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Teleportation of quantum states and entangled state pairs of arbitrary dimensions

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Abstract

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Teleportation of qubits is proposed in the well known paper of Bennet *et al.* 1993. It is shown how quantum states may be teleported without physically moving it or infringing the no-cloning theorem. This report explores generalizations of the original teleportation paper. Multiple qudit teleportation is described and furthermore it is worked out how higher dimensional qubits are teleported. Beyond this, it is shown that qudit teleportation is not the only variation to be done. As there are many variations, one variation is shown where an entangled Bell state pair shared between two parties Alice and Bob is teleported to two different parties, say Charlie and Dan. From here on, many more variations are imaginable.

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Contents

List of Figures

List of Abbreviations

BSM	**B**ell **S**tate **M**easurement
CNOT	**C**ontrolled **NOT**
EPR	**E**instein-**P**odolsky-**R**osen
GHZ	**G**reenberger-**H**orne-**Z**eilinger
POVM	**P**ositive **O**perator **V**alued **M**easure
QC	**Q**uantum **C**ryptography
QKD	**Q**uantum **K**ey **D**istribution

Chapter 1

Introduction

1.1 Motivation

With the development of quantum mechanics a unit of information storage was introduced. The so called qubit is the quantum analog to a classical bit which is quiet different from its classical variant. First of all the qubit may be a superposition of its states. A unit of information is no longer only zero or one, true or false or yes or no. It may be a combination of its possible states.

Furthermore as a consequence from quantum mechanics we know that measuring a quantum bit (qubit) will set the state of this quantum bit. Consequently, it is easily proven that one cannot simply clone one qubit to another qubit. This so called no-cloning theorem may sound as a restriction. However, using qubits another phenomenon is available. Quantum teleportation will allow us to transfer information to another location without physically moving it.

Quantum teleportation has advantages in information security. The state to be teleported cannot be measured directly. The teleportation protocol lets you create a combined state with the location where to teleport to and therefore changes the original state. However, this new state is used to get information on how to decode the original information from the used entangled state on the new location.

Protocols are worked out on how to teleport simple qubits from one location to another using entangled pairs of qubits. These protocols are well described in two-dimensional Hilbert space. This allows for teleportation of two-dimensional qubits. The aim of this project is to get a description on how to teleport quantum information in a higher-dimensional Hilbert space. This generalization should work for the case of multiple two-dimensional qubits or the case of a single d-dimensional bit of quantum information (qudit).

1.2 Background information

Theoretical background
First research on quantum computing already started in the 70s of the twentieth century. Published in 1973, Holevo's theorem states an upper bound on the amount of information inside a quantum state (Holevo, 1973). Late

twentieth century theory came into practice with one of the first working preliminary quantum computers. On Oxford University a real two qubit system was used to solve problems using quantum mechanics (Jones, Mosca, and Hansen, 1998).

Bennett et al. (1993) were the first to propose a protocol on transferring quantum information without physically replacing it. By this time, it was already known that information cannot travel instantaneously from one place to another. Furthermore, it was known that a quantum state cannot be copied. In the paper of Bennet *et al.* a new method was introduced to be able to transfer information using quantum mechanical phenomena.

From here, research groups came up with several variations on the original proposed teleportation scheme by Bennett et al. (1993). During the twenty first century the known teleportation protocols were generalized from single-state to multistate teleportation using entangled state quantum channels (Yang and Guo, 2000; Lee, Min, and Oh, 2002; Zhang, 2006).

Protocols proposed in these early years were mostly mathematical descriptions of quantum teleportation. It were Brassard, Braunstein, and Cleve (1998) who proceeded first with a description of a quantum circuit proficient for single qubit teleportation. This circuit consisting of quantum gates is used and named after Brassard.

Experimental timeline
In the period Brassard, Braunstein, and Cleve (1998) introduced their quantum circuit for teleportation, the first quantum teleportation experiments succeeded (Bouwmeester et al., 1997; Boschi et al., 1998). The teams implemented quantum teleportation with the use of polarization states of photons.

Meanwhile, around the same time quantum teleportation became more prominent, a subgroup of Quantum Cryptography (QC), namely Quantum Key Distribution (QKB), became prominent. Many experiments in QKB are using photons for long range quantum communication. Photons however, are limited by there fiber/photon loss. Last year, Korzh et al. (2015) broke the quantum communication distance record of 307km using photons.

As a consequence of the no-cloning theorem, to achieve long distance quantum communication, quantum repeaters are required to extend the limited distance when using photons (Simon et al., 2010). Quantum repeaters use quantum teleportation to divide a given long distance into separable shorter parts. Different types of quantum repeaters may use photons and/or matter depending on their way of using quantum memories and their performance (Simon et al., 2010).

Years after the first teleportation with photons, teleportation between light and matter (Sherson et al., 2006) and teleportation between single ions has been shown (Olmschenk et al., 2009).

D-dimensional teleportation
Quantum teleportation experiments until today have not brought any results in teleportation of higher dimensional quantum states. Theoretically, few people have come up with methods of teleporting higher dimensional quantum states (Goyal and Konrad, 2013; Goyal et al., 2014).

1.3 Scientific questions

Although quantum teleportation have been demonstrated several times over the last two decades, these experiments were limited to two dimensions. Due to the technical limitations in quantum communication, low transmission rates are an obstruction in the communicating using quantum states (Goyal and Konrad, 2013). As QC is focussing on enlarging the quantum communication distance, how the communication may be increased or worked around are important unanswered questions.

Extending the experimental two-dimensional quantum teleportation used today to higher dimensions may be an answer to the lower transmission rates. Other questions arise when extending to d-dimensional teleportation. What will this do to the teleportation features. I.e., will transforming multiple qubits into a single dimensional qudit improve its teleportation fidelity and how will the total information transfer time improve when using higher dimensional qudits?

Chapter 2

Higher dimensional quantum state teleportation

2.1 Single qubit teleportation

Teleportation was first introduced by Bennett et al. (1993). They came up with a method on information transfer within the limitations of quantum mechanics described as a conceptual and mathematical exercise. In their description of teleportation they made use of long range correlations between quantum states debated by and named after Einstein, Podolsky, and Rosen (1935): Einstein-Podolsky-Rosen (EPR) pairs.

Bennett et al. (1993) described first preparing an EPR singlet state between two particles which will act as a quantum channel $|\Psi^-\rangle_{23}$ and the quantum mechanical information to be teleported $|\psi\rangle_1$.

$$|\psi\rangle_1 = a\,|0\rangle_1 + b\,|1\rangle_1 \tag{2.1}$$

$$\left|\Psi^-\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(|01\rangle_{23} - |10\rangle_{23}\right) \tag{2.2}$$

As the no-cloning theorem tells us, we cannot measure the state $|\psi\rangle$ and copy it elsewhere. Braunstein, Mann, and Revzen (1992) introduced the measurement in the Bell operator basis which is the basis of Bell states, Bell State Measurement (BSM) (appendix A).

To see how the system transforms after a Bell State Measurement M, one example is given.

$$|\Lambda\rangle_{123} = |\psi\rangle_1 \otimes \left|\Psi^-\right\rangle_{23} \tag{2.3}$$

$$M^{\Phi^+}|\Lambda\rangle_{123} = \frac{|\Phi^+\rangle_{12}\langle\Phi^+|}{\sqrt{\langle\Phi^+|\Lambda\rangle_{12}\langle\Lambda|\Phi^+\rangle}}\,|\Lambda\rangle_{123} \tag{2.4}$$

$$\begin{aligned} &= \frac{1}{1/2}\frac{1}{2}\left|\Phi^+\right\rangle_{12}[a\,\langle 00|00\rangle_{12}\otimes|1\rangle_3 - a\,\langle 00|01\rangle_{12}\otimes|0\rangle_3 + \\ &\quad b\,\langle 00|10\rangle_{12}\otimes|1\rangle_3 - b\,\langle 00|11\rangle_{12}\otimes|0\rangle_3 \\ &\quad a\,\langle 11|00\rangle_{12}\otimes|1\rangle_3 - a\,\langle 11|01\rangle_{12}\otimes|0\rangle_3 \\ &\quad b\,\langle 11|10\rangle_{12}\otimes|1\rangle_3 - b\,\langle 11|11\rangle_{12}\otimes|0\rangle_3] \end{aligned} \tag{2.5}$$

$$= \left|\Phi^+\right\rangle_{12}\otimes\left(a\,|1\rangle - b\,|0\rangle\right)_3 \tag{2.6}$$

