Libor market model with stochastic volatility

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Abstract:
In this thesis, we investigate the Libor Market Model (LMM) with Displaced Diffusion and Stochastic Volatility (LMM-DDSV) for pricing of interest rate derivatives. In particular, we derive pricing formulas for caplets and swaptions under the LMM-DDSV model with constant parameters, and we extend the model with time-dependent parameters. The time-dependent LMM-DDSV model is approximated by a time-homogeneous model with averaged parameters, obtained by using the parameter averaging method. We show that the skewness and curvature of the implied volatility curves can be controlled by the LMM-DDSV model parameters, which fits the market implied volatility curves better compared to the standard model without stochastic volatility, which implies a flat and unrealistic volatility curve. A well-known drawback of stochastic volatility models is that a so-called moment explosion can occur in some parameter settings, which we demonstrate by using an in-arrears swap. Despite of a potential moment explosion issue, we show that the LMM-DDSV model with parameter averaging generally yields accurate price approximations.

Keywords:
Interest rate derivatives, pricing of caplets and swaptions, LIBOR, Libor Market Model (LMM), LMM-DDSV model, displaced diffusion, stochastic volatility, implied volatility, skewness and curvature, time-dependent parameters, parameter averaging method, moment explosion, in-arrears swaps.
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Chapter 1

Introduction

The concept of interest rates is widely known in everybody’s daily life when people save their money in a bank account, lend money to others, or when they borrow capital from a bank for their mortgage: an interest should be paid by the borrower to the lender. Interest rates are also a vital tool for monetary policy and within the financial environment. However, managing interest rate risk (i.e., the control of uncertain cash flows due to fluctuations in interest rates) is an issue with greater complexity. In particular, pricing and hedging of securities which depend on interest rates, create the necessity for financial mathematical interest rate models.

In this thesis we investigate mathematical models which model the forward rates, especially the forward LIBOR rates (London Interbank Offered Rates), which are one of the most important interest rates quoted in the market; it is widely used as a reference rate for many securities in financial markets. LIBOR is a rate at which banks are willing to lend to each other, and it is also used to determine a term structure for the interest rates, also known as a yield curve. The yield curve reflects the relation between the cost of borrowing (interest rate) and the term (time-to-maturity) at which time the money should be paid back to the lender. It is also an important variable in the interest rate (IR) derivatives market, which is the largest derivatives market in the world. For pricing IR derivatives, it is important to develop a model which is able to describe the evolution of a yield curve realistically, and remains manageable from a practical point-of-view.

Despite the popularity of LIBOR, the research on interest rate models have been mainly focused on describing the short-rate $r(t)$, which represents the interest rate valid for an infinitesimally short period of time after timepoint $t$. Short-rate models have been extensively studied in the literature, see Vasicek [31], Ingersoll and Ross [14], Ho and Lee [18]. A popular short-rate model is the Hull-White model [20], for which analytic formulas are available to price plain vanilla IR products such as bond options, caps and swaptions.

Models for forward rates have been proposed by Heath, Jarrow and Merton (HJM) [16], and Brace, Gatarek and Musiela (BGM) [9]. The HJM model is a general frame-
work to model instantaneous forward rates, which are not observable in the market, and captures the full dynamics of the entire forward rate curve, while the short-rate models only capture the dynamics of a point on the curve. When the volatility and drift of the instantaneous forward rate are assumed to be deterministic, this model is known as the Gaussian HJM model of forward rates, see Musiela and Rutkowski [28].

The BGM model [9] is a mathematical model describing the evolution of LIBOR rates, which belongs to the class of Libor Market Models (LMM). The model assumes that forward rates have a lognormal distribution and has served as a benchmark model for interest rate derivatives. However, a limitation of the BGM model is that it implies flat implied volatility curves, whereas the market implied volatility curves observed in the LIBOR markets exhibit some skew and curvature.

To overcome the shortcoming of the BGM model, in this thesis we investigate a LMM model with stochastic volatility. The most popular stochastic volatility model is the Heston model [17], well-known from equities and is used to incorporate skewness and curvature in the implied volatility. The advantage of the Heston model is that the skewness and curvature can be controlled by the model parameters and the model is analytically tractable for closed-form pricing of European style of derivatives. For IR derivatives, this model has been modified to specific characteristics of IR markets, see Andersen and Andreasen [2], Andersen and Brotherton-Ratcliffe [3], Piterbarg [29], and Wu and Zhang [32].

In this thesis we are particularly interested in pricing plain vanilla interest rate derivatives, like swaptions and caplets, since there exist closed-form pricing formulas for these instruments and they serve as calibration instruments to calibrate the model parameters.

The outline of this thesis is as follows: In Chapter 2, we give some background information on interest rate modelling and give an overview of fixed income instruments and the most common interest rate derivatives. In Chapter 3 we discuss the BGM model, and we introduce the LMM model with stochastic volatility. In Chapter 4, we derive pricing formulas for caplets and swaptions under the LMM model with Displaced Diffusion and Stochastic Volatility, and we extend our model with time-dependent parameters. The parameter averaging method is used to approximate the dynamics of the time-dependent model by dynamics with time-homogenous parameters. In Chapter 5 we present numerical results and we investigate the so-called moment explosion, a well-known problem for stochastic volatility models. Finally, in Chapter 6 we give the conclusion and discuss directions for further research.
Chapter 2

Background

In this chapter we introduce several basic concepts of financial instruments, which we will use throughout this thesis. We assume an economy where non-dividend paying securities are traded continuously from time 0 to time $T$. We assume that the prices of these securities are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the filtration $\mathcal{F} = \{\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\} : 0 \leq t \leq T\}$ satisfies the usual conditions and $W$ is a $m$-dimensional Brownian motion. The economy is also assumed to be free of arbitrage and complete. A random variable $V$ is called a contingent claim or a $T$-maturity derivative security if $V(T)$ is $\mathcal{F}_T$-measurable. The $T$-maturity derivative security pays out an amount of $V(T)$ at time $T$ and makes no payments before $T$.

In this thesis we use annualized time-scales, and this can possibly be extended with day-count conventions. There are different types of day-count conventions which measure the difference between two dates such as Actual/365 and Actual/360. For more detailed information about day-count conventions, we refer to Brigo and Mercurio [10].

2.1 Zero-coupon bonds and interest rates

One of the simplest contingent claims is a $T$-maturity zero-coupon bond, which is defined as follows.

**Definition 2.1.1. (zero-coupon bond)** A $T$-maturity zero-coupon bond is a security paying one unit of a given currency at time $T$, its value at time $t$ is denoted by $P(t,T)$, $t \in [0, T]$. Clearly, we have that $P(T,T) = 1$.

We define a money-market account (or bank account) where the value at time $t$ is denoted by $\beta(t)$. In this account, profits are accrued continuously at the prevailing risk-free rate in the market. Its dynamics are assumed to be given by

$$d\beta(t) = r(t)\beta(t)dt, \quad \beta(0) = 1,$$
where $r(t)$ is the short-rate or instantaneous rate (for the exact definition of short-rate, see Definition 2.1.5). Under this assumption, it can be easily verified that $\beta(t)$ satisfies:

$$\beta(t) = e^{\int_0^t r(u) du}.$$ 

We also note that any positive traded non-dividend paying security can be used as a numeraire. In general, for any numeraire $N$, there exists a probability measure $P^N$ such that the price of any contingent claim $V$ normalized by $N$ is a martingale under $P^N$. In the following definition we introduce such a measure corresponding to $\beta(t)$.

**Definition 2.1.2. (risk-neutral measure)** We define the risk-neutral measure $Q$ to be the measure such that $V(t)/\beta(t)$ is a martingale under the measure $Q$. It holds that, for all $t \leq T$,

$$V(t) = \beta(t) \mathbb{E}_t^Q \left[ \frac{V(T)}{\beta(T)} \right],$$

where $V(t)$ denotes the price at time $t$ of a derivative security making an $\mathcal{F}_T$-measurable payment of $V(T)$ and $\mathbb{E}_t^Q$ is the conditional expectation with respect to $\mathcal{F}_t$ under the measure $Q$. The money-market account $\beta(t)$ is called the numeraire corresponding to the measure $Q$.

Under the risk-neutral measure, we can derive the value at time $t$ of a $T$-maturity zero-coupon bond. This result is given in the following lemma.

**Lemma 2.1.3. (zero-coupon bond price)** The price of a zero-coupon bond with maturity $T$ at time $t$ is given by

$$P(t, T) = \mathbb{E}_t^Q \left[ e^{-\int_t^T r(u) du} \right].$$

**Proof.** By definition of the risk-neutral measure and $P(T, T) = 1$, we have that

$$P(t, T) = \beta(t) \mathbb{E}_t^Q \left[ \frac{P(T, T)}{\beta(T)} \right] = \mathbb{E}_t^Q \left[ e^{-\int_t^T r(u) du} \right].$$

Under the arbitrage-free assumption, the following relationship should hold:

$$P(t, T + \tau) = P(t, T) \mathbb{E}_t^Q[P(T, T + \tau)], \quad \tau > 0. \quad (2.1)$$

Otherwise, if the above relationship does not hold, then there exists a self-financing trading strategy that results in a risk-less profit. To see this, suppose that the inequality $P(t, T + \tau) > P(t, T) \mathbb{E}_t^Q[P(T, T + \tau)]$ holds for some $t \leq T \leq T + \tau$. Then, at time $t$, we can sell $P(t, T) \mathbb{E}_t^Q[P(T, T + \tau)]/P(t, T + \tau)$ time $(T + \tau)$-maturity bonds and buy $\mathbb{E}_t^Q[P(T, T + \tau)]$ time $T$-maturity bonds. Furthermore, at maturity time $T$, we receive $\mathbb{E}_t^Q[P(T, T + \tau)]$ units of currency and buy one time $(T + \tau)$-maturity bond. Finally, at time $T + \tau$ we pay $P(t, T) \mathbb{E}_t^Q[P(T, T + \tau)]/P(t, T + \tau)$ units of currency and receive one
unit of currency. Following this strategy results in a profit of \( 1 - P(t, T)\mathbb{E}_t^Q[P(T, T + \tau)]/P(t, T + \tau) \) at time \( T + \tau \). A similar trading strategy with a risk-less profit could be established in case the other inequality \( P(t, T + \tau) < P(t, T)\mathbb{E}_t^Q[P(T, T + \tau)] \) holds.

The time interval \( \tau \) is often called a tenor. The financial interpretation of Equation (2.1) is that given a cash-flow of 1 unit of currency at \( T + \tau \), its corresponding value at anytime \( t < T + \tau \) must be the same if we discount back in a single step from \( T + \tau \) to \( t \) or in two steps. It is known as the no-arbitrage relation. Next, we define the forward price of the zero-coupon bond.

**Definition 2.1.4. (zero-coupon bond forward price)** For \( \tau > 0 \), the time \( t \) forward price of the zero-coupon bond spanning \([T, T + \tau]\), is given by

\[
P(t, T + \tau) := \mathbb{E}_t^Q[P(T, T + \tau)].
\]

and from Equation (2.1), it follows that

\[
P(t, T, T + \tau) = \frac{P(t, T + \tau)}{P(t, T)}.
\] (2.2)

It is also important to define rates and yields related to a zero-coupon bond price. We give the following definitions:

- continuously compounded forward yields
- compounded forward rates
- instantaneous forward rates
- short rates

**Definition 2.1.5. (continuously compounded forward yield)** For a given tenor \( \tau > 0 \), the continuously compounded forward yield \( y(t, T, T + \tau) \) is defined by

\[
e^{-y(t, T, T + \tau)\tau} = P(t, T, T + \tau), \text{ for any } t < T.
\]

**Definition 2.1.6. (compouned forward rate)** The simply compounded forward rate \( L(t, T, T + \tau) \) is given by

\[
1 + \tau L(t, T, T + \tau) = \frac{1}{P(t, T, T + \tau)}.
\]

The simply compounded forward rate is a constant rate for the time interval \([T, T + \tau]\) prevailing at time \( t \). We assume that the spot rates \( L(T, T, T + \tau) \) are the London Interbank Offered Rates (LIBOR) quoted in the interbank market. We also denote \( L(t, T, T + \tau) \) as the forward LIBOR rates.
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Note that by taking the limit $\tau \to 0$, we have the following equations:

\[
f(t, T) := \lim_{\tau \to 0^+} L(t, T, T + \tau) = \lim_{\tau \to 0^+} \frac{1}{\tau} \left( \frac{1}{P(t, T, T + \tau)} - 1 \right) = \lim_{\tau \to 0^+} \frac{1}{\tau} \frac{P(t, T) - P(t, T + \tau)}{P(t, T)} = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} = -\frac{\partial \ln P(t, T)}{\partial T}.\]

**Definition 2.1.7. (instantaneous forward rate)** The quantity defined by $f(t, T)$ is the time $t$ instantaneous forward rate to time $T$, and it satisfies that

\[
P(t, T) = \exp \left( -\int_t^T f(t, u) du \right).\]

**Definition 2.1.8. (short-rate)** The short-rate $r(t)$ at time $t$ is defined by

\[
r(t) := f(t, t) = \lim_{T \downarrow t} f(t, T).\]

2.2 Measures

Previously, we introduced the risk-neutral measure corresponding to $\beta(t)$. By choosing different numerares, there exist different measures. In this section, we introduce numerares and measures used in pricing interest rates derivatives for later reference.

2.2.1 $T$-Forward measure

We introduce a measure which takes the $T$-maturity zero-coupon bond as the numeraire.

**Definition 2.2.1. (T-forward measure)** The $T$-forward measure $Q^T$ is the measure which makes $V(t)/P(t, T)$ a martingale, that is,

\[
V(t) = P(t, T) \mathbb{E}^T_t \left[ \frac{V(T)}{P(T, T)} \right] = P(t, T) \mathbb{E}^T_t [V(T)], \quad \text{for all } t \leq T,
\]

where $\mathbb{E}^T_t$ is the conditional expectation w.r.t. $\mathcal{F}_t$ under the $T$-forward measure $Q^T$.

The forward LIBOR rate $L(t, T, T + \tau)$ with $t \leq T$ and $\tau > 0$ is a martingale under the $(T + \tau)$-forward measure $Q^{T+\tau}$. This result is proven by the following lemma.
Lemma 2.2.2. (martingale under forward measure) The forward LIBOR rate 
$L(t, T, T + \tau)$ is a martingale under the $(T + \tau)$-forward measure $Q^{T+\tau}$. Hence, we have the following equation:
\[
L(t, T, T + \tau) = \mathbb{E}^{T+\tau}_t[L(T, T, T + \tau)], \quad t \leq T,
\]
where $\tau > 0$.

Proof. Recall the definition of a forward LIBOR rate $L(t, T, T + \tau)$,
\[
1 + \tau L(t, T, T + \tau) = \frac{1}{P(t, T, T + \tau)}.
\]
From Equation (2.2), we have that
\[
1 + \tau L(t, T, T + \tau) = \frac{P(t, T)}{P(t, T + \tau)}, \tag{2.3}
\]
which is equivalent to
\[
L(t, T, T + \tau) = \frac{1}{\tau} \left[ \frac{P(t, T)}{P(t, T + \tau)} - 1 \right]. \tag{2.4}
\]
To show that $L(t, T, T + \tau)$ is a martingale under the $(T + \tau)$-forward measure, we need to show that
\[
L(t, T, T + \tau) = \mathbb{E}^{T+\tau}_t[L(s, T, T + \tau)],
\]
holds for any $t \leq s \leq T$. Equation (2.4) yields that
\[
\mathbb{E}^{T+\tau}_t[L(s, T, T + \tau)] = \mathbb{E}^{T+\tau}_t \left[ \frac{1}{\tau} \left( \frac{P(s, T)}{P(s, T + \tau)} - 1 \right) \right] = \frac{1}{\tau} \left\{ \mathbb{E}^{T+\tau}_t \left[ \frac{P(s, T)}{P(s, T + \tau)} \right] - 1 \right\}.
\]
Since $P(s, T)$ is a traded asset and a traded asset normalized by the numeraire is a martingale, $P(s, T)/P(s, T + \tau)$ is a martingale under the $(T + \tau)$-forward measure. Hence, we have that
\[
\mathbb{E}^{T+\tau}_t[L(s, T, T + \tau)] = \frac{1}{\tau} \left( \frac{P(t, T)}{P(t, T + \tau)} - 1 \right) = L(t, T, T + \tau).
\]
\[
\square
\]

2.2.2 Spot measure

The spot measure is defined in this subsection. We consider a set of dates
\[
0 = T_0 < T_1 < \cdots < T_N, \quad \tau_n = T_{n+1} - T_n.
\]
and define an asset price process $B(t)$ by
\[
B(t) = \begin{cases} 
P(t, T_{n+1}) \prod_{n=0}^n (1 + \tau_n L_n(T_n)), & \text{for } t > 0, \\ 1, & \text{for } t = 0, \end{cases} \tag{2.5}
\]
where \( i \) is such that \( T_i < t \leq T_{i+1} \) and
\[
L_n(T_n) := L(T_n, T_n, T_{n+1}).
\]
(2.6)

Note that Equation (2.5) is equivalent to
\[
B(t) = P(t, T_{i+1}) \prod_{n=0}^{i} 1/P(T_n, T_{n+1}),
\]
by the definition of compounded forward rate \( L_n(t) \). The asset price process \( B(t) \) represents the time \( t \) value of an asset starting from one unit of currency at time 0 and reinvesting at each tenor date \( T_n \) in the zero-coupon bonds for the next tenor \( T_{n+1} \). It is a discrete-time equivalent of the continuously compounded money market account \( \beta(t) \).

**Definition 2.2.3. (spot measure)** The spot measure \( Q^B \) is the measure which makes \( V(t)/B(t) \) a martingale, that is,
\[
V(t) = B(t) \mathbb{E}_t^B \left[ \frac{V(T)}{B(T)} \right],
\]
where \( \mathbb{E}_t^B \) is the conditional expectation w.r.t. \( \mathcal{F}_t \) under the spot measure \( Q^B \).

The advantage of using the spot measure over a forward measure in the simulation is that a possible bias coming from discretizing the drift is more evenly distributed over the different rates. We will discuss this in further detail in Chapter 3.

### 2.3 Black formula

In this section we discuss the Black formula, which is a theoretical pricing model for options. First, we give a brief introduction of options.

**Definition 2.3.1. (options)** An option is a financial instrument which gives the holder the right to buy or sell the underlying security \( F \) for a certain price \( K \) at a future time \( T \). The price \( K \) is also known as the strike price and the time \( T \) is known as the maturity. A call (put) option is an option which gives the right to buy (sell) the underlying asset.

Given the strike price \( K \) and the underlying level \( F \), options are said to be in-the-money (ITM), at-the-money (ATM), or out-of-the-money (OTM). In-the-money options are options which have intrinsic value if they were exercised. Out-the-money options have zero intrinsic value. In other words: if \( F > K \), then the call option is ITM and its corresponding put option is OTM, and vice versa. The option is said to be ATM when \( F = K \).

If the holder of an option can only exercise at maturity, then the option is called a European option. For American options, the holder may exercise the option at any time before the maturity date. American options are generally computed numerically with
CHAPTER 2. BACKGROUND

for example binomial tree methods, Monte Carlo methods, or finite-difference methods.

For European options, the analytical pricing formula is well-known under certain assumptions as derived by Black and Scholes [8] and Merton [27]. Here we will introduce the pricing formula derived by Black.

**Definition 2.3.2. (Black formula)** At maturity $T$, let $F(T)$ be the underlying price and it is assumed to be log-normally distributed. We assume that the volatility of $F(T)$ is given by $\sigma$ and we denote $\mu(0) := \mathbb{E}(F(T))$. The call price (under the risk-neutral measure) at time 0, for a European call option with strike $K$ and maturity $T$ is given by the formula:

$$c(0) = P(0, T)(\mu(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2)), \quad (2.7)$$

where

$$d_1 = \frac{\log(\mu(0)/K) + \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}},$$

$$d_2 = d_1 - \sigma \sqrt{T},$$

and $\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$ is the standard normal cumulative distribution function.

