \mathbb{C} \times \nu\backslash\{0\} \to 0
\end{aligned}
$$

and indeed $\mu_\nu|_{\nu\backslash\{0\}} = \sum_{i=0}^n (-1)^i \wedge^i p^*\nu|_{\nu\backslash\{0\}} = 0 \in K(\nu\backslash\{0\}$.
We will first define a long exact sequence of vector spaces,

$$0 \to \wedge^n \mathbb{C}^n \xrightarrow{d^n} \wedge^{n-1}\mathbb{C}^n \xrightarrow{d^{n-1}} \cdots \xrightarrow{d^3} \wedge^2\mathbb{C}^n \xrightarrow{d^2} \mathbb{C}^n \xrightarrow{d^1} \mathbb{C} \to 0$$

Let $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the standard inner product. Pick an element $v \in \mathbb{C}^n$. Define the map $v^* : \mathbb{C}^n \to \mathbb{C}$ by $v^*(\alpha) = \langle v, \alpha \rangle$. Now we define $d^i_v(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=0}^i (-1)^k e_{j_1} \wedge \cdots \wedge v^*(e_{j_k}) \wedge \cdots \wedge e_{j_i}$ on basis elements, and extend this map linearly to all of $\wedge^i \mathbb{C}^n$. This gives a long exact sequence for all $v \neq 0$.
The fiber of ν is $\mathbb{C}^n$, so the fiber of $p^*\nu$ is $\mathbb{C}^n$ as well. The bundle $\wedge^i p^*\nu$ is obtained by replacing all fibers by $\wedge^i \mathbb{C}^n$ while maintaining the identifications. We define differentials $d^i : \wedge^i p^*\nu|_{\nu\backslash\{0\}} \to \wedge^{i-1} p^*\nu|_{\nu\backslash\{0\}}$ by $d^i(u, v) = (u, d^i_u(v))$. We obtain long exact sequences on the fibers. We still need to check that the d^i are continuous. This is a local question, so we may pick a subspace $U \in \nu\backslash\{0\}$ where $p^*\nu|_U = U \times \mathbb{C}^n$. Then $\wedge^i p^*\nu|_U = U \times \wedge^i \mathbb{C}^n$ and $d^i(u, v) = (u, d^i_u(v))$ is continuous. $\square$
So the Thom morphism is given by

$$
\begin{array}{ccc}
K^*(X) & \to & K^*_c(\nu) \\
\eta & \mapsto & \displaystyle\sum_{i=0}^n p^*\eta \otimes \wedge^i (p^*\nu)
\end{array}
$$

To be able to generalize this definition to the case of twisted K-theory we want to split the morphism up in three maps.

$$
\begin{array}{ccc}
K^*(X) & \xrightarrow{-\otimes\mu_\nu} & K^*(X) \otimes K^*_c(\nu) \\
\downarrow{\scriptstyle \text{Thom}} & & \downarrow{\scriptstyle p^*\otimes id} \\
K^*_c(\nu) & \xleftarrow{\text{module mult.}} & K^*(\nu) \otimes K^*_c(\nu)
\end{array}
$$

Then we can define the Thom morphism in the twisted case as

$$\begin{array}{ccc} K^*(X,H) & \xrightarrow{-\otimes \mu_\nu} & K^*(X,H) \otimes K^*_c(\nu) \\ \Big\downarrow{\scriptstyle \text{Thom}} & & \Big\downarrow{\scriptstyle p^*\otimes id} \\ K^*_c(\nu, p^*H) & \xleftarrow{\text{module mult}} & K^*(\nu, p^*H) \otimes K^*_c(\nu) \end{array}$$

The statement that this is an isomorphism is equivalent to the Bott periodicity theorem and thus hard to prove. There are proofs in the literature [], [] but these are all given in the setting of C^*-algebras which the author is unfamiliar with.

Unfortunately the bundles for which we want to construct Thom morphisms won't always be complex. They will be spin$^{\mathbb{C}}$-bundles, so next we give a definition of spin$^{\mathbb{C}}$ vector bundles and we will construct a Thom class μ_ν for these bundles. In this case there can be a dimension shift.

Definition 3.2.7. *A vector bundle $\nu \to X$ is a $spin^{\mathbb{C}}(n)$ bundle when the classifying map factors as follows;*

$$\begin{array}{ccc} & & Bspin^{\mathbb{C}}(n) \\ & \nearrow & \Big\downarrow \\ X & \longrightarrow & BO(n) \end{array}$$

Lemma 3.2.8. *The following are equivalent*

1. *$\nu \to X$ is a $spin^{\mathbb{C}}(n)$ bundle*
2. *there is a principal $spin^{\mathbb{C}}(n)$-bundle P_ν such that $\nu = P_\nu \times_{spin^{\mathbb{C}}(n)} \mathbb{R}^n$*

To prove this lemma we need the lemma and fact below.

Lemma 3.2.9. *Let $\pi : E \to B$ be a principal G-bundle, Y a G-space and $f : X \to B$ a map. Then $f^*E \times_G Y \cong f^*(E \times_G Y)$.*

Proof: The map $\pi' : E \times_G Y \to X$ is given by $\pi'([(e,y)]) = \pi(e)$[1]. The right hand side becomes,

$$f^*(E \times_G Y) = \{(x, [(e,y)]) | f(x) = \pi'([(e,y)]) = \pi(e)\}$$

[1] We discussed this in section 2.1

where $(x, [(e, y)]) = (x', [(e', y')])$ if and only if $e' = e.g$, $y' = y.g$ and $x = x'$. On the other hand, the left hand side is given by,

$$f^*E \times_G Y = \{[((x, e), y)] | f(x) = \pi(e)\}$$

and $[((x, e), y)] = [((x', e'), y')]$ if and only if $(x', e') = (x, e).g = (x, e.g)$, and $y' = y.g$.

Fact 3.2.10. *The universal vector bundle $E_n \to Gr_n$ is constructed as*

$$EO(n) \times_{O(n)} \mathbb{R}^n \to BO(n)$$

where $BO(n) \cong Gr_n$.[2]

Proof of lemma 3.2.9: $\nu \to X$ is a $\mathrm{spin}^{\mathbb{C}}(n)$-bundle. Denote its classifying map by $c_\nu : X \to BO(n)$. Then

$$\nu = c_\nu^*(E_n) = c_\nu^*(EO(n) \times_{O(n)} \mathbb{R}^n) = c_\nu^* EO(n) \times_{O(n)} \mathbb{R}^n.$$

The map c_ν factors through $B\mathrm{spin}^{\mathbb{C}}(n)$ by definition of a $\mathrm{spin}^{\mathbb{C}}(n)$-bundle. We write c'_ν for the map from $X \to \mathrm{spin}^{\mathbb{C}}(n)$. Define $P_\nu := c'^*_\nu(E\mathrm{spin}^{\mathbb{C}}(n))$. Then

$$P_\nu \otimes_{\mathrm{spin}^{\mathbb{C}}(n)} \mathbb{R}^n = c'^*_\nu E\mathrm{spin}^{\mathbb{C}}(n) \otimes_{\mathrm{spin}^{\mathbb{C}}(n)} \mathbb{R}^n = c_\nu^* EO(n) \otimes_{O(n)} \mathbb{R}^n$$

since $\tau \circ c'_\nu = c_\nu$, with $\tau : B\mathrm{spin}^{\mathbb{C}}(n) \to BO(n)$.

To construct the Thom class for $\pi : \nu \to X$, first assume that ν is an even dimensional bundle. So $\nu = P_\nu \times_{\mathrm{spin}^{\mathbb{C}}(2m)} \mathbb{R}^{2m}$.
The lemma below implies that $\mathrm{spin}^{\mathbb{C}}(2k)$ acts on $\wedge(\mathbb{C}^k)$, since $\mathrm{spin}^{\mathbb{C}}(2k)$ is a subgroup of C^c_{2k}.

