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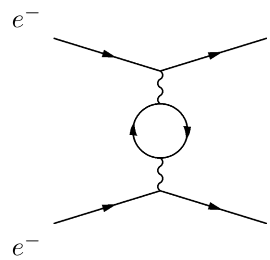
Unitarity methods for loop calculations

Bachelor's thesis



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Abstract

In this thesis, methods for the calculation of Feynman diagrams containing one loop are studied. These methods are based on the unitarity property of the scattering process, which follows from conservation of probability during an interaction. From unitarity, a relation for the imaginary part of a one loop amplitude is derived. This relation states that the imaginary part can be calculated by looking at the separate tree diagrams that contribute to the loop. The full amplitude of the loop can be reconstructed using a dispersion relation, which is based on Cauchy's Integral Theorem. Another way to reconstruct the full amplitude is to expand the amplitude in a basis of four scalar diagrams, and use unitarity to acquire the coefficients of this basis. This method is called generalized unitarity.

Examples of the application of the unitarity methods are studied by the hand of vacuum polarization in QED. In the first place, this amplitude is calculated using dimensional regularization (a method without unitarity). After that, both unitarity methods are also applied to vacuum polarization, where it turns out that these methods give the same amplitude as the method without unitarity does.

Preface

This thesis is the result of my bachelor's project in the third year of my bachelor in Physics and Astronomy at the University of Amsterdam. The project was part of the bachelor's projects offered by the Institute for Theoretical Physics Amsterdam (ITFA). The actual place I worked on my project was at the theory group of Nikhef.

In the first place I would like to thank Eric Laenen for supervising my bachelor's project. And I would also like to thank all the people of the Nikhef theory group for all the nice after-lunch-parties on the rooftop terrace of the Nikhef building. Especially Lisa Hartgring for teaching me how to use the computer program Form and how to draw Feynman diagrams with Latex. Also the master's thesis of Ori Yudilevich [10] was of a lot of use to me. The calculations in this thesis are performed with a little bit different strategy than in Ori's thesis, but to me it was a great introduction to the subject of unitarity and all the calculations, and sometimes gave me hints when I was stuck somewhere.

Prerequisite knowledge for reading this thesis includes the Feynman rules of Quantum Electrodynamics, listed in appendix C.1. For an introduction to Feynman rules, see a textbook on Quantum Field Theory, like [1]. Some derivations of solutions to integrals are made in the appendix.

Contents

1 Introduction

In high energy physics, the link between theory and experiment is the differential cross section. This value can both be measured in particle colliders, and calculated from the theory behind an interaction. In this way, theories can be tested or developed in order to explain observations in experiments. As an example we take the case of electron-electron scattering. The simplest Feynman diagram in this process is:

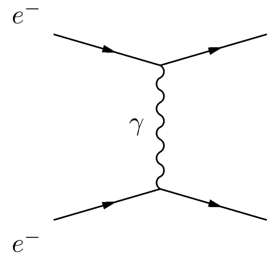


Figure 1: Feynman diagram for e^-e^--scattering

When this process takes place in a collider, the electrons are scattered at a certain angle due to this process. It would be nice if we could calculate the distribution of scattering angles somehow. The differential cross section in the center of mass frame for two incoming and two outcoming particles of the same kind, is given by [1]:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{CM}} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 \tag{1.1}$$

The parameter s is only an experimental value: it is the squared center of mass energy. $\mathcal{M}$ is the amplitude of the Feynman diagram. This amplitude follows completely from the theory we are using to describe the interaction. Each line inside the diagram corresponds to a certain Feynman rule. When the rules are applied to the diagram, the amplitude $\mathcal{M}$ can be calculated. The Feynman rules of Quantum Electrodynamics are listed in appendix C.1. With these rules, and some experimental values which are known from the properties of a collider, we are able to do theoretical predictions on the scattering processes.

As the energy of colliders (such as the LHC in Geneva) becomes higher, more complicated interactions containing loops will be important in interpreting experimental results and developing underlying theories. At the same time, theoretical predictions of the behavior of interactions between fundamental particles are also becoming far more complicated, so efficient calculational techniques are needed. One way to make a loop calculation easier, is to imply conservation of probability during the interaction. This is the main point of the unitarity method. From this rather simple statement, a powerful tool for evaluating loop amplitudes follows.

Troughout this thesis, several methods of calculating loops inside Feynman diagrams are studied, including methods with or without the use of unitarity. As an example, these methods are applied on the vacuum polarization amplitude. One of the unitarity methods is also applied on a more non trivial example, where the answer was not known in advance.

2 Loop diagrams

2.1 Vacuum polarization

When loops are allowed in the Feynman diagrams, the diagram in figure 1 is not the only scattering possibility for the case of electron-electron scattering. Other higher order diagrams containing loops have to be taken into account. A loop diagram contributing to the full amplitude is for example:

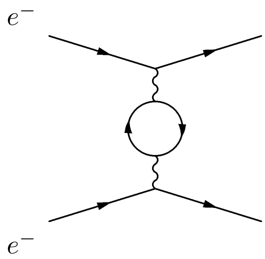


Figure 2: Higher order diagram for e^-e^--scattering

The loop arising inside the photon propagator is called vacuum polarization. The photon creates a fermion and anti-fermion pair, and they will annihilate later on. When the loop exists for a short amount of time, the Heisenberg uncertainty principle allows violation of energy conservation during that time. As a consequence, the invariant momentum of the fermions in the loop is allowed to be higher than the invariant momentum of the propagator. The loop is called to be off mass shell when this happens. For this case, $q^2 < 4m^2$, with m the mass of the fermions inside the loop and q^2 the invariant momentum of the photon propagator. When the loop is on mass shell, $q^2 > 4m^2$. Whether the loop is on or off shell will be important in using the unitarity method in section 4.

Each additional vertex in a diagram adds a factor of $\sqrt{\alpha}$ to the amplitude. As $\alpha \approx \frac{1}{137}$, each higher order diagram gets a lower amplitude. Calculating the contribution of each diagram, and adding all the amplitudes later on, is called perturbation theory. In first order perturbation theory with one loop inside the propagator, the photon propagator (from the Feynman rules, appendix C.1) changes in:

$$\frac{-ig^{\mu\nu}}{q^2+i\epsilon} \Rightarrow \frac{-ig^{\mu\nu}}{q^2+i\epsilon} + \frac{-ig^{\mu\rho}}{q^2+i\epsilon}\,\Pi_{\rho\sigma}(q^2)\,\frac{-ig^{\sigma\nu}}{q^2+i\epsilon} \tag{2.1}$$

With $\Pi^{\mu\nu}(q^2)$ the contribution of the loop. To get the first order contribution in perturbation theory, we need to calculate this contribution somehow.

2.2 Regularization and renormalization

It turns out that there is a problem concerning the calculation of the vacuum polarization amplitude $\Pi^{\mu\nu}$. Requiring conservation of momentum at each vertex in figure 3 does not constrain the internal loop momentum p, so it can take on every value. Therefore, it's not the case that we can just pick any value for p, but we have to include all possible values. This can be done by integrating the amplitude over the internal loop momentum, extending from $-\infty$ to $+\infty$. But when the Feynman rules are applied onto this diagram, there will be two factors of p^{-2} in the amplitude caused by the two fermions inside the loop. A four dimensional integral like $\int \frac{d^4p}{p^4}$ will

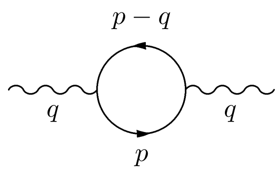


Figure 3: Flow of momentum inside the loop

be present, and this integral will be divergent. The final amplitude $\mathcal{M}$ will contain a singularity due to this divergent integral, which is going to give a problem. The differential cross section is a physically measurable quantity, so it is not possible to be infinity.

A solution to this problem is to renormalize the amplitude, by removing the singular terms from the amplitude and absorb it into some renormalization constant. This constant will be a part of the theory, and can be present in the corresponding Lagrangian. When this is possible, the theory is called renormalizable. Quantum Electrodynamics, the theory where vacuum polarization can be calculated from, is such a renormalizable theory.

Dimensional regularization

Given the fact that the amplitude of vacuum polarization is renormalizable, there is still the need to find out which terms of the amplitude are singular. A technique called regularization calculates the amplitude, where the singularities will be isolated in separate terms inside the amplitude. There are several regularization schemes available, but the most powerful scheme is dimensional regularization. Integrals that are divergent in the usual four space-time dimensions, are possible to be finite in an arbitrary complex number of d dimensions. There is absolutely no physical meaning behind a calculation in, for example, three and a half dimensions. But the advantage is a convergent integral. As a consequence of a calculation in an arbitrary dimension, the amplitudes will also be functions of dimension in the end.

The dimension can be set back to four in the end. When this is done by taking a limit to four dimensions, it will be clear which terms are delivering singularities. A practical way to do is, is by working in $d = 4 - 2\epsilon$ dimensions, and take the limit $\epsilon \to 0$ in the end.

3 Vacuum polarization by dimensional regularization

This section is devoted to the calculation of the amplitude of vacuum polarization by means of dimensional regularization. Constructing the vacuum polarization amplitude $\Pi^{\mu\nu}(q^2)$ from the Feynman rules (appendix C.1):

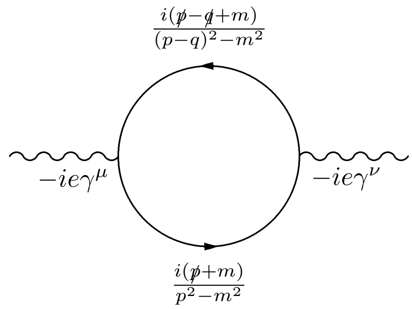


Figure 4: Corresponding Feynman rules of each line and vertex

Inserting each Feynman rule into the amplitude, and integrating over internal loop momentum:

$$\Pi^{\mu\nu}(q^2) = -\int \frac{d^d p}{(2\pi)^d} \mathrm{tr}\left[(-ie\gamma^\mu)\frac{i(\not{p}-\not{q}+m)}{(p-q)^2-m^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\right] \tag{3.1}$$

The point of applying dimensional regularization is that the integration over the internal loop momentum takes places in d dimensions.

Equation 3.1 can be rewritten:

$$\Pi^{\mu\nu}(q^2) = -e^2 \int \frac{d^d p}{(2\pi)^d} \frac{N^{\mu\nu}(p,q)}{((p-q)^2-m^2)(p^2-m^2)} \tag{3.2}$$

Where all the terms in the numerator are combined to $N^{\mu\nu}$:

$$N^{\mu\nu} = \mathrm{tr}\left[\gamma^\mu(\not{p}-\not{q}+m)\gamma^\nu(\not{p}+m)\right]$$

The first step in the calculation will be to evaluate the trace.

