Robustly minimal iterated function systems on compact manifolds

Master Thesis
Student: Daniël Faruk Younis
Supervisor: Ale Jan Homburg
KdV Institute for Mathematics
University of Amsterdam
Science park 904
1098 XH Amsterdam
The Netherlands
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Abstract

An iterated function system generated by diffeomorphisms is the collection of all possible compositions of the diffeomorphisms. An iterated function system on a manifold, generated by a finite number of diffeomorphisms is called minimal if for each point its orbit under all possible compositions of the diffeomorphisms lies dense in the manifold. In this thesis we investigate minimal iterated function systems generated by as few diffeomorphisms as possible. Furthermore, these iterated function systems are $C^1$ robustly minimal. That is, small perturbations in the $C^1$ topology of the diffeomorphisms are also minimal.
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1 Introduction

An iterated function system on a manifold, generated by a finite number of diffeomorphisms is called minimal if for each point its orbit under all possible compositions of the diffeomorphisms lies dense in the manifold.

An iterated function system generated by \(\{g_1, \ldots, g_k\}\) diffeomorphisms on \(M\) is called \(C^1\) robustly minimal if there is a neighborhood \(U \subset (\text{Diff}^1(M))^k\) of \(g_1, \ldots, g_s\) such that each element in \(U\) forms a minimal system on \(M\).

My thesis is developed around the following theorem contained in [1] by Homburg, Ghane and Sarizadeh.

**Theorem 1.1.** There exist diffeomorphisms \(T_1, \ldots, T_{n+3}\) on an \(n\)-dimensional compact manifold and a neighborhood \(U \subset \text{Diff}^1(M) \times \ldots \times \text{Diff}^1(M)\) \(n+3\) times of \((T_1, \ldots, T_{n+3})\) such that each element in \(U\) forms a minimal iterated function system on \(M\).

This result raises the question of the minimal number of generators of \(C^1\) robustly minimal iterated function systems. Here the number of generators is dependent of the dimension of the manifold. For example, in three dimensions this theorem provides a minimal iterated function system where the number of generators is independent of the dimension of the manifold. It lowers the number to three diffeomorphisms. Recent results (see the discussion in the final chapter) show that the minimal number of generators can even be reduced to two. As an iterated function system generated by one diffeomorphism only contains the collection of all possible compositions with itself and its inverse and as there are compact manifolds where the orbit of some points under the composition of only one diffeomorphism can never lie dense in the manifold (see for an example [3]), two is the smallest number of generators.

Throughout this text, \(M\) will stand for an \(n\)-dimensional smooth (infinitely differentiable) compact manifold and diffeomorphism will stand for \(C^1\)-diffeomorphism. A \(C^1\)-diffeomorphism is a map between manifolds which is continuously differentiable and has a continuously differentiable inverse.

We start of by introducing the \(C^1\) topology on \(M\). Consider the space \(\text{Diff}^1(M)\) of diffeomorphisms of \(M\) to \(M\). For \(r > 0\) the \(C^1\) topology on \(M\) is generated by open balls \(B_1(f, r) = \{g \in \text{Diff}^1(M) | d_1(f, g) < r\}\) where the metric \(d_1\) is given by

\[
d_1(f, g) = \sup_{x \in M} \| f(x) - g(x) \| + \sup_{x \in M} \| Df(x) - Dg(x) \|
\]

and \(Df\) stands for the derivative of \(f\).

Let \(g : X \to X\) be a map. Define \(g^n\) as the \(n\)th iterate of \(g\) by : \(g^0 = \text{id}_X\) and \(g^n = g \circ g^{n-1}\), where \(\text{id}_X\) is the identity function on \(X\). Consider a finite collection of maps \(g_i : X \to X, i = 1, \ldots, s\). Then the iterated function system \(G(X; g_1, \ldots, g_s)\) on \(X\) generated by a collection of maps \(\{g_1, \ldots, g_s\}\) on \(X\) is given by iterates \(g_i \circ \cdots \circ g_h\) with \(i_j \in \{1, \ldots, s\}\).
The proof of Theorem 1.1 is found in [1]. It is based on two parts, one locally and one globally. First an iterated function system of \(n + 1\) diffeomorphisms is constructed which is locally (on a subset of \(M\)) minimal. Furthermore, this subset has no empty interior.

Secondly, there exists a diffeomorphism that iterates a dense subset of \(M\) to a point in this non-empty interior. This diffeomorphism together with its inverse and the \(n + 1\) diffeomorphisms form an ifs that is minimal on the manifold. It is proven that an ifs consisting of generators that are taken sufficiently \(C^1\)-close is also minimal.

The first part of the proof is based on the fact that manifolds are locally Euclidean. For some iterated function systems that consists of contractions on the Euclidean space it is fairly easy to prove it is minimal on a subset with non-empty interior. Therefore, it makes sense to construct an iterated function system of diffeomorphisms on \(M\) that behave like these contractions on a small subset of \(M\). Iterated function systems that consists of contractions on the Euclidean space often create fractals.

2 Fractals and some of their characteristics

The name fractal was coined by the famous mathematician Mandelbrot to describe objects that were too irregular to fit into a traditional geometrical setting [6]. Often a fractal has some sort of self-similarity (they are made up of parts that resemble the whole in some way), perhaps approximate or statistical. In most cases of interest a fractal is defined in a very simple way, perhaps recursively [6].

One of the best known and most easily constructed fractals is the middle third Cantor set. It is constructed from a unit interval by a sequence of deletion operations; see figure 1. Let \(E_0\) be the interval \([0, 1]\). Let \(E_1\) be the set obtained by deleting the middle third of \(E_0\), so that \(E_1\) consists of the two intervals \([0, \frac{1}{3}]\) and \([\frac{2}{3}, 1]\). We continue in this way, with \(E_k\) obtained by deleting the middle third of each interval in \(E_{k-1}\). Thus \(E_k\) consists of \(2^k\) intervals each of length \(3^{-k}\). The middle third Cantor set \(F\) consists of the numbers that are in \(E_k\) for all \(k\), that is, \(F\) is the intersection \(\bigcap_{k=0}^{\infty} E_k\). The Cantor set \(F\) may be thought of as the limit of the sequence of sets \(E_k\) as \(k\) tends to infinity.

\[
\begin{array}{cccccc}
0 & \frac{1}{3} & \frac{2}{3} & 1 \\
\hline
& \ & \ & \ & \ & \ & E_0 \\
\hline
\ & \ & \ & \ & \ & \ & E_1 \\
\ & \ & \ & \ & \ & \ & E_2 \\
\ & \ & \ & \ & \ & \ & E_3 \\
\ & \ & \ & \ & \ & \ & E_4 \\
\ & \ & \ & \ & \ & \ & E_5 \\
\ & \ & \ & \ & \ & \ & \vdots \\
\ & \ & \ & \ & \ & \ & F \\
\ & \ & \ & \ & \ & \ & F_R \\
\end{array}
\]

**Figure 1:** Construction of the middle third Cantor set \(F\), by repeated removal of the middle third of intervals. Note that \(F_L\) and \(F_R\), the left and right parts of \(F\), are copies of \(F\) scaled by a factor \(\frac{1}{3}\).