Accordingly, the price of a European put option is given by

$$p(0) = P(0, T)(KN(-d_2) - \mu(0)\mathcal{N}(-d_1)). \quad (2.8)$$

The Equations (2.7) and (2.8) are known as the Black formula, here it is denoted by $Black(0, F, T, K, \sigma)$.

**Remark 2.3.3. (implied volatility)** The Black formula depends on the input parameter $\sigma$ (and strike $K$ and maturity $T$). However, the volatility $\sigma$ is typically extracted from market data, which gives rise to a notion of implied volatility. The implied volatility of an option is the volatility that is implied by the market price of the option based on a pricing model like the Black’s model. Given the current market price $V$ of the option price, the implied volatility $\sigma_{imp}$ is the level of volatility such that the following equation is satisfied:

$$Black(0, F, T, K, \sigma_{imp}) - V = 0.$$  

**Remark 2.3.4. (volatility surface)** For different strikes and maturities, the implied volatilities can be calculated numerically by observing the market prices. The mapping from strike $K$ and maturity $T$ to implied volatility is known as the volatility surface. The implied volatilities for options which are not directly observed in the market are determined by using interpolation methods and using the information from the volatility surface. From this volatility surface, the price of options which may not be observed in the market can be calculated by using the Black formula.
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If we use the Black formula throughout this thesis, then we refer to Equation (2.7) or Equation (2.8) depending on the type of options.

2.4 Fixed income instruments

In this section, we will introduce the definitions and pricing formulas of fixed income securities which are traded in the financial market. First, we introduce forward rate agreements (FRA), plain vanilla swaps, and in-arrears swaps. Later in this section, we discuss interest rate caps, floors and swaptions.

2.4.1 Forward rate agreements (FRA)

A forward rate agreement is a fixed income instrument which allows one to lock-in the interest rate for a future time interval at a certain fixed rate. The formal definition of a forward rate agreement is as follows.

Definition 2.4.1. (Forward Rate Agreement) A Forward Rate Agreement (FRA) for the period \([T, T + \tau]\) is a contract to exchange a payment based on a fixed rate \(K\) against a payment based on the time \(T\) spot Libor rate with tenor \(\tau\). The exchange happens at time \(T + \tau\).

In other words, the fixed rate payer pays out the amount \(\tau K\) and receives \(\tau L(T, T, T + \tau)\) at time \(T + \tau\). Hence, the value of a FRA at time \(T + \tau\) from the perspective of the fixed rate payer is given by:

\[ V_{FRA}(T + \tau) = \tau(L(T, T, T + \tau) - K). \]

Accordingly, the value of the contract at time \(t < T\) is given by

\[ V_{FRA}(t) = P(t, T + \tau)E_{t}^{T+\tau}[\tau(L(T, T, T + \tau) - K)]. \]

Since \(L(t, T, T + \tau)\) is a martingale under the \((T + \tau)\)-forward measure \(Q^{T+\tau}\), we have that

\[ V_{FRA}(t) = \tau P(t, T + \tau)(L(t, T, T + \tau) - K), \]

and Equation (2.4) implies the following relation:

\[ V_{FRA}(t) = P(t, T) - P(t, T + \tau) - \tau KP(t, T + \tau). \]
2.4.2 Plain vanilla swaps

We introduce the plain vanilla swap, which is a generalization of the FRA.

**Definition 2.4.2. (swap)** A swap is an agreement between two counterparties to exchange cash at each pre-defined time date. A plain vanilla interest rate swap is a swap in which one leg is a stream of fixed rate payments and the other a stream of payments based on a floating rate (which is often the LIBOR rate).

More formally, consider a vanilla interest rate swap with a fixed rate $K$ and assume that both legs follow the tenor structure (i.e, the set of pre-defined time dates for an exchange of cash):

$$0 \leq T_0 < T_1 < T_2 < \cdots < T_{N-1} < T_N, \quad \tau_n = T_{n+1} - T_n. \quad (2.9)$$

At time $T_{n+1}$, the net cash flow of the fixed rate payer (receiver) is given by (on a unit notional)

$$\tau_n w(L_n(T_n) - K),$$

where $L_n(t) = L(t, T_n, T_{n+1})$ for $n = 0, \cdots, N-1$. Here, we introduce the payer/receiver indicator $w$ which is defined by

$$w = \begin{cases} 
1, & \text{for the payer,} \\
-1, & \text{for the receiver.} 
\end{cases}$$

Hence, for any $t \in [0, T_0]$, the price of a payer (receiver) swap is given by

$$V_{\text{swap}}(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}) E_{t}^{T_{n+1}} \left[ w(L_n(T_n) - K) \right]$$

$$= \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}) w(L_n(t) - K)$$

$$= w \left( \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}) \right) \left( \sum_{n=0}^{N-1} \frac{\tau_n P(t, T_{n+1}) L_n(t)}{\sum_{n=0}^{N-1} \tau_n P(t, T_{n+1})} - K \right). \quad (2.10)$$

where we define $E_{t}^{T_{n+1}} := E_{t}^{T_{n+1}}$. If we define $A(t)$ and $S(t)$ by

$$A(t) := A_{0,N}(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}),$$

$$S(t) := S_{0,N}(t) = \sum_{n=0}^{N-1} \frac{\tau_n P(t, T_{n+1}) L_n(t)}{\sum_{n=0}^{N-1} \tau_n P(t, T_{n+1})},$$

then we can simplify the formula of a payer (receiver) swap price to

$$V_{\text{swap}}(t) = wA(t)(S(t) - K). \quad (2.11)$$
The quantity $A(\cdot)$ is called the annuity of the swap or Present Value of a Basis Point (PVBP), and the quantity $S(t)$ is the forward swap rate. The forward swap rate is the fixed rate $K$ which makes $V_{swap}(t)$ equal to 0. Therefore, the swap rate $S(t)$ is often called the par or break-even rate of the swap.

Another definition of the forward swap rate can also be given with the result in the next Remark.

**Remark 2.4.3.** Since Equation (2.4) can be rewritten as

$$\tau_n L_n(t) = \frac{P(t, T_n)}{P(t, T_{n+1})} - 1,$$

we have that

$$S(t) = \frac{\sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}) L_n(t)}{A(t)}$$

$$= \frac{\sum_{n=0}^{N-1} P(t, T_{n+1}) \left( \frac{P(t, T_n)}{P(t, T_{n+1})} - 1 \right)}{A(t)}$$

$$= \frac{\sum_{n=0}^{N-1} (P(t, T_n) - P(t, T_{n+1}))}{A(t)}$$

$$= \frac{P(t, T_0) - P(t, T_N)}{A(t)}.$$

We introduce the generalized definition of the annuity $A(t)$ and the forward swap rate $S(t)$ for later reference.

**Definition 2.4.4. (annuity rate and swap rate)** For any integers $k, m$ satisfying $0 \leq k < N$, $m > 0$ and $k + m \leq N$, the annuity rate $A_{k,m}$ is defined by

$$A_{k,m}(t) = \sum_{n=k}^{k+m-1} P(t, T_{n+1}) \tau_n,$$

and the swap rate $S_{k,m}(t)$ at time $t \leq T_k$ is defined by

$$S_{k,m}(t) = \frac{P(t, T_k) - P(t, T_{k+m})}{A_{k,m}(t)}.$$

Note that the annuity rate is also used as a numeraire because it is the linear combination of zero-coupon bonds. Therefore there exist a measure corresponding to the annuity rate, which is defined as follows.

**Definition 2.4.5. (swap measure)** The swap measure $Q^{k,m}$ is the measure, which is induced by the annuity factor $A_{k,m}(t)$ as a numeraire. It gives that

$$V(t) = A_{k,m}(t) \mathbb{E}_t^{k,m} \left[ \frac{V(T)}{A_{k,m}(T)} \right],$$

where $\mathbb{E}_t^{k,m}$ is a conditional expectation w.r.t. $\mathcal{F}_t$ under the swap measure $Q^{k,m}$. 

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An advantage of using the swap measure is that the swap rate is a martingale under the swap measure. This is explained in the following lemma.

**Lemma 2.4.6.** The swap rate \( S_{k,m}(t) \) is a martingale under the measure \( Q_{k,m} \).

**Proof.** Since \( P(\cdot, T_k) \) and \( P(\cdot, T_{k+m}) \) are traded assets, the difference \( P(\cdot, T_k) - P(\cdot, T_{k+m}) \) is also a traded asset. It yields that \( (P(\cdot, T_k) - P(\cdot, T_{k+m})) / A_{k,m}(\cdot) \) is a martingale under the swap measure \( Q_{k,m} \). For any \( T \leq T_k \), we have that

\[
E_{t,m}^k[S_{k,m}(T)] = E_{t,m}^k\left[ \frac{P(T, T_k) - P(T, T_{k+m})}{A_{k,m}(T)} \right] = \frac{P(t, T_k) - P(t, T_{k+m})}{A_{k,m}(t)} = S_{k,m}(t).
\]

\( \square \)

### 2.4.3 In-arrears swaps

In this subsection, we introduce the in-arrears swaps which are interest rate swaps that sets and pays interest in arrears. These swaps are complex than the plain vanilla swaps (see Section 2.4.2). The formal definition is given below.

**Definition 2.4.7. (in-arrears swap)** An in-arrears swap is an interest rate swap in which the floating rate is observed and paid on the same day. Consider the tenor structure:

\[
0 \leq T_0 < T_1 < \cdots < T_N, \quad \tau_n = T_{n+1} - T_n,
\]

where \( n \in \{0, \cdots, N-1\} \). At time \( T_n \), the net cash flow of the fixed rate payer (receiver) is given by (on a unit notional):

\[
\tau_n w(L_n(T_n) - K),
\]

where \( K \) is the fixed rate and \( w \) is the payer/receiver indicator.

At time \( t \leq T_0 \), the price of an in-arrears payer swap is given by

\[
V(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_n) E_{t}^{n}[L_n(T_n) - K]
= \sum_{n=0}^{N-1} \tau_n P(t, T_n)(E_{t}^{n}[L_n(T_n)] - \tau_n P(t, T_n)K).
\]

Note that \( L_n(T_n) \) is not a martingale under the \( T_n\)-forward measure. By using the Change of Numeraire theorem (see Theorem A.1.2 in Appendix A), we have that

\[
E_{t}^{n}(L_n(T_n)) = E_{t}^{n+1}\left[ L_n(T_n) \frac{P(T_n, T_{n+1})/P(0, T_{n+1})}{P(T_n, T_{n+1})/P(0, T_{n+1})} \right]
= E_{t}^{n+1}\left[ L_n(T_n) \frac{P(T_n, T_{n+1})/P(0, T_{n+1})}{P(0, T_n)/P(0, T_{n+1})} \right]
= E_{t}^{n+1}\left[ L_n(T_n) \frac{1 + \tau_n L_n(T_n)}{1 + \tau_n L_n(0)} \right]
= L_n(t) + \tau_n E_{t}^{n+1}[L_n^2(T_n)]
\]

\( \frac{1}{1 + \tau_n L_n(0)} \)

(2.13)
where Equation (2.4) is used in the third equality. From Equation (2.12) and (2.13), we obtain the following relation:

$$V(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_n) \frac{L_n(t) + \tau_n \mathbb{E}^{n+1}[L_n^2(T_n)]}{1 + \tau_n L_n(0)} - \tau_n P(t, T_n) K$$

$$= \sum_{n=0}^{N-1} \tau_n P(t, T_n) \left( \frac{L_n(t) + \tau_n \mathbb{E}^{n+1}[L_n^2(T_n)]}{1 + \tau_n L_n(0)} - K \right).$$

(2.14)

Hence, the price of an in-arrears swap at time $t = 0$ is given by

$$V(0) = \sum_{n=0}^{N-1} \tau_n P(0, T_n) \left( \frac{L_n(0) + \tau_n \mathbb{E}^{n+1}[L_n^2(T_n)]}{1 + \tau_n L_n(0)} - K \right).$$

(2.15)

**Remark 2.4.8. (convexity adjustment)** To derive an analytical formula of $\mathbb{E}^{n+1}[L_n^2(T_n)]$, we need to make an assumption about the distribution of $L_n(t)$. We assume that $L_n(t)$ is log-normally distributed, i.e., $L_n(t)$ follows the SDE given by

$$dL_n(t) = \sigma_n(t) L_n(t) dW^{n+1}(t),$$

where $\sigma_n(t)$ is a deterministic function and $W^{n+1}(t)$ is a Brownian motion under the $T_{n+1}$-forward measure.

The assumption is equivalent to

$$L_n(t) = L_n(0) \exp \left( -\frac{1}{2} \int_0^t \sigma_n^2(s) \, ds + \int_0^t \sigma_n(s) \, dW^{n+1}(s) \right).$$

One can show that:

$$\mathbb{E}^{n+1}[L_n^2(T_n)] = \mathbb{E}^{n+1} \left[ L_n^2(0) \exp \left( -\int_0^{T_n} \sigma_n(s)^2 \, ds + 2 \int_0^{T_n} \sigma_n(s) \, dW^{n+1}(s) \right) \right]$$

$$= L_n^2(0) \exp \left( -\int_0^{T_n} \sigma_n^2(s) \, ds \right) \mathbb{E}^{n+1} \left[ \exp \left( 2 \int_0^{T_n} \sigma_n(s) \, dW^{n+1}(s) \right) \right].$$

Note that $\int_0^{T_n} \sigma_n(s) \, dW^{n+1}(s)$ is normally distributed with zero mean and variance $\int_0^{T_n} \sigma_n^2(s) \, ds$. For a random variable $X$ having a Normal distribution with mean $\mu$ and variance $\sigma^2$, the moment generating function of $X$ is well-known to be given by

$$\mathbb{E}[e^{\mu X}] = e^{\mu t + \frac{1}{2} \sigma^2 t^2}.$$  

(2.16)

Together with Equation (2.16) this yields that:

$$\mathbb{E}^{n+1}[L_n^2(T_n)] = L_n^2(0) \exp \left( -\int_0^{T_n} \sigma_n^2(s) \, ds \right) \mathbb{E}^{n+1} \left[ \exp \left( 2 \int_0^{T_n} \sigma_n(s) \, dW^{n+1}(s) \right) \right]$$

$$= L_n^2(0) \exp \left( -\int_0^{T_n} \sigma_n^2(s) \, ds \right) \exp \left( 2 \int_0^{T_n} \sigma_n^2(s) \, ds \right)$$

$$= L_n^2(0) \exp \left( \int_0^{T_n} \sigma_n^2(s) \, ds \right).$$
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Hence, the price of an in-arrears swap at time \( t = 0 \) is given by the formula:

\[
V(0) = \sum_{n=0}^{N-1} \tau_n P(0, T_n) \left( \frac{L_n(0) + \tau_n E^{\tau+1}(L_n^2(T_n))}{1 + \tau_n L_n(0)} - K \right) = \sum_{n=0}^{N-1} \tau_n P(0, T_n) \left( \frac{L_n(0) + \tau_n L_n^2(0) \exp\left(\int_0^{T_n} \sigma_n^2(s) \, ds\right)}{1 + \tau_n L_n(0)} - K \right) = \sum_{n=0}^{N-1} \tau_n P(0, T_n) \left( L_n(0) + \frac{\tau_n L_n^2(0)}{1 + \tau_n L_n(0)} \left( \exp\left(\int_0^{T_n} \sigma_n^2(s) \, ds\right) - 1 \right) - K \right).
\]

Here, the term \( C_{\text{adj}} \) is called the convexity adjustment. For more information on convexity adjustments, we refer to Hunt and Kennedy [21].

2.4.4 Interest rate caps and floors

In this subsection we introduce a cap or floor which is a popular interest rate option. A cap or floor provides insurance against the interest rates rising above or falling below a certain level. A cap/floor is a portfolio of call/put options which are also called caplets/floorlets.

Definition 2.4.9. (caplets and floorlets) A caplet (floorlet) is a European call (put) option on a LIBOR rate. A caplet (floorlet) on LIBOR rate \( L(t, T, T + \tau) \) with strike \( K \) pays out at time \( T + \tau \)

\[
\tau \{ w(L(T, T, T + \tau) - K) \}^+,
\]

per unit notional where \( w \) is a caplet/floorlet indicator such that \( w = 1 \) for a caplet and \( w = -1 \) for a floorlet.

Since a caplet (floorlet) is a European call (put) option on a LIBOR rate, we can price a caplet (floorlet) by the Black formula under the following assumptions. We assume that \( L(T, T, T + \tau) \) is log-normally distributed with the forward price \( \mu(t) \) and volatility \( \sigma \). Note that \( \mu(0) = L(0, T, T + \tau) \) because \( L(t, T, T + \tau) \) is a martingale under the \( (T + \tau) \)-forward measure. The price of a caplet (floorlet) at time 0 is given in the following form:

\[
V_{\text{caplet/floorlet}}(0) = \tau P(0, T + \tau)(wL(0, T, T + \tau)N(wd_1) - wKN(wd_2)),
\]

\[
d_1 = \frac{\log(L(0, T, T + \tau)/K) + \sigma^2T/2}{\sigma\sqrt{T}},
\]

\[
d_2 = d_1 - \sigma\sqrt{T}.
\]

For later reference, we call the above equation as the Black formula for caplets (floorlets). Now, we define caps and floors as follows.
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Definition 2.4.10. A cap is a strip of caplets and a floor is a strip of floorlets. The price of an $N$-period cap (floor) at time 0 is simply the sum of time 0-value of $N$ caplets. Hence, using the Black formula, we price a cap(floor) with strike $K$ at time 0 by

$$V_{\text{cap/floor}}(0) = \sum_{n=0}^{N-1} P(0,T_{n+1})(wL_n(0)\mathcal{N}(wd_1) - wKN(wd_2))$$

$$d_1 = \frac{\log(L_n(0)/K) + \sigma^2 T_n/2}{\sigma \sqrt{T_n}},$$

$$d_2 = d_1 - \sigma \sqrt{T_n},$$

where $\sigma_n$ is volatility corresponding to $L_n(t)$. To price a cap (floor), we consider two approaches. One approach is to use different volatility for each caplet (floorlet), which is referred to as spot volatility. The other approach is to use the same volatility for all caplets (floorlets) comprising any particular cap (floor) but to vary this volatility according to the life of the cap (floor). These volatilities are referred to as flat volatilities. The flat volatilities are usually quoted in the market. The spot volatilities can be stripped from these quotes.

2.4.5 European swaptions

In this subsection, we introduce another popular interest option called the (European) swaptions. The definition of a European swaption is given below.

Definition 2.4.11. (swaptions) A European swaption gives the holder the right to enter into a swap at a future date at a given fixed rate $K$. A payer (receiver) swaption is an option where the underlying is a payer (receiver) swap.

From Equation (2.11) and assuming that the underlying swap starts on the expiry date $T_0$ of the option, the payoff of a swaption at time $T_0$ is given by

$$(V_{\text{swap}}(T_0))^+ = (wA(T_0)(S(T_0) - K))^+.$$

Under the swap measure $Q^{0,m}$, the price of a swaption at time 0 can be calculated by

$$V_{\text{swaption}}(0) = A(0)\mathbb{E}^{0,m}\left[\frac{wA(T_0)(S(T_0) - K))^+}{A(T_0)}\right] = A(0)\mathbb{E}^{0,m}\left[(w(S(T_0) - K))^+\right].$$

For a swaption, there exists a Black-like formula. More precisely, the price of a swaption with strike $K$ at time 0 is given by

$$V_{\text{swaption}}(0) = A(0)(wS(0)\mathcal{N}(wd_1) - wKN(wd_2)),$$

$$d_1 = \frac{\log(S(0)/K) + \sigma^2 T_0/2}{\sigma \sqrt{T_0}},$$

$$d_2 = d_1 - \sigma \sqrt{T_0}.$$

For later reference, we call the above equation as the Black formula for swaptions.
Chapter 3

The LIBOR Market Model (LMM)

In this chapter, we introduce the LIBOR Market Model which is one of the most popular interest rate models. We give an overview of the relevant chapters in Andersen and Piterbarg (see [5]). In Section 3.1 we establish notations and some preliminary results to describe the LMM and briefly discuss the standard LMM or BGM model [9]. A drawback of the BGM model is that is not able to capture skew and curvature in the implied volatility. Therefore, in Section 3.2, we introduce a LMM with stochastic volatility in order to overcome the shortcoming of the standard BGM model.