3.1 Calculation of the trace

Combining the brackets inside the trace:

$$N^{\mu\nu} = \mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu\not{p} - \gamma^\mu\not{q}\gamma^\nu\not{p} + \gamma^\mu m\gamma^\nu\not{p} + \gamma^\mu\not{p}\gamma^\nu m - \gamma^\mu\not{q}\gamma^\nu m + \gamma^\mu m\gamma^\nu m] \tag{3.3}$$

There are several identities concerning the trace of gamma matrices, see [8] for an extended list:

$$\mathrm{tr}[\gamma^\mu] = 0 \tag{3.4}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{3.5}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0 \tag{3.6}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{3.7}$$

Because $\not{p} = \gamma^\mu p_\mu$, the thirth, fourth and fifth term of (3.3) all contain three gamma matrices, and the trace of these terms will become zero according to identity 3.6. Now we can use the linearity property of the trace to evaluate the other terms seperately:

The first term follows from identity 3.7:

$$\begin{aligned} tr[\gamma^\mu \not{p} \gamma^\nu \not{p}] &= tr[\gamma^\mu \gamma^\alpha p_\alpha \gamma^\nu \gamma^\beta p_\beta] \\ &= tr[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta] p_\alpha p_\beta \\ &= 4(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}) p_\alpha p_\beta \\ &= 4(p^\mu p^\nu - g^{\mu\nu} p_\alpha p^\alpha + p^\nu p^\mu) \\ &= 4(2p^\mu p^\nu - g^{\mu\nu} p^2) \end{aligned} \tag{3.8}$$

The second term also follows from identity 3.7:

$$\begin{aligned} tr[\gamma^\mu \not{q} \gamma^\nu \not{p}] &= tr[\gamma^\mu \gamma^\alpha q_\alpha \gamma^\nu \gamma^\beta p_\beta] \\ &= tr[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta] q_\alpha p_\beta \\ &= 4(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}) q_\alpha p_\beta \\ &= 4(q^\mu p^\nu - g^{\mu\nu} q_\alpha p^\alpha + q^\nu p^\mu) \end{aligned} \tag{3.9}$$

The sixth term follows from 3.5:

$$tr[\gamma^\mu m \gamma^\nu m] = 4g^{\mu\nu} m^2$$

Putting this all together:

$$N^{\mu\nu}(p,q) = 2p^\mu p^\nu - q^\mu p^\nu - q^\nu p^\mu + g^{\mu\nu}(-p^2 + m^2 + q \cdot p) \tag{3.10}$$

3.2 Feynman parameter integrals

A powerful tool in calculating integrals containing products in the denominator, are Feynman parameter integrals.

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2} \tag{3.11}$$

Introducing the Feynman parameter x in equation 3.10 and rewriting the denominator:

$$\begin{aligned} \Pi^{\mu\nu}(q^2) &= -4e^2 \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N^{\mu\nu}(p,q)}{\left[x\left((p-q)^2 - m^2\right) + (1-x)(p^2 - m^2)\right]^2} \\ &= -4e^2 \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N^{\mu\nu}(p,q)}{\left[(p-xq)^2 + x(1-x)q^2 - m^2\right]^2} \end{aligned} \tag{3.12}$$

With the introduction of a Feynman parameter, the integration over p can become possible. The integration over x can be performed after the integration over p. Now perform the substitution $p - xq = l$. This is useful in the sense that we change the integration variable. And just to keep the expressions tidy, we define

$$D = -x(1-x)q^2 + m^2 \tag{3.13}$$

So $\Pi^{\mu\nu}$ becomes:

$$\Pi^{\mu\nu}(q^2) = -4e^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{N^{\mu\nu}(l,q)}{\left[l^2 - D\right]^2} \tag{3.14}$$

Now we have to perform the same substitution on the numerator:

$$N^{\mu\nu}(p,q) = 2p^\mu p^\nu - q^\mu p^\nu - q^\nu p^\mu + g^{\mu\nu}(-p^2 + m^2 + q \cdot p)$$

The first part becomes:

$$\begin{aligned} 2p^\mu p^\nu &= 2(l+xq)^\mu (l+xq)^\nu \\ &= 2l^\mu l^\nu + 2xl^\mu q^\nu + 2xq^\mu l^\nu + 2x^2 q^\mu q^\nu \end{aligned}$$

The second part becomes:

$$\begin{aligned} -q^\mu p^\nu - q^\nu p^\mu &= -q^\mu (l+xq)^\nu - q^\nu (l+xq)^\mu \\ &= -q^\mu l^\nu - xq^\mu q^\nu - l^\mu q^\nu - xq^\mu q^\nu \end{aligned}$$

The last part becomes:

$$\begin{aligned} g^{\mu\nu}(-p^2 + m^2 + q \cdot p) &= g^{\mu\nu}(-(l+xq)^2 + m^2 + q \cdot (l+xq)) \\ &= g^{\mu\nu}(-l^2 - x^2 q^2 - 2xl \cdot q + m^2 + q \cdot l + xq^2) \end{aligned}$$

We can immediately throw away all the terms that are linear in l. They integrate to zero because they are odd functions of l. Sum up all the remaining terms:

$$N^{\mu\nu}(l,q) = 2l^\mu l^\nu + 2x(x-1)q^\mu q^\nu + g^{\mu\nu}(-l^2 + x(1-x)q^2 + m^2)$$

3.3 l integrals

The part with $l^\mu l^\nu$ can be simplified using the following relation: (For a proof, see for example [3])

$$\int d^d l \; l^\mu l^\nu f(l^2) = \frac{1}{d} g^{\mu\nu} \int d^d l \; l^2 f(l^2) \tag{3.15}$$

This relation justifies the substitution of $l^\mu l^\nu \to \frac{1}{d} g^{\mu\nu} l^2$ in the numerator:

$$N^{\mu\nu}(l,q) = 2x(x-1)q^\mu q^\nu + g^{\mu\nu}\left[\left(\frac{2}{d} - 1\right) l^2 + x(1-x)q^2 + m^2\right]$$

Now use a relation derived in appendix E.1:

$$D \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2} = \left(1 - \frac{2}{d}\right) \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{\left[l^2 - D\right]^2}$$

The result of this relation is that we can change $(1 - \frac{2}{d})l^2 \to D$ in the numerator, because the numerator is also integrated over l:

$$\begin{aligned} N^{\mu\nu}(l,q) &= 2x(x-1)q^\mu q^\nu + g^{\mu\nu}\left[-D + x(1-x)q^2 + m^2\right] \\ &= -2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left[2x(1-x)q^2\right] \\ &= 2x(1-x)(g^{\mu\nu} q^2 - q^\mu q^\nu) \end{aligned} \tag{3.16}$$

Plugging this back in equation 3.14:

$$\Pi^{\mu\nu}(q^2) = -4e^2(g^{\mu\nu} q^2 - q^\mu q^\nu) \int_0^1 dx \; 2x(1-x) \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2}$$

The l-integral doesn't have a tensor structure in the integrand anymore, and can be evaluated using a prove from [3]. The result of this prove is:

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \tfrac{d}{2}\right) D^{\frac{d}{2}-2}$$

And:

$$\Pi^{\mu\nu}(q^2) = \frac{-8ie^2}{(4\pi)^{\frac{d}{2}}}(g^{\mu\nu}q^2 - q^\mu q^\nu) \int_0^1 dx\, x(1-x)\, \Gamma\left(2 - \tfrac{d}{2}\right) D^{\frac{d}{2}-2} \tag{3.17}$$

Ward identity

At this point, the integration over the internal loop momentum is performed completely. The only integration left, is the Feynman parameter x. So at this moment, we can clearly see the tensor structure of the amplitude, which is $g^{\mu\nu}q^2 - q^\mu q^\nu$. There is a check that we can perform on this tensor structure, by the means of the Ward identity. This identity in Quantum Field Theory is an expression of current conservation, which is a result of gauge-invariance [5]. For the vacuum polarization amplitude, the Ward identity yields:

$$q_\mu \Pi^{\mu\nu}(q^2) = 0 \tag{3.18}$$

By inserting the tensor structure of $\Pi^{\mu\nu}$ into the Ward identity:

$$\begin{aligned} q_\mu(g^{\mu\nu}q^2 - q^\mu q^\nu) &= q^\nu q^2 - q_\mu q^\mu q^\nu \\ &= q^\nu q^2 - q^2 q^\nu \\ &= 0 \end{aligned}$$

The result at this moment (equation 3.17) matches the tensor structure required by the Ward identity.

3.4 Integration of the Feynman parameter

Untill now we've worked with an arbitrary dimension d. If we want to apply dimensional regularization to this calculation, we have to take the limit to 4 dimensions. A way to do this is by substituting $d = 4 - 2\epsilon$, and take the limit $\epsilon \to 0$ later on, as discussed in section 2.

The aim of this subsection is to solve the x-integral.

Approximation

The starting point is the integral over x in equation 3.17:

$$I = \int_0^1 x(1-x)\, \Gamma(\epsilon)\, D^{-\epsilon} dx \tag{3.19}$$

This integral is very hard to solve exactly, because D is also a function of x. A way out is to expand D and drop higher order terms. The expansion is allowed because we are working close to four dimensions, so ϵ is small anyway.

$$D^{-\epsilon} \approx 1 - \epsilon \ln D$$

By the same argument, the Gamma function can also be expanded, see appendix D.2:

$$\Gamma(\epsilon) \approx \frac{1}{\epsilon}$$

Applying the expansions and using the definition of D in equation 3.13:

$$I = \int_0^1 x(1-x)\left(\frac{1}{\epsilon} - \ln[-x(1-x)q^2 + m^2]\right) dx \tag{3.20}$$

The logarithm can be manipulated as:

$$\ln[-x(1-x)q^2 + m^2] = \ln\left[-x(1-x)\frac{q^2}{m^2} + 1\right] + \ln m^2$$

And the integral of equation 3.20 becomes:

$$\begin{aligned} I &= \left(\frac{1}{\epsilon} - \ln m^2\right)\int_0^1 x(1-x)dx - \int_0^1 x(1-x)\ln\left[-x(1-x)\frac{q^2}{m^2} + 1\right] dx \\ &= \frac{1}{6}\left(\frac{1}{\epsilon} - \ln m^2\right) - \int_0^1 x(1-x)\ln\left[-x(1-x)\frac{q^2}{m^2} + 1\right] dx \end{aligned} \tag{3.21}$$

3.5 Branch cuts

Logarithms contain branch cuts when the argument is real and negative. The logarithm of equation 3.21 has a branch cut when

$$x(1-x)\frac{q^2}{m^2} > 1$$

In the domain of the Feynman parameter x between 0 and 1, the possible values of $x(1-x)$ are $0 < x(1-x) < \frac{1}{4}$. The maximum value is $\frac{1}{4}$, so the location of the branch cut in the amplitude will be

$$q^2 > 4m^2$$

Recall from section 2.1 that this is exactly the case where the loop is on mass shell. Here we can make the observation that there is a branch cut in the amplitude when the loop is on mass shell, or in other words, when the ratio between q^2 and m^2 is greater than 4. This branch cut property when the loop is on mass shell will be in our advantage when using the unitarity method in section 4.