At first glance it might appear that we have removed so much of the interval \([0, 1]\) during the construction of \(F\), that nothing remains [6]. This is not the case, in fact, \(F\) is an uncountable-infinite
set, which contains infinitely many numbers in every neighborhood of each of its points. The middle third Cantor set \( F \) consists precisely of those numbers in \([0, 1]\) whose base-3 (or ternary) expansion does not contain the digit 1, i.e. all numbers \( a_13^{-1} + a_23^{-2} + a_33^{-3} + \cdots \) with \( a_i = 0 \) or 2 for each \( i \).

To see this, note that to get \( E_1 \) from \( E_0 \) we remove those numbers with \( a_1 = 1 \), to get \( E_2 \) from \( E_1 \) we remove those numbers with \( a_2 = 1 \), and so on. The fraction \( \frac{1}{3} \) in \( F \) can also be represented as a ternary expansion that does not contain the digit 1. In the decimal system 1 is equal to 0.99999... \(^1\). Likewise, in the ternary notation 1 is equal to 0.22222... \( \_3 \) and so \( \frac{1}{3} \) can be expressed as 0.1, or 0.02222... \( \_3 \).

It may appear that only the endpoints are left, but that is not the case either. The number \( \frac{1}{4} \), for example, is in the left-third interval, so it is not removed at the first step, and in the right-third of that interval in the second step, and in the left-third of that, in the third step. Continuing in this fashion, ad infinitum, alternating between left- and right-thirds. Since it is never in one of the middle thirds, it is never removed, and yet it is also not one of the endpoints of any middle third \([8]\). In ternary notation \( \frac{1}{4} \) can be written as 0.02020202020... \( \_3 \). It can be proved that there are as many points in the Cantor set as there are in \([0, 1]\). To see this, for any number \( y \) in \([0, 1]\), its binary representation can be translated into a ternary representation of a number \( x \) in \( F \) by replacing all the 1’s by 2’s. For instance if \( y = 3/10 = 0.0100110011001..._2 \), we write \( x = 0.0200220022002..._3 = 7/30 \). Consequently, we have a surjective image from \( F \) to the unit interval \([8]\). This set is uncountably infinite, while the endpoints of all deleted intervals are countably infinite, so in the sense of cardinality, most members of the Cantor set are not endpoints of deleted intervals.

Although \( F \) contains uncountable infinitely many points its length is \( 1 - \frac{1}{3} \sum_{k=0}^{\infty}(\frac{2}{3})^k = 1 - 1 = 0 \).

We list some features of the middle third Cantor set \( F \) because similar features are found in many different fractals.

(a) \( F \) is self-similar: the part of \( F \) in the interval \([0, \frac{1}{3}]\) and the part of \( F \) in \([\frac{2}{3}, 1]\) are both geometrically similar to \( F \), scaled by a factor \( \frac{1}{3} \), and so on. The Cantor set contains copies of itself at many different scales.

(b) The set \( F \) has a ‘fine structure’; that is, it contains detail at arbitrarily small scales. The more we ‘enlarge’ the figure of the Cantor set, the more gaps become apparent to the eye.

(c) Although \( F \) has an intricate detailed structure, the actual definition of \( F \) is very straightforward.

(d) \( F \) is obtained by a recursive procedure. Our construction consisted of repeatedly removing the middle thirds of intervals. Successive steps give increasingly good approximations \( E_k \) to the set \( F \).

Another example of a fractal that shows similar features as the middle third Cantor set is the von Koch curve, see figure 2. We let \( E_0 \) be a line segment of unit length. The set \( E_1 \) consists of the four segments obtained by removing the middle third of \( E_0 \) and replacing it by the other two sides of the equilateral triangle based on the removed segment. We continue in this way, with \( E_k \) obtained by removing the middle third of \( E_{k-1} \) and replacing it by the other two sides of the equilateral triangle. \( E_k \) approaches a limiting curve \( F \), called the von Koch curve.

\(^1x = 0, 907 \iff 10x = 9, 907 \iff 10x - x = 9 \iff x = 1.\)
Figure 2: (a) Construction of the von Koch curve $F$. At each step, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three von Koch curves fitted together to form a snowflake curve.

Calculating the length of $E_k$ shows it is of length $(\frac{4}{3})^k$ and letting $k$ tend to infinity implies $F$ has infinite length. The self-similarity of a fractal can be used to define the fractal. Iterated function systems do this in a unified way.

3 Iterated function systems generated by contractions

Let $D$ be a closed subset of $\mathbb{R}^n$ (often $D = \mathbb{R}^n$) and $d$ a metric on $D$. A mapping $S : D \to D$ is called a contraction on $D$ if there is a number $q$ with $0 < q < 1$ such that $d(S(x), S(y)) \leq qd(x, y)$ for all $x, y \in D$. Clearly any contraction is continuous.

From hereon, we call an iterated function system $\mathcal{G}(X; S_1, \ldots, S_m)$ on $X$ generated by a finite family of contractions $\{S_1, \ldots, S_m\}$, with $m \geq 2$, an IFS. We call a non-empty compact subset $F$ of $D$ an invariant set for the IFS if

$$F = \bigcup_{i=1}^{m} S_i(F).$$

We call a non-empty compact subset $F$ of $D$ an attractor for the IFS if, for every $x \in X$ and for all $1 \leq i \leq m : d((S_{i_1} \circ \cdots \circ S_{i_n})(x), F) \to 0$ as $n \to \infty$. So, points that get close enough to the attractor remain close even if slightly disturbed.
The fundamental property of an IFS is that it determines a unique attractor, which is an invariant set and usually a fractal. For a simple example, take $F$ to be the middle third Cantor set. Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be given by

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$ 

Then $S_1(F)$ and $S_2(F)$ are just the left and right 'halves' of $F$, so that $F = S_1(F) \cup S_2(F)$; thus $F$ is an invariant set of the IFS consisting of the contractions $\{S_1, S_2\}$, the two mappings, which represent the basic self-similarities of the Cantor set.

We shall prove the fundamental property that an IFS has a unique (non-empty compact, i.e. closed and bounded) invariant set that is an attractor. We will give the proof from the book 'Fractal Geometry' [6].

We start by defining a metric or distance $d_H$ between the closed subsets $D$ of $\mathbb{R}^n$, often $D = \mathbb{R}^n$. Let $S$ denote the class of all non-empty compact subsets of $D$.

**Definition 3.1.** Recall that the $\delta$-neighborhood of a set $A$ is the set of points within distance $\delta$ of $A$, i.e. $A_\delta = N(A, \delta) = \{x \in D : d(x, a) < \delta \text{ for some } a \in A\}$. We make $S$ into a metric space by defining the distance between two sets $A$ and $B$ to be the least $\delta$ such that the $\delta$-neighborhood of $A$ contains $B$ and vice versa:

$$d_H(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\}$$

(see figure 3).

A simple check shows that $d_H$ is a metric or distance function, that is, satisfies the three requirements

(i) $d_H(A, B) \geq 0$ with equality if and only if $A = B$,  
(ii) $d_H(A, B) = d_H(B, A)$, 
(iii) $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$ for all $A, B, C \in S$. The metric $d_H$ is known as the *Hausdorff metric* on $S$. In particular, if $d_H(A, B)$ is small, then $A$ and $B$ are close to each other as sets.

