Analysis of combined HERA data with the Impact-Parameter dependent Saturation model

Author: Merijn van de Klundert

Supervisors: Dr. Raju Venugopalan
            Prof. dr. Eric Laenen

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1 Abstract

We analyze the recent combined HERA [1] high precision data set on electron proton and positron proton scattering with the impact parameter dependent saturation (IP-Sat) model. The IP-Sat model is a dipole model that incorporates the concept of gluon saturation. The IP-Sat model has been tested successfully to the low $x$ regime of a previous data set obtained by the ZEUS collaboration ([2],[3]) in 2000. We confront the model with the 2010 high precision combined HERA data set (which combines the data from both the ZEUS and H1 experiment). With a proper set of cuts we were able to successfully fit three free parameters of a parameterized gluon distribution function to the reduced cross section. A fourth parameter, related to the impact parameter dependence, is obtained from a fit to the distribution of exclusive $J/\psi$ production. Next we fix these parameters and test predictions for $F_2$ and the charm structure function with the data. Also we test the predictions of our model to the H1 and ZEUS data for exclusive $J/\psi$ production. Our predictions match the data very well. We provide as well the saturation scale $Q_s$ as function of $x$ and the impact parameter; the latter provides nontrivial insight in saturation. We conclude there are strong hints for the universality of the ip-sat model. The results of our research serve as direct input for various models that describe LHC data on pp, pA and AA collisions. Our paper [4] has been accepted for publication by phys. rev. D.
2 Introduction

For over decades physicists have tried to gain insight in the structure of the proton (we will not pursue a historical introduction here on the various discoveries that lead to our current understanding of its structure though). The structure of the proton is described by parton distribution functions which describe the number of quarks, antiquarks or gluons as function of their momentum fraction $x$. These functions can’t be calculated from scratch from QCD since we’re in the low energy domain where perturbation theory breaks down due to the running coupling.

The quark and antiquark distribution functions have been estimated experimentally, and these tend to zero for decreasing $x$. The gluonic distribution function however seems only to rise with decreasing $x$. At one hand the experimental data suggest an unbound increase of the gluonic pdf, while from theoretical arguments we know there has to be an upper limit (otherwise one gets for example unitarity violations; the probability for a particular interaction becomes higher than the probability for any reaction). The goal of our research (as will be outlined below) is to provide a well-tested parameterization of the low $x$ gluon distribution function. We will try to obtain the gluonic parameters by fitting data on $e^\pm p$ scattering with a phenomenological model that describes cross sections, the so called IP-Sat (impact parameter dependent saturation) model.

Note that the IP-Sat model has already been tested successfully on data as available in 2005. However, the H1 and Zeus experiment published in 2010 a combined data set. These data have a much smaller statistical error. Therefore in this thesis we investigate if the conclusions from the previous research still hold for the new data set.

The rough outline of this thesis is as follows: since the IP-Sat model uses the running coupling as one of its ingredients we first derive the so called renormalization group equation. We apply this equation to the theory of QCD and obtain an equation for the running coupling. We apply this equation to the theory of QCD and obtain an equation for the running coupling.

Since the data we investigate are for $e^\pm p$ scattering, we need to obtain an expression for the hadronic tensor $W^{\mu\nu}$. We first outline the idea of gluon saturation and how this leads to the perception of a classical, non-Abelian background field. We show how to derive from a few basic assumptions a (complicated) expression for $W^{\mu\nu}$. This expression contains, aside the coupling constant, the gluon distribution function, for which we provide a parameterized distribution. Although we cannot derive the QCD processes directly from theory, in this way we obtain a model that describes these interactions as a function of the parameters of the gluon distribution.

In the next section we consider the scattering process of an electron with the proton since this is our experimental observable. We explore the concept and kinematics of the dipole model. In this model, the virtual photon which is emitted by the electron, interacts not directly with the proton. Instead it fluctuates into a quark anti-quark dipole. This dipole is a colored object which can interact and probe the gluons in the proton. From the geometry of the model it follows the model must have a dependence on the impact parameter between the photon and the proton, this is why the model we investigate is called the IP-Sat model. Just as with the gluon distribution function we make an educated guess for the dependence on the impact parameter and later test it.

Subsequently we’ll derive expressions for various experimentally accessible observables
like the total cross section and the cross section for exclusive diffractive processes.

In the results section we test the model on the high precision data set. We first fit the parameters of the parameterized gluon distribution function to the total cross section. With the obtained distribution we make self consistent predictions to the other processes and the saturation scale.

We conclude with a discussion on the observed effects and differences with previous research. Also we give a suggestion for a further improvement. Also we discuss some examples of the broad range of applicability of the model to for example LHC data.
3  Renormalization and the Renormalization Group

The fact that the coupling constant for QCD (and in general for non-Abelian gauge theories under certain conditions) becomes weak at high energy scales provides us with one of the fundamental assumptions needed to construct the IP-Sat model. Therefore we’ll derive first a general equation that describes dynamics of physical quantities w.r.t. the energy scale, this is called the Callan-Symanzic equation. Next we describe how this affects the behavior of the coupling constant in QCD.

We first discuss briefly the division of quantum field theories in three classes: super renormalizable-, renormalizable- and non-renormalizable theories. Subsequently we discuss how, using dimensional renormalization, we can absorb the infinities arising in loop diagrams in the bare parameters of the Lagrangian. Emphasis is put on the role of the renormalization condition in this process. As toy model we use $\phi^4$ theory to leading order throughout this chapter (and occasionally point out differences with gauge theories). Next we turn to a more technical derivation of the renormalization group equation (also called the Callan-Symanzic equation) and some equations that follow from it. As an example we’ll apply these equations to derive the running of the coupling of the $\phi^4$ theory. Finally we derive the coupling constant for a non-Abelian gauge theory and apply the equation to obtain the formula for the strong coupling constant we’ll use in our model.

3.1  Determining Renormalizability by Power Counting

In diagrams containing a loop unconstrained momentum $k$ can flow through the loop. Since we must integrate over this momentum in calculating the amplitude of the diagram these amplitudes potentially are divergent.

The superficial degree of divergence $D$ is the highest power of $k$ in the numerator minus the power of $k$ in the denominator (note that a term $\int d^4k$ in the numerator counts as four powers of $k$ in this definition). So if $D > 0$ the integral (integrated up to a cutoff $\Lambda$ diverges like $\Lambda^D$). When $D = 0$ we expect a divergence of $\ln(\Lambda)$ and when $D > 0$ we don’t expect a divergence. We will express $D$ for $\phi^n$ theory in terms of the number of vertices $V$, the number of external lines $N$, the number of propagators $P$ and the dimension of the field $d$. A propagator adds two powers of $k$ to the denominator, while a loop adds $d$ powers of $k$ to the numerator. So we obtain:

$$D = dL - 2P.$$  \hfill (3.1)

In a $\phi^n$ theory the number of loops in a diagram is given by:

$$L = P - V + 1.$$  \hfill (3.2)

At a vertex we encounter $n$ lines, so the total number of lines is $n \cdot V$. Both external lines (denoted $N$) and propagators can connect to a vertex, but a propagator will always connect to two vertices (in an amputated diagram). So $n \cdot V$ is as well given by $N + 2P$. So we obtain:

$$P = \frac{nV - N}{2}.$$  \hfill (3.3)
Using these identities we can express the degree of divergence of a diagram as well as:

\[ D = d + n \cdot V \left( \frac{d-2}{2} - 1 \right) - \frac{N}{2} (d-2) \]  

(3.4)

Note that for different theories (for example QED) such rules get more complicated (since one has to take into account both electron and photon propagators for example) but the procedure of defining \( D \) in terms of \( V, d \) and \( N \) is just the same.

We’re now in a position to classify theories. Theories for which only a finite number of diagrams have \( 0 \leq D \) are super renormalizable, since all infinities can be absorbed in the Lagrangian with a finite number of extra parameters. There exist theories in which we have a finite number of diagrams with \( 0 \leq D \), but these diagrams occur in all orders of the theory. These theories are renormalizable. Theories that don’t have a finite number of superficially divergent diagrams are non-renormalizable.

A closer inspection of eq. 3.4 reveals that this theory is super renormalizable for \( d > 4 \) (since then the term multiplying \( V \) is negative and from a certain order onwards diagrams won’t diverge anymore). For \( d = 4 \) the theory is renormalizable, and for \( d > 4 \) the theory is non-renormalizable since we can always make a diagram which diverges and all higher order diagrams (an infinite number of diagrams) diverge as well.

We can perform an analysis as well how the dimension of the parameters of the Lagrangian affect the renormalizability of the theory; this will allow us to constrain the possible forms of renormalizable Lagrangians enormously. In natural units, the action has to be dimensionless. The integral \( \int d^d x \) adds minus \( d \) powers of units of mass to the terms in the action. From the kinetic term in the Lagrangian \( (k^2 \phi^2) \) we can thus infer that the dimension of \( \phi \) must be \( \frac{d-2}{2} \). So for a term \( \lambda \phi^n \) the required dimension of \( \lambda \) is: \( d - n \cdot \left( \frac{d-2}{2} \right) \) (note we can perform a similar analysis to all parameters of the Lagrangian). We consider an amputated diagram with \( N \) external lines arising from a \( \lambda \phi^N \) interaction. Since \( \lambda \) has dimension \( d - n \cdot \left( \frac{d-2}{2} \right) \) the dimensionality of the diagram is \( d - N \frac{d-2}{2} \). The (evt.) divergent part of a diagram with \( V \) vertices will be proportional to \( \lambda^V \Lambda^D \). Using dimensional analysis we find that

\[ d - N \cdot \left( \frac{d-2}{2} \right) - V \cdot (d - n \cdot \left( \frac{d-2}{2} \right)) = D. \]  

(3.5)

Note that the term that multiplies \( V \) is just the dimensionality of \( \lambda \). Therefore we conclude that for a positive mass dimension of the coupling constant the theory is super renormalizable, if the dimension is 0 it is renormalizable and when the dimension is smaller than 0, every higher order diagram diverges more so the theory isn’t renormalizable.

### 3.2 Dimensional Renormalization of \( \phi^4 \) Theory

We’ll present here briefly the formal steps and calculations for the dimensional renormalization of \( \phi^4 \) theory to leading order. Emphasis will be on the results and the role of the renormalization condition (we assume some familiarity with the technical details of the calculations). We start with the Lagrangian for \( \phi^4 \) theory with bare fields and parameters \( m_0 \) and \( \lambda_0 \) (bare parameters are, in general, infinite):

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4 \]  

(3.6)
From equation 3.4 we see that for this theory in four dimensions $D = 4 - N$. The theory is invariant under $\phi \rightarrow -\phi$ so amplitudes with an odd number of external legs vanish. The only (possible) divergent amplitudes are displayed in fig. 3.1:

![Figure 3.1: The three divergent amplitudes in $\phi^4$ theory, from left to right: the vacuum energy, the two point Greens function and the four point Greens function. Note that the blobs may still contain contributions to arbitrary order in $\lambda$.](image)

We’ll ignore the unobservable shift in the vacuum energy and absorb the other two infinities arising in the amplitudes in the field strength, mass and coupling. We start by absorbing the field strength $Z$ using the formula for the two point correlation function:

$$\int d^4x \langle \Omega | T(\phi(x)\phi(0)) | \Omega \rangle e^{ip \cdot x} = \frac{iZp^2 - m^2}{p^2 - m^2}. \tag{3.7}$$

In this $m$ is the physical mass of the particle. In order to get rid of the residue $Z$ we rescale the field:

$$\phi = Z^{1/2} \phi_r \tag{3.8}$$

We express the Lagrangian in terms of the renormalized field (we do this step first such that we can perform the shifts of the parameters w.r.t. the renormalized field):

$$\mathcal{L} = \frac{Z}{2} (\partial_\mu \phi_r)^2 - \frac{Z}{2} m_0^2 \phi_r^2 - \frac{Z}{2} \lambda_0 Z^2 - \frac{4!}{\lambda} \phi_r^4 \tag{3.9}$$

We define the following so called counterterms:

$$\delta Z = Z - 1, \quad \delta_m = m_0^2 Z - m^2, \quad \delta_\lambda = \lambda_0 Z^2 - \lambda \tag{3.10}$$

At this point we can perform the actual renormalization of the coupling and mass: we set the coupling and mass to their physical value $\lambda$ and $m$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4 + \frac{Z}{2} (\partial_\mu \phi_r)^2 - \frac{m^2}{2} \phi_r^2 - \frac{\lambda}{4!} \phi_r^4. \tag{3.11}$$

This Lagrangian gives rise to the Feynman rules as given in fig. 3.2.