Integration

Next we can factorize the term inside the logarithm, and split the logarithm into three parts:

$$-x(1-x)a + 1 = a\left(x - \frac{1}{2}(1+b)\right)\left(x - \frac{1}{2}(1-b)\right)$$

Where we define a as the ratio between q^2 and m^2, and b as:

$$a = \frac{q^2}{m^2} \tag{3.22}$$

$$b = \sqrt{1 - \frac{4}{a}} \tag{3.23}$$

The integral of the first part of the factorization containing $\ln a$ is straightforward, and the answer is $-\frac{1}{6}\ln a$. Two integrals containing $\ln[x - \frac{1}{2}(1 \pm b)]$ are left.

The explicit derivation of the solution to the following integral is made in appendix E.2, using partial integration. This derivation treats the branch cut property directly.

$$- \int_0^1 x(1-x)\ \ln\left[x - \frac{1}{2}(1 \pm b)\right] dx =$$
$$\underbrace{-\frac{1}{6}\ln\left[\frac{1}{2} \pm \frac{1}{2}b\right]}_{\text{Part A}} + \underbrace{\left[\frac{1}{12} \pm \frac{b}{8} \mp \frac{b^3}{24}\right]\ln\left[\frac{1 \mp b}{-1 \mp b}\right]}_{\text{Part B}} + \underbrace{\frac{5}{36} + \frac{1}{6}(1 \pm b) - \frac{1}{12}(1 \pm b)^2}_{\text{Part C}}$$

Now add the two cases for the plus and the minus sign, because those correspond to the two remaining parts of the factorization.

Part A:

$$\begin{aligned} -\frac{1}{6}\ln\left[\frac{1}{2} + \frac{1}{2}b\right] - \frac{1}{6}\ln\left[\frac{1}{2} - \frac{1}{2}b\right] &= -\frac{1}{6}\ln\left[\frac{1}{4} - \frac{1}{4}b^2\right] \\ &= -\frac{1}{6}\ln\frac{1}{a} \\ &= \frac{1}{6}\ln a \end{aligned}$$

This term will cancel with the term coming from the integral of the first part of the factorization.

Part B:

$$\begin{aligned} \frac{1}{12}\ln\left[\frac{1-b}{-1-b}\right] + \frac{1}{12}\ln\left[\frac{1+b}{-1+b}\right] &= \frac{1}{12}\ln\left[\frac{(1-b)(1+b)}{(-1-b)(-1+b)}\right] \\ &= \frac{1}{12}\ln\left[\frac{b-1}{b-1}\right] = 0 \\ \frac{b}{8}\ln\left[\frac{1-b}{-1-b}\right] - \frac{b}{8}\ln\left[\frac{1+b}{-1+b}\right] &= \frac{b}{8}\ln\left[\frac{(b-1)^2}{(b+1)^2}\right] \\ &= -\frac{b}{4}\ln\left[\frac{b+1}{b-1}\right] \\ -\frac{b^3}{24}\ln\left[\frac{1-b}{-1-b}\right] + \frac{b^3}{24}\ln\left[\frac{1+b}{-1+b}\right] &= \frac{b^3}{24}\ln\left[\frac{(b+1)^2}{(b-1)^2}\right] \\ &= \frac{b^3}{12}\ln\left[\frac{b+1}{b-1}\right] \end{aligned}$$

Part C:

$$\frac{5}{36} + \frac{1}{6}(1+b) - \frac{1}{12}(1+b)^2 + \frac{5}{36} + \frac{1}{6}(1-b) - \frac{1}{12}(1-b)^2 = \frac{16}{36} - \frac{b^2}{6}$$

Next we can combine all the terms.

$$
\begin{aligned}
I &= \frac{1}{6}\left(\frac{1}{\epsilon} - \ln m^2\right) + \frac{16}{36} - \frac{b^2}{6} + \left(\frac{b^3}{12} - \frac{b}{4}\right) \ln\left[\frac{b+1}{b-1}\right] \\
&= \frac{1}{6}\left(\frac{1}{\epsilon} - \ln m^2\right) + \frac{10}{36} + \frac{2}{3a} - \frac{1}{6}\left(1 + \frac{2}{a}\right)\sqrt{1 - \frac{4}{a}} \ln\left[\frac{b+1}{b-1}\right] \\
&= \frac{1}{6}\left[\frac{1}{\epsilon} - \ln m^2 + \frac{5}{3} + \frac{4}{a} - \left(1 + \frac{2}{a}\right)\sqrt{1 - \frac{4}{a}} \ln\left[\frac{b+1}{b-1}\right]\right]
\end{aligned}
\tag{3.24}
$$

3.6 Result

Now we set back the result of the Feynman parameter integral I in equation 3.24 into equation 3.17. Use that $\alpha = \frac{e^2}{4\pi}$ in natural units, and get the result of Π up to $\mathcal{O}(\epsilon)$:

$$
\Pi^{\mu\nu}(q^2) = -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \ln m^2 + \frac{5}{3} + \frac{4}{a} + \left(1 + \frac{2}{a}\right)\sqrt{1 - \frac{4}{a}} \ln\left[\frac{b-1}{b+1}\right]\right] \tag{3.25}
$$

When the dimension is set back to four, ϵ goes to zero, and the only divergence in amplitude is due to the $\frac{1}{\epsilon}$ term. This term can be absorbed into some renormalization constant, to make the final amplitude convergent.

Equation 3.25 is the final amplitude achieved by dimensional regularization. This technique of regularization, where the calculation is performed close to four dimensions, turns out to be very powerful. But the problem is that it requires integration of the internal loop momentum. For more complicated loop diagrams with more external lines, the calculation is going to be much longer, as more external momenta are entering the calculation. Until this moment, the advantages of unitarity were not used yet. In the next sections, unitarity will be introduced, and applied onto the calculation of the vacuum polarization.

4 Unitarity

In quantum theory, the absolute value squared of an operator defines the probability of the physical process that is represented by the operator. In other words, when a scattering event is defined by an operator S, the corresponding total probability is $SS^\dagger$. The main assumption of the unitarity method is conservation of probability during the interaction. The total probability of an interaction during a scattering event, has to be equal to one all the time. Applying this assumption on the scattering matrix:

$$SS^\dagger = 1 \tag{4.1}$$

This is exactly the requirement for a unitary matrix. By implying conservation of probability, the scattering matrix must by unitary. The unitarity method for one loop amplitudes is based on this statement.

4.1 Optical Theorem

By a scattering event, we distinguish the case of forward scattering and the case of a transition between the particle states. The scattering matrix can be defined as:

$$S = 1 + iT \tag{4.2}$$

Where the identity matrix 1 represents the forward scattering, the case where basically nothing happens. The matrix T represents the transition matrix, and is related to the amplitude of Feynman diagrams as:

$$\mathcal{M}_{i\to f} = \langle f|T|i\rangle \tag{4.3}$$

It would be nice to turn equation 4.1 around for a requirement on Feynman diagrams. This is also done in [5]. Let's see what the requirement on the transition matrix T will be when the scattering matrix S is unitary:

The Hermitian transpose of S is (using equations B.4 and B.5 from the appendix):

$$S^\dagger = (I + iT)^\dagger = I - iT^\dagger$$

Inserting this in equation 4.1:

$$\begin{aligned} (I + iT)(I - iT^\dagger) &= I \\ I - iT^\dagger + iT + T^\dagger T &= I \\ -i(T - T^\dagger) &= T^\dagger T \end{aligned} \tag{4.4}$$

Thus, applying unitarity on the scattering matrix S, gives a requirement on the transition matrix T. This requirement is also called the Optical Theorem.

To find the optical theorem for $\mathcal{M}$, we can contract the Optical Theorem on both sides with initial and final states as in equation 4.3. The left-hand side of 4.4 is straightforward:

$$\begin{aligned} -i\langle f|(T - T^\dagger)|i\rangle &= -i(\mathcal{M} - \mathcal{M}^*) \\ &= 2\mathrm{Im}\mathcal{M} \end{aligned}$$

The important step comes in rewriting the right-hand side. Because we are dealing with a product of two operators, it is allowed to insert a complete set of intermediate eigenstates $\sum_k |k\rangle\langle k|$

between these two operators.

$$
\begin{aligned}
\langle f|T^\dagger T|i\rangle &= \sum_k \langle f|T^\dagger|k\rangle\langle k|T|i\rangle \\
&= \sum_k \mathcal{M}^*_{f\to k}\mathcal{M}_{i\to k}
\end{aligned}
$$

Combining the left and right hand side again, the condition on $\mathcal{M}$ becomes:

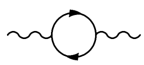
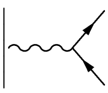
$$2\text{Im}\ \mathcal{M}_{i\to f} = \sum_k \mathcal{M}^*_{f\to k}\mathcal{M}_{i\to k} \tag{4.5}$$

Because it is allowed to insert intermediate states k, any loop inside a Feynman diagram can be split up into an amplitude of the initial and final state going to some intermediate states of the loop. This is the power of the unitarity method. By cutting the loop into seperate tree-diagrams, the imaginary part can be calculated.

The remaining tree-amplitudes of intermediate states don't contain any internal loop-momentum, and no integration over fourmomentum will be needed. The summation over k also contains all the possible spin states and the phase space of the intermediate states. So in words, equation 4.5 means that the imaginary part of any amplitude is proportional to the sum of all possible tree-amplitudes that contribute to the loop, including all spin states and all phase space.

In the case of the vacuum polarization amplitude, the intermediate states k are the fermion and antifermion of the loop. In this case equation 4.5 tells that:

$$2\text{Im}\ \Pi^{\mu\nu}(q^2) = \sum_{\text{spin}} \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} d\text{Lips} \tag{4.6}$$

2Im  $= \sum_{\text{spin}} \Bigg|$  $\Bigg|^2 d\text{Lips}$ (4.7)

An example of how to deal with the intermediate states k explicitly for the vacuum polarization amplitude can be found in section 5.

