Tashkinov trees and the Goldberg–Seymour conjecture

Lotte de Jonker

April 21, 2014

Master Thesis Mathematics

Supervisors: Dr. R.J. (Ross) Kang, Dr. J.H. (Jan) Brandts

KdV Institute for Mathematics
Faculty of Science
University of Amsterdam
Abstract

In the field of edge-colorings of graphs, a famous conjecture for the chromatic index is the Goldberg–Seymour conjecture. In many results towards this conjecture, Tashkinov trees and Tashkinov’s Theorem are used. Augmenting paths are used to extend edge-colorings and Tashkinov trees can be used in a similar way with the aid of Tashkinov’s Theorem. We give the proof of Tashkinov’s Theorem and show how the theorem is used in results towards the Goldberg–Seymour conjecture.

Tashkinov trees are transformed into vertex-Tashkinov trees on line graphs by using a characterisation theorem for line graphs. Tashkinov’s Theorem is also transferred to the line graph setting, resulting in Tashkinov’s Theorem for line graphs. Vertex-Kierstead paths are a substructure of vertex-Tashkinov trees and Kierstead’s Theorem for line graphs is a subcase of Tashkinov’s Theorem for line graphs. We shall see that one potential extension of Kierstead’s Theorem for line graphs to a wider class of graphs than line graphs is not possible due to counterexamples obtained with the aid of Mycielskians. A structure theorem for quasi-line graphs, which is used to extend results for line graphs to results for quasi-line graphs, is investigated. Also the Goldberg–Seymour conjecture is transformed into an equivalent conjecture for line graphs. A generalization for claw-free graphs is stated and discussed.

Master Thesis

Title: Tashkinov trees and the Goldberg–Seymour conjecture
Author: Lotte de Jonker, lottedejonker@gmail.com, 5773369
Daily supervisor: Dr. R.J. (Ross) Kang
First examiner: Dr. J.H. (Jan) Brandts
Second examiner: Prof. Dr. A. (Lex) Schrijver
Date: April 21, 2014

Korteweg–de Vries Institute for Mathematics
University of Amsterdam
Science Park 904, 1098 XH Amsterdam
The Netherlands
Acknowledgments

First of all I would like to thank my daily supervisor Ross Kang. Without his guidance I would not be where I am now, writing my acknowledgements after an exciting project. Ross was always there to help me and immediately answered my emails. I am thankful for his time and effort during the past months. On some crucial moments he just pointed me in the right direction by suggesting the perfect article. He also arranged that I could read a preprint of a chapter about edge-colorings by Jessica McDonald. I would also like to thank her for allowing me to read this chapter. It was helpful during the start of my project and I really enjoyed reading it. Also my second supervisor Jan Brandts deserves a word of thanks. His feedback on the structure of my thesis as a whole as well on small details definitely helped improving my thesis. I would also like to thank Lex Schrijver in advance for being second assessor.

This project was interesting and exciting. During the few more difficult moments, my family and friends gave me some mental support. Hereby I would like to thank them. Writing a thesis also requires discipline and that is where I was very happy to have Karin Monster by my side. Karin and I wrote our theses during the same period and alternately worked at her and my place. This structure definitely had a positive influence on the progression of my thesis.
## Contents

**Introduction**  2

1  **Introduction to Tashkinov trees**  4
   1.1  Basic graph theory and Tashkinov trees  4
   1.1.1  Vertex-colorings  6
   1.1.2  Edge-colorings  8
   1.1.3  Kierstead paths and Tashkinov trees  12
   1.2  Tashkinov trees in results towards the Goldberg–Seymour conjecture  15

2  **Tashkinov’s Theorem**  19
   2.1  Preliminary work  19
   2.2  The proof of Tashkinov’s Theorem  21

3  **Vertex-Tashkinov trees**  29
   3.1  Kierstead’s Theorem for line graphs  32
   3.2  The structure of quasi-line graphs  39

**Discussion**  46

**Conclusion and suggestions**  49

**Bibliography**  50
Introduction

Graph coloring is a popular subject in graph theory. There are different ways to color a graph. For instance, we can color the vertices of a graph such that adjacent vertices are not colored with the same color. Naturally, the following question arises. What is the least number of colors needed to color the vertices of a graph this way? If we focus on planar graphs it has been proven that only four colors are needed. This is the famous Four Color Theorem which was first conjectured in 1852. After a number of attempted proofs and false counterexamples, Appel and Haken [1, 2] gave the first correct proof in 1976, with the aid of computer case analysis.

Another way of coloring a graph is to color its edges. This can be used for scheduling problems. A simple example is given in Figure 2. Again, one may wonder what is the least number of colors needed to color the edges of a graph such that adjacent edges receive different colors. An open problem called the Goldberg–Seymour conjecture states that this number is equal to either the maximum degree, or the maximum degree plus one or the density. The Goldberg–Seymour conjecture was stated in the seventies and in 1996
Kahn [16] showed that it is true in an asymptotic sense. Several other partial results in support of the Goldberg–Seymour conjecture have been obtained. Many of them use a class of combinatorial objects called Tashkinov trees.

![Diagram of Tashkinov trees](image)

**Figure 2:** Ross has to teach classes 1 and 2, Jan has to teach classes 2, 3 and 4, and Lex has to teach classes 1, 3 and 4. A graph that represents this situation is given on the left. In the middle the edges of this graph are colored. Let blue be the classes given at nine, green the classes given at ten and red the classes given at eleven. Then on the right a correct schedule is given.

**Outline of the thesis**

In Chapter 1 we will give a short introduction into graph theory, focusing in particular on colorings. Definitions are given and short proofs of related results are shown. We will introduce a construction by Mycielski, which will be of use in the third chapter when we construct some counterexamples. After introducing Kierstead paths, which are a subclass of Tashkinov trees, Tashkinov trees are defined. Tashkinov’s Theorem is stated and we give an outline of the proof of Kierstead’s Theorem, which is equal to Tashkinov’s Theorem but then for Kierstead paths instead of Tashkinov trees. In Section 1.2 the Goldberg–Seymour conjecture and some results towards this conjecture are discussed. In the proofs of all these results, Tashkinov’s Theorem is used. We finish Chapter 1 by giving the structure of a few of these proofs. The complete proof of Tashkinov’s Theorem will be given in Chapter 2.

In Chapter 3 we will transform Tashkinov trees into vertex-Tashkinov trees on line graphs. This transformation into the line graph setting is also made for the subcase of Kierstead paths and Kierstead’s Theorem. The possibility of extending Kierstead’s Theorem for line graphs into Kierstead’s Theorem for graphs that are not necessarily line graphs is investigated. For a straightforward approach, counterexamples will be given. In Section 3.2 a structure theorem for quasi-line graphs, which is used in extending results for line graphs to results for quasi-line graphs, will be discussed. In the discussion we will return to the Goldberg–Seymour conjecture and transform it into the line graph setting. A stronger conjecture will be stated and discussed.
Chapter 1

Introduction to Tashkinov trees

In this chapter some basic graph theoretical definitions that will be used throughout this thesis are given. The concept of Kierstead paths and Tashkinov trees will be introduced. After that, we will have a look at the Goldberg–Seymour conjecture and how Tashkinov trees are used in results towards this conjecture.

1.1 Basic graph theory and Tashkinov trees

A graph $G = (V(G), E(G))$ is a set of vertices $V(G)$ combined with a set of edges $E(G) \subseteq \{\{u, v\} \mid u, v \in V(G) \text{ and } u \neq v\}$. A vertex $u \in V(G)$ and an edge $e \in E(G)$ are incident if $u \in e$, we also say $u$ is an endpoint of $e$ in this case. Two different vertices $u, v \in V(G)$ are adjacent if $\{u, v\} \in E(G)$ and two different edges $e, f \in E(G)$ are adjacent if $e \cap f \neq \emptyset$. Later, when we discuss edge-colorings we need the definition of a more general class of graphs, that is the class of multigraphs. A multigraph $G$ is a graph where multiple edges are allowed; thus two different edges may have the same set of endpoints and $E(G)$ is a multiset in this case. The multiplicity $\mu(G)$ of a multigraph $G$ is the maximum number of edges between two vertices. The following definitions are defined for graphs but are naturally suited for multigraphs also.

The neighbourhood of a vertex $v \in V(G)$ is given by $N(v) = \{u \in V(G) \mid u \text{ and } v \text{ are adjacent}\}$. A subgraph $H$ of $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any subset $U \subseteq V(G)$, the induced subgraph $G[U]$ of $G$ is the graph with vertex set $U$ and edge set $\{e \in E(G) \mid e = \{u, v\} \text{ and } u, v \in U\}$. The degree of a vertex $v \in V(G)$ in a graph $G$ is denoted and defined by
\[ d_G(v) := |\{e \in E(G) \mid v \text{ is an endpoint of } e\}| \]

and the maximum degree of \( G \) is denoted by

\[ \Delta(G) := \max\{d_G(v) \mid v \in V(G)\}. \]

Let \( \{v_0, v_1, ..., v_n\} \in V(G) \) be a set of distinct vertices, then \( P = (v_0, v_1, ..., v_n) \) is a path on \( G \) of length \( n \) if \( \{v_i, v_{i+1}\} \in E(G) \) for all \( 0 \leq i < n \). A graph \( G \) is connected if there exists a path between every pair of distinct vertices. A connected subgraph of \( G \) that is maximal is a component of \( G \). For \( n \geq 2 \) \( C = (v_0, v_1, ..., v_n) \) is a cycle of length \( n \) if \( v_0, ..., v_{n-1} \) are distinct vertices, \( v_0 = v_n \) and \( \{v_i, v_{i+1}\} \in E(G) \) for all \( 0 \leq i < n \). A graph \( G \) is a tree if \( G \) is connected and does not contain a cycle. Given an integer \( n \geq 0 \), the complete graph on \( n \) vertices is a graph with all vertices pairwise adjacent and it is denoted by \( K_n \).

A set \( C \subseteq V(G) \) is called a clique if \( G[C] \) is a complete graph. Sometimes we will say that a subgraph \( H \) of \( G \) is a clique, this means that \( V(H) \) is a clique. The clique number \( \omega(G) \) of a graph \( G \) is the maximum cardinality of a subset of \( V(G) \) that is a clique. A subset \( U \subseteq V(G) \) is called an independent set if \( E(G[U]) = \emptyset \). The maximum size of a subset of \( V(G) \) that is an independent set is the independence number and denoted by \( \alpha(G) \). A matching of a graph \( G \) is a subset \( M \) of \( E(G) \), with no pair of adjacent edges. The maximum size

\[
\begin{align*}
\text{Figure 1.1:} & \quad P \text{ is a path of length 5 with a 2-edge-coloring.} \\
& \quad C \text{ is a cycle of length 5 with a 3-vertex-coloring.} \\
& \quad K_5 \text{ is a complete graph on 5 vertices with a partial 3-vertex-coloring.} \\
& \quad T \text{ is a tree and } G \text{ is a multigraph.}
\end{align*}
\]
of a subset of $E(G)$ that is a matching in $G$ is called the \textit{matching number} and denoted by $\tau(G)$. To summarize,

\begin{align*}
\omega(G) &= \max\{|C| \mid C \subseteq V(G) \text{ and } G[C] \text{ is a clique}\}, \\
\alpha(G) &= \max\{|U| \mid U \subseteq V(G) \text{ is an independent set}\} \quad \text{and} \\
\tau(G) &= \max\{|M| \mid C \subseteq E(G) \text{ is a matching}\}.
\end{align*}

Figure 1.2: A graph $G$ with $\omega(G) = 4$, $\alpha(G) = 3$ and $\tau(G) = 3$ is shown three times. The first with a blue clique, the second with a green independent set and the third with a red matching.

1.1.1 \textbf{Vertex-colorings}

Given a positive integer $k$, a \textit{k-vertex-coloring} on a graph $G$ is a function

$$\varphi : V(G) \to \{1, 2, \ldots, k\}$$

such that for any two different vertices $u, v \in V(G)$ that are adjacent $\varphi(u) \neq \varphi(v)$. Note that if we color the vertices of a multigraph, the multiple edges between two vertices may be ignored and seen as a single edge, resulting in a simple graph (i.e. a graph $G$ with $\mu(G) \leq 1$). This simple graph has the same vertex-colorings as the original multigraph and that is why we only consider vertex-colorings on simple graphs. A k-vertex-coloring $\varphi$ of a subgraph $H$ of $G$ with $V(H) < V(G)$ is called a \textit{partial k-vertex-coloring} of $G$ and has \textit{domain} $\text{dom}(\varphi) = V(H)$. For each color $i \in [k] := \{1, 2, \ldots, k\}$, we call the set $\{v \in V(G) \mid \varphi(v) = i\}$ a \textit{color class}. The \textit{chromatic number} $\chi(G)$ of a graph $G$ is the smallest $k$ such that there exists a k-vertex-coloring on $G$.