**Figure 3:** The Hausdorff distance between the sets $A$ and $B$ is the least $\delta > 0$ such that the $\delta$-neighborhood $A_\delta$ of $A$ contains $B$ and the $\delta$-neighborhood $B_\delta$ of $B$ contains $A$.

We give the proof of the fundamental result on IFS that depends on the following two theorems. The first shows that the Hausdorff metric inherits completeness. The second one shows that every contraction on a nonempty complete metric space admits a unique fixed point.
Theorem 3.1. If $(X,d)$ is a complete metric space, then the Hausdorff metric $d_H$ induced by $d$ is also complete.

Proof. Suppose $(A_n)$ is a Cauchy sequence with respect to the Hausdorff metric. That is,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n, m > N : d_H(A_n, A_m) < \varepsilon.$$ 

By selecting a subsequence if necessary, we may assume that $d_H(A_n, A_{n+1}) < 2^{-n}$. This is in the notation of Definition 3.1 the same as $A_n \subset \mathcal{N}(A_{n+1}, 2^{-n})$ and $A_{n+1} \subset \mathcal{N}(A_n, 2^{-n})$.

Claim 3.1. For any $N \in \mathbb{N}$, there is a sequence $(x_n)_{n \geq N}$ in $X$ such that $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Any such sequence is Cauchy with respect to $d$ and thus converges to some $x \in X$.

By applying the triangle inequality, we see that for any $n \geq N$, $d(x_n, x) < 2^{-n+1}$.

Define $A$ to be the set of all $x \in X$ such that $x$ is the limit of a sequence $(x_n)_{n \geq 0}$ with $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Then $A$ is nonempty.

Furthermore, by the definition of $A$ and the claim it follows, that for any $n$, if $x \in A$, then there is some $x_n \in A_n$ such that $d(x_n, x) < 2^{-n+1}$, and so

$$A \subset \{ x \in A : d(x, x_n) < 2^{-n+1} \text{ for some } x_n \in A_n \} = \mathcal{N}(A_n, 2^{-n+1}). \quad (3.1)$$

Consequently, the set $A$ is bounded. Hence $\overline{A} \in \mathcal{S}$.

Suppose $\varepsilon > 0$. Thus $\varepsilon > 2^{-N} > 0$ for some $N \in \mathbb{N}$. Let $n > N + 1$. Then by applying the claim we have that for any $x_n \in A_n$, there is some $x \in X$ with $d(x_n, x) < 2^{-n+1}$. That is, $A_n \subset \{ x_n \in A_n : d(x_n, x) < 2^{-n+1} \text{ for some } x \in \overline{A} \subseteq X \} = \mathcal{N}(\overline{A}, 2^{-n+1})$. With (3.1) this shows that $d_H(A_n, \overline{A}) < 2^{-n+1} < \varepsilon$. Hence the sequence $(A_n)$ converges to $A$ in the Hausdorff metric. $\square$

Proof of Claim 3.1. Because the $A_i$’s are non-empty compact subsets, for any $N \in \mathbb{N}$, there is a sequence $(x_n)_{n \geq N}$ in $X$ such that $x_n \in A_n$ and $d(x_n, x_{n+1}) < d_H(A_n, A_{n+1}) < 2^{-n}$.

If $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Thus, if $N < m < n$ we have

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + \ldots + d(x_{n-1}, x_n) < \sum_{k=m}^{n-1} 2^{-k} < 2^{-m+1} \sum_{k=1}^{\infty} 2^{-k} = 2^{-m+1}(1) \leq 2^{-N} < \varepsilon.$$ 

Hence the sequence $(x_n)$ is Cauchy. By applying the triangle inequality again, we see that for any $n \geq N$

$$d(x_n, x) \leq d(x_n, x_{n+1}) + \ldots + d(x_l, x) \leq \sum_{k=n}^{l} 2^{-k} \leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n}(2) = 2^{-n+1}.$$ 

$\square$

Theorem 3.2 (Banach’s contraction mapping theorem). Let $(X,d)$ be a nonempty complete metric space. Let $T : X \to X$ be a contraction on $X$. Then the map $T$ admits an unique fixed point $x^*$ in $X$. That is, there is one and only one $x^* \in X$ such that $T(x^*) = x^*$.

The proof can be found in the Appendix. Now we can state the fundamental result on IFS and prove it.
Theorem 3.3. Consider the IFS given by the contractions \( \{S_1, \ldots, S_m\} \) on \( D \subset \mathbb{R}^n \), so that
\[
|S_i(x) - S_i(y)| \leq c_i|x - y| \quad (x, y) \in D
\] (3.2)
with \( c_i < 1 \) for each \( i \). Then there is a unique invariant set \( F \), i.e. a non-empty compact set such that
\[
F = \bigcup_{i=1}^m S_i(F).
\] (3.3)
Moreover, if we define a transformation \( S \) on the class \( S \) of non-empty compact sets in \( D \) by
\[
S(E) = \bigcup_{i=1}^m S_i(E)
\] (3.4)
for \( E \in S \), and write \( S^k \) for the \( k \)th iterate of \( S \) (so \( S^0(E) = E \) and \( S^k(E) = S(S^{k-1}(E)) \) for \( k \geq 1 \), then
\[
F = \lim_{k \to \infty} S^k(E).
\] (3.5)
If \( E \in S \) such that \( S_i(E) \subset E \) for all \( i \), then
\[
F = \bigcap_{k=0}^\infty S^k(E).
\] (3.6)

Proof. Note that sets in \( S \) are transformed by \( S \) into other sets of \( S \). If \( A, B \in S \) then
\[
d_H(S(A), S(B)) = d_H\left(\bigcup_{i=1}^m S_i(A), \bigcup_{i=1}^m S_i(B)\right) \leq \max_{1 \leq i \leq m} d_H(S_i(A), S_i(B))
\]
using the definition of the metric \( d_H \) and noting that if the \( \delta \)-neighborhood \( (S_i(A))_\delta \) contains \( S_i(B) \) for all \( i \) then \( (\bigcup_{i=1}^m S_i(A))_\delta \) contains \( \bigcup_{i=1}^m S_i(B) \), and vice versa. By (3.2)
\[
d_H(S(A), S(B)) \leq (\max_{1 \leq i \leq m} c_i) d_H(A, B).
\] (3.7)

By Theorem 3.1 \( d_H \) is a complete metric on \( S \), that is every Cauchy sequence of sets in \( S \) is convergent to a set in \( S \). Since 0 < \( \max_{1 \leq i \leq m} c_i < 1 \), (3.7) states that \( S \) is a contraction on the complete metric space \((S, d_H)\). By Banach’s contraction mapping theorem there is a unique set \( F \in S \) such that \( S(F) = F \), which is (3.3), and moreover \( S^k(E) \to F \) as \( k \to \infty \).

If \( S_i(E) \subset E \) for all \( i \) then \( S(E) \subset E \), so that \( S^k(E) \) is a decreasing sequence of non-empty compact sets. Hence the limit of this sequence, which is \( F \), must equal the intersection \( \bigcap_{k=0}^\infty S^k(E) \).