Lemma 3.2.11. *C^c_{2k} acts on $\wedge\mathbb{C}^k$.*

Give $\mathbb{C}^k$ the standard Hermitian metric and let $e_1, ...e_k$ be an orthonormal basis. Then we have an inherited metric on the exterior algebra and

$$\{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_n} | 0 \le n \le k, i_1 < ... < i_n\}$$

is an orthonormal basis for it. For each $v \in \mathbb{C}^k$ we define $d_v : \wedge\mathbb{C}^k \to \wedge\mathbb{C}^k$ by $d_v(w) = v \wedge w$. Let δ_v be the adjoint with respect to the metric. We define

$$\mathbb{C}^k \otimes_{\mathbb{R}} \wedge\mathbb{C}^k \to \wedge\mathbb{C}^k$$

[2]See lemma B.12

by $v \otimes w \to d_v(w) - \delta_v(w)$. Now $(d_v - \delta_v)^2(w) = -|v|^2 w$, [4]. So we have a corresponding bilinear map $\psi : \mathbb{R}^{2k} \to End(\wedge \mathbb{C}^k)$. This map has the property that $\psi(v)^2 = -|v|^2 id = Q(v)id$ so by the universal property [4] of the Clifford algebra there is a linear map $\bar{\psi} : C_{2k} \to End(\wedge \mathbb{C}^k)$ such that $\bar{\psi} \circ i = \psi$ with $i : \mathbb{R}^{2k} \to C_{2k}$ the embedding into the tensor algebra, followed by the quotient map. So $\wedge \mathbb{C}^k$ is a complex C_{2k} module, or in other words a $C_{2k} \otimes_{\mathbb{R}} \mathbb{C}$ module. This exactly means that C_{2k}^c acts on $\wedge \mathbb{C}^k$.
M is $\mathbb{Z}_2$ graded, where

$$
\begin{aligned}
M^0 &= \oplus_{i=0}^{\lfloor m/2 \rfloor} \wedge^{2i} \mathbb{C}^m \\
M^1 &= \oplus_{i=0}^{\lfloor (m-1)/2 \rfloor} \wedge^{2i+1} \mathbb{C}^m
\end{aligned}
$$

Define $E := P_\nu \times_{\text{spin}^{\mathbb{C}}(2m)} M$. This bundle is $\mathbb{Z}_2$ graded since $\text{spin}^{\mathbb{C}}(2m)$ acts on M leaving the grading in tact.

Definition 3.2.12. *We define the Thom class as* $\mu_\nu = \pi^* E^0 - \pi^* E^1$.

We need to show that $\pi^* E^0 - \pi^* E^1 \in K_c^\bullet(\nu)$.
When ν is odd dimensional, there is an embedding $e : \text{spin}^{\mathbb{C}}(n) \hookrightarrow \text{spin}^{\mathbb{C}}(n+1)$, and $\nu \oplus \mathbb{R} \cong (e \circ f)^* E\text{spin}^{\mathbb{C}}(n+1)$. So $\nu \oplus \mathbb{R}$ is a $\text{spin}^{\mathbb{C}}(n+1)$-bundle. We can define a Thom class in $K_c^\bullet(\nu \oplus \mathbb{R}) \cong K^{\bullet+1}(\nu)$, thus for odd dimensional bundles we get a dimension shift.

Lemma 3.2.13. *When ν is a complex vector bundle, the two definitions of μ_ν agree.*

3.3 Umkehr maps

An Umkehr map in a homology or cohomology theory is an induced map in the 'wrong' direction. In twisted K-theory this means that for a morphism $f : X \to Y$ we wish to construct a map

$$f_! : K^\bullet(X, f^*H) \to K^{\bullet+d}(Y, H)$$

with $d = \dim(Y) - \dim(X)$, as in property 5. It is not always possible to construct the Umkehr map, the map f needs to have some nice properties. In the case of K-theory we require that $f^*TY \oplus TX$ is a spin$^{\mathbb{C}}$ bundle. Furthermore we only consider the following two cases, our map is a smooth map between smooth manifolds and

Case A: $i : X \hookrightarrow Y$ is an embedding.

Case B: $p : X \twoheadrightarrow Y$ is a fiber bundle with fiber F, a smooth manifold.

Lemma 3.3.1. *When $\xi \to X$ and $\eta \to X$ are both spin$^{\mathbb{C}}$-bundles, and $\nu \oplus \eta = \xi$, then $\nu \to X$ is a spin$^{\mathbb{C}}$-bundle as well.*

Proof: There is a open cover $\{U_\alpha\}$ such that all three bundles are trivial over the sets U_α. The clutching functions $\phi^\xi_{\alpha,\beta}, \phi^\eta_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ both factor through spin$^{\mathbb{C}}$. The clutching function of $\xi \oplus \nu$ are given by $\begin{pmatrix} \phi^\xi_{\alpha,\beta}(u) & 0 \\ 0 & \phi^\nu_{\alpha,\beta} \end{pmatrix}$. So since the clutching functions of η factor through spin$^{\mathbb{C}}$ the clutching functions of ν do as well.

We have $i : X \hookrightarrow Y$, let $\nu \to X$ be the normal bundle over X to Y. Now $\nu \cong i^*TY - TX = i^*TY \oplus TX - 2TX$, now $2TX$ can be given a complex structure and thus is in particular a spin$^{\mathbb{C}}$-bundle. Lemma 3.3.1 assumption now tells us that ν is a spin$^{\mathbb{C}}$-bundle. Let d be the dimension of ν. We can construct a Thom class and a Thom morphism, $K^\bullet(X, i^*H) \to K_c^{\bullet+d}(\nu, p^*i^*H)$. By the tubular neighborhood theorem we know that there is a neighborhood $X \subset T \subset Y$ such that ν is homeomorphic to T. Let $f : T \to \nu$ be this homeomorphism. To obtain the umkehr map we define an extension map

$$K_c^{\bullet+d}(\nu, p^*i^*H) \to K^{\bullet+d}(Y, H)$$

We have $\nu \cong T \subset Y$, a compactly supported section $\sigma : \nu \to E_{p^*i^*H} \times_{PU} \Omega^\bullet\mathrm{Fred}(\mathcal{H})$ is zero in a neighborhood of the boundary δT, so we can extend it by zero on $Y \backslash T$. Since the following diagram commutes up to homotopy

$$\begin{array}{ccc} \nu & \xrightarrow{\cong} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[i]{} & Y \end{array}$$

we have $E_{p^*i^*H} = E_H$. So we have in fact defined a section $\bar{\sigma} : Y \to E_H \times_{PU} \Omega^i\mathrm{Fred}(\mathcal{H})$ and we are done.

We have found the morphism

$$K^\bullet(X, i^*H) \to K^{\bullet+d}(Y, H)$$

and indeed $d = \dim(\nu) = \dim(i^*TY) - \dim(TX) = \dim(Y) - \dim(X)$.
Now we discuss case B where $p : X \to Y$ is a fibration with smooth fibre F. For N big enough we can find an embedding $i : X \to \mathbb{C}^N$. Then $g = (f, i) : X \to Y \times \mathbb{C}^N$ is an embedding as well and $\nu_g = p^*TY \oplus \mathbb{C}^N - TX$ is a *spin*-bundle, we now use the same construction as before and obtain

$$\begin{aligned} K^\bullet(X, p^*H) \to K^{\bullet+\dim(\nu_g)}(\nu_g, \pi^*p * H) &\to K^{\bullet+\dim(\nu_g)}(Y \times \mathbb{C}^N, H) \\ &= K^{\bullet+\dim(\nu_g)-N}(Y, H) \end{aligned}$$

And $\dim(\nu_g) - N = \dim(Y) + N - \dim(X) - N = \dim(Y) - \dim(X)$.

3.4 Calculations of twisted K-theory

In this section we use spectral sequences to compute twisted K-theory groups. The Atiyah-Hirzebruch spectral sequence for K-theory, is the spectral sequence in proposition 2.2.6, with $h^* = K^*$ and a fibration $* \to X \xrightarrow{\text{id}} X$. This sequence converges to K-theory. In [1], Atiyah and Segal construct different differentials and obtain a spectral sequence with the same E_2 page, that converges to twisted K-theory.
Before treating our first example we need some general facts about the Atiyah-Hirzebruch spectral sequence for (twisted) K-theory. The pages of a spectral sequence are depicted as a table with q on the vertical axes and p on the horizontal axes.
All rows corresponding to odd q are zero since $E_2^{p,q} = H^p(X, K^q(*))$ and for odd q, $K^q(*) = 0$. For even q, $K^q(*) = \mathbb{Z}$ so the E_2 page is given by;

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	0	0	0	0	0	0
2	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$H^4(X,\mathbb{Z})$	$H^5(X,\mathbb{Z})$
1	0	0	0	0	0	0
0	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$H^4(X,\mathbb{Z})$	$H^5(X,\mathbb{Z})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
E_2	0	1	2	3	4	5

In the cases of our interest it is possible to find these cohomology groups. To find the differentials is harder, at least the odd differentials. The even differentials are all zero since they go down an odd number of rows, they have either domain or range zero. To find the terms of the third page E_3, we take $\ker(d_2)/\mathrm{im}(d_2)$ at every point. Since $d_2 = 0$ everywhere, we obtain $E_3^{p,q} = E_2^{p,q}$.
Luckily in some cases the odd differentials for normal and twisted K-theory have been calculated.

Our first computation is the twisted K-theory of the three sphere, S^3. The twist H is an element of $H^3(S^3,\mathbb{Z}) = Z$.

Example 3.4.1 (S^3)**.** *The cohomology of S^3 is given by:*

$$H^i(S^3) = \begin{cases} \mathbb{Z} & i = 0,3 \\ 0 & \textit{otherwise} \end{cases}$$

so the E_3 page is

2	$\mathbb{Z}$	0	0	$\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$
E_3	0	1	2	3

The only differential that can be non-trivial is d_3 from the zeroth column to the third column two rows below. This is because higher differentials will go four or more rows to the right and thus have trivial image. It turns out that d_3 is taking the cup product with H [1]. When $H = 0$ the d_3 map is zero as well, so $E_\infty = E_2$. Using part (c) of the proposition we calculate $K^0(S^3) = K^1(S^3) = \mathbb{Z}$. When $H \neq 0$ however, the $E_4 = E_\infty$ page looks like

2	0	0	0	$\mathbb{Z}_H$
1	0	0	0	0
0	0	0	0	$\mathbb{Z}_H$
E_3	0	1	2	3

Now using (c) we find $K^0(S^3, H) = 0$ *and* $K^1(S^3, H) = \mathbb{Z}_H$.