So, we are coloring the vertices of a graph $G$ and $\chi(G)$ is the least number of colors that we need to color every vertex such that adjacent vertices receive different colors. A trivial lower bound for the chromatic number of a graph is its clique number, because the vertices in a clique are pairwise adjacent they must receive different colors. Note that the color classes are independent.
sets so their sizes are bounded by $\alpha(G)$. This results in another trivial lower bound that is $\frac{|V(G)|}{\alpha(G)}$. For a trivial upper bound, suppose we have $\Delta(G) + 1$ colors. Then iteratively assigning the lowest available color to the vertices is always possible since we have more than $\Delta(G)$ colors. Doing this results in a $(\Delta(G) + 1)$-vertex-coloring of $G$. Combining these facts results in the following lemma.

**Lemma 1.1.** Let $G$ be a graph. Then

$$\max\left\{ \omega(G), \frac{|V(G)|}{\alpha(G)} \right\} \leq \chi(G) \leq \Delta(G) + 1.$$  \hfill (1.1)

We will now give a specific construction by Mycielski [26] which provides a graph with any desired chromatic number, while also guaranteeing certain graph properties.

Given a graph $G$ with $V(G) = \{v_1, ..., v_n\}$, its Mycielskian $M(G)$ is the graph which is constructed as follows. We copy all the vertices of $G$ and add edges between every original vertex and the copies of its neighbours. Then an extra vertex is added and this vertex is connected to all vertices in the copy of $V(G)$. This leads to the following vertex set and edge set:

- $V(M(G)) = V(G) \cup \{v'_1, ..., v'_n\} \cup \{v\}$ and
- $E(M(G)) = E(G) \cup \{\{v_i, v'_j\} \mid \{v_i, v_j\} \in E(G)\} \cup \{\{v, v'_i\} \mid 1 \leq i \leq n\}$.

An example is given in Figure 1.3. Note that the set of copies $\{v'_1, ..., v'_n\}$ is an independent set. We will first show that taking the Mycielskian increases the chromatic number by precisely one.

**Lemma 1.2.** Let $G$ be a graph and $M(G)$ its Mycielskian. Then

$$\chi(M(G)) = \chi(G) + 1.$$  

\hfill 7
**Proof:** Let $G$ be a graph and $M(G)$ its Mycielskian. We will use the same notation for the vertex and edge set of $M(G)$ as before. In order to prove $\chi(M(G)) \leq \chi(G) + 1$, we extend a $k$-vertex-coloring on $G$ to a $(k + 1)$-vertex-coloring on $M(G)$. Let $\varphi$ be a $k$-vertex-coloring on $G$, we define the $(k + 1)$-vertex-coloring $\varphi'$ on $M(G)$ as follows:

- For all vertices $v_i \in V(G)$, $\varphi'(v_i) = \varphi(v_i)$.
- For all copies $v'_i \in V(M(G))$, $\varphi'(v'_i) = \varphi(v_i)$.
- The extra vertex $v$ receives a new color: $\varphi'(v) = k + 1$.

Clearly, no adjacent vertices received the same color. Thus we conclude $\chi(M(G)) \leq \chi(G) + 1$. Next, we assume $\chi(M(G)) < \chi(G) + 1$.

Let $\varphi$ be a $k$-vertex-coloring on $M(G)$, $k = \chi(G)$. Let $C$ be the set of all vertices $v_i \in V(G)$ such that $\varphi(v_i) = \varphi(v)$. We know for the copy $v'_i$ of each $v_i \in C$ that $\varphi(v'_i) \notin \varphi(N(v_i))$ and $\varphi(v'_i) \neq \varphi(v)$. Thus we can change the coloring $\varphi$ into $\varphi'$ by coloring each $v_i \in C$ with $\varphi(v'_i)$. Now $\varphi'$ does not use $\varphi(v)$ on $V(G)$ and therefore $\chi(G) < k$, this is a contradiction. ■

Mycielskians have another useful property, that is, the clique number is preserved conditionally upon $E(G) \neq \emptyset$.

**Lemma 1.3.** Let $G$ be a graph and $M(G)$ its Mycielskian. If $E(G) \neq \emptyset$, then $\omega(M(G)) = \omega(G)$.

**Proof:** Let $G$ be a graph and $M(G)$ its Mycielskian. We will use the same notation for the vertex and edge set of $M(G)$ as before. Because $G$ is a subgraph of $M(G)$, $\omega(M(G)) \geq \omega(G)$.

Suppose $\omega(M(G)) > \omega(G)$ and let $C$ be a maximum clique on $M(G)$. The vertex $v$ is not in $C$, since its neighbourhood $N(v)$ is an independent set and $\omega(M(G)) > \omega(G) \geq 2$. Suppose $v'_i \in C$.

Then it is the only copy that is contained in $C$, since the copies form an independent set. All vertices in the clique $C \setminus \{v'_i\}$ are adjacent to $\{v'_i\}$ and therefore also adjacent to $v_i$. But then $(C \setminus \{v'_i\}) \cup \{v_i\}$ is a clique on $G$ with size $\omega(M(G))$, which is a contradiction. ■

### 1.1.2 Edge-colorings

Up to this point in the chapter we have only discussed vertex-colorings. Now we discuss the concept of edge-colorings which is central to this thesis.
We will see that in this case we cannot ignore multiple edges as we did for vertex-colorings, thus here we consider multigraphs. A \textit{k-edge-coloring} on a multigraph \( G \) is a function
\[
\phi : E(G) \to \{1, 2, \ldots, k\}
\]
such that any two different edges that are adjacent receive a different value. Let \( \phi \) be a \( k \)-edge-coloring. A \( k \)-edge-coloring \( \phi \) of a subgraph \( H \) of \( G \) with \( E(H) < E(G) \) is called a \textit{partial} \( k \)-edge-coloring of \( G \) and has domain \( \text{dom}(\phi) = E(H) \). For each color \( i \in [k] \), we call the set \( \{ e \in E(G) \mid \phi(e) = i \} \) a \textit{color class}. Note that a color class is a matching and thus \( \tau(G) \) is an upper bound for the size of a color class. Analogously to the chromatic number, the \textit{chromatic index} \( \chi'(G) \) of a multigraph \( G \) is the smallest integer \( k \) such that there exists a \( k \)-edge-coloring on \( G \). For a vertex \( v \) the set of colors used on edges incident to \( v \) is denoted by \( \phi(v) \) and \( \overline{\phi}(v) := [k] \setminus \phi(v) \). We will use lower case Greek letters such as \( \gamma \) and \( \delta \) instead of integers to indicate the colors of an edge-coloring. Also we will use phrases such as, ”the color \( \gamma \) is used on \( e \)” or ”\( e \) is colored with \( \gamma \)” if \( \phi(e) = \gamma \).

A trivial lower bound for the chromatic index of a graph \( G \) is its maximum degree \( \Delta(G) \) since all edges incident to the same vertex must receive different colors. For other bounds, it is useful to view edge-colorings as vertex-colorings on an auxiliary graph. Therefore we introduce line graphs.

\textbf{Definition 1.1.} Let \( G \) be a multigraph. The line graph \( L(G) \) of \( G \) is the simple graph with
\begin{itemize}
  \item vertex set \( V(L(G)) = E(G) \) and
  \item two vertices of \( L(G) \) are adjacent if and only if their corresponding edges in \( G \) are adjacent.
\end{itemize}

An example of a graph and its line graph is given in Figure 1.4. Note that each edge-coloring of a multigraph \( G \) is a vertex-coloring of \( L(G) \) and vice versa. Hence,
\[
\chi'(G) = \chi(L(G)).
\]

This is a useful fact, for now we can translate every bound for the chromatic number to a bound for the chromatic index. Note that for a multigraph \( G \), \( \Delta(L(G)) \leq 2\Delta(G) - 2 \). Then inequality (1.1) leads to the following inequality:
\[
\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1. \tag{1.2}
\]
Hence,
\[ \Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1. \tag{1.3} \]

Next we give two improvements on the upper bound of the chromatic index. The first is due to Shannon [33].

**Theorem 1.1** (Shannon’s Theorem). Let \( G \) be a multigraph with maximum degree \( \Delta(G) \) and \( \chi'(G) \) its chromatic index. Then
\[ \chi'(G) \leq \left\lfloor \frac{3}{2} \Delta(G) \right\rfloor. \]

The second is known as Vizing’s Theorem [36] and was independently discovered by Gupta [15].

**Theorem 1.2** (Vizing’s Theorem). Let \( G \) be a multigraph with multiplicity \( \mu(G) \), \( \Delta(G) \) its maximum degree and \( \chi'(G) \) its chromatic index. Then
\[ \chi'(G) \leq \Delta(G) + \mu(G). \]

Vizing’s Theorem shows that for simple graphs, the gap of inequality (1.3) is down to one. Both results are tight and this is shown by a so-called Shannon graph in Figure 1.5.

There are even simple graphs \( G \) for arbitrary \( \Delta(G) \) with \( \chi'(G) > \Delta(G) \), but let us see next how the trivial lower bound \( \Delta(G) \) is attained for multigraphs \( G \) under the condition of bipartiteness. This result is due to König [22] and the proof uses the concept of augmenting paths.
A graph is bipartite if its vertex set can be partitioned into two independent sets (for an example see Figure 1.6). If $\varphi$ is a $k$-edge-coloring of $G$, then a path $P$ is a $(\gamma, \delta)$-alternating path if $\varphi(e) \in \{\gamma, \delta\}$ for every edge $e$ of $P$. $P$ is a maximal $(\gamma, \delta)$-alternating path if it is a $(\gamma, \delta)$-alternating path that is not a strict subgraph of a $(\gamma, \delta)$-alternating path. If $P$ is a maximal $(\gamma, \delta)$-alternating path and $u, v \in P$, then we define $\varphi \setminus P$ to be the coloring that is exactly the same as $\varphi$ except on $P$ where the colors are switched. We say that the coloring $\varphi \setminus P$ is obtained from $\varphi$ by augmenting the path $P$. We will now state König’s Theorem and give its proof.

**Theorem 1.3** (König’s Theorem). Let $G$ be a bipartite multigraph, $\Delta(G)$ its maximum degree and $\chi'(G)$ its chromatic index. Then

$$\chi'(G) = \Delta(G).$$

**Proof:** Let $\varphi$ be a partial $\Delta(G)$-edge-coloring of $G$ and $e$ an uncolored edge, thus $e \not\in dom(\varphi)$. Let $u$ and $v$ be the endpoints of $e$. Since $e$ is not colored, we know $\varphi(u), \varphi(v) \neq \emptyset$. We will show that $\varphi$ can be extended such that $e$ is also colored.

![Figure 1.6: A bipartite graph on which a path is augmented such that $e$ can be colored with blue, the bold lines illustrate this.](image)

Suppose $\varphi(u) \cap \varphi(v) \neq \emptyset$. Let $\alpha \in \varphi(u) \cap \varphi(v)$ and extend $\varphi$ by coloring $e$ with $\alpha$. This results in a coloring $\varphi'$ with $dom(\varphi') = dom(\varphi) \cup \{e\}$.

Suppose $\varphi(u) \cap \varphi(v) = \emptyset$. Let $\alpha$ and $\beta$ be colors such that $\alpha \in \varphi(u)$ and $\beta \in \varphi(v)$. We consider the maximal $(\alpha, \beta)$-alternating path $P$ starting in $u$. 


Suppose $P$ ends in $v$. Then $|E(P)|$ is even and $P \cup \{e\}$ forms an odd cycle (i.e. a cycle of odd length). This contradicts the fact that $G$ is bipartite. Hence we know that $P$ does not contain $v$. We augment $P$ and obtain the coloring $\varphi' = \varphi \setminus P$. Now $\beta \in \overline{\varphi}(u) \cap \overline{\varphi}(v)$ and we color $e$ with $\beta$. Again, we have obtained a coloring with an extended domain. Figure 1.6 illustrates this.

Since we can change and extend every partial $\Delta(G)$-edge-coloring of $G$, it is $\Delta(G)$-edge colorable.

\subsubsection{Kierstead paths and Tashkinov trees}

We saw in the proof of König’s Theorem that finding a maximal alternating path that has an uncolored edge incident to one of the endpoints of this path tells us that there exists a coloring with a greater domain. More complex structures that we can also use in this way are Kierstead paths and Tashkinov trees. A Kierstead path with respect to an edge $e$ and edge-coloring $\varphi$ is a path $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ such that $e_1 = e$ is uncolored and for all $1 < j \leq n$, there is an $i < j$ such that $\varphi(e_j) \in \overline{\varphi}(v_i)$. Recall that $\overline{\varphi}(v)$ denotes the set of colors that are not used on an edge incident to the vertex $v$. Note that the notation for a Kierstead path differs from our notation of a path. It also contains the edges for convenience. An example of a Kierstead path is given in Figure 1.7.