There is one important remark on cutting a loop in tree diagrams. Tree diagrams are always on mass shell, because the final states of a tree cannot be virtual states. This means that whenever a loop is cut by the unitarity method, the loop is placed on mass shell. We have already seen in section 3.5 that the amplitude has a branch cut for the on shell values of the external momentum. This branch cut is going to be in our favor when we want to calculate the full amplitude from the imaginary part.

4.2 Dispersion relations

Now that the imaginary part of the amplitude along the onshell values of the momentum can be calculated, the next step will be to extract the complete amplitude. A way to do this, is using so called dispersion relations. By knowing the imaginary part and the analytic properties of an arbitrary function, it is possible to calculate the complete function everywhere. Dispersion relations are an important tool in physics and an example are the Kramers-Kronig relations(see [4] p333), which gives relations between the real and imaginary part of a function which is analytic in the upper half plane. These dispersion relations find there applications in, for example, classical electrodynamics and wave mechanics. In this section we will derive a dispersion relation for a function with a branch cut along the real axis, making advantage of the imaginary value of the function on this branch cut.

Complex analysis tells us that an arbitrary complex function is determined by it's behaviour at the poles of the function. In this specific case, we are dealing with a function $F(z)$ with a branch cut located on the real axis running from M to ∞. Elsewhere, the function is analytic. Constructing an integration contour enclosing the branch cut as in figure 5, and using Cauchy's Integral Formula (D.1), gives the following relation for $F(z)$ in terms of it's behaviour along the branch cut:

$$F(z) = \frac{1}{2\pi i}\int_M^\infty \frac{F(x+i\epsilon)}{x-z}dx - \frac{1}{2\pi i}\int_M^\infty \frac{F(x-i\epsilon)}{x-z}dx + C_\infty \tag{4.8}$$

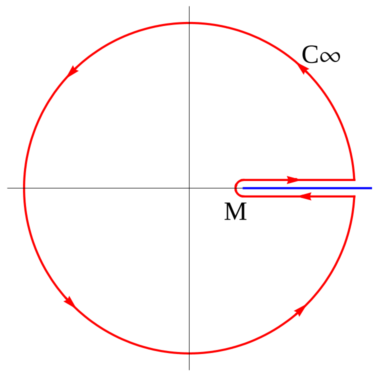


Figure 5: Integration contour in the complex plane, with a branch cut located on the real axis

$F(x-i\epsilon)$ is the contour slightly above the branch cut, while $F(x+i\epsilon)$ is the contour slightly below the branch cut. These two parts of the contour are in opposite direction, so one of the two contours gets a negative contribution.

We can further extend equation 4.8 to derive a useful dispersion relation:

$$F(z) = \frac{1}{2\pi i}\int_M^\infty \frac{F(x+i\epsilon) - F(x-i\epsilon)}{x-z}dx + C_\infty$$

Schwarz reflection principle tells that:

$$F(z) = \overline{F(\bar{z})}$$

For any complex number $z = x + iy$ with $x, y \in \mathbb{R}$:

$$z - \bar{z} = x + iy - (x - iy) = 2iy = 2i\text{Im}(z)$$

Combining these two statements gives:

$$F(x+i\epsilon) - F(x-i\epsilon) = F(x+i\epsilon) - \overline{F(x+i\epsilon)} = 2i\text{Im}F(x) \tag{4.9}$$

Now equation 4.8 becomes:

$$F(z) = \frac{1}{\pi} \int_M^{\infty} \frac{\text{Im}F(x)}{x - z} dx + C_{\infty} \tag{4.10}$$

With this equation it is possible to reconstruct the whole function $F(z)$ by integrating the imaginary part of the function along the branch cut. C_{∞} has the possibility to vanish for $z \to \infty$. This happens when z decays to zero fast enough. When z does not decay fast enough, it is possible to subtract a value $F(z_0)$ from $F(z)$. This is done in appenix D.1. When one subtraction is not enough, further subtractions can be made.

4.3 Overview of the unitarity method

The imaginary part of an amplitude can be evaluated by calculating the tree-amplitudes that contribute to the interaction from the initial and final state to the intermediate states. This statement follows from the constraint on the S-matrix to be unitary. Splitting loop-amplitudes into tree-amplitudes requires the momentum to be onshell, yielding a branch cut in the full amplitude along these values. Exploiting features of complex analysis gives us a dispersion relation, from which we can extract the full amplitude of the interaction. The advantage upon direct calculation from the Feynman diagram, is that there is no internal loop momentum integration needed in the calculation. In the next section this method will be applied on the amplitude of vacuum polarization.

5 Vacuum polarization by unitarity

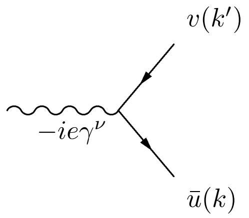
The vacuum polarization amplitude is already known from dimensional regularization, and now it is time to see how the unitarity method deals with this important example. The first step will be to calculate the imaginary part from the contributing tree diagrams (equation 4.6). Therefore, the tree diagrams have to be constructed from the Feynman rules. After that, the sum over all spin states will be done by using completeness relations, and the integration over Lorentz invariant phase space will be performed by exploiting the Ward identity.

This example is also discussed explicitly in [10]. The first part of the calculation that is performed right here basically follows the same strategy, but contains adjustments and refinements on the calculation of [10].

5.1 Imaginary part

Tree amplitude

The contributing tree-amplitude $\mathcal{M}_{\text{tree}}$ can be constructed from the Feynman rules:



$$i\mathcal{M}_{\text{tree}} = \bar{u}(k)(-ie\gamma^\mu)v(k')$$

The hermitian conjugate of $\mathcal{M}_{\text{tree}}$ is:

$$\begin{aligned} i\mathcal{M}^\dagger_{\text{tree}} &= [\bar{u}(k)(-ie\gamma^\mu)v(k')]^\dagger \\ &= -ie\,[v(k')]^\dagger(\gamma^\mu)^\dagger[\bar{u}(k)]^\dagger \\ &= -ie\,[v(k')]^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 u(k) \\ &= \bar{v}(k')(-ie\gamma^\mu)u(k) \end{aligned}$$

Where in the first step we've used property B.2, in de second step C.13 and C.14, and in de last step C.12.

$$\sum_{\text{spin}} \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} = e^2 \sum_{\text{spin}} \bar{u}(k)\gamma^\mu v(k')\bar{v}(k')\gamma^\nu u(k) \tag{5.1}$$

Summing over spin states

With the completeness relations: (see [1])

$$\sum_{\text{spin}} u(p)\bar{u}(p) = \not{p} + m$$
$$\sum_{\text{spin}} v(p)\bar{v}(p) = \not{p} - m$$

the sum over the spins becomes:

$$\begin{aligned}\sum_{\text{spin}} \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} &= e^2 \text{ tr}\left[(\not{k}+m)\gamma^\mu(\not{k}'-m)\gamma^\nu\right] \\ &= e^2\left[\text{tr}\left[\gamma^\nu\gamma^\mu\gamma^\beta\gamma^\alpha\right]k_\alpha k'_\beta - m^2\text{tr}\left[\gamma^\mu\gamma^\nu\right]\right] \\ &= 4e^2\left[\left(g^{\nu\alpha}g^{\mu\beta} - g^{\nu\mu}g^{\alpha\beta} + g^{\nu\beta}g^{\mu\alpha}\right)k_\alpha k'_\beta - g^{\mu\nu}m^2\right] \\ &= 4e^2\left(k^\nu k'^\mu + k'^\nu k^\mu - g^{\mu\nu}(k\cdot k' + m^2)\right)\end{aligned} \tag{5.2}$$

Where we've dropped the cross-terms in the first step, because those contain an odd number of gamma-matrices, and the trace becomes zero. The other steps ared performed by the identies on the trace of gamma matrices, used before in the calculation of the one loop amplitude using dimensional regularization.

Phase space integration

Lorentz invariant phase space for two outgoing particles in the center of mass frame: For a derivation, see [7] p83.

$$d\text{Lips} = \frac{|\vec{k}|}{16\pi^2\sqrt{s}}\, d\Omega_{\text{CM}} \tag{5.3}$$

From kinematics follows:

$$|\vec{k}| = \sqrt{\frac{q^2}{4} - m^2} \tag{5.4}$$

And:

$$\begin{aligned} d\text{Lips} &= \frac{\sqrt{\frac{q^2}{4} - m^2}}{16\pi^2\sqrt{q^2}}\, d\Omega_{\text{CM}} \\ &= \frac{1}{32\pi^2}\sqrt{1-\frac{4}{a}}\, d\Omega_{\text{CM}}\end{aligned}$$

Use this to calculate the complete imaginary part by equation 4.6 and inserting the sum over the spin states in equation 5.2:

$$\begin{aligned}\text{Im }\Pi^{\mu\nu} &= \frac{1}{2}\sum_{\text{spin}}\mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} d\text{Lips} \\ &= \frac{1}{2}\frac{4e^2}{32\pi^2}\sqrt{1-\frac{4}{a}}\int d\Omega_{\text{CM}}\left(k^\nu k'^\mu + k'^\nu k^\mu - (k\cdot k' + m^2)g^{\mu\nu}\right) \\ &= \frac{\alpha}{4\pi}\sqrt{1-\frac{4}{a}}\int d\Omega_{\text{CM}}\left(k^\nu k'^\mu + k'^\nu k^\mu - (k\cdot k' + m^2)g^{\mu\nu}\right)\end{aligned} \tag{5.5}$$

Fixing the tensor structure

Now recall from section 3.3 that the amplitude of vacuum polarization has a certain tensor structure, which satisfies the Ward identity. The structure that was found by dimensional regularization is not just a solution to the Ward identity, it is the only possible tensor structure [5]. So the amplitude should look like:

$$\Pi^{\mu\nu}(a) = \Pi(a)(q^\mu q^\nu - q^2 g^{\mu\nu}) \tag{5.6}$$

The angular integration must contain this tensor structure in the final answer:

$$\int \frac{d\Omega_{\text{CM}}}{4\pi}\left(k^\nu k'^\mu + k'^\nu k^\mu - \frac{q^2}{2}g^{\mu\nu}\right) = F(a)(q^\mu q^\nu - q^2 g^{\mu\nu})$$

Where $F(a)$ is some scalar function. Now contract both sides with the metric $g_{\mu\nu}$:

$$\begin{aligned}\int \frac{d\Omega_{\text{CM}}}{4\pi}\left(g_{\mu\nu}(k^\nu k'^\mu + k'^\nu k^\mu) - (k\cdot k' + m^2)g_{\mu\nu}g^{\mu\nu}\right) &= F(a)(g_{\mu\nu}q^\mu q^\nu - q^2 g_{\mu\nu}g^{\mu\nu}) \\ \int \frac{d\Omega_{\text{CM}}}{4\pi}(2k\cdot k' - 4(k\cdot k' + m^2)) &= F(a)(q^2 - 4q^2) \\ \int \frac{d\Omega_{\text{CM}}}{4\pi}(-2k\cdot k' - 4m^2) &= -3q^2F(a)\end{aligned} \tag{5.7}$$