Clearly, (2.6) shows with a possibility of $\frac{1}{4}$, that the $|\Phi^+\rangle_{12}$ Bell state outcome gives one a quantum state relative to $|\psi\rangle$ on the third particle. After one applies a certain unitary operation onto it, in this situation $U_{\Phi+} = \sigma_z\sigma_x$, the quantum state $|\psi\rangle$ is projected onto the third particle. This procedure can be done for all of the possible BSM outcomes, hence

$$\begin{aligned}|\Lambda\rangle_{123} = \frac{1}{2}\big[&\left|\Phi^+\right\rangle_{12} \otimes (-b\,|0\rangle + a\,|1\rangle)_3 + \left|\Phi^-\right\rangle_{12} \otimes (b\,|0\rangle + a\,|1\rangle)_3 + \\ &\left|\Psi^+\right\rangle_{12} \otimes (-a\,|0\rangle + b\,|1\rangle)_3 + \left|\Psi^-\right\rangle_{12} \otimes (-a\,|0\rangle + -b\,|1\rangle)_3\big] \quad (2.7)\end{aligned}$$

Clearly, each of the four outcomes on the measurement joint system of qubits 1 and 2 have equal probability. This outcome is classically transferred from Alice to Bob. With this information, Bob knows what unitary operation to apply to get the state of $|\psi\rangle$ on his qubit.

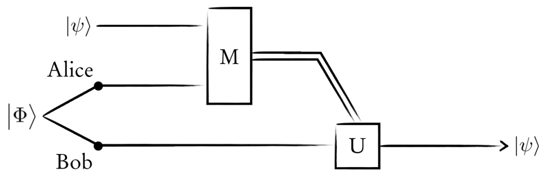

FIGURE 2.1: An abstract diagram of the teleportation protocol described. Quantum states are presented by the single lines and classical information is represented by the double lines.

All unitary operations belonging to a certain measurement are given:

$$\frac{|\Phi^+\rangle_{12}\langle\Phi^+|}{\sqrt{\langle\Phi^+|\Lambda\rangle_{12}\langle\Lambda|\Phi^+\rangle}}\,|\Lambda\rangle_{123} = \left|\Phi^+\right\rangle_{12} \otimes (a\,|1\rangle - b\,|0\rangle)_3$$
$$U_{\Phi+} = \sigma_z\sigma_x \quad (2.8)$$

$$\frac{|\Phi^-\rangle_{12}\langle\Phi^-|}{\sqrt{\langle\Phi^-|\Lambda\rangle_{12}\langle\Lambda|\Phi^-\rangle}}\,|\Lambda\rangle_{123} = \left|\Phi^-\right\rangle_{12} \otimes (a\,|1\rangle + b\,|0\rangle)_3$$
$$U_{\Phi-} = \sigma_x \quad (2.9)$$

$$\frac{|\Psi^+\rangle_{12}\langle\Psi^+|}{\sqrt{\langle\Psi^+|\Lambda\rangle_{12}\langle\Lambda|\Psi^+\rangle}}\,|\Lambda\rangle_{123} = \left|\Psi^+\right\rangle_{12} \otimes (-a\,|0\rangle + b\,|1\rangle)_3$$
$$U_{\Psi+} = -\sigma_z \quad (2.10)$$

$$\frac{|\Psi^-\rangle_{12}\langle\Psi^-|}{\sqrt{\langle\Psi^-|\Lambda\rangle_{12}\langle\Lambda|\Psi^-\rangle}}\,|\Lambda\rangle_{123} = \left|\Psi^-\right\rangle_{12} \otimes (-a\,|0\rangle - b\,|1\rangle)_3$$
$$U_{\Psi-} = -I \quad (2.11)$$

2.2 Qudit teleportation

After stating the teleportation scheme for a two dimensional qubit, Bennett et al. (1993) briefly noted it is possible to generalize the protocol. The state teleported $|\psi\rangle$ is taken to a higher dimension $d > 2$ and used together with two entangled particles with $d > 2$ states, in the generalized Bell state.

Hence,

$$|\psi\rangle_1 = \sum_{i=0}^{d-1} c_i \, |i\rangle \tag{2.12}$$

$$|\Psi\rangle_{23} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle \tag{2.13}$$

$$|\Lambda\rangle_{123} = |\psi\rangle_1 \otimes |\Psi\rangle_{23} \tag{2.14}$$

Similar to the two-dimensional case, see how to system transforms after measuring in the generalized Bell state on the d-dimensional qubits (qudits) 1 and 2. This joint measurement will be the eigenvalue of eigenstates which may be written as

$$|\Phi_{nm}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i j n/d} \, |j\rangle \otimes |j \oplus m\rangle \tag{2.15}$$

Hence

$$M_{12}^{\Phi_{mn}} \, |\Lambda\rangle_{123} = \frac{|\Phi_{nm}\rangle_{12} \langle \Phi_{nm}|}{\sqrt{\langle \Phi_{nm}|\Lambda\rangle_{12} \langle \Lambda|\Phi_{nm}\rangle}} \tag{2.16}$$

$$= \frac{1}{1/d} \, |\Phi_{nm}\rangle_{12} \left[\sum_{j,k,l=0}^{d-1} \frac{c_k}{d} e^{2\pi i j n/d} \; {}_{12}\langle j, j \oplus m | k, l, l\rangle_{123} \right] \tag{2.17}$$

$$= |\Phi_{nm}\rangle_{12} \, \frac{d}{d} \sum_{j,k,l=0}^{d-1} e^{2\pi i j n/d} c_k \, |l\rangle_3 \, \delta_{j,k} \delta_{j \oplus m, l} \tag{2.18}$$

$$= |\Phi_{nm}\rangle_{12} \sum_{j=0}^{d-1} e^{2\pi i j n/d} c_j \, |j \oplus m\rangle_3 \tag{2.19}$$

The classical information to be sent in this generalized situation will therefore be the found $\{n, m\}$ classical dit value. These dits $\{n, m\}$ are used to determine the unitary operation U on Bob's qudit of the entangled pair.

To eventually retrieve the $|\psi\rangle$ value on the third qudit following (2.19), one applies the unitary operator

$$U_3^{n'm'} M_{12}^{\Phi_{mn}} \, |\Lambda\rangle_{123} = \sum_{j,k=0}^{d-1} e^{2\pi i k (n-n')/d} c_j \left| k \oplus m' \right\rangle \left\langle k \middle| j \oplus m \right\rangle_3 \tag{2.20}$$

$$= \sum_{j,k=0}^{d-1} c_j \, |j\rangle_3 \, \delta_{n,n'} \delta_{k,k \oplus m} \delta_{k \oplus m', j} \tag{2.21}$$

$$= |\psi\rangle_3 \tag{2.22}$$

To clarify, the unitary operator U in terms of the classical dits $\{n, m\}$ is

$$\begin{aligned} U^{nm} &= \sum_{k=0}^{d-1} e^{-2\pi ikn/d} \, |k \oplus (d-m)\rangle\langle k| \\ &= \sum_{k=0}^{d-1} e^{-2\pi ikn/d} \, |k \ominus m\rangle\langle k| \end{aligned} \tag{2.23}$$

2.3 Generalized quantum channel

The introduction of information teleportation within the rules of quantum mechanics created many opportunities in fields such as quantum dense coding, a technique using quantum mechanics to compress data (Mattle et al., 1996). The protocol, however, describes the use of the state to be teleported and the state acting as a quantum channel within a Hilbert space of same dimensions. To further improve the teleportation protocol, we could generalize it by using a d-dimensional quantum channel to teleport a single f-dimensional qudit ($d > f$).