![Figure 1.7: A graph and partial 4-edge-coloring. The bold path is a Kierstead path and the colored blocks next to $v_0$ and $v_1$ denote their missing colors.](image_url)

Given a graph $G$ with partial $k$-edge-coloring $\varphi$, a subset $U \subseteq V(G)$ is elementary if $\overline{\varphi}(u) \cap \overline{\varphi}(u') = \emptyset$ for any two different $u, u' \in U$. We say that a Kierstead path is elementary if its vertex set is elementary. Kierstead’s Theorem, which is stated below, shows us that Kierstead paths can also be used to extend an edge-coloring.
Theorem 1.4 (Kierstead’s Theorem). Let $G$ be a multigraph and $\varphi$ a partial $(\Delta(G) + s)$-edge-coloring of $G$, with $s \geq 1$ and $e \in E(G)$ uncolored. If there exists a non-elementary Kierstead path with respect to $e$ and $\varphi$, then there exists a $(\Delta(G) + s)$-edge-coloring of $\text{dom}(\varphi) \cup e$.

For the formal proof of Kierstead’s Theorem we refer the reader to [17]. Here we only give an outline of the proof, since we will see the same kind of arguments in detail in the proof of Tashkinov’s Theorem given in the next chapter. The proof of Kierstead’s Theorem uses a double induction.

Take a non-elementary Kierstead path $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ and two indices $i, j$ such that $i < j$. The first induction is on $j$. If $j = 1$, then $e_0$ can be colored with $\alpha$. In case $j > 1$, a second induction on $j - i$ is used. If $j - i = 1$, then $\beta = \varphi(e_j)$ is missing at a vertex $v_k$ such that $k < i$. Change the color of $e_j$ into $\alpha$. Then $(v_0, e_1, v_1, \ldots, e_i, v_i)$ is a non-elementary Kierstead path and the first induction hypothesis is used. If $j - i > 1$, $\alpha \in \varphi(v_{i+1})$ and $\gamma \in \overline{\varphi}(v_{i+1})$, then let $C$ be the maximal $(\alpha, \gamma)$-alternating path starting in $v_{i+1}$. It is shown that $C$ does not end at a vertex $v_i$ and does not contain an edge $e \in \{e_0, \ldots, e_i\}$. Then two cases are distinguished, one where $C$ ends at $v_i$ and one where $C$ does not end at $v_i$. In the first case the first induction hypothesis is used and in the second case the second induction hypothesis is used to complete the proof.

A Kierstead path is a special type of a Tashkinov tree, which we now informally describe before giving the formal definition.

Given a graph $G$ with partial $k$-edge-coloring $\varphi$. Start with an uncolored edge. Then add edges such that each time you add an edge,

- the new structure is a tree and
- the added edge is colored with a color that is missing at a vertex of the current tree.

In Figure 1.8 an example of a Tashkinov tree is given. Here is the formal definition of a Tashkinov tree:

**Definition 1.2.** Let $G$ be a multigraph and $\varphi$ a partial $k$-edge-coloring such that $e \in E(G)$ is not colored. A Tashkinov tree in $G$ with respect to $e$ and $\varphi$ is a tree $T = (v_0, e_1, v_1, \ldots, e_n, v_n)$ such that

1. $e_1 = e$ has endpoints $v_0$ and $v_1$,
2. for all $1 < j \leq n$, $e_j$ has endpoints $v_j$ and $v_i$ for some $i < j$, and
3. for all $1 < j \leq n$, there is an $i < j$ such that $\varphi(e_j) \in \overline{\varphi}(v_i)$. 

13
A Tashkinov tree is maximal if it is not a strict subgraph of a Tashkinov tree. The next theorem states that if a partially colored graph contains a non-elementary Tashkinov tree, the coloring can be extended.

**Theorem 1.5** (Tashkinov’s Theorem). *Let $G$ be a multigraph and $\varphi$ a partial $(\Delta(G) + s)$-edge-coloring of $G$, with $s \geq 1$ and $e \in E(G)$ uncolored. If there exists a non-elementary Tashkinov tree with respect to $e$ and $\varphi$, then there exists a $(\Delta(G) + s)$-edge-coloring of $\text{dom}(\varphi) \cup e$.*

The original proof of Tashkinov’s Theorem by Tashkinov [35] is in Russian. A preprint of Favrlholdt, Stiebitz, and Toft [10] contains an almost complete proof in English. A gap in the proof was noticed by McDonald [25] and she gave the missing part. In the next chapter these two are combined into a complete exposition of the proof of Tashkinov’s Theorem.

The earlier mentioned result by Vizing, that $\chi'(G) \leq \Delta(G) + \mu(G)$ for any multigraph $G$, can be proved using Tashkinov’s Theorem. This proof is much shorter than Vizing’s original proof if we ignore the length of the proof of Tashkinov’s Theorem itself.

**Proof of Vizing’s Theorem:** Let $G$ be a multigraph with a maximal partial $k$-edge-coloring $\varphi$ and $e \in E(G)$ is uncolored. Let $T$ be a maximal Tashkinov tree with respect to $e$ and $t = |V(T)|$. By Tashkinov’s Theorem, we know that $T$ is elementary. Let $v$ be a vertex of $T$ that is not incident to $e$. For each color missing at $V(T) \setminus \{v\}$ there must be an edge with that color incident to $v$ and because of the maximality of $T$, that edge must have an endpoint in $V(T) \setminus \{v\}$. There are at most $\mu(G)(t - 1)$ different edges between $V(T) \setminus \{v\}$ and $v$. This results in the following inequality:

$$\mu(G)(t - 1) \geq \left| \bigcup_{u \in V(T) \setminus \{v\}} \varphi(u) \right| \geq 2 + (t - 1)(k - \Delta(G)).$$

Figure 1.8: A graph and partial 4-edge-coloring. The bold tree is a Tashkinov tree and the colored blocks next to $v_0, v_1$ and $v_2$ denote their missing colors.
And thus

$$(t - 1)(\Delta(G) + \mu(G) - k) \geq 2. \quad (1.4)$$

Filling in $k = \Delta(G) + \mu(G)$ in inequality 1.4 results in a contradiction. ■

The last proof uses Tashkinov trees in a nice way. In the next section the use of Tashkinov trees in the Goldberg–Seymour conjecture will be discussed.

### 1.2 Tashkinov trees in results towards the Goldberg–Seymour conjecture

In Section 1.1 we already mentioned $\Delta(G)$ as a trivial lower bound for $\chi'(G)$. Before stating the Goldberg–Seymour conjecture, we will introduce another lower bound that is used in the conjecture.

As noted before, the matching number $\tau(G)$ is an upper bound for the cardinality of each color class of an edge-coloring. This means that there are at least $\frac{|E(G)|}{\tau(G)}$ colors needed to color the edges of $G$. Note that because the edges of a matching are incident to at most $|V(G)| - 1$ vertices if $|V(G)|$ is odd we know

$$\tau(G) \leq \frac{1}{2}(|V(G)| - 1)$$

and thus

$$\chi'(G) \geq \frac{|E(G)|}{\tau(G)} \geq \frac{|E(G)|}{\frac{1}{2}(|V(G)| - 1)}. \quad (1.5)$$

Clearly inequality (1.5) needs to hold for any induced subgraph on an odd number of vertices. By taking the maximum over all these induced subgraphs we obtain the following lower bound for the chromatic index called the density of $G$;

$$\rho(G) = \max \left\{ \frac{2|E(G[S])|}{|S| - 1} \mid S \subseteq V(G), |S| \geq 3 \text{ odd} \right\}.$$

Since the chromatic index is an integer, the density rounded up is also a lower bound;

$$\chi'(G) \geq \lceil \rho(G) \rceil.$$

The Goldberg–Seymour conjecture, posed independently by Goldberg [11] and Seymour [32] in the 1970s, states that if $\chi'(G) > \Delta(G) + 1$ then $\chi'(G) = \lceil \rho(G) \rceil$. 
Conjecture 1.1 (Goldberg–Seymour conjecture). For any multigraph $G$,

$$\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta(G) + 1\}.$$ 

The Goldberg–Seymour conjecture would imply that for any graph $G$ the chromatic index is equal to $\Delta(G)$, $\Delta(G) + 1$ or $\lceil \rho(G) \rceil$. Although it is still not known whether the conjecture holds, the conjecture is well studied and there are results towards the conjecture. One important result is due to Kahn [16] who proved an asymptotic version of the Goldberg–Seymour conjecture. He proved the following theorem that tells us that the chromatic index and the fractional chromatic index $\chi'_f(G)$, which will be defined after stating the theorem, asymptotically agree as $\chi'(G)$ becomes arbitrarily large.

**Theorem 1.6.** For any $\epsilon \geq 0$ there exists $\Delta_\epsilon$ such that every graph $G$ with $\chi'(G) > \Delta_\epsilon$ satisfies $\chi'(G) < (1 + \epsilon)\chi'_f(G)$.

A *fractional edge-coloring* is a function $c$ from the set of matchings $\mathcal{M}(G)$ to the interval $[0, 1]$ such that $\sum_{M \in \mathcal{M}(G), e \in M} c(M) \geq 1$ for each $e \in E(G)$. The *fractional chromatic index* is given by

$$\chi'_f(G) = \min \left\{ \sum_{M \in \mathcal{M}(G)} c(M) \mid c \text{ is a fractional edge-coloring} \right\}.$$ 

It is clear that $\chi'_f(G) \leq \chi'(G)$, since any edge-coloring is a fractional edge-coloring. The following equality follows from Edmonds’ polytope Theorem [8], for the explicit implication we refer to Seymour [31].

$$\chi'_f(G) = \max\{\Delta(G), \rho(G)\} \quad (1.6)$$

Equality (1.6) and Theorem 1.6 imply that the Goldberg–Seymour conjecture is true asymptotically with respect to $\chi'(G)$.

More results towards the Goldberg–Seymour have been obtained, for instance inequality (1.7) is proven for $m = 9$ by Goldberg [12][13], $m = 11$ by Nishizeki and Kashiwagi [27] and independently Tashkinov [35], $m = 13$ by Stiebitz, Favrlholdt and Toft [34], $m = 15$ by Scheide [30] and $m = 25$ by Kurt [24].

$$\chi'(G) \leq \max \left\{ \lceil \rho(G) \rceil, \Delta(G) + 1 + \frac{\Delta(G) - 2}{m - 1} \right\}. \quad (1.7)$$

All of these proofs use Tashkinov trees and Tashkinov’s Theorem. McDonald [25] also proved some results towards the Goldberg–Seymour conjecture.
Her proofs follow a similar structure using Tashkinov trees. After stating a few of McDonald’s results we will give the structure she used in their proofs. This will provide some insight in how Tashkinov trees and Tashkinov’s Theorem can be used.

First we need to give some definitions. Let $G$ be a multigraph with $k$-edge-coloring $\varphi$ and let $T$ be a Tashkinov tree with respect to $\varphi$ in $G$. A color $\alpha \in \varphi(E(G))$ is defective if it occurs on more than one edge leaving $T$ (i.e. an edge that has exactly one endpoint in $V(T)$). Let $C_\varphi$ be the set of all edge-colorings $\pi$ that can be obtained from $\varphi$ by augmenting paths and such that $T$ is a Tashkinov tree with respect to $\pi$. $T$ is augmenting-maximal if $T$ is maximal for all colorings in $C_\varphi$. The following two theorems were used by McDonald and show us immediately how Tashkinov trees can be used in order to get results towards the Goldberg–Seymour conjecture.

Theorem 1.7. Let $G$ be a multigraph and let $\varphi$ be a partial $(\Delta(G) + s)$-edge-coloring of $G$, for some $s \geq 1$. Let $T$ be a maximal elementary Tashkinov tree in $G$ and suppose that $T$ has no defective colors. Then

$$\lceil \rho(G) \rceil \geq \Delta(G) + s + 1.$$ 

Theorem 1.8. Let $G$ be a multigraph with a partial $(\chi'(G) - 1)$-edge-coloring and let $t \geq 3$ be an integer. Suppose that there exists an elementary Tashkinov tree with at least $t$ vertices. Then

$$\chi'(G) \leq \max \left\{ \lceil \rho(G) \rceil, \Delta(G) + 1 + \frac{\Delta(G) - 3}{t + 1} \right\}.$$ 

The last theorem tells us that we only need to find large enough elementary Tashkinov trees in order to obtain results towards the Goldberg–Seymour conjecture. McDonald constructed large elementary Tashkinov trees with the use of odd girth and girth. Here she improved the following result by Goldberg [13].

Theorem 1.9. Let $G$ be a multigraph which contains an odd cycle. Then

$$\chi'(G) \leq \max \left\{ \lceil \rho(G) \rceil, \Delta(G) + 1 + \frac{\Delta(G) - 2}{g_o - 1} \right\}.$$ 

The (odd) girth $g(g_o)$ of a graph $G$ is the size of the largest (odd) cycle that is a subgraph of $G$. McDonald’s improvement of the upper bound in Theorem 1.9 and also an upper bound with respect to the girth are stated below.
Theorem 1.10. Let $G$ be a multigraph which contains an odd cycle. Then
\[\chi'(G) \leq \max \left\{ \lceil \rho(G) \rceil, \Delta(G) + 1 + \frac{\Delta(G) - 3}{g_o + 3} \right\}.\]

Theorem 1.11. Let $G$ be a multigraph which contains a cycle. Then
\[\chi'(G) \leq \max \left\{ \lceil \rho(G) \rceil, \Delta(G) + 1 + \frac{\Delta(G) - 3}{\left\lfloor \frac{3g}{2} \right\rfloor + 2} \right\}.\]

Both of the proofs of Theorems 1.10 and 1.11 follow the same structure. We will give a succinct overview of this structure now in order to provide the reader with a small insight in how Tashkinov trees can be used.