It is tempting to say we’ve added counterterms to the original Lagrangian, but actually we’ve just started with the Langrangian in its bare fields and parameters and subsequently split out the bare parameters in the physical parameters and counterterms. These counterterms shift the bare coupling and mass down to the physical value, so the counterterms
Figure 3.2: The Feynman rules for renormalized perturbation theory, as obtained from the Langrangian in eq. 3.11. From left to right the first two terms are the renormalized two point and four point Greens function. The third and fourth term are the counterterms for these functions. Note the vertex factors for the counterterms may contain contributions to all orders in $\lambda$ (except to $\lambda^0$ order). When we do a calculation in renormalized perturbation theory to a certain order in $\lambda$ we have to carefully take all contributions from the renormalized terms and counterterms to that order into account to obtain a consistent result.

themselves are (usually) infinite. We must define the physical mass and coupling constant by specifying the renormalization condition and there is no unique way to do this. We choose here a set of “natural” renormalization conditions, as displayed in fig. 3.3.

The freedom of defining the renormalization conditions will allow us to derive a.o. the running coupling constant in section 3.4.

3.2.1 The Counterterms to Leading Order

To clarify how the dependence on the renormalization conditions enters the renormalized field strength, coupling and mass we renormalize our toy theory.

We first calculate the four point Greens function to second order in $\lambda$. To calculate the (infinite) contribution from the loop diagrams we use dimensional renormalization (we presume some familiarity with the technical steps, a good introduction can be found in [7]). Next we impose the second renormalization condition from fig. 3.3 on this result. This defines the counterterm $\delta_\lambda$.

We repeat the same steps for the two point Greens function (although we’ll perform this calculation to linear order in $\lambda$). This will yield the result for $\delta_Z$ and $\delta_m$.

The counterterms can in principle be calculated to arbitrary order using the renormalization conditions, but the calculations become much more complicated and the dependence of the various Greens functions on $\delta_Z$, $\delta_Z$ and $\delta_\lambda$ intertwines (i.e. one can’t deduce the value of one counterterm from just one Greens function).

The Four Point Greens Function To second order in $\lambda$ we get four contributions from the renormalized part of the Lagrangian and one contribution from a counterterm, see the diagrams in fig. 3.4.

The amplitude for the first diagram is $-i\lambda$. For the three loop diagrams it is convenient to define $p = p_1 + p_2$. This allows us to define:

$$V(p^2) = \frac{i}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{(k + p)^2 - m^2}.$$  

(3.12)
Figure 3.3: A “natural” set of renormalization conditions. The left hand condition specifies both the mass of the particle and the location of the pole in the propagator and its residue at this pole. The right hand diagram specifies the coupling in the limit of zero incoming momentum: $s = 4m^2, t = u = 0$.

Figure 3.4: The various contributions to second order in $\lambda$ arising from the right hand diagram of fig. 3.1.

The amplitude for each of the loop diagrams is:

$$\frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \equiv (-i\lambda)^2 \cdot iV(p^2).$$

So the total amplitude to order $\lambda^2$ is:

$$iM = -i\lambda + \frac{(-i\lambda)^2}{2} \left(iV(s) + iV(u) + iV(t)\right) - i\delta\lambda.$$

We calculate $V(p^2)$ in $d$ dimensions and at the end of the calculation take $\lim_{d \to 4}$. We focus on the integrand. It can be written as an integral:

$$\frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2} = \int_0^1 dx \frac{1}{k^2 + 2xp \cdot k + xp^2 - m^2}.$$

We focus at this point on the momentum integral. We can perform a variable substitution $k = l + xp$. This doesn’t affect the boundaries of the momentum integral and has unit Jacobian w.r.t. $k$:

$$V(p^2) = \frac{i}{2} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + x(1-x)p^2 - m^2)^2}.$$
We perform a Wick rotation to obtain an Euclidean metric \((l_E^0 = -il_0)\):

\[
V(p^2) = -\frac{1}{2} \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + x(1-x)p^2 - m^2)^2}.
\]

(3.17)

We write the integral as \(\int d^d l_E = \int d^{d-1}\Omega \int dl_E\). We can perform the integration over the solid angle in an arbitrary dimension. The result is:

\[
\int d^d \Omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}.
\]

(3.18)

In the remaining one dimensional momentum integral we perform again a change of the integration variable and substitute \(l^2\) for \(l\). Further we define \(\Delta = x(1-x)p^2 - m^2\). Then the remaining momentum integral becomes:

\[
\int \frac{dl^2}{(2\pi)^d} \frac{(l^2)^{d/2-1}}{(l^2 + \Delta)^2}
\]

(3.19)

We make again a variable substitution to obtain an integral over a finite domain. We define

\[
y = \frac{\Delta}{l^2 + \Delta}.
\]

(3.20)

Therefore \(l^2 = \Delta\left(\frac{1}{y^2} - 1\right)\). Note that in this change of variables \(\lim_{l^2 \to \infty}\) is replaced by \(\lim_{y \to 0}\) (for some values of \(d\) this integral can be expected to diverge so we must be careful in defining the lower bound) and \(\lim_{l^2 \to 0}\) becomes \(y = 1\). We obtain for the momentum integral (we add in the piece for the angular integration and the integral over \(x\) in a moment):

\[
\frac{1}{2} \Delta^{2-d} \int_0^1 dy \ y^{1-d/2} (1-y)^{d/2-1}.
\]

(3.21)

The definition of the \(B\) function is given by:

\[
\int_0^1 \ dx \ x^{\alpha-1} (1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}.
\]

(3.22)

The \(\Gamma\) function has poles for \(z = 0, -1, -2, \ldots\) corresponding to \(d = 4, 6, \ldots\). We define \(\epsilon = d - 4\) (since we’re interested in the value of \(V(p^2)\) in four dimensions). We can approximate the singular \(\Gamma\) function as:

\[
\Gamma(2 - d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma
\]

(3.23)

in which \(\gamma\) is the Euler-Mascheroni constant.

The prefactor \(\frac{1}{2(\Delta)^{2-d/2}}\) can be written as \(\frac{1}{2} \Delta^{-\epsilon/2}\). We rewrite it as \(\frac{1}{2} e^{-\frac{\epsilon}{2} \ln(\Delta)}\). Since \(\epsilon\) is small we Taylor expand to order \(\epsilon\):

\[
\frac{1}{2(\Delta)^{2-d/2}} \approx \frac{1}{2} \left(1 - \frac{\epsilon}{2} \ln(\Delta)\right)
\]

(3.24)
Inserting the value of $\Delta$ (see eq. 3.20) we obtain (neglecting terms $O(\epsilon)$ and higher):

$$V(p^2) = -\frac{1}{32\pi^2} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma - \ln(\Delta) \right) = -\frac{1}{32\pi^2} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma - \ln(m^2 - x(1-x)p^2) \right)$$  \hspace{1cm} (3.25)

Our normalization condition demands the amplitude equals $-i\lambda$ at $s = 4m^2, t = u = 0$. When we impose this condition upon eq. 3.14 we obtain for the counterterm $\delta\lambda$:

$$\delta\lambda = -\lambda^2 \left( V(4m^2) + 2V(0) \right)$$

$$= \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left( \frac{6}{\epsilon} - 3\gamma - \ln(m^2 - x(1-x)4m^2) - 2\ln(m^2) \right).$$  \hspace{1cm} (3.26)

Note that implicitly the normalization condition has entered our formula for $\delta\lambda$ and that $\lim_{d \to 4} \delta\lambda$ is divergent (as expected).

To obtain the four point Greens function in eq. 3.14 we calculate the contribution from the loop diagrams (using the formula for $V(p^2)$ from eq. 3.25 and subtract the $\delta\lambda$ counterterm. We obtain for our amplitude then again a finite (though complicated) result:

$$iM = -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[ \log \left( \frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} \right) + \log \left( \frac{m^2 - x(1-x)t}{m^2} \right) + \log \left( \frac{m^2 - x(1-x)u}{m^2} \right) \right].$$  \hspace{1cm} (3.27)

Note that the second order contribution to the amplitude is dependent on the kinematics of the process while the counter terms are fixed w.r.t. the momenta due to the renormalization condition. Therefore the second order result for the four point Greens function depends on the momenta as well, contrary to the leading order process for which the amplitude is just the coupling $\lambda$.

**The Two Point Greens Function**  In order to determine the contributions (to leading order in $\lambda$) to $\delta Z$ and $\delta m$ we must compute the two point Greens function to leading order.

We start with defining a new class of diagrams: the one particle irreducible ($1PI$) diagrams. An informal definition of these diagrams is that these are the diagrams which we can’t divide into two lower order members of the class by just cutting one line of the diagram. The sum of these diagrams for the two point Greens function are represented in fig. 3.5 and we denote the amplitude of this diagram $-iM^2(p^2)$. We can express the two point Greens function as a geometric series in the $1PI$ diagrams and thereby express the amplitude in terms of $-iM^2(p^2)$, see fig. 3.6. To determine the value of the counterterms we apply the renormalization conditions as defined in fig. 3.3. These give us immediately two expressions for $M^2(p^2)$:

$$-iM^2(p^2)|_{p^2=m^2} = 0 \hspace{1cm} \frac{d}{dp^2} M^2(p^2)|_{p^2=m^2} = 0$$  \hspace{1cm} (3.28)
Figure 3.5: The representation of the one particle irreducible diagrams for the two point Greens function. The amplitude of the diagram is $-iM^2(p^2)$.

Figure 3.6: The two point Greens function as geometric series in $1PI$ diagrams. The amplitude of the two point Greens function is given by $\frac{i}{p^2-m^2-M^2(p^2)}$.

To first order we have two contributions to $-iM^2(p^2)$: one is given by the counterterm for the two point Greens function (already seen previously in fig. 3.2), the other is given by a loop diagram. Both contributions are depicted below in fig. 3.7.

The amplitude for the loop diagram is given by $-i\lambda \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2-m^2}$. The contribution due to the counterterm is $i(p^2\delta Z - \delta m)$.

With a calculation similar to the calculation for the loop contribution to the loop corrections to the four point Greens functions we obtain for the loop diagram $-\frac{i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$. So we obtain:

$$-iM^2(p^2) = 0 = -\frac{i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}} + i(p^2\delta Z - \delta m).$$\hspace{1cm} (3.29)

Since $\delta Z$ is multiplied by $p^2$ and the contribution due to the loop diagram is independent.

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of $p^2$, $\delta Z$ is trivially 0 to first order and we obtain:

$$\delta m = -\frac{\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1 - d/2)}{(m^2)^{1-d/2}}.$$

(3.30)

Before we turn to the renormalization group we stress the importance of the consistency of perturbation theory. In the first order, no counterterms come in. In second order they come in and counter the divergences that occur as this order due to loop diagrams, at a certain scale defined by the renormalization conditions. The point is that this method works to all orders if one consistently keeps working to a certain order in both the renormalized fields and the counterterms.