Suppose Alice and Bob share an entangled state $|\Psi\rangle$ just as before in a d-dimensional Hilbert space. Furthermore, Alice has a second d-dimensional qudit in state $|\psi\rangle$.

$$|\psi\rangle_1 = \sum_{i=0}^{f-1} a_i \, |i\rangle_1 \, , \qquad \sum_i |a_i|^2 = 1 \tag{2.24}$$

$$|\Psi\rangle_{23} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_2 \otimes |j\rangle_3 \tag{2.25}$$

The total system of the generalized quantum channel and qudit combined is therefore written as

$$\begin{aligned} |\Lambda\rangle_{123} &= |\psi\rangle_1 \otimes |\Psi\rangle_{23} \\ &= \frac{1}{\sqrt{d}} \sum_{i=0}^{f-1} \sum_{j=0}^{d-1} a_i \, |i\rangle_1 \otimes |jj\rangle_{23} \end{aligned} \tag{2.26}$$

You-Bang et al. (2010) described this same system acts in a teleportation scheme for an arbitrary quantum channel, however, in a less general case. Following the usual procedure, Alice now performs a generalized Bell state measurement on her particles 1 and 2.

$$|\Psi_{kl}\rangle_{12} = \frac{1}{\sqrt{f}} \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} e^{2\pi il\beta/d} \, |\alpha\rangle_1 \otimes |\beta \oplus k\rangle_2 \tag{2.27}$$

The total system after the measurement M may be written as

$$M = \frac{|\Psi_{kl}\rangle_{12}\langle\Psi_{kl}|}{\sqrt{\langle\Psi_{kl}|\Lambda\rangle_{12}\langle\Lambda|\Psi_{kl}\rangle}} \tag{2.28}$$

$$\begin{aligned} M\,|\Lambda\rangle_{123} &= \frac{1}{1/\sqrt{fd}} \left(\frac{1}{\sqrt{f}} \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} e^{2\pi i l\beta/d} \; {}_1\langle\alpha| \otimes \; {}_2\langle\beta \oplus k| \right) \times \\ & \quad \left(\frac{1}{\sqrt{d}} \sum_{\gamma=0}^{f-1} \sum_{\delta=0}^{d-1} a_\gamma \, |\gamma\rangle_1 \otimes |\delta\delta\rangle_{23} \right)^1 \end{aligned} \tag{2.29}$$

$$= \frac{\sqrt{fd}}{\sqrt{fd}} \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} \sum_{\gamma=0}^{f-1} \sum_{\delta=0}^{d-1} a_\gamma e^{2\pi i l\beta/d} \, \langle\alpha|\gamma\rangle_1 \, \langle\beta \oplus k|\delta\rangle_2 \, |\delta\rangle_3 \tag{2.30}$$

$$= \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} \sum_{\gamma=0}^{f-1} \sum_{\delta=0}^{d-1} a_\gamma e^{2\pi i l\beta/d} \delta_{\alpha,\gamma} \delta_{\beta\oplus k,\delta} \, |\delta\rangle_3 \tag{2.31}$$

$$= \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} a_\alpha e^{2\pi i l\beta/d} \, |\beta \oplus k\rangle_3 \tag{2.32}$$

$$= \left|\psi'_{kl}\right\rangle_3 \tag{2.33}$$

After measurement, Alice sends Bob the information k and l, so Bob may perform the unitary operation $U_{k'l'}$ (equation 2.20) to eventually obtain the state $|\psi\rangle_3$.

$$\begin{aligned} |\psi\rangle &= U_{k'l'} \left|\psi'_{kl}\right\rangle_3 \\ &= \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} \sum_{\varepsilon=0}^{d-1} a_\alpha e^{2\pi i(l\beta + l'\varepsilon)/d} \left|\varepsilon \oplus k'\right\rangle\!\left\langle\varepsilon\middle|\beta \oplus k\right\rangle \end{aligned} \tag{2.34}$$

$$= \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} a_\alpha \delta_{l\cdot\beta,-l'\cdot\varepsilon} \delta_{\varepsilon,\beta\oplus k} \left|\beta \oplus (k' + k)\right\rangle \tag{2.35}$$

$$= \sum_{\alpha=0}^{f-1} \sum_{\beta=0}^{d-1} a_\alpha \, |\alpha\rangle \, \delta_{\alpha,\beta\oplus(k'+k)} \tag{2.36}$$

$$= |\psi\rangle_3 \tag{2.37}$$

From the kronicker delta's in equations (2.35) and (2.36), the relation between k' and k, and l' and l follow, hence $k' = d - k$ and $l' = -l$. Hence the unitary operation U_{kl}, Bob has to perform is

$$U_{kl} = \sum_{\varepsilon=0}^{d-1} e^{-2\pi i l\varepsilon/d} \, |\varepsilon \ominus k\rangle\langle\varepsilon| \tag{2.38}$$

Summarizing, this generalization seems similar to the qudit teleportation (§2.2). Note, how the unitary operation Bob has to perform on its qudit, works on a f-dimensional subspace. More importantly, we implicitly assumed the quantum channel's Hilbert space greater than the quantum

[1] Please, note $(d > f)$

information to be teleported. Let's stress that a teleportation protocol will not work for a system where $d < f$.

2.4 Two qubit teleportation

Lee, Min, and Oh (2002) worked out how the teleportation scheme changes if extended from one two-dimensional qubit to a two-qubit teleportation. They stated how a perfect teleportation requires the maximally entangled quantum channel $|\Psi\rangle$ of two pairs of qubits. In this situation, Alice wants to teleport two qubits U_1 and U_2 to Bob. This composition is written as

$$|\psi\rangle_U = \sum_{i,j=0}^{1} c_{ij} |i\rangle_{U_1} \otimes |j\rangle_{U_2} \tag{2.39}$$

The protocol uses two maximally entangled Bell states shared between Alice and Bob, $|\varphi\rangle_{AB}$. Nielsen and Chuang (2000) have shown how a quantum gate on two of the four qudits transform the two Bell states to an inseparable quantum channel $|\Psi\rangle_{AB}$.

$$|\varphi\rangle_{AB} = \left(\frac{1}{\sqrt{2}} \sum_{i=0}^{1} |ii\rangle_{A_1B_1}\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{j=0}^{1} |jj\rangle_{A_2B_2}\right) \tag{2.40}$$

$$= \frac{1}{2} \sum_{i,j=0}^{1} |ij\rangle_{A_1A_2} \otimes |ij\rangle_{B_1B_2} \tag{2.41}$$

The transformation to an inseparable quantum channel is done with the use of two nonlocal unitary operations $\hat{U}$ and $\hat{V}$.

$$|\Psi\rangle_{AB} = \frac{1}{2} \sum_{i,j=0}^{1} |\phi_{ij}\rangle_A \otimes |\varphi_{ij}\rangle_B \tag{2.42}$$

$$= \frac{1}{2} \sum_{i,j=0}^{1} \hat{U} |ij\rangle_{A_1A_2} \otimes \hat{V} |ij\rangle_{B_1B_2} \tag{2.43}$$

$$= \left(I_A \otimes \hat{V}\hat{U}^T\right) |\varphi\rangle_{AB} \tag{2.44}$$

Lee, Min, and Oh (2002) state that a maximally entangled pure state must satisfy the relation, to which the above case of two EPR qubit pairs as a quantum channel suffice:

$$\mathrm{Tr}_{B(A)} \left(|\Psi\rangle_{AB}\langle\Psi|\right) = \frac{1}{4} I_{A(B)} \tag{2.45}$$

Therefore, the quantum channel of two maximal entangled qubit pairs may be used. Another possibility may be the generalized Greenberger-Horne-Zeilinger (GHZ) state of four qubits

$$|\Phi_4\rangle_{AB} = \sum_{i=0}^{1} \lambda_i |\alpha_i\rangle_{A_1} \otimes |\beta_i\rangle_{A_2} \otimes |\gamma_i\rangle_{B_1} \otimes |\delta_i\rangle_{B_2} \tag{2.46}$$

The density matrix of the quantum channel used may tell more about entanglement of the given system. The partial trace over the density matrix of a given subsystem tells more about the other subsystems within the combined system. Tracing over qubit A in a system combining qubits A and B tells us about the properties over qubit B. However, in an entangled system, the subsystems are correlated. When tracing out information about the one subsystem, this leads to a result without any information on the other subsystem. This is shown in the requirement of (2.45). More on this in §4.2 and §4.3.

With this four qubit GHZ description we may see that this system is not sufficient for teleportation. If one takes the partial trace of the density matrix of this system, so measures for example the qubits A_1 and A_2, one may see how information about the other subsystem is not proportional to I_B, but to a projector into subspace $\{|\alpha_i, \beta_i\rangle_A\}$.

$$\mathrm{Tr}_B\left(|\Phi_4\rangle_{AB}\langle\Phi_4|\right) = \sum_{i=0}^{1} |\lambda_i|^2 |\alpha_i, \beta_i\rangle_A \langle\alpha_i, \beta_b| \tag{2.47}$$

2.5 Multiparticle teleportation

It is seen that two qubits can not be teleported from Alice to Bob with the use of four qubits in GHZ state. Multiparticle teleportation is possible and is described by Yang and Guo (2000). In essence, this teleportation protocol is not different from the original described in §2.1. It is described how N-qubits are teleported from Alice to Bob using N ERP pairs as quantum channels.