- First a simple argument is given for the case $\chi(G) - 1 < \Delta(G) + 1$ and then the focus is on the case where $\chi(G) - 1 \geq \Delta(G) + 1$.
- A maximal partial ($\chi'(G) - 1$)-edge-coloring is taken and a maximal Tashkinov tree $T$ is created.
- Tashkinov’s Theorem tells that $T$ is elementary and Theorem 1.7 is used to prove that $T$ has a defective color.
- It is proven that $T$ is not augmenting-maximal.
- With augmenting paths a larger elementary Tashkinov tree is created.
- Now Theorem 1.8 is used to define the new upper bound for $\chi'(G)$.

Now we have seen that Tashkinov’s Theorem is of use in results towards the Goldberg–Seymour conjecture. In the next chapter we will give the complete proof of Tashkinov’s Theorem.
Chapter 2

Tashkinov’s Theorem

In this chapter we will present the proof of Tashkinov’s Theorem. We already mentioned that the original proof is in Russian by Tashkinov [35]. A preprint of Favrholdt, Stiebitz, and Toft [10] contains an almost complete proof in English. A gap in this proof was noticed by McDonald [25] and she gave the missing part. The proof we give in this chapter is complete and follows from combining the English proof with the gap by Favrholdt et al and the missing part that Jessica McDonald gave. In the first section we will prove a lemma that will be repeatedly used during the second section where the complete proof of Tashkinov’s Theorem is given.

2.1 Preliminary work

Recall that for a $k$-edge-coloring $\varphi$ of a graph $G$ and a maximal $(\gamma, \delta)$-alternating path $P$ on $G$, the $k$-edge-coloring $\varphi' = \varphi \setminus P$ is equal to $\varphi$ except for the edges of $P$, the colors on those edges are switched. For an endpoint $v$ of $P$ we have $\overline{\varphi}(v) = \varphi(v) \triangle \{\gamma, \delta\}$, where $\triangle$ denotes the symmetric difference. For all other vertices the sets of missing colors remain the same. For $U \subseteq V(G)$, $\overline{\varphi}(U) = \{\overline{\varphi}(u) \mid u \in U\}$. If $u, v$ are vertices such that $\gamma \in \overline{\varphi}(u)$ and $\delta \in \overline{\varphi}(v)$, then $(u, v)$ is called a $(\gamma, \delta)$-pair. In this chapter only edge-colorings are used, so sometimes we will say coloring instead of edge-coloring.

First we repeat the formal definition of a Tashkinov tree.

Definition 2.1. Let $G$ be a multigraph and $\varphi$ a partial $k$-edge-coloring such that $e \in E(G)$ is not colored. A Tashkinov tree in $G$ with respect to $e$ and $\varphi$ is a tree $T = (v_0, e_1, v_1, \ldots, e_n, v_n)$ such that

1. $e_1 = e$ has endpoints $v_0$ and $v_1$, 

2. for all \(1 < j \leq n\), \(e_j\) has endpoints \(v_j\) and \(v_i\) for some \(i < j\), and

3. for all \(1 < j \leq n\), there is an \(i < j\) such that \(\varphi(e_j) \in \varphi(v_i)\).

Remember that a Tashkinov tree with respect to \(e\) and \(\varphi\) is elementary if \(\varphi(u) \cap \varphi(v) = \emptyset\) for every pair of distinct vertices \(u, v \in V(T)\). The path number \(p(T)\) of a Tashkinov tree \(T = (v_0, e_1, ..., e_n, v_n)\) is the smallest index such that \((v_i, e_{i+1}, ..., e_n, v_n)\) is a path. If \(T = (v_0, e_1, v_1, ..., e_n, v_n)\) is a Tashkinov tree, then for an integer \(0 < j \leq n\), \(T_{v_j} := (v_0, e_1, v_1, ..., e_j, v_j)\) is also a Tashkinov tree.

**Lemma 2.1.** Let \(G\) be a multigraph and \(\varphi\) a partial \((\Delta(G) + s)\)-edge-coloring of \(G\), with \(s \geq 1\). If \(T = (v_0, e_1, ..., e_n, v_n)\) is a Tashkinov tree with respect to \(e\) and \(\varphi\) such that \(V(T)\) is not elementary, \(p(T)\) is minimal and \(|V(T)|\) is minimal subject to \(p(T)\), then the following hold.

a. \(p(T) \geq 3\).

b. If \((v_i, v_j)\) is a \((\gamma, \delta)\)-pair, \(0 \leq i < j < n\) and \(\gamma\) is not used on \(T_{v_j}\), then there is a maximal \((\gamma, \delta)\)-alternating path \(Q\) from \(v_i\) to \(v_j\). (\(T\) is a Tashkinov tree with respect to \(e\) and \(\varphi' = \varphi \setminus Q\).)

c. If \(j\) is an integer such that \(1 \leq j < n\), then there are at least four colors in \(\varphi(V(T_{v_j}))\) that are unused on \(T_{v_j}\).

**Proof:**

a. Suppose \(p(T) = 2\). Then we have the situation in the picture below.

![Figure 2.1: A Tashkinov tree with path number 2.](image)

We see that for \(T' = (v_0' = v_1, e_1', v_1' = v_0, e_2, v_2, ..., e_n, v_n)\), \(p(T') = 0 < p(T)\). This is a contradiction since \(p(T)\) is minimal. If \(p(T) = 1\), then \(T\) is a path and therefore \(p(T) = 0\). If \(p(T) = 0\), then \(T\) is a Kierstead path and Kierstead’s Theorem tells us that \(V(T)\) is elementary. This is a contradiction.

b. Let \((v_i, v_j)\) be a \((\gamma, \delta)\)-pair, \(0 \leq i < j < n\) and assume that \(\gamma\) is not used on \(T_{v_j}\). We look at \(Q\), the \((\gamma, \delta)\)-alternating path starting in \(v_j\). \(V(T_{v_j})\) is
elementary because $p(Tv_j) \leq p(T)$ and $|V(Tv_j)| < |V(T)|$. Since $V(Tv_j)$ is elementary, we know $\delta \notin \overline{\varphi}(v)$ for $v \in \{v_0, ..., v_{j-1}\}$ and because of the third condition of Definition 2.1 we know $\delta$ is also not used on the edges of $Tv_j$. Now we know $E(Q) \cap E(Tv_j) = \emptyset$. Note that if $Q$ enters $v_i$, then it ends in $v_i$ since $\gamma \in \overline{\varphi}(v_i)$. If $Q$ does not end at $v_i$, the situation is shown in the picture below.

![Figure 2.2: Case b, before augmenting $Q$. When the structure between two vertices is not clear, their connection is indicated with waving lines. A color that is missing by a vertex is denoted with a bar above that color.](image)

We augment $Q$ in order to obtain the coloring $\varphi' = \varphi \setminus Q$. $Tv_j$ is a Tashkinov tree with respect to $e$ and $\varphi'$. Since $\gamma \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j) \neq \emptyset$, $V(Tv_j)$ is not elementary. This is a contradiction since $p(Tv_j) \leq p(T)$ and $|V(Tv_j)| < |V(T)|$. Therefore $Q$ ends at $v_i$ and is a maximal $(\gamma, \delta)$-alternating path from $v_i$ to $v_j$.

c. Let $j$ be an integer such that $1 \leq j < n$. For $i \in \{2, 3, ..., j\}$ we know $|\overline{\varphi}(v_i)| \geq \Delta(G) + s - d_G(v_i) \geq 1$ and for $i \in \{0, 1\}$ we know $|\overline{\varphi}(v_i)| \geq \Delta(G) + s - d_G(v_i) + 1 \geq 2$. $V(Tv_j)$ is elementary because $p(Tv_j) \leq p(T)$ and $|V(Tv_j)| < |V(T)|$, therefore we know $\overline{\varphi}(V(Tv_j)) = \sum_{i=0}^{j} \overline{\varphi}(v_i) \geq j + 3$. $Tv_j$ has $j$ edges, of which only $j - 1$ are colored. Therefore, we conclude that there are at least four colors in $\overline{\varphi}(V(Tv_j))$ that are unused on $Tv_j$.

### 2.2 The proof of Tashkinov’s Theorem

In Section 1.2 we saw that Tashkinov trees and Tashkinov's Theorem are used often in results towards the Goldberg–Seymour conjecture. So now we investigate Tashkinov’s Theorem in order to get completely familiar with it. Here we present the detailed proof of Tashkinov’s Theorem.

**Theorem 2.1** (Tashkinov’s Theorem). Let $G$ be a multigraph and $\varphi$ a partial $(\Delta(G) + s)$-edge-coloring of $G$, with $s \geq 1$ and $e \in E(G)$ uncolored. If there exists a non-elementary Tashkinov tree with respect to $e$ and $\varphi$, then there exists a $(\Delta(G) + s)$-edge-coloring of $\text{dom}(\varphi) \cup e$. 
Proof: Let $G$ be a multigraph and suppose there exists a non-elementary Tashkinov tree with respect to $e$ and $\varphi$, where $\varphi$ is a partial $(\Delta(G) + s)$-edge-coloring of $G$, $s \geq 1$. Choose $T$ and $\varphi$ such that $T = (v_0, e_1, ..., e_n, v_n)$ is a non-elementary Tashkinov tree with respect to $e$ and $\varphi$, and such that

- $p(T)$ is minimal and
- $|V(T)|$ is minimal subject to $p(T)$.

We define

$$C^T = \{ \pi \text{ a $(\Delta(G) + s)$-edge-coloring of } \text{dom}(\varphi) \mid T \text{ is a non-elementary Tashkinov tree with respect to } \pi \}.$$ 

Note that for all $v \in V(T)$ we have $\overline{\varphi}(v) \neq \emptyset$ because $\Delta(G) + s > d_G(v)$.

We assume that $\text{dom}(\varphi) \cup e$ is not $(\Delta(G) + s)$-edge colorable and work towards a contradiction. We will handle two cases according to the path number of $T$.

Case 1: $p(T) = n$. First we note that in this case $e_n$ has endpoints $v_n$ and $v_i$ for some $i < n - 1$. We will successively prove the following three statements.

(a) There exists a coloring $\psi \in C^T$ such that $\overline{\psi}(v_j) \cap \overline{\psi}(v_n) \neq \emptyset$ for some $j < n - 1$ and some $\beta \in \overline{\psi}(v_j) \cap \overline{\psi}(v_n)$ is not used on $T$.

(b) There exists a coloring $\psi \in C^T$ such that $\psi(e_n) \in \psi(v_{n-1})$.

(c) There exists a coloring $\psi \in C^T$ such that $\psi(e_n) \in \psi(v_{n-1})$ and $\overline{\psi}(v_j) \cap \overline{\psi}(v_n) \neq \emptyset$ for some $j < n - 1$.

Statement (a) is needed to prove statement (b) and statement (b) is needed to prove statement (c). Then, statement (c) will be used to create a smaller Tashkinov tree which will imply a contradiction. We will show this in our finishing argument of Case 1, but first we give the proofs of the three essential statements.

(a) Let $\psi$ be any coloring in $C^T$. Let $\alpha$ be a color and $j \leq n - 1$ an integer such that $\alpha \in \overline{\psi}(v_j) \cap \overline{\psi}(v_n)$. This is possible since $V(T)$ is not elementary and $V(Tv_{n-1})$ is not elementary. By Lemma 2.1c, there are at least two colors in $\overline{\psi}(V(Tv_{n-2}))$ that are unused on $T$. Let $\beta$ be a color and $m \leq n - 2$ an integer such that $\beta \in \overline{\psi}(v_m)$ is not used on $T$ and $\alpha \neq \beta$.

Suppose $j = n - 1$. Since $Tv_{n-1}$ is an elementary Tashkinov tree and $\alpha \in \overline{\psi}(v_{n-1})$, we know $\alpha \in \psi(v_k)$ for all $0 \leq k < n - 1$. Because of the third condition of Definition 2.1, $\alpha$ is not used on $T$. The pair $(v_j, v_m)$ is
an \((\alpha, \beta)\)-pair, \(m < j\) and \(\beta\) is not used on \(T\), so we use Lemma 2.1b. Let \(Q\) be the \((\alpha, \beta)\)-alternating path from \(v_m\) to \(v_j\) and \(\psi' = \psi \setminus Q \in C^T\). Now \(\alpha \in \overline{\psi'}(v_m) \cap \overline{\psi'}(v_n), m < n - 1\) and \(\alpha\) is not used on \(T\) and we found the coloring we were looking for.

Figure 2.3: Case (a), before augmenting \(Q\) and \(j = n - 1\).