If the invariant set \( F \) has property (3.5) we call \( F \) an attractor.

4 Minimality

A map \( f : X \to X \) is minimal if each closed subset \( A \subset X \) such that \( f(A) \subset A \) is empty or coincides with \( X \). Likewise, we can define minimality for an iterated function system.

Definition 4.1. The iterated function system \( \mathcal{G}(X; f_1, \ldots, f_s) \) is minimal if each closed subset \( A \subset X \) such that \( f_i(A) \subset A \) for all \( i \), is empty or coincides with \( X \).

Equivalently, for a minimal iterated function system \( \mathcal{G}(X; f_1, \ldots, f_s) \), for any \( x \in X \) the collection of iterates \( f_{i_1} \circ \cdots \circ f_{i_k}(x) \), \( i_j \geq 0 \), is dense in \( X \).
For \( X = \{x\} \) the two definitions are trivially equivalent. To see that the two definitions of minimality are equivalent for \( X \setminus \{x\} \neq \emptyset \), suppose the first definition holds. A point \( \{x\} \in X \) is a closed nonempty subset and \( \{x\} \neq X \) so by assumption of minimality therefore \( f_i(x) \neq x \), for some \( i \geq 0 \). Define \( A \) as the closure of the collection of all iterates \( f_i \circ \cdots \circ f_{i_k}(x) \), \( i_j \geq 0 \). Then \( A \) is a closed subset that is nonempty (contains \( x \)), and every iteration of this subset is by definition contained in itself. Hence by minimality \( A = X \).

On the other hand, suppose that the first definition is false. Then there is a closed subset \( A \) that is not equal to the empty set or \( X \) such that \( f_i(A) \subset A \) for all \( i \). Then the closure of \( f_i \circ \cdots \circ f_{i_k}(x) \), \( i_j \geq 0 \) for any \( x \in A \) is at most \( A \). Hence also the second definition is false.

We can now prove the following lemma:

**Lemma 4.1.** The unique attractor \( F \) from Theorem 3.3 is the unique non-empty compact set such that the iterated function system \( \mathcal{G}(F; S_1, \ldots, S_m) \) is minimal.

**Proof.** Let \( E \subset F \) be an arbitrary non-empty closed subset such that \( S_i(E) \subset E \) for all \( i \). By (3.6) of Theorem 3.3, \( F \) can then be written as \( \bigcap_{k=0}^{\infty} S^k(E) \) and so, \( F \subset S^k(E) \) for all \( k \). So in particular, \( F \subset S^0(E) = E \). With our assumption this implies that \( E = F \). Hence by the first definition of definition 4.1 \( \mathcal{G}(F; S_1, \ldots, S_m) \) is minimal. \( \square \)

## 5 Invariant sets with nonempty interior generated by two contractions

In this chapter we will give examples of \textbf{IFS} with two affine contractions that have an invariant set that is minimal and with nonempty interior on \( \mathbb{R}^n \), for every \( n \geq 0 \).

Take the following two affine maps \( S_1, S_2: \mathbb{R} \to \mathbb{R} \) given by

\[
S_1(x) = \left( \frac{1}{2} + 2\varepsilon \right)x - \varepsilon; \quad S_2(x) = \left( \frac{1}{2} + 2\varepsilon \right)x + \frac{1}{2} - \varepsilon.
\]

They are turned into contractions if we define \( q = \frac{1}{2} + 2\varepsilon \) where \( 0 \leq \varepsilon < \frac{1}{4} \). Because \( q \in (0, 1) \) and \( d(S_1(x), S_1(y)) = d(S_2(x), S_2(y)) \leq q d(x, y) \) for all \( x, y \in \mathbb{R} \) (in the Euclidean metric \( d \)).

If we take \( B \) to be the unit interval and \( S(B) = S_1(B) \cup S_2(B) \), with \( S^k \) for the \( k \)th iterate of \( S \) for \( k \geq 1 \)(like in (3.4) of Theorem 3.3), then by construction:

\[
B \subset S(B) \subset S^2(B) \subset \ldots \subset S^k(B) \subset \ldots
\]

by (3.5) of Theorem 3.3 this means that \( B \subset F \) where \( F \) is the unique attractor for the iterated function system given by the contractions \( S_1 \) and \( S_2 \). This attractor is a minimal set by Lemma 4.1 that contains the unit interval. Hence we have a minimal set with a nonempty interior.

In this case it is easy to compute \( F \). It is the interval \([-p, 1+p] \) where,

\[
p = \varepsilon \sum_{k=0}^{\infty} \left( \frac{1}{2} + \varepsilon \right)^k = \frac{\varepsilon}{1 - \left( \frac{1}{2} + \varepsilon \right)} = \frac{2\varepsilon}{1 - 4\varepsilon}.
\]

For instance, if we choose \( \varepsilon = \frac{1}{4} \) then,

\[
p = \frac{1}{8} \sum_{k=0}^{\infty} \left( \frac{3}{4} \right)^k = \frac{1}{8} \frac{1}{1 - \frac{3}{4}} = \frac{1}{2}
\]
hence $F$ is $[-\frac{1}{2}, \frac{3}{2}]$.

It is possible to create minimal sets with nonempty interior for higher dimensions in a similar way. That is, by using only two contractions. We define two mappings $S_1, S_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$S_1(x_1, x_2) = \begin{pmatrix}
0 & \frac{1}{a_1}(\frac{1}{2} + 2\varepsilon) \\
-a_1 - 2\varepsilon & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
-\varepsilon \\
a_1 + \varepsilon
\end{pmatrix}
$$

and

$$S_2(x_1, x_2) = \begin{pmatrix}
0 & \frac{1}{a_1}(-\frac{1}{2} - 2\varepsilon) \\
a_1 + 2\varepsilon & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
1 + \varepsilon \\
-\varepsilon
\end{pmatrix}
$$

and turn them into contractions by choosing $\frac{1}{2} < \frac{1}{2} + 2\varepsilon < a_1 < a_1 + 2\varepsilon < 1$.

If we start with a 2-box (in two dimensions) with vertices $(0, 0), (1, 0), (0, a_1)$ and $(1, a_1)$ then the image of this box under $S_1$ is a shrunken and tilted version that covers the left half of the original box. The contraction $S_2$ does the same but tilts in the other direction such that it covers the other side of the box. In every iteration step, the union of the images under $S_1$ and $S_2$ cover the previous images (see figure 4).

![Figure 4:](image_url) To ensure that $S_1$ and $S_2$ are contractions such that the union of images under $S_1$ and $S_2$ cover the images of previous iterations, the length and the width of the boxes have to shrink in every iteration step, such that its union cover the previous boxes. If we choose $\frac{1}{2} < \frac{1}{2} + 2\varepsilon < a_1 < a_1 + 2\varepsilon < 1$, this is ensured. In the figure $a_1 = \frac{3}{4}$ and $a_1 + 2\varepsilon = \frac{17}{30}$.

If we define $B$ to be the 2-box with lengths 1 and $a_1$ then again by construction of $S_1, S_2$:

$$B \subset S(B) \subset S^2(B) \subset \ldots \subset S^k(B) \subset \ldots$$

such that $B \subset F$ where $F$ is the unique attractor for the iterated function system given by the contractions $S_1$ and $S_2$. Hence we have a minimal set with a nonempty interior in $\mathbb{R}^2$.