The next example is rather general and includes the previous example as a special case. We want to calculate the twisted K-theory of principal circle bundles E over M^g, the closed surface of genus g.

Example 3.4.2 (circle bundles over M^g)**.** *These bundles are classified by their first Chern class*[3]*. The first Chern class is an element of* $H^2(M^g) = \mathbb{Z}$ *so is given by an integer. We will denote the circle bundle corresponding to* $k \in \mathbb{Z}$ *by* ν_k*. To calculate the twisted K-theory groups we need the cohomology groups of* E_g *which were calculated in 2.2.8 . For the trivial bundle we found*

$$H^i(M^g \times S^1) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^{2g+1} & i = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

and when $k \neq 0$ *we have*

$$H^i(E_k) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^{2g} & i = 1 \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_k & i = 2 \\ 0 & \textit{otherwise} \end{cases}$$

So we find that the E_2 *and thus the* E_3 *page of the Atiyah Hirzebruch spectral sequence for* $k = 0$ *is given by*

2	$\mathbb{Z}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}$
E_3	0	1	2	3

[3] Or rather the first Chern class of the complex line bundle corresponding to the circle bundle $\gamma_E = E \times_{U(1)} \mathbb{C}$.

again the only differential that is possibly nontrivial is d_3 and by [1] it is given by multiplication by $H \in H^3(E_0) = \mathbb{Z}$. We find

(E,H)	$(E_0,0)$	(E_0,H) $0 \neq H \in \mathbb{Z}$
$K^0(E,H)$	$\mathbb{Z}^{2g+2}$	$\mathbb{Z}^{2g+1} \oplus \mathbb{Z}_H$
$K^1(E,H)$	$\mathbb{Z}^{2g+2}$	$\mathbb{Z}^{2g+1}$

Finally when $k \neq 0$ the E_3 page is

2	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_k$	$\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_k$	$\mathbb{Z}$
E_3	0	1	2	3

again $d_3 = H \in H^3(E_k) = \mathbb{Z}$ and all other differentials vanish. We find

(E,H)	$(E_k,0)$	(E_k,H) $0 \neq H \in \mathbb{Z}$
$K^0(E,H)$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_H$
$K^1(E,H)$	$\mathbb{Z}^{2g+1} \oplus \mathbb{Z}_k$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_k$

In the next chapter we will show that the above calculations are in accordance with T-duality. We will also calculate some more twisted K-theory groups. To do so we need the T-duality isomorphism, this is why we defer these calculations.

Chapter 4

T-duality

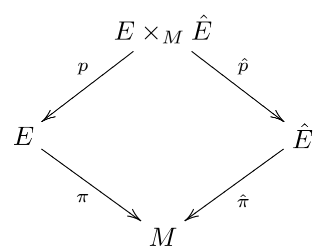
4.1 The T-duality theorem

In this section we will state and prove the T-duality theorem [7]. The T-duality theorem is a statement about pairs (E, H) where E is a principal circle bundle and $H \in H^3(E, \mathbb{Z})$ a cohomology class.

Theorem 4.1.1. *Let $\pi : E \to M$ be a principal circle bundle, with M a smooth manifold and let H be an element of $H^3(E, \mathbb{Z})$. Then there exists a unique principal circle bundle $\hat{\pi} : \hat{E} \to M$ and a unique element $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$ such that*

$$c_1(E) = \hat{\pi}_! \hat{H} \quad c_1(\hat{E}) = \pi_! H \quad p^* H = \hat{p}^* \hat{H}$$

where the maps $p, \hat{p}$ are defined by



and we have the following isomorphism in twisted K-theory.

$$K^\bullet(E, H) \cong K^{\bullet+1}(\hat{E}, \hat{H})$$

For the first part of this statement we will need to define the Chern classes and make the maps in the Gysin sequence 2.2.7 explicit.

Theorem 4.1.2 (Chern classes)**.** *There is a unique sequence of functions* $c_1, c_2, ...$ *assigning to each complex vector bundle* $E \to B$ *a class* $c_i(E) \in H^{2i}(B, \mathbb{Z})$ *depending only on the isomorphism class of* E*, such that*

1. $c_i(f^*(E)) = f^*(c_i(E))$ *for a pullback* $f^*(E)$
2. $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$ *for* $c = 1 + c_1 + c_2 + ... \in H^*(B, \mathbb{Z})$
3. $c_i(E) = 0$ *when* $i > dim(E)$
4. *for the canonical line bundle* $E \to \mathbb{C}P^\infty$*,* $c_1(E)$ *is a generator of* $H^2(\mathbb{C}P^\infty, \mathbb{Z})$ *specified in advance.*

For a proof of this theorem/definition we refer to [10]. Let $\pi : E \to M$ be a principal circle bundle. We can obtain an oriented complex vector bundle by changing all the fibers from S^1 to $\mathbb{C}$ using the associated bundle construction. We obtain a Gysin sequence for E where the dimension shift is 1 and the maps are given by [6];

$$\cdots \longrightarrow H^k(M) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\pi_!} H^{k-1}(M) \overset{\cup c_1(E)}{\longrightarrow} H^{k+1}(S(E)) \longrightarrow \cdots$$

The map $\pi_!$ is the Umkehr map in cohomology, it is given by a construction similar to the one we discussed for K-theory in the previous section.

Proof of 4.1.1: Let $\hat{E}$ be the principal circle bundle associated to the complex line bundle with first Chern class $\pi_! H$. By abuse of notation we write $c_1(\hat{E})$ for this Chern class. Looking at the following part of the Gysin sequence for $\pi : E \to M$

$$H^3(M) \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_!} H^2(M) \overset{\cup c_1(E)}{\longrightarrow} H^4(M)$$

we see that $c_1(\hat{E}) \cup c_1(E)$ is zero since the sequence is exact and $c_1(\hat{E}) \in \mathrm{im}(\pi_!)$. Next we look at the same part of the Gysin sequence for $\hat{E}$

$$H^3(M) \xrightarrow{\hat{\pi}^*} H^3(\hat{E}) \xrightarrow{\hat{\pi}_!} H^2(M) \overset{\cup c_1(\hat{E})}{\longrightarrow} H^4(M)$$

we just saw that $c_1(\hat{E}) \cup c_1(E) = 0$ so by exactness there must be an element $\hat{H} \in H^3(\hat{E})$ such that $\hat{\pi}_! \hat{H} = c_1(E)$. This element is not unique since we could add any homology class $\alpha \in H^3(\hat{E})$ with $\hat{\pi}_!\alpha = 0$, i.e. $\alpha = \widehat{pi}^*(\beta)$ for $\beta \in H^3(M)$. But we require $p^*H = \hat{p}^*\hat{H}$ making the choice unique.

To prove that there is an isomorphism in twisted K-theory we make use of spectral sequences.

Theorem 4.1.3. *Let $\pi : E \to M$ be a circle bundle. There is a twisted version of the Atiyah-Hirzebruch-Serre spectral sequence with $E_2^{p,q} = H^p(M, K^q(S^1))$ converging to $K^{p+q}(E, H)$. When $\pi_1(M)$ acts trivially on $K^*(S^1)$ we have $E_2^{p,q} = H^p(M) \otimes K^q(S^1)$.*

We want to prove T-duality using the following diagram

$$\begin{array}{ccccc} & & \mathcal{PD} \circlearrowright & & \\ H^*(M)\otimes K^*(S^1) & \xrightarrow{\mathrm{id}\otimes\mathrm{proj}_1^*} & H^*(M)\otimes K^*(S^1\times\hat{S}^1) & \xrightarrow{\mathrm{id}\otimes\mathrm{proj}_{2!}} & H^*(M)\otimes K^*(\hat{S}^1) \\ \Downarrow & & \Downarrow & & \Downarrow \\ K^*(E,H) & \xrightarrow{p^*} & K^*(E\times\hat{E},p^*H) & \xrightarrow{\hat{p}_!} & K^*(\hat{E},\hat{H}) \\ & & \circlearrowright \Lambda_{\mathcal{B}} & & \end{array}$$