From kinematics follows that:

$$\begin{aligned}q^2 = (k+k')^2 &= k^2 + k'^2 + 2k\cdot k' \\ &= 2m^2 + 2k\cdot k' \\ 2k\cdot k' &= q^2 - 2m^2\end{aligned}$$

By inserting this statement into equation 5.7, there is no k-dependence left in the integrand, and the angular integral cancels against the factor $\frac{1}{4\pi}$.

$$F(a) = -\frac{1}{3q^2}\left(-q^2 - 2m^2\right) = \frac{1}{3}\left(1 + \frac{2}{a}\right) \tag{5.8}$$

Result

Plugging everything in equation 5.5:

$$\begin{aligned}\text{Im}\Pi^{\mu\nu} &= \frac{\alpha}{3}\sqrt{1-\frac{4}{a}}\left(1+\frac{2}{a}\right)(q^\mu q^\nu - q^2 g^{\mu\nu}) \\ &= \text{Im}\Pi(a)(q^\mu q^\nu - q^2 g^{\mu\nu})\end{aligned}$$

With:

$$\text{Im}\Pi(a) = \frac{\alpha}{3}\sqrt{1-\frac{4}{a}}\left(1+\frac{2}{a}\right) \tag{5.9}$$

This result is the imaginary part of the vacuum polarization amplitude along the on mass shell values of q^2. It is achieved by using the statement that the imaginary part can be calculated from the separate contributing tree-diagrams. This statement followed from unitarity.

5.2 Full amplitude by dispersion relations

The full amplitude can be reconstructed by using the dispersion relation derived from Cauchy's Integral Theorem in section 4.2. The problem we encounter is that the contour at infinity does not vanish, which is actually required for the dispersion relations to work. A solution is to subtract two dispersion relations from each other, which is done in appendix D.1. As a result of the subtraction, the integrand inside the dispersion relation decays one degree faster at infinity, and the contour at infinity will vanish in this case. So for this case, we should use the once-subtracted dispersion relation D.6. Take z_0 . The branch cut runs from $a = 4$ to infinity, which was already known from the observations made in section 3.5.

$$\begin{aligned} \Pi(a) - \Pi(0) &= \frac{a}{\pi} \int_4^\infty \frac{\mathrm{Im}\Pi(a)}{(a' - a)a'} da' \\ &= \frac{\alpha}{3\pi} a \int_4^\infty \sqrt{1 - \frac{4}{a'}} \left(1 + \frac{2}{a'}\right) \frac{1}{(a' - a)a'} da' \end{aligned} \tag{5.10}$$

Only integration over the scalar a' is needed to derive the solution. This requires a series of substitutions, which is done into very detail in [10], page 53-54.

Result:

$$\Pi(a) - \Pi(0) = \frac{\alpha}{3\pi} \left[\frac{5}{3} + \frac{4}{a} - \left(2 + \frac{4}{a}\right) \sqrt{1 - \frac{4}{a}} \ln \left[\frac{b - 1}{b + 1} \right] \right] \tag{5.11}$$

The divergent $\frac{1}{\epsilon}$ term in the amplitude achieved by dimension regularization (equation 3.25) is not present when using the unitarity method. This is due to the fact that the we already subtracted a value from the amplitude by using the once-subtracted dispersion relation. Here we can make the observation that this result is implicitly renormalized.

This unitarity method still requires an amount of work to derive the amplitude. However the calculation is simpler than direct calculation by Feynman diagrams, because no integration over internal loop momentum is needed.

6 Generalized unitarity

As described in the previous sections, it is possible to reconstruct the imaginary part of a loop amplitude by taking unitarity cuts. After that, the real part can be reconstructed by using dispersion relations. The disadvantage of the dispersion method is that there is still an integral to perform. Despite the fact that this integral doesn't have a tensor structure as in dimensional regularization, we are still looking for more sophisticated methods to calculate loop amplitudes. By exploiting another property of loop amplitudes described in this section, it is not needed to perform dispersion integrals anymore.

The main point of the generalized unitarity method is that a one loop diagram can be written in a basis of four types of scalar diagrams:

$$\mathcal{M}_{\text{loop}} = d_I \text{Box} + c_I \text{Tri} + b_I \text{Bub} + a_I \text{Tad} + Q \tag{6.1}$$

These diagrams are scalar boxes, triangles, bubbles and tadpoles, from which the solutions to the corresponding integrals are known. An overview of all possible scalar integrals and solutions can be found in [6].

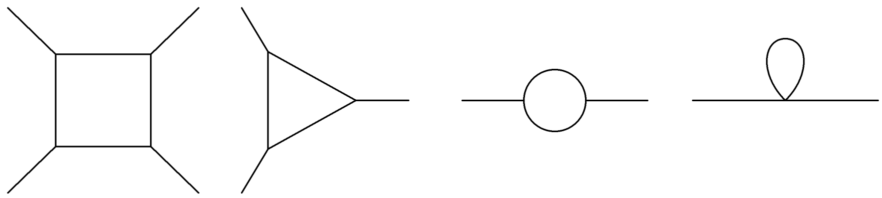


Figure 6: Box, triangle, bubble and tadpole contributions

Passarino-Veltman reduction [2] is a technique where tensor loop integrals with an arbitrary number of internal lines are reduced to scalar loop integrals. The number of internal lines present in these scalar integrals will be dependent on the power of the internal loop momentum l in the numerator. Another result from [9], says that scalar integrals with five or more internal lines can be reduced to scalar integrals with four internal lines. This follows from the fact that in four dimensions, there are only four independent momenta.

When these two statements are combined, the expansion of any one-loop integral in the basis of equation 6.1 is justified.

Given the statement that any loop amplitude can be written as the sum of scalar integrals, the problem reduces to the calculation of the coefficients in front of the integrals, and some rational term Q. This is the place where unitarity comes in. We've seen before that cutting a diagram corresponds with taking the imaginary part. Applying unitarity cuts on equation 6.1 gives constraints on the coefficients, and for some cases it will be possible to read off the coefficients directly after unitarity cuts.

7 Vacuum polarization by generalized unitarity

In this section, the amplitude of vacuum polarization is calculated by applying the method of generalized unitarity, as described in section 6.

When the amplitude is written in the basis of scalar integrals, the box and triangle diagrams fall out. These diagrams have three and four external particles, where vacuum polarization only has two external particles. The amplitude can be written as:

$$\Pi(a) = b_I \text{Bub} + a_I \text{Tad} + Q \tag{7.1}$$

where the bubble and tadpole integrals are known from [6]:

$$\text{Bub} = \frac{1}{\epsilon} + 2 - \ln m^2 + \sqrt{1 - \frac{4}{a}} \ln \left[\frac{b-1}{b+1} \right] + \mathcal{O}(\epsilon) \tag{7.2}$$

$$\begin{aligned} \text{Tad} &= m^2 \frac{1}{m^{2\epsilon}} \left(\frac{1}{\epsilon} + 1 \right) + \mathcal{O}(\epsilon) \\ &= m^2 (1 - \epsilon \ln m^2) \left(\frac{1}{\epsilon} + 1 \right) + \mathcal{O}(\epsilon) \\ &= m^2 \left(\frac{1}{\epsilon} + 1 - \ln m^2 \right) + \mathcal{O}(\epsilon) \end{aligned} \tag{7.3}$$

The important step of the calculation by generalized unitarity, is to perform the unitarity cut on equation 7.1. The tadpole integral will fall out, as this contribution does not contain branch cuts.

$$\Pi_{\text{cut}}(a) = b_I \text{Bub}_{\text{cut}} \tag{7.4}$$

Π_{cut} is just the imaginary part of vacuum polarization, which is calculated already in section 5.1. Bub_{cut} can be calculated by taking the discontinuity of the branch cut of equation 7.2. By plugging in these two values, it is easy to read off the bubble coefficient:

$$\begin{aligned} \frac{\alpha}{3} \sqrt{1 - \frac{4}{a}} \left(1 + \frac{2}{a} \right) &= \pi b_I \sqrt{1 - \frac{4}{a}} \\ b_I &= \frac{\alpha}{3\pi} \left(1 + \frac{2}{a} \right) \end{aligned}$$

Setting back this coefficient in the full amplitude:

$$\begin{aligned} \Pi(a) =& \frac{\alpha}{3\pi} \left[\frac{1}{\epsilon} - \ln m^2 + \frac{4}{a} + \left(1 + \frac{2}{a} \right) \sqrt{1 - \frac{4}{a}} \ln \left[\frac{b-1}{b+1} \right] + \underbrace{\frac{2}{a} \left(\frac{1}{\epsilon} - \ln m^2 \right) + 2} \right] \\ &+ a_I m^2 \left(\frac{1}{\epsilon} - \ln m^2 + 1 \right) + Q + \mathcal{O}(\epsilon) \end{aligned} \tag{7.5}$$

Compared to the answer achieved by dimensional regularization (equation 3.25), the underbraced terms should not be there. However, these terms can easily be absorbed in the rational constants a_I and Q, because the tadpole contribution also contains the underbraced terms.

$$a_I = -\frac{\alpha}{3\pi} \frac{2}{q^2}$$

The remainder can be absorbed in Q,

$$Q = \frac{\alpha}{3\pi}\left(\frac{2}{a} - \frac{1}{3}\right)$$

Apparently this is not the right way to calculate the remaining coefficients a_I and Q, as in cases in which we actually want to apply this method the final answer is not known. It only illustrates that calculating the bubble-coefficient using generalized unitarity works.

8 Box diagram by generalized unitarity

Now the unitarity method is applied onto an answer that was already known from other techniques, it is time to see how generalized unitarity deals with a more non-trivial example.

The diagram that is going to be calculated, is shown in figure 8. The diagram contains a box as a loop, where the internal lines of the loop are fermions.