Alice owns all qubits to be teleported $\{|\psi_i\rangle_1\}, \psi_i \in \{0,1\}$. As, for this explanation, it is not relevant to describe all Bell states we say all N quantum channels are prepared in the singlet state, $|\Psi^-\rangle_{23}$, by Alice.

After preparation of the quantum channel, Alice sends one of the two particles to Bob. Following, Alice does her measurement on the combined ERP particle and her first state to be teleported $|\psi_1\rangle_1$ as see did in §2.1. The result is sent to Bob using a classical channel and Bob performs his corresponding unitary operation. At this point Bob has to take additional action. He sends back a bit of classical information to inform Alice he is done with its procedure. After receiving this conformation, Alice repeats the process for particle $i = 2$. The teleportation thus requires the use of an additional classical bit of information for every qubit but the last one, $N - 1$.

In line with multiparticle generalization, Zhang (2006) worked out a mathematical description of how to teleport N qudit states from Alice to Bob. Alice has N, d-dimensional quantum states $|\Phi\rangle$.

$$|\Lambda\rangle_{X_1 X_2 \ldots X_N} = \prod_{i=1}^{n} \sum_{j_i=0}^{d-1} c_{j_i} |j_i\rangle_{X_i} \tag{2.48}$$

To teleport the qubits, Alice and Bob share N generalized ERP states $|\Phi_{00}\rangle_{23}$.

$$|\Psi_{0000...00}\rangle_{A_1B_1A_2B_2...A_NB_N} = \prod_{i=1}^{N} |\Psi_{00}\rangle_{A_iB_i} \tag{2.49}$$

$$= \frac{1}{d^{N/2}} \prod_{i=1}^{N} \sum_{j=0}^{d-1} |jj\rangle_{A_iB_i} \tag{2.50}$$

$$|\Gamma\rangle = |\Lambda\rangle_{X_1X_2...X_N} \otimes |\Psi_{0000...00}\rangle_{A_1B_1A_2B_2...A_NB_N} \tag{2.51}$$

For each set of qudits in ERP state there exists an unitary operator U_{kl} and V_{kl} such that any arbitrary ERP state may be described (Zhang, 2006).

$$U_{kl} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-2\pi i (j - l \bmod d)k/d} |j - l \bmod d\rangle\langle j| \tag{2.52}$$

$$V_{kl} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i (j + l \bmod d)k/d} |j + l \bmod d\rangle\langle j| \tag{2.53}$$

As in the usual protocol, Alice starts measuring. She measures the qudit pairs $\{(X_i, A_i)\}, \forall i$. This results in the total state measured

$$|\Psi_{k_1l_1k_2l_2...k_Nl_N}\rangle_{X_1A_1X_2A_2...X_NA_N} = \prod_{i=1}^{N} U_{A_i}^{k_il_i} |\Psi_{00}\rangle_{X_iA_i} \tag{2.54}$$

$${}_{X_1A_1X_2A_2...X_NA_N}\langle\Psi_{k_1l_1k_2l_2...k_Nl_N}|\Gamma\rangle = \prod_{i=1}^{N} V_{B_i}^{(k_il_i)\dagger} |\psi\rangle_{B_i} \tag{2.55}$$

$$\left(\prod_{i=1}^{N} V_{B_i}^{(k_il_i)}\right) {}_{X_1A_1X_2A_2...X_NA_N}\langle\Psi_{k_1l_1k_2l_2...k_Nl_N}|\Gamma\rangle = \prod_{i=1}^{N} |\psi\rangle_{B_i} \tag{2.56}$$

Now Alice knows all k_i and l_i and sends this classical information to Bob. With these $2N$ classical bits, Bob can retrieve the state $|\Lambda\rangle$ on it's qudits.

$$|\psi\rangle_{B_1B_2...B_N} = \prod_{i=1}^{N} V_{B_i}^{k_il_i} |\Psi_{00}\rangle_{B_i} \tag{2.57}$$

Chapter 3

Entangled state teleportation

3.1 GHZ channel entanglement teleportation

In §2.4 it is shown how a generalized GHZ channel of four qubits is not suitable as a channel for quantum teleportation for dual qubit teleportation. However, a GHZ channel with three qubits is suitable to act as a quantum channel in a new teleportation protocol. This protocol will teleport an entangled pair of qudits $|\xi\rangle$, instead of a single qudit. The entangled pair $|\xi\rangle$ is shared between Alice and Bob. The three particle GHZ state quantum channel $|\Phi\rangle$ is shared between Bob, Charlie and Dan.

$$|\xi\rangle_{AB} = \alpha' \left|\phi 0'\right\rangle + \beta' \left|\phi' 1'\right\rangle \tag{3.1}$$

$$|\Phi\rangle_{BCD} = \frac{1}{\sqrt{2}} \left(|0\phi 0\rangle + \left|1\phi' 1\right\rangle\right)^{1} \tag{3.2}$$

This quantum channel $|\Phi\rangle$ would allow for the teleportation of entangled state $|\xi\rangle$ (Ghosh et al., 2002). The combined system of the entangled state and the GHZ quantum channel may be written as:

$$\begin{align} |\Gamma\rangle &= \left|\xi'\right\rangle_{AB_1} \otimes |\Phi\rangle_{B_2CD} \tag{3.3} \\ &= \frac{1}{\sqrt{2}} \left(\alpha \left|\phi 0'\right\rangle + \beta \left|\phi' 1'\right\rangle\right)_{AB_1} \otimes \left(|0\phi 0\rangle + \left|1\phi' 1\right\rangle\right)_{B_2CD} \tag{3.4} \end{align}$$

Before teleportation, the sender, Bob, has to do a unitary operation U on its qubit of the entangled state, so $U : |\xi\rangle \to |\xi'\rangle$, and on its qubit of the quantum channel, so $U : |\Phi\rangle \to |\Phi'\rangle$.

$$U = \begin{cases} |0'\rangle & \to |0\rangle \\ |1'\rangle & \to e^{-i\varepsilon}\,|1\rangle \end{cases}, \quad \left\langle \phi \middle| \phi' \right\rangle = re^{i\varepsilon} \tag{3.5}$$

The complete system of five particles, where $|\phi''\rangle = e^{-i\varepsilon}\,|\phi'\rangle$, now is transformed to

$$\begin{align} \left|\Gamma'\right\rangle &= U\,|\Gamma\rangle \tag{3.6} \\ &= \frac{1}{\sqrt{2}} \left(\alpha\,|\phi 0\rangle + \beta \left|\phi'' 1\right\rangle\right)_{AB_1} \otimes \left(|0\phi 0\rangle + \left|1\phi'' 1\right\rangle\right)_{B_2CD} \tag{3.7} \end{align}$$

[1]The states $|\phi\rangle$ and $|\phi'\rangle$ not need to be orthogonal.

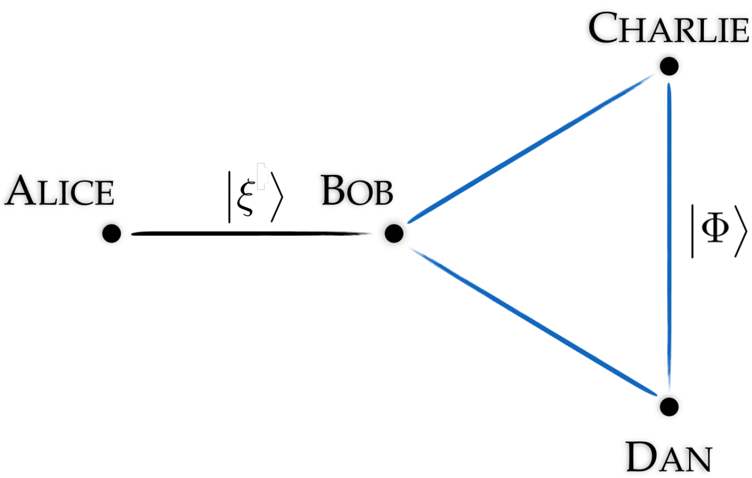

FIGURE 3.1: An abstract diagram of the teleportation protocol for teleporting an entangled state $|\xi\rangle$ before teleportation. The blue line between Bob, Charlie and Dan represents the GHZ state shared between them.