Now, suppose \(j < n - 1\). If \(\beta \in \overline{\psi}(v_n)\), then \(\beta \in \overline{\psi}(v_m) \cap \overline{\psi}(v_n)\) and \(\beta\) is not used on \(T\) so we are done. Now we assume \(\beta \in \psi(v_n)\). Let \(Q\) be the maximal \((\alpha, \beta)\)-alternating path starting in \(v_n\). The path \(Q\) does not contain vertices of \(Tv_{n-2}\). Otherwise, let \(v_k \in Q \cap V(Tv_{n-2})\) and \(Q'\) be the \((\alpha, \beta)\)-alternating path from \(v_k\) to \(v_n\). Then \(T' = (v_0, ..., v_{n-2}, Q', v_n)\) is a non-elementary Tashkinov tree with \(p(T') < p(T)\), which contradicts the minimality of \(p(T)\). Since \(e_n\) and \(e_{n-1}\) have endpoints in \(Tv_{n-2}\), we know that \(Q\) does not contain any edge of \(T\). Let \(\psi' = \psi \setminus Q \in C^T\), then \(\beta \in \overline{\psi'}(v_m) \cap \overline{\psi'}(v_n), m \leq n - 2\) and \(\beta\) is not used on \(T\). This completes the proof of statement (a).

Figure 2.4: Case (a), before augmenting \(Q\) and \(j < n - 1\).

(b) Let \(\psi \in C^T\) be an edge-coloring such that statement (a) is fulfilled, that is \(\alpha \in \overline{\psi}(v_j) \cap \overline{\psi}(v_n), j < n - 1\) and \(\alpha\) is not used on \(T\). If \(\beta = \psi(e_n) \in \overline{\psi}(v_{n-1})\), then \((j, n - 1)\) is a \((\alpha, \beta)\)-pair and we use Lemma 2.1b. Let \(Q\) be the maximal \((\alpha, \beta)\)-alternating path between \(v_j\) and \(v_{n-1}\) and \(\psi' = \psi \setminus Q \in C^T\). Since
\( \alpha \in \overline{\psi}(v_n) \) and \( Q \) does not end in \( v_n \), we know \( \psi'(e_n) = \psi(e_n) \). Thus \( \psi'(e_n) = \beta \in \psi'(v_{n-1}) \) and \( \psi' \) is our desired coloring. This completes the proof of statement (b).

\[ \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.5}
\caption{Case (b), before augmenting \( Q \).}
\end{figure}

\[(c)\] We start with \( \psi \in C^T \) such that statement (b) is fulfilled, that is \( \psi(e_n) \in \psi(v_{n-1}) \). Suppose \( \overline{\psi}(v_j) \cap \overline{\psi}(v_n) = \emptyset \) for all \( j < n - 1 \). Since \( T \) is not elementary and \( T v_{n-1} \) is elementary, we know \( \in \overline{\psi}(v_{n-1}) \cap \overline{\psi}(v_n) \neq \emptyset \). Let \( \alpha \in \overline{\psi}(v_{n-1}) \cap \overline{\psi}(v_n) \). Because of Lemma 2.1c, there exists a color \( \beta \) and an integer \( m \leq n - 2 \) such that \( \beta \in \overline{\psi}(v_m) \) and \( \beta \) is not used on \( T \). Then \( \beta \neq \psi(e_n) \) and \( \alpha \neq \beta \) because otherwise \( \overline{\psi}(v_m) \cap \overline{\psi}(v_n) \neq \emptyset \). The pair \( (v_{n-1}, v_m) \) is an \((\alpha, \beta)\)-pair, \( m < n - 1 \) and \( \beta \) is not used on \( T \), so we use Lemma 2.1b. Let \( Q \) be the maximal \((\alpha, \beta)\)-alternating path from \( v_m \) to \( v_{n-1} \) and \( \psi' = \psi \setminus Q \in C^T \). Again, we have \( \alpha \in \overline{\psi}(v_m) \), so \( \psi'(e_n) = \psi(e_n) \neq \alpha, \beta \).

Now \( \alpha \in \overline{\psi}(v_m) \cap \overline{\psi}(v_n) \neq \emptyset \), \( m < n - 1 \), \( \psi'(e_n) \in \psi'(v_{n-1}) \) and we have proven statement (c).

\[\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.6}
\caption{Case (c), before augmenting \( Q \).}
\end{figure}
Recall that $T$ and $\varphi$ are chosen such that $T$ is a non-elementary Tashkinov tree with respect to $e$ and $\varphi$, and such that $p(T)$ is minimal, $|V(T)|$ is minimal subject to $p(T)$ and $p(T) = n$. Using statement (c), we will now derive a contradiction for Case 1.

Let $\psi$ be a coloring such that statement (c) is fulfilled. By the fact $\psi(e_n) \in \psi(v_{n-1})$ and the third condition of Definition 2.1, $\psi(e_n) \in \overline{\psi}(v_j)$ for some $j < n - 1$. Then removing $v_{n-1}$ gives us $T' = (v_0, e_1, v_1, ..., v_{n-2}, e_n, v_n)$, which is a Tashkinov tree with respect to $e$ and $\psi$. The vertex set $V(T')$ is not elementary, $p(T') \leq p(T)$ and $|V(T')| < |V(T)|$. This yields a contradiction, completing Case 1.

During the second case we will use the following notation for $\psi \in C_T$,

$$I_\varphi = \{i < n \mid \overline{\varphi}(v_i) \cap \overline{\varphi}(v_n) \neq \emptyset\}.$$ 

We note that $V(Tv_{n-1})$ is elementary and $V(T)$ is not elementary for $\psi \in C_T$ and thus $I_\psi \neq \emptyset$.

**Case 2:** $p(T) \leq n - 1$.

Suppose $\max(I_\psi) \geq p(T)$ for some $\psi \in C_T$. Let $\psi \in C_T$ be a coloring of $G$ such that $i = \max(I_\psi)$ is maximal. We know $\overline{\psi}(v_i) \cap \overline{\psi}(v_n) \neq \emptyset$ and take $\alpha \in \overline{\psi}(v_i) \cap \overline{\psi}(v_n)$. Note that for each integer $i < j \leq n$ the edge $e_j$ has endpoints $v_{j-1}$ and $v_j$ and if $j \neq n$ then $\overline{\psi}(v_j) \cap \overline{\psi}(v_n) = \emptyset$. First we show $i = n - 1$ by assuming the contrary.

![Figure 2.7: Case $i < n - 1$, before augmenting $Q$ and under the assumption $\max(I_\psi) \geq p(T)$](image)

Suppose $i < n - 1$. Let $\beta \in \overline{\psi}(v_{i+1})$, then $\beta \neq \alpha$ because $\overline{\psi}(v_{i+1}) \cap \overline{\psi}(v_n) = \emptyset$. $V(Tv_{i+1})$ is elementary, therefore $\alpha \in \psi(v_j)$ for all $j < i$ and thus no edge
of $Tv_i$ is colored with $\alpha$. Since $e_{i+1}$ has endpoints $v_i$ and $v_{i+1}$, the color $\alpha$ is unused on $Tv_{i+1}$. The pair $(v_i, v_{i+1})$ is an $(\alpha, \beta)$-pair and we use Lemma 2.1b. Let $Q$ be the maximal $(\alpha, \beta)$-alternating path from $v_i$ to $v_{i+1}$ and $\psi' = \psi \setminus Q \in C^T$. Now $\alpha \in \overline{\psi'}(v_{i+1}) \cap \overline{\psi'}(v_n)$. This contradicts the maximality of $\max(I_\psi)$ and we conclude $i = n - 1$.

Let $\gamma = \psi(e_n)$. Because $T$ is a Tashkinov tree with respect to $\psi$, $\gamma \in \overline{\psi}(v_j)$ for an integer $j < n - 1$. Recolor $e_n$ with $\alpha$ in order to obtain $\psi'$. Now $T$ need not be a Tashkinov tree with respect to $e$ and $\psi'$, but $Tv_{n-1}$ does. $Tv_{n-1}$ is not elementary since $\gamma \in \overline{\psi'}(v_j) \cap \overline{\psi'}(v_{n-1})$. The fact that $p(Tv_{n-1}) = p(T)$ and $|V(Tv_{n-1})| < |V(T)|$ gives a contradiction in how we have chosen $T$ and $\varphi$.

Suppose $\max(I_\psi) < p(T)$ for all $\psi \in C^T$. Let $j = p(T)$. Then $v_{j-1}$ is not an endpoint of $e_j$ and $j \geq 3$ by Lemma 2.1a. Let $\psi \in C^T$ be a coloring such that $i = \min(I_\psi)$ is minimal and let $\alpha \in \overline{\psi}(v_i) \cap \overline{\psi}(v_n)$. We know $i \leq j - 1$, but we now show that this inequality is strict.

Assume $i = j - 1$. By Lemma 2.1c there are at least four colors in $\overline{\psi}(V(Tv_{j-2}))$ that are not used on $Tv_{j-2}$. Let $\beta$ be a color and $h \leq j - 2$ an integer such that $\beta \in \overline{\psi}(v_h)$ is not used on $Tv_{j-1}$. The pair $(v_{j-1}, v_h)$ is an $(\alpha, \beta)$-pair and we use Lemma 2.1b. Let $Q$ be the maximal $(\alpha, \beta)$-alternating path from $v_h$ to $v_{j-1}$ and $\psi' = \psi \setminus Q \in C^T$. Now $\alpha \in \overline{\psi'}(v_h) \cap \overline{\psi'}(v_{n-1})$ and thus $\min(I_\psi) \leq h < i = \min(I_\psi)$. This contradicts the minimality of $\min(I_\psi)$ and we conclude $i < j - 1$.

![Figure 2.8: Case $i = j - 1$, before augmenting $Q$ and under the assumption $\max(I_\psi) < p(T)$.](image)

Let $\delta \in \overline{\psi}(v_j)$. Because of Lemma 2.1c, there exists a color $\gamma \in \overline{\psi}(V(Tv_{j-2}))$ that is not used on $Tv_j$ and $\gamma \neq \alpha$. Let $h \leq j - 2$ be an integer such that $\gamma \in \overline{\psi}(v_h)$. Since $V(Tv_j)$ is elementary, $\gamma \neq \delta$. 

26
Suppose $\gamma \in \overline{\psi}(v_n)$. Let $Q$ be the maximal $(\gamma, \delta)$-alternating path from $v_h$ to $v_j$ and $\psi' = \psi \setminus Q \in C^T$, their existence is guaranteed by Lemma 2.1b. Then $\psi' = \psi \setminus Q \in C^T$, their existence is guaranteed by Lemma 2.1b. Then $\gamma \in \overline{\psi}(v_j) \cap \overline{\psi}(v_n) \implies \max(I_{\psi'}) \geq j = p(T)$, this is a contradiction. Thus $\gamma \in \psi(v_n)$.

![Figure 2.9: Q before it is augmented and P in case $i_0 < j - 1$ and under the assumption $\max(I_{\psi}) < p(T)$.

Let $P$ be the $(\alpha, \gamma)$-chain starting in $v_n$. We note that $P$ is a path because $\alpha \in \overline{\psi}(v_n)$. Suppose $V(P) \cap V(Tv_{j-1}) \neq \emptyset$. Let $v_{i_0}$ be the first vertex of $P$ that is in $Tv_{j-1}$ and $P' = (v_{i_0}, f_1, z_1, ..., f_m, z_m = v_n)$, which is the $(\alpha, \gamma)$-alternating path from $v_{i_0}$ to $v_n$. If $i_0 < j - 1$ then $T' = (v_{i_0}, e_1, ..., e_{j-2}, v_{j-2}, f_1, z_1, ..., f_m, z_m = v_n)$ is a Tashkinov tree with respect to $e$ and $\psi$, since $\alpha \in \overline{\psi}(v_i)$, $\gamma \in \overline{\psi}(v_h)$ and $i, h \leq j - 2$. $V(T')$ is not elementary because $\gamma \in \overline{\psi}(v_h) \cap \overline{\psi}(v_n)$ and $p(T') < j = p(T)$. This contradicts the choice of $T$ and $\varphi$. If $i_0 = j - 1$, then $T' = (v_{i_0}, e_1, ..., e_{j-1}, v_{j-1}, f_1, z_1, ..., f_m, z_m)$ is a Tashkinov tree with respect to $e$ and $\psi$. Again, $V(T')$ is not elementary and $p(T') < j = p(T)$, which is a contradiction.

![Figure 2.10: T' in case $i_0 = j - 1$.](image)
Now we know $V(P) \cap V(Tv_{j-1}) = \emptyset$ and $P$ ends at $z \notin V(Tv_{j-1})$. For all $v \in \{v_j, \ldots, v_{n-1}\}$ we have that $\alpha, \gamma \in \psi(v)$, because otherwise $Tv_{n-1}$ would not be elementary ($\alpha \in \overline{\psi}(v_i)$, $\gamma \in \overline{\psi}(v_h)$). Thus, $z \notin V(T)$ and by augmenting $P$ we obtain $\psi' = \psi \setminus P \in \mathcal{C}^T$. Note that $T$ is also a Tashkinov tree with respect to $e$ and $\psi'$ and also for $\psi'$ the following holds: $\gamma \in \overline{\psi'}(v_h)$, $\delta \in \overline{\psi'}(v_j)$, $h \leq j - 2$ and $\gamma$ is unused on $Tv_j$. Let $Q$ be the maximal $(\gamma, \delta)$-alternating path from $v_h$ to $v_j$ and $\psi'' = \psi' \setminus Q \in \mathcal{C}^T$, their existence is guaranteed by Lemma 2.1b. Then $\gamma \in \overline{\psi''}(v_j) \cap \overline{\psi''}(v_n)$. Thus $\max(I_{\psi''}) \geq j = p(T)$, which is a contradiction.