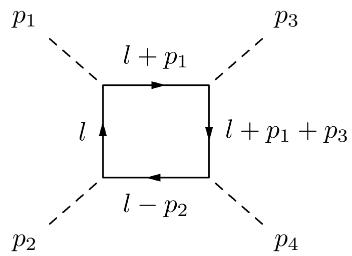


Figure 7: Box diagram with momentum flow

The momentum flow is chosen in such a way that all momentum are incoming, so:

$$p_1 + p_2 + p_3 + p_4 = 0$$

The main point of the method of generalized unitarity is to write the original diagram in terms of the basis of scalar loop integrals, as in equation 6.1. This time we perform two unitarity cuts because of the box structure of the loop. The first cut is taken on the propagator between p_1 and p_2, and the second cut is taken on the propagator between p_1 and p_3. As an immediate consequence of taking a double cut, the triangle, bubble and tadpole contributions will be zero after the unitarity cuts:

$$I_{\text{doublecut}} = d_I \text{Box}_{\text{doublecut}} \tag{8.1}$$

The main goal is to search for the box coefficient d_I. Therefore, need perform the double unitarity cut on both the diagram itself and the scalar box integral.

To simplify the calculation a little bit in order to focus on the method of generalized unitarity only, we take all lines to be massless. The consequences are:

Masses of all external lines are zero:

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0 \tag{8.2}$$

Masses of all internal lines are zero:

$$l^2 = 0 \tag{8.3}$$

$$(l + p_1)^2 = l^2 + 2l \cdot p_1 + p_1^2 = 0 \ \to l \cdot p_1 = 0 \tag{8.4}$$

$$(l + p_1 + p_3)^2 = l^2 + 2l \cdot p_1 + p_1^2 + p_3^2 + 2l \cdot p_3 + 2p_1 \cdot p_3 = 0 \ \to l \cdot p_3 = -p_1 \cdot p_3 \tag{8.5}$$

$$(l - p_2)^2 = l^2 - 2l \cdot p_2 + p_2^2 = 0 \ \to l \cdot p_2 = 0 \tag{8.6}$$

8.1 Unitarity cuts on the diagram itself

From the Feynman rules of QED, the amplitude is equal to:

$$-\int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{tr}[\not{l}(\not{l}+\not{p}_1)(\not{l}+\not{p}_1+\not{p}_3)(\not{l}-\not{p}_2)]}{l^2(l+p_1)^2(l+p_1+p_3)^2(l-p_2)^2} \tag{8.7}$$

When a unitarity cut is performed on the Feynman rules, the propagators can be upgraded to delta functions:

$$-\int \frac{d^d l}{(2\pi)^d} \,\mathrm{tr}[...]\delta(l^2)\delta((l+p_1)^2)\delta((l+p_1+p_3)^2)\delta((l-p_2)^2) \tag{8.8}$$

8.1.1 Trace

$$\mathrm{tr}[\not{l}(\not{l}+\not{p}_1)(\not{l}+\not{p}_1+\not{p}_3)(\not{l}-\not{p}_2)]$$

Write out the slash notation and split the gamma matrices from the four-momenta:

$$\mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] l_\mu(l_\nu+p_{1\nu})(l_\rho+p_{1\rho}+p_{3\rho})(l_\sigma-p_{2\sigma})$$

Use:

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

And the trace becomes:

$$4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})l_\mu(l_\nu+p_{1\nu})(l_\rho+p_{1\rho}+p_{3\rho})(l_\sigma-p_{2\sigma})$$

Working out the brackets and contracting the metrics with all terms of four momenta will turn each term into two dot-products. Terms with three or four times l inside, contain at least one l^2 and will drop out. All other terms without p_3 will also drop out, because $l\cdot p_1 = l\cdot p_2 = 0$.

Now the only remaining terms containing p_3 are:

$$\begin{aligned}&- l_\mu l_\nu p_{3\rho} p_{2\sigma}\\ &+ l_\mu p_{1\nu} p_{3\rho} l_\sigma\\ &- l_\mu p_{1\nu} p_{3\rho} p_{2\sigma}\end{aligned}$$

The first and the second one are also zero, because each of the two dot-products are either l^2 or $l\cdot p_1$,$l\cdot p_2$.
The third term:

$$-4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})l_\mu p_{1\nu}p_{3\rho}p_{2\sigma} = -4(\underbrace{l_\mu p_1^\mu}_{0} p_{3\rho}p_2^\rho - l_\mu p_3^\mu p_{1\nu}p_2^\nu + \underbrace{l_\mu p_2^\mu}_{0} p_{1\nu}p_3^\nu)$$

Inside this term, there is only one non-zero contribution: $4(l\cdot p_3)(p_1\cdot p_2)$. Now use equation 8.5 to rewrite this contribution: $-4(p_1\cdot p_3)(p_1\cdot p_2)$. And the final answer to the trace is:

$$\mathrm{tr}[\not{l}(\not{l}+\not{p}_1)(\not{l}+\not{p}_1+\not{p}_3)(\not{l}-\not{p}_2)] = -4(p_1\cdot p_3)(p_1\cdot p_2) \tag{8.9}$$

This sets the complete integral to:

$$4(p_1\cdot p_3)(p_1\cdot p_2)\int \frac{d^d l}{(2\pi)^d}\;\delta(l^2)\delta((l+p_1)^2)\delta((l+p_1+p_3)^2)\delta((l-p_2)^2) \tag{8.10}$$

8.1.2 Jacobian

Because the delta functions contain invariant momentum of l, it is not possible to solve the delta functions at once. We need to write l^μ in a basis of the incoming momentum:

$$l^\mu = \alpha p_1^\mu + \beta p_2^\mu + \gamma p_3^\mu + \delta k^\mu \tag{8.11}$$

The only problem is that the incoming momenta are not completely independent. So there is the need for a fourth momentum k^μ, where $k_\mu p_i^\mu = 0$, for $i = 1, 2, 3$. A convenient choice for k_μ is:

$$k_\mu = \frac{1}{p_1 \cdot p_2} \epsilon_{\mu\nu\rho\sigma} p_1^\nu p_2^\rho p_3^\sigma$$

The trick is to integrate over the coefficients of l^μ, instead of integrating over l^μ itself. To turn the integral around into an integral over the four coefficients, we need to deal with the Jacobian of the coordinate transformation.

$$\begin{aligned} d^4 l &= J d\alpha d\beta d\gamma d\delta \\ &= \epsilon^{\mu\nu\rho\sigma} k_\mu p_{1\nu} p_{2\rho} p_{3\sigma} d\alpha d\beta d\gamma d\delta \\ &= \frac{1}{p_1 \cdot p_2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\alpha\beta\gamma} p_1^\alpha p_2^\beta p_3^\gamma p_{1\nu} p_{2\rho} p_{3\sigma} d\alpha d\beta d\gamma d\delta \end{aligned}$$

Where the choice for k_μ is used in the last step. Contracting the two Levi-Civita tensors gives:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = \underbrace{\delta_\nu^\alpha \delta_\rho^\beta \delta_\sigma^\gamma}_{1} - \underbrace{\delta_\nu^\beta \delta_\rho^\alpha \delta_\sigma^\gamma}_{2} - \underbrace{\delta_\nu^\alpha \delta_\rho^\gamma \delta_\sigma^\beta}_{3} + \underbrace{\delta_\nu^\beta \delta_\rho^\gamma \delta_\sigma^\alpha}_{4} + \underbrace{\delta_\nu^\gamma \delta_\rho^\alpha \delta_\sigma^\beta}_{5} - \underbrace{\delta_\nu^\gamma \delta_\rho^\beta \delta_\sigma^\alpha}_{6} \tag{8.12}$$

And:

$$\delta_\nu^\alpha p_1^\alpha p_{1\nu} = p_1^\alpha p_{1\alpha} = p_1^2$$

So all the seperate terms of equation 8.12 are:
1: $p_1^2 p_2^2 p_3^2 = 0$ (massless)
2: $-(p_1 \cdot p_2)^2 p_3^2 = 0$ (massless)
3: $-p_1^2 (p_2 \cdot p_3)^2 = 0$ (massless)
4: $(p_1 \cdot p_2)(p_2 \cdot p_3)(p_1 \cdot p_3)$
5: $(p_1 \cdot p_3)(p_1 \cdot p_2)(p_2 \cdot p_3)$
6: $-(p_1 \cdot p_2)^2 p_2^2 = 0$ (massless)

The total integral will be:

$$\frac{8}{(2\pi)^4}(p_1 \cdot p_3)^2 (p_1 \cdot p_2)(p_2 \cdot p_3) \int d\alpha d\beta d\gamma d\delta \quad \delta(l^2)\delta((l+p_1)^2)\delta((l+p_1+p_3)^2)\delta((l-p_2)^2)$$

8.1.3 Delta functions

Now we have an integral over the coefficients of l^μ. The next step is to write the expressions inside the delta functions in terms of these coefficients. Work out the squares inside the delta functions:

$$\delta(l^2)\delta(l^2 + 2l \cdot p_1)\delta(l^2 + 2l \cdot p_1 + 2l \cdot p_3 + 2p_1 \cdot p_3)\delta(l^2 - 2l \cdot p_2)$$

The first delta function sets l^2 to zero inside the other delta functions. With the same argument, $2l \cdot p_1$ is set to zero inside the third delta function.

$$\frac{1}{2^3}\delta(l^2)\delta(l \cdot p_1)\delta(l \cdot p_3 + p_1 \cdot p_3)\delta(l \cdot p_2)$$

All the separate delta functions are equal to:

$$\begin{aligned}
\delta(l^2) &= \delta(2\alpha\beta p_1\cdot p_2 + 2\alpha\gamma p_1\cdot p_3 + 2\beta\gamma p_2\cdot p_3 + \delta^2 k^2)\\
\delta(l\cdot p_1) &= \delta(\beta p_1\cdot p_2 + \gamma p_1\cdot p_3)\\
&= \frac{1}{p_1\cdot p_2}\delta\left(\beta + \gamma\frac{p_1\cdot p_3}{p_1\cdot p_2}\right)\\
\delta(l\cdot p_3 + p_1\cdot p_3) &= \delta(\alpha p_1\cdot p_3 + \beta p_2\cdot p_3 + p_1\cdot p_3)\\
\delta(l\cdot p_2) &= \delta(\alpha p_1\cdot p_2 + \gamma p_2\cdot p_3)\\
&= \frac{1}{p_1\cdot p_2}\delta\left(\alpha + \gamma\frac{p_2\cdot p_3}{p_1\cdot p_2}\right)
\end{aligned}$$

Where we used the relation:

$$\delta(Ax - B) = \frac{1}{A}\delta\left(x - \frac{B}{A}\right)$$

Now plug the delta functions in the complete integral:

$$\frac{8}{(2\pi)^4}\frac{1}{2^3}\frac{(p_1\cdot p_3)^2}{(p_1\cdot p_2)}(p_2{\cdot}p_3)\int d\alpha d\beta d\gamma d\delta\ \delta(l^2)\delta\left(\beta + \gamma\frac{p_1\cdot p_3}{p_1\cdot p_2}\right)\delta(\alpha p_1{\cdot}p_3 + \beta p_2{\cdot}p_3 + p_1{\cdot}p_3)\delta\left(\alpha + \gamma\frac{p_2\cdot p_3}{p_1\cdot p_2}\right)$$