First we prove that the composition $T := \mathrm{proj}_{2!} \circ \mathcal{PD} \circ \mathrm{proj}_1^*$ in the top row is an isomorphism from $K^*(S^1)$ to $K^*(\hat{S}^1)$. To do so notice that the spectral sequence with E_2 page $H^*(X) \otimes K^*(pt)$ converges to $K^*(X)$ so it is enough to show that we obtain an isomorphism on the E_2-pages;

$$H^*(S^1) \otimes K^*(pt) \xrightarrow{\pi^*} H^*(S^1 \times \hat{S}^1) \otimes K^*(pt) \xrightarrow{\mathcal{PD}} H^*(S^1 \times \hat{S}^1) \otimes K^*(pt)$$

$$\xrightarrow{\hat{\pi}_!} H^*(\hat{S}^1) \otimes K^*(pt)$$

Notice that at first sight there is a problem, $proj_{2!}$ has a dimension shift of one. This means that $T : H^i(S^1) \to H^{i+1}(\hat{S}^1)$ and this could never be an isomorphism. But the tensoring with $K^*(\mathrm{pt})$ solves this problem, $K^*(\mathrm{pt}) \cong \mathbb{Z}[t, t^{-1}]$ with $\deg(t) = -2$. Let $H^*(S^1) = \mathbb{Z}[1, u]$ and $H^*(\hat{S}^1) = \mathbb{Z}[1, \hat{u}]$. We have

$$H^*(S^1) \otimes K^* = \begin{cases} \mathbb{Z}[1 \otimes t^n] & \text{for even dimensions } 2n \\ \mathbb{Z}[u \otimes t^n] & \text{for odd dimensions } 2n+1 \end{cases}$$

When we calculate the Poincaré duality map coming from dual cell-structures we obtain;

$$\begin{aligned}
\mathcal{PD}: H^*(T,\mathbb{Z}) &\rightarrow H^*(T,\mathbb{Z})\\
1 &\mapsto u\otimes\hat{u}\\
u &\mapsto \hat{u}\\
\hat{u} &\mapsto u\\
u\otimes\hat{u} &\mapsto 1
\end{aligned}$$

The integration over the circle $\pi_!: H^*(T,\mathbb{Z}) \rightarrow H^*(\hat{S}^1)$ is given by

$$\begin{aligned}
1 &\mapsto 0\\
u &\mapsto 1\\
\hat{u} &\mapsto 0\\
u\otimes\hat{u} &\mapsto \hat{u}
\end{aligned}$$

So we obtain

$$\begin{aligned}
T(1) &= \mathrm{proj}_{2!}\circ\mathcal{PD}\circ\mathrm{proj}_1^*(1) = \mathrm{proj}_{2!}(u\otimes\hat{u}) = \hat{u}\\
T(u) &= \mathrm{proj}_{2!}\circ\mathcal{PD}\circ\mathrm{proj}_1^*(u) = \mathrm{proj}_{2!}(u) = 1
\end{aligned}$$

This indeed gives an isomorphism. Notice that the Poincaré duality is crucial for the isomorphism, if we wouldn't use it we would obtain $1\mapsto 0$ and $u\mapsto 1$. The toprow of our diagram is an isomorphism on the E_2-page of the spectral sequences converging to the bottom row of our diagram, this gives the T-duality isomorphism.

4.2 Examples

In this section we would like to discuss some examples of T-dual pairs and the corresponding T-duality isomorphisms in twisted K-theory. This section is based on [7] and [5] and discusses examples from both papers.

Example 4.2.1 (trivial example)**.** *Let M be our base space and $S^1 \times M$ the trivial circle bundle over M. This bundle has Chern class 0. Equip this bundle with trivial H-flux, so $(E, H) = (S^1 \times M, 0)$. The T-dual has to have Chern class zero, so is again the trivial bundle, and the H-flux is 0. So $(M \times S^1, 0)$ is self T-dual. The T-duality isomorphism tells us that*

$$K^0(M \times S^1) \cong K^1(M \times S^1).$$

Another way to see this is by the following proof. Let M_+ denote $M \cup \{\infty\}$, the following equalities hold

$$\begin{aligned}(X \times S^1)_+ &= M_+ \wedge S^1_+ \\ &= M_+ \wedge (S^1 \vee S^0) \\ &= (M_+ \wedge S^1) \vee (M_+ \wedge S^0) \\ &= (M_+ \wedge S^1) \vee M_+\end{aligned}$$

Now indeed;

$$K^i(M \times S^1) = \tilde{K}^i((M \times S^1)_+) = \tilde{K}^i(M_+ \wedge S^1) \oplus \tilde{K}^i(M_+) = K^{i+1}(M) \oplus K^i(M).$$

So indeed, $K^i(M \times S^1) \cong K^{i+1}(M \times S^1)$.
The other main examples found in the literature are those of circle bundles over

1. a two-dimensional oriented compact manifold, so $M = M^g$ is characterized by its genus g.

2. a real projective space $M = \mathbb{R}P^n$.

3. a complex projective space $M = \mathbb{C}P^n$.

We will discuss a couple of interesting cases of type (1) separately, before treating this class in its complete generality. Interesting examples arise when $g = 0$ so $M = S^2$. These include the trivial circle bundle, the Hopf bundle and Lens space bundles over S^2. For the reader unfamiliar with Lens spaces L_k, in appendix C we define them and show that $L_0 = S^2 \times S^1$, $L_1 = S^3$ and $c_1(L_k) = k$.

Example 4.2.2 (Trivial circle bundle)**.** *Let $E = S^2 \times S^1$ be the trivial principal circle bundle over S^2. The H-flux is an element in $H^3(S^2 \times S^1)$, we can calculate* *this group by using the Künneth formula* [1]*.*

$$H^3(S^2 \times S^1) = H^2(S^2) \otimes H^1(S^1) = \mathbb{Z}$$

So the H-flux is given by an integer $k = kd\theta$, where $d\theta$ is a choice of generator for $H^1(S^1)$. We have seen the example where $k = 0$ above. More interesting is the case where $k \neq 0$, in this case the first Chern class of $\hat{E}$ is $\pi_!(kd\theta) = k$. So $\hat{E}$ is the Lens space (or dual Lens space) L_k. As we have calculated in 2.2.8, the cohomology of this space is given by

$$H^n(L_k) = \begin{cases} \mathbb{Z} & n = 0, n = 3 \\ \mathbb{Z}_k & n = 2 \\ 0 & \textit{otherwise} \end{cases}$$

The H-flux on L_k is an element in $H^3(L_k) = \mathbb{Z}$, so it is an integer as well. We know that $\hat{\pi}_!\hat{H} = 0$ and $\hat{\pi}_!(n) = n$ so we find that $(S^2 \times S^1, k)$ is T-dual to $(L_k, 0)$.
Now our T-duality isomorphism tells us that

$$K^\bullet(S^2 \times S^1, k) \cong K^{\bullet+1}(L_k)$$

We can verify these statements by comparing the results of our calculations in 3.4.2. Our results are shown again in the table below and they indeed agree with T-duality.

(E,H)	$(S^2 \times S^1, k)$	$(L_k, 0)$
$K^0(E,H)$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_k$
$K^1(E,H)$	$\mathbb{Z} \oplus \mathbb{Z}_k$	$\mathbb{Z}$

Next we discuss the example of Lens space bundles with non-trivial twist. For the reader unfamiliar with Lens spaces and Lens space bundles we refer to appendix C. Since L_1 is the Hopf bundle $S^1 \to S^3 \to S^2$, this example is automatically included.

Example 4.2.3 (Lens space bundles)**.** *In this example we have a lens space bundle $L_k \to S^2$ together with non-trivial (nonzero) H-flux given by a integer*

[1]The Künneth formula states $H^n(X \times Y) = \oplus_{i+j=n} H^i(X) \otimes H^j(Y)$

$l \in \mathbb{Z} = H^3(L_k)$. We find that the T-dual space is L_l and $\hat{H} = k$.
The T-duality isomorphism tells us that

$$K^\bullet(L_k, l) \cong K^{\bullet+1}(L_l, k)$$

Again this corresponds to the calculations done in 3.4.2, as you can check in the following table.

(E,H)	(L_k,l)	(L_l,k)
$K^0(E,H)$	$\mathbb{Z}_k$	$\mathbb{Z}_l$
$K^1(E,H)$	$\mathbb{Z}_l$	$\mathbb{Z}_k$

Now we will discuss the case where $M = M^g$, a closed oriented surface of genus g, in full generality.

Example 4.2.4 (Bundles over M^g). *In 2.2.8 we saw that $H^2(M^g) = \mathbb{Z}$, so the principal circle bundles over M^g are classified by an integer. We will denote the bundle corresponding to k by E_k. We have also seen that $H^3(E_k) = \mathbb{Z}$ for all k and that $\pi_!(k) = k$, so the H-flux is given by an integer l. Thus a pair is given as (E_k, l), and the T-dual pair is (E_l, k). T-duality gives*

$$K^\bullet(E_k, l) \cong K^{\bullet+1}(E_l, k)$$

this corresponds to our calculation of the twisted K-theory of E_k as summarized in the table.