![Figure 2.11: After augmenting $P$ and before augmenting $Q$.](image)

In both cases we have obtained a contradiction and therefore there is no Tashkinov tree $T$ such that $V(T)$ is not elementary. We conclude that if $\text{dom}(\varphi) \cup \{e\}$ is not $(\Delta(G) + s)$-edge-colorable and $T$ is a Tashkinov tree with respect to $e$ and $\varphi$, then $V(T)$ is elementary. This proves the theorem.

We will now point out the gap in the proof of Favrholdt et al. [10]. In Case 1, three statements are proven of which the last one is essential for the concluding statement of Case 1. In the proof of Favrholdt et al. the second statement is not proven. After statement (a), they immediately made an attempt to proof statement (c). In their proof of statement (c), they were augmenting a path and forgot that the color of $e_n$ might also be changed. This mistake was only made in the preprint and a complete version of Tashkinov’s Theorem can also be found in [34].
Chapter 3

Vertex-Tashkinov trees

In this chapter we will transform Tashkinov trees on graphs into vertex-Tashkinov trees on line graphs, using a straightforward characterisation of line graphs due to Bermond and Meyer [4]. From this, Tashkinov’s Theorem can immediately be transformed into an equivalent theorem for line graphs. Next, we investigate the possibility of using vertex-Tashkinov trees on graphs that are not necessarily line graphs. We will follow a treatment of Kierstead’s Theorem by McDonald [25]. After this we have a look at a structure theorem for quasi-line graphs and how this can be of use in extending results for line graphs to results for quasi-line graphs.

First we discuss some facts about line graphs. In Chapter 1 we already saw that each edge-coloring of a graph $G$ is a vertex-coloring of its line graph $L(G)$ and vice versa. Moreover, each vertex of degree $d_v$ in a graph $G$ corresponds to a clique of size $d_v(G)$ in the line graph. Each matching on $G$ is an independent set in $L(G)$ and vice versa. Overall, for each graph $G$ the following hold:

\[
\begin{align*}
\chi'(G) &= \chi(L(G)), \\
\Delta(G) &\leq \omega(L(G)) \quad \text{and} \\
\tau(G) &= \alpha(L(G)).
\end{align*}
\]

For each clique of size $k$ in a line graph $L(G)$, there must be $k$ edges that are pairwise adjacent in the graph $G$. Therefore, $G$ contains a vertex $v$ with degree $d_v \geq k$, except for $k = 3$ (see Figure 3.1). In case $k = 3$, $K_3$ is the line graph of $K_3$ and $\Delta(K_3) = 2 < 3 = \omega(L(K_3))$. We notice that a complete graph on three vertices is the only connected simple graph for which the inequality in the middle is strict. Therefore, the inequality in the middle is an equality for simple graphs that are not a disjoint union of triangles.
Not all simple graphs are line graphs. For example, $M(M(G))$ for any graph $G$ is not a line graph. Recall that for a Mycielskian graph $M(G)$ of $G$, $\chi(M(G)) = \chi(G) + 1$ and $\omega(M(G)) = \omega(G)$. Now suppose the contrary and let $M(M(G))$ be the line graph of a graph $H$. By Vizing’s Theorem, we obtain the following inequality which is a contradiction:

$$\Delta(H) + 1 \geq \chi'(H) = \chi(M(M(G))) = \chi(G) + 2 \geq \omega(G) + 2 \geq \Delta(H) + 2.$$ 

Line graphs have a well understood structure. For instance Beineke’s Theorem [3] shows that they can be characterised using forbidden induced subgraphs.

Figure 3.1: $K_2, K_3, K_4$ and simple graphs having these as their line graphs.

Figure 3.2: The nine forbidden subgraphs of Beineke’s Theorem.
Theorem 3.1 (Beineke’s Theorem). Let $G$ be a simple graph. $G$ is the line graph of a simple graph if and only if $G$ does not contain an induced subgraph equal to one of the graphs in Figure 3.2.

Bermond and Meyer [4] obtained a multigraph analogue of Beineke’s Theorem, using only seven forbidden subgraphs. Also, they proved the following theorem, which is a multigraph analogue of a theorem of Krausz [23].

Theorem 3.2. A simple graph $G$ is a line graph of a multigraph if and only if there exists a family of subgraphs $W = \{W_i \mid i \in I\}$ of $G$, such that the following three conditions hold.

(L1) Every edge of $G$ occurs in at least one $W_i$.

(L2) Every vertex of $G$ occurs in exactly two $W_i$.

(L3) Each $W_i$ is a clique.

Theorem 3.2 will be useful when we transform Tashkinov trees into vertex-Tashkinov trees on the line graph. We will see this in the next section, but here we already give some intuition for Theorem 3.2. Each $W_i$ corresponds to a vertex $v_i$ in the multigraph. The vertices of the clique $W_i$ are precisely the edges in the multigraph that are incident to $v_i$. Since every edge $e$ in the multigraph has two endpoints, the cliques corresponding to these endpoints are precisely the only two cliques that contain the vertex corresponding to $e$.

![Figure 3.3](image-url)  

Figure 3.3: Illustration of the intuition for Theorem 3.2. On the left side a multigraph and on the right side its line graph with an appropriate family of subgraphs.

Both endpoints of an edge in a graph are associated to a clique in its line graph. So, intuitively it is easy to see that line graphs have the following property:

(A) The neighbourhood of any vertex can be partitioned into two sets $N_1$ and $N_2$, such that $G[N_1]$ and $G[N_2]$ are cliques.
Formally, property (A) follows immediately from (L2) and (L3) of Theorem 3.2. Graphs for which property (A) holds are called *quasi-line graphs*. All graphs in Figure 3.2 except for $G_1$ and $G_2$ are quasi-line graphs but not line graphs. Naturally, one might wonder about what results that hold for line graphs could also hold for the wider class of quasi-line graphs.

Another class of graphs that attracted a lot of interest the past few years, is the class of claw-free graphs. These are graphs which do not contain a *claw* (i.e. the graph $G_1$ in Figure 3.2) as an induced subgraph. It is easy to see that a claw is not a quasi-line graph. Therefore, a quasi-line graph cannot contain an induced subgraph that is a claw. Thus, quasi-line graphs are claw-free graphs. But not all claw-free graphs are quasi-line graphs. See, for example, $G_2$ in Figure 3.2. Again, one may wonder what results that hold for line graphs could also hold for claw-free graphs. To summarize, if $C_{\text{line}}, C_{\text{quasi}}$ and $C_{\text{clawfree}}$ denote the classes of respectively line graphs, quasi-line graphs and claw-free graphs, then the following holds:

$$C_{\text{line}} \subset C_{\text{quasi}} \subset C_{\text{clawfree}}.$$  

In 2005, Chudnovsky and Seymour gave an overview [7] of a series of articles about claw-free graphs. In this series, they provided a structure theorem for claw-free graphs and also a structure theorem for quasi-line graphs is given. With the help of these structure theorems one can extend results for line graphs to results for quasi-line graphs and claw-free graphs. For instance, King and Reed [19] used the structure theorem for quasi-line graphs in order to prove that for any quasi-line graph the chromatic number and fractional chromatic number asymptotically agree. Maybe this way we can also obtain some results for vertex-Tashkinov trees on quasi-line graphs. We will discuss the Structure Theorem for quasi-line graphs in the second section of this chapter. But first, we will make the transformation to vertex-Tashkinov trees and try a different approach that was mentioned by McDonald [25].

### 3.1 Kierstead’s Theorem for line graphs

First we give the definition of a vertex-Tashkinov tree on a line graph. This definition was given by McDonald [25] and seems to be the most compact.

**Definition 3.1.** Let $G$ be a simple graph such that there exists a family of subgraphs $W = \{W_i | i \in I\}$ of $G$ such that (L1), (L2) and (L3) hold. Let $\varphi$ be a partial vertex-coloring. $T = (v_1, \ldots, v_n)$ is a vertex-Tashkinov tree with respect to $\varphi$ if there are distinct $W_0, \ldots, W_n \in W$ such that the following two conditions hold.
(V1) \( v_1 \) is uncolored and for all \( 1 < j \leq n \), there is an \( i < j \) such that 
\[ \varphi(v_j) \in \varphi(W_i) := \varphi(V(G)) \setminus \varphi(V(W_i)). \]

(V2) \( v_1, \ldots, v_n \) are distinct and for all \( 1 \leq i \leq n \), \( v_i \in V(W_j) \cap V(W_i) \) for some \( 0 \leq j < i \).

![Figure 3.4: A Tashkinov tree and its corresponding vertex-Tashkinov tree are represented by the bold edges.](image)

A Tashkinov tree on \( G \) can be easily transformed into a vertex-Tashkinov tree on \( \mathcal{L}(G) \). We just take the vertices of \( \mathcal{L}(G) \) that correspond with the edges of the Tashkinov tree. Vice versa, from a vertex-Tashkinov tree on \( \mathcal{L}(G) \) we can obtain a Tashkinov tree on \( G \) by taking the edges of \( G \) that correspond with the vertices of the vertex-Tashkinov tree. Figure 3.4 illustrates this and we say that the Tashkinov tree and its transformation into a vertex-Tashkinov tree communicate. During this section we will focus on a specific type of a vertex-Tashkinov tree, that is, a vertex-Kierstead path. Remember that we introduced Kierstead paths in Chapter 1. When defining a vertex-Kierstead path, we only need to replace condition (V2) of Definition 3.1 by condition (V2'), which is given below.

(V2') \( v_1, \ldots, v_n \) are distinct and there are distinct \( W_0, \ldots, W_n \) such that for all \( 1 \leq i \leq n \), \( v_i \in V(W_{i-1}) \cap V(W_i) \).

All kinds of properties and conditions of Tashkinov trees can be easily transformed to the line graph setting. During this section we will only use the definition of a vertex-Kierstead path to be elementary.

**Definition 3.2.** A vertex-Kierstead path \( P = (v_1, \ldots, v_n) \) with respect to the coloring \( \varphi \) is elementary if for all \( 0 \leq i, j \leq n \) such that \( i \neq j \) the following holds: 
\[ \overline{\varphi}(W_i) \cap \overline{\varphi}(W_j) = \emptyset. \]

Note that if a Kierstead path and a vertex-Kierstead path communicate, then they are both elementary or they are both not elementary. Kierstead’s Theorem can be transformed to a line graph version now [25].
Theorem 3.3 (Kierstead’s Theorem for line graphs). Let $G$ be a simple graph with a family of subgraphs $W = \{W_i \mid i \in I\}$ of $G$, such that (L1), (L2) and (L3) hold. Let $\varphi$ a partial $(\omega(G) + s)$-vertex-coloring of $G$, with $s \geq 1$. Suppose that there exists a vertex-Kierstead path $P = \{v_1, ..., v_n\}$ that is not elementary. Then there exists a partial $(\omega(G) + s)$-vertex-coloring of $G$ with domain $\text{dom}(\varphi) \cup \{v_1\}$.

Theorem 3.3 follows directly from the original Kierstead’s Theorem, but if we drop one of the conditions (L1), (L2) or (L3), then it does not. Therefore, McDonald gave a proof of Theorem 3.3 in the line graph setting that follows the structure of the proof of Kierstead’s Theorem. While doing this, she highlighted the places where (L1), (L2) and (L3) are used. She commented that it was not easy and maybe not even possible to drop one of the conditions (L1), (L2) or (L3) for the conclusion to hold. The arguments she gave were convincing and that is why we tried to find a counterexample. We found one with clique number four in case $s = 1$ and also a way to create counterexamples of arbitrarily high clique number. After this, counterexamples for each $s \geq 1$ were easy to deduce. For now, the focus is on dropping (L3). Dropping conditions (L1) and (L2) are easy cases which will be handled at the end of this section.

Now we will present our first counterexample for Theorem 3.3 where condition (L3) is dropped. We take $s = 1$ and we want to have a graph $G$ with $\chi(G) \geq \omega(G) + 2$. In order to get such a graph, we use the Mycielskian construction since it increases the chromatic number while preserving the clique number. We start with a clique of size two and take the Mycielskian of it, resulting in a cycle of length five. Next, we take the Mycielskian graph of this cycle. The obtained graph is known as the Grötzsch graph [14] and the construction is shown in Figure 3.5. Recall that for the Mycielskian graph $M(G)$ of a graph $G$, $\chi(M(G)) = \chi(G) + 1$ and $\omega(M(G)) = \omega(G)$. Therefore, by construction we know that $\omega(G) = 2$ and $\chi(G) = 4$.

