Abstract

Deformation quantization is a way to quantize a classical mechanical system described by a Poisson manifold and characterized by its Poisson structure. In contrast to most methods of quantization, in which a Hilbert space of states with operators acting on it is somehow constructed, deformation quantization allows for the definition of a non-commutative star product directly on the classical phase space. This product then allows for the description of quantum behaviour in algebraic terms.

First, as an introduction to the subject, the simplest star product is considered. It is called the Moyal product and was already known of in the 1940s as it naturally emerged in the Weyl-Wigner correspondence of phase space distributions. When quantizing a classical theory, there is a certain freedom of ordering of non-commutative operators. Two such orderings are discussed. Moreover, the Moyal product is seen to correspond to the star product induced by a symmetric ordering.

After this, some symplectic theory is introduced as a natural framework for classical mechanics. In particular, Hamilton’s equations of motion are cast in symplectic language. Anticipating a more general theorem, every symplectic manifold is shown to carry a natural Poisson structure.

Then, quantization of a (classical mechanical) Poisson structure by deformation is defined in terms of a so-called star product. This type of product is studied using a certain group action and Hochschild cohomology, which is explained in the appendix. The concept of differential graded Lie algebra is then defined and two examples are considered. The Formality theorem, due to Maxim Kontsevich, is a mathematical result formulated in terms of these objects. It is interpreted, and leads to a one-one correspondence between classes of formal deformations of the null Poisson structure and equivalence classes of star products. As for its application in physics, it follows from this theorem that every classical mechanical system can essentially be quantized uniquely.

Finally, an explicit formula for the construction of a star product is given. It is in terms of a certain class of graphs, reminiscent of the calculation of transition amplitudes in field theory using Feynman diagrams. This formula is applied to the constant Poisson structure. The corresponding star product is found to be the Moyal product.