This same procedure as done in §2.2 results in a rewritten system as:

$$\begin{aligned} |\Gamma'\rangle =& \frac{1}{2}\Big[\big(\alpha\,|\phi\phi 0\rangle + \beta\,|\phi''\phi''1\rangle\big)_{ACD}\otimes|\phi^+\rangle_{B_1B_2} + \\ &\big(\alpha\,|\phi\phi 0\rangle - \beta\,|\phi''\phi''1\rangle\big)_{ACD}\otimes|\phi^-\rangle_{B_1B_2} + \\ &\big(\alpha\,|\phi\phi''1\rangle + \beta\,|\phi''\phi 0\rangle\big)_{ACD}\otimes|\psi^+\rangle_{B_1B_2} + \\ &\big(\alpha\,|\phi\phi''1\rangle - \beta\,|\phi''\phi 0\rangle\big)_{ACD}\otimes|\psi^-\rangle_{B_1B_2}\Big]^{2} \end{aligned} \tag{3.8}$$

From here on, the analogy to the teleportation scheme in §2.1 seems clear. After his measurement, Bob sends the classical information to distinguish between the four possible states. However, as the generalization of the entangled state (3.1) and the quantum channel (3.2) stated, $\langle\phi|\phi'\rangle = r$. Therefore, the result obtained here is still ambiguous, which is easily seen in the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ where the result measured is mutually exclusive.

$$|\xi'\rangle_{AB} = \alpha\,|\phi 0\rangle_{AB} + \beta\,|\phi''1\rangle_{AB} \tag{3.9}$$

$$\begin{aligned} &= \alpha\left(\cos\frac{\theta}{2}|u_1\rangle + \sin\frac{\theta}{2}|u_2\rangle\right)_A\otimes|0\rangle_B + \\ &\quad\beta\left(\cos\frac{\theta}{2}|u_1\rangle - \sin\frac{\theta}{2}|u_2\rangle\right)_A\otimes|0\rangle_B \end{aligned} \tag{3.10}$$

$$= \cos\frac{\theta}{2}|u_1\rangle_A\otimes(\alpha\,|0\rangle + \beta\,|1\rangle)_B + \tag{3.11}$$

$$\sin\frac{\theta}{2}|u_2\rangle_A\otimes(\alpha\,|0\rangle - \beta\,|1\rangle)_B{}^{3} \tag{3.12}$$

[2]The states $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are the well-known EPR states, see Appendix A.
[3]Note that the value of θ is known: $\langle\phi|\phi''\rangle = 1 - 2\sin^2\frac{\theta}{2}, \theta\in[0,\frac{\pi}{2}]$.

This will ultimately give Alice the ability to send the last bit of information, necessary for this teleportation process. After Alice has distinguished on the new basis which she has measured, she sends this result to the two receivers Charlie and Dan. Note, at this time, that Charlie and Bob both received three bits at this time, in order to veritably retrieve the state $|\xi'\rangle$.

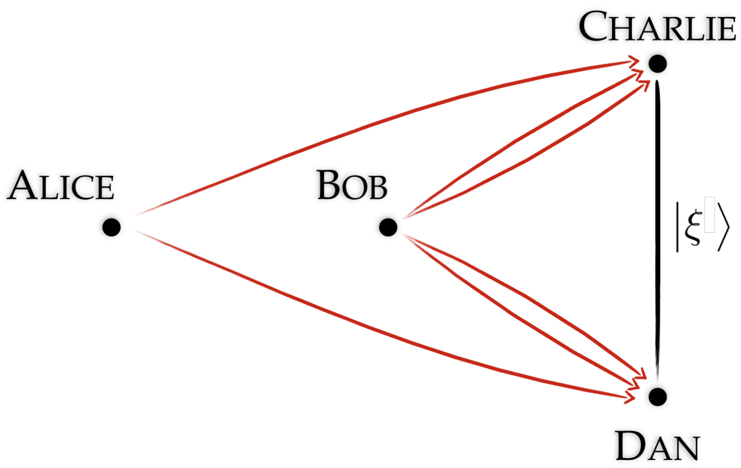

FIGURE 3.2: An abstract diagram of the teleportation protocol for teleporting an entangled state $|\xi\rangle$ after teleportation. The line between Charlie and Dan represents the teleported state. The red lines are the classical bits sent during the teleportation process.

The three bits of information are a result of the eight possible operations Charlie and Dan have to perform. The eight possible measurements and matching operations are listed in Appendix B.

Chapter 4

Successful teleportation

4.1 Quantum ensembles

Hughston, Jozsa, and Wootters (1993) gave a new interpretation on quantum states when they showed how a density matrix ρ of an entangled state may be created from any true ensemble. Such an ensemble may be defined as a collection of states $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$. Hence

$$p_i = \langle\psi_i|\psi_i\rangle \tag{4.1}$$

$$\rho = \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i| \tag{4.2}$$

In their theorem (Hughston, Jozsa, and Wootters, 1993), they defined an ensemble $\{|\psi_1\rangle, \ldots, |\psi_r\rangle\}$ for a certain density matrix ρ, with $k = \mathrm{rank}(\rho)$. This ensemble is constructed from a $r \times k$-matrix M consisting of k columns in $\mathbb{C}^r$, made from the eigen-ensemble $\{|e_1\rangle, \ldots, |e_k\rangle\}$.

$$|\psi_i\rangle = \sum_{j=1}^{k} M_{ij}\,|e_j\rangle\,, \quad (r \geq k) \tag{4.3}$$

The second part of their theorem shows that one can create any ensemble consistent with the related density matrix. They proof how $\{|\phi_1\rangle, \ldots, |\phi_s\rangle\}$ may be any ensemble of pure states for a density matrix with $k = \mathrm{rank}(\rho)$. Their theorem states how there exists a matrix N_{ij} of k orthonormal vectors in $\mathbb{C}^s$.

$$|\phi_i\rangle = \sum_{j=1}^{k} N_{ij}\,|e_j\rangle \tag{4.4}$$

From this second part of the theorem it follows that any ensemble can only create the density matrix ρ if $s \geq k$. If $s = k$, the ensemble of states is linear independent.

Measuring one half of an entangled state pair may therefore be interpreted as creating or choosing a specific ensemble of states for the corresponding density matrix ρ.

Now, since we know that for a given density matrix there may exist more than one ensemble, the receiver within the teleportation protocol is not able to distinguish between all possible ensembles. The sender Alice 'creates' an ensemble after she does her measurement. She then sends information to Bob which ensemble is being used. With this information, Bob can reconstruct the ensemble from the density matrix to find the state he possesses.

4.2 Positive operator value measure

A generalization of projection measurement is done by the positive operator value measure (POVM). These POVM's are a set of Hermitian operators $\{A_1, \ldots, A_r\}$ working on the Hilbert space. They satisfy the condition

$$\sum_i A_i = I \tag{4.5}$$

As the number of results from the POVM may be larger than the dimension of the Hilbert space, the operators are $A_i = |\alpha_i\rangle\langle\alpha_i|$ with $|\alpha_i\rangle$ not necessarily being normalized or orthogonal. The operators itself however, are a compete set of orthogonal subspaces (Hughston, Jozsa, and Wootters, 1993).

4.3 Conclusive teleportation

One problem of the proposed teleportation methods proposed above is that reality is not as perfect as theory describes. Fidelity is an index value that describes the degree of overlap between two quantum states. Thus, in teleportation it describes the degree in which the state to be teleported at the sender side before teleportation, is equal to the state teleported at the receiver side. If teleportation has worked out perfectly, fidelity is one.

This reduction in fidelity in an experimental environment may for example due to a nonperfect entangled quantum channel or imperfections in the BSM (Bao et al., 2012). Other common errors come from the technical challenges in a quantum bit, as the energy between the quantum levels are of very low quantity. Any electromagnetic noise or heat inflicting, make it hard to control the quantum level desired (IBM, 2016).

To improve the teleportation's fidelity one may use an alternative protocol of the original protocol of Bennett et al. (1993). This conclusive teleportation may also be used when Bob cannot perform a certain unitary operation. For example, if Bob is technically limited to only states in which he does not have to do any rotation, in the Bell state he has a $\frac{1}{4}$ change he will receive the correct quantum state.

In conclusive teleportation, Alice will send only one bit instead of two bits. This bit does not tell Bob what unitary operation to apply, but only whether the measured basis is the basis Bob wants to receive. If true, the teleportation is successful. This method may also be used when the shared entangled states $|\Psi\rangle_{23}$, used to teleport a single state $|\phi\rangle_1$, are not fully entangled.

$$|\phi\rangle_1 = \alpha|0\rangle + \beta|1\rangle \tag{4.6}$$

$$|\Psi\rangle_{23} = a|00\rangle + b|11\rangle + a|01\rangle + b|10\rangle \tag{4.7}$$

In the two-dimensional protocol one starts with a Bell measurement in two steps. The first step of measurement is checking whether the entangled pair is in the subspace $\{|00\rangle, |11\rangle\}$ or $\{|01\rangle, |01\rangle\}$.