![Figure 3.5: Construction of the Grötzsch graph $G$.](image-url)
In Figure 3.6 the Grötzsch graph is drawn with a partial 3-vertex-coloring. Since only the vertex \( v_1 \) in the middle is not colored and \( \chi(G) = 4 \), we know that the given coloring is maximal. Four subgraphs of \( V(G) \), which are denoted by \( W_0, W_1, W_2, \) and \( W_3 \) are surrounded by the thin lines. Let \( V(W_4) \) be the set of the uppermost red vertex together with its two blue neighbours and the left green vertex. One can easily check that (L1) and (L2) hold for \( W_0, W_1, W_2, W_3, W_4 \). That is, each edge is contained in \( E(W_i) \) for some \( i \in \{0, \ldots , 4\} \) and each vertex is contained in exactly two sets of \{\( V(W_0) \), \( V(W_1) \), \( V(W_2) \), \( V(W_3) \), \( V(W_4) \)\}.

![Figure 3.6: Counterexample with clique number 2.](image)

A path \( P = (v_1, v_2, v_3) \) of length two is given by the bold edges in Figure 3.6. \( P \) is a vertex-Kierstead path since \( \varphi(v_2), \varphi(v_3) \in \varphi(W_0) \) and for all \( 1 \leq i \leq 3 \), \( v_i \in V(W_{i-1}) \cap V(W_i) \). Because \( W_1 \) and \( W_2 \) both miss the color green, we know \( P \) is not elementary. So, we found a vertex-Kierstead path that is not elementary, but we know \( \varphi \) cannot be extended. Therefore, the condition (L3) cannot be dropped in Kierstead’s Theorem for line graphs.

We will show that a counterexample with arbitrarily high clique number can be created using a similar construction. For \( i \geq 2 \) we construct the graph \( G \) with a partial \((i + 1)\)-vertex-coloring \( \varphi \) as follows:

1. Let \( G'' \) be a clique of size \( i \) and color its vertices with colors \( \{1, \ldots , i\} \).

2. Let \( G' \) be the Mycielskian of \( G'' \). Color the copies of \( V(G'') \) with the
same color as their original vertices and color the extra vertex $x_1$ with color $i + 1$.

3. Expand $G'$ to $G$ by taking the Mycielskian of $G'$. Color the copies with the same color as their original vertices. The copy of $x_1$ is called $x_2$ and the extra (uncolored) vertex is called $x_0$.

In Figure 3.7 an example for the case $i = 3$ is given, the edges between $G'$ and the copies of $G'$ are not drawn.

![Figure 3.7: Construction of $G$ and $\varphi$ in case $i = 3$, with the edges between $G'$ and its copy not drawn.](image)

Because we took the Mycielskian two times, we know by construction that $G$ has clique number $i$ and chromatic index $i + 2$. Note that the color class $i + 1$ consists of only two vertices; $x_1$ and $x_2$. Next we define five subsets of $V(G)$:

- $U_0 := \{x_0, x_2\}$,
- $U_1 := \{x_0\} \cup (N(x_0) \setminus \{x_2\}) \cup V(G'')$,
- $U_2 := V(G'') \cup \{v \in V(G) \mid \text{there is a vertex } v'' \in V(G'') \text{ such that } v \text{ is a copy of } v''\}$,
- $U_3 := \{x_1, x_2\} \cup N(x_1)$ and
- $U_4 := \{x_1\}$. 

36
For $0 \leq i \leq 4$ let $W_i$ be the induced subgraph $G[U_i]$ of $G$. The graph $G$ and subgraphs $W_0, ..., W_4$ are illustrated in Figure 3.8. In this figure, $V_1$ is the set of vertices of $G''$, which forms a clique. The set $V_2$ is the first copy of $V_1$ and $V_1 \cup V_2 \cup \{x_1\} = V(G')$. The set $V_3 \cup V_4 \cup \{x_2\}$ is the copy of $V(G'')$, thus $V_2$ and $V_3 \cup V_4 \cup \{x_2\}$ are both independent sets. If there are edges between two sets, then a dotted line connects those two sets. Note that there are no edges between $V_2$ and $V_3$ because $V_2$ is an independent set. A blue path is also given, which we discuss later.

One can check by observing Figure 3.8 that (L1) and (L2) hold for the family $\{W_i \mid 0 \leq i \leq 4\}$, but we will also write this down formally:

(L1): We divide the edges in three types and for each type we give the $W_i$ which contains that particular type of edge.
For every edge $e$ adjacent to $x_0$, $e$ is contained in $W_0$ or $W_1$.
For every edge $e$ adjacent to $x_1$ or $x_2$, $e$ is contained in $W_0$ or $W_3$.
For every edge $e$ not adjacent to $x_0$, $x_1$ or $x_2$, $e$ is contained in $W_1$ or $W_2$.

(L2): First we show that $x_0$, $x_1$ and $x_2$ are contained in exactly two $W_i$. The rest of the vertices will be divided in three types.
\(x_0\) is only in \(W_0\) and \(W_1\), \(x_1\) is only in \(W_3\) and \(W_4\) and \(x_2\) is only in \(W_0\) and \(W_3\).

For all vertices \(v \in V_1 \cup V_3\), \(v\) is only in \(W_1\) and \(W_2\).

For all vertices \(v \in V_2\), \(v\) is only in \(W_2\) and \(W_3\).

For all vertices \(v \in V_4\), \(v\) is only in \(W_1\) and \(W_3\).

Now we are able to define a vertex-Kierstead path \(P\) on \(G\) of length two that is not elementary. Let \(P = \{x_0,v_1,v_2\}\) such that \(v_1 \in V_3\) and \(v_2 \in N(v_1) \cap V_2\). This is illustrated by the blue edges in figure 3.3. Such a path is a vertex-Kierstead path because \(x_0 \in V(W_0) \cap V(W_1)\), \(v_1 \in V(W_1) \cap V(W_2)\), \(v_2 \in V(W_2) \cap V(W_3)\) and \(\varphi(v_1), \varphi(v_2) \in \overline{\varphi}(W_0)\). The color \(i + 1\) is only used on \(x_1\) and \(x_2\), thus \(i + 1 \in \overline{\varphi}(W_1) \cap \overline{\varphi}(W_2)\) and therefore we know that \(P\) is not elementary.

Now we know that there exist graphs with arbitrarily large clique number such that Theorem 3.2 where (L3) is dropped does not hold in case \(s = 1\). We can find such a counterexample for any \(s \geq 1\). When we constructed our \(G\) we started with a clique and we were taking the Mycielskian two times. If we continue taking the Mycielskian and do this \(s + 1\) times in total, we obtain a graph \(G\) with chromatic number \(\omega(G) + s + 1\). Now call the graph before we took the last Mycielskian \(G'\) and the graph before that one \(G''\).

The same family of subsets of \(V(G)\) and path \(P\) as in Figure 3.8 will suffice.

By using the counterexample from above we can easily see that (L1) and (L2) cannot be dropped either. Suppose we only want to drop (L1), then the graph is still a quasi-line (and thus claw-free) graph. We take the same graph \(G\) and path \(P = \{x_0,v_1,v_2\}\) as before, but we define a different family of subgraphs of \(G\). We start with the following subgraphs:

- \(W_0 = G[\{x_0,v_1\}]\),
- \(W_1 = G[\{v_1,v_2\}]\) and
- \(W_2 = G[\{v_2\}]\).

Then we complete our family by adding singleton graphs (i.e. subgraphs containing exactly one vertex) such that all vertices occur in exactly two subgraphs. Obviously, (L2) and (L3) hold and it is easy to check that \(P\) is a vertex-Kierstead path that is not elementary.
Suppose we only want to drop (L2). Again, we take the same graph $G$ and path $P = \{x_0, v_1, v_2\}$. This time our family of subgraphs of $G$ is defined as follows:

- $W_0 = G[\{x_0, v_1\}]$ and
- $W_1, W_2 = G[\{v_1, v_2\}]$.

We complete our family by adding for each edge $e \in E(G)$ the subgraph $G[e]$. It is clear that (L1) and (L3) hold and that $P$ is a vertex-Kierstead path that is not elementary.

We have confirmed what McDonald expected, that Kierstead’s Theorem (and thus Tashkinov’s Theorem) for line graphs cannot be straightforwardly generalized by relaxing one of the structural conditions in Bermond and Meyer’s characterization for line graphs of multigraphs. In the next section we discuss an alternative approach to generalizing Tashkinov’s Theorem, in the spirit of other work which has extended line graph results into quasi-line or claw-free graph results.

### 3.2 The structure of quasi-line graphs

In the last section we saw that Kierstead’s Theorem for line graphs cannot be extended to quasi-line graphs. We have the idea of creating a Tashkinov kind of structure on quasi-line graphs. Maybe we can create a structure such that a result analogous to Tashkinov’s Theorem for line graphs holds. Then we would like to extend this result to quasi-line graphs. As mentioned before, the series of papers by Chudnovsky and Seymour[7] builds a structure theorem for quasi-line graphs. We will now present that theorem and show how it is used to extend results for line graphs to results for quasi-line graphs.

**Theorem 3.4** (Structure Theorem for quasi-line graphs). Any quasi-line graph containing no clique cutset and no nonlinear homogeneous pair of cliques is either a circular interval graph or a composition of linear interval strips.

We have not yet defined clique cutsets, nonlinear homogeneous pairs of cliques, circular interval graphs and compositions of linear interval strips, so we will treat these definitions now. A *clique cutset* of a graph $G$ is a clique $C \subseteq V(G)$ such that $G[V(G) \setminus C]$ consists of more components than $G$. 

39
Definition 3.3. Let $G$ be a graph and $A, B \subseteq V(G)$ cliques. The pair $(A, B)$ is a homogeneous pair of cliques if

- for all $v \in V(G)\setminus (A \cup B)$, $v$ is adjacent to all vertices of $A$ or to none of $A$ and $v$ is adjacent to all vertices of $B$ or to none of $B$,
- $A \cap B = \emptyset$ and
- $|V(A)| + |V(B)| \geq 3$.

Definition 3.4. A graph $G$ is a linear interval graph if

- there exists a function $\psi : V(G) \to \mathbb{R}$ and
- there exists a set of intervals $\mathcal{I}$ on $\mathbb{R}$ such that $u, v \in V(G)$ are adjacent if and only if there is an interval $I \in \mathcal{I}$ such that $\psi(u), \psi(v) \in I$.

![Figure 3.9: A linear interval strip.](image)

A linear interval strip $(S_e, X_e, Y_e)$ is a linear interval graph with vertex set $S_e$ and specified endcliques $X_e$ and $Y_e$. An example is shown in Figure 3.9. A directed graph $H$ is a graph where the edges are directed. An edge of $H$ which starts at $u \in V(H)$ and ends at $v \in V(H)$ is denoted by $u \overset{e}{\rightarrow} v$. A composition of linear interval strips is made as follows:

1. Start with a directed graph $H$ and a set of linear interval strips
   $\{(S_e, X_e, Y_e) \mid e \in E(H)\}$.
2. For each $e = u \overset{e}{\rightarrow} v \in E(H)$ replace $e$ by $S_e$, such that $X_e$ is at $u$ and $Y_e$ is at $v$.
3. For all $v \in V(H)$, let $C_v$ be the following set:

$$C_v = (\bigcup\{V(X_e) \mid e = \overset{e}{\rightarrow} u \text{ for some } u \in V(H)\}) \cup (\bigcup\{V(Y_e) \mid e = \overset{e}{\rightarrow} v \text{ for some } u \in V(H)\})$$

and add edges such that $C_v$ is a clique.
Figure 3.10: Composition of linear interval strips.

Figure 3.10 illustrates the construction of a composition of linear interval strips. The last definition, which follows, is a more advanced case of a linear interval graph.

**Definition 3.5.** A graph $G$ is a circular interval graph if

- there exists a function $\psi : V(G) \rightarrow C$, where $C$ denotes a circle, and
- there exists a set of intervals $\mathcal{I}$ on $\mathbb{R}$ such that $u, v \in V(G)$ are adjacent if and only if there is $I \in \mathcal{I}$ such that $\psi(u), \psi(v) \in I$.

Now we have all the ingredients that are needed to understand Theorem 3.4 and we will give some results in which the Structure Theorem for quasi-line graphs is used.

The first result, due to King and Reed [19], was already mentioned at the beginning of this chapter and will now be formally stated.

**Theorem 3.5.** Let $G$ be a quasi-line graph. Then

$$\chi(G) \leq \lceil \chi_f(G) + 3\sqrt{\chi_f(G)} \rceil.$$
Another result has to do with Reed’s conjecture \cite{29} which states that for any graph $G$ the following holds:

$$\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil.$$ 

Reed’s conjecture was proven for line graphs by King, Reed and Vetta \cite{20}. King and Reed \cite{21} extended this result to quasi-line graphs using the Structure Theorem for quasi-line graphs. Also a local strengthening of Reed’s conjecture holds for quasi-line graphs, Chudnovsky, King, Plumettaz and Seymour \cite{5} proved this using the Structure Theorem for quasi-line graphs.