Likewise, we can generalize our definitions of $S_1, S_2$ to $n$ dimensions (see figure 5). We define
two affine maps $S_1, S_2 : \mathbb{R}^n \to \mathbb{R}^n$ by

$$
\begin{pmatrix}
0 & \frac{1}{a_1}(\frac{1}{2} + 2\varepsilon) & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{a_2}(-a_1 - 2\varepsilon) & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \vdots & \ddots & \ddots & 0 \\
-a_{n-1} - 2\varepsilon & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
-x \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

for $S_1$ and

$$
\begin{pmatrix}
0 & \frac{1}{a_1}(-\frac{1}{2} - 2\varepsilon) & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{a_2}(a_1 + 2\varepsilon) & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \vdots & \ddots & \ddots & 0 \\
a_{n-1} + 2\varepsilon & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
1 + \varepsilon \\
-\varepsilon \\
\vdots \\
-\varepsilon
\end{pmatrix}
$$

for $S_2$.

If we choose $a_1, \ldots, a_{n-1}$ and $\varepsilon$ such that $\frac{1}{2} < \frac{1}{2} + 2\varepsilon < a_1 < a_1 + 2\varepsilon < a_2 < a_2 + 2\varepsilon < \ldots < a_{n-1} < a_{n-1} + 2\varepsilon < 1$ then $S_1$ and $S_2$ are contractions.

If we take $B$ to be the $n$-box with lengths $1, a_1, \ldots, a_{n-1}$ for $n \geq 3$ then again by construction:

$$B \subset S(B) \subset S^2(B) \subset \ldots \subset S^k(B) \subset \ldots$$

such that $B \subset F$.

![Figure 5: A 3-box with vertices (0,0,0), (1,0,0), (0,a_1,0), (1,a_1,0), (0,0,a_2), (1,0,a_2), (0,a_1,a_2) and (1,a_1,a_2). If we choose $a_1$, $a_2$ and $\varepsilon$ such that $\frac{1}{2} < \frac{1}{2} + 2\varepsilon < a_1 < a_1 + 2\varepsilon < a_2 < a_2 + 2\varepsilon < 1$ then $S_1$ and $S_2$ are contractions and the image(s) of the box(es) under $S_1$ cover(s) one side of the previous image(s) and $S_2$ does the same but covers the other side. Together they cover all the images of previous iterations.](image)

Similar to the affine contractions $S_1$ and $S_2$ we can construct an affine expansion $T$ (that is, $T^{-1}$ is an affine contraction) which has its repelling fixed point contained in the interior of $B$ (and therefore in $F$). For example, the affine expansion $T$ on $\mathbb{R}^n$ given by

$$T(x_1, x_2, \ldots, x_n) = \left(2x_1 - \frac{1}{2}, 2x_2 - \frac{a_1}{2}, \ldots, 2x_n - \frac{a_{n-1}}{2}\right),$$

has its repelling fixed point in the center of the $n$-box.

Collecting the results we have the following: from Lemma 4.1 we know that the iterated function system $G(F; S_1, S_2)$ is minimal. Equivalently, by Definition 4.1, for any $x \in F$ the collection of
iterates of \( x \) under \( S_1 \) and \( S_2 \) is dense in \( F \). The iterates of this collection under \( T \) is dense in \( \mathbb{R}^n \). Resulting in the following lemma.

**Lemma 5.1.** The iterated function system \( G(\mathbb{R}^n; S_1, S_2, T) \) is minimal.

Recall the \( C^1 \) topology that is generated by the open balls \( B_1 \). By definition, if \( f \) is an affine contraction, every map that is contained in the open ball \( B_1(f, r) \) must be an affine contraction too. Because the contractions are affine, the collinearity relation between points and the ratios of distances along a line are preserved. This along with the lemma gives the following corollary.

**Corollary 5.1.** There exists a neighborhood \( W \subset \text{Diff}^1(\mathbb{R}^n) \times \text{Diff}^1(\mathbb{R}^n) \) of \((S_1, S_2)\) such that each element \( F = (f_1, f_2) \) in this neighborhood admits an invariant set which is minimal and has nonempty interior.

## 6 Vector fields, Transversality and Critical elements

As is mentioned in the introduction, there exists a diffeomorphism that iterates a dense subset of \( M \) to a point. This uses a gradient Morse-Smale vector field on \( M \), which has a unique hyperbolic repellor and attractor. What is meant by this is explained in this chapter and the next. Therefore, those who are familiar with this concept, or only interested in the global idea can skip this chapter and the next and continue from thereon.

We start with some basic concepts. Recall that for a point \( x \) in \( M \) we can attach a copy of \( \mathbb{R}^n \) tangential to \( M \) at \( x \). The resulting structure \( T_x M \) is the tangent space of \( M \) at \( x \). Then for a smooth curve in \( M \) passing through \( x \), the derivative of the curve at \( x \) is a vector in \( T_x M \).

Since a tangent space \( T_p M \) is the set of all tangent vectors to \( M \) at \( p \), the tangent bundle \( TM \) is the collection of all tangent vectors, along with the information of the point to which they are tangent

\[
TM = \{(p, v) : p \in M, v \in T_p M\}.
\]

A section of the tangent bundle, meaning that to every point \( x \) in a manifold \( M \), a vector \( X(x) \in T_x M \) is associated, is called a vector field.

We shall only consider vector fields of class \( C^1 \) and define \( \mathfrak{X}(M) \) to be the space of \( C^1 \) vector fields on \( M \) endowed with the \( C^1 \) topology. We will now give a proposition whose proof can be found in [15].

**Proposition 6.1.** Let \( M \) be a compact manifold and \( X \in \mathfrak{X}(M) \). There exists on \( M \) a global \( C^1 \) flow for \( X \). That is, there exists a \( C^1 \) map \( \varphi : \mathbb{R} \times M \to M \) such that \( \varphi(0, p) = p \) and \( (\partial/\partial t)\varphi(t, p) = X(\varphi(t, p)) \).

From this we obtain the following:

Let \( X \in \mathfrak{X}(M) \) and let \( \varphi : \mathbb{R} \times M \to M \) be the flow determined by \( X \). For each \( t \in \mathbb{R} \) the map \( X_t : M \to M, X_t(p) = \varphi(t, p) \), is a \( C^1 \) diffeomorphism. Moreover, \( X_0 = \text{identity} \) and \( X_{t+s} = X_t \circ X_s \) for all \( t, s \in \mathbb{R} \).

Two submanifolds of a given finite dimensional smooth manifold are said to intersect transversally if at every point of intersection, their separate tangent spaces at that point together span the tangent
space of the ambient manifold at that point. Formally, two submanifolds $Y$ and $Z$ in a space $M$ intersect transversally if, for all $p \in Y \cap Z$,

$$T_p Y + T_p Z = \{ v + w : v \in T_p Y, w \in T_p Z \} = T_p M.$$  

*Transversality* can be seen as the “opposite” of tangency. If two submanifolds do not intersect, then they are automatically transversal. For example, two curves in $\mathbb{R}^3$ are transversal only if they do not intersect at all.