### 3.3 The Renormalization Group Equation

The bare Greens functions (the Greens functions expressed in the bare fields) are given by some functions of $\lambda_0$ and a momentum cutoff $\Lambda$. The scale $M$ at which we renormalize our theory is arbitrary and does not enter in the bare Greens function. The dependence on the renormalization scale does enter when we remove the cutoff dependence and renormalize our fields and couplings. The renormalized Greens functions are equal to the bare Greens functions up to the field strength rescaling:

$$Z^{-n/2}\langle \Omega | T(\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n))|\Omega \rangle = \langle \Omega | T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|\Omega \rangle$$

(3.31)

A variation of the renormalization scale leads to a variation in other parameters of the theory (namely the coupling and the field strength) to compensate for this change. We elaborate this statement. Let $G^{(a)}(x_1, \cdots x_n)$ be the connected $n$-point Greens function in renormalized perturbation theory. A shift in the renormalization scale results in a shift in the coupling and field strength:

$$M \to M + \delta M$$

$$\lambda \to \lambda + \delta \lambda$$

$$\phi \to (1 + \delta \eta)\phi.$$

(3.32)
The effect of the shift in the field strength on the Greens function due to a shift in the scale $M$ is straightforward:

$$\phi \to (1 + \delta \eta)\phi \quad G^{(n)} \to (1 + n\delta \eta)G^{(n)}$$

We therefore obtain:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M} \delta M + \frac{\partial G^{(n)}}{\partial \lambda} \delta \lambda = \eta \delta \eta G^{(n)}$$

It is convenient to define two dimensionless parameters $\beta$ and $\gamma$:

$$\beta \equiv \frac{M}{\delta M} \delta \lambda \quad \gamma \equiv -\frac{M}{\delta M} \delta \eta$$

Using these we obtain the Callan-Symanzic equation:

$$[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + n\gamma] G^{(n)}(x_1, \cdots, x_n, M, \lambda) = 0$$

Since this is an equation for a Greens function for a renormalized theory, the $\beta$ and $\gamma$ functions can’t depend on the cutoff $\Lambda$. So, by dimensional analysis, these can’t depend on $M$ either, so these only depend on $\lambda$. Note for more complicated theories this equation may have various $\beta$ and $\gamma$ functions; in QED for example we have photon and electron fields with their own $\gamma$ functions.

To obtain some intuition for these functions we calculate these for our toy theory (this was partially the reason for the calculation of the counterterms). However, our previously chosen “natural” normalization condition might pose problems if we would consider a massless theory (the counterterms are ill defined then, see eq. 3.26 and 3.30). Therefore we start with a new (unphysical) set of renormalization conditions and recalculate the counterterms we need for a massless $\phi^4$ theory (using the intermittent results of the preceding section this is straightforward). The new renormalization conditions at an arbitrary scale $M$ are given in fig. 3.8.

We’ll first apply the Callan-Symanzic equation to $G^{(4)}$ and successively to $G^{(2)}$.

$$G^{(4)} = \left[ -i\lambda + \frac{(-i\lambda)^2}{2} \left( iV(s) + iV(u) + iV(t) \right) - i\delta \lambda \right] \cdot \prod_{i=1,\cdots,4} \frac{i}{p_i^2}$$

When we apply the new renormalization condition to this equation we obtain for $\delta \lambda$:

$$\delta \lambda = (-i\lambda)^2 \cdot 3V(-M^2) = \frac{3\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2 - d/2)}{(x(1-x)M^2)^{(2 - d/2)}}$$

In the limit of four dimensions we obtain:

$$\lim_{d \to 4} \delta \lambda = \frac{3\lambda^2}{2(4\pi)^2} \left[ \frac{1}{2 - d/2} - \ln(M^2) + \text{finite} \right].$$

The reason we don’t elaborate the finite term further is because aside not being divergent it is independent of $M$, so it won’t contribute to $\gamma$ or $\beta$. The counterterm $\delta \lambda$ gives $G^{(4)}$
its $M$ dependence (so we see explicit how the renormalization condition enters the Greens function).

The first term in the Callan-Symanzic equation (eq. 3.36) is a differential w.r.t. $M$ which we can evaluate at this point:

$$M \frac{\partial}{\partial M} G^{(4)} = \frac{3i\lambda^2}{(4\pi)^2} \prod_i \frac{i}{p_i^2}. \quad (3.40)$$

The counterterms for the two point Greens function of the massless theory are zero (to leading order in $\lambda$) and therefore certainly have no dependence on $M$ or to this order (which follows from setting $m = 0$ in eq. 3.30 and observing $\delta Z = 0$). From this it follows that $\gamma$ is (at least to order $\lambda$) zero.

We know $\gamma$ won’t contribute up to first order in $\lambda$, therefore the contributions of the last term in eq. 3.36 are at least of order $\lambda^3$. The derivative of $G^{(4)}$ w.r.t. $\lambda$ gives a contribution up to first order in $\lambda$. Therefore $\beta$ can’t have a dependence linear in $\lambda$ since there would be no term to counter it. So $\beta$ must be at least of order $\lambda^2$ and cancel the term in eq. 3.40. We obtain:

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \quad (3.41)$$

### 3.4 The Running Coupling Constant

We have obtained a general renormalization equation and the $\beta$ and $\gamma$ function to first and second order in $\lambda$ for the two and four point Greens functions of $\phi^4$ theory. In this section we evaluate how this affects the coupling $\lambda$ and the Greens functions. To describe the evolution of the coupling and the Greens function (which is given by the solution of a
partial differential equation) we’ll use an analogy with bacterial growth. This makes the mathematics more approachable and gives insight in the physics behind the equations.

We start by observing the dimension of $G^{(2)}$ is minus 2 (in mass dimension units) so we can formulate the function as:

$$G^{(2)}(p) = \frac{i}{p^2} g(-p^2/M^2). \quad (3.42)$$

In the Callan-Symanzic equation a derivative w.r.t. $M$ was taken. We can replace this derivative with a derivative to $p$ and obtain a modified equation:

$$[p \frac{\partial}{\partial p} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2 - 2\gamma] G^{(2)}(p) = 0. \quad (3.43)$$

We can rewrite this equation to suit a situation for which we have a better physical intuition. Let

$$\log(p/m) \to t, \quad \lambda \to x, \quad -\beta(\lambda) \to v(x)$$

$$2\gamma(\lambda - 2) \to \rho(x), \quad G^{(2)}(p, \lambda) \to D(t, x). \quad (3.44)$$

Then the Callan-Symanzic equation for the two point function takes the following form:

$$[\frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - \rho(x)] D(t, x) = 0. \quad (3.45)$$

We can identify this equation with for example bacteria inhabiting a pipe where water flows through with a speed $v(x)$. The density of bacteria at a given space and time is given by $D(t, x)$ and the rate at which they grow is $\rho(x)$. For this system eq. 3.45 describes the future behavior. The position of a fluid element at $t = 0$ is given by $\bar{x}(t, x)$. This element satisfies:

$$\frac{d}{dt} \bar{x}(t', x) = -v(\bar{x}). \quad (3.46)$$

In this equation $\bar{x}(0, x) = x$. The general solution to the equation for $D(t, x)$ is an expression in the initial concentration distribution $D(0, x) \equiv D_i(x)$. The solution is given by:

$$D(t, x) = D_i\left(\bar{x}(t, x)\right) \cdot \exp\left(\int_0^t dt' \rho[\bar{x}(t', x)]\right)$$

$$= D_i\left(\bar{x}(t, x)\right) \cdot \exp\left(\int_x^{\bar{x}(t)} \frac{dx'}{v(x')} \rho(x')\right) \quad (3.47)$$

We can gain insight in our field theory by replacing the parameters $t, x, \rho$ and $v$ with their equivalents from eq. 3.44. For $t = 0$ we obtain $-p^2 = M^2$ and $D_i(x)$ becomes $G_i(\lambda)$ (that is, the value of the Greens function at the initial condition is an unknown function). We obtain an equation for the Greens function and for the coupling $\lambda(p, \lambda)$:
The initial function \( G_i \) can’t be calculated from renormalization theory. To obtain it we must expand the Greens function in \( \lambda \) and match it to its corresponding order of an expansion of 3.48.

For the four point Greens function we have to leading order (we define \( p_i^2 = -P^2 \) and \( p_i p_j = 0 \): \( G^{(4)}(P) = \left( \frac{i}{P^2} \right)^2 (-i \lambda) \). We then find, using 3.48 for the four point Greens function that \( G_i(\bar{\lambda}(p, \lambda)) = -i \bar{\lambda} \).

We interpret our result as follows: the first factor in the first eq. of 3.48 is the coupling at the appropriate momentum scale; for \( p = M \) this coupling becomes our ordinary coupling \( \lambda \). The exponential factor is the field strength of the Greens function integrated over the momentum from \( M \) to \( p \). At the intermediate energies the field strength is evaluated at the properly adjusted running coupling \( \bar{\lambda}(p, \lambda) \).

Now we’re finally in position to calculate the running coupling for our toy theory. We derived the \( \beta \) function for \( \phi^4 \) theory in eq. 3.41. Using eq. 3.48 we can obtain the evolution of the coupling:

\[
\frac{d}{d \log(p/M)} \bar{\lambda}(p) = \beta(\bar{\lambda})
\]

(3.49)

(in this equation the boundary condition is \( \bar{\lambda}(M, \lambda) = \lambda \)). Integrating the equation we find:

\[
\bar{\lambda}(p) = \frac{\lambda}{1 - (3\lambda/16\pi^2) \log(p/M)}
\]

(3.50)

An important observation is that when we expand \( \bar{\lambda} \) in \( \lambda \) (observe \( \bar{\lambda} \) is a geometric series in \( \lambda \)), we find that \( \lambda^n \) is multiplied by a logarithm of order \( n \) – 1:

\[
\bar{\lambda} = \sum_{n=0}^{\infty} \lambda^{n+1} \left( \frac{3}{16\pi^2} \log \left( \frac{p}{M} \right) \right)^n
\]

(3.51)

This has a very important consequence: for \( p \) much greater or smaller than \( M \) perturbation theory won’t work anymore.

The general principle of the evolution of the coupling and the Greens function has been outlined now and applied in detail to a toy theory. We can now evaluate the effect of the renormalization group on more complicated, non-Abelian theories.

3.5 The \( \beta \) Function and Coupling Constant of non-Abelian Gauge Theories

As we have seen in the previous section the counterterms arising in the renormalization have a dependence on the renormalization scale \( M \). At the same time the renormalized
Greens functions must be independent of this scale. Via the Callan-Symanzic equation we could compute how a change of renormalization scale affected the coupling and field strength renormalization. As an example we calculated the behavior of the coupling and the Greens function for \( \phi^4 \) theory to leading order, see eq. 3.51 and 3.48.

Since the QCD coupling constant enters directly in the model under consideration in this thesis we investigate here the form of the \( \beta \) function of a non-Abelian gauge theory and calculate the coupling constant. We’ll see the number of fermions and the symmetry group of the theory are very important in determining the behavior of the theory. We’ll have to calculate three counter terms, their diagrams are displayed in fig. 3.9. Below we calculate consecutively the diagrams contributing to the self energy of the gauge bosons and fermions and the contributions to the vertex factor to leading order. Then we can calculate the counterterms and the \( \beta \) function. Note that in the previous section some clarifications have been put to the technical details of the calculations. In this section the emphasis is on obtaining an overview rather than the technical details. We’ll work throughout in Feynman-‘t Hooft gauge.

\[ i(q^2g^{\mu\nu} - q^\mu q^\nu)\delta^{ab} \left( \frac{-g^2}{4\pi} \cdot \frac{4n_f}{3} C(r) \Gamma(2 - d/2) \cdots + \right) \]  

3.5.1 The Gauge Boson- and Fermion Self Energy and the Contribution to the Vertex Factor

There are four divergent amplitudes contributing to the gauge boson self energy, see fig. 3.10.

\[ i(q^2g^{\mu\nu} - q^\mu q^\nu)\delta^{ab} \left( \frac{-g^2}{4\pi} \cdot \frac{4n_f}{3} C(r) \Gamma(2 - d/2) \cdots + \right) \]  

In this equation \( n_f \) is the number of fermions the gauge boson can fluctuate into. The term \( C(r) \) is the trace over the matrices of the symmetry group in the representation \( r \): \( C(r) = \text{Tr}(t^a t^b) \).
The amplitudes for the other three diagrams are rather complicated. It is essential to note that the amplitudes per diagram are not very intuitive and contain for example as well poles for \( d = 2 \). However when all amplitudes are added up these cancel again. The remaining divergent part of these three diagrams is

\[
ig^{\mu \nu} - q^{\mu} q^{\nu}) \delta^{ab} \left( \frac{g^2}{(4\pi)^2} \frac{5}{3} C_2(G) \Gamma(2 - d/2) + \cdots \right).
\]

(3.53)

In this equation \( C_2(G) \) is the contraction of the symmetry groups structure constants: \( f^{acd} f^{bcd} = C_2(G) \delta^{ab} \). The sum of eq. 3.52 and eq. 3.53 gives the counterterm \( \delta_3 \).

The diagram for the fermion self energy is given in fig. 3.11. The divergent part of this diagram is:

\[
\frac{ig^2}{(4\pi)^2} k C_2(r) \Gamma(2 - d/2).
\]

(3.54)

This divergent part is absorbed by the counterterm \( \delta_2 \).