(E,H)	$(E_k,0)$ $k \neq 0$	(E_0,l) $l \neq 0$	(E_k,l) $k \neq 0, l \neq 0$
$K^0(E,H)$	$\mathbb{Z}^{2g+1} \oplus \mathbb{Z}_k$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_k$
$K^1(E,H)$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}^{2g+1} \oplus \mathbb{Z}_l$	$\mathbb{Z}^{2g} \oplus \mathbb{Z}_l$

The next example is that of circle bundles over $\mathbb{R}P^n$. In 2.2.9 we saw that for $n \leq 1$ there are no non-trivial bundles, so we dismiss these two cases. When $n > 1$ there are two circle bundles, the trivial bundle and $E_n \to \mathbb{R}P^n$ the bundle with first Chern class $1 \in \mathbb{Z}_2$. We also calculated the cohomology of E_n there.

$$H^0(E_n) = H^1(E_n) = \mathbb{Z}, \qquad H^{2n+1}(E_{2n}) = \mathbb{Z}_2,$$
$$H^{2n+1}(E_{2n+1}) = \mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad H^{2n+2}(E_{2n+1}) = \mathbb{Z}$$

In order to determine $H^{2n+1}(E_{2n+1})$ we use an explicit model for the non-trivial bundle E_n, namely $E_n = S^n \times I/(x,0) \sim (-x,1)$, where $(x,t) \mapsto [x] \in \mathbb{R}P^n =$

$S^n/x \sim -x$. Now we can calculate the cohomology of E_n using the Mayer-Vietoris long exact sequence. Let $A = S^n\times[0,1/3]\sqcup S^n\times[2/3,1]/(x,0)\sim(-x,1)$ and let $B = S^n\times[1/3,2/3]$. Then both $A\simeq S^n$ and $B\simeq S^n$, $E_n = A\cup B$ and $A\cap B = S^n\sqcup S^n$. The Mayer-Vietoris sequence becomes

$$\cdots\to H^{n-1}(A\cap B)\to H^n(E_n)\to H^n(A)\oplus H^n(B)\to H^n(A\cap B)\to\cdots$$

$$\cdots\to 0\to H^n(E_n)\to\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\to\cdots$$

we cannot inject $\mathbb{Z}_2$ in $\mathbb{Z}$, since there is no element of order 2 in $\mathbb{Z}$, so $H^n(E_n)$ has to be $\mathbb{Z}$ and not $\mathbb{Z}\oplus\mathbb{Z}_2$.

We deferred the calculation of the twisted K-theory groups of bundles over $\mathbb{R}P^n$ to this section, because we will need to use T-duality to compute them. We also need the cohomology groups, $H^i(\mathbb{R}P^n\times S^1) = H^i(\mathbb{R}P^n)\oplus H^{i-1}(\mathbb{R}P^n)$ for this calculation. For n odd they are given by

$$H^i(\mathbb{R}P^n\times S^1) = \begin{cases}\mathbb{Z} & i=0,1,n+1\\ \mathbb{Z}\oplus\mathbb{Z}_2 & i=n\\ \mathbb{Z}_2 & 2\le i\le n-1\end{cases}$$

and n even

$$H^i(\mathbb{R}P^n\times S^1) = \begin{cases}\mathbb{Z} & i=0,1\\ \mathbb{Z}_2 & 2\le i\le n+1\end{cases}$$

Let us first look at the case $n=2$.

Example 4.2.5 **($\mathbb{R}P^2$).** *We have $E_2\to\mathbb{R}P^2$ with H-flux in $H^3(E_2)=\mathbb{Z}_2$ and $\mathbb{R}P^2\times S^1\to\mathbb{R}P^2$ with H-flux in $H^3(\mathbb{R}P^2\times S^1)=\mathbb{Z}_2$. We have the following (non-trivial) T-dual pairs, $(E_2,0)\overset{T}{\sim}(\mathbb{R}P^2\times S^1,1)$ and $(E_2,1)\overset{T}{\sim}(E_2,1)$. So T-duality tells us that*

$$\begin{aligned}K^\bullet(E_2,0) &\cong K^{\bullet+1}(\mathbb{R}P^2\times S^1,1)\\ K^0(E_2,1) &\cong K^1(E_2,1)\end{aligned}$$

To check this is indeed the case we calculate these groups using the Atiyah-Hirzebruch spectral sequence. The third page for E_2 is given by

2	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$
E_3	0	1	2	3

When $H = 1$ the d_3 differential is the surjection onto $\mathbb{Z}_2$ so the E_4 page is given by

2	$\mathbb{Z}$	$\mathbb{Z}$	0	0
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	0
E_4	0	1	2	3

There are no non-trivial higher differentials so this is the E_∞-page and thus $K^0(E_2, 1) = K^1(E_2, 1) = \mathbb{Z}$, this corresponds with T-duality. If H is zero instead, then the E_∞-page equals the E_3 page and we find $K^0(E_2, 0) = \mathbb{Z}$ and $K^1(E_2, 0) = \mathbb{Z} \oplus \mathbb{Z}_2$.
For $\mathbb{R}P^2 \times S^1$ the E_3 page is given by

2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
E_3	0	1	2	3

when $H = 1$ the d_3 differential is the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ and the E_∞-page is given by

2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
E_4	0	1	2	3

So indeed $K^0(\mathbb{R}P^2 \times S^1, 1) = \mathbb{Z} \oplus \mathbb{Z}_2 = K^1(E_2, 0)$ and $K^1(\mathbb{R}P^2 \times S^1, 1) = \mathbb{Z} = K^0(E_2, 0)$.

The next case we discuss is $n = 3$, the circle bundles over $\mathbb{R}P^3$.

Example 4.2.6 ($\mathbb{R}P^3$)**.** *As usual we start by computing the second cohomology of our base space to see how many different bundles we have. $H^2(\mathbb{R}P^3) = \mathbb{Z}_2$ so again there are two bundles, $\mathbb{R}P^3 \times S^1$ and E. The cohomology of $\mathbb{R}P^3$ is given by*

$$H^i(\mathbb{R}P^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}_2 & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

So by the Künneth formula we find that the third cohomology group of $\mathbb{R}P^3 \times S^1$ *is* $\mathbb{Z} \oplus \mathbb{Z}_2$*. The H-flux is an element of the form* (k, l)*. Notice that* $\pi_!(k, l) = l$*. The cohomology of* E *is given by*

$$H^i(E, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1, 3, 4 \\ 0 & otherwise \end{cases}$$

So we obtain $H^3(E) = \mathbb{Z}$ *and we see that* $\pi_!$ *is the zero map. Thus we find that* $(\mathbb{R}P^3, (k, 0))$ *is self T-dual and* $(\mathbb{R}P^3, (k, 1))$ *is T-dual to* (E, k)*. The* k*'s have to be equal since* $p^*(k, 0) = \hat{p}^*(k) = k$*. The T-duality isomorphism gives*

$$K^\bullet(\mathbb{R}P^3 \times S^1, (k, 0)) \cong K^{\bullet+1}(\mathbb{R}P^3 \times S^1, (k, 0))$$

$$K^\bullet(\mathbb{R}P^3 \times S^1, (k, 1)) \cong K^{\bullet+1}(E, k)$$

The first equality actually reduces to the trivial case. For the second equality we calculate the right hand side in order to obtain the twisted K-theory groups of $\mathbb{R}P^3 \times S^1$*. The* E_3 *page is given by*

2	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$
E_3	0	1	2	3	4

The differentials vanish according to [7]. So we obtain $K^0(E, k) = K^1(E, k) = \mathbb{Z}^2$ *and T-duality gives the unknown K-groups as depicted in the table.*

(E, H)	$(\mathbb{R}P^3 \times S^1, (k, 1))$	(E, k)
$K^0(E, H)$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$K^1(E, H)$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

Finally we discuss the general case with $n > 3$.

Example 4.2.7 ($\mathbb{R}P^n, n > 3$)**.** *For* $n > 3$ *we can never have H-flux on* E_n *since* $H^3(E_n) = 0$*. We also know that the H-flux on* $\mathbb{R}P^n \times S^1 \to \mathbb{R}P^n$ *is an element of* $H^3(\mathbb{R}P^n \times S^1) = \mathbb{Z}_2$*. So we find that* $(E_n, 0) \overset{T}{\sim} (\mathbb{R}P^n \times S^1, 1)$*.*
Now let n *be even, the* E_3*-page of the spectral sequence for* E_n *is given by*

2	$\mathbb{Z}$	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}_2$
1	0	0	0	$\cdots$	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}_2$
E_3	0	1	2	$\cdots$	n	$n+1$

The only possibly nontrivial differential is $d_n : E^{1,n} \to E^{n+1,0}$. According to [7] this map is trivial. So the K-theory groups are $K^0(E) = \mathbb{Z}$ and $K^1(E) = \mathbb{Z} \oplus \mathbb{Z}_2$. This gives us the K-theory groups of $\mathbb{R}P^3 \times S^1$ as well. Now when n is odd the E_3 page of the spectral sequence is

2	$\mathbb{Z}$	$\mathbb{Z}$	0	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}$
1	0	0	0	$\cdots$	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}$
E_3	0	1	2	$\cdots$	n	$n+1$

Again the differentials are non-zero, so we find that $K^0(E_n) = K^1(E_n) = \mathbb{Z}^2$. By T-duality we have the following results.