An integral curve of $X \in \mathfrak{X}(M)$ through a point $p \in M$ is a $C^2$ map $a : \mathbb{R} \to M$, where $I$ is an interval containing 0, such that $a(0) = p$ and $a'(t) = X(a(t))$ for all $t \in I$. The image of an integral curve is called an *orbit* or trajectory.

Let $X \in \mathfrak{X}(M)$ and let $X_t$, $t \in \mathbb{R}$, be the flow of $X$. The orbit of $X$ through $p \in M$ is the set $\{X_t(p) : t \in \mathbb{R}\}$. If $X(p) = 0$ the orbit of $p$ reduces to $p$. In this case we say that $p$ is a *singularity* of $X$. Otherwise, the map $a : \mathbb{R} \to M$, $a(t) = X_t(p)$, is an immersion, that is, the derivative of $a$ is everywhere injective.

If $a$ is not injective there exists $\omega > 0$ such that $a(\omega) = a(0) = p$ and $a(t) \neq p$ for $0 < t < \omega$. In this case the orbit of $p$ is diffeomorphic to the circle $S^1$ and we say that it is a *closed orbit* with *period* $\omega$. If the orbit is not singular or closed it is called *regular*. Thus a regular orbit is the image of an injective immersion of the line.

We call singularities and closed orbits the *critical elements*.

The *$\omega$-limit set* of a point $p \in M$, $\omega(p)$, is the set of those points $q \in M$ for which there exists a sequence $t_n \to \infty$ with $X_{t_n}(p) \to q$. Similarly, we define the *$\alpha$-limit set* of $p$ as $\alpha(p) = \{ q \in M : X_{t_n}(p) \to q \text{ for some sequence } t_n \to -\infty \}$. We note that $\omega(p) = \omega(\tilde{p})$ if $\tilde{p}$ belongs to the orbit $p$. Thus we can define the $\omega$-limit of the orbit of $p$ as $\omega(p)$. Intuitively $\omega(p)$ is where the orbit of $p$ is “born” and $\omega(p)$ is where it “dies”. See figure 6 for an example.

![Figure 6](image)

**Figure 6:** We consider the unit sphere $S^2 \subset \mathbb{R}^3$ with centre at the origin and standard coordinates $(x, y, z)$ in $\mathbb{R}^3$. We call $p_N = (0, 0, 1)$ and $p_S = (0, 0, -1)$ in $S^2$. We define the vector field $X$ on $S^2$ by $X(x, y, z) = (-xz, -yz, x^2 + y^2)$. Then $X$ is of class $C^\infty$ and the singularities of $X$ are $p_N, p_S$. As $X$ is tangent to the meridians of $S^2$ and points upwards, $\alpha(p) = p_S$ and $\omega(p) = p_N$ if $p \in S^2 \setminus \{ p_N, p_S \}$.

**Definition 6.1.** Let $X \in \mathfrak{X}(M)$ and let $p \in M$ be a singularity of $X$ (so $X(p) = 0$). We say that $p$ is a hyperbolic singularity if $DX_p : T_p M \to T_p M$ is a hyperbolic linear vector field, that is, $DX_p$ has no eigenvalue on the imaginary axis.
Definition 6.2. Let $p \in M$ be a fixed point of $f \in \text{Diff}^1(M)$ (so $f(p) = p$). We say that $p$ is a hyperbolic fixed point if $Df_p : T_p M \to T_p M$ is a hyperbolic isomorphism, that is, if $Df_p$ has no eigenvalue of modulus $1(|\lambda| \neq 1)$.

Let $\gamma$ be a closed orbit of a vector field $X \in \mathfrak{X}(M)$. Through a point $x_0 \in \gamma$ we consider a section $\Sigma$ transversal to the field $X$. The orbit through $x_0$ returns to intersect $\Sigma$ at time $\tau$, where $\tau$ is the period of $\gamma$. By the continuity of the flow of $X$ the orbit through a point $x \in \Sigma$ sufficiently close to $x_0$ also returns to intersect $\Sigma$ at a time near to $\tau$. Thus if $V \subset \Sigma$ is a sufficiently small neighborhood of $x_0$ we can define a map $P : V \to \Sigma$ which to each point $x \in V$ associates $P(x)$, the first point where the orbit of $x$ returns to intersect $\Sigma$. This map is called the Poincaré map associated to the orbit $\gamma$ (and the section $\Sigma$). Notice that $P$ is $C^1$.

Definition 6.3. Let $p \in \gamma$ where $\gamma$ is a closed orbit of $X$. Let $\Sigma$ be a section transversal to $X$ through the point $p$. We say that $\gamma$ is a hyperbolic closed orbit of $X$ if $p$ is a hyperbolic fixed point of the Poincaré map $P : V \subset \Sigma \to \Sigma$.

7 Gradient Morse-Smale vector fields

Before defining the set of Morse-Smale vector fields we still need some new concepts and notation.

Let $X \in \mathfrak{X}(M)$. Consider the sets $L_\alpha(X) = \{p \in M : p \in \alpha(q) \text{ for some } q \in M\}$ and $L_\omega(X) = \{p \in M : p \in \omega(q) \text{ for some } q \in M\}$. These sets are invariant by the flow generated by $X$ and the orbit of any point is “born” in $L_\alpha(X)$ and “dies” in $L_\omega(X)$.

Let $M$ be a compact manifold. Two vector fields $X, Y \in \mathfrak{X}(M)$ are topologically equivalent if there exists a homeomorphism $h : M \to M$ which takes orbits of $X$ to orbits of $Y$ preserving their orientation. So if $p \in M$ and $\delta > 0$ there exists an $\varepsilon > 0$ such that, for $0 < t < \delta$, $hX_t(p) = Y_{t'}(h(p))$ for some $0 < t' < \varepsilon$.

We now define $X$ and $Y$ to be topologically equivalent at $p$ and $q$ respectively if there exists neighborhoods $V_p$ and $W_q$ and a homeomorphism $h : V_p \to W_q$, with $h(p) = q$, which takes orbit pieces in $V_p$ of $X$ to orbit pieces in $W_q$ of $Y$ preserving their orientation.

Definition 7.1. Let $X \in \mathfrak{X}(M)$. We say that $p \in M$ is a wandering point for $X$ if there exists a neighborhood $V$ of $p$ and a number $t_0 > 0$ such that $X_t(V) \cap V = \emptyset$ for $|t| > t_0$. Otherwise we say that $p$ is nonwandering.

We write $\Omega(X)$ for the set of nonwandering points of $X$. The following properties follow easily from this definition:

(a) $\Omega(X)$ is compact and invariant by the flow $X_t$;
(b) $\Omega(X) \supset L_\alpha(X) \cup L_\omega(X)$. In particular, $\Omega(X)$ contains the critical elements of $X$;
(c) if $X, Y \in \mathfrak{X}(M)$ and $h : M \to M$ is a topological equivalence between $X$ and $Y$ then $h(\Omega(X)) = \Omega(Y)$.

In general $\Omega$ contains $L_\alpha \cup L_\omega$ strictly.