The two diagrams contributing to the correction of the vertex factor are given in fig. 3.55. The divergent part of these diagrams is:

\[
\frac{g^2}{(4\pi)^2} \Gamma(2 - d/2) \left( C_2(r) + C_2(G) \right).
\]

(3.55)

This yields the result for counterterm \( \delta_1 \).

We obtained all counterterms (note that for the logarithmic divergences encountered here the renormalization scale always enters via the form \( \Gamma(2 - d/2) \)) and proceed with calculating the running coupling.

### 3.5.2 The Running Coupling

The \( \beta \) function is given by:

\[
\beta(g) = g M \frac{\partial}{\partial M} (-\delta_1 + \delta_2 + \delta_3/2)
\]

(3.56)
(in this equation the factor $1/2$ for $\delta_3$ comes from the fact we’re considering a two point Greens function). Upon plugging in our previous obtained results for the counterterms we obtain our final (lowest order) result for $\beta(g)$:

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right].$$  \hspace{1cm} (3.57)

Observe that this is a very general applicable result; it applies to any non-Abelian gauge theory and expresses the $\beta$ function entirely in terms of some fundamental properties of the symmetry group and the number of fermions.

Having obtained the $\beta$ function we can apply the renormalization group equation 3.48 to calculate the coupling for a non-Abelian gauge theory at an arbitrary scale $Q$. We obtain:

$$\alpha(Q) = \frac{\alpha}{1 + \left( b_0(\alpha/2\pi) \right) \log(Q/M)}$$  \hspace{1cm} (3.58)

(in the equation $b_0 = \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)$ and $\alpha$ is the coupling at scale $M$). We see that for any non-Abelian theory (with a sufficiently small number of fermions) the coupling becomes weak at high energies; this phenomenon is called asymptotic freedom. But the reverse may happen as well: for low energies the coupling may become large and invalidate perturbation theory (although we should remember we often expand these theories in the coupling times $\ln(P/M)$).

Since we’ll be studying the strong interaction due to QCD (an SU(3) gauge theory in the fundamental representation) we evaluate the strong coupling $\alpha_s$ (to leading order) for this theory:

$$\alpha_s(Q) = \frac{\alpha_s}{1 + \alpha_s(11 - 2n_f/3) \log(Q/M)}$$  \hspace{1cm} (3.59)

In QCD the scale $M$ is usually replaced with $\Lambda_{QCD}$, which is defined as:

$$1 = g^2 \left( b_0/(2\pi) \right) \ln(M/\Lambda_{QCD})$$  \hspace{1cm} (3.60)

Then the coupling becomes:

$$\alpha_s(Q) = \frac{2\pi}{b_0 \ln(Q/\Lambda_{QCD})}$$  \hspace{1cm} (3.61)

The characteristic scale $\Lambda_{QCD}$ corresponds to approximately the scale at which the coupling becomes strong, which is roughly the size of a hadron ($\approx 200$ MeV). An important consequence is that perturbation theory only is valid for somewhat higher values of $Q$, approximately 1 GeV. Various experiments have made more elaborate estimates of $Q_{CD}$ (for example via analysis of the total cross section $e^+e^- \rightarrow$ hadrons, see [6]).

Now that we have obtained a leading order expression for the coupling we will investigate in the next section the effect of this on the gluon distribution in a proton.
4 Fermions in a Classical non-Abelian Background Field

In this section we will see how, due to the running coupling constant, the low \( x \) gluons in the proton can be perceived as a classical background field. We first outline how fermions behave in this classical non-Abelian background field since we will need this to calculate the hadronic tensor \( W^{\mu\nu} \). We cannot take the complete gluon distribution as starting point for our calculation since the hard gluons are strongly coupled. We can however compute the effect of the soft gluons on the fermions and make predictions for \( W^{\mu\nu} \) incorporating their effect. The strength of this approach is that it doesn’t rely on a twist expansion to a certain order but that all twists are included. Also the model doesn’t suffer from so-called unitarity violations. The emphasis of this section is on the physical arguments behind the assumptions and the main flow of the derivation. For the mathematical details of the derivation of various equations (which can be laborious) we refer to the appendices. We will first derive the form of the hadronic tensor \( W^{\mu\nu} \) and subsequently express the structure functions using our result.

In this and the subsequent chapters we will work in the so-called infinite momentum frame and use variables suitable to describe inelastic scattering processes. Usually the four vectors are expressed in light cone coordinates. We refer to appendix E for a brief explanation on this.

4.1 The Hadronic Tensor \( W^{\mu\nu} \)

In this subsection we will derive an expression for the hadronic tensor \( W^{\mu\nu} \). We will later show that we can derive the gluon distribution from this tensor. \( W^{\mu\nu} \) depends on the Lorentz invariants \( q^2 \) (the momentum transfer) and \( P \cdot q \) (\( P \) is the momentum of the target hadron). \( W^{\mu\nu} \) relates to \( T^{\mu\nu} \) (the forward Compton scattering amplitude) via the optical theorem. It is the Fourier transform of the current-current correlator between the hadronic current at 0 and \( x \). With the hadronic current \( J \) the current of the quarks is implied:

\[
W^{\mu\nu}(q^2, P \cdot q) = 2 \text{Disc} T^{\mu\nu}(q^2, P \cdot q) = \frac{1}{2\pi} \text{Im} \int d^4x \exp(iq \cdot x) \cdot \langle P|T\left(J^\mu(x)J^\nu(0)\right)|P\rangle
\]  

(4.1)

We start by rewriting this amplitude to an expectation value using the fact that \( \langle O \rangle = \langle P|O|P\rangle/\langle P|P\rangle \) (for any operator \( O \)). We see we need to provide a normalization for our states. We chose to normalize \( \langle P|P\rangle = (2\pi)^3 E m^3 (P - P') \) (\( m \) is the target mass of the hadron). Then \( \langle P|P\rangle = (2\pi)^3 E m^3 (0) = (2\pi)^3 E m V \). We replace \( V \) (the spatial volume) by \((1/(2\pi)^3)\sigma\) times an integral over the longitudinal extend of the states (denoted by \( X_- \)). This is necessary to include contributions of all quarks at all \( X_- \). We obtain:

\[
W^{\mu\nu}(q^2, P \cdot q) = \frac{1}{2\pi} \frac{P^+}{m} \sigma \text{Im} \int d^4x dX_- \exp(iq \cdot x) \cdot \langle P|T\left(J^\mu(X_- + x/2)J^\nu(X_- - x/2)\right)|P\rangle
\]  

(4.2)

(we have symmetrized the integrand w.r.t. \( x \), which can be accomplished by a shift in \( X_- \)).

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The fermionic currents are given by:

\[
\langle T(J^\mu J^\nu) \rangle = \langle T(\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(y)\gamma^\nu\psi(y)) \rangle
\]  

(4.3)

In the limit where we treat the low \(x\) gluons as a classical background field we can define the fermionic propagator \(S_A(x)\) in this background field (note that the fermionic field is still a quantum field which can be expanded in creation and annihilation operators).

The assumption that the sum of the gluons can be treated as a classical field is highly non-trivial. The gluon density increases with decreasing \(x\) (without taking recombination into account the number of gluons grows approximately as \(x^{-0.3}\) due to splitting). The occupation number becomes large for gluons with a small momentum fraction of the hadron, and therefore (by the uncertainty principle) the momentum (in the transverse direction) is large and the coupling becomes weak. By these two consequences (large density and small coupling) we can describe the wee low \(x\) gluons as a classical field (for a more in depth treatment we refer to [8]). The higher \(x\) gluons will be treated as sources and we can consider various distributions of these. We will apply a Gaussian distribution below.

Under this assumption the propagator is

\[
S_A(x, y) = -i\langle \psi(x)\bar{\psi}(y) \rangle_A
\]  

(4.4)

We apply Wick’s theorem to eq. 4.1. We obtain all contractions times normal ordered products (for the terms that are not contracted). The last die on the vacuum, so we’re left with only fully contracted terms. Since we have an expectation value we take the trace (note the trace is both over color and spinor indices!):

\[
\langle T(J^\mu J^\nu) \rangle = \text{Tr}(\gamma^\mu S_A(x))\text{Tr}(\gamma^\nu S_A(y)) + \text{Tr}(\gamma^\mu S_A(x, y)\gamma^\nu S_A(y, x))
\]  

(4.5)

The first part doesn’t contribute (it is a tadpole contribution and contains no imaginary part). So we obtain:

\[
W^{\mu\nu}(q^2, P \cdot q) = \frac{1}{2\pi} \sigma \frac{P^+}{m} \text{Im} \int d^4x dX_- \exp(iq \cdot x) \cdot \\
\text{Tr}(\gamma^\mu S_A(X_+ + x/2, X_- - x/2)\gamma^\nu S_A(X_- - x/2, X_+ + x/2))
\]  

(4.6)

Before we can continue the calculation of the hadronic tensor we need to calculate \(S_A(x, y)\). We outline the steps briefly below.

The Fermionic Propagator in the Background Field  Before we can construct the propagator (see eq. 4.4) we need to construct the wavefunction \(\psi\) in the classical background field. For \(\psi\) we need thus to formulate the background field \(A\).

There are two different approaches to derive the form of the background field. We outline here how to construct a solution to the inhomogeneous Maxwell equations from physical assumptions (note that we work here with a classical non-Abelian field). We point out another way to derive the same solution is to start with a (classical) Lienard-Wiechert
potential for two particles moving along the z direction. Upon making a relativistic approximation for the potentials for fast moving sources one obtains so called William-Weizacker fields.

We provide here an overview, the complete derivation is done in appendix A. The sources (embedded in the classical gluonic field) have momentum $P^{++}$ (with $k^+$ the momentum of the gluons). The sources are lightlike so $x^- = 0$; the sources “sit” at $x^- = 0$. Since they move (approximately) just in the $x^+$ direction the current also just has a $+$ component:

$$D_\nu F_{\alpha\mu}^\nu(x) = \delta^{\mu+}_\nu \rho_\alpha$$  \hspace{1cm} (4.7)

($\alpha$ denotes the color index of the potential due to the source). We assume $A^-$ to be zero (there are physical motivations for this explained in the appendix) and look for a consistent solution to the equation. We can derive the form of the field modulo the gauge freedom: $F^{ij} = 0$.

A field that fulfills these conditions is a so called “pure gauge”:

$$A^i = U \partial^i U^\dagger$$  \hspace{1cm} (4.8)

We have two field degrees of freedoms at this point: $U$ and $A^+$. By picking a gauge we can eliminate either of the two. For the derivation of the eigenstates of $\psi$ (see below) it will be convenient to work in the so called singular gauge ($U = 0$). However to calculate the fermionic propagator from the wavefunction $\psi$, it is more practical to work in the light cone gauge ($A^+ = 0$). Therefore we will provide as well a gauge transformation relating the two. Further it is explained in the appendix that the fields are approximated independent of $x_-$.