After the first measurement, we will be left with the subspace that is spanned by $\{00; 11\}$ or $\{01; 01\}$. Following normally, at this point one would do a second Bell measurement to distinguish between the two possible states left. The subspace after the first measurement is one of these two possibilities:

$$\begin{gathered}\left\{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ -b \end{pmatrix}\right\}_{\{00;11\}} \\ \left\{\begin{pmatrix} b \\ a \end{pmatrix}, \begin{pmatrix} b \\ -a \end{pmatrix}\right\}_{\{01;01\}}\end{gathered} \tag{4.8}$$

Instead of doing a second Bell measurement, one would do a POVM to differentiate between the states in the given subspace. Mor and Horodecki (1999) considered the following POVM matrices A_i to perform.

$$A_1 = \begin{pmatrix} b^2 & ba \\ ba & a^2 \end{pmatrix} \tag{4.9}$$

$$A_2 = \begin{pmatrix} b^2 & -ba \\ -ba & a^2 \end{pmatrix} \tag{4.10}$$

$$A_3 = \begin{pmatrix} 1 - \left(\frac{b}{a}\right)^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.11}$$

A nice way to think about these measurements is that when Alice performs a POVM to her part of the shared entangled state, she chooses a specific density matrix ρ-ensemble for the entangled state as described in §4.1. After measurement, the other particle of the entangled pair will be projected to the orthogonal state of the measurement. This way, Alice tells Bob what state he is in and what operator to apply to get the desired $|\phi\rangle_3$ state on his particle. Note, how an outcome of the A_3 POVM does not distinguish between the two possible states in the basis left after the first measurement (4.8) and so the teleportation has failed.

Mor and Horodecki (1999) stated that using this method the teleportation fidelity will be one. However, since not all teleportations succeed the probability of a successful teleportation is less than one if the quantum channel used is not a perfectly entangled. They stated the probability of a successful teleportation to be

$$P = 1 - \left(|a|^2 - |b|^2\right) \tag{4.12}$$

4.4 Partially entangled GHZ channel

For a successful teleportation with the protocols described before, the quantum channels are assumed maximally entangled. In practice, creating and maintaining such an entangled state is very hard and may therefore limit the success rate of teleportation. To overcome this problem, only recently Xiong et al. (2016) introduced a new protocol on teleporting qubits using a non maximally entangled quantum channel.

The protocol uses a partially entangled GHZ state

$$|\Phi_{GHZ}\rangle_{234} = \sqrt{1-n^2}\,|000\rangle_{234} + n\,|111\rangle_{234} \tag{4.13}$$

shared between Alice and Bob, where qubits 3 and 4 belong to Bob and qubit 2 to Alice together with the qubit 1 which is to be teleported. Hence

$$|\psi\rangle_1 = \alpha\,|0\rangle_1 + \beta\,|1\rangle_1 \tag{4.14}$$

As is customary, Alice performs a combined measurement to distinguish between the four possible outcomes, namely the Bell states. Additional, Bob has to measure his qubit 3 to be able to see what operations to do next. Explicitly written the system is

$$\begin{aligned}
|\Gamma\rangle_{1234} &= |\psi\rangle_1 \otimes H_3\,|\Phi_{GHZ}\rangle_{234} &(4.15)\\
&= \frac{1}{2}\Big\{\big|\Phi^+\big\rangle_{12} \otimes |0\rangle_3 \otimes \Big(\sqrt{1-n^2}\alpha\,|0\rangle_4 + n\beta\,|1\rangle_4\Big)\\
&\quad + \big|\Phi^+\big\rangle_{12} \otimes |1\rangle_3 \otimes \Big(\sqrt{1-n^2}\alpha\,|0\rangle_4 - n\beta\,|1\rangle_4\Big)\\
&\quad + \big|\Phi^-\big\rangle_{12} \otimes |0\rangle_3 \otimes \Big(\sqrt{1-n^2}\alpha\,|0\rangle_4 - n\beta\,|1\rangle_4\Big)\\
&\quad + \big|\Phi^-\big\rangle_{12} \otimes |1\rangle_3 \otimes \Big(\sqrt{1-n^2}\alpha\,|0\rangle_4 + n\beta\,|1\rangle_4\Big)\\
&\quad + \big|\Psi^+\big\rangle_{12} \otimes |0\rangle_3 \otimes \Big(n\alpha\,|1\rangle_4 + \sqrt{1-n^2}\beta\,|0\rangle_4\Big)\\
&\quad - \big|\Psi^+\big\rangle_{12} \otimes |1\rangle_3 \otimes \Big(n\alpha\,|1\rangle_4 - \sqrt{1-n^2}\beta\,|0\rangle_4\Big)\\
&\quad + \big|\Psi^-\big\rangle_{12} \otimes |0\rangle_3 \otimes \Big(n\alpha\,|1\rangle_4 - \sqrt{1-n^2}\beta\,|0\rangle_4\Big)\\
&\quad - \big|\Psi^-\big\rangle_{12} \otimes |1\rangle_3 \otimes \Big(n\alpha\,|1\rangle_4 + \sqrt{1-n^2}\beta\,|0\rangle_4\Big)\Big\} &(4.16)
\end{aligned}$$

Subsequently, Bob performs the required Pauli operators σ_x and/or σ_z and performs a Hadamard gate on its qubit 3. At this point of the protocol, when all operations and measurements, say N, are performed, Bob starts using its extra qubit 5. This qubit is in ground state $|0\rangle_5$.

$$N\big(\big|\Psi^\pm\big\rangle_{12}\big)\,|\Gamma\rangle_{1234} \otimes |0\rangle_5 = |\Theta\rangle_{123} \otimes |\Xi\rangle^{\Psi}_{45} \tag{4.17}$$

$$|\Xi\rangle^{\Psi}_{45} = \Big(\sqrt{1-n^2}\alpha\,|0\rangle_4 + n\beta\,|1\rangle_4\Big) \otimes |0\rangle_5 \tag{4.18}$$

$$N\big(\big|\Phi^\pm\big\rangle_{12}\big)\,|\Gamma\rangle_{1234} \otimes |0\rangle_5 = |\Theta\rangle_{123} \otimes |\Xi\rangle^{\Phi}_{45} \tag{4.19}$$

$$|\Xi\rangle^{\Phi}_{45} = \Big(n\alpha\,|0\rangle_4 + \sqrt{1-n^2}\beta\,|1\rangle_4\Big) \otimes |0\rangle_5 \tag{4.20}$$

The state $|\psi\rangle$ still not completely retrieved, but it is getting close. For both of the two solutions of $|\Xi\rangle$, there exists an unitary operator U to finalize the teleportation. Explicitly:

$$U^{\Psi} = \begin{pmatrix} n & \sqrt{1-n^2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sqrt{1-n^2} & -n & 0 & 0 \end{pmatrix} \tag{4.21}$$

$$U^{\Phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{1-n^2} & n & 0 \\ 0 & -n & \sqrt{1-n^2} & 0 \end{pmatrix} \tag{4.22}$$

$$U^{\Psi} |\Xi\rangle_{45}^{\Psi} = (\alpha |0\rangle_4 + \beta |1\rangle_4) \otimes n |0\rangle_5 + \alpha |1\rangle_4 \otimes \alpha \sqrt{1-n^2} |1\rangle_5 \tag{4.23}$$

$$U^{\Phi} |\Xi\rangle_{45}^{\Phi} = (\alpha |0\rangle_4 + \beta |1\rangle_4) \otimes n |0\rangle_5 + \alpha |1\rangle_4 \otimes \beta \sqrt{1-n^2} |1\rangle_5 \tag{4.24}$$

As seen in the equations (4.23) and (4.24), the desired state $|\psi\rangle_4$ is retrieved if the qubit 5 is in state $|0\rangle_5$. So, after Bob performed the unitary operator U, he measures the fifth qubit to see if teleportation succeeded. Explicitly, the measurements where the fifth qubit is found on $|0\rangle_5$ are written out:

$$\begin{aligned} M_5^0 U^{\Psi} |\Xi\rangle_{45}^{\Psi} &= \frac{|0\rangle_5 \langle 0|}{\sqrt{P_0}} \left[(\alpha |0\rangle_4 + \beta |1\rangle_4) \otimes n |0\rangle_5 + \alpha |1\rangle_4 \otimes \beta \sqrt{1-n^2} |1\rangle_5 \right] \\ &= \frac{1}{\sqrt{|n^2|}} (\alpha |0\rangle_4 + \beta |1\rangle_4) \, n |0\rangle_5 && (4.25) \\ &= |\psi\rangle_4 \otimes |0\rangle_5 && (4.26) \end{aligned}$$

Thus, with the additional qubit 5 and its measurements and operations, Bob is able to determine whether the teleportation is successful even with the non maximally entangled quantum channel.