Finally, we can collect everything so far in the following definition:
Definition 7.2. Let $M$ be a compact manifold of dimension $n$ and let $X \in \mathfrak{X}(M)$. We say that $X$ is a Morse-Smale vector field if:

1. $X$ has a finite number of critical elements (singularities and closed orbits) all of which are hyperbolic;
2. If $\sigma_1$ and $\sigma_2$ are critical elements of $X$ then $W^S(\sigma_1)$ is transversal to $W^U(\sigma_2)$;
3. $\Omega(X)$ is equal to the union of the critical elements of $X$.

Definition 7.3. Let $X \in \mathfrak{X}(M)$ and $p \in M$. We say that $X$ is locally stable at $p$ if for any given neighborhood $U(p) \subset M$ there exists a neighborhood $V_X$ of $X$ in $\mathfrak{X}(M)$ such that for each $Y \in V_X$, $X$ at $p$ is topologically equivalent to $Y$ at $q$ for some $q \in U$.

A proposition in [15] now says the following: If $h$ is a topological equivalence between $X, Y \in \mathfrak{X}(M)$, then

(a) $p \in M$ is a singularity of $X$ if and only if $h(p)$ is a singularity of $Y$,

(b) the orbit of $p$ for the vector field $X$ is closed if and only if the orbit of $h(p)$ for the vector field $Y$ is closed,

(c) the image of the $\omega$-limit set of the orbit of $p$ for the vector field $X$ by $h$, is the image of the $\omega$-limit set of the orbit of $h(p)$ for the vector field $Y$ and similarly for the $\alpha$-limit set.

So a vector field is structurally stable if the topological behavior of its orbits does not change under small perturbations of the vector field. Formally we say that $X \in \mathfrak{X}(M)$ is structurally stable if there exists a neighborhood $V$ of $X$ in $\mathfrak{X}(M)$ such that every $Y \in V$ is topologically equivalent to $X$.

An attractor is a set towards which a dynamical system evolves over time. In contrast, a repeller is a set from where a dynamical system evolves over time.

We now state an important proposition and theorem whose proof can be found in [13], [14] and [16].

Proposition 7.1. A vector field $X \in \mathfrak{X}(M)$ is Morse-Smale if and only if:

(a) $X$ has a finite number of critical elements, all hyperbolic;

(b) There are no saddle-connections (A saddle-connection is an orbit whose $\alpha$- and $\omega$-limits are saddles.) and

(c) each orbit has a unique critical element as its $\alpha$-limit and has a unique critical element as its $\omega$-limit.

Theorem 7.1. If $X \in \mathfrak{X}(M)$ is Morse-Smale vector field then $X$ is structurally stable.

Consider $M \subset \mathbb{R}^k$. At each point $p \in M$ we take in $T_p M$ the inner product $< , >_p$ induced by $\mathbb{R}^k$. We denote the norm induced by this inner product by $\| \|_p$. If $X$ and $Y$ are $C^1$ Morse-Smale vector fields on $M$ then the function $g : M \to \mathbb{R}$, $g(p) = < X(p), Y(p) >_p$ is of class $C^1$. Let $f : M \to \mathbb{R}$ be a $C^2$ map. For each $p \in M$ there exists a unique vector $X(p) \in T_p M$ such that $Df_p v = < X(p), v >_p$ for all $v \in T_p M$. This defines a (Morse-Smale) vector field $X$ which is of class $C^1$. It is called the gradient (Morse-Smale) vector field of $f$ and written as $X = \text{grad } f$. 


Some basic properties of gradient fields are, firstly, that \( \text{grad } f(p) = 0 \) if and only if \( Df_p = 0 \). Secondly, along nonsingular orbits of \( X = \text{grad } f \) we have \( f \) strictly increasing because \( Df_p X(p) = \| X(p) \|_p^2 \). In particular \( \text{grad } f \) does not have closed orbits.

Moreover, the \( \omega \)-limit of any orbit consists of singularities. For let us suppose that \( X(q) \neq 0 \) and \( q \in \omega(p) \) for some \( p \in M \). Let \( S \) be the intersection of \( f^{-1}(f(q)) \) with a small neighborhood of \( q \). We see that \( S \) is a submanifold of dimension \( m - 1 \) orthogonal to \( X = \text{grad } f \) and, by the continuity of the flow, the orbit through any point near \( q \) intersects \( S \). As \( q \in \omega(p) \) there exists a sequence \( p_n \) in the orbit of \( p \) converging to \( q \). Thus the orbit of \( p \) intersects \( S \) in more than one point (in fact, in infinitely many points) which is absurd since \( f \) is increasing along orbits. If the \( \omega \)-limit of an orbit of a gradient vector field contains more than one singularity, it must contain infinitely many.

## 8 \( C^1 \) Robustly minimal iterated function systems

Before we state our main result and prove it we make a remark on orientation. The bases \( b_1 \) and \( b_2 \) in \( \mathbb{R}^n \) are said to have the same orientation if the linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) that takes \( b_1 \) to \( b_2 \) has positive determinant.

If \( f \) is a diffeomorphism from \( M \) to \( M \) that is a contraction on a small neighborhood \( U \subset M \), \( f \) is homotopic to every contraction \( A \) on \( U \) if \( A \) has positive determinant (see chapter on homotopy in [4]). Therefore we can create a homotopy \( g \) by ‘pasting’ \( f \) on \( M \setminus U \) to \( A \) on \( V \subset U \) such that \( g \) is a diffeomorphism (see left figure 7). If \( A \) has no positive determinant \( g \) does not have to be a diffeomorphism (see right figure 7).

\[ \text{Figure 7:} \quad \text{In the left figure, the continuous map } g : M \to M \text{ consisting of the union of } f \text{ on } M \setminus U, \text{ a diffeomorphism that is a contraction on } (M \setminus V) \cap U \text{ and the affine contraction } A \text{ on the smaller neighborhood } V, \text{ is a diffeomorphism. In the right figure, the continuous map } g : M \to M \text{ is not a diffeomorphism(not injective) because } A \text{ has no positive determinant.} \]

Recall the affine contractions \( S_1 \) and \( S_2 \) and the affine expansion \( S_3 \) in Chapter 5. The affine expansion \( T \) has obviously positive determinant. Further, notice that the direction \( S_1 \) tilts the \( n \)-box is of no importance, as long as \( S_2 \) tilts the box in the opposite direction in every dimension. So by reversing directions, where necessary, it can be assured that the determinant of both contractions are positive in \( n \geq 3 \). (The determinant of \( S_1 \) and \( S_2 \) are positive for \( n < 3 \) as is easily seen.)

**Theorem 8.1.** Let \( M \) be a compact connected \( m \)-dimensional manifold. Then there exist diffeomorphisms \( g_1, g_2 \) and \( g_3 \) on \( M \) and a neighborhood

\[ U \subset \text{Diff}^1(M) \times \text{Diff}^1(M) \times \text{Diff}^1(M) \]
of \((g_1, g_2, g_3)\) such that each element in \(U\) generates a minimal iterated function system on \(M\).