In the singular gauge it is shown that

$$\tilde{A}^\mu = \delta^{\mu+}_\nu \alpha^a(x_t)$$  \hspace{1cm} (4.9)

(the tilde is to denote we’re in singular gauge). For the solution in the light cone gauge we obtain the following solution for $U$:

$$U^\dagger(x_t) = \exp \left( ig \int_{-\infty}^{\infty} dz_- \alpha(z_-, x_t) \right)$$  \hspace{1cm} (4.10)

With the equation for the potential in the singular gauge we can construct the wave function $\psi$. The rough outline is that one starts with the free field equation for $z_- < 0$. Using the form of the potential as given above, by means of a Greens function we construct the wavefunction for $z_+ > 0$ from the free field spinor. With applying the boundary condition to the Greens function (it must vanish for $z_- < 0$) we obtain the solution. We can conveniently obtain the wavefunction in the lightcone gauge by simply gauge transforming our spinor with the appropriate gauge transformation. The full calculation can be found in appendix B, we state here just the part of the result for $z_- > 0$ in the light cone gauge:

$$\psi(x) = \frac{1}{\sqrt{2}} U(x_t) \int \frac{d^2 p_t}{(2\pi)^2} d^2 z_t U^\dagger(z_t) \exp \left( ip_t \cdot x_t - iq_--x_- \right) \exp \left( iz_t (q_t - p_t) \right) \exp \left( -i \frac{(p_t^2 + M^2 - \lambda) x_--}{2q_-} \right) \left[ 1 + \frac{\alpha q_t + \beta m}{\sqrt{2q_-}} \right] \alpha_- u(q)$$  \hspace{1cm} (4.11)
Now that we obtained the wavefunction we can construct the fermionic propagator in the background field. The propagator is given by the usual equation:

\[
S_A(x, y) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} \bar{\psi}(x) \psi(y)
\] (4.12)

Therefore we have to calculate \(\bar{\psi}\psi\). This is a laborious calculation (see appendix C). However (a.o. using the gauge invariance of the propagator to simplify the expression) we can obtain a relatively simple expression:

\[
S_A(x, y) = S_0(x-y) - i \int d^4z \left( \theta(x-) \theta(-y-)(V^\dagger(z_t) - 1) - \theta(-x-) \theta(y-)(V(z_t) - 1) \right) S_0(x-z) \gamma_- \delta(z-) S_0(z-y)
\] (4.13)

(in this equation \(S_0 = \int \frac{d^4q}{(2\pi)^4} \frac{-m}{q^2 + m^2 - i\epsilon} e^{iq(x-y)}\))

We see we have obtained the free field propagator with an additional, complicated term which depends on the gluonic field through \(V\). The Feynman diagrams for the old and new propagator are displayed in fig. 4.1.

![Figure 4.1: A diagrammatic representation of the propagator in eq. 4.13. We can separate the propagator in the free propagator plus one adapted term.](diagram.png)

Having obtained the expression for the propagator we can continue our computation of the hadronic tensor \(W^{\mu\nu}\). First we need to introduce a new term. In the computation of \(W^{\mu\nu}\) we'll encounter often terms like \(\frac{1}{N_c} \text{Tr}(U(x_t)U^\dagger(y_t))\). We define the Fourier transform of \(\gamma(x_t)\) as:

\[
\tilde{\gamma}(p_t) = \int d^2x_t \exp^{-ip_t \cdot x_t} (\gamma(x_t) - 1)
\] (4.14)

1Note however that to calculate the hadronic tensor use is made of various intermittent forms of the propagator and not only the fully simplified result.
(note that it follows that $\int \frac{d^4p}{(2\pi)^4} \tilde{\gamma}(p_t) = 0$). The index $\rho$ means we have to average this function over a distribution of sources. We can make various assumptions on this, we'll work with a Gaussian distribution. This is reasonable for very large nuclei or deep inelastic scattering, see [8].

We're now able to calculate the hadronic tensor as given in eq. 4.6. The calculation (in which a.o. the Landau Cutcosky rule is applied) is done in appendix D. The result is:

$$W_{\mu\nu}(x,Q^2) = \frac{-N_c \sigma P^+}{16\pi^2 m} \frac{1}{(q^2_{-})} \text{Im} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \int_{-\infty}^{-M_{p-q}^2} \frac{dp}{2\pi} \tilde{\gamma}(k_t)$$

\[4.15\]

in which $M_{p-q} = (p_t - q - t)^2 + M^2$. $I$ is defined as:

$$I(k_t, p_t q, p_+) = \frac{1}{p_+ - \frac{(M_{-q}^2 - M_{p-q}^2)}{2q_{-}}} \cdot \frac{1}{p_+ - \frac{M_{p-q}^2 - M_{p+k-q}^2}{2q_{-}}}$$

\[4.16\]

The above result is complicated. However, it was shown [11] that when we take the limit of $q^2 \rightarrow \infty$ and $M \rightarrow 0$ we obtain a very familiar leading twist result:

$$F_2(x,Q^2) = \frac{\sigma N_c}{2\pi^4} \int \frac{d^2k}{(2\pi)^2} \tilde{\gamma}(k_t) \frac{p_t^2 + k_t \cdot p_t}{k_t^2 + 2k_t \cdot t} \ln \left(\frac{(k_t + p_t)^2}{p_t^2}\right)$$

\[4.17\]

This result can be obtained by other means as well. We can construct a light cone number operator by constructing creation and annihilation operators (which work on the Fock space):

$$\frac{dN}{d^3k} = \frac{1}{(2\pi)^3} b_s^\dagger b_s$$

\[4.18\]

The creation and annihilation operators can be constructed from the field $\psi$ in the background field $A$ (which we already derived), see [11].

It is shown in a paper by Jaffe [9] that the distribution $\frac{dN}{d^3k}$ relates to $F_2$ (up to leading twist) via:

$$F_2(x,Q^2) = \int_0^{Q^2} dk_t^2 \frac{dN}{d^2k_t dx}$$

\[4.19\]

When inserting the proper form of the operators and working out the resulting equation we find again eq. 4.17. This confirms that a limit of the general formula (including all twists) reproduces a well known (leading twist) result.

### 4.2 The Structure Functions at Small $x$

The next step is to express the structure functions (which are experimental observables) in terms of the hadronic tensor $W^{\mu\nu}$. We note that on general arguments (Lorentz and gauge invariance and symmetry in $\mu, \nu$) we can express the hadronic tensor in terms of structure functions as:
\[ mW^{\mu\nu} = -(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2})F_1(x, Q^2) + (P^{\mu} - \frac{q^{\mu}(P \cdot q)}{q^2})(P^{\nu} - \frac{q^{\nu}(P \cdot q)}{q^2}) \frac{F_2(x, Q^2)}{P \cdot q}. \] (4.20)

Since we work in the infinite momentum frame, \( P^+ \to \infty \) and \( P^-, P^t \approx 0 \). In our framework \( q^+ = 0 \) as well. We can therefore contract both expressions with \( q^- \) and a normal vector \( n^\mu = \delta^\mu_+ \). We obtain:

\[ F_1(x, Q^2) = \frac{F_2(x, Q^2)}{2x} + \frac{q^2}{(q^-)^2} W^- \]
\[ \frac{F_2(x, Q^2)}{2x} = -\frac{(q^-)^2}{q^2} W^+ \] (4.21)

It is motivated in [10] that from these equations we can deduce the following equation for \( F_2(x, Q^2) \):

\[ F_2(x, Q^2) = \frac{Q^2\sigma N_c}{2\pi^3} \int_0^1 dz \int_0^{\Lambda_{QCD}} d^2 x_t (1 - \gamma(x_t, y)) \cdot \left[ K_0^2(x_t A)(4z^2(1 - z)Q^2 + M^2) + K_1^2(x_t A)A^2(z^2 + (1 - z)^2) \right] \] (4.22)

For the function \( \gamma(x_t, y) \) it is shown in [11] and [10] that (for a Gaussian distribution) it can be written as:

\[ \exp \left( -\frac{\alpha_s \pi^2}{2\sigma N_c^2} x^2 G(x, \frac{1}{x_t^2}) \right) \] (4.23)

In this equation \( xG \) is the gluon distribution multiplied by the Bjorken-\( x \) and \( x_t \) is the longitudinal separation between the two quarks of the dipole.

A saturation scale \( Q_s(x) \) is associated with eq. 4.23. It is defined as the virtuality for which the argument of the exponent becomes unit. Per \( x \) value \( Q_s(x) \) can be obtained by solving

\[ Q_s^2(x) = -\frac{\alpha_s(Q_s^2)\pi^2}{2\sigma N_c^2} x g(x, Q_s^2) \] (4.24)

The saturation scale indicates the cross over from a logarithmic to a power law distribution (see [12]).
5 The Dipole Model

5.1 Introduction

We start here with a description of the dipole model. This is, in principle, a general model into which various phenomenological dipole cross sections $N_{q\bar{q}}$ can be implemented and tested. We need to implement a phenomenological cross section for $N_{q\bar{q}}(x,r,b)$ (the QCD part of the interaction) since we can’t compute this cross section from theory in the non-perturbative regime. We discuss in this section the general aspects and kinematics of the dipole model. In section 6 we will discuss the IP-Sat model which is the phenomenological dipole cross section that we investigate in this thesis. We will show how to derive this cross section from the hadronic tensor derived in section 4.2.

Consider the process in which an electron interacts with the low $x$ content of the proton, which mainly consists of gluons. The virtual photon $\gamma^*$ (with virtuality $Q^2$) can fluctuate into a $q\bar{q}$ dipole. This colored object can interact with the gluonic content of the proton via color singlet exchange. After this interaction the dipole recombines to a virtual photon (or another well defined final state), see fig. 5.1.

A clear advantage of the dipole model is that it self consistently describes various experimentally accessible observables. At one hand it describes the reduced cross section $\frac{d\sigma}{dxdQ^2}$ and the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ (the first is a direct, unambiguous experimental observable. The reduced cross section and $F_2(x,Q^2)$ are kinematically related though). But at the other hand two experimental observables can be described as well: particular structure functions and exclusive diffractive processes. The particular structure functions describe inclusive processes in which for example minimally a $c\bar{c}$ is formed (completely independent of the final state). One can think also of different inclusive processes like $F_{b\bar{b}}(x,Q^2)$. An exclusive diffractive process is a process in which the initial state is an electron and proton and the final state after the collision is an electron, proton and a single additional particle. In the language of the dipole picture the virtual photon fluctuates in a dipole and the dipole deflects the proton via color singlet exchange. Afterwards the dipole fluctuates into a new particle. The particles we consider are vector mesons ($J/\psi, \rho$ and $\phi$) but the dipole can fluctuate into a real photon as well. The latter process is called deeply virtual Compton scattering.

Usually one fits the parameters of $N_{q\bar{q}}$ (to be discussed below) to $F_2(x,Q^2)$ or $\sigma_r$. Subsequently one uses the parameters to make predictions for the other processes. These predictions can then be tested to experimental data. In this way one can scrutinize the model to various data sets. We’ll follow this approach as well (we’ll use $\sigma_r$ to determine the parameters of $N_{q\bar{q}}$). Below we give descriptions for the structure functions and exclusive processes in terms of $N_{q\bar{q}}$. 

Figure 5.1: A diagrammatic representation of the dipole model. The distance between the photon and the proton is the impact parameter $b$. 

![Diagram of dipole model](image)
5.2 The Dipole Cross Section and the Structure Functions $F_2$ and $F_L$

The lifetime of the $q\bar{q}$ is much longer than the interaction time with the target. Therefore the cross section can be factorized as:

$$\sigma_{L,T}^{\gamma p}(Q^2, x) = 2 \sum_f \int \int d^2b d^2r \int_0^1 dz |\psi_f^{L,T}(r, z, Q^2)|^2 N_{q\bar{q}}(x, r, b)$$

(5.1)

In this equation $r$ is the size of the dipole, $z$ is the momentum fraction of the photon carried by one of the quarks and $b$ is the impact parameter between the photon and the proton. We integrate over all dipole sizes and impact parameters. $|\psi_f^{L,T}(r, z, Q^2)|^2$ is the product of the overlap function between the photon and the dipole (this can be directly computed from QED), see for example [13].

We can express the experimental observables $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in terms of $\sigma_{L,T}^{\gamma p}(x, Q^2)$ as:

$$F_2(Q^2, x) = \frac{Q^2}{4\pi^2\alpha_{EM}} \left[ \sigma_{L}^{\gamma p}(x, Q^2) + \sigma_{T}^{\gamma p}(x, Q^2) \right]$$

(5.2)

$$F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{EM}} \left[ \sigma_{L}^{\gamma p}(x, Q^2) \right]$$

(5.3)

The reduced cross section $\sigma_r(x, y, Q^2)$ is a directly experimentally accessible observable and given by:

$$\sigma_r(x, y, Q^2) = F_2(x, Q^2) - \frac{y^2}{1 + (1 - y)^2} F_L(x, Q^2)$$

(5.4)

(in which $y = Q^2/(sx)$). We can use these equations to construct $F_2^{c\bar{c}}(x, Q^2)$ as well by just using charm quarks in the equations above (instead of summing over all flavours).