3.1 LIBOR market model dynamics and measures

Before we study the LIBOR market model, we will define a tenor structure. For a fixed \( N \in \mathbb{N} \), we define the following tenor structure:

\[
0 = T_0 < T_1 < \cdots < T_N, \quad \tau_n = T_{n+1} - T_n, \tag{3.1}
\]

where \( n = 0, \cdots, N - 1 \). The distance \( \tau_n \) is usually set to 0.25 or 0.5 (corresponding to 3 or 6 months). We focus on a finite set of zero-coupon bonds \( P(t, T_n) \), for the set \( \{ n : t < T_n \leq T_N \} \). As \( t \) moves forward this set shrinks. It is useful to introduce the so-called index function \( q(t) \) which satisfies

\[
T_{q(t)-1} \leq t < T_{q(t)}.
\]

The index function \( q(t) \) can be seen as the index of the first zero-coupon bond that has not expired by time \( t \). For any \( T_n > t \), the price of a zero-coupon bond is given by

\[
P(t, T_n) = P(t, T_{q(t)}) \prod_{i=q(t)}^{n-1} \frac{1}{(1 + \tau_i L_i(t))}.
\]

Note that we can also rewrite the asset price process \( B(t) \) as

\[
B(t) = P(t, T_{q(t)}) \prod_{n=0}^{q(t)-1} (1 + \tau_n L_n(T_n)). \tag{3.2}
\]
Now, we consider the set of forward Libor rates $L_q(t), L_{q(t)+1}(t), \cdots, L_{N-1}(t)$. Recall that for all $n \geq q(t)$, $L_n(t)$ is a martingale under the $T_{n+1}$-forward measure $Q^{T_{n+1}}$. Hence, the dynamics have to be driftless. It leads us to assume that the dynamics of forward rates are of the following form:

$$dL_n(t) = \sigma_n(t)\tau_n\sigma_n(t)\left(1 + \tau_nL_n(t)\right)dt + dW^n(t),$$

(3.3)

where $W^{n+1}(t) = W^{T_{n+1}}(t)$ is an $m$-dimensional standard Brownian motion under the $T_{n+1}$-forward measure $Q^{T_{n+1}}$ and $\sigma_n$ is an $m$-dimensional process adapted to the filtration generated by $W^{n+1}(t)$. In Subsection 3.1.1, we will discuss the details of the process $\sigma_n$.

In the following Lemma, we establish the dynamics of the Libor forward rate $L_n(t)$ under the neighbouring $T_n$-forward measure.

**Lemma 3.1.1.** Assume that $L_n(t)$ is given by Equation (3.3). The dynamics of $L_n(t)$ under the $T_n$-forward measure $Q^{T_n}$ are given by

$$dL_n(t) = \sigma_n(t)\tau_n\sigma_n(t)\left(1 + \tau_nL_n(t)\right)dt + dW^n(t),$$

(3.4)

where $W^n(t)$ is an $m$-dimensional standard Brownian motion under the $T_n$-forward measure $Q^{T_n}$.

**Proof.** By the Change of Numeraire theorem (see Theorem A.1.2 in Appendix A), we have that

$$\zeta(t) = \mathbb{E}_t^{(n+1)}\left(\frac{dQ^{T_n}}{dQ^{T_{n+1}}}\right) = \frac{P(t, T_n)P(0, T_n)}{P(t, T_{n+1})/P(0, T_{n+1})} = (1 + \tau_nL_n(t)) \frac{P(0, T_{n+1})}{P(0, T_n)},$$

where the last equality follows from Equation (2.4). Since $P(0, T_{n+1})$, $P(0, T_n)$ and $\tau_n$ are known for a given $n$, we have that

$$d\zeta(t) = \frac{P(0, T_{n+1})}{P(0, T_n)}\tau_n dL_n(t) = \frac{P(0, T_{n+1})}{P(0, T_n)}\tau_n\sigma_n(t)\left(1 + \tau_nL_n(t)\right)dt,$$

(3.5)

Dividing Equation (3.5) by $\zeta(t)$, we obtain the relation

$$d\zeta(t) = \frac{\tau_n\sigma_n(t)\left(1 + \tau_nL_n(t)\right)dt}{\zeta(t)},$$

and Girsanov’s theorem (see Theorem A.1.3 in Appendix A) implies that

$$dW^n(t) = dW^{n+1}(t) - \frac{\tau_n\sigma_n(t)}{1 + \tau_nL_n(t)}dt,$$

(3.6)

where $dW^n(t)$ is a Brownian motion under the $T_n$-forward measure $Q^{T_n}$. Hence, together with Equation (3.3) it yields that

$$dL_n(t) = \sigma_n(t)\tau_n\sigma_n(t)\left(1 + \tau_nL_n(t)\right)dt + dW^n(t).$$

$\square$
CHAPTER 3. THE LIBOR MARKET MODEL (LMM)

Lemma 3.1.1 gives an idea how to connect the $T_{n+1}$-forward measure $Q_{T_{n+1}}$ to the $T_n$-forward measure $Q_{T_n}$. Using Lemma 3.1.1 iteratively, we can represent the dynamics of $L_n(t)$ under the terminal measure $Q_{T_N}$ which is the $T_N$-forward measure.

**Lemma 3.1.2.** Assume that the dynamics of $L_n(t)$ are given by Equation (3.3). Under the terminal measure $Q_{T_N}$, the dynamics of $L_n(t)$ are given by

$$
    dL_n(t) = \sigma_n(t)^\top \left( - \sum_{j=n+1}^{N-1} \frac{\tau_j \sigma_j(t)}{1 + \tau_j L_j(t)} + dW^N(t) \right).
$$

(3.7)

**Proof.** Since Equation (3.6) holds for all $n = q(t), \cdots, N-1$, we can apply this recursively such that

$$
    dW^N(t) = dW^{N-1}(t) + \frac{\tau_{N-1} \sigma_{N-1}(t)}{1 + \tau_{N-1} L_{N-1}(t)} dt
$$

$$
    = dW^{N-2}(t) + \sum_{j=N-2}^{N-1} \frac{\tau_j \sigma_j(t)}{1 + \tau_j L_j(t)} dt
$$

$$
    \vdots
$$

$$
    = dW^{n+1}(t) + \sum_{j=n+1}^{N-1} \frac{\tau_j \sigma_j(t)}{1 + \tau_j L_j(t)} dt,
$$

which proves Equation (3.7).

The dynamics of $L_n(t)$ can also be written under the spot measure (see Section 2.2.2). The result is stated in the following lemma.

**Lemma 3.1.3.** Under the spot measure $Q^B$, the process $L_n(t)$ is given by

$$
    dL_n(t) = \sigma_n(t)^\top \left( \sum_{j=q(t)}^{n} \frac{\tau_j \sigma_j(t)}{1 + \tau_j L_j(t)} dt + dW^B(t) \right),
$$

(3.8)

where $W^B(t)$ is an $m$-dimensional Brownian motion under the spot measure $Q^B$.

**Proof.** For convenience, recall that the asset price process $B(t)$ is given by (see Equation (3.2))

$$
    B(t) = P(t, T_{q(t)}) \prod_{n=0}^{q(t)-1} (1 + \tau_n L_n(T_n)).
$$

At any time $t$, the random part of $B(t)$ is $P(t, T_{q(t)})$. Hence, we need to derive the dynamics under the $T_{q(t)}$-forward measure $Q^{T_{q(t)}}$. Applying Lemma 3.1.1 recursively, we obtain that

$$
    dW^{n+1}(t) = \sum_{j=q(t)}^{n} \frac{\tau_j \sigma_j(t)^\top}{1 + \tau_j L_j(t)} dt + dW^{q(t)}(t).
$$

For the more rigorous proof, we refer to Jamshidian [23].

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Remark 3.1.4. (Terminal measure and spot measure) For the simulation of $L_n(t)$, we can consider the dynamics under the terminal measure or under the spot measure. Under the terminal measure, the number of terms in the drift summation of $L_n(t)$ is always $N - n - 1$ for all $t \leq T_n$. This means that for $m < n$, $L_m(t)$ is possibly more biased than $L_n(t)$. However, the number of terms in the drift summation of $L_n(t)$ under the spot measure is given by $n - q(t) + 1$. It means that the number of terms in the drift summation of $L_n(t)$ decreases as $t$ increases. Hence, the possible bias from the discretization of the drift is more evenly distributed among the forward rates. This motivates us to consider the dynamics of $L_n(t)$ under the spot measure instead of the terminal measure.

3.1.1 The choice of $\sigma_n(t)$

We assume that the volatility process $\sigma_n(t)$ is separable in a deterministic part and a non-deterministic part. The reason why we assume the separable form is that it makes the model analytically tractable, which allows for the derivation of closed-form pricing formulas for plain vanilla options. The existence of closed-form formulas allows for an efficient calibration of the model to plain vanilla derivatives.

Under the separability assumption, the process $\sigma_n(t)$ is given by:

$$\sigma_n(t) = \lambda_n(t)\varphi(L_n(t)), \quad (3.9)$$

where $\lambda_n(t)$ is a bounded vector-valued deterministic function and $\varphi : \mathbb{R} \to \mathbb{R}$ is a time-homogeneous local volatility function.

Proposition 3.1.5. Assume that Equation (3.9) holds with $\varphi(0) = 0$ and that $L_n(0) \geq 0$ for all $n$. Furthermore, assume that $\varphi$ is locally Lipschitz continuous and satisfies the growth condition

$$\varphi(x)^2 \leq C(1 + x^2), \quad x > 0,$$

or equivalently,

$$|\varphi(x)| \leq C'(1 + x), \quad x > 0,$$

where $C$ and $C'$ are some positive constants. Then, non-explosive, pathwise unique solutions of the no-arbitrage SDEs for $L_n(t)$, $q(t) \leq n \leq N - 1$, exist under all measures $Q^T_i$, $q(t) \leq i \leq N$. If $L_n(0) > 0$, then $L_n(t)$ stays positive at all $t$.

Proof. We refer the reader to Piterbarg [5].

There are some standard parameterizations of $\varphi$. The most popular one is $\varphi(x) = x$ which implies a log-normal formulation of the Libor market model and is also known as the BGM model. We will briefly discuss this model in the next subsection. The other possible choice of $\varphi(x)$ is $bx + a$ with $a \neq 0$ and $0 < b \leq 1$. It is called the displaced log-normal formulation. We will assume the displaced log-normal formulation for the LIBOR market model with stochastic volatility. The stability properties related to the displaced log-normal specification are stated in the following lemma.
Lemma 3.1.6. Consider a local volatility Libor market model with local volatility function \( \varphi(x) = bx + a \), where \( 0 < b \leq 1 \) and \( a \neq 0 \). Assume that \( bL_n(0) + a > 0 \) and \( a/b < \tau_n^{-1} \) for all \( n = 1, 2, \cdots, N-1 \). Then non-explosive, pathwise unique solutions of the no-arbitrage SDEs for \( L_n(t) \), \( q(t) \leq n \leq N - 1 \), exist under all measure \( Q^{T_i} \), \( q(t) \leq i \leq N \). All \( L_n(t) \) are bounded from below by \( -a/b \).

Proof. We refer the reader to Piterbarg [5].

More specifically, if we assume \( b > 0 \), \( a = (1-b)L_n(0) \) and \( (1-b)L_n(0)/b < \tau_n^{-1} \), then we have non-explosive, pathwise unique solutions of the SDEs for \( L_n(t) \).

3.1.2 Brace-Gatarek-Musiela (BGM) model

As we briefly discussed in the previous section, we call the Libor market model with local volatility function \( \varphi(x) = x \) the “log-normal forward LIBOR model”. It is also known as the Brace-Gatarek-Musiela (BGM) model [9]. The dynamics of the log-normal forward rate are given by

\[
dL_n(t) = L_n(t)\lambda_n(t)^T dW^{n+1}(t),
\]

(3.10)

where \( \lambda_n : \mathbb{R} \rightarrow \mathbb{R}^m \) is a bounded deterministic function and \( W^{n+1} \) is a \( m \)-dimensional standard Brownian motion under the \( T_{n+1} \)-forward measure. For the BGM model, the pricing formula for a caplet is known as follows.

Theorem 3.1.7. The time 0 price of a caplet on forward rate \( L_n(t) \) is given by

\[
V_{\text{caplet}}(0) = \tau_n P(0, T_{n+1})(L_n(0)N(d_1) - KN(d_2)),
\]

(3.11)

where

\[
d_1 = \frac{\log(L_n(0)/K) + v^2/2}{v},
\]

\[
d_2 = \frac{\log(L_n(0)/K) - v^2/2}{v},
\]

and

\[
v^2 = \int_0^{T_n} ||\lambda_n(t)||^2 dt.
\]

Proof. For the proof, we refer to Brace, Gatarek and Musiela [9].

Note that Equation (3.11) corresponds to the Black formula for caplets with \( \sigma^2 T_n = v^2(t) \).
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3.2 LMM with displaced diffusion and stochastic volatility (LMM-DDSV)

A drawback of the BGM model is that it is less flexible in recovering essential characteristics of interest rate markets, in particular, the volatility smile. More precisely, it implies a flat implied volatility curve (see Figure 3.1a), but in many markets we observe that the volatility smile has a skewed shape such that the volatility curve is downward sloping for smaller strikes and has some curvature for larger strikes (see Figure 3.1b). Capturing the volatility skew is important in pricing derivatives, especially for exotic products. To overcome the shortcomings of the BGM model, we introduce the Libor market model with stochastic volatility.

In order to generate curvature in the implied volatilities we introduce a mean-reverting scalar process $z(t)$, with dynamics of the form

$$dz(t) = \theta(z_0 - z(t))dt + \eta \sqrt{z(t)} dZ^B(t), \quad z(0) = z_0 = 1, \quad (3.12)$$

where $\theta$, $z_0$, and $\eta$ are positive constants, and $Z^B$ is a Brownian motion under the spot measure $\mathbb{Q}^B$. This process is also known as the Cox-Ingersoll-Ross (CIR) process. The process $z$ has the role of a scaling factor so typically $z(0) = z_0 = 1$. Given the variance process, we assume that the forward rate processes $L_n(t)$ in $\mathbb{Q}^B$ follows, for all $n \geq q(t)$,

$$dL_n(t) = \sqrt{z(t)}\varphi(L_n(t))\lambda_n(t)^\top (\sqrt{z(t)}\mu_n(t)) dt + dW^B(t),$$

$$\mu_n(t) = \sum_{j=q(t)}^{n} \tau_j \varphi(L_j(t))\lambda_j(t) \frac{1}{1 + \tau_j L_j(t)}. \quad (3.13)$$

Note that the above formulation can be seen as Equation (3.8) with

$$\sigma_n = \sqrt{z(t)}\varphi(L_n(t))\lambda_n(t).$$

\[\text{Source: ING}\]
Now we turn to pricing of interest rate derivatives under the following model (called the ϕ
model since
the possibility to control the slope of the implied volatility skew with the parameter
forward rates under the
T
forward measure. Since
L
(t) is a martingale (driftless) under the
T
forward measure Q
T
, we should have
\[ dL_n(t) = \sqrt{z(t)}\varphi(L_n(t))\lambda_n(t)^\top dW^{n+1}(t). \]
It means that under the
T
forward measure, the dynamics of
L
(t) are given by:
\[ dW^{n+1}(t) = \sqrt{z(t)}\mu_n(t)\,dt + dW^B(t), \]
where \( W^{n+1}(t) \) is a \( m \)-dimensional Brownian motion under the
T
forward measure.

By defining the \( m \)-dimensional vector
\[ a(t) = d\langle Z, W^B \rangle(t)/dt, \]
we can write
\[ dZ(t) = a(t)^\top dW^B(t) + \sqrt{1-\|a(t)\|^2}\,d\tilde{W}(t), \]
where \( \tilde{W}(t) \) is a scalar Brownian motion independent of \( W^B(t) \). Under the
T
forward measure \( Q^{T_{n+1}} \), we have that
\[ dZ(t) = a(t)^\top (dW^{n+1}(t) - \sqrt{z(t)}\mu_n(t)\,dt + \sqrt{1-\|a(t)\|^2}\,d\tilde{W}(t) \]
\[ = dZ^{n+1}(t) - a(t)^\top \sqrt{z(t)}\mu_n(t)\,dt, \]
where \( dZ^{n+1}(t) = a(t)^\top dW^{n+1}(t) + \sqrt{1-\|a(t)\|^2}d\tilde{W}(t) \).

The process \( z(t) \) under \( T_{n+1} \)-forward measure \( Q^{T_{n+1}} \), \( n \geq q(t) - 1 \), is given by:
\[ dz(t) = \theta(z_0 - z(t))\,dt + \eta\sqrt{z(t)}(-\sqrt{z(t)}\mu_n(t)^\top d\langle Z, W^B \rangle(t) + dZ^{n+1}(t)), \tag{3.14} \]
where \( Z^{n+1}(t) \) is a Brownian motion in measure \( Q^{T_{n+1}} \). Note that if \( Z(t) \) and \( W^B(t) \) are independent, that is, \( d\langle Z, W^B \rangle(t) = 0 \), then the term \( \sqrt{z(t)}\mu_n(t)^\top d\langle Z, W^B \rangle(t) \) in Equation (3.14) is equal to 0. It simplifies the dynamics of \( z(t) \):
\[ dz(t) = \theta(z_0 - z(t))\,dt + \eta\sqrt{z(t)}\,dZ^{n+1}(t), \]
since \( dZ(t) = dZ^{n+1}(t) \). We assume that \( Z(t) \) and \( W^B(t) \) are independent, and we use the displaced log-normal formulation for the local volatility function \( \varphi(x) \) (specifically, \( \varphi(L_n(t)) = bL_n(t) + (1 - b)L_n(0) \) for \( 0 < b \leq 1 \)) because it is tractable and it gives the possibility to control the slope of the implied volatility skew with the parameter \( b \).

Now we turn to pricing of interest rate derivatives under the following model (called the LMM-DDSV: LMM with Displaced Diffusion and Stochastic Volatility).

**Model: The LMM-DDSV**

\begin{align*}
    dL_n(t) &= \sqrt{z(t)}(bL_n(t) + (1 - b)L_n(0))\lambda_n(t)^\top dW^{n+1}(t), \\
    dz(t) &= \theta(z_0 - z(t))\,dt + \eta\sqrt{z(t)}\,dZ^{n+1}(t), \quad z_0 = z(0) = 1 \\
    d\langle Z^{n+1}, W^{n+1} \rangle(t) &= 0
\end{align*}

\[ (3.15) \]
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3.2.1 Rank reduction

Short rate models for interest rates imply the perfectly correlated movement of forward rates. However, in practice, various points on the forward rates curve do not move co-monotonically with each other. To incorporate this observation, the LIBOR market model uses a multi-dimensional Brownian motion where the components drives each forward rate. In general, three or four dimensions are sufficient to capture the most of variation of the points on the forward curve (see Andersen and Piterbarg [5]). It gives us the motivation to choose the dimension \( m \) of Brownian motion of forward rate dynamics (see Equation (3.3)) smaller than the total number \( N - 1 \) of forward rates in the tenor structure.