Proof. Take a gradient Morse-Smale vector field \(X = \text{grad} f\) on \(M\) with a unique hyperbolic repelling equilibrium \(q\) and a unique hyperbolic attracting equilibrium \(p\) (see for example [13, Theorem 3.35] for the existence of Morse functions \(F\) with unique extrema) and let \(f\) be its time-1 map. Furthermore, \(f\) is an element of \(\text{Diff}^1(M)\). Below we use that the stable manifold of \(p\) and the unstable manifold of \(q\) lie dense in \(M\).

First we consider \(f\) near \(p\). Working in a coordinate chart on a small open neighborhood of \(p\), we may assume that \(f\) acts on \(\mathbb{R}^n\) and \(f\) is a contraction on a ball \(B_o\) in \(\mathbb{R}^n\). Since \(f\) is homotopic to \(S_1\) on \(B_o\), we may alter \(f\) to a diffeomorphism that remains a contraction on \(B_o\) and is equal to \(S_1\) on a smaller ball \(B_1 \subset B_o\). The resulting diffeomorphism on \(M\) will be \(g_1\) (see the remark on orientation). Likewise, we can alter \(f\) near \(p\) into a diffeomorphism that equals \(S_2\) on a neighborhood of \(p\). Write \(g_2\) for the resulting diffeomorphism on \(M\). Note that \(g_1\) and \(g_2\) have finitely many hyperbolic fixed points, among which a unique hyperbolic attracting fixed point. The stable manifold of this attracting fixed point lies dense in \(M\). We now consider \(f^{-1}\) near \(p\) and alter it to a diffeomorphism that equals \(T\) on a neighborhood of \(p\). The resulting diffeomorphism \(f_3\) also has finitely many hyperbolic fixed points. Finally, with a small perturbation, we perturb \(f_3\) to a diffeomorphism \(g_3\) so that each of the fixed points of \(g_3\) is contained in the stable manifold of the attracting fixed point of \(g_1\).

By construction of \(S_1, S_2\) and \(T\) the ifs generated by \(g_1\) and \(g_2\) has an attractor \(F\) with nonempty interior near \(p\) and the repelling fixed point of \(g_3\) in its interior. The unstable manifold of the repelling fixed point \(g_3\) lies dense in \(M\). Iterations of \(F\) under \(g_3\) therefore cover a dense subset of \(M\). Now iterations of a single point under \(g_3\) will converge to one of the fixed points of \(g_3\), which lie in the stable set of the attracting fixed point of \(g_1\) and are mapped into \(F\) under iteration by \(g_1\). This implies that the iterated function system \(\{g_1, g_2, g_3\}\) is minimal.

The construction is robust for perturbations in the \(C^1\) topology, as is clear from the following observations. By Corollary 5.1, if \(h_1, h_2\) and \(h_3\) are diffeomorphisms \(C^1\) close to \(g_1, g_2\) and \(g_3\), the iterated function system generated by \(h_1\) and \(h_2\) possesses an attractor with nonempty interior that contains the repelling fixed point of \(h_3\). Small \(C^1\) perturbations of \(g_1\) and \(g_2\) have nearby fixed points and bounded parts of its stable manifolds are also nearby. The same applies to \(g_3\). For both diffeomorphisms the stable and unstable manifolds of their attracting respectively repelling fixed points are dense in \(M\). In particular, the fixed points of \(h_3\) are contained in the stable manifold of the attracting fixed point of \(h_1\), for \(h_1\) and \(h_3\) sufficiently small \(C^1\) perturbation of \(g_1\) and \(g_3\). \(\square\)

9 Discussion

My supervisor Homburg strengthens this theorem further, by using a different attractor \(S\) and repeller \(T\). By constructing an affine contraction from the composition \(S \circ T\) such that the fixed point of \(T\) lies in the nonempty interior of the invariant set generated by \(\{S, S \circ T\}\), it is possible to omit the \(g_2\) from the prove of Theorem 8.1. Ending up with an ifs generated by two diffeomorphisms that is robustly minimal. For details of the construction I refer to [2].

After attachment of this paper on his web-site, Homburg received an e-mail from M. Nassiri who had also proven robust minimality of iterated function systems for two generators [3]. The papers are different in approach and the construction of a proof.
A Appendix

The proof of Banach fixed point theorem. Let \((X,d)\) be a non-empty, complete metric space, and let \(T\) be a contraction mapping on \((X,d)\) with constant \(0 < q < 1\) such that

\[
d(T(x),T(y)) \leq qd(x,y) \quad \text{for all } x,y \in X.
\]

(A.1)

Pick an arbitrary \(x_0 \in X\), and define the sequence \((x_n)_{n \geq 0}\) by \(x_n := T^n x_0\) and \(T^0 x_0 = x_0\). Let \(a := d(Tx_0, x_0)\). We first show by induction that for any \(n \geq 0\),

\[
d(T^n x_0, x_0) \leq \frac{1 - q^n}{1 - q} a.
\]

For \(n = 0\), this is obvious. For any \(n \geq 1\), suppose that \(d(T^{n-1} x_0, x_0) \leq \frac{1 - q^{n-1}}{1 - q} a\). Then by the triangle inequality and repeated application of (A.1) we get

\[
d(T^n x_0, x_0) \leq d(T^n x_0, T^{n-1} x_0) + d(T^{n-1} x_0, x_0)
\leq q^{n-1} d(Tx_0, x_0) + \frac{1 - q^{n-1}}{1 - q} a
\leq q^{n-1} q a + \frac{1 - q^{n-1}}{1 - q} a
= \frac{1 - q^n}{1 - q} a.
\]

Because \(q^n \to 0\) as \(n \to \infty\), we have that for all \(\varepsilon > 0\) there is a number \(N \in \mathbb{N}\) such that \(\frac{q^n}{1-q} a < \varepsilon\) for all \(n \geq N\). Now for any \(m, n \geq N\) and \(m \geq n\),

\[
d(x_m, x_n) = d(T^n x_0, T^m x_0)
\leq q^n d(T^m-n x_0, x_0)
\leq q^n \frac{1 - q^{m-n}}{1 - q} a
< \frac{q^n}{1 - q} a < \varepsilon,
\]

so the sequence \((x_n)\) is a Cauchy sequence. Because \((X,d)\) is complete this implies the sequence has a limit, say \(x^*\) in \(X\).

We now prove that \(x^*\) is a fixed point of \(T\). Suppose it is not, then \(T(x^*) \neq x^*\) and so \(d(T(x^*), x^*) \geq \delta > 0\). However, \((x_n)\) converges to \(x^*\), so there is a natural number \(N\) such that \(d(x_n, x^*) < \frac{\delta}{2}\) for all \(n \geq N\). Then

\[
d(Tx^*, x^*) \leq d(Tx^*, x_{N+1}) + d(x_{N+1}, x^*)
= d(Tx^*, Tx_N) + d(x_{N+1}, x^*)
\leq qd(x^*, x_N) + d(x_{N+1}, x^*)
< \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

gives a contradiction. So \(x^*\) is a fixed point of \(T\). Suppose there is another fixed point \(x'\) of \(T\). Because \(x^* \neq x'\), \(d(x^*, x') > 0\), but then,

\[
d(x^*, x') = d(Tx^*, Tx') \leq qd(x^*, x') < d(x^*, x'),
\]

gives a contradiction. Therefore, \(x^*\) is the unique fixed point of \(T\).
References