Chapter 5

Discussion

5.1 Discussion

Concluding, described is the teleportation scheme for a qubit using a Bell state between Alice and Bob. This was theoretically introduced in 1993. New teleportation schemes deliberately were introduced as they are mostly a generalization of the original paper. In this thesis, the generalization was made in different ways.

It is shown how a higher dimensional quantum channel may not a priori be suitable for multi-particle teleportation. A higher dimensional quantum channel, however, is suitable for other teleportation schemes as introduced in 1993. Teleporting an entangled pair of particles between more than three parties is discussed. This is only one alternative teleportation scheme, but there are many more schemes to think of and many more constructions containing quantum information to be teleported. For example, think of a way of teleporting generalized Bell states of more than two particles. Furthermore, there can be thought of entangled states shared between multiple parties where it can be decided which of the parties to teleport to at a later period of time.

Further, it is shown how a two dimensional qubit teleportation is taken to a higher arbitrary dimension with the use of an higher dimensional quantum channel. The quantum channel is initially taken with a equal dimensions as the quantum state to be teleported. In the following section, it is shown how the quantum channel not necessarily has to have to the same dimensions as the quantum state to teleport. As long as the quantum channel has any dimension greater than or equal to its quantum state to be teleported, teleportation will be possible. Finalizing this section, will take the original twenty three year old paper to one of the most general systems conceivable.

5.1.1 Experiments

Since Bouwmeester et al. (1997) brought theory into action for the first time, many parties followed (Takesue et al., 2015; Ma et al., 2012). Nowadays, long distance teleportation is technically restricted to a bit further than 100km. The generalizations from the theory in this report are mostly not yet taken to successful experiments. Many teleportation experiments are focussed on enlarging the physical distance between Alice and Bob. The last years, a few papers propose some techniques on teleportation of higher dimensional qudits (Goyal and Konrad, 2013).

The variation on the traditional teleportation scheme described in §3.1 on how to teleport entangled states are still not more than theory. Experiments of teleportation of entangled states were not found at the moment of writing.

Occluding, a brief experiment is done on the first open access quantum computer built by IBM. In Appendix C it is described how the single qubit teleportation is implemented on this open access circuit.

Appendix A

Entangled states

A.1 Einstein-Podolsky-Rosen states

$$\left|\Phi^{+}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right) \tag{A.1}$$

$$\left|\Phi^{-}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B \right) \tag{A.2}$$

$$\left|\Psi^{+}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \right) \tag{A.3}$$

$$\left|\Psi^{-}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right) \tag{A.4}$$

A.2 Greenberger-Horne-Zeilinger states

The entangled GHZ quantum state with $M > 2$ subsystems is written

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes M} + |1\rangle^{\otimes M} \right) \tag{A.5}$$

Appendix B

GHZ teleportation system

The teleportation of an entangled state $|\xi\rangle$ is written out in §3.1 which is mainly originating from (Ghosh et al., 2002). The eventual complete system after measurement by Alice and Bob may be written out as:

$$
\begin{aligned}
|\Gamma'\rangle &= \frac{1}{\sqrt{2}}\left(\alpha\,|\phi 0\rangle + \beta\,|\phi'' 1\rangle\right)_{AB_1} \otimes \left(|0\phi 0\rangle + |1\phi'' 1\rangle\right)_{B_2CD} && \text{(B.1)}\\
&= \frac{1}{2}\,[\\
&\quad \left(\alpha\,|\phi\phi 0\rangle + \beta\,|\phi''\phi'' 1\rangle\right)_{ACD} \otimes |\phi^+\rangle_{B_1B_2} +\\
&\quad \left(\alpha\,|\phi\phi 0\rangle - \beta\,|\phi''\phi'' 1\rangle\right)_{ACD} \otimes |\phi^-\rangle_{B_1B_2} +\\
&\quad \left(\alpha\,|\phi\phi'' 1\rangle + \beta\,|\phi''\phi 0\rangle\right)_{ACD} \otimes |\psi^+\rangle_{B_1B_2} +\\
&\quad \left(\alpha\,|\phi\phi'' 1\rangle - \beta\,|\phi''\phi 0\rangle\right)_{ACD} \otimes |\psi^-\rangle_{B_1B_2}\\
&\quad] && \text{(B.2)}\\
&= \frac{1}{2}\,[\\
&\quad \cos\frac{\theta}{2}\,|u_1\rangle_A \otimes \left(\alpha\,|\phi 0\rangle + \beta\,|\phi'' 1\rangle\right)_{CD} \otimes |\Phi^+\rangle_{B_1B_2} +\\
&\quad \sin\frac{\theta}{2}\,|u_2\rangle_A \otimes \left(\alpha\,|\phi 0\rangle - \beta\,|\phi'' 1\rangle\right)_{CD} \otimes |\Phi^+\rangle_{B_1B_2} +\\
&\quad \cos\frac{\theta}{2}\,|u_1\rangle_A \otimes \left(\alpha\,|\phi 0\rangle - \beta\,|\phi'' 1\rangle\right)_{CD} \otimes |\Phi^-\rangle_{B_1B_2} +\\
&\quad \sin\frac{\theta}{2}\,|u_2\rangle_A \otimes \left(\alpha\,|\phi 0\rangle + \beta\,|\phi'' 1\rangle\right)_{CD} \otimes |\Phi^-\rangle_{B_1B_2} +\\
&\quad \cos\frac{\theta}{2}\,|u_1\rangle_A \otimes \left(\alpha\,|\phi'' 0\rangle + \beta\,|\phi 1\rangle\right)_{CD} \otimes |\Psi^+\rangle_{B_1B_2} +\\
&\quad \sin\frac{\theta}{2}\,|u_2\rangle_A \otimes \left(\alpha\,|\phi'' 0\rangle - \beta\,|\phi 1\rangle\right)_{CD} \otimes |\Psi^+\rangle_{B_1B_2} +\\
&\quad \cos\frac{\theta}{2}\,|u_1\rangle_A \otimes \left(\alpha\,|\phi'' 0\rangle + \beta\,|\phi 1\rangle\right)_{CD} \otimes |\Psi^-\rangle_{B_1B_2} +\\
&\quad \sin\frac{\theta}{2}\,|u_2\rangle_A \otimes \left(\alpha\,|\phi'' 0\rangle - \beta\,|\phi 1\rangle\right)_{CD} \otimes |\Psi^-\rangle_{B_1B_2} +\\
&] && \text{(B.3)}
\end{aligned}
$$

At this point, it is clear how both Alice and Bob have to do their measurement and Charlie and Dan only can distinguish between the eight states when they have received all of the three classical bits.

To complete clarification, all eight unitary operations U' which will help to eventually retrieve the state $|\xi\rangle_{CD}$ are written out below.

$$U'_{u_1,\Phi^+} = I_C \otimes U_D^{-11} \tag{B.4}$$
$$U'_{u_2,\Phi^+} = I_C \otimes U_D^{-1}\sigma_D^z \tag{B.5}$$
$$U'_{u_1,\Phi^-} = I_C \otimes U_D^{-1} \tag{B.6}$$
$$U'_{u_2,\Phi^-} = I_C \otimes U_D^{-1}\sigma_D^z \tag{B.7}$$
$$U'_{u_1,\Psi^+} = V_C \otimes U_D^{-1} \tag{B.8}$$
$$U'_{u_2,\Psi^+} = V_C \otimes U_D^{-1}\sigma_D^z \tag{B.9}$$
$$U'_{u_1,\Psi^-} = V_C \otimes U_D^{-1} \tag{B.10}$$
$$U'_{u_2,\Psi^-} = V_C \otimes U_D^{-1}\sigma_D^z \tag{B.11}$$

where

$$V: \quad \begin{cases} |\phi\rangle \to |\phi''\rangle \\ |\phi''\rangle \to |\phi\rangle \end{cases} \tag{B.12}$$

[1]The unitary operator U^{-1} is the inverse operator of the operator performed by Bob, see equation (3.5).