Note that from the dynamics given by (3.13), we have

\[
d(L_k, L_j)(t) = z(t)\varphi(L_k(t))\varphi(L_j(t))\lambda_k(t)\lambda_j(t)^\top dt.
\]

This implies that the instantaneous correlation between \( L_k(t) \) and \( L_j(t) \) is given by

\[
\rho_{k,j}(t) = \frac{d(L_k, L_j)(t)}{\sqrt{d(L_k, L_k)(t)d(L_j, L_j)(t)}} = \frac{\lambda_k(t)^\top \lambda_j(t)}{|\lambda_k(t)| |\lambda_j(t)|}.
\] (3.16)

When \( m = 1 \), we can see that \( \rho_{k,j}(t) = 1 \), that is, \( L_k(t) \) and \( L_j(t) \) are fully correlated. As the number of Brownian motion increases, we are able to capture more complicated correlation structures, but it also increases the complexity of the model. Once we get a sample variance-covariance matrix from empirical data, the smallest \( m \) which is sufficient to duplicate correlation can be found by the tools of principal components analysis (PCA). In the next section we present the PCA briefly.

3.2.2 The principal components analysis (PCA)

In this section we present a general discussion on PCA. For more details, we refer to Andersen and Piterbarg [4]. Consider a \( p \)-dimensional Gaussian variable \( Z \) with a given covariance matrix \( \Sigma \). Without loss of generality, we assume that the mean of \( Z \) is 0 and that \( \Sigma \) has full rank (positive definite). Our aim is to find a \( (p \times m) \)-dimensional matrix \( D \) such that

\[
Z \approx DX,
\] (3.17)

where \( X \) is an \( m \)-dimensional vector of independent standard Gaussian variables, \( m \leq p \). Define

\[
D^* := \arg \min f(D) = tr \left( (\Sigma - DD^\top)(\Sigma - DD^\top)^\top \right),
\]

where \( tr(A) \) is the trace of a matrix \( A \). It is shown that [4]:

\[
D^* = V_m \sqrt{E_m},
\]
where $E_m$ is an $m \times m$ diagonal matrix containing the $m$ largest eigenvalues of $\Sigma$, and $V_m$ is a $p \times m$ matrix of $m$ $p$-dimensional eigenvectors corresponding to the eigenvalues in $E_m$. Using $D^*$ in Equation (3.17), we have

$$Z \approx \tilde{Z} := V_m \sqrt{E_m} X = \sqrt{e_1} v_1 X_1 + \sqrt{e_2} v_2 X_2 + \cdots + \sqrt{e_m} v_m X_m,$$

where $v_i$ denotes the $i$-th column of $V_m$ and the $e_j$'s are the eigenvalues, sorted in decreasing order of magnitude. The $v_i$ is called the $i$-th principal component of $Z$, and the variable $\sqrt{e_i} X_i$ as the $i$-th principal factor.

A sample variance-covariance matrix

In this subsection, we introduce some ways to obtain the empirical instantaneous correlations between forward rates from market data, described in Andersen and Piterbarg [5]. For a fixed $\tau$, we define “sliding” forward rates $l(t, x)$ with tenor $x$ as

$$l(t, x) = L(t, t + x, t + x + \tau).$$

For a given set of tenors $x_1, \cdots, x_{N_x}$ and a given set of calendar times $t_0, t_1, \cdots, t_{N_t}$, we define the $N_x \times N_t$ matrix $O$ with

$$O_{i,j} = \frac{l(t_j, x_i) - l(t_{j-1}, x_i)}{\sqrt{t_j - t_{j-1}}}, \quad i = 1, \cdots, N_x, j = 1, \cdots, N_t$$

where $\sqrt{t_j - t_{j-1}}$ is the annualized factor. Assuming time-homogeneity and ignoring small drift terms, a sample variance-covariance matrix $C$ can be obtained by

$$C = \frac{OO^T}{N_t}.$$

Introducing the diagonal matrix $c$ such that

$$c_{i,j} = \begin{cases} \sqrt{C_{i,j}}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the empirical forward rate correlation matrix $R$ is given by: $R = c^{-1} C c^{-1}$. In general, we expect that the correlation matrix $R = [r_{ij}]$, $i, j = 1, \cdots, N_x$ satisfies the following:

- The matrix $R$ is real and symmetric
- $r_{ii} = 1$
- The matrix $R$ is positive semi-definite
- $i \mapsto r_{ij}$ is decreasing
- $i \mapsto r_{i+p,i}$ is increasing for fixed $p \in \{1, \cdots, N_x - 1\}$
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The empirical correlation matrix $R$ is often relatively noisy in a sense that it might not satisfy one of these conditions. Hence, it is convenient to work with a parametric form. A possible parametric form (see Rebonato [30]) is given by

$$
\rho_{j,k}(t) = q(T_k - t, T_j - t),
$$

where

$$
q(x, y) = \rho_\infty + (1 - \rho_\infty) \exp(-|x - y| (\beta \exp(\alpha (\min(x, y))))).
$$

(3.18)

where $-1 \leq \rho_\infty \leq 1$, $\beta > 0$ and $\alpha \in \mathbb{R}$. We can use the parameters $\xi^* = (\alpha^*, \beta^*, \rho_\infty^*)$ which satisfies

$$
\xi^* = \arg \min \left( \text{tr} \left( (R - R_1(\xi))(R - R_1(\xi)^\top) \right) \right),
$$

where $R$ is the empirical correlation matrix, $R_1(\xi)$ is the correlation matrix generated by $\xi$, and the solution for $\xi^*$ minimizes the Frobenius norm. Variations of this parametric form are also possible. Another alternative to obtain the matrix $R$ is to calibrate $R$ to the prices of spread options in the market (see [5]).

3.2.3 Construction of $\lambda_k(t)$

As we have seen in Equation (3.16), $\lambda_k(t)$ is closely related to correlation and volatility $||\lambda_k(t)||$ of forward rates. It implies that we need to find $\lambda_k(t)$ which determine the overall correlation and volatility structure of forward rates in the model. We assume that

$$
\lambda_k(t) = h(t, T_k - t), \quad ||\lambda_k(t)|| = g(t, T_k - t),
$$

where $h : \mathbb{R}_+^2 \to \mathbb{R}^m$ and $g : \mathbb{R}_+^2 \to \mathbb{R}_+$. For the function $g$, we use

$$
g(t, x) = g(x) = (a + bx)e^{-cx} + d, \quad a, b, c, d \in \mathbb{R}_+.
$$

(3.19)

Usually, the function $||\lambda_k(t)||$ is considered to be piecewise constant in $t$, with discontinuities at $T_n$, $n = 1, \cdots, N - 1$,

$$
||\lambda_k(t)|| = \sum_{n=1}^k 1_{\{T_{n-1} \leq t < T_n\}} ||\lambda_{n,k}|| = \sum_{n=1}^k 1_{\{q(t) = n\}} ||\lambda_{n,k}||,
$$

To find $||\lambda_{n,k}||$, we use a bilinear interpolation method. Given a grid of times and tenors $\{t_j\} \times \{x_j\}$, $i = 1, \cdots, N_t$, $j = 1, \cdots, N_x$, we introduce a matrix $G$ such that

$$
G_{i,j} = (a + bx_j)e^{-cx_j} + d.
$$

Then, $||\lambda_{n,k}||$ is given by

$$
||\lambda_{n,k}|| = w_{++}G_{i,j} + w_{+-}G_{i,j-1} + w_{-+}G_{i-1,j} + w_{--}G_{i-1,j-1},
$$

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and by denoting \( \tau_{n,k} = T_k - T_{n-1} \), with

\[
\begin{align*}
\tau_{n,k} &= T_k - T_{n-1} - 1, \\
(i) &= \min \{ a : t_a \geq T_{n-1} \}, \\
(j) &= \min \{ b : x_b \geq \tau_{n,k} \}, \\
(w_{++}) &= (T_{n-1} - t_{i-1})(\tau_{n,k} - x_{j-1}) / (t_i - t_{i-1})(x_j - x_{j-1}), \\
(w_{+-}) &= (T_{n-1} - t_{i-1})(x_j - \tau_{n,k}) / (t_i - t_{i-1})(x_j - x_{j-1}), \\
(w_{-+}) &= (t_i - T_{n-1})(\tau_{n,k} - x_{j-1}) / (t_i - t_{i-1})(x_j - x_{j-1}), \\
(w_{--}) &= (t_i - T_{n-1})(x_j - \tau_{n,k}) / (t_i - t_{i-1})(x_j - x_{j-1}).
\end{align*}
\]

For each \( T_n \), let the matrix \( R(T_n) \) be an \((N-n) \times (N-n)\) instantaneous correlation matrix such that

\[
(R(T_n))_{i,j} = \rho_{i,j}(T_{n-1}), \quad i, j = n, \ldots, N - 1,
\]

and define a diagonal volatility matrix \( c(T_n) \) with elements

\[
(c(T_n))_{i,j} = \begin{cases} 
\| \lambda_{n,n+j-1} \|, & \text{for } i = j, \\
0, & \text{for } i \neq j,
\end{cases}
\]

where \( j = 1, \ldots, N - n \). Given \( R(T_n) \) and \( c(T_n) \), an instantaneous covariance matrix \( C(T_n) \) for forward rates is given by

\[
C(T_n) = c(T_n)R(T_n)c(T_n).
\] (3.20)

Suppose that the \((N-n) \times m\) matrix \( H(T_n) \) consists of elements

\[
(H(T_n))_{j,i} = h_i(T_n, T_{n+j-1} - T_n),
\]

for \( j = 1, \ldots, N - n \) and \( i = 1, \ldots, m \). Then, it follows that

\[
C(T_n) = H(T_n)H(T_n)^\top.
\]

Together with Equation (3.20), it yields

\[
H(T_n)H(T_n)^\top = c(T_n)R(T_n)c(T_n).
\]

Applying the PCA decomposition, the matrix \( R(T_n) \) can be written as

\[
R(T_n) = D(T_n)D(T_n)^\top.
\] (3.21)

Hence, we get

\[
H(T_n) = c(T_n)D(T_n).
\]

Assuming piecewise constant interpolation of \( \lambda_k(t) \) for \( t \), the full set of factor volatilities \( \lambda_k(t) \) can be constructed for all \( t \) and \( T_k \).
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Remark 3.2.1. When we apply PCA decomposition on a correlation matrix, the approximated matrix is in general not a valid correlation matrix anymore, meaning that the diagonal elements are not equal to 1. Hence, we need to rescale the approximated matrix to make sure that it has diagonal elements equal to 1. More precisely, the matrix $D(T_n)$ in Equation (3.21) is

$$D(T_n) = \begin{pmatrix}
  v_{11} \sqrt{e_1} & v_{21} \sqrt{e_2} & \cdots & v_{m1} \sqrt{e_m} \\
  v_{12} \sqrt{e_1} & v_{22} \sqrt{e_2} & \cdots & v_{m2} \sqrt{e_m} \\
  \vdots & \vdots & \ddots & \vdots \\
  v_{1N-n} \sqrt{e_1} & v_{2N-n} \sqrt{e_2} & \cdots & v_{mN-n} \sqrt{e_m}
\end{pmatrix} \cdot \begin{pmatrix}
  \sqrt{e_1} & 0 & \cdots & 0 \\
  0 & \sqrt{e_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \sqrt{e_m}
\end{pmatrix}$$

where $v_i = (v_{i1}, v_{i2}, \ldots, v_{iN-n})^T$ is an eigenvector corresponding to the eigenvalue $e_i$.

Then, the approximating matrix is given by

$$R(T_n) = D(T_n)D(T_n)^T$$

$$= \begin{pmatrix}
  \sum_{k=1}^{m} v_{k1}^2 e_k & \sum_{k=1}^{m} v_{k1} v_{k2} e_k & \cdots & \sum_{k=1}^{m} v_{k1} v_{kN-n} e_k \\
  \sum_{k=1}^{m} v_{k1} v_{k2} e_k & \sum_{k=1}^{m} v_{k2}^2 e_k & \cdots & \sum_{k=1}^{m} v_{k2} v_{kN-n} e_k \\
  \vdots & \vdots & \ddots & \vdots \\
  \sum_{k=1}^{m} v_{kN-n} v_{k1} e_k & \sum_{k=1}^{m} v_{kN-n} v_{k2} e_k & \cdots & \sum_{k=1}^{m} v_{kN-n}^2 e_k
\end{pmatrix}.$$
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The matrix form of forward rate dynamics

From the definition of \( \lambda_n(t) \), we have

\[
\lambda_n(t) = \begin{pmatrix}
||\lambda_n(t)||v_{1n}\sqrt{\epsilon_1} \\
||\lambda_n(t)||v_{2n}\sqrt{\epsilon_2} \\
\vdots \\
||\lambda_n(t)||v_{mn}\sqrt{\epsilon_m}
\end{pmatrix},
\]

and this implies that

\[
\lambda_n^\top(t)\lambda_j(t) = \begin{pmatrix}
||\lambda_n(t)||v_{1n}\sqrt{\epsilon_1} \\
||\lambda_n(t)||v_{2n}\sqrt{\epsilon_2} \\
\vdots \\
||\lambda_n(t)||v_{mn}\sqrt{\epsilon_m}
\end{pmatrix} \begin{pmatrix}
||\lambda_j(t)||v_{1j}\sqrt{\epsilon_1} \\
||\lambda_j(t)||v_{2j}\sqrt{\epsilon_2} \\
\vdots \\
||\lambda_j(t)||v_{mj}\sqrt{\epsilon_m}
\end{pmatrix} = ||\lambda_n(t)||\rho_{n,j}||\lambda_j(t)||,
\]

where \( \rho_{n,j} \) is the correlation between \( L_n \) and \( L_j \). The dynamics of \( L_n(t) \) are equivalent to

\[
dL_n(t) = \sqrt{z(t)}(bL_n(t) + (1-b)L_n(0)||\lambda_n(t)||}
\cdot (\sqrt{z(t)}\mu'_n(t) dt + (v_{1n}\sqrt{\epsilon_1} \ v_{2n}\sqrt{\epsilon_2} \ \cdots \ v_{mn}\sqrt{\epsilon_m}) dW^B(t),
\]

where

\[
\mu'_n(t) = \sum_{j=q(t)}^n \frac{\tau_j(bL_j(t) + (1-b)L_j(0))\rho_{n,j}||\lambda_j(t)||}{1 + \tau_jL_j(t)}.
\]

We can also consider the set of forward rates dynamics. The matrix form of the dynamics are given by

\[
\begin{pmatrix}
\frac{dL_1(t)}{dt} \\
\vdots \\
\frac{dL_{N-1}(t)}{dt}
\end{pmatrix} = \sqrt{z(t)} \begin{pmatrix}
\varphi(L_1(t)||\lambda_1(t)|| \\
\vdots \\
0
\end{pmatrix} \begin{pmatrix}
\mu'_1(t) \\
\vdots \\
\mu'_{N-1}(t)
\end{pmatrix} dt + \begin{pmatrix}
v_{11} & \cdots & v_{m1} \\
\vdots & \ddots & \vdots \\
v_{1N-1} & \cdots & v_{mN-1}
\end{pmatrix} \begin{pmatrix}
\sqrt{\epsilon_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sqrt{\epsilon_m}
\end{pmatrix} \begin{pmatrix}
dW^B_1(t) \\
\vdots \\
dW^B_m(t)
\end{pmatrix},
\]

where

\[
\varphi(L_n(t)) = bL_n(t) + (1-b)L_n(0),
\]

and \( W^B_k \) is the \( k \)-th component of an \( m \)-dimensional Brownian motion \( W^B \).
Chapter 4

Pricing IR Derivatives under the LMM-DDSV Model

In this chapter, we derive pricing formulas under the Libor Market Model with Displaced Diffusion and Stochastic Volatility (see Andersen and Piterbarg [5]). The LMM-DDSV model is closely related to the Heston model [17], which is one of the most well-known stochastic volatility models for equities. A benefit of using the Heston model is that in case of constant parameters there exist analytical pricing formulas for European options. Before we present the pricing formula for caplets and swaptions, we first introduce the Heston model in Section 4.1. In Sections 4.2 and 4.3 we derive pricing formulas for caplets and swaptions under the LMM-DDSV model with constant (time-homogeneous) parameters, and in Section 4.4 we extend the LMM-DDSV model with time-dependent parameters. The time-dependent LMM-DDSV model is approximated by a time-homogeneous model with averaged parameters, obtained by a parameter averaging method. Finally, in Section 4.5 we investigate the impact of parameters on the implied volatility curve, and in Section 4.6 we discuss simulation methods for the LMM-DDSV model.

4.1 The Heston model

We start with the dynamics of an asset price process $X(t)$ under the Heston model. They are given by:

\[
\begin{align*}
    dX(t) &= \mu X(t)dt + \sqrt{z(t)}X(t)dW(t), \\
    dz(t) &= \theta(z_0 - z(t))dt + \eta \sqrt{z(t)}dZ(t), \\
    z(0) &= z_0 = 1,
\end{align*}
\]

(4.1)

where $Z(t)$ and $W(t)$ are Brownian motions under a probability measure $\mathbb{P}$ and with $d\langle Z, W \rangle(t) = \rho \, dt$. This model has been extensively studied in the literature and Heston [17] suggested a technique to derive a closed-form solution to price of a European (call) option under the stochastic volatility model, where the asset price process is driven by
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Before we present the result, we define the moment generating function of log $X(t)$ by:

$$\Psi_X(u,t) := \mathbb{E}^P \left( e^{u \log X(t)} \right).$$

Since $dX(t)/X(t) = \sqrt{z(t)} \lambda(t) dW(t)$, we have that

$$\ln X(t) = \int_0^t \lambda(s) \sqrt{z(s)} dW(s) - \frac{1}{2} \int_0^t \lambda(s)^2 z(s) ds,$$

and this implies that

$$\Psi_X(u,t) = \mathbb{E}^P \left( e^{u \log X(t)} \right)$$

$$= \mathbb{E}^P \left( e^{u \int_0^t \lambda(s) \sqrt{z(s)} dW(s) - \frac{1}{2} \int_0^t \lambda(s)^2 z(s) ds} \right)$$

$$= \mathbb{E}^P \left( e^{u \int_0^t \lambda(s) \sqrt{z(s)} dW(s) - \frac{1}{2} u^2 \int_0^t \lambda(s)^2 z(s) ds} e^{\frac{1}{2} u (u-1) \int_0^t \lambda(s)^2 z(s) ds} \right)$$

$$= \mathbb{E}^P \left( \varsigma(t) e^{\frac{1}{2} u (u-1) \int_0^t \lambda(s)^2 z(s) ds} \right),$$

where

$$\varsigma(t) = e^{u \int_0^t \lambda(s) \sqrt{z(s)} dW(s) - \frac{1}{2} u^2 \int_0^t \lambda(s)^2 z(s) ds}.$$

The function $\varsigma(t)$ is in fact an exponential martingale. From Girsanov’s theorem (see A.1.3 in Appendix A), we have

$$\Psi_X(u,t) = \mathbb{E}^P \left( \varsigma(t) e^{\frac{1}{2} u (u-1) \int_0^t \lambda(s)^2 z(s) ds} \right) = \Psi\varsigma(u) \left( \frac{1}{2} u (u-1), t \right),$$

$$= \Psi\varsigma(u) \left( \frac{1}{2} u (u-1), t \right).$$
where
\[ \Psi_{\lambda^2 z}(v, t) := \mathbb{E}^P \left( e^{v \int_0^t \lambda(s)^2 z(s) \, ds} \right), \] (4.3)

Let us define
\[ G(t, z) := \mathbb{E}^P \left( e^{v \int_0^T \lambda(s)^2 z(s) \, ds} \mid z(t) = z \right), \]
then, by applying the Feynman-Kac formula, we can derive the PDE which is affine in \( z \) (see [4]). The solution of the PDE can be made such that
\[ G(t, z) = \exp(A(t, T) + z B(t, T)), \]
where \( A(t, T) \) and \( B(t, T) \) satisfy the following relationship:
\[ \frac{d}{dt} A(t, T) + \frac{\theta z_0}{2} B(t, T) + \frac{\eta^2}{2} B(t, T)^2 + v \lambda(t)^2 = 0. \]

Since \( G(0, z_0) = \Psi_{\lambda^2 z}(v, u; T) \), we have that
\[ \Psi_{\lambda^2 z}(v, T) = \exp(A(0, T) + z_0 B(0, T)) \]
where
\[ \frac{d}{dt} A(t, T) + \theta z_0 B(t, T) = 0, \]
\[ \frac{d}{dt} B(t, T) - \theta B(t, T) + \frac{\eta^2(t)}{2} B(t, T)^2 + v \lambda(t)^2 = 0, \]
with the terminal condition \((A(T, T), B(T, T)) = (0, 0)\). Under the assumption that \( \lambda(t) = \lambda \) and \( \eta(t) = \eta \) are both constant, the Riccati ODE’s can be solved analytically. It will be convenient to address the result here.