In Chapter 1 we saw Shannon’s Theorem which states $\chi'(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$ for any multigraph $G$. An immediate consequence is that for a line graph $G$, $\chi(G) \leq \frac{3}{2}\omega(G)$. Chudnovsky and Ovetsky \cite{6} extended this to the following theorem, again by using the Structure Theorem for quasi-line graphs.

**Theorem 3.6.** Let $G$ be a quasi-line graph. Then

$$\chi(G) \leq \frac{3}{2}\omega(G).$$

They also provided an infinite family of quasi-line graphs that are not line graphs for which the bound in Theorem 3.6 is tight. We will also give this family. Let $k > 0$ be an even integer and let $G$ be a circular interval graph with $|V(G)| = 3k - 1$ and such that for every $k$ consecutive vertices on the circle there is an interval containing precisely these $k$ vertices. Clearly, the clique number is equal to $k$ and $\alpha(G) = 2$, since for any three vertices there must be two of them that are less than $k - 1$ apart on the circle. Thus

$$\chi(G) \geq \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil = \left\lceil \frac{3k - 1}{2} \right\rceil \geq \frac{3}{2}\omega(G).$$

The results we just gave were all giving an upper bound for the chromatic number of quasi-line graphs. They all used the Structure Theorem for quasi-line graphs in a similar way which we will now discuss.

Suppose that we have an upper bound $x$ for the chromatic number of line graphs and we want to prove that this upper bound also holds for quasi-line graphs. We will show that it is only needed to prove $\chi(G) \leq x$ for each circular interval graph $G$.

Let $G$ be a minimal counterexample. Thus $G$ is a quasi-line graph with $\chi(G) > x$. We will show that $G$ cannot contain a clique cutset or a nonlinear homogeneous pair of cliques and $G$ is not a composition of linear strips.
Then, by Theorem 3.4 we know that $G$ is a circular interval graph and therefore we only need to prove $\chi(G) \leq x$ for each circular interval graph $G$.

Suppose $G$ contains a clique cutset $C$. For each component $H$ of $G[V\setminus C]$ we color the vertices of $G_H := G[V(H) \cup N_G(V(H))]$ with $x$ colors. This is possible since $G$ is the smallest graph with $\chi(G) > x$. Then, we independently permute colors if necessary such that for each coloring and each vertex $v \in C$, $v$ is colored with the same color. This way, $G$ can be colored with $x$ colors, which is a contradiction.

Suppose $G$ contains a nonlinear homogeneous pair of cliques. The following lemma, which is used by King and Reed [19], tells us straight away that $G$ cannot be a smallest counterexample.

**Lemma 3.1.** Let $G$ be a quasi-line graph on $n$ vertices containing a nonlinear homogeneous pair of cliques $(A, B)$. We can find a strict subgraph $G'$ of $G$ such that $G'$ is a quasi-line graph, $\chi(G') = \chi(G)$ and given a $k$-vertex-coloring of $G'$ we can find a $k$-vertex-coloring of $G$.

Suppose $G$ is a composition of linear interval graphs. We will have a closer look at compositions of linear interval graphs and we will see that if $G$ is not a line graph, then it must admit a canonical interval 2-join.

**Definition 3.6.** A graph, $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ is an interval 2-join if

- $V(G) = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$,
- $X_1, X_2, Y_1, Y_2$ are cliques, $X_1, Y_1 \subseteq V_1$ and $X_2, Y_2 \subseteq V_2$,
- for $v_1 \in V_1$ and $v_2 \in V_2$, $\{v_1, v_2\} \in E(G)$ if and only if $v_1 \in X_1$ and $v_2 \in X_2$ both hold or $v_1 \in Y_1$ and $v_2 \in Y_2$ both hold, and
- $(V_2, X_2, Y_2)$ is a linear interval strip.

If $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ is an interval 2-join and $X_2 \cap Y_2 = \emptyset$ then it is called a canonical interval 2-join. We say that a graph $G$ admits an interval 2-join if there exists an induced subgraph of $G$ that has an interval 2-join.

Now we return to our minimal counterexample $G$ which we assumed is a composition of linear interval strips. If for each strip $(S_e, X_e, Y_e)$, $X_e = Y_e$ then $G$ is a clique and thus a line graph. So, if $G$ is not a line graph, there exists a strip $(S_e, X_e, Y_e)$ such that $X_e \neq Y_e$. Let $e = \{u, v\}$, $X_1 = C_u \setminus X_e$, $X_2 = C_v \setminus X_e$, $Y_1 = X_e \cap C_u$, $Y_2 = X_e \cap C_v$.
\[ Y_1 = C_v \setminus Y_e \text{ and } C = X_e \cap Y_e. \text{ Then } ((V(G) \setminus (X_e \cup Y_e)) \cup C, X_1 \cup C, Y_1 \cup C), (S_e \setminus C, X_e \setminus C, Y_e \setminus C)) \text{ is a canonical interval 2-join of } G[C_u \cup S_e \cup C_v], \text{ see Figure 3.12.} \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_12.png}
\caption{Illustration of a canonical interval 2-join of \( G[C_u \cup S_e \cup C_v] \).}
\end{figure}

We showed that \( G \) admits a canonical interval 2-join and by Lemma 3.2, which is proven by King [18], we know that \( G \) cannot be a smallest counterexample.

\textbf{Lemma 3.2.} \textit{Let} \( G \) \textit{be a quasi-line graph on} \( n \) \textit{vertices that admits a canonical interval 2-join} \( ((V_1, X_1, Y_1), (V_2, X_2, Y_2)) \). \textit{Given a} \( k \)-\textit{vertex-coloring of} \( G[V_1] \), \textit{we can find a} \( k \)-\textit{vertex-coloring of} \( G \).

We conclude that \( G \) must be a circular interval graph. So, if we have an upper bound \( x \) for the chromatic number of line graphs and we can prove that \( x \) is also an upper bound for the chromatic number of circular interval graphs, then we immediately know that \( x \) is an upper bound for the chromatic number of quasi-line graphs.

At last, we return to our idea of creating a Tashkinov tree kind of structure on quasi-line graphs. If we want to use the structure theorem it makes sense trying to find a feasible structure on circular interval graphs. To start, one
could first focus on finding a structure such as a vertex-Kierstead path. Here we give a suggestion for such a structure:

Let $G$ be a quasi-line graph and $\varphi$ a partial vertex-coloring. Let $P = (v_1, ..., v_n)$ and $W_0, ..., W_n$ subgraphs such that the following hold.

(K1) $v_1$ is uncolored and $v_1, ..., v_n$ are distinct,

(K2) $W_0 = \{v_1\}$,

(K3) $W_i = \{v_i\} \cup N(v_i)$ for all $1 \leq i \leq n$,

(K4) $v_i \in W_{i-1} \cap W_i$ for all $1 \leq i \leq n$ and

(K5) for all $1 < j \leq n$, there is an $i < j$ such that $\varphi(v_j) \in \varphi(W_i)$.

Then $P = (v_1, ..., v_n)$ is elementary if for all $1 < i, j \leq n$ such that $i \neq j$ the following holds: $\varphi(W_i) \cap \varphi(W_j) = \emptyset$.

We definitely think it could be interesting to continue this search for a feasible structure on quasi-line graphs. And maybe the Structure Theorem for quasi-line graphs can be of use once a good structure is found.
Discussion

In Chapter 1 we considered the Goldberg–Seymour conjecture. Let us see next what happens if we transform this conjecture to the line graph setting, as we did for Tashkinov trees in Chapter 3. Recall the conjecture.

**Goldberg–Seymour conjecture.** For any multigraph $G$

$$
\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta(G) + 1\},
$$

where the density $\rho(G)$ is

$$
\rho(G) = \max \left\{ \frac{2|E(G[S])|}{|S|-1} \mid S \subseteq V(G), |S| \geq 3 \text{ odd} \right\}.
$$

Remember that $\lceil \rho(G) \rceil$ is a natural lower bound for the chromatic index of a graph $G$, as discussed in Chapter 1. In the same way a lower bound for the chromatic number can be obtained. In Lemma 1.1 we saw that $\frac{V(G)}{\alpha(G)}$ is a lower bound for the chromatic number of a graph $G$. Clearly, this also holds for each subgraph of $G$ and therefore

$$
\rho'(G) = \max_{S \subseteq V} \left\{ \frac{|S|}{\alpha(G[S])} \right\}
$$

is a lower bound for the chromatic number of $G$. The chromatic number is an integer and thus

$$
\chi(G) \geq \lceil \rho'(G) \rceil.
$$

Now an equivalent statement of the Goldberg–Seymour conjecture is as follows.

**Conjecture 3.1.** For any line graph $G$

$$
\chi(G) \leq \max\{\lceil \rho'(G) \rceil, \omega(G) + 1\}.
$$

Next we give an even more comprehensive statement, the truth of which can also be questioned.
**Question 3.1.** For any claw-free graph $G$, does the following inequality hold?

$$\chi(G) \leq \max\{\lceil \rho'(G) \rceil, \omega(G) + 1 \}.$$ 

For the answer to Question 3.1 to be yes, we need

$$\chi_f(G) \leq \max\{\lceil \rho'(G) \rceil, \omega(G) + 1 \},$$

since $\chi_f(G) \leq \chi(G)$. In Chapter 1 we mentioned the fact

$$\chi'_f(G) = \max\{\Delta(G), \rho(G)\},$$

which follows from Edmonds’ matching polytope Theorem [8]. An immediate consequence is $\chi_f(G) = \max\{\rho'(G), \omega(G)\}$ for $G$ a line graph. Perhaps this can be proven for quasi-line graphs using the Structure Theorem from Section 3.4. Since the fact $\chi'_f(G) = \max\{\Delta(G), \rho(G)\}$ is used to prove an asymptotic version of the Goldberg–Seymour conjecture, proving $\chi_f(G) = \max\{\rho'(G), \omega(G)\}$ for claw-free graphs could also help in an asymptotic version of Question 3.1. We already saw in Section 3.2 that King and Reed [19] proved that the chromatic number and the fractional chromatic number asymptotically agree.

Edmonds’ matching polytope gives an explicit description of the convex hull of characteristic vectors of matchings. As mentioned in Chapter 1, Kahn [16] proved an asymptotic version of the Goldberg–Seymour conjecture, by using $\chi'_f(G) = \max\{\Delta(G), \rho(G)\}$ which follows from Edmonds’ matching polytope. Edmonds’ matching polytope can be transformed to an explicit description of the stable set polytope for line graphs. But generalizing this to the stable set polytope for claw-free graphs is an open problem. Although there have been attempts which used the structure theorem of last section, there is not even a corresponding conjecture. There is a conjecture for the quasi-line graph case due to Rebea [28]. This conjecture was partially proven by Chudnovsky and Seymour [7]. The missing piece, which is about fuzzy circular interval graphs, was proven by Eisenbrand, Oriolo, Stauffer and Ventura [9]. So, in linear optimization where a lot is known about the matching polytope there is still a lot to find out about the stable set polytope. Also this different approach to the Goldberg–Seymour conjecture (conjecture 3.1) could be explored more.

Proving that the answer to Question 3.1 is yes might be too much to ask for, since it implies the Goldberg–Seymour conjecture. So we made a small attempt to find a counterexample for Question 3.1. In order to find such a counterexample, we could try to find a graph $G$ such that
• $G$ is a claw-free graph but not a line graph,
• $\chi(G) \geq \omega(G) + 2$ and
• $\frac{V(G)}{\alpha(G)}$ small.

Using the Mycielskian construction appeared to be successful in Section 3.1, but here it does not work since it generates claws. The family of circular interval graphs provided after Theorem 3.6 is also not satisfying. But perhaps Beineke’s Theorem can be of use here. Making compositions of forbidden subgraphs could be a way to create a counterexample for Question 3.1 and is worth to investigate.
Conclusion and suggestions

Tashkinov trees are of great use in results towards the Goldberg–Seymour conjecture. Tashkinov’s Theorem is essential when using Tashkinov trees and its proof is given in detail. Tashkinov trees were easy to transform to vertex-Tashkinov trees on line graphs, and this was also the case for Tashkinov’s Theorem. Then the idea of extending Tashkinov’s Theorem for line graphs to Tashkinov’s Theorem for quasi-line and claw-free graphs arose. A straightforward approach to prove Kierstead’s Theorem for quasi-line graphs was halted by giving counterexamples. We concluded that in order to create a theorem such as Tashkinov’s Theorem for quasi-line graphs, a different structure is needed. We suggest finding such a structure as future work and gave a possible structure.

The Goldberg–Seymour conjecture was also stated in an equivalent version for line graphs. Then we stated an even stronger question.

**Question 3.2.** For any claw-free graph $G$, does the following inequality hold?

$$\chi(G) \leq \max\{\lceil \rho'(G) \rceil, \omega(G) + 1\}.$$ 

In the Discussion we gave a suggestion for how to find a counterexample for Question 3.2 and it might be interesting to resolve this.
Bibliography