(E, H)	$(\mathbb{R}P^{2n} \times S^1, 1)$	$(E_{2n}, 0)$	$(\mathbb{R}P^{2n+1} \times S^1, 1)$	$(E_{2n+1}, 0)$
$K^0(E, H)$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$K^1(E, H)$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

Our final example is the case where the base space is the n-dimensional complex projective space. Again we make one distinct example of the low-dimensional case $n = 1$.

Example 4.2.8 ($\mathbb{C}P^1$)**.** *We know that $H^2(\mathbb{C}P^1) = \mathbb{Z}$ so we can denote our circle bundles by E_r, for E_r the bundle with first Chern class r. The H-flux on the trivial bundle is an element of $H^3(\mathbb{C}P^1 \times S^1) = \mathbb{Z}$. Using the Gysin sequence we find $H^3(E_r) = \mathbb{Z}$, thus (E_k, l) is T-dual to (E_l, k). Notice that this includes the trivial bundle since $E_0 = \mathbb{C}P^1 \times S^1$. T-duality dictates*

$$K^\bullet(E_k, l) \cong K^\bullet(E_l, k)$$

We can only calculate the untwisted K-theory in this case, which gives us a result for the twisted K-theory of the trivial bundle by T-duality. These groups are given in [5].

Example 4.2.9. *To classify the bundles over* $\mathbb{C}P^n$ *we calculate* $H^2(\mathbb{C}P^n) = \mathbb{Z}$, *denote the generator by* α. *Denote by* $\pi_n : E_{n,r} \to \mathbb{C}P^n$ *the principal circle bundle with Chern class* $c_1(E_{n,r}) = r\alpha$. *The third cohomology group* $H^3(\mathbb{C}P^n \times S^1) = \mathbb{Z}$. *Using the Gysin sequence one can calculate* $H^3(E_{n,r}) = 0$ *when* $n > 1$. *So there is no H-flux possible on* $E_{n,r}$. *We obtain* $(\mathbb{C}P^n \times S^1, r)$ *is T-dual to* $(E_{n,r}, 0)$. *Again* $K^\bullet(\mathbb{C}P^n \times S^1, r)$ *can be calculated by T-duality. See [5] for the results.*

Chapter 5

Comparision with literature and further reading

This thesis is mainly based on [7] and [5]. The former has a strong physical background. It discusses examples of T-duality known from physics literature and develops a systematic approach for the topological effect of T-duality in string theory. It discusses the T-duality isomorphism in twisted K-theory as well as a similar result in twisted cohomology that we did not mention. The main differences between the approach in [7] and our discussion of T-duality are that we ignore the physics and that the T-duality isomorphism is constructed in a different way. In [7] the twist H is an element of de Rahm cohomology and the T-duality isomorphism is constructed using a particular bundle gerbe [18].

In [5] on the other hand, T-duality is discussed in a purely mathematical setting. The authors discuss T-dual pairs and analyse them in a very abstract manner. They later on use their mathematical machinery to set up the T-duality isomorphism. The authors use a formal approach where they define twisted generalized cohomology theories. They show that when such a theory is T-admissible, then it has a T-duality isomorphism. Examples of these are twisted de Rham cohomology and twisted K-theory.

We work with the definition of twisted K-theory as the homotopy classes of sections of a certain bundle. There is another way to describe twisted K-theory using C^*-algebras. Unfortunately the study of these lie outside the scope of this thesis. In the draft of the book 'Topology, C^*-Algebras, and String Duality' by Jonathan M. Rosenberg one can find an introduction to C^*-algebras. The book discusses the physics and algebraic topology of T-duality as well. In the C^*-algebra setting, T-duality can be extended to more general spaces. In the appendix of [12] one can find a short history of twisted K-theory.

Appendix A

Complex K-theory

In this first appendix we briefly discuss complex K-theory as the Grothendieck group completion of classes of vector bundles. We also introduce the Grassmannian manifolds and Bott periodicity. Most of the theorems in this section are stated without a proof. For a more elaborate discussion of these topics see for example [10].

Definition A.1. *Let* $\mathit{Vect}^n_{\mathbb{C}}(X)$ *be the set of isomorphism classes of* n*-dimensional complex vector bundles over* X*. And let* $\mathit{Vect}_{\mathbb{C}} = \cup_n \mathit{Vect}^n_{\mathbb{C}}(X)$*.*

$\mathrm{Vect}_{\mathbb{C}}$ is a monoid with the Whitney sum as operation and the zero-dimensional vector bundle $X \times 0$ as zero.

Definition A.2. *The complex K-theory group* $K(X)$ *is defined as the Grothendieck group completion of* $\mathit{Vect}_{\mathbb{C}}$*.*

The elements of $K(X)$ are given by formal differences $[E]-[E']$ of isomorphism classes of vector bundles. We will write n for the trivial bundle of dimension n, i.e. $n = X \times \mathbb{C}^n$. Recall that for any vector bundle E over a compact Hausdorff space, there exists a vector bundle E' such that $E \oplus E' \simeq m$. This implies that any element of $K(X)$ can be put in the form $[E] - n$, since for any element $[E] - [E']$ there is a isomorphism class $[E'']$ such that $[E' \oplus E''] = n$, thus $[E] - [E'] = [E \oplus E''] - m$.

Definition A.3. *Let $i^* : K(X) \to K(x_0)$ be the map that sends $[E] - n$ to $\dim(E) - n$. The reduced complex K-theory group of a space X is defined to be the kernel of this map.*

$$\tilde{K}(X) := \ker(K(X) \xrightarrow{i^*} K(x_0))$$

The elements of $\tilde{K}(X)$ can be written in the form $[E] - n$ where $\dim(E) = n$. In general if $[E] - [E'] \in \tilde{K}(X)$ then $\dim(E) = \dim(E')$.

Fact A.4. *For paracompact spaces X there is a bijection between $Vect_{\mathbb{C}}(X) = [X, BU(n)]$.*

Here $BU(n)$ is the *classifying space* of $U(n)$, the group of unitary matrices in $\mathbb{C}^n$. More information about classifying spaces can be found in appendix B. For a proof of this fact we refer to [10], where one can find a proof of $\text{Vect}_{\mathbb{C}}(X) = [X, Gr(n)]$. $Gr(n)$ is the n-dimensional Grassmannian, as defined in appendix B. In that appendix we also prove that $BU(n) \simeq Gr(n)$.
Let BU be defined as $\cup_{n=0}^{\infty} BU(n)$ with the limit topology.

Theorem A.5. *If X is a paracompact space, the groups $[X, BU]$ and $\tilde{K}(X)$ are isomorphic.*

Proof: Since X is paracompact and BU is defined to be a infinite union with the limit topology we have

$$[X, BU] = lim_{n\to\infty}[X, BU(n)].$$

We use that $Vect^n_{\mathbb{C}}(X) = [X, BU(n)]$, so we are left to show that $\tilde{K}(X) \cong \lim_{n\to\infty} Vect^n_{\mathbb{C}}(X)$. In order to show this we define the following maps.

$$\begin{aligned} f_n : Vect^n_{\mathbb{C}}(X) &\to \tilde{K}(X) \\ \xi^n &\mapsto \xi - n \end{aligned}$$

We obtain a map $f : \lim_{n\to\infty} Vect^n_{\mathbb{C}}(X) \to \tilde{K}(X)$ by commutativity of the following diagram.

$$\begin{array}{ccc} Vect^n_{\mathbb{C}}(X) & \longrightarrow & Vect^{n+1}_{\mathbb{C}}(X) \\ & \searrow & \downarrow \\ & & \tilde{K}(X) \end{array} \qquad \begin{array}{ccc} \xi & \longmapsto & \xi \oplus 1 \\ & \searrow & \downarrow \\ & & \xi - n \sim \xi \oplus 1 - (n+1) \end{array}$$

This map is surjective since any element of $\tilde{K}(X)$ can be written as $E - n$ with $n = \dim(E)$, so $f(E) = f_n(E) = E - n$. The map is injective since $f(E) = 0$ implies that $f_n(E) = 0$ for $n = \dim(E)$. So $E - n = 0$, thus $E = n$ this is the zero element in the limit.

Corollary A.6. $K(X) \cong [X, BU \times \mathbb{Z}]$.