**Lemma 4.1.1.** Assuming that \( \lambda(t) = \lambda \) and \( \eta(t) = \eta \), \( \Psi_{\lambda^2 z}(v, T) \) is given by
\[ \Psi_{\lambda^2 z}(v, T) = \Psi_z(v \lambda^2, T), \]
where \( \Psi_z(v, T) := \mathbb{E}^P (e^{v \int_0^T z(s) \, ds}) = \exp(A(v, T) + z_0 B(v, T)) \), with
\[ A(v, T_n) = \frac{\theta z_0}{\eta^2} \left[ 2 \log \left( \frac{2 \gamma}{\theta + \gamma - e^{-\gamma T_n}(\theta - \gamma)} \right) + (\theta - \gamma) T_n \right], \]
\[ B(v, T_n) = \frac{2v(1 - e^{-\gamma T_n})}{(\theta + \gamma)(1 - e^{-\gamma T_n}) + 2 \gamma e^{-\gamma T_n}}, \]
and \( \gamma = \gamma(v) = \sqrt{\theta^2 - 2 \eta^2 v}. \)

**Proof.** We refer to Andersen and Piterbarg [4].
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Note that the \( z(t) \) is an affine process. The process \( X = \{ X_t : 0 \leq t \leq T \} \) is called affine if the \( \mathcal{F}_t \) conditional characteristic function of \( X(T) \) is exponential affine in \( X(t) \), for all \( t \leq T \). That is, there exist functions \( A(u, t) \) and \( B(u, t) \) satisfying
\[
E[e^{uX(T)}|\mathcal{F}_t] = e^{A(u, T-t) + B(u, T-t)X(t)}.
\]

If the dynamics of the process \( X(t) \) are assumed to be
\[
dX(t) = (b(t) + \beta(t)X(t))dt + \sqrt{a(t) + \alpha(t)X(t)}dW(t),
\]
where \( a(t), b(t), \alpha(t) \) and \( \beta(t) \) are deterministic continuous functions, then the functions \( A \) and \( B \) satisfy the Riccati ODE’s,
\[
\frac{d}{dt}A(u, T) = \frac{1}{2}a(t)B^2(t, T) - b(t)B(t, T)
\]
\[
\frac{d}{dt}B(t, T) = -\frac{1}{2}\alpha(t)B^2(t, T) + \beta(t)B(t, T) - 1
\]
with the terminal conditions \( (A(T, T), B(T, T)) = (0, 0) \).

Now, we state the analytical pricing formula of the European call option under the Heston model in the following Theorem. This theorem will be used in deriving pricing formula of caplets and swaptions because caplets and swaptions are European call option types.

**Theorem 4.1.2.** Let \( X(t) \) be the asset price process defined by Equation (4.2). The theoretical fair price for a European call option with strike \( K \) and maturity \( T \) under the Heston model is given by
\[
\mathbb{E}^P(X(T) - K)^+ = \text{Black}(0, X(T), K, \sigma) - \frac{K}{\pi} \int_0^\infty \Re\left( \frac{e^{-(i\omega + \alpha)\log(K)}}{\alpha + i\omega(1 - \alpha - i\omega)}q(\alpha + i\omega) \right) d\omega,
\]
where \( \alpha \) is usually set to \( 1/2 \) and with
\[
q(u) = \Psi_X(u, T) - \Psi_X^0(u, T),
\]
\[
\Psi_X^0(u, T) = \exp\left( \frac{1}{2}z_0u(u - 1) \int_0^T \lambda(s)^2 ds \right).
\]
The Black formula is given by
\[
\text{Black}(0, X(T), K, \sigma) := X(0) \mathcal{N}(d_1) - K \mathcal{N}(d_2),
\]
with \( \nu^2 = z_0 \int_0^T \lambda^2(t) dt, \ d_1 = \frac{\log(X(0)/K) + \frac{1}{2}\nu^2}{\nu}, \ d_2 = d_1 - \nu, \) and \( \sigma = \frac{\nu}{\sqrt{t}} \).

**Proof.** For the proof, we refer the reader to Appendix A.2.
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Remark 4.1.3. By setting $\alpha = 1/2$, the indefinite integral in Theorem 4.1.2 can be simplified as follows. Note that:

$$q \left( \frac{1}{2} + iw \right) = \Psi_X \left( \frac{1}{2} + iw, T \right) - \Psi_X^0 \left( \frac{1}{2} + iw, T \right)$$

$$= \Psi_{\lambda^2 z} \left( \frac{1}{2} \left( \frac{1}{2} + iw \right), T \right) - e^{\frac{1}{2} z_0 \left( \frac{1}{2} + iw \right) \left( \frac{1}{2} + iw - 1 \right)} \int_0^T \lambda^2(s) \, ds$$

$$= \Psi_{\lambda^2 z} \left( -\frac{1}{2} \left( \frac{1}{4} + w^2 \right), T \right) - e^{-\frac{1}{2} z_0 \left( \frac{1}{2} + w^2 \right)} \int_0^T \lambda^2(s) \, ds.$$ 

Hence, $q(1/2 + iw)$ is a real-valued function, and it implies that

$$\frac{K}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-\left( \frac{1}{2} + iw \right) \log(K)}}{(w^2 + 1/4)} q(1/2 + iw) \right) \, dw$$

$$= \frac{K}{\pi} \int_0^\infty \sqrt{\frac{1}{K}} \cos \left( \frac{w \log \left( \frac{1}{K} \right)}{w^2 + 1/4} \right) q(1/2 + iw) \, dw.$$ 

According to Attari [7], the simplified equation has the benefit in terms of computational time.

Remark 4.1.4. (Numerical implementation) In order to compute the improper integral, we can approximate the improper integral by a proper integral, such that the absolute difference is smaller than a given tolerance $\epsilon$. From the result by Hoorens [19], we know that if we set

$$w_{\text{max}} = \frac{1}{2} \tan \left( \frac{\pi}{2} - \frac{\epsilon}{4} \right),$$

then we have that

$$\left| \int_{w_{\text{max}}}^\infty \frac{\cos \left( w \log \left( \frac{1}{K} \right) \right)}{(w^2 + 1/4)} q(1/2 + iw) \, dw \right| \leq \epsilon.$$

Hence, we can approximate the improper integral by a proper integral:

$$\frac{K}{\pi} \int_0^{w_{\text{max}}} \sqrt{\frac{1}{K}} \cos \left( \frac{w \log \left( \frac{1}{K} \right)}{w^2 + 1/4} \right) q(1/2 + iw) \, dw$$

$$\approx \frac{K}{\pi} \int_0^{w_{\text{max}}} \sqrt{\frac{1}{K}} \cos \left( \frac{w \log \left( \frac{1}{K} \right)}{w^2 + 1/4} \right) q(1/2 + iw) \, dw.$$ 

In our implementation, we use the MATLAB function quadgk, also known as the adaptive Gauss-Kronrod quadrature, because it is an efficient method for high accuracies and oscillatory integrands. It truncates the region of integration by an internal criteria.


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4.2 Pricing of caplets

In this subsection, we derive the pricing formula for caplets under the LMM-DDSV model. For convenience, we repeat the dynamics of forward rate \( L_n(t) \) here:

\[
\begin{align*}
    dL_n(t) &= \sqrt{z(t)}(bL_n(t) + (1 - b)L_n(0))\lambda_n(t)\top dW^{n+1}(t), \\
    dz(t) &= \theta(z_0 - z(t)) \, dt + \eta \sqrt{z(t)} \, dZ^{n+1}(t), \\
    z_0 &= z(0) = 1,
\end{align*}
\]

(4.4a)

where \( W^{n+1}(t) \) and \( Z^{n+1}(t) \) is an \( m \)-dimensional Brownian motion and one-dimensional Brownian motion under \( T_{n+1} \)-forward measure, respectively. Equation (4.4a) is equivalent to

\[
dL_n(t) = \sqrt{z(t)}(bL_n(t) + (1 - b)L_n(0))\lambda_n(t)\top ||\lambda_n(t)|| dW^{n+1}(t).
\]

In order to derive the pricing formula, we want to use the results from Section 4.1. Therefore, we have to map the LMM-DDSV model onto the Heston model. Since the LMM-DDSV model assumes an \( m \)-dimensional Brownian motion \( W^{n+1}(t) \), the first step is to replace this by one-dimensional Brownian motion. Hence, we define \( Y^{n+1}(t) \) by

\[
Y^{n+1}(t) = \int_0^t \frac{\lambda_n(s)}{||\lambda_n(s)||} dW^{n+1}(s).
\]

The differential equation of \( Y^{n+1}(t) \) is given by

\[
dY^{n+1}(t) = \frac{\lambda_n(t)}{||\lambda_n(t)||} dW^{n+1}(t).
\]

Since \( \lambda_n(t) \) is bounded, the Itô integral \( Y^{n+1} \) is martingale with mean 0 and variance \( t \). By the Lévy’s characterization of Brownian motions (see Karatzas and Shreve [25]), \( Y^{n+1} \) can be identified as a Brownian motion. Substituting \( \lambda_n(t)\top/||\lambda_n(t)|| dW^{n+1}(t) \) by \( dY^{n+1}(t) \), the dynamics are rewritten as

\[
\begin{align*}
    dL_n(t) &= \sqrt{z(t)}(bL_n(t) + (1 - b)L_n(0))\lambda_n(t)\top dY^{n+1}(t), \\
    dz(t) &= \theta(z_0 - z(t)) \, dt + \eta \sqrt{z(t)} \, dZ^{n+1}(t), \\
    z_0 &= z(0) = 1,
\end{align*}
\]

(4.5a)

where \( Y^{n+1}(t) \) and \( Z^{n+1}(t) \) are independent Brownian motions under the \( T_{n+1} \)-forward measure \( Q^{T_{n+1}} \). The next step in order to map the LMM-DDSV model onto the Heston model is to define a coordinate transformation. For a given strike \( K \), we define the shifted forward process \( \bar{L}_n \) and the shifted strike \( \bar{K} \) by

\[
\begin{align*}
    \bar{L}_n(t) &= bL_n(t) + (1 - b)L_n(0), \\
    \bar{K} &= bK + (1 - b)L_n(0),
\end{align*}
\]

and the dynamics of \( \bar{L}_n(t) \) are given by:

\[
\begin{align*}
    d\bar{L}_n(t) &= b \sqrt{z(t)}\bar{L}_n(t)\lambda_n(t)\top dY^{n+1}(t), \\
    \bar{L}_n(0) &= L_n(0), \\
    d\bar{z}(t) &= \theta(z_0 - \bar{z}(t)) \, dt + \eta \sqrt{z(t)} \, dZ^{n+1}(t), \\
    \bar{z}_0 &= \bar{z}(0) = 1.
\end{align*}
\]

(4.6)
Note that the initial value $\tilde{L}_n(0)$ may not be equal to 1. As the last step to map the LMM-DDSV model onto the Heston model defined by Equation (4.2), we need to introduce a process $X(t) = \tilde{L}_n(t)/\tilde{L}_n(0)$ with $X(0) = 1$.

**Theorem 4.2.1.** Assuming that $||\lambda_n(t)||$ is constant, the price of a caplet under the LMM-DDSV model at time $t = 0$ is given by

$$V_{\text{caplet}}(0) = \frac{1}{b} P(0, T_{n+1}) \tau_n \mathbb{E}^{n+1}(\tilde{L}_n(T_n) - \bar{K})^+.$$  \hspace{1cm} (4.7)

where

$$\mathbb{E}^{n+1}(\tilde{L}_n(T_n) - \bar{K})^+ = \text{Black}(0, \tilde{L}_n, T_n, \bar{K}, ||\lambda_n||b) - \frac{\bar{K}}{\pi} \int_{0}^{\infty} \text{Re} \left( \frac{e^{-(iw+1/2)\log(\bar{K}/\tilde{L}_n(0))}}{(w^2 + 1/4)} q(1/2 + iw) \right) dw,$$  \hspace{1cm} (4.8)

with the function $q(u)$ defined by

$$q(u) = \Psi_z \left( \frac{1}{2} (||\lambda_n||b)^2 u(u - 1), T_n \right) - e^{\frac{1}{2}||\lambda_n||^2 b^2 z_0 T_n u(u - 1)},$$

and also $\Psi_z(u, t)$ is defined in Lemma 4.1.1 and Black$(0, X, T, K, \sigma)$ is given by Theorem 4.1.2.

**Proof.** Note that

$$\mathbb{E}^{n+1}(\tilde{L}_n(T_n) - \bar{K})^+ = \tilde{L}_n(0) \mathbb{E}^{n+1} \left( \frac{\tilde{L}_n(T_n)}{\tilde{L}_n(0)} - \frac{\bar{K}}{\tilde{L}_n(0)} \right)^+.$$

Define $X(t) = \tilde{L}_n(t)/\tilde{L}_n(0)$, the dynamics of $X(t)$ given by

$$dX(t) = b\sqrt{z(t)} X(t)||\lambda_n|| dY^{n+1}(t), \quad X(0) = 1,$$

$$dz(t) = \theta(z_0 - z(t)) dt + \eta \sqrt{z(t)} dZ(t), \quad z_0 = z(0) = 1,$$

which are the dynamics considered in Theorem 4.1.2. Hence, it implies that

$$\tilde{L}_n(0) \mathbb{E}^{n+1} \left( \frac{\tilde{L}_n(T_n)}{\tilde{L}_n(0)} - \frac{\bar{K}}{\tilde{L}_n(0)} \right)^+ = \text{Black}(0, \tilde{L}_n, T, \bar{c}, \sigma) - \frac{\bar{K}}{\pi} \int_{0}^{\infty} \text{Re} \left( \frac{e^{-(iw+1/2)\log(\bar{K}/\tilde{L}_n(0))}}{1/4 + w^2} q(1/2 + iw) \right) dw,$$

where $\sigma^2 = z_0(b||\lambda_n||)^2 = (b||\lambda_n||)^2$ and

$$q(u) = \Psi_X(u, T_n) - \Psi^0_X(u, T_n) = \Psi_z \left( \frac{1}{2} u(u - 1)(b||\lambda_n||)^2, T_n \right) - e^{\frac{1}{2} u(u - 1)z_0 b^2 ||\lambda_n||^2 T_n}.$$  

\qed
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4.3 Pricing of swaptions

Consider a swaption and the tenor structure given by (3.1). Recall that the price of the swaption with maturity \(T_j\) at time \(t\) is given by

\[ V_{\text{swaption}}(t) = A(t)E^{j,k-j}(S(T_j - K)^+), \]

where

\[ S(t) = S_{j,k-j}(t) = \frac{P(t,T_j) - P(t,T_k)}{A(t)}, \]

and

\[ A(t) = A_{j,k-j}(t) = \sum_{n=j}^{k-1} P(t,T_{n+1})\tau_n, \]

with the underlying security being a swap, making payments at time \(T_{j+1}, T_{j+2}, \ldots, T_k\), \(j < k \leq N\). It can be seen as a European call option on the swap rate, so that it is convenient to derive the dynamics of the forward swap rate \(S(t)\).

**Proposition 4.3.1.** Assume that the forward rate dynamics are given by Equation (3.15). Under the swap measure \(Q^{j,k-j}\), the dynamics of the swap rate are given by

\[ dS(t) = \sqrt{z(t)}(bs(t) + (1-b)s(0))\sum_{n=j}^{k-1} w_n(t)\lambda_n(t)^{\top}dW^{j,k-j}(t), \quad (4.9) \]

where \(W^{j,k-j}\) is a Brownian motion under the swap measure \(Q^{j,k-j}\) and the stochastic weights \(w_n\) are given by

\[ w_n(t) = \frac{bL_n(t) + (1-b)L_n(0)}{bs(t) + (1-b)s(0)} \times \frac{\partial S(t)}{\partial L_n(t)} = \frac{\partial S(t)}{\partial L_n(t)} \times \left( \alpha_n(t) + \frac{\tau_n}{1 + \tau_nL_n(t)} \sum_{i=j}^{n-1} \alpha_i(t)(L_i(t) - s(t)) \right), \]

where

\[ \alpha_n(t) = \frac{\tau_nP(t,T_{n+1})}{\sum_{r=j}^{n-1} \tau_r P(t,T_{r+1})}. \]

**Proof.** For the proof, we refer the reader to Appendix A.3. \qed

In order to derive the pricing formula for swaptions, we map the dynamics of the swap rate given by Equation (4.9) onto the Heston model. However, it can immediately be seen that the dynamics of the swap rate given in Equation (4.9) are not analytically tractable, because the weights are stochastic. One possible way to overcome this problem is to freeze the weights at their time 0 values, which is accurate when \(\partial S(t)/\partial L_n(t)\) is a near-constant, see Andersen and Piterbarg [5].
Define
\[ \lambda_S(t) = \sum_{n=j}^{k-1} w_n(0) \lambda_n(t), \]
where \( w_n(0) \) is the value of the stochastic weights \( w_n(t) \) at time 0. The swap rate dynamics can be approximated by
\[
\begin{align*}
    dS(t) &= \sqrt{z(t)}(bS(t) + (1 - b)S(0)) ||\lambda_S(t)|| dY^{j,k-j}(t), \\
    dz(t) &= \theta(z_0 - z(t))dt + \eta \sqrt{z(t)}dZ(t),
\end{align*}
\]
where \( Y^{j,k-j} \) and \( Z(t) \) are independent scalar Brownian motions under the swap measure \( Q^{j,k-j} \), and
\[
||\lambda_S(t)|| dY^{j,k-j}(t) = \sum_{n=j}^{k-1} w_n(0) \lambda_n(t) \top dW^{j,k-j}(t),
\]
Analogous to Section 4.2, we can map the dynamics of the swap rate given by Equation (4.10) onto Heston model by using a coordinate transformation. The pricing formula of a swaption is stated in the following theorem.

**Theorem 4.3.2.** Assume that \( ||\lambda_S(t)|| = ||\lambda_S|| \), for all \( t \). If we define \( \tilde{S}(t) \) and \( \tilde{K} \) by
\[
\begin{align*}
    \tilde{S}(t) &= bS(t) + (1 - b)S(0), \\
    \tilde{K} &= bK + (1 - b)S(0),
\end{align*}
\]
then the time 0 price of the payer swaption is given by
\[
V_{swaption}(0) = A(0) \mathbb{E}^A((S(T_j) - K)^+) = 1/b A(0) \mathbb{E}^A((\tilde{S}(T_j) - \tilde{K})^+),
\]
and where the expectation is given by
\[
\mathbb{E}^A((\tilde{S}(T_j) - \tilde{K})^+) = \text{Black}(0, \tilde{S}, T_j, \tilde{K}, ||\lambda_S||b) - \frac{\tilde{K}}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i(u+1/2) \log(\tilde{K}/\tilde{S}(0))}}{(w^2 + 1/4)} q(1/2 + iu) \right) dw.
\]
The function for \( q(u) \) is given by
\[
q(u) = \Psi_z \left( \frac{1}{2} ||\lambda_S||b^2 u(u - 1), T_j \right) - e^{1/2 ||\lambda_S||b^2 z_0 T_j u(u - 1)},
\]
where \( \Psi_z(u,t) \) is defined in Lemma 4.1.1 and \( \text{Black}(0, X, T, K, \sigma) \) is given in Theorem 4.1.2.