Appendix C

Single qubit teleportation on IBM's quantum circuit

To conclude the theory, single qubit teleportation is shown on IBM's quantum circuit. In May 2016, IBM Research launched the first open access quantum circuit (IBM, 2016). This five qubit circuit is open to the public, after taking a brief online quantum course. With the aspirations of launching a fifty to hundred qubit open access quantum circuit within the next ten years, they hope to accelerate quantum computation research by giving access to quantum computers to more and more people.

In consonance, it is shown how the teleportation protocol works on the IBM Quantum Experience.
Three of five qubits in the system are used, starting in state $|0\rangle$. The circuit is built into three sections. The first section is the part where qubits 2 and 3 are brought into an entangled state. The second section is where a combined measurement of qubits 1 and 2 is done. The last part is the rotation of the third qubit to retrieve the original state of qubit 1. However, due to technical limitations, it is not possible to do the σ_z rotation. Furthermore, is it not possible to have conditional rotations on classical signals. Therefore only σ_x rotations are done by using a $CNOT$ gate.

This takes us to the two possible qubit teleportations:

$$|0\rangle_1 \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{23} \rightarrow \frac{1}{\sqrt{2}}\left(|00\rangle_{12} \otimes |0\rangle_3 + |01\rangle_{12} \otimes |1\rangle_3\right)_{12} \tag{C.1}$$

$$|1\rangle_1 \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{23} \rightarrow \frac{1}{\sqrt{2}}\left(|10\rangle_{12} \otimes |0\rangle_3 + |11\rangle_{12} \otimes |1\rangle_3\right)_{12} \tag{C.2}$$

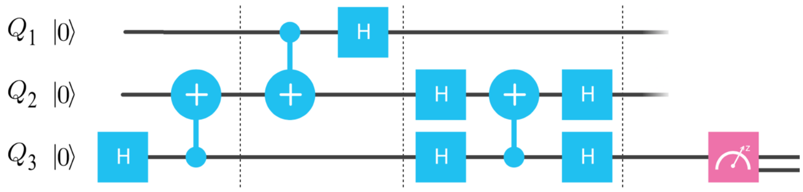

FIGURE C.1: The quantum circuit built in the IBM Quantum Experience. The first Hadamard gate and CNOT gate create an entangled state between Q_2 and Q_3. The Second CNOT and Hadamard gates are the measurement by fictitious Alice. Due to technical limitations, on the third part the CNOT gate is flipped by the four Hadamard gates.

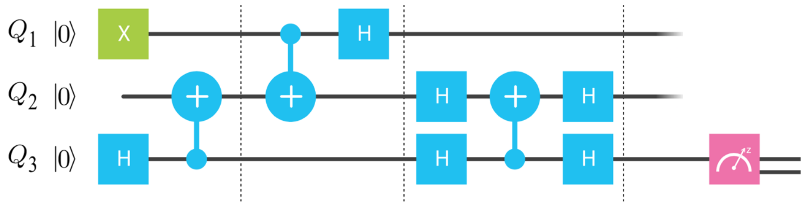

FIGURE C.2: Similar to figure C.1, the state of Q_1 is teleported to Q_3. The initial state is prepared onto $|1\rangle$ by a σ_x rotation.

Each of the two quantum circuits are run 8192 times in the IBM lab. Results show how the teleportation process is successful in most of the runs, but fails in about 10% of the times.

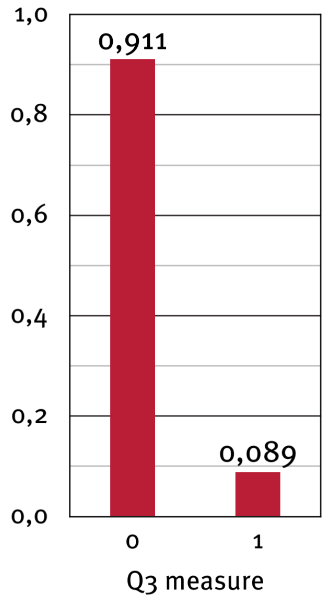

FIGURE C.3: The results of 8192 runs of the quantum circuit teleporting the state $|0\rangle$ shown in figure C.1.

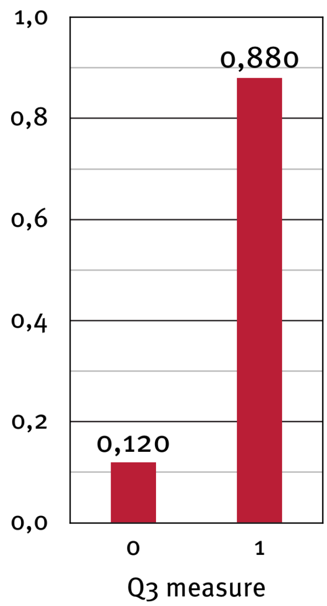

FIGURE C.4: The results of 8192 runs of the quantum circuit teleporting the state $|1\rangle$ shown in figure C.2.

Appendix D

Popular scientific summary (Dutch)

Kwantummechanica werkt met andere informatie stromen dan wij intuitíef verwachten. Waar wij van de traditionele computer een nul óf een één gebruiken als informatie kan een kwantum toestand ook uit een combinatie van deze twee bestaan, de superpositie. Ten gevolge van de regels uit de kwantummechanica is het echter niet mogelijk om informatie te kopiëren. Om toch een kwantummechanische informatiestroom te creëren bestaat er wel de optie om de informatie te teleporteren. Hoewel dit vooral klinkt als science-fiction, is dit fenomeen wel onderdeel van een zeer populair en succesvol onderzoeksgebied.

De basis van kwantummechanische teleportatie ligt in het delen van een verstrengelde toestand. Deze toestand bestaat uit twee deeltjes welke beide bestaan uit een superpositie van nul en één, maar waarvan de toestanden op een bepaalde manier aan elkaar gerelateerd zijn. Dit betekent dat wanneer de toestand van het ene deeltje bekend is, ook duidelijk is in welke toestand het andere deeltje zich bevindt zonder dat men het hoeft te meten.

Het gehele protocol van teleportatie begint met twee partijen, genaamd Alice en Bob. Alice heeft één deeltje van het verstrengelde paar, Bob de andere. Alice doet een meting om de rotatie van het te teleporteren deeltje ten opzichte van haar deeltje van de verstrengelde toestand te meten. In de kwantummechanica, vervalt de toestand van een deeltje na de meting hiervan. Na de gecombineerde meting is het dus niet meer mogelijk de losse twee deeltjes te meten, maar is enkel informatie beschikbaar over de combinatie. In het systeem van een enkel kwantummechanisch deeltje in een superpositie van nul en één zijn er vier mogelijke rotaties mogelijk.

Na meting wordt doorgegeven aan Bob welke van de vier mogelijke rotaties hij moet uitvoeren op zijn deel van de verstrengelde toestand. Bob roteert zijn deel van de verstrengelde toestand op een dusdanige manier dat de originele toestand van Alice wordt teruggehaald op zijn deeltje waarmee de teleportatie is voltooid.

In dit rapport wordt besproken hoe dit teleportatie protocol kan worden veralgemeniseerd. Deze generalisatie beschrijft de teleportatie van meerdere deeltjes, maar ook die van de teleportatie van hoger dimensionale kwantummechanische toestanden. Wanneer een deeltje niet enkel uit de superpositie van nul en één bestaat, maar uitgebreid met twee, drie, vier, etc, wordt beschreven in hoe dit deeltje ook kan worden geteleporteerd.

Verder blijkt dat de verstrengelde toestanden waarvan gebruik wordt gemaakt bij de teleportatie ook kunnen worden geteleporteerd. Dit teleportatie protocol bestaat uit meerdere partijen die betrokken zijn, zoals Alice, Bob, Charlie en Dan. De teleportatie vindt daarbij plaats van Alice en Bob naar Charlie en Dan.

In de praktijk wordt dit technisch veelal uitgevoerd door bijvoorbeeld fotonen te polariseren. Waar een foton een kwantummechanisch verschijnsel is, kan polarisatie van licht fungeren als een kwantumtoestand. Daarnaast bieden fotonen de mogelijkheid dat zij relatief goed afstanden kunnen overbruggen door ze door speciale fiber kabels te sturen. Nadat Dik Bouwmeester en zijn team in 1997 voor het eerst het protocol experimenteel uit wisten te voeren is het huidige afstandsrecord van meer dan 100km is eigendom van Hiroki Takesue uitgevoerd in 2015.

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