Proof: $\tilde{K} = \ker(K(X) \to K(x_0))$ and $K(x_0) = \mathbb{Z}$ since all vector bundles over a point are trivial so they are classified by there dimension which gives the natural numbers $\mathbb{N}$ and the Grothendieck group completion with respect to addition then gives the integers $\mathbb{Z}$. So $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$. We have the following short exact sequence of groups

$$0 \to [X, BU] \xrightarrow{i} [X, BU \times \mathbb{Z}] \xrightarrow{p} \mathbb{Z} \to 0$$

where $i(f) = \tilde{f}$ and $\tilde{f}(x) = (f(x), 0)$. The map p is given by $p(g) = \pi_{\mathbb{Z}} \circ g$ this is well defined since a continuous map must have its image in one and the same path component and a homotopy stays in the same path component as well and $\mathbb{Z}$ is a discrete group. Exactness is now obvious and thus

$$[X, BU \times \mathbb{Z}] \cong [X, BU] \oplus \mathbb{Z} \cong \tilde{K}(X) \oplus \mathbb{Z} \cong K(X).$$

Often we write $K^0(X)$ instead of $K(X)$ and similarly $\tilde{K}^0$. Notice how $K^0(X) = \tilde{K}(X) \oplus \mathbb{Z} = \tilde{K}(X_+)$, where $X_+ = X \cup \{\infty\}$. K-theory extends to a cohomology theory by defining

$$\tilde{K}^{-n}(X) = [S^n \wedge X, BU] = [X, \Omega^n BU]$$

for $n \in \mathbb{N}$ and

$$K^k(X) = K^k(X_+) \quad \forall\, k \in \mathbb{Z}$$

The following theorem is very important for K-theory and defines the positive K-theory groups.

Theorem A.7 (Bott periodicity)**.**

$$K(X) \cong K^{i+2}(X)$$

For a proof of this theorem we refer to [3] or [10].

Appendix B

Classifying Spaces

In this appendix we discuss the classifying spaces, EG and BG of a group G. The reason these spaces are called classifying spaces is that every principal G-bundle can be obtained as a pullback of $\pi : EG \to BG$. This has been discussed in section 2.1. The spaces BG and EG are defined up to homotopy equivalence only.

Definition B.1. *The space EG is a contractible space with free action of G. The space BG is defined to be EG/G.*

The question arises whether such spaces EG and BG exist for every group G. The answer is affirmative and we will give an explicit construction here. A useful application of these explicit models is that they give us information about the classifying space 'up to homotopy'.
In order to give this explicit description, we need some machinery. We define semisimplicial sets (s.s.s.), a functor, $X_\bullet : \mathbf{Grp} \to \mathbf{SSS}$, that associates a semisimplicial set to a group and the geometric realization functor $|\cdot| : \mathbf{SSS} \to \mathbf{Top}$.

Definition B.2. *A semisimplicial set is a graded set, indexed on the integers together with maps $\partial_i : X_k \to X_{k-1}$ where $k \geq 1$ and $i \in \{0, 1, ..., k\}$ such that for $j > i$ we have*

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$$

A map of semi simplicial sets $f : X_\bullet \to Y_\bullet$ is given by a set of maps $f_n : X_n \to Y_n$ such that the following diagram commutes for all n and i.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array}$$

We now define the s.s.s. functor, $X_\bullet : \mathbf{Grp} \to \mathbf{SSS}$.

Definition B.3. *For a group G let $X_n(G) = G^{\times n}$ and define $\partial_i : X_{n+1} \to X_n$ for $0 \leq i \leq n$ by*

$$(g_0, ..., g_n) \mapsto \begin{cases} (g_1, ..., g_n) & i = 0 \\ (g_0, ..., g_{i-1}, g_i.g_{i+1}, g_{i+2}, ..., g_n) & 0 < i < n \\ (g_0, .., g_{n-1}) & i = n \end{cases}$$

When $f : G \to H$ is a group homomorphism we define $X_n(f) : X_n(G) \to X_n(H)$ by $f(g_0, ..., g_{n-1}) = (f(g_0), ..., f(g_{n-1}))$.

To define the geometric realization we make use of the standard simplices Δ^n, consisting of n-tuples of non-negative real numbers that sum up to 1. We define maps $d_i : \Delta^n \to \Delta^{n+1}$ by $(x_0, x_1, ..., x_n) \mapsto (x_0, x_1, ..., x_{i-1}, 0, x_i, ...x_n)$ for $0 \leq i \leq n$.

Definition B.4. *The geometric realization of a s.s.s. $X_\bullet$ is defined to be*

$$\left(\coprod_{i \in \mathbb{N}_0} X_i \times \Delta^i \right) / \sim$$

where the equivalence is given by $(\partial_i(x), y) \sim (x, d_i(y))$. A morphism f of semisimplicial sets gets sent to $|f| : (x, y) \mapsto (f_n(x), y)$ for $x \in X_n$. This is well defined under the equivalence relation since we require that $\partial_i \circ f_n = f_{n-1} \circ \partial_i$.

Now we define the classifying space BG as $|X_\bullet(G)|$. EG is constructed in a similar manner, it is the geometric realization of the following semisimplicial set.

Definition B.5. *Let $Y_i = G^{\times i+1}$ and define maps $\partial_i : Y_n \to Y_{n-1}$ for $0 \leq i \leq n$ as follows*

$$(g_0, g_1, ..., g_n) \mapsto (g_0, ..., g_{i-1}, g_{i+1}, ..., g_n)$$

Then $EG = |Y_\bullet|$

Now let us check that these models are in accordance with the definition above. So we need to prove the following lemma

Lemma B.6. *EG as constructed above has a free G action and is contractible. Furthermore*

$$BG = EG/G$$

Proof (from [17]): To prove that EG is contractible we define a retraction of EG to a point, $h_t : EG \to EG$.

$$h_t : [(g_0, g_1, ..., g_n), (y_0, y_1, ..., y_n)] \mapsto [(e, g_0, g_1, ..., g_n), ((1-t), ty_0, ty_1, ..., ty_n)]$$

where e is the unit of G. When $t = 1$ we have

$$[(g_0, g_1, ..., g_n), (y_0, y_1, ..., y_n)] \mapsto [(e, g_0, g_1, ..., g_n), (0, y_0, y_1, ..., y_n)]$$

but the right hand side is equal to

$$[(e, g_0, g_1, ..., g_n), d_0(y_0, y_1, ..., y_n)] \sim [\partial_0(e, g_0, g_1, ..., g_n), (y_0, y_1, ..., y_n)]$$

so $h_1 = \mathrm{id}_{EG}$.
When $t = 0$ we find

$$[(g_0, g_1, ..., g_n), (y_0, y_1, ..., y_n)] \mapsto [(e, g_0, g_1, ..., g_n), (1, 0, 0, ..., 0)] = [(e), (1)]$$

So indeed EG is contractible.
The action of G on EG is given by $g.[(g_0, g_1, ..., g_n), y] = [(g_0.g, g_1.g, ..., g_n.g)]$. To see that the action is free, note that the only way to 'get rid of' terms $g_i g$ is when there are zeroes in y, but even in the most extreme case where $y = (0, ..., 1, ..., 0)$ we still obtain $[(g_i g), (1)] = [(g_i), (1)]$ implying $g = e$.
Finally to see that $BG = EG/G$ we check this on the level of s.s.s. so we will prove that $X_\bullet = Y_\bullet/G$. To do so we construct explicit maps. Let $[(g_0, g_1, g_2, ..., g_n)]$ be an equivalence class of Y_n under the action of G. There is a unique representative of the form $(e, g'_1, ..., g'_n)$ and this element can be written as $(e, g''_1, g''_1 g''_2, ..., g''_1 g''_2 ... g''_n)$. Let $(g''_1, g''_2, ..., g''_n)$ be the image in X_n. There is an obvious inverse map sending $(h_0, ..., h_{n-1})$ to $[(e, h_0, h_0 h_1, ..., h_0 h_1 ... h_n)]$. So we just need to check if this behaves well with respect to the maps ∂_i. Indeed deleting a g_i in Y_n corresponds exactly to multiplying g''_i and g''_{i+1}. This finishes the proof.

Fact B.7. *If G is an abelian group then so is BG.*

The following property will be needed in the defintion of twisted K-theory.

Lemma B.8.

$$K(\mathbb{Z}, n) \simeq B^n\mathbb{Z}$$

Proof: Since $\mathbb{Z}$ is an abelian group we find that $B\mathbb{Z}$ is again an abelian group, so it makes sense to repeat the bar construction. Also since in general $BG = EG/G$ we have a fibration $G \to EG \to BG$ which gives a long exact sequence in homotopy. So since EG is contractible we obtain isomorphisms $\pi_n(BG) \cong \pi_{n-1}(G)$. Now the only nontrivial homtopy group of $\mathbb{Z}$ is $\pi_0(\mathbb{Z}) = \mathbb{Z}$, the lemma follows.

Lemma B.9. *$PU(\mathcal{H})$ is a $BK(\mathbb{Z}, 1) = K(\mathbb{Z}, 2)$.*

Proof: We know that S^1 is a model for $K(\mathbb{Z}, 1)$ and $\mathbb{C}^\times$ the units in $\mathbb{C}$ is equivalent to S^1, thus we need to show that $PU(\mathcal{H})$ is a classifying space for $\mathbb{C}^\times$. By Kuiper's theorem[1] [13] we know that $U(\mathcal{H})$ is contractible. The following action is free

$$\begin{aligned} \mathbb{C}^\times \times U(\mathcal{H}) &\to U(\mathcal{H}) \\ (\alpha, A) &\mapsto \alpha A \end{aligned}$$

$\alpha \cdot A$ is again a unitary matrix since $(\alpha A)^\dagger \alpha A = \alpha\bar{\alpha}A^\dagger A = 1$. So $B\mathbb{C}^\times \cong U(\mathcal{H})/\mathbb{C}^\times = PU(\mathcal{H})$.