**Proof.** Analogous to Theorem 4.2.1. □

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4.4 Parameter averaging method

The dynamics given by Equation (4.9) are less flexible to capture the variability in volatility skews across maturities. Therefore, we consider the forward rate dynamics with time-dependent parameters, which are given by

\[
\begin{align*}
    dL_n(t) &= \sqrt{z(t)}(b_n(t)L_n(t) + (1 - b_n(t))L_n(0))\lambda_n(t)\top dW^{n+1}(t), \\
    dz(t) &= \theta(z_0 - z(t))dt + \eta(t)\sqrt{z(t)}dZ^{n+1}(t).
\end{align*}
\]  

(4.12)

where \(W^{n+1}(t)\) is an \(m\)-dimensional Brownian motion under the \(Q_{T^{n+1}}\)-forward measure. We call these dynamics the LMM-DDSV model with time-dependent parameters. Piterbarg [29] derived the dynamics of the swap rate for the above model and the result is given in the following theorem.

**Theorem 4.4.1.** The approximate dynamics of the swap rate \(S(t)\) under the swap measure \(Q^{j,k-j}(t)\) are given by

\[
    dS(t) \approx \sqrt{z(t)}(b_S(t)S(t) + (1 - b_S(t))S(0))\lambda_S\top dW^{j,k-j}(t),
\]

where

\[
    \lambda_S(t) = \sum_{n=j}^{k-1} \frac{\partial S(0)}{\partial L_n(0)} \frac{L_n(0)}{S(0)} \lambda_n(t),
\]

\[
    b_S(t) = \sum_{n=j}^{k-1} p_n(t)b_n(t),
\]

and for \(n = j, j+1, \ldots, k-1\), we have

\[
    p_n(t) = \frac{\lambda_n(t)\top \lambda_S(t)}{(k-j)|||\lambda_S(t)|||^2}.
\]

**Proof.** We refer the reader to Piterbarg [29].

Note that as we discussed in the Section 4.3, we can derive the dynamics of the swap rate which is driven by a scalar Brownian motion. The dynamics are given by:

\[
\begin{align*}
    dS(t) &= \sqrt{z(t)}(b_S(t)S(t) + (1 - b_S(t))S(0))|||\lambda_S(t)||| dY^{j,k-j}(t), \\
    dz(t) &= \theta(z_0 - z(t))dt + \eta(t)\sqrt{z(t)}dZ(t).
\end{align*}
\]  

(4.13)

Unless otherwise stated we will use the dynamics from Equation 4.13. A benefit of considering time-dependent parameters is that it can describe the market volatility skews.
CHAPTER 4. PRICING IR DERIVATIVES UNDER THE LMM-DDSV MODEL

for different maturities more accurately. Therefore, we want to replace the dynamics, given in Equation (4.13), by

\[
\begin{align*}
\bar{d}S(t) &= \sqrt{\bar{z}(t)(\bar{b}_S \bar{S}(t) + (1 - \bar{b}_S)\bar{S}(0))||\bar{\lambda}_S||} \, dY^{j,k-j}(t), \\
\bar{d}\bar{z}(t) &= \theta(\bar{z}_0 - \bar{z}(t)) \, dt + \bar{\eta}\sqrt{\bar{z}(t)} \, dZ(t).
\end{align*}
\]

or similarly, we want to replace the dynamics of the forward rates given in Equation (4.12) by

\[
\begin{align*}
\bar{d}\bar{L}_n(t) &= \sqrt{\bar{z}(t)(\bar{b}_n \bar{L}_n(t) + (1 - \bar{b}_n)\bar{L}_n(0))||\bar{\lambda}_n||} \, dY^{n+1}(t), \\
\bar{d}\bar{z}(t) &= \theta(\bar{z}_0 - \bar{z}(t)) \, dt + \bar{\eta}\sqrt{\bar{z}(t)} \, dZ(t).
\end{align*}
\]

These can be done by three steps, as suggested in Andersen and Piterbarg [4]. The procedure is presented below for the swap rate dynamics. The procedure is similar for the forward rate dynamics.

**Parameter averaging method:**

**Step 1:** We approximate the stochastic variance process in Equation (4.13) by a process with constant parameters:

\[
\begin{align*}
\bar{d}\bar{z}(t) &= \theta(\bar{z}_0 - \bar{z}(t)) \, dt + \bar{\eta}\sqrt{\bar{z}(t)} \, dZ(t), \quad \bar{z}_0 = z_0 = 1. \tag{4.14}
\end{align*}
\]

Since the curvature of the implied volatility is related to the kurtosis of the distribution of \(S(T_j)\) and controlled by the variance of

\[
\int_0^{T_j} ||\lambda_S(t)||^2 z(t) \, dt,
\]

we choose the parameter \(\bar{\eta}\) such that the first and the second moments of

\[
\int_0^{T_j} ||\lambda_S(t)||^2 z(t) \, dt, \quad \text{and} \quad \int_0^{T_j} ||\lambda_S(t)||^2 \bar{z}(t) \, dt,
\]

are equal. The constant parameter \(\bar{\eta}\) such that the process given by Equation (4.14) satisfies these conditions, is given by

\[
\bar{\eta}^2 = \frac{\int_0^{T_j} e^{2\theta r} \rho_{T_j}(r) \, dr}{\int_0^{T_j} e^{2\theta r} \rho_{T_j}(r) \, dr},
\]

where

\[
\rho_{T_j}(r) = \int_r^{T_j} e^{-\theta s} ||\lambda_S(s)||^2 \int_s^{T_j} e^{-\theta t} \, dt \, ds.
\]

**Step 2:** In this step, we replace the time-dependent skew parameter \(b_S(t)\) by a constant
parameter \( \tilde{b}_S \), that is, the dynamics of the swap rate, given by Equation (4.13), are approximated by

\[
d\tilde{S}(t) = \sqrt{\tilde{z}(t)}(\tilde{b}_S \tilde{S}(t) + (1 - \tilde{b}_S)\tilde{S}(0))||\lambda_S(t)||dY^{i,k-j}(t), \quad \tilde{S}(0) = S(0),
\]

in a way that the parameter \( \bar{b}_S \) minimizes

\[
\mathbb{E}(S(T_j) - \tilde{S}(T_j))^2.
\]

From Andersen and Piterbarg [4], it is known that the constant parameter \( \bar{b}_S \) is given by

\[
\bar{b}_S = \int_0^{T_j} b_S(t)w(t)\,dt,
\]

and

\[
w(t) = \frac{\nu^2(t)||\lambda_S(t)||^2}{\int_0^{T_j} \nu^2(t)||\lambda_S(t)||^2\,dt},
\]

\[
\nu^2(t) = \bar{z}_0^2(t) \int_0^t ||\lambda_S(s)||^2\,ds + \bar{z}_0\eta^2 e^{-\theta t} \int_0^t ||\lambda_S(s)||^2 e^{\theta s} - e^{-\theta s}\frac{2\theta}{2\theta} \,ds.
\]

**Step 3:** The last step to make the model time-homogeneous is to approximate the dynamics of \( \tilde{S}(t) \) defined by

\[
d\tilde{S}(t) = \sqrt{\tilde{z}(t)}(\tilde{b}_S \tilde{S}(t) + (1 - \tilde{b}_S)\tilde{S}(0))||\lambda_S(t)||dY^{i,k-j}(t), \quad \tilde{S}(0) = S(0)
\]

with a process \( \tilde{S}(t) \) which dynamics are given by

\[
d\tilde{S}(t) = \sqrt{\tilde{z}(t)}(\tilde{b}_S \tilde{S}(t) + (1 - \tilde{b}_S)\tilde{S}(0))||\tilde{\lambda}_S||dY^{i,k-j}(t), \quad \tilde{S}(0) = S(0).
\]

This is done in a way such that

\[
\mathbb{E}^{i,k-j}(\tilde{S}(T_j) - \tilde{S}(0))^+ = \mathbb{E}^{i,k-j}(S(T_j) - S(0))^+.
\]

The effective volatility parameter \( \tilde{\lambda}_S \) which satisfies this condition is obtained by solving the equation:

\[
\Psi_{||\lambda_S||^2\tilde{z}(c,T_j)} = \Psi_{\tilde{z}(c\tilde{\lambda}_S^2,T_j)},
\]

where \( \Psi_{\lambda\tilde{z}}(u,t) \) is defined in Equation (4.3) and \( \Psi_{\tilde{z}}(u,t) \) is defined in Lemma 4.1.1.
4.5 The impact of the parameters on implied volatility curves

In this section, we investigate the impact of the parameters $b$, $\eta$ and $||\lambda_n(t)||$ of the LMM-DDSV model on the implied volatility curve. We consider caplets starting in 3 year and maturing in 3.5 year (3 year caplets for short) and the analytical approximating formula for caplets is available for the LMM-DDSV model. For swaptions the impact of the parameters on implied volatility curves are expected to be similar to caplet case. This is because the dynamics of swap rates are equivalent to the Heston model.

The basic parameter settings for the simulations are:

$$b = 0.2, \quad ||\lambda_n(t)|| = 0.15, \quad \theta = 0.4, \quad \eta = 0.8.$$ 

In each test case, we vary the value of only one parameter, while the other parameters are kept constant.

Firstly, the implied volatility curves of 3 year caplets with different skew parameters $b = 0.1, 0.2, 0.5, 1$ are depicted in Figure 4.1. As it can be seen in Figure 4.1, the smaller the skew parameter $b$ the steeper the implied volatility curves. The result can be interpreted as follows. As $b$ becomes close to 1, the dynamics of forward rates become similar to the log-normal dynamics, which imply a flat volatility curve. On the other hand, for the dynamics of forward rates with $b$ close to 0, the implied distributions become close to a normal distribution, which implies that we can control the skew/slope of the implied volatility curve by adjusting the value of $b$.

![Figure 4.1: Implied volatility curves for different skew parameters $b \in \{0.1, 0.2, 0.5, 1\}$](image)

Secondly, Figure 4.2 presents implied volatilities of 3 year caplets. The volatility-of-volatility parameter is varied between $\eta = 0.2, 0.8, 1.3$ and 2. The smaller the value of $\eta$ the flatter implied volatility curve is. The parameter $\eta$ controls the variance of the variance for the forward rate (see Section 4.4) and it results in controlling the kurtosis of the distribution of forward rates, which is related to the curvature of the implied volatility curve.

![Figure 4.2: Implied volatility curves for different volatility-of-volatility parameters $\eta$](image)
volatility curve. Note that we can also control the curvature of the implied volatility curve by the parameter $\theta$ in the variance process, but it works in the opposite way as the parameter $\eta$, that is, the larger the value for $\theta$ gives the flatter implied volatility curve. This can be explained roughly as follows. The long-term variance of the variance process is given by
\[
\frac{2\sigma^2 \eta^2}{2\theta}.
\]
and as $\theta$ increases the long-term variance of the variance process decreases.

Finally, we vary the volatility parameter $|\lambda(t)|$%. The variance of the dynamics at time $T$ is given by
\[
b \int_0^T z(t)|\lambda(t)|^2 dt.
\]
As we increase the value of $|\lambda(t)|$, the terminal volatility of the forward rate increases. Hence, the larger $|\lambda(t)|$ results in a larger level of the implied volatility curve. This can be seen in Figure 4.3 which shows implied volatilities for 3 year caplets with different volatility parameters.

4.6 Simulation of the LMM-DDSV model

In this section, we discuss efficient Monte Carlo methods for simulating the LMM-DDSV model. To get a certain acceptable accuracy, Monte Carlo methods generally need a large number of sample paths which mean that they are typically computational intensive. Hence, an efficient implementation is desirable. We will consider simulations methods for the variance process and the forward rates processes in the following subsections.
CHAPTER 4. PRICING IR DERIVATIVES UNDER THE LMM-DDSV MODEL

0.025 0.03 0.035 0.04 0.045 0.05 0.055
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Strike
Implied Volatility

\[ |\lambda_n(t)| \in \{0.1, 0.15, 0.2, 0.4\} \]

Figure 4.3: Implied volatility curves for volatility parameter \( |\lambda_n(t)| \in \{0.1, 0.15, 0.2, 0.4\} \)

4.6.1 Simulation of the variance process

In this subsection, methods for simulating the variance process given by Equation (3.12) will be discussed. The most straightforward and well-known way to simulate the variance process is by applying the Euler scheme which (equally) discretizes the time interval \([0, T]\) into \(N\) sub-intervals with time increment \(\Delta\). Then, at each time \(i\Delta, i \in [0, \cdots, N - 1]\), \(\hat{z}_{i+1}\) is obtained by

\[
\hat{z}_{i+1} = \hat{z}_i + \theta(z_0 - \hat{z}_i)\Delta + \eta\sqrt{\hat{z}_i}\sqrt{\Delta}W_i, \quad \hat{z}_0 = z_0,
\]

where \(W_i\) is a standard normal random variable. This scheme is simple to implement, but a drawback is that the value \(z(t)\) can become negative, since \(W\) follows the standard normal distribution. This is an undesired property because \(z(t)\) is a variance process and by definition of the variance it should remain positive. To overcome this problem, we introduce the log-Euler scheme.

1. **The log-Euler scheme**: Although the log-Euler scheme is intuitive and easy to implement, discretization introduces a bias to the simulation results and a large number of time steps may be needed to reduce the discretization bias. It is an alternative scheme of the Euler scheme by introducing an invertible transformation \(z(t) = f(y(t))\), with \(f : \mathbb{R} \to \mathbb{R}_+\), and then applying the Euler scheme to \(y\) at each step recovering \(z\) as \(f(y)\). More precisely, set \(z(t) := \exp(y(t))\), such that \(y(t) = \log(z(t))\). By Itô’s lemma, it follows that:

\[
dy(t) = \left(\frac{\theta(z_0 - z(t))}{z(t)} - \frac{1}{2} \frac{\eta^2z(t)}{z^2(t)}\right) dt + \eta\sqrt{z(t)}dZ(t).
\]

By applying the Euler scheme on the above SDE and using the definition of \(y(t)\), we obtain

\[
\hat{z}_{i+1} = \hat{z}_i \exp \left( \left(\frac{\theta(z_0 - \hat{z}_i)}{\hat{z}_i} - \frac{1}{2} \frac{\eta^2\hat{z}_i}{\hat{z}_i^2}\right) dt + \eta\sqrt{\hat{z}_i}\sqrt{\Delta}W_i\right),
\]

(4.15)
which gives a positive value.

2. **Sampling from a \( \chi^2 \) distribution:** This method proposed by Broadie and Kaya [11] samples from the exact distribution of \( z(t) \) given \( z(u) \) for \( u < t \). The transition law of \( z(t) \) is given by:

\[
z(t) = \frac{\eta^2 (1 - e^{-\theta(t-u)})}{4\theta} \chi^2_d \left( \frac{4\theta e^{-\theta(t-u)}}{\eta^2(1 - e^{-\theta(t-u)})} z(u) \right),
\]

where \( \chi^2_d(\lambda) \) denotes the non-central chi-squared random variable with \( d = 4z_0\theta/\eta^2 \) degrees of freedom, and non-centrality parameter

\[
\lambda = \frac{4\theta e^{-\theta(t-u)}}{\eta^2(1 - e^{-\theta(t-u)})} z(u).
\]

Sampling from a non-central chi-squared distribution can be computationally expensive (see [1]), especially when \( 0 < d < 1 \).

3. **The Quadratic-Exponential(QE) Scheme:** It is a moment-matching scheme proposed by Andersen [1], and the procedure is as follows:

(a) Compute the first and the second moments of \( z(t + \Delta) \) given \( z(t) \). Since \( z(t + \Delta) \) is distributed as \( e^{-\theta\Delta}/\lambda \) time a non-central chi-square distribution with \( d \) degrees of freedom and non-centrality parameter \( \lambda \) defined above, the first and the second moments of \( z(t + \Delta) \) given \( z(t) \) are given by

\[
m := E[z(t + \Delta)|z(t)] = z_0 + (z(t) - z_0)e^{-\theta\Delta},
\]

\[
s^2 := Var[z(t + \Delta)|z(t)] = \frac{z(t)\eta^2 e^{-\theta\Delta}}{\theta} \left( 1 - e^{-\theta\Delta} \right) + \frac{z_0\eta^2}{2\theta} \left( 1 - e^{-\theta\Delta} \right)^2.
\]

(b) Compute \( \psi = \frac{s^2}{m^2} \).

(c) Pick \( \psi_c \in [1,2] \).

(d) If \( \psi \leq \psi_c \), then

\[
z(t + \Delta) \approx d(c + Z)^2, \quad Z \sim N(0,1)
\]

where

\[
c^2 = \frac{2}{\psi} - 1 + \sqrt{\frac{2}{\psi}} \sqrt{\frac{2}{\psi}} - 1,
\]

\[
d = \frac{m}{1 + c^2}.
\]
Otherwise,
\[ z(t + \Delta) \approx \psi^{-1}(U; p; q) \]
where
\[ \psi^{-1}(u; p; q) = \begin{cases} 0, & 0 \leq u \leq p \\ \frac{1}{q} \log \left( \frac{1-p}{1-u} \right), & p < u \leq 1 \end{cases} \]
with \( p = \frac{\psi^{-1}}{\psi + 1} \), \( q = \frac{2}{m(\psi + 1)} \), and for \( U \sim \text{Uniform}(0, 1) \).

Figure 4.4: Distribution of the variance process when the Feller condition is satisfied

Figure 4.5: Distribution of the variance process when the Feller condition fails

Now, we will test the performance and show the test results here. Computationally, the QE method allows one to use a larger time-step, which is computationally more
efficient. It is known that the process \( z(t) \) remains positive if
\[
2\theta z_0 \leq \eta^2,
\]
which is called the Feller condition. If the Feller condition is violated, then the origin is accessible and strongly reflecting (see Proposition 2 in Andersen [1]). Figures 4.4a and 4.4b show the distribution of \( z(t + \Delta) \) given \( z(t) = 0.09 \) with different \( \Delta \) when the Feller condition is satisfied. In both cases, the distribution of \( z(t + \Delta) \) from the QE scheme fits the exact distribution better than the one from the log-Euler scheme.

Figures 4.5a and 4.5b also represent the distribution of \( z(t + \Delta) \) given \( z(t) = 0.09 \) with different \( \Delta \) when the Feller condition is violated. Same as in Figure 4.4, the QE scheme gives a better match to the exact distribution than the log-Euler scheme. In addition, we can see that the process \( z(t) \) has a strong affinity of the area around the origin. However, once \( z(t) \) becomes zero, log-Euler scheme cannot be used because \( z(t) \) is placed in the denominator, see Equation (4.15). In practice, the Feller condition can be violated under certain market conditions. Hence, the QE scheme is more suitable for our simulations.