Working out the delta function integrals over α and β has as result that we can fill in:

$$\begin{aligned}
\alpha &= -\gamma\frac{p_2\cdot p_3}{p_1\cdot p_2}\\
\beta &= -\gamma\frac{p_1\cdot p_3}{p_1\cdot p_2}
\end{aligned}$$

$$\frac{1}{(2\pi)^4}\frac{(p_1\cdot p_3)^2}{(p_1\cdot p_2)}(p_2\cdot p_3)\int d\gamma d\delta\ \delta(l^2)|_{\alpha,\beta=\dots}\ \delta(-\gamma\frac{p_2\cdot p_3}{p_1\cdot p_2}p_1\cdot p_3 - \gamma\frac{p_1\cdot p_3}{p_1\cdot p_2}p_2\cdot p_3 + p_1\cdot p_3)$$

$$\frac{1}{(2\pi)^4}\frac{(p_1\cdot p_3)}{(p_1\cdot p_2)}(p_2\cdot p_3)\int d\gamma d\delta\ \delta(l^2)|_{\alpha,\beta=\dots}\ \delta(\gamma - \frac{p_1\cdot p_2}{2p_2\cdot p_3})$$

And the values for α,β and γ are:

$$\begin{aligned}
\alpha &= -\frac{1}{2}\\
\beta &= -\frac{p_1\cdot p_3}{2p_2\cdot p_3}\\
\gamma &= \frac{p_1\cdot p_2}{2p_2\cdot p_3}
\end{aligned}$$

Now we can use this to plug everything into $\delta(l^2)$:

$$\frac{1}{(2\pi)^4}\frac{(p_1\cdot p_3)}{(p_1\cdot p_2)}(p_2\cdot p_3)\int\delta\left(\delta^2 k^2 - \frac{p_1\cdot p_3}{2p_2\cdot p_3}p_1\cdot p_2\right)d\delta$$

$$\frac{1}{(2\pi)^4}\frac{(p_1\cdot p_3)}{(p_1\cdot p_2)}(p_2\cdot p_3)\frac{1}{k^2}\int\delta\left(\delta^2 - \frac{1}{k^2}\frac{p_1\cdot p_3}{2p_2\cdot p_3}p_1\cdot p_2\right)d\delta$$

where we know that: $\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x+a) + \delta(x-a)\right]$

$$\frac{1}{(2\pi)^4}\frac{(p_1\cdot p_3)}{(p_1\cdot p_2)}(p_2\cdot p_3)\frac{1}{k^2}\sqrt{k^2\frac{2p_2\cdot p_3}{(p_1\cdot p_3)(p_1\cdot p_2)}}\underbrace{\int\frac{1}{2}\left[\delta(\delta+a) + \delta(\delta-a)\right]d\delta}_{1}$$

where (from the definition of k^μ)

$$k^2 = \frac{2(p_1 \cdot p_3)(p_2 \cdot p_3)}{p_1 \cdot p_2}$$

$$\frac{1}{(2\pi)^4}\frac{1}{2}\sqrt{k^2 \frac{2p_2 \cdot p_3}{(p_1 \cdot p_3)(p_1 \cdot p_2)}}$$

$$\frac{1}{(2\pi)^4}\frac{1}{2}\sqrt{4\frac{(p_2 \cdot p_3)^2}{(p_1 \cdot p_2)^2}}$$

And the final answer to the double unitarity cut on the diagram is:

$$I_{\text{double cut}} = \frac{1}{(2\pi)^4}\left|\frac{p_2 \cdot p_3}{p_1 \cdot p_2}\right| \tag{8.13}$$

8.2 Unitarity cuts on the scalar box integral

The next step will be to perform the double unitarity cut on the scalar box integral. From [6] this expression is known:

$$\text{Box} = \frac{1}{(2\pi)^4}\frac{1}{s_{12}s_{13}}\left[\frac{2}{\epsilon^2}\left((-s_{12})^{-\epsilon} + (-s_{13})^{-\epsilon}\right) - \ln^2\left(\frac{-s_{12}}{-s_{13}}\right) - \pi^2\right] + \mathcal{O}(\epsilon) \tag{8.14}$$

Where the definition is used that $s_{ij} = (p_i + p_j)^2$. This is equal to $2p_i \cdot p_j$ in the massless case.

Now it is time to perform the unitarity cut on this equation. The first unitarity cut is taken in the s_{12}-channel, where the second cut is taken in the s_{13}-channel, in the same way as the cuts are performed at diagram 8. From the previous sections we know that this is possible by taking the imaginary part at the branch cuts of this equation. The only point where branch cuts are present is at the logarithm squared. Rewriting this term as:

$$\ln^2(-s_{12}) + \ln^2(-s_{13}) - 2\ln(-s_{12})\ln(-s_{13})$$

When we take the branch cut in the s_{12}-channel, the squared logarithm containing s_{13} vanishes. In the cross term, $\ln(-s_{13})$ remains as a coefficient in front of the branch cut of $\ln(-s_{12})$. The logarithm of equation 8.14 after the s_{12}-channel cut is:

$$\pi\ln(-s_{12}) - 2\pi\ln(-s_{13})$$

In the s_{13}-channel, there is only one branch cut left in the second term, and logarithm squared becomes $-2\pi^2$. Now the complete scalar box integral after double unitarity cut is:

$$\begin{aligned}\text{Box}_{\text{double cut}} &= \frac{1}{(2\pi)^4}\frac{2\pi^2}{s_{12}s_{13}} \\ &= \frac{1}{(2\pi)^4}\frac{\pi^2}{2(p_1 \cdot p_2)(p_1 \cdot p_3)}\end{aligned} \tag{8.15}$$

8.3 Solving for the box coefficient

Achieving the box coefficient d_I is not complicated anymore. We can just fill in the answers from equation 8.13 and 8.15 into equation 8.1.

$$\begin{aligned} \left|\frac{p_2 \cdot p_3}{p_1 \cdot p_2}\right| &= d_I \frac{\pi^2}{2(p_1 \cdot p_2)(p_1 \cdot p_3)} \\ d_I &= \frac{2}{\pi^2} \left|\frac{p_2 \cdot p_3}{p_1 \cdot p_2}\right| (p_1 \cdot p_2)(p_1 \cdot p_3) \end{aligned} \tag{8.16}$$

This is the coefficient that is standing in front of the scalar box integral in the basis of scalar loop integrals. This coefficient is only achieved by using the method of generalized unitarity, and gives a great part of the final answer of the amplitude of the diagram we where looking for. Of course there are also the triangle, bubble and tadpole contributions left to do, but these require another way of looking at unitarity than just performing a double unitarity cut.

9 Conclusion

By assuming conservation of probability during an interaction, a whole calculational method of evaluating loop amplitudes based on the unitarity property of the scattering matrix followed. We took the example of vacuum polarization to see an application of the unitarity method.

Three different methods were applied on the vacuum polarization amplitude. The first one was dimensional regularization, a method without the use of unitarity. With dimensional regularization, the amplitude was calculated by integrating over the internal loop momentum in an arbitrary dimension. The divergences in the amplitude became evident when the dimension was set back to four in the end. The divergences can be absorbed in the theory afterwards by renormalization.

After this calculation, unitarity was introduced. Unitarity gives a relation between the imaginary part of the loop amplitude and the contributing tree amplitudes. In the first studied unitarity method, the full amplitude was reconstructed by using the advantage of the branch cut in the amplitude. Cauchy's integral theorem gives a dispersion relation, where the imaginary part is integrated over the branch cut in order to reconstruct the full amplitude.

The last discussed method was generalized unitarity, where the loop amplitude was written in a basis of scalar integrals. Performing unitarity cuts makes the calculation of the coefficients in front of the scalar integrals possible. Afterwards, the coefficients can be set back in the basis of scalar integrals to retrieve the full amplitude.

A not too hard assumption like conservation of probability gives physicists an elegant calculational method for evaluating the amplitude of all kind of one loop processes playing a role in high energy physics.

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A Units and conventions

Units and conventions used throughout this thesis:

Heaviside-Lorentz units

In this unit system, some fundamental constants of nature are normalized:

$$c = \hbar = \epsilon_0 = \mu_0 = 1 \tag{A.1}$$

This normalization is often used in Quantum Field Theory. In this convention, the finestructure constant α becomes:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} = \frac{e^2}{4\pi} \tag{A.2}$$

Feynman slash notation

The Feynman slash notation $\not{p}$ means that the Dirac gamma matrix γ^μ is contracted with the four vector p^μ, such that

$$\not{p} = \gamma^\mu p_\mu \tag{A.3}$$

B Linear Algebra

Hermitian transpose

For any matrix A and B with the same dimensions:

$$(A + B)^\dagger = A^\dagger + B^\dagger \tag{B.1}$$

For A and B:

$$(AB)^\dagger = B^\dagger A^\dagger \tag{B.2}$$

For any matrix A and $\lambda \in \mathbb{C}$

$$(\lambda A)^\dagger = \bar{\lambda} A^\dagger \tag{B.3}$$

Unitarity

A matrix U is called unitary when it satisfies the following condition:

$$UU^\dagger = I \tag{B.4}$$

This implies the following properties of a unitair matrix U:

$$U \text{ is invertible} \tag{B.5}$$

$$U^{-1} = U^\dagger \tag{B.6}$$

C Quantum Electrodynamics

C.1 Feynman rules

For an overview and introduction to Feynman rules, see for example: [1].