Definition B.10. *The (complex) Grassmannian manifolds are defined as follows*

$$Gr_n(\mathbb{C}^k) = \left\{n\text{-dimensional subspaces of } \mathbb{C}^k\right\}$$

the topology on them is given as a quotient topology of the Stiefel-Whitney manifolds. The inclusions $\mathbb{C}^k \subset \mathbb{C}^{k+1}$ give inclusions $Gr_n(\mathbb{C}^k) \subset Gr_n(\mathbb{C}^{k+1})$ and we define

$$Gr_n(\mathbb{C}) = \cup_k Gr_n(\mathbb{C}^k)$$

[1]This is a different theorem from 2.3.10, both go under the name Kuiper's theorem.

Definition B.11. *The Stiefel manifolds are defined as*

$$\begin{aligned} W_{k,n} &= k\text{-dimensional frames in } \mathbb{C}^n \\ &= \{(v_1, v_2, ..., v_k) \in \mathbb{C}^n | v_i \text{ are orthonormal}\} \end{aligned}$$

for $k < n$.

Lemma B.12. $BU(n) = Gr_n$.

Proof: First let us prove that the lemma holds for $n = 1$. We know that $U(1) = S^1$ and that $Gr_1 = \mathbb{C}P^\infty$. Now $\mathbb{C}P^n = S^{2n+1}/S^1$ where S^1 acts freely on S^{2n+1} by viewing both as subspaces of $\mathbb{C}^{\,n}$. Now

$$\mathbb{C}P^\infty = \lim_{n\to\infty} S^{2n+1}/S^1 = S^\infty/S^1$$

So we are done if we prove that S^∞ is contractible, because then we have constructed $\mathbb{C}P^\infty$ as the quotient of a contractible space with free $U(1)$ action. To see this note that we can interchange homotopy groups with the limit

$$\pi_i(\lim_{n\to\infty}(S^{2n+1})) = \lim_{n\to infty} \pi_i(S^{2n+1})$$

since S^1 is compact and thus maps into a finite subcomplex of S^{2n+1}. Indeed this limit is zero.

Now for the general case, $n \geq 1$ we would like to do something similar. We will replace S^∞ by the Stiefel manifolds. $U(n)$ acts on the Stiefel manifolds by regarding an element of $U(n)$ as a unitary $n \times n$ matrix A, and $A.(v_0, ..., v_k) = (A.v_0, ..., A.v_k)$. And $Gr_n = \lim_{k\to\infty} W_{k,n}/U(n)$. So it remains to show that $\lim_{k\to\infty} W_{k,n}$ is contractible. To do so we will prove that $W_{n,k}$ is $2(k-n)$-connected, i.e. $\pi_i(W_{n,k}) = 0$ for all $i \leq 2(k-n)$, which will finish the proof.

We use induction on to prove that $W_{n,k}$ is $2(k-n)$-connected. When $n = 1$ we have $W_{1,k} = \{v_0 \in \mathbb{C}^k | |v_0| = 1\} = S^{2k-1}$ and this is indeed $2(k-1)$-connected. For the induction step we use the fact that there is a fibration

$$S^{2k-2n+1} \to W_{n,k} \to W_{n-1,k}$$

since the fibre over an element $(v_0, ...v_{n-1})$ is a vector in $\mathbb{C}^{k-n-1}$ of length 1. This gives a long exact sequence in homotopy. And since $W_{n-1,k}$ is $2k-2n+2$-connected we have isomorphisms

$$\pi_i(S^{2k-2n+1}) \cong \pi_i(W_{n,k})$$

for $i \leq 2k - 2n + 2$. So $\pi_i(W_{n,k})$ equals zero when $i \leq 2k - 2n$ and $\mathbb{Z}$ when $i = 2k - 2n + 1$, so indeed $W_{n,k}$ is $2(k - n)$ connected.

Appendix C

Lens Spaces

In this appendix we define the Lens space bundles and calculate their first Chern classes.

Definition C.1. *A Lens space bundle $L_k \to S^2$, consists of the Lens space*

$$L_k = S^3/\mathbb{Z}_k$$

where the action of $\mathbb{Z}_k$ on S^3 is given by $(n, v) \mapsto e^{2n\pi/k}.v$, considering $S^3 \subset \mathbb{C}$. And a projection map

$$\pi : L_k \to S^2.$$

We consider $S^2 \cong \mathbb{C} \cup \{\infty\}$, π is then given by $[(z_1, z_2)] \mapsto z_1/z_2$. It is left to the reader to show that this map is well defined. The action of S^1 on L_k is

$$\begin{array}{ccc} S^1 \times L_k & \longrightarrow & L_k \\ (e^{i\theta}, [(z_1, z_2)]) & \longrightarrow & \left[(e^{i\theta/k} z_1, e^{i\theta/k} z_2)\right] \end{array}$$

and turns $\pi : L_k \to S^2$ into a principal S^1-bundle.

Notice that for $k = 1$ the Lens space bundle is the Hopf bundle $S^1 \to S^3 \to S^2$.

Fact C.2 (example 1.13 [10])**.** *Let $f, g : S^{k-1} \to GL_n(\mathbb{C})$ be clutching functions for the n-dimensional complex vector bundles $E_f, E_g \to S^k$. Then $E_{fg} \oplus n$ is isomorphic to $E_f \oplus E_g$.*

In the lemma below we use properties of the Chern class that can be found in theorem 4.1.2.

Lemma C.3. *Let $\mathcal{L} \to S^k$ be a line bundle, then the following identity holds.*

$$c_1(\mathcal{L}^{\otimes n}) = nc_1(\mathcal{L})$$

We will prove this by induction on n, it is definitely true for $n = 1$. So let us assume that $c_1(\mathcal{L}^{\otimes n}) = nc_1(\mathcal{L})$. By fact C.2 and the fact that $E_{f^n} = E_f^{\otimes n}$ for line bundles we have

$$\mathcal{L}^{\otimes n+1} \oplus 1 \approx \mathcal{L}^{\otimes n} \oplus \mathcal{L}.$$

Now since $\dim(\mathcal{L}) = 1$ we have $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$ and thus

$$c(\mathcal{L}^{\otimes n+1} \oplus 1) = c(\mathcal{L}^{\otimes n} \oplus \mathcal{L})$$

But since the total Chern class has the properties; $c(E \oplus E') = c(E)c(E')$ and $c(1) = 1$, we find the following equality.

$$\begin{aligned} 1 + c_1(\mathcal{L}^{\otimes n+1}) &= (1 + n\, c_1(\mathcal{L}))(1 + c_1(\mathcal{L})) \\ &= 1 + n + 1c_1(\mathcal{L}) + (n+1)c_1(\mathcal{L})c_1(\mathcal{L}) \end{aligned}$$

So indeed $c_1(\mathcal{L}^{\otimes n+1}) = (n+1)c_1(\mathcal{L})$ since $c_1(\mathcal{L})c_1(\mathcal{L}) \in H^4(S^2, \mathbb{Z}) = 0$.

Lemma C.4. $c_1(L_k) = k \in \mathbb{Z} = H^2(S^2, \mathbb{Z})$

To prove this lemma we will prove the following statements.

1. The Hopf bundle $S^3 \to S^2$ corresponds to the canonical line bundle $H \to \mathbb{C}P^1 \cong S^2$ under the bijection

$$\left\{\text{principal } S^1\text{-bundles over } S^2\right\} \leftrightarrow \left\{\text{complex line bundles over } S^2\right\}.$$

2. The principal S^1 bundle corresponding to $H^{\otimes n}$ is L_n.

3. $c_1(H) = 1$.

In order to prove (1) notice that $H \subset \mathbb{C}P^1 \times \mathbb{C}^2$ and that $\mathbb{C}P^1 = S^3/v \sim \lambda v$ for $\lambda \in S^1 \subset \mathbb{C}$. So $H = \left\{((z_1, z_2), v) \in \mathbb{C}P^1 \times \mathbb{C}^2 | v = \lambda(z_1, z_2), \lambda \in \mathbb{C}^\times\right\}$, the associated circle bundle $S(H)$ is then given by

$$S(H) = \left\{((z_1, z_2), v) \in \mathbb{C}P^1 \times \mathbb{C}^2 | v = \lambda(z_1, z_2), \lambda \in S^1\right\}$$

But that is exactly S^3, so indeed H corresponds to the Hopf bundle. For part (3) notice that $H = i^*(E)$ the pullback of the canonical line bundle $E \to \mathbb{C}P^\infty$ over the inclusion $i : \mathbb{C}P^1 \to \mathbb{C}P^\infty$. So $c_1(H) = i^*(1) = 1$. And for part (2) notice that $H^{\otimes n}$ corresponds to $(S^3)^{\otimes n} = L_n$.

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