4.6.2 Simulation of the forward rates processes

Once we simulate the variance process \( z(t) \), we can simulate the forward rates processes. For the simulation of forward rates, we can use the Euler method. More precisely, for \( n = q(t + \Delta), q(t + \Delta) + 1, \cdots, N - 1 \), the forward rates are simulated by
\[
L_n(t + \Delta) = L_n(t) + \sqrt{z(t)}[bL_n(t) + (1 - b)L_n(0)]\lambda_n(t) + \sqrt{z(t)}\mu_n(t)\Delta + \sqrt{\Delta}\begin{pmatrix} Z_1 \\ \vdots \\ Z_m \end{pmatrix},
\]
where \( q(t) \) is the index function. However, instead of simulating forward rates one by one, we can simulate the vector of forward rates at once by using the matrix form, defined in Equation (3.22). The step of the Euler method from \( t \) to \( t + \Delta \) is given by
\[
\begin{pmatrix}
L_1(t + \Delta) \\
\vdots \\
L_{N-1}(t + \Delta)
\end{pmatrix} =
\begin{pmatrix}
L_1(t) \\
\vdots \\
L_{N-1}(t)
\end{pmatrix} + \sqrt{z(t)}
\begin{pmatrix}
\varphi(L_1(t))||\lambda_1(t)|| & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \varphi(L_{N-1}(t))||\lambda_{N-1}(t)||
\end{pmatrix}
\begin{pmatrix}
v_{11} & \cdots & v_{m1} \\
\vdots & \ddots & \vdots \\
v_{1N-1} & \cdots & v_{mN-1}
\end{pmatrix}
\begin{pmatrix}
\sqrt{\tau_1} \\
\vdots \\
\sqrt{\tau_{N-1}}
\end{pmatrix}
\begin{pmatrix}
\sqrt{\Delta} \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
\vdots \\
Z_m
\end{pmatrix}.
\]
It turns out that the simulation of the vector of forward rates is faster than the simulation of forward rates one by one. The other remark is that for the simulation of the forward
rates process we can reduce the computational time by using the recursive relation of the drift term $\mu_n(t)$. To see this, note that

$$
\mu_n(t) = \sum_{j=q(t)}^{n} \frac{\tau_j \varphi(L_j(t)) \lambda_j(t)}{1 + \tau_j L_j(t)} \\
= \sum_{j=q(t)}^{n-1} \frac{\tau_j \varphi(L_j(t)) \lambda_j(t)}{1 + \tau_j L_j(t)} + \frac{\tau_n \varphi(L_n(t)) \lambda_n(t)}{1 + \tau_n L_n(t)} \\
= \mu_{n-1}(t) + \frac{\tau_n \varphi(L_n(t)) \lambda_n(t)}{1 + \tau_n L_n(t)}.
$$

There are other simulation methods for the BGM model proposed in the literature, such as the lagging predictor-corrector method. In this method the terminal values of the forward rates are first predicted, and then these values are used to correct the approximation in the drift coefficient, see Hunter, Jäckel, and Joshi [22] for more information.
Chapter 5

Numerical Results

The goal of this chapter is to discuss the numerical results of the LMM-DDSV model by pricing IR derivatives. In Chapter 4, we considered the pricing formulas of caplets and swaptions under the LMM-DDSV model. Therefore, the simulation will be mainly done for caplets and swaptions, so that we are able to compare the analytical prices to the Monte Carlo simulation results. In Section 5.1, we describe the general simulation settings. The results will be shown for two different cases in Section 5.2 and Section 5.3, respectively:

- The LMM-DDSV model with time-homogeneous parameters: see Equation (3.15)
- The LMM-DDSV model with time-dependent parameters: see Equation (4.12)

For the LMM-DDSV model with time-dependent parameters, we use the parameter averaging method as an approximation. In Section 5.4, we will also discuss a shortcoming of the stochastic volatility models which is the possibility of a so-called moment explosion, see [6]. This can be seen for interest rate derivatives for which prices are not only dependent on the first moment, but also on higher moments of the forward rate process. Typically, LMM is used for products that depend on higher moments. An in-arrears swap is such a derivative, therefore we investigate the moment explosion problem using this product.

5.1 General simulation settings

We consider a semi-annual tenor structure up to 8 years (i.e., \(T_N = 8\)). Since the last forward rate fixes at 7.5 years, there are 15 forward rates to be considered. The correlation between forward rates follows a parametric form given by Equation (3.18) with parameters: \(\rho_\infty = 0.3\), \(\alpha = 0.05\) and \(\beta = 0.05\). As mentioned in Section 3.2.1, we assume that there are three Brownian motions which drive the forward rates (instead of 15). We choose the following constant parameters in the time-homogeneous model:

\[
\begin{align*}
b &= 0.2, & L_n(0) &= 0.4, & \theta &= 0.4, & \eta &= 0.8, & ||\lambda_n(t)|| &= 0.15.
\end{align*}
\]
CHAPTER 5. NUMERICAL RESULTS

To see the performance of the parameter averaging method in the time-dependent model, we consider piecewise constant parameters (see Figure 5.1):

- \( \eta(t) = \sum_{j=1}^{15} 1_{[T_{j-1}, T_j)}(t)(2.2 - 0.5j) \)
- \( b_n(t) = \sum_{j=1}^{15} 1_{[T_{j-1}, T_j)}(t)(0.6 - j/50 + 0.01(n-1)) \)

and for \( ||\lambda_n(t)|| \) we use the parametric form in Equation (3.19) with \( a = 0.1, b = 0, c = 0.3 \) and \( d = 0.1 \). As an initial condition we take \( L_n(0) = 0.04 \). These parameters were chosen to keep the level of implied volatility around realistic levels 15% \( \sim \) 20%.

![Graph](image_url)

Figure 5.1: (Piece-wise constant) time-dependent parameters

For the Monte Carlo simulations, we use \( 10^6 \) sample paths and time increments \( \Delta = 0.01 \). The Euler method is used for the simulation of forward rates and the Quadratic Exponential is used to simulate the stochastic variance process (see Section 4.6). The strikes we consider are equidistant around the ATM level, i.e., we take \( ATM \pm d \), for \( d \in \{0, 0.005, 0.01, 0.015\} \).

5.2 Time-homogeneous LMM-DDSV model

In this section, we consider the LMM-DDSV model with time-homogeneous parameters. Under this model, we price caplets with different maturities and swaptions with different maturities and different tenors. We denote a swaption with a \( x \)-year option maturity that gives the right to enter into a \( z \)-year swap as a \( xy \times zy \) swaption.

5.2.1 Caplets

First, we consider the caplet starting in 1-year and maturing in 1.5-year (the 1-year caplet for short). Figure 5.2a shows the prices and implied (caplet) volatilities.
CHAPTER 5. NUMERICAL RESULTS

(a) 1-year caplet prices and implied volatilities

(b) Black price as function of implied volatility

Figure 5.2: Caplet prices and its implied volatility

For the 1-year caplet, the top right graph in Figure 5.2a shows the difference (in basis-points or bps) between the analytical price and the MC price. The bottom right graph in Figure 5.2a shows the difference in terms of implied volatilities. Although we see a small difference (0.04 bps) for the ITM caplets, we see a large difference of 43.75 bps in terms of implied volatility.

(a) 3-year caplet price

(b) Black price as function of implied volatility

Figure 5.3: Caplet prices and its implied volatility

For comparison purposes, we consider the caplet starting in 3-year and maturing in 3.5-year (the 3-year caplet for short). Figure 5.3a shows also small difference in terms of price similarly to the 1-year caplet case. Since there is no approximation involved to derive the analytical formula for caplets, this error comes from the Monte Carlo sim-
ulation (and/or from the numerical integration errors). However, we observe that for the 3-year caplet the error of the ITM implied volatility is much smaller than the one for the 1-year caplet. This can be explained by the fact that a caplet with a short maturity is less sensitive to changes in volatility (i.e., the vega sensitivity $\frac{\partial V}{\partial \sigma}$ is smaller). This implies that a small difference in price results in a big difference in implied volatility.

To understand these results better, we show the ITM caplet prices (strike $K = 0.025$), as function of the implied volatility, in Figures 5.2b and 5.3b. The ATM implied volatility is shown in the Figures with a vertical dashed line. For the OTM region of the 1-year caplet, we can observe in Figure 5.2b that the Black price is “nearly flat” as a function of the implied volatility, i.e., $\frac{\partial V}{\partial \sigma} \approx 0$. On the other hand, for the OTM of the 3-year caplet we can see in Figure 5.3b that the Black price is not “fully flat” as a function of the implied volatility, i.e., $\frac{\partial V}{\partial \sigma} \neq 0$. Hence, the large errors in terms of implied volatilities for the 1-year caplet can be explained by the small vega sensitivity.

### 5.2.2 Swaptions

For caplets there exists an analytical formula, however we have to approximate the swap rate dynamics in order to derive closed-form pricing formulas for swaptions. Hence, the difference between the analytical prices and Monte Carlo prices comes not only from the Monte Carlo simulation error, but also from the assumptions made for the approximation method.

In Table 5.1, we present the performance of the approximation of swaptions for different maturities and tenors. “MC” refers to Monte Carlo simulation results and “Approx” refers to the results obtained by Equation (4.11). Overall, we can see that the approximation performs reasonably well (maximum difference in price is $-0.65$ bps and in implied volatility it is $-147.19$ bps for the smallest strikes). The large difference in implied volatility for the $1y \times 7y$ swaption can be explained similarly as for the 1-year caplet (see Figure 5.4), and see section 5.2.1 (small vega for options with short maturity).

![Figure 5.4: Swaption (Black) price as function of implied volatility](image-url)
# Chapter 5. Numerical Results

Table 5.1: The swaption prices and their implied volatilities obtained by the Monte Carlo simulation and pricing formula

<table>
<thead>
<tr>
<th>Swap.</th>
<th>Strike</th>
<th>Price(bps)</th>
<th>Implied vol(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MC(Error)</td>
<td>Approx</td>
</tr>
<tr>
<td>1y × 1y</td>
<td>0.025</td>
<td>140.15(0.05)</td>
<td>140.12</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>94.47(0.05)</td>
<td>94.46</td>
</tr>
<tr>
<td></td>
<td>0.035</td>
<td>52.70(0.05)</td>
<td>52.72</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>21.83(0.03)</td>
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60
CHAPTER 5. NUMERICAL RESULTS

Table 5.1: The swaption prices and their implied volatilities obtained by the Monte Carlo simulation and pricing formula

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<th>Implied vol(%)</th>
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5.3 Time-dependent LMM-DDSV model and parameter averaging method

In this section, we consider the LMM-DDSV model with time-dependent parameters. The time-dependent model is approximated by a time-homogeneous model with constant and “averaged” parameters. The parameter averaging method is used to determine the averaged parameters, and for the constant parameter model we can apply the analytical pricing formulas.

5.3.1 Caplets

Figure 5.5 presents the performance of the parameter averaging method for a caplet starting in 7-year and maturing in 7.5-year. The maximum difference in implied volatility is $-22.83$ bps. We observe the smallest difference, between the MC implied volatility and the analytical volatility, around the ATM level. This can be explained by the fact that the parameter averaging method is designed to give the most accurate results for ATM-caplets.
CHAPTER 5. NUMERICAL RESULTS

5.3.2 Swaptions

Table 5.2 presents swaption prices with different maturities and tenors, obtained from Monte Carlo simulations with time-dependent parameters and the approximation formula given by Theorem 4.3.2. The constant parameters for the approximation formula are calculated by using the parameter averaging method. For the 1-year maturity ITM-swaptions with a 7-year tenor, we see the largest difference as we have observed in the time-homogeneous case.

Table 5.2: The swaption prices and their implied volatilities obtained by the Monte Carlo simulation and pricing formula

<table>
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<th>Implied vol(%)</th>
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<td>Approx</td>
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## CHAPTER 5. NUMERICAL RESULTS

Table 5.2: The swaption prices and their implied volatilities obtained by the Monte Carlo simulation and pricing formula

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<th>Implied vol(%)</th>
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CHAPTER 5. NUMERICAL RESULTS

5.4 Moment explosion for stochastic volatility models

As discussed in Section 4.2, we can map the LMM-DDSV model on the Heston model. However, it is well-known that the Heston model suffers from the possibility of a so-called moment explosion [6]. More precisely, for a process $X(t)$ following the Heston dynamics, the moment $E(X^w(T))$, $w \geq 2$ can become infinite within a finite time $T$. Since the LMM-DDSV model can be mapped onto the Heston model, the LMM-DDSV model can suffer from the moment explosion as well.

More formally, we consider the process $L_n(t)$, given by Equation (4.6), with $||\lambda_n(t)|| = \lambda_n$. Fix $k = \frac{(b\lambda_n)^2}{2}(w - 1)$, such that $k > 0$ and define

$$c = \frac{2k}{\eta^2} > 0, \quad a = -\frac{2\theta}{\eta^2}, \quad D = a^2 - 4c,$$

then $E(L_n^w(T))$ will be finite for $T < T^*$ and infinite for $T \geq T^*$, where the critical time $T^*$ is given by:

- For $D \leq 0, a < 0$: $T^* = \infty$
- For $D \geq 0, a > 0$: $T^* = \frac{\gamma^{-1}\eta^{-2}\log\left(\frac{a/2+\gamma}{a/2-\gamma}\right)}{2\sqrt{D}}$, with $\gamma := \frac{1}{2}\sqrt{D}$
- For $D < 0$: $T^* = 2\beta^{-1}\eta^{-2}(\pi_1_{a<0} + \arctan(2\beta/a))$, with $\beta := \frac{1}{2}\sqrt{-D}$

For more information we refer to [6]. Given the parameters, we can calculate the critical time $T^*$ by using the above equations.

Recall from Section 2.4.3, the pricing formula for in-arrears swaps with reset dates $\{T_0, T_1, \ldots, T_{N-1}\}$:

$$V(0) = \sum_{n=0}^{N-1} \tau_n P(0, T_n) \left( \frac{L_n(0) + \tau_n E^{n+1}(L_n^2(T_n))}{1 + \tau_n L_n(0)} - K \right), \quad (5.1)$$

which implies that the price of in-arrears swaps is also related to the second moment of the forward rates. The moment explosion implies that $E^{n+1}(L_n^2(T_n))$ can become infinite if $T_n > T^*$. In contrary, the pricing formula for swaps depends only on the first moment, so that swaps do not suffer from moment explosion. To assess the moment explosion, we price payer swaps and in-arrears payer swaps. Therefore, we simulate forward rates under the LMM-DDSV model with parameters defined in Table 5.3 (with semi-annually tenor structure up to 5 years, i.e., $T_N = 5$). An in-arrears swap with strike $K$ resents on dates $\{T_1, T_2, \ldots, T_8, T_9\}$.

For the parameters of case A in Table 5.3, the critical time is $T^* = 39.62$ which is larger than $T_N$, and for the parameters of case B, the critical time is $T^* = 3.65$. Therefore, we expect that the in-arrears swap price will not diverge and remains stable for case A. For case B, the second moment may become infinite so that the Monte Carlo price
can become distorted and unstable. To determine whether the price from Monte Carlo simulations are reasonable stable, we also price swaps and in-arrears swaps with the analytical formulas. The analytical swap price is obtained by using Equation (2.10) and the analytical in-arrears swap price is obtained by using Equation (2.17) with the implied volatility $\sigma_n$. The results are presented in Table 5.4.

<table>
<thead>
<tr>
<th></th>
<th>case A</th>
<th>case B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda_n$</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>$K$</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>$T^*$</td>
<td>39.62</td>
<td>3.65</td>
</tr>
</tbody>
</table>

Table 5.3: Simulation parameters and the critical time $T^*$

From Table 5.4 we observe that the Monte Carlo prices and the analytical prices differ the most from each other for the in-arrears swap under settings of case B. For the other cases, the MC and analytical prices are generally close to each other. The difference between the MC prices and the analytical prices for both swaps and in-arrears swaps are small for case A. The difference between swap price and in-arrears swap price is also small, as this difference comes from the convexity adjustment. For case B, we observe larger differences between the analytical in-arrears swap price and the MC price. Thus, this example shows that the model suffers from moment explosion, which is clearly visible for an in-arrears swap.

To understand this better, we calculate the Net Present Value (NPV) for the sample paths with diverging forward rates, as shown in Table 5.5. The NPV is given by the sum of discounted cashflows:

$$NPV = \sum_{n=1}^{9} \frac{CashFlow_n}{Numeraire_n}.$$
CHAPTER 5. NUMERICAL RESULTS

More precisely, for swaps the cash-flow at $T_{n+1}$ is dependent on the forward rate fixing at $T_n$, and the (numeraire) bank account at $T_{n+1}$ is dependent on the forward rate fixing at $T_n$. Hence, we have for swaps that the discounted cash-flows equals:

$$\text{Discounted } CF = \frac{CF_{T_{n+1}}(F(T_n))}{B_{T_{n+1}}(F(T_n))}. \quad (5.2)$$

On the other hand, for in-arrears swaps the cash-flow at $T_{n+1}$ is dependent on the forward rate fixing at $T_{n+1}$, and the bank account at $T_{n+1}$ is dependent on the forward rate fixing at $T_n$, so that for in-arrears swap, the discounted cash-flows equals:

$$\text{Discounted } CF = \frac{CF_{T_{n+1}}(F(T_{n+1}))}{B_{T_{n+1}}(F(T_n))}. \quad (5.3)$$

The major difference between Equation (5.2) and Equation (5.3) is that when a forward rate explodes at a certain time, then for the calculation of the discounted cash-flow for swaps, the effect offsets each other in the numerator and denominator of Equation (5.2). For in-arrears swaps, if the forward rate explodes at time $T_{n+1}$, then the effect is not offset by the denominator of Equation (5.3). As a consequence, the NPV of a swap is smaller than the NPV of an in-arrears swap.

<table>
<thead>
<tr>
<th>n</th>
<th>$L_n(T_n)$</th>
<th>$B(T_n)$</th>
<th>Cash Flow</th>
<th>Discounted</th>
<th>Cash Flow</th>
<th>Discounted</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.040</td>
<td>1.000</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>0.043</td>
<td>1.020</td>
<td>0.0100</td>
<td>0.00980</td>
<td>0.0114</td>
<td>0.0112</td>
</tr>
<tr>
<td>2</td>
<td>0.088</td>
<td>1.042</td>
<td>0.0114</td>
<td>0.01095</td>
<td>0.0338</td>
<td>0.0325</td>
</tr>
<tr>
<td>3</td>
<td>0.72</td>
<td>1.088</td>
<td>0.0338</td>
<td>0.0311</td>
<td>0.35</td>
<td>0.321</td>
</tr>
<tr>
<td>4</td>
<td>9.64</td>
<td>1.48</td>
<td>0.35</td>
<td>0.236</td>
<td>4.809</td>
<td>3.253</td>
</tr>
<tr>
<td>5</td>
<td>1267.21</td>
<td>8.60</td>
<td>4.809</td>
<td>0.559</td>
<td>633.6</td>
<td>73.64</td>
</tr>
<tr>
<td>6</td>
<td>72165.5</td>
<td>5459.90</td>
<td>633.6</td>
<td>0.116</td>
<td>36083.74</td>
<td>6.609</td>
</tr>
<tr>
<td>7</td>
<td>8100358</td>
<td>1.96E+08</td>
<td>36083.74</td>
<td>0.0002</td>
<td>4050179</td>
<td>0.0206</td>
</tr>
<tr>
<td>8</td>
<td>4.43E+08</td>
<td>7.98E+14</td>
<td>4050179</td>
<td>5.08E-09</td>
<td>2.22E+08</td>
<td>2.78E-07</td>
</tr>
<tr>
<td>9</td>
<td>9.85E+08</td>
<td>1.77E+23</td>
<td>2.22E+08</td>
<td>1.25E-15</td>
<td>4.93E+08</td>
<td>2.78E-15</td>
</tr>
<tr>
<td>10</td>
<td>1.3E+09</td>
<td>8.71E+31</td>
<td>4.93E+08</td>
<td>5.65E-24</td>
<td>6.49E+08</td>
<td>7.45E-24</td>
</tr>
</tbody>
</table>

Table 5.5: Net Present Value (NPV) for swaps and in-arrears swaps

The above table illustrates that the stochastic volatility model suffers from a moment explosion, which is clearly visible with an in-arrears swap. Note that the moment explosion is not clearly visible for an ordinary swap.
Chapter 6

Conclusion

In this thesis we investigated the Libor Market Model (LMM) with Displaced Diffusion and Stochastic Volatility (LMM-DDSV). The reason to investigate the LMM-DDSV model is to better fit the market implied volatility curve. A drawback of the standard BGM model [9] is that the BGM model implies a flat volatility curve, which is not consistent with the market implied volatility curve, see Figures 3.1a and 3.1b.

In Sections 4.2 and 4.3 we have derived pricing formulas for caplets and swaptions under the LMM-DDSV model, which is closely related to the Heston model [17] for Stochastic Volatility (see Section 4.1). We have seen that there are analytical pricing formulas available for European-style of options for the model with constant (time-homogeneous) parameters. We implemented the LMM-DDSV model and extended the model with time-dependent parameters in Section 4.4. We approximated the time-dependent model by a time-homogeneous model and by using a parameter averaging method.

In Chapter 5 we have seen that the parameter averaging theorems yield an accurate approximation of the caplet prices, see Figure 5.5. In addition, we have seen that the approximations made to derive closed-form pricing formulas for swaptions are accurate, see Table 5.2. The numerical and simulation results also show that the skewness and curvature of the implied volatility curve can be controlled by the model parameters (see Figures 5.2, 5.3 and 5.5, and Tables 5.1 and 5.2). The numerical approximation results in terms of implied volatilities can become relatively large in some parameter cases, if the prices are nearly insensitive to changes in volatility (i.e., when the vega sensitivity $\frac{\partial V}{\partial \sigma}$ is small). In these cases, a small approximation error in terms of prices may result into a relative larger approximation error in terms of implied volatilities.

A drawback of stochastic volatility models is that they can suffer from the so-called moment explosion, meaning that the higher moments of a stochastic process can become infinite within a finite time. In Section 5.4 we have shown an example where a moment explosion can occur. It is possible to determine whether the moment explosion can occur or not for a given parameter setting, as the characterization of the critical
time is given by the results in [6]. This phenomenon is shown by using in-arrears swaps, since the pricing formula for an in-arrears swap depends on the second moment of the forward rates. Comparison between swaps and in-arrear swaps reveals that the Net Present Value (NPV) for an in-arrears swap can become much larger than the NPV of a swap.
CHAPTER 6. CONCLUSION

**Future research direction**

We recommend the following directions for further research.

- In this thesis we have investigated the LMM-DDSV model for pricing of interest rate derivatives. A logical next step is to investigate an efficient calibration of the Libor Market Model with stochastic volatility to actual market data. An approach could be based on a weighted Monte Carlo technique (see Chen, Grzelak and Oosterlee [13]). Under this approach, the Monte Carlo weights are not uniformly chosen, but are chosen to replicate the market instruments. However, there is currently no consensus how the LMM-DDSV model has to be calibrated in a stable and efficient way. The choice of the constant model parameters is non-trivial for the LMM-DDSV model. We propose to do future research on the optimal choice of the model parameters in order to improve the robustness of the calibration.

- We have seen that the LMM-DDSV model is not robust, as there is a potential issue of moment explosion under this model. It may be interesting to do more research on solving this problem or to use models which avoid this explosion issue.

- In this thesis we have only considered the DDSV model, which is a flexible model for generating skewness and curvature in the implied volatilities. There are other potential models which are able to generate skewness and curvature in the implied volatilities, while still remain practically manageable. For instance, other forms of Stochastic Volatility models with a different local volatility formulation may be worthwhile to investigate, instead of the displaced diffusion formulation. Another way is to use LMM with different correlation parameterisations. The main calibration instrument are usually the swaptions, but these instruments do not depend significantly on the forward rate correlations [24]. In case we want to price exotic products, then considering other correlation structures can become more important as the prices of exotic products typically depend on the full dynamics of the model.
Appendix A

Theorems and Proofs

A.1 Theorems

In this section, we present some well-known and fundamental theorems which are used in this thesis. We omit the proofs of these theorems, these can be found in Andersen and Piterbarg [4] and in Karatzas and Shreve [25].

**Theorem A.1.1. (Radon-Nikodym Theorem)** Let $P$ and $\hat{P}$ be equivalent probability measures on a common measure space $(\Omega, F)$. There exists a unique (a.s.) non-negative random variable $R$ with $\mathbb{E}^P(R) = 1$, such that

$$\hat{P}(A) = \mathbb{E}^P(R1_A), \quad \text{for all } A \in F.$$ 

The random variable $R$ is called the Radon-Nikodym derivative of $\hat{P}$ w.r.t. $P$ and is denoted by $d\hat{P}/dP$.

**Theorem A.1.2. (Change of Numeraire)** Consider two numeraires $N(t)$ and $M(t)$, inducing equivalent martingale measures $Q^N$ and $Q^M$, respectively. If the market is complete, then the density process of Radon-Nikodym derivatives relating the two measures is uniquely given by

$$\varsigma(t) = \mathbb{E}_t^{Q^N} \left( \frac{dQ^M}{dQ^N} \right) = \frac{M(t)/M(0)}{N(t)/N(0)}.$$ 

For the following Theorem A.1.3 (Girsanov’s Theorem), we consider two measures $P$ and $P(\theta)$ related by a density $\varsigma^\theta(t) = \mathbb{E}_t^P(dP(\theta)/dP)$, where $\varsigma^\theta(t)$ is an exponential martingale given by the Itô process

$$d\varsigma^\theta(t)/\varsigma^\theta(t) = -\theta(t)^\top dW(t),$$
APPENDIX A. THEOREMS AND PROOFS

and where $W(t)$ is a $d$-dimensional $P$-Brownian motion and $\theta(t)$ is a $d$-dimensional process. By an application of Itô’s lemma, we can write

$$\varsigma^\theta(t) = \exp \left( - \int_0^t \theta(s) \, dW(s) - \frac{1}{2} \int_0^t \theta(s) \, \theta(s) \, ds \right) = \mathcal{E} \left( - \int_0^t \theta(s) \, dW(s) \right),$$

where $\mathcal{E}(\cdot)$ is the Doleans exponential. If $\theta(t)$ satisfies the Novikov condition, i.e.,

$$\mathbb{E}^P \left[ \exp \left( \frac{1}{2} \int_0^t \theta(s) \, \theta(s) \, ds \right) \right] < \infty, \text{ for all } t \in [0, \infty),$$

or, if the process is defined only on finite time interval $[0, T]$,

$$\mathbb{E}^P \left[ \exp \left( \frac{1}{2} \int_0^T \theta(s) \, \theta(s) \, ds \right) \right] < \infty$$

then it defines a proper martingale on $[0, \infty)$, or on $[0, T]$.

**Theorem A.1.3.** (Girsanov’s Theorem) Assume that $\varsigma^\theta(t)$ defined above is a martingale on $[0, T]$. Then for all $t \in [0, T]$,

$$W^\theta(t) = W(t) + \int_0^t \theta(s) \, ds,$$

defines a Brownian motion under the measure $P(\theta)$ on $[0, T]$.

A.2 Proof of Theorem 4.1.2

**Proof.** We include the proof of Theorem 4.1.2 given by Andersen and Piterbarg [4] for the reader. First note that:

$$\mathbb{E}^P (X(T) - K)^+ = \mathbb{E}^P \left( e^{\log X(T)} - e^{\log K} \right)^+ = \mathbb{E}^P \left( e^{\log X(T)} - e^{\log K} \min(e^{\log X(T)-\log K}, 1) \right) = \mathbb{E}^P X(T) - K\mathbb{E}^P \min(e^{\log X(T)-\log K}, 1).$$

Following Carr and Madan [12], we have

$$\mathbb{E}^P (\min(e^{\log X(T)-\log K}, 1)) = e^{-\alpha \log K} \int_{-\infty}^{\infty} \left[ \min(e^{-\log K - x}, 1) e^{\alpha \log K - x} \right] [e^{\alpha x} p(x)] \, dx,$$
APPENDIX A. THEOREMS AND PROOFS

where $\alpha$ is a dampening constant which lies in $(0, 1)$ and $p(t)$ is the density of $\log X(T)$.

For notational convenience, we define

$$f_1(x) = \min(e^{-x}, 1)e^{\alpha x},$$
$$f_2(x) = e^{\alpha x}p(x).$$

If we let $\mathcal{F}$ be the Fourier transform and $\mathcal{F}^{-1}$ its inverse, then we have that

$$\mathbb{E}^p(\min(e^{\log X(T) - \log K}, 1)) = e^{-\alpha \log K}(f_1 * f_2)(\log K)$$
$$= e^{-\alpha \log K}(\mathcal{F}^{-1}((\mathcal{F}(f_1 * f_2))(\log K))$$
$$= e^{-\alpha \log K}(\mathcal{F}^{-1}(\mathcal{F}(f_1)\mathcal{F}(f_2))(\log K),$$

where $*$ is the convolution operator. Next, we calculate $\mathcal{F}f_1$ and $\mathcal{F}f_2$:

$$\mathcal{F}f_1(w) = \int_{-\infty}^{\infty} e^{iwx} \min(e^{-x}, 1)e^{\alpha x} \, dx = \int_{-\infty}^{0} e^{(\alpha+iw)x} \, dx + \int_{0}^{\infty} e^{(\alpha+iw-1)x} \, dx$$
$$= \frac{1}{\alpha + iw} - \frac{1}{\alpha - 1 + iw} = \frac{1}{(\alpha + iw)(1 - \alpha - iw)},$$

and

$$\mathcal{F}f_2(w) = \int_{-\infty}^{\infty} e^{iwx} e^{\alpha x}p(x) \, dx = \int_{-\infty}^{\infty} e^{(\alpha+iw)x}p(x) \, dx = \mathbb{E}(e^{(\alpha+iw)\log X(T)})$$
$$= \Psi_X(\alpha + iw, T).$$

Hence, we have that

$$\mathbb{E}^p(\min(e^{\log X(T) - \log K}, 1)) = e^{-\alpha \log K}\mathcal{F}^{-1}(\mathcal{F}f_1(w)\mathcal{F}f_2(w))(\log K)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+iw)\log K} \frac{\Psi_X(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw \quad (A.1)$$

Combining the fact that $X(T)$ is martingale under measure $\mathbb{P}$ and the result from Equation (A.1), we have

$$\mathbb{E}^p(X(T) - K)^+ = X(0) - \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+iw)\log K} \frac{\Psi_X(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw. \quad (A.2)$$

We proceed as follows to derive a type of control variate method. Consider the case $\eta(t) = 0$. Then, the dynamics of $z(t)$ are given by

$$dz(t) = \theta(z_0 - z(t)) \, dt$$

which implies that $z(t) = z(0)$ for all $t$ and the dynamics of $X(t)$ simplify to

$$dX(t) = \sqrt{z(0)}\lambda(t)X(t) \, dW(t)$$
APPENDIX A. THEOREMS AND PROOFS

It yields that $X(t)$ is log-normally distributed. Hence, we have that

$$E_P(X(T) - K)^+ = X(0)N(d_1) - KN(d_2) := \text{Black}(0, X, T, K, \sigma)$$  \hspace{1cm} (A.3)

where

$$\sigma^2 = \frac{\sigma_0}{\sqrt{T}} \int_0^T \lambda^2(t) \, dt$$

$$d_1 = \frac{\log(X(0)/K) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}.$$  \hspace{1cm} (A.5)

We can also use Equation (A.2) to solve the expectation which is given by

$$E_P(X(T) - K)^+ = X(0) - \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K} \frac{\Psi_X^0(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw,$$  \hspace{1cm} (A.4)

where

$$\Psi_X^0(u, T) = E_P(e^{iu \log X(T)}) = E_P(e^{u \sqrt{z_0} \int_0^T \lambda(t) \, dW(t) - \frac{1}{2} z_0 \int_0^T \lambda^2(t) \, dt}).$$

From Equation (A.3) and (A.4), we get

$$\text{Black}(0, X, T, K, \sigma) = X(0) - \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K} \frac{\Psi_X^0(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw,$$  \hspace{1cm} (A.5)

which implies that

$$\Delta := \text{Black}(0, X, T, K, \sigma) - X(0) + \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K} \frac{\Psi_X^0(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw$$

equals to 0. Adding Equation (A.5) to Equation (A.2) yields the following

$$E^P(X(T) - K)^+ = E^P(X(T) - K)^+ + \Delta$$

$$= X(0) - \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K} \Psi_X(\alpha + iw, T) \, dw$$

$$+ \text{Black}(0, X, T, K, \sigma) - X(0) + \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K} \frac{\Psi_X^0(\alpha + iw, T)}{(\alpha + iw)(1 - \alpha - iw)} \, dw$$

$$= \text{Black}(0, X, T, K, \sigma) - \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-(-\alpha + iw) \log K q(\alpha + iw)} \, dw,$$
APPENDIX A. THEOREMS AND PROOFS

where \( q(u) := \Psi_X(u, T) - \Psi_X^H(u, T) \).

If we define
\[
H(w) = e^{-(\alpha + iw) \log K} \frac{q(\alpha + iw)}{(\alpha + iw)(1 - \alpha - iw)},
\]
then we can easily check that \( H(-w) = \overline{H(w)} \) from the properties of complex conjugates. Such a function satisfies
\[
\int_{-\infty}^{\infty} H(w) \, dw = 2 \int_0^{\infty} \text{Re}(H(w)) \, dw,
\]
and this implies that
\[
\mathbb{E}^P(X(T) - K)^+ = \text{Black}(0, X, T, K, \sigma) - \frac{K}{\pi} \int_0^{\infty} \text{Re} \left( e^{-(\alpha + iw) \log K} \frac{q(\alpha + iw)}{(\alpha + iw)(1 - \alpha - iw)} \right) \, dw.
\]

\( \square \)

A.3 Proof of Proposition 4.3.1

Proof. Since the swap rate \( S(t) \) is a martingale under the swap measure \( \mathbb{Q}^{j,k-j} \), the drift term of the process for \( S(t) \) must be zero under this measure. From the definition, \( S(t) \) is a function of \( L_j(t), L_{j+1}(t), \ldots, L_{k-1}(t) \) and an application of Itô’s lemma shows that
\[
dS(t) = \sum_{n=j}^{k-1} \sqrt{z(t)(bL_n(t) + (1-b)L_n(0))} \frac{\partial S(t)}{\partial L_n(t)} \lambda_n(t) \top dW^{j,k-j}(t)
\]
\[
= \sqrt{z(t)(bS(t) + (1-b)S(0))} \sum_{n=j}^{k-1} \frac{bL_n(t) + (1-b)L_n(0)}{bS(t) + (1-b)S(0)} \frac{\partial S(t)}{\partial L_n(t)} \lambda_n(t) \top dW^{j,k-j}(t)
\]
\[
= \sqrt{z(t)(bS(t) + (1-b)S(0))} \sum_{n=j}^{k-1} w_n(t) \lambda_n(t) \top dW^{j,k-j}(t),
\]
where
\[
w_n(t) = \frac{bL_n(t) + (1-b)L_n(0)}{bS(t) + (1-b)S(0)} \frac{\partial S(t)}{\partial L_n(t)}.
\]

Now we will derive an expression for \( \frac{\partial S(t)}{\partial L_n(t)} \). Note that \( S(t) \) can be written by
\[
S(t) = \sum_{i=j}^{k-1} \tau_i P(t, T_{i+1}) L_i(t) \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) = \sum_{i=j}^{k-1} \frac{\tau_i P(t, T_{i+1})}{\sum_{r=j}^{k-1} \tau_r P(t, T_{r+1})} L_n(t) = \sum_{i=j}^{k-1} \alpha_i(t) L_i(t),
\]
where
\[
\alpha_i(t) = \frac{\tau_i P(t, T_{i+1})}{\sum_{r=j}^{k-1} \tau_r P(t, T_{r+1})}.
\]

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Then, we can easily see that:

\[ \frac{\partial S(t)}{\partial L_n(t)} = \alpha_n(t) + \sum_{i=j}^{k-1} \frac{\partial \alpha_i(t)}{\partial L_n(t)} L_i(t). \]  

(A.6)

Note that we have

\[ P(t, T_{i+1}) = \frac{P(t, T_i)}{1 + \tau_i L_i(t)} \]

\[ = \frac{P(t, T_{i-1})}{(1 + \tau_i L_i(t))(1 + \tau_{i-1} L_{i-1}(t))} \]

\[ = P(t, T_j) \prod_{l=j}^{i-1} \frac{1 + \tau_l L_l(t)}{1 + \tau_l L_l(t)}. \]

Then, the partial derivative w.r.t. \( L_n(t) \) is given by

\[ \frac{\partial P(t, T_{i+1})}{\partial L_n(t)} = \begin{cases} \frac{P(t, T_j)}{1 + \tau_n L_n(t) \prod_{l=j}^{i-1} (1 + \tau_l L_l(t))}, & i \geq n \\ 0, & \text{otherwise} \end{cases} \]

\[ = \frac{-\tau_n}{1 + \tau_n L_n(t) \prod_{l=j}^{i-1} (1 + \tau_l L_l(t))} \cdot H(i - n) \]

\[ = \frac{-\tau_n}{1 + \tau_n L_n(t)} P(t, T_{i+1}) H(i - n), \]

where \( H(\cdot) \) is the Heaviside function: \( H(x) = 1 \) for 0 ≤ \( x \) and \( H(x) = 0 \) for \( x < 0 \). By using the above result, for \( n > j \), we can show that:

\[ \frac{\partial \alpha_i(t)}{\partial L_n(t)} = \frac{\partial}{\partial L_n(t)} \left( \frac{\tau_i P(t, T_{i+1})}{\sum_{r=j}^{k-1} \tau_r P(t, T_{r+1})} \right) \]

\[ = \tau_i \left\{ \frac{\partial P(t, T_{i+1})}{\partial L_n(t)} \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) - P(t, T_{i+1}) \frac{\partial}{\partial L_n(t)} \left( \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) \right) \right\} \]

\[ = \tau_i \left( \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) \right)^2 \left\{ \frac{\partial P(t, T_{i+1})}{\partial L_n(t)} \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) - P(t, T_{i+1}) \sum_{r=j}^{k-1} \tau_r \frac{\partial P(t, T_{r+1})}{\partial L_n(t)} \right\} \]

\[ = \frac{\tau_i}{1 + \tau_n L_n(t)} P(t, T_{i+1}) H(i - n) \]

\[ - \frac{\tau_i}{1 + \tau_n L_n(t)} \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) H(i - n) \]

\[ - \frac{\tau_i}{1 + \tau_n L_n(t)} \left( \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) \right) \sum_{r=j}^{k-1} \tau_r \frac{\partial P(t, T_{r+1})}{\partial L_n(t)} H(r - n) \]

\[ - \frac{\tau_i}{1 + \tau_n L_n(t)} \left( \sum_{r=j}^{k-1} \tau_r P(t, T_{r+1}) \right) \sum_{r=j}^{k-1} \tau_r \frac{\partial P(t, T_{r+1})}{\partial L_n(t)} H(r - n) \]
which simplifies to

\[
\frac{\partial \alpha_i(t)}{\partial L_n(t)} = \frac{\tau_n}{1 + \tau_n L_n(t)} \alpha_i(t) \left( -H(i - n) + 1 - \sum_{r=j}^{n-1} \alpha_r(t) \right). \tag{A.7}
\]

Hence, substituting the result from Equation (A.7) in Equation (A.6) shows:

\[
\frac{\partial S(t)}{\partial L_n(t)} = \alpha_n(t) + \sum_{i=j}^{k-1} \frac{\partial \alpha_i(t)}{\partial L_n(t)} L_i(t)
\]

\[
= \alpha_n(t) + \sum_{i=j}^{k-1} \frac{\tau_n}{1 + \tau_n L_n(t)} \alpha_i(t) \left( -H(i - n) + 1 - \sum_{r=j}^{n-1} \alpha_r(t) \right) L_i(t)
\]

\[
= \alpha_n(t) + \frac{\tau_n}{1 + \tau_n L_n(t)} \sum_{i=j}^{k-1} \alpha_i(t) \left( -H(i - n) + 1 - \sum_{r=j}^{n-1} \alpha_r(t) \right) L_i(t)
\]

\[
= \alpha_n(t) + \frac{\tau_n}{1 + \tau_n L_n(t)} \left( \sum_{i=j}^{k-1} \alpha_i(t)(-H(i - n) + 1)L_i(t) - \sum_{r=j}^{n-1} \alpha_r(t) \sum_{i=j}^{k-1} \alpha_i(t)L_i(t) \right)
\]

\[
= \alpha_n(t) + \frac{\tau_n}{1 + \tau_n L_n(t)} \left( \sum_{i=j}^{n-1} \alpha_i(t)L_i(t) - \sum_{r=j}^{n-1} \alpha_r(t)S(t) \right)
\]

\[
= \alpha_n(t) + \frac{\tau_n}{1 + \tau_n L_n(t)} \sum_{i=j}^{n-1} \alpha_i(t)(L_i(t) - S(t)).
\]
Bibliography


BIBLIOGRAPHY