The photon dipole overlap functions $|\psi_f^{L,T}(r, z, Q^2)|^2$ are given by:

$$|\psi_f^{T}(r, z, Q^2)|^2 = \frac{\alpha_{EM} N_c}{2\pi^2} \sum_f \epsilon_f^2 \left\{ \epsilon_f^2 [K_1(\epsilon_f r)]^2 [z^2 + (1 - z)^2] + m_f^2 [K_0(\epsilon_f r)]^2 \right\}$$

(5.5)

$$|\psi_f^{L}(r, z, Q^2)|^2 = \frac{\alpha_{EM} N_c}{2\pi^2} \sum_f \epsilon_f^2 \left\{ 4Q^2 z^2 (1 - z)^2 [K_0(\epsilon_f r)]^2 \right\}$$

(5.6)

in which $\epsilon_f$ is defined as:

$$\epsilon_f^2 = z(1 - z)Q^2 + m_f^2$$

(5.7)

Note we used the same mass parameter for the $u$, $d$ and $s$ quark throughout this thesis.

5.3 Exclusive Diffractive Processes and Deeply Virtual Compton Scattering

We briefly outline here how we can calculate various other processes from our model using the optical theorem. The optical theorem relates the amplitude of the total cross section to exclusive forward processes (for a good derivation of this application of the optical theorem see for example [7]). We obtain:
\[
\frac{d\sigma_{\gamma^p T,L}^{*p}}{dt} = \frac{1}{16\pi} |A_{\gamma^p T,L}^{*p \rightarrow Ep}|^2 (1 + \beta^2)
\] (5.8)

With the scattering amplitude:

\[
A_{\gamma^p T,L}^{*p \rightarrow Ep} = 2i \sum_f \int \int d^2b d^2r \int_0^1 dz (\psi^*_E \psi)_{L,T} \exp \left( i[b - (1 - z)r] \cdot \Delta \right) N_{qq}(x, r, b)
\] (5.9)

In this equation $\Delta^2 = -t$. The factor $(1 + \beta^2)$ is a correction term, compensating for the fact that the dipole S matrix possibly isn’t purely real. For a more elaborate discussion we refer to [4]. Also a discussion on the skewedness effect is included there (a phenomenological correction to the gluon distribution). The phase factor $\exp \left( i[b - (1 - z)r] \cdot \Delta \right)$ is for the non-forward part of the wavefunctions. Effectively $\exp \left( i[b - (1 - z)r] \cdot \Delta \right)$ and $\exp \left( i[b - (1 - z)r] \cdot \Delta \right)$ take into account the size of the vector mesons in impact parameter and dipole transverse size. These were first introduced in [5].

For the real photon the overlap function is well defined from QED and has only a transverse component. For the photon vector meson overlap functions various solutions exist however, see for example [14]. There are indications that vector mesons can rather be described by Gaussian fluctuations in the transverse plane than with Bessel functions. Therefore we used the so called Boosted Gaussians [14]. The parameters of this model are fixed by a normalization condition and the leptonic decay width of the vector meson.
6 The IP-Sat Model

In this section we start to discuss the model for $N_{q\bar{q}}$ that we investigated. If we use the model in a different way than was done previously we’ll point this out.

Combining eq. 5.2 with eq. 4.22 we obtain the following model for $N_{q\bar{q}}$, based on the theoretical framework of saturation:

$$N_{q\bar{q}}(x, r) = 1 - \exp \left( -\frac{\pi^2 r^2}{2 N_c} \alpha_s(\mu^2) x g(x, \mu^2) \right)$$  \hspace{1cm} (6.1)

In this equation $r$ is the dipole size (as discussed above), $\alpha_s$ is the strong coupling and $x g$ is the gluon distribution function (evaluated at $x$ and the scale $\mu^2$) multiplied by Bjorken $x$. In this equation the scale $\mu$ is related to the dipole size $r$ via

$$\mu^2 = \frac{C}{r^2} + \mu_0^2$$  \hspace{1cm} (6.2)

(In principle $C$ is a free parameter of the model. We investigated its dependency and came to the same conclusion as previous research that $C=4$ GeV is an optimal value, which is used throughout this thesis.) We have to make an educated guess for the gluon distribution function at the initial scale $\mu_0$. We used the following form (based on the parameterization in [5], which is based on experimental observations):

$$x g(x, \mu_0^2) = A_g x^{-\lambda_g} (1 - x)^{5.6}$$  \hspace{1cm} (6.3)

Further for $x$ we use the Bjorken $x$ for the light quarks ($u, d, s$) and $x = x(1 + 4 m^2_c/Q^2)$ for the charm quark (the reason for this is that it allows for a smooth evolution, it was first applied when the DGLAP evolution was performed with a Mellin transformation).

This gluon distribution function (as well as the strong coupling) has to be evolved to a scale $\mu^2$. The evolution of the gluon distribution function was performed by using the LO DGLAP equation (we neglect coupling to quarks for the low-$x$ gluon distribution):

$$\frac{\partial x g(x, \mu^2)}{\partial \ln(\mu^2)} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 dz P_{gg}(z) \frac{x}{z} g(\frac{x}{z}, \mu^2)$$  \hspace{1cm} (6.4)

We used a fixed number of flavor evolution scheme with $N_f = 4$ since we sum over $u, d, s$ and $c$ quarks (previous research [5] used $N_f = 3$). For both the gluon evolution as the strong coupling a value of $\Lambda_{QCD} = 0.156$ GeV was used (as determined from experimental measurements of $\alpha_s$ at the $Z_0$ mass; in [5] a value of 0.2 GeV was used).

However eq. 6.1 has no dependence on the impact parameter (and the dependence doesn’t follow from the derivation in section 4.1). This dependence has to be put in “by hand”. Below we insert a thickness function depending on the impact parameter $b$ in the model:

$$N_{q\bar{q}}(x, r, b) = 1 - \exp \left( -\frac{\pi^2 r^2}{2 N_c} \alpha_s(\mu^2) x g(x, \mu^2) T_G(b) \right)$$  \hspace{1cm} (6.5)

We used a Gaussian for the thickness function, normalized at unity:

$$T_G(b) = \frac{1}{2\pi B_G} \exp\left(-\frac{b^2}{2B_G^2}\right)$$  \hspace{1cm} (6.6)
\( B_G \) is a free parameter. It is related to the Hofstadter proton radius (named after the researcher who measured it). The latter however is an electromagnetic quantity while \( B_G \) is rather related to the gluonic radius of the proton. Other forms can (and have) been considered though, for example a step function [5].

This particular choice for \( N_{q\bar{q}} \) is called the impact-parameter dependent saturation model (IP-Sat model). Overall it has four free parameters: \( A_g, \lambda_g, B_G \) and the initial scale \( \mu_0 \). We’d like to point out some other models were used prior to the IP-Sat model. The so-called GBW model describes \( N_{q\bar{q}} \) (integrated over the impact parameter) using the dipole radius and the saturation scale \( Q_s \):

\[
N_{q\bar{q}}(x,r,b) = \sigma_0(1 - \exp \left( -r^2Q_s^2(x)/4 \right)) \tag{6.7}
\]

(\( \sigma_0 \) is a constant). Also a so-called CGC-sat (color glass condensate saturation) model was developed, we refer to [5] for a more detailed description.

6.1 Results

As a test of our model we first reproduced the parameters given by [5] by fitting the old \( F_2(x,Q^2) \) data. Next we confronted the model with the data on the reduced cross section from the new combined data set. We successfully fitted \( \sigma_r \). The emphasis in this thesis will be on the analysis of the gluon distribution we extracted from \( \sigma_r \) and the predictions for the structure functions \( F_2(x,Q^2) \) and \( F_{c\bar{c}}(x,Q^2) \). Aside we include data and predictions on exclusive \( J/\psi \) production as a “proof of principle”. In our paper ([4]) we include as well data and successful predictions for \( F_L(x,Q^2) \) (using the old Hera data), deeply virtual Compton scattering and exclusive \( \phi \) and \( \rho \) vector meson production.

6.1.1 The Gluonic Distribution Function

The parameters of the gluonic distribution function were determined by fits to \( F_2(x,Q^2) \) in [5]. The parameter \( B_G \) was determined separately though by fitting the slope of the \( t \) distribution of \( J/\psi \) mesons and a value of 4 GeV was found (as a reference we put these values in table 6.1). We reproduced the parameters that were quoted and the value of \( B_G \). The latter we used for all our results.

A difference between the approach from [5] and our approach is that we determined our parameters by a fit to the reduced cross section of the combined high precision data set from H1 and Zeus [?]? and used the parameters to make predictions for a.o. \( F_2(x,Q^2) \). The reason for this is that the \( \sigma_r \) data are unbiased w.r.t. theoretical assumptions needed to extract the structure functions from the data.

We treated the systematic and statistical errors as independent. An eventual Z boson contribution to the reduced cross section is neglected since it only becomes important at very high \( Q^2 \) values. We point out that the errors in the data are of the order of a percent and subsequently we demanded a high level of precision in the numerical evaluation of our model. Therefore we parallelized our code and analyzed the data on a cluster.

We applied a cut on the \( Q^2 \) values of the data points: \( Q^2_{min.} \leq Q^2 \leq 650 \text{ GeV}^2 \) and \( x \leq 0.01 \). We investigated how the fit parameters and the fit quality behaved as function of \( Q^2_{min.} \), see fig. 6.1. We see the fit quality deteriorates for \( Q^2_{min.} < 0.75 \text{ GeV}^2 \) and stabilizes
for higher values. The gluonic parameters saturate as well from this value on. Therefore we took $Q_{\text{min.}}^2 = 0.75 \text{ GeV}^2$ (note in [5] a lower bound of $Q_{\text{min.}}^2 = 0.25 \text{ GeV}^2$ was applied). We investigated as well the dependence of the fit quality on the $u,d,s$ and charm mass. The results are displayed in fig. 6.2. Clearly the data prefer very light masses for the $u,d$ and $s$ quarks, which is theoretically appealing. With a fit, in which $m_u = m_d = m_s$ was a free parameter, a mass of $\approx 10^{-4} \text{ GeV}$ was found. Analyses of previous data required a much higher mass (50 to 140 MeV, see [5]) in contrast.

![Graph](image)

Figure 6.1: The parameters of the IP-Sat model as function of $Q_{\text{min.}}^2$. In the right upper plot we display the corresponding overall reduced $\chi^2$.

On this $Q^2$ range we were successfully able to fit $\sigma_r$. We used two values for the charm mass, see table 6.1 (we also display the results of [5] here). The errors in the fit parameters are negligibly small (less than a percentage). The errors displayed in this thesis for various data are due to the freedom in choosing the charm mass ($1.27 \leq m_c \leq 1.4 \text{ GeV}$). We see the values for the reduced $\chi^2$ for the new data are slightly worse than the previous values of [5] but these are still of good quality. We tested the old parameter values of [5] as well on the new data and found a reduced $\chi^2$ of 3.19. The earlier fit results should therefore be superseded by the new fit results in table 6.1.
In fig. 6.3 we display the gluon distribution function we obtained for various values of $Q^2$. These were plotted together with the same distributions as extracted by the CT10 [15] and MSTW [16] collaborations for comparison. These collaborations used a leading twist collinear factorization approach with NNLO DGLAP evolution. The uncertainties in the CT10 and MSTW distributions come from uncertainties in obtaining a fit from a global analysis while the error band in the IP-Sat parameterization is due to the freedom in choosing a charm quark mass. At large virtualities we are in the color transparency region (for a discussion on color transparency see [17]) where saturation has little effect and we see only small differences between the IP-Sat and the standard perturbative predictions. The differences are mainly caused by the fact that in the IP-Sat model LO DGLAP evolution without quarks was applied, while the other predictions include quark degrees of freedom. For low virtualities at low $x$ we see the IP-Sat model gives different and much more stable results than CT10 and MSTW. Here the higher twist contributions (which are included in the IP-Sat model) prove to be important in extracting the gluonic parameters.

Since we have successfully obtained a parameterization of the gluon distribution we can apply eq. 4.24 to determine the saturation scale. The saturation scale is by definition dependent on $x$ but in our model it has as well a dependence on $b$. In fig. 6.4 we display the saturation scale as function of $1/x$ for a few impact parameters (obtained with the fit parameters from table 6.1) together with the scale as determined previously by [5]. We see the old and new scales are consistent with each other (although the new scale seems a bit lower for the lowest $x$ values). In the right hand plot we see the saturation scale as function of the impact parameter $b$ for a few $x$ values for the parameters of table 6.1.

Table 6.1: The parameters of the initial gluon distribution. In the first row are the parameters from [5] fitted to the Zeus data set ([2], [3]). As cuts $x \leq 0.01$ and $0.25 \leq Q^2 \leq 650$ GeV were applied. We confirmed these parameters on the given data set. In the second and third row are the values obtained from our fit to the combined data set ([?]). As cuts $x \leq 0.01$ and $0.75 \leq Q^2 \leq 650$ GeV were applied.

<table>
<thead>
<tr>
<th>Data</th>
<th>$B_G$/GeV</th>
<th>$m_{u,d,s}$/GeV</th>
<th>$m_c$/GeV</th>
<th>$\mu_0^2$/GeV$^2$</th>
<th>$A_g$</th>
<th>$\lambda_g$</th>
<th>$\chi^2$/d.o.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_2$ ([2], [3])</td>
<td>4</td>
<td>0.14</td>
<td>1.4</td>
<td>1.17</td>
<td>2.55</td>
<td>0.02</td>
<td>193/160</td>
</tr>
<tr>
<td>$\sigma_r$ ([?])</td>
<td>4</td>
<td>$\approx0$</td>
<td>1.27</td>
<td>1.51</td>
<td>2.308</td>
<td>0.058</td>
<td>298.89/259</td>
</tr>
<tr>
<td>$\sigma_r$ ([?])</td>
<td>4</td>
<td>$\approx0$</td>
<td>1.4</td>
<td>1.428</td>
<td>2.373</td>
<td>0.052</td>
<td>316.61/259</td>
</tr>
</tbody>
</table>
The bands represent the uncertainty in the choice of $m_c$. We observe that the saturation scale grows faster with $1/x$ for central than peripheral collisions and can differ up to an order of magnitude between different impact parameters. This is non-trivial and clearly indicates the importance of the impact parameter dependence on the saturation scale.

6.1.2 Tests of the Model and the Obtained Parameters

With the parameters (see table 6.1) we obtained by fitting $\sigma_r$ (and the $t$-distribution of the $J/\psi$ distribution for obtaining $B_G$) we can make predictions for $F_2(x, Q^2)$ (see fig. 6.5) and $F_{c\bar{c}}(x, Q^2)$ (see fig. 6.6). For these predictions we used the high precision data from the combined Hera-Zeus set [?]. The data points for $F_{c\bar{c}}(x, Q^2)$ were obtained from $\sigma_r^c$ under the assumption that the contribution of $F_L^{c\bar{c}}$ to the reduced cross section is negligible in the range considered here. For predictions to the old Zeus $F_L(x, Q^2)$ data we refer to [4]. The predictive power for describing the data is striking. Also for $x$ values up to $10^{-1}$ the predictions for the $F_2(x, Q^2)$ data remain good. We have extended our predictions beyond the range of the data as a future reference.
Figure 6.4: The saturation scale as function of $1/x$ for the gluon distribution function as extracted from the old (in red) an new data set (see table 6.1). Right: the saturation scale as function of $b$ for various $x$ values. The lower and upper curves of the bands in the plots represent the parameterization with $m_c$ 1.27 GeV and 1.4 GeV respectively.
Next we confront our model with exclusive $J/\psi$ production as function of $t$, $Q^2$ and $W$ in fig. 6.7, 6.8 left and 6.8 right respectively, for the parameter sets belonging to the two values of $m_c$. The data are from both the Hera and Zeus collaboration (but not from an overall combined data set as with the $F_2(x,Q^2)$ data). For the $W$ distribution we integrated over $t$ up to 1 GeV. It is observed the agreement between data and predictions is excellent. The same analysis is applied to exclusive $\rho$ and $\phi$ production and deeply virtual Compton scattering in [4]. Generally it was observed for these processes as well that our predictions match the data well.

Figure 6.5: The $F_2(x,Q^2)\times 2^i$ data from the high precision combined data set [?] with the prediction for the parameter set from table 6.1 with $m_c=1.27$. To separate the data points and the model for each $Q^2$ value we multiplied the results by a factor $2^i$. 

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Figure 6.6: The $F_{cc}(x, Q^2)$ data from the high precision combined data set [?] with the prediction for the parameter set from table 6.1. The solid and dashed line represent $m_c=1.27$ and 1.4 GeV respectively. It is assumed $\sigma_{cc} \approx F_{cc}$ (see main text).

Figure 6.8: Left: the differential $J/\psi$ cross section as function of $Q^2 + M_{J/\psi}^2$. The data are from the H1 and Zeus collaboration ([18] to [22]). The solid and dashed lines represent the model for the parameter set from table 6.1 with $m_c=1.27$ and 1.4 GeV respectively. The ZEUS data points are scaled to the H1 $Q^2$ bins using the $Q^2$ dependence measured by ZEUS of the form $\sigma \propto (Q^2 + M_{J/\psi}^2)^{-2.44}$.
Figure 6.7: The differential $J/\psi$ cross section as function of $t$. The data are from the H1 and Zeus collaboration ([18] to [22]). The solid and dashed lines represent the model for the parameter set from table 6.1 with $m_c = 1.27$ and 1.4 GeV respectively.
7 Conclusion and Applications

7.1 Conclusion

In this thesis we successfully tested the validity of the IP-Sat model to the new HERA combined high precision data set [1]. The data in this set are significantly more precise than the Zeus data set ([2] and [3]) on which the model was tested previously [5].

We successfully fitted $\sigma_r$ with our model which determined the three free parameters of the gluon distribution and showed the old parameter set has to be superseded with the parameters as extracted from the new data set. We applied a cut on the data points and showed lowering the lower cutoff on $Q^2$ below 0.75 GeV$^2$ deteriorates the fit quality. The fits preferred light $u,d,s$ quarks and we could make successful fits for $1.27 \leq m_c \leq 1.4$.

The obtained gluon distribution function provides a non-trivial insight in the saturation scale and its dependency on $b$; the saturation scale might be an order of magnitude higher for central than for peripheral collisions. We used our obtained gluon distribution functions to make predictions for the structure functions $F_2(x,Q^2)$ and $F_{c\bar{c}}(x,Q^2)$. Our predictions were shown to match the high precision data very well.

Also we compared our model with data (from both the H1 and Zeus experiment, see [18] to [22]) on exclusive $J/\psi$ production as function of $t$, $Q^2$ and $W$. The $W$ dependence shows the energy dependence is modeled well while the $t$ dependence suggests the shape of the proton is correct. The $Q^2$ dependence indicates the virtuality dependence of the gluon distribution, and its evolution via the DGLAP formalism with the transverse dipole size, are modeled correctly. We point out that only $B_G$ was fixed for describing the exclusive processes. With this parameter we could describe simultaneously the exclusive production of $J/\psi$, $\rho$ and $\phi$ (as function of various kinematical variables) and deeply virtual Compton scattering (for the latter see [4]), while the wavefunctions for these are very different. This strongly suggests the extracted impact parameter dependent distribution is universal.

We’d like to propose a suggestion for future improvements still. We have not included $b$ quarks in our analysis. However it appears these have a non-negligible contribution on at least half of the kinematic range (in $x$) considered here, see fig. 7.1 left. It would be interesting to include $b$ quarks in our fit to the reduced cross section (perhaps this influences the boundaries as well on which we can successfully fit the data and the overall fit quality). Further we could compare predictions for $F_{b\bar{b}}$ with experimental data, see fig. 7.1 right. This would provide another cross check of the validity of the IP-Sat model.

7.2 Applications

We outline a few applications of the IP-Sat model in hadronic physics. One example is the so-called IP-Glasma model (glasma is a contraction of glass (which stems from the theory of the Colored Glass Condensate) and plasma, the remnant of a heavy ion collision). The IP-Glasma models the initial state of a high energy heavy ion collision. It makes use of the superposition principle and distributes nucleons according to the Wood-Saxon distribution in the ion, see fig. 7.2 from [23]. The gluonic content of the nucleons is described by the parameterization we determined in this thesis. The IP-Glasma has so far made successful predictions for the coefficients of harmonics in the momentum distributions of particles in heavy ion collisions (the second coefficient describes the so-called well known
Figure 7.1: Left: the contributions of charm ($f_{cc}$) and beauty ($f_{bb}$) to the total cross section, as function of $Q^2$ for various $x$ values. The inner error bars indicate the statistical error. The statistical and systematic errors are added in quadrature and represented by the outer error bar. The prediction is for NNLO QCD. The charm data point at $x = 0.005$ and $Q^2 = 300$ GeV$^2$s has been interpolated from $x_s = 0.008$. Figure from [26]. Right: The structure function $F_{2}^{bb}$ as a function of $Q^2$ for various $x$ values. The inner error bars indicates the statistical error. The statistical and systematic errors added in quadrature are represented by the outer error bar. The predictions of QCD calculations are also shown (note we haven’t made any predictions with the IP-Sat model for these data yet). Some points have been interpolated in $x$ for visual clarity. Figure from [26].

elliptic flow). Since we showed the existing parameterization has to be superseded with the parameters we determined in this thesis our results directly will contribute to future IP-Glasma predictions.

Considerable efforts are ongoing for the design of a future EIC (an Electron Ion Collider), both an LHeC [24] and eRHIC [25] are considered as upgrades of existing experiments. A key observable in such an experiment is the saturation scale as discussed and predicted in this thesis. It would be experimentally accessible at such a collider due to the so called nuclear “oomph” factor, which increases the scale according to $(Q^A_s)^2 \approx A^{1/3}(Q^p_s)^2$. For the nucleus of a gold atom this factor would be approximately 6.
The dipole model applies directly to $eA$ scattering as well and currently predictions on the possibilities of the physics program of such an experiment are investigated, see a.o. [24] and [25]. Our newly found parameterization of the gluon distribution function will directly input these predictions. We look forward to see the impact of our parameters on predictions related to $p+p$, $p+A$ and $A+A$ collisions.

Figure 7.2: The initial condition of a heavy ion collision by the IP-Glasma model. Figure from [23].
8 Acknowledgement

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References


Appendix A  The Form of the Classical non-Abelian Gauge Field

The starting point of the derivation is the equation motivated on physical grounds in the main text:

\[ D_\nu F^\nu\mu(x) = J^\mu = \delta^{\mu+} \rho_\alpha. \]  

(A.1)

In this equation and the subsequent derivation the source can have any spread in the \( x_- \) direction. After providing a general solution, we will motivate (on physical arguments) a solution that has a very simple \( x_- \) dependence.

We will throughout the calculation absorb the non-Abelian generators of the QCD field in the field: \( A = A_\alpha \tau_\alpha \). This equation is in this form only correct if we assume \( A^- = 0 \). We try to obtain a self consistent solution for this assumption (otherwise the source gets rotated by Wilson lines build from \( A^- \)). We will prove below this equation leads to \( F^{ij} = 0 \) and that \( A^+, A^{ij} \) are static (i.e.independent of \( x_+ \)).

We stress the importance of paying attention to the location of the + or – index, since \( A^- = A_+ \). We start with considering the continuity equation:

\[ D_\mu J^\mu = 0 \]  

(A.2)

But \( J^\mu \) has only a + component, so we obtain:

\[ (\partial_+ + A_+) J^+ = 0 \]  

(A.3)

Observe that we can write \( A_+ = A^- = 0 \). We therefore conclude that \( J^+ \) is static (independent of \( x_+ \)), and therefore so must be the fields. The fields from here on will have only a dependence on \( x_- \) and \( x_i \) but we’ll suppress the argument mostly for notational convenience. We investigate one of the inhomogeneous Maxwell equations, for the \( i^{th} \) component of the source (note that usually we indicate the \( i^{th} \) components with index \( t \) conform the light cone notation, but for some applications here \( i \) and \( j \) are more convenient):

\[ D_\nu F^\nu i = 0 \]  

\[ D_\nu F^{i\nu} = D_- F^{-i} + D_+ F^{+i} + D_j F^{ji} \]  

(A.4)

Note that \( D_+ = D^- = \partial^- - A^- \). This vanishes. The terms arising in \( D_- F^{-i} \) vanish similarly. Therefore we are left with:

\[ D_j F^{ji} = 0 \]  

(A.5)

which gives us \( F^{ij} = 0 \). A solution to this equation is given by \( A^i = \frac{1}{g} U \partial^i U^\dagger \) where \( U \) is an element of the SU(3) gauge group.

We have two independent degrees of freedom for our gauge field now, \( A^+ \) and \( U \). We can eliminate either of the two by picking a gauge. Consider:

\[ \partial_\mu A^\mu = 0 \]  

(A.6)
This either implies \( \partial_t A^t = 0 \) or \( U = 0 \). For the first we obtain the field in the singular gauge:
\[
\tilde{A}_a^\mu = \delta^{\mu+} \alpha_a
\] (A.7)

In this equation \( \alpha_a \) is related to the charge via:
\[
-\nabla_2^2 \alpha_a = \tilde{\rho}_a
\] (A.8)

The solution in the light cone gauge can be obtained from the solution in the singular gauge. We transform the solution with a gauge transformation \( U \) s.t. \( \tilde{A}^+ = 0 \):
\[
A^\mu = U (\tilde{A}^\mu + \frac{i}{g} \partial^\mu) U^\dagger
\] (A.9)

We can invert the last equation to obtain (note the dependence on \( x_- \)):
\[
U^\dagger(x_-, x_t) = \exp \left( ig \int_{-\infty}^{x_-} dz_- \alpha(z_-, x_t) \right)
\] (A.11)

(If the lower limit in the integral is chosen to give a “retarded” boundary condition, see [8]).

However for the scattering model we’ll investigate we consider the sources to sit in \( x_- = 0 \) (see the main text). A more precise analysis reveals that this is not completely accurate and actually the source has support at positive \( x_- \) s.t. \( 0 \leq x_- \leq 1/k_+ \). The result is that the gauge field is independent of \( x_- \) everywhere except on \( 0 \leq x_- \leq 1/k_+ \), see fig. A.1. Therefore in the derivations in the main text we’ll make the following approximation:
\[
A^t(x_-, x_t) \approx \theta(x_-) \frac{ig}{g} V(\partial^t V^\dagger \equiv \theta(x_-) A^t(x_t)
\] (A.12)

In which the following form of \( V \) is implied:
\[
V^\dagger(x_t) = \exp \left( ig \int_{-\infty}^{\infty} dz_- \alpha(z_-, x_t) \right)
\] (A.13)

For \( x_- > 1/k_+ \) we have \( U(x_-, x_t) = V(x_t) \), and in the derivation in the main text we will work with eq. A.12 for the gauge rotation (which is approximated independent of \( x_- \)).

Appendix B The Fermionic field in the Background of the Classical non-Abelian Gauge Field

In this appendix we derive the form of the Fermionic wave function in the classical non-Abelian background field \( A \). The form of \( A \) was derived in appendix A in two gauges.
B.1 Inserting the Potential in the Dirac Equation

Our calculation starts with picking the gauge field in the Singular Gauge (see eq. A.7):

$$\tilde{A}_a^\mu = \delta^{\mu+} \alpha_a$$  \hspace{1cm} (B.1)

We note that the charge distribution “sits” in $\delta(x_-) = 0$. It can be shown (see for example [27]) that in the singular gauge the potential itself has this $\delta$ function as well. We will provide a derivation for the wavefunction assuming the following form of the potential, conform [27] and [28]:

$$A^+ = \delta(x_-) P^+ \Lambda \equiv V$$  \hspace{1cm} (B.2)

In this $P^+$ is a projection operator: $P^+ = P_+ = 1 - \alpha_z$.

We start with the Dirac equation rewritten in terms of $\alpha$ matrices ($\beta$ has the conventional meaning of $\gamma^0$, see appendix E):

$$(\alpha_t p_t + \beta m - V - E)\psi = 0$$  \hspace{1cm} (B.3)

We will solve the Dirac equation by using a Greens function. We start by considering the solution for $x_- < 0$, which is given by plane waves times the free spinor:

$$(\alpha_t p_t + \beta m - E)\phi = 0$$  \hspace{1cm} (B.4)

For future reference we give the free electron spinor $u$ in light cone coordinates:

$$\phi = u(q) \exp(i(q_t x_t - q_+ x_+ - q_- x_-))$$  \hspace{1cm} (B.5)

So we obtain:

$$(\alpha_t p_t + m - E)\psi = V\psi + (\alpha_t p_t + \beta m - E)\phi$$  \hspace{1cm} (B.6)
The Greens function $G_0$ for the free field spinor is well known:

$$G_0(r, r') = \frac{1}{(2\pi)^4} \int d^4k (\alpha_t k_t + \beta m + k_0) \exp(ik \cdot (r - r')) \frac{\exp(ik \cdot (r - r'))}{k^2 + m^2}$$  \hspace{1cm} (B.7)

For future purposes we rewrite the Greens function in light cone coordinates:

$$G_0(r, r') = \frac{1}{(2\pi)^4} \int dk_+ dk_- dk_t (\alpha_t k_t + m + k_0) \cdot \exp(-i(k_+ \cdot (r - r') - k_- \cdot (r - r')_+ - k_t \cdot (r - r')_t))$$

$$-2k_+ k_- + k_t^2 + m^2$$  \hspace{1cm} (B.8)

For $x_- < 0$ the wavefunction is given by $\phi$. For $x_+ > 0$ we construct the wavefunction using a Greensfunction:

$$\psi = \theta(-x_-) \phi + \theta(x_-) \int d^4z G_0(x, z) V(z) \psi(z)$$  \hspace{1cm} (B.9)

### B.2 Obtaining an Effective Potential

Our potential is given in eq. B.2. However before proceeding we will rewrite it, using perturbation theory. We will solve the time dependence of equation B.3 using the Schrodinger equation:

$$i\frac{\partial \psi}{\partial t} = H \psi = \left( H_0 + \delta(z - t)(1 - \alpha_z)\Lambda \right) \psi$$  \hspace{1cm} (B.10)

(we have written out $x_-$ explicitly for reasons that will become clear shortly). We solve the time dependence in the usual way, i.e. by inserting

$$\psi_j(r, t) = \sum_k \alpha^j_k(t) \phi_k(r) \exp(-iE_k t).$$

However we just insert this in the left hand side of B.10. Upon closing the equations at both sides with $\phi_f$ and doing the spatial integration we obtain:

$$i\frac{\partial \alpha^j_f(t)}{\partial t} = -i \exp(iE_j t) \langle \phi_f | \delta(z - t)(1 - \alpha_z)\Lambda | \psi^j(r, t) \rangle$$  \hspace{1cm} (B.11)

Next we consider again eq. B.10, but first multiply both sides with the projection operator $1 - \alpha_z$ (remembering $(1 - \alpha_z)^2 = 1 - \alpha_z$). In this way we can obtain a solution for $(1 - \alpha_z)\psi$ which we just need in eq. B.11:

$$i\frac{\partial \psi}{\partial t} = (1 - \alpha_z) \left( H_0 + \delta(z - t)\Lambda \right) \psi$$  \hspace{1cm} (B.12)

We express both sides of the equation in light cone coordinates ($x_- = \frac{1}{\sqrt{2}}(t - z)$ and $x_+ = \frac{1}{\sqrt{2}}(t + z)$):

$$i\frac{\partial \psi}{\partial x_-} = (1 - \alpha_z) \left( H_0 + \delta(x_-)\Lambda \right) \psi$$  \hspace{1cm} (B.13)

Integrating both sides over $x_-$ we obtain the following solution for $(1 - \alpha_z)\psi$:

$$(1 - \alpha_z)\psi^j(r, t) = (1 - \alpha_z) \exp \left(-i\theta(t - z)\Lambda \right) \phi^j(r) \exp(-iE_j t)$$  \hspace{1cm} (B.14)
Note we used the boundary condition $\psi^j(r, t \to -\infty) = \phi^j \exp(-iE_j t_{\rightarrow -\infty})$. We substitute B.14 into B.11, and integrate both sides over $t$:

$$\alpha^j_f(t \to \infty) = \delta_{fi} - i \int dt \langle \phi_f | \delta(z - t) \exp \left(i(E_f - E_i) t - i\theta(z - t)\Lambda \right) (1 - \alpha_z) \Lambda | \phi_j \rangle$$  \hspace{1cm} (B.15)

The integration of the right hand side of eq. B.15 requires some attention. In some of the next steps we'll have to integrate by parts and therefore we rewrite the $\delta$ function as a derivative of the $\theta$ function. Ignoring $\delta_{fi}$ for now:

$$\alpha^j_f(t \to \infty) = \int dt \exp \left(i(E_f - E_i) t\right) \frac{\partial}{\partial t} \langle \phi_f | \exp \left(-i\theta(z - t)\Lambda \right) (1 - \alpha_z) \Lambda | \phi_j \rangle$$  \hspace{1cm} (B.16)

Next change variables to $u = t - z$:

$$\alpha^j_f(t \to \infty) = \exp \left(i(E_f - E_i) z\right) \int du \exp \left(i(E_f - E_i)u\right) \frac{\partial}{\partial u} \langle \phi_f | \exp \left(-i\theta(u)\Lambda \right) (1 - \alpha_z) \Lambda | \phi_j \rangle$$  \hspace{1cm} (B.17)

For notational convenience rewrite $\langle \phi_f | \exp \left(-i\theta(u)\Lambda \right) (1 - \alpha_z) \Lambda | \phi_j \rangle = g(u)$:

$$\alpha^j_f(t \to \infty) = \exp \left(i(E_f - E_i) z\right) \int du \exp \left(i(E_f - E_i)u\right) \frac{\partial}{\partial u} g(u)$$  \hspace{1cm} (B.18)

Split the integration region in $(-\infty, 0)$ and $(0, \infty)$:

$$\alpha^j_f(t \to \infty) = \exp \left(i(E_f - E_i) z\right) \left[ \int_{-\infty}^0 du \exp \left(i(E_f - E_i)u\right) \frac{\partial}{\partial u} g(u) + \int_0^\infty du \exp \left(i(E_f - E_i)u\right) \frac{\partial}{\partial u} g(u) \right]$$  \hspace{1cm} (B.19)

Observe that the derivative gives $\delta(u)$, and we can therefore change the boundaries to an infinitesimal $\epsilon$. Next perform integration by parts. We work out the $\int_{-\epsilon}^0$ term.

$$\int_{-\epsilon}^0 du \exp \left(i(E_f - E_i)u\right) \frac{\partial}{\partial u} g(u) = \left[ \int_{-\epsilon}^0 g(u) \right] - \int_{-\epsilon}^0 du i(E_f - E_i) \exp \left(i(E_f - E_i)u\right) g(u)$$  \hspace{1cm} (B.20)

The integrand in the second term doesn’t contain singularities. In the limit $\epsilon \to 0$ therefore it disappears. For the $\int_0^\epsilon$ term we obtain a similar term. Added up we obtain:

$$\alpha^j_f(t \to \infty) = \delta_{fi} + \exp \left(i(E_f - E_i) z\right) \left[ \int_{-\epsilon}^\epsilon \exp \left(i(E_f - E_i)u\right) g(u) \right]$$  \hspace{1cm} (B.21)

Filling in the boundaries, the definition of $g(u)$ and using the properties of the step function we obtain:

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\[ \alpha_j^f(t \to \infty) = \delta_{f_i} + \langle \phi_f | \exp(i(E_f - E_i)z)(\exp(-i\Lambda) - 1)(1 - \alpha_z)|\phi_j \rangle \]  
(B.22)

This we can in turn write again as integral over \( t \):

\[ \alpha_j^f(t \to \infty) = \delta_{f_i} + \int dt \langle \phi_f | \delta(z - t) \exp(i(E_f - E_i)t)(\exp(-i\Lambda) - 1)(1 - \alpha_z)|\phi_j \rangle \]  
(B.23)

Although it might seem a bit mysterious first to get rid of the \( t \) integral and later fill it in again, in this way we have rewritten the potential to an effective first order result. Associating the term between brackets with the potential in first order perturbation theory, we can rewrite the term in our original equation using the effective term in eq. B.24:

\[ V = \delta(z - t)(1 - \alpha_z)\Lambda = \delta(z - t)(\exp(-i\Lambda) - 1)(1 - \alpha_z) \]  
(B.24)

**B.3 Calculating \( \psi \) with the Effective Potential and \( G_0 \)**

We will use the effective form of \( V \) derived above to solve our original problem and insert it in eq. B.9. Next we have to carry out the resulting integrals (expressed fully in lightcone coordinates):

\[ \theta(x_-) \int d^4 z G_0(x,z)V(z)\psi(z) = -\theta(x_