External lines

boson,antiboson (spin 0)

$$1 \tag{C.1}$$

incoming,outgoing fermion(spin$\frac{1}{2}$)

$$u(p), \bar{u}(p) \tag{C.2}$$

with $\bar{u} = u^\dagger \gamma^0$

incoming,outgoing antifermion(spin$\frac{1}{2}$)

$$v(p), \bar{v}(p) \tag{C.3}$$

with $\bar{v} = v^\dagger \gamma^0$

Internal lines

boson propagator (spin 0)

$$\frac{i}{p^2 - m^2 + i\epsilon} \tag{C.4}$$

fermion propagator (spin $\frac{1}{2}$)

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{C.5}$$

photon propagator (spin 1)

$$\frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \tag{C.6}$$

Vertex factors

photon-boson (spin 0) vertex

$$-ie(p + p')^\mu \tag{C.7}$$

photon-fermion (spin $\frac{1}{2}$) vertex

$$-ie\gamma^\mu \tag{C.8}$$

Other

closed loop

$$-1 \text{ and take the trace of the } \gamma \text{ matrices} \tag{C.9}$$

C.2 Completeness relations

For a discussion, see [1].

$$\sum_{spin} u(p)\bar{u}(p) = \not{p} + m \tag{C.10}$$

$$\sum_{spin} v(p)\bar{v}(p) = \not{p} - m \tag{C.11}$$

C.3 Dirac gamma matrices

$$(\gamma^0)^2 = 1 \tag{C.12}$$
$$\bar{u} = u^\dagger \gamma^0 \tag{C.13}$$
$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \tag{C.14}$$

D Complex Analysis

Cauchy integral formula

$$F(z) = \frac{1}{2\pi i} \oint_C \frac{F(z')}{z'-z} dz' \tag{D.1}$$

ML-inequality

$$\left| \int_\gamma f(z) dz \right| \leq ML \tag{D.2}$$

Where $M \equiv \max_{z\in\gamma} f(z)$ and L is the length of γ

D.1 Once subtracted dispersion relation

$F(z)$ and $F(z_0)$ can be rewritten using the dispersion relation 4.10. When these two expressions are subtracted:

$$\begin{aligned} F(z) - F(z_0) =& \frac{1}{\pi}\int_M^\infty \frac{\mathrm{Im}F(z')}{z'-z} dz' - \frac{1}{\pi}\int_M^\infty \frac{\mathrm{Im}F(z')}{z'-z_0} dz' \\ &+ \frac{1}{2\pi i}\oint_{C_\infty} \frac{F(z')}{z'-z} dz' - \frac{1}{2\pi i}\oint_{C_\infty} \frac{F(z')}{z'-z_0} dz' \end{aligned} \tag{D.3}$$

$$\begin{aligned} =& \frac{1}{\pi}\int_M^\infty \left[\frac{1}{z'-z} - \frac{1}{z'-z_0}\right] \mathrm{Im}F(z') dz' \\ &+ \frac{1}{2\pi i}\oint_{C_\infty} \left[\frac{1}{z'-z} - \frac{1}{z'-z_0}\right] F(z') dz' \end{aligned} \tag{D.4}$$

$$\begin{aligned} =& \frac{1}{\pi}\int_M^\infty \frac{z'-z_0-z'+z}{(z'-z)(z'-z_0)} \mathrm{Im}F(z') dz' \\ &+ \frac{1}{2\pi i}\oint_{C_\infty} \frac{z'-z_0-z'+z}{(z'-z)(z'-z_0)} F(z') dz' \end{aligned} \tag{D.5}$$

$$\begin{aligned} =& \frac{z-z_0}{\pi}\int_M^\infty \frac{\mathrm{Im}F(z')}{(z'-z)(z'-z_0)} dz' \\ &+ \frac{z-z_0}{2\pi i}\oint_{C_\infty} \frac{F(z')}{(z'-z)(z'-z_0)} dz' \end{aligned} \tag{D.6}$$

The last term of this equation explains why the subtraction of $F(z_0)$ solves the problem. This term has a factor of z'^2 in the denominator, so by using this method, the whole integrand will decay one degree faster at infinity and C_∞ has the possibility to vanish.

D.2 Gamma function

$z \in \mathbb{C}$ and $n \in \mathbb{Z}$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{D.7}$$

$$\Gamma(n) = (n-1)! \tag{D.8}$$

$$\Gamma(z+1) = z\Gamma(z) \tag{D.9}$$

$$\Gamma(z) = \frac{1}{z} + \mathcal{O}(z) \tag{D.10}$$

E Integrals

E.1 Dimensional regularization

Integrals arising in dimensional regularization:

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right) D^{\frac{d}{2}-2} \tag{E.1}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{\left[l^2 - D\right]^2} = \frac{-i}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \Gamma\left(1 - \frac{d}{2}\right) D^{\frac{d}{2}-1} \tag{E.2}$$

For a proof, see [3].

The relation between these integrals can be shown when we devide equation E.1 by equation E.2 and make use of property D.9 of the Gamma function in the second step:

$$\begin{aligned}
\frac{\displaystyle\int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2}}{\displaystyle\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{\left[l^2 - D\right]^2}} &= \frac{\frac{i}{(4\pi)^{\frac{d}{2}}}}{\frac{-i}{(4\pi)^{\frac{d}{2}}}} \frac{1}{\frac{d}{2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(1-\frac{d}{2})} \frac{D^{\frac{d}{2}-2}}{D^{\frac{d}{2}-1}} \\
&= -\frac{2}{d} \frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1-\frac{d}{2})} \frac{1}{D} \\
&= \left(1 - \frac{2}{d}\right) \frac{1}{D} \\
D \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[l^2 - D\right]^2} &= \left(1 - \frac{2}{d}\right) \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{\left[l^2 - D\right]^2}
\end{aligned} \tag{E.3}$$

E.2 Log integrals

The integrals needed in some Feynman parameter integrals are of the following form:

$$\int x^n \ln(x-c) dx$$

with $n = 0, 1, 2$ and $c \in \mathbb{C}$ are derived below:

These integrals can be evaluated using partial integration.

For $n = 0$:

$$\begin{aligned}
\int \ln(x-c)dx &= \int (x)' \ln(x-c)dx \\
&= x\ln(x-c) - \int \frac{x}{x-c}dx \\
&= x\ln(x-c) - \int \left(1 - \frac{c}{x-c}\right) dx \\
&= x\ln(x-c) - x - c\ln(x-c) \\
&= (x-c)\ln(x-c) - x
\end{aligned}$$

For $n = 1$:

$$\begin{aligned}
\int x\ln(x-c)dx &= \int x\left[(x-c)\ln(x-c) - x\right]' dx \\
&= x(x-c)\ln(x-c) - x^2 - \int (x-c)\ln(x-c)dx + \int xdx \\
2\int x\ln(x-c)dx &= x(x-c)\ln(x-c) - \frac{1}{2}x^2 + c\int \ln(x-c)dx \\
\int x\ln(x-c)dx &= \frac{1}{2}(x^2-c^2)\ln(x-c) - \frac{1}{2}cx - \frac{1}{4}x^2
\end{aligned}$$

For $n = 2$:

$$\begin{aligned}
\int x^2\ln(x-c)dx =& \int x\left[\frac{1}{2}(x^2-c^2)\ln(x-c) - \frac{1}{2}cx - \frac{1}{4}x^2\right]' dx \\
=& x\left[\frac{1}{2}(x^2-c^2)\ln(x-c) - \frac{1}{2}cx - \frac{1}{4}x^2\right] \\
&- \int \frac{1}{2}(x^2-c^2)\ln(x-c)dx + \int \frac{1}{2}cxdx + \int \frac{1}{4}x^2dx \\
\frac{3}{2}\int x^2\ln(x-c)dx =& \frac{1}{2}x(x^2-c^2)\ln(x-c) - \frac{1}{4}cx^2 - \frac{1}{6}x^3 + \frac{1}{2}c^2\int \ln(x-c) \\
=& \frac{1}{2}x(x^2-c^2)\ln(x-c) - \frac{1}{4}cx^2 - \frac{1}{6}x^3 \\
&+ \frac{1}{2}c^2(x-c)\ln(x-c) - \frac{1}{2}c^2x \\
=& \frac{1}{2}(x^3-c^3)\ln(x-c) - \frac{1}{6}x^3 - \frac{1}{4}cx^2 - \frac{1}{2}c^2x \\
\int x^2\ln(x-c)dx =& \frac{1}{3}(x^3-c^3)\ln(x-c) - \frac{1}{9}x^3 - \frac{c}{6}x^2 - \frac{c^2}{3}x
\end{aligned}$$

Feynman parameter integral

Using the results of the derivation of $n = 1$ and $n = 2$, it is possible to solve the Feynman parameter integral in section 3.4:

$$-\int_0^1 x(1-x)\ \ln\left[x - \frac{1}{2}(1 \pm b)\right] dx$$

First part:

$$
\begin{aligned}
&-\int_0^1 x\ \ln\left[x-\frac{1}{2}(1\pm b)\right]dx\\
&=\left[-\frac{1}{2}[x^2-\frac{1}{4}(1\pm b)^2]\ln[x-\frac{1}{2}(1\pm b)]+\frac{1}{4}(1\pm b)x+\frac{1}{4}x^2\right]_0^1\\
&=-\frac{1}{2}[1-\frac{1}{4}(1\pm b)^2]\ln[1-\frac{1}{2}(1\pm b)]\\
&\quad+\frac{1}{4}(1\pm b)+\frac{1}{4}+\frac{1}{2}[-\frac{1}{4}(1\pm b)^2]\ln[-\frac{1}{2}(1\pm b)]\\
&=-\frac{1}{2}\ln[1-\frac{1}{2}(1\pm b)]+\frac{1}{8}(1\pm b)^2\ln\left[\frac{1-b}{-1-b}\right]+\frac{1}{4}(1\pm b)+\frac{1}{4}
\end{aligned}
$$

Second part:

$$
\begin{aligned}
&\int_0^1 x^2\ \ln\left[x-\frac{1}{2}(1\pm b)\right]dx\\
&=\left[\frac{1}{3}[x^3-\frac{1}{8}(1\pm b)^3]\ln[x-\frac{1}{2}(1\pm b)]-\frac{1}{9}x^3-\frac{1}{12}(1\pm b)x^2-\frac{1}{12}(1\pm b)^2x\right]_0^1\\
&=\frac{1}{3}[1-\frac{1}{8}(1\pm b)^3]\ln[1-\frac{1}{2}(1\pm b)]-\frac{1}{9}-\frac{1}{12}(1\pm b)-\frac{1}{12}(1\pm b)^2\\
&\quad-\frac{1}{3}[-\frac{1}{8}(1\pm b)^3]\ln[-\frac{1}{2}(1\pm b)]\\
&=\frac{1}{3}\ln[1-\frac{1}{2}(1\pm b)]-\frac{1}{24}(1\pm b)^3\ln\left[\frac{1-b}{-1-b}\right]-\frac{1}{9}-\frac{1}{12}(1\pm b)-\frac{1}{12}(1\pm b)^2
\end{aligned}
$$

Complete integral:

$$
\begin{aligned}
&-\int_0^1 x(1-x)\ \ln\left[x-\frac{1}{2}(1\pm b)\right]dx=\\
&-\frac{1}{6}\ln\left[\frac{1}{2}\pm\frac{1}{2}b\right]+\left[\frac{1}{12}\pm\frac{b}{8}\mp\frac{b^3}{24}\right]\ln\left[\frac{1\mp b}{-1\mp b}\right]+\frac{5}{36}+\frac{1}{6}(1\pm b)-\frac{1}{12}(1\pm b)^2
\end{aligned}
\tag{E.4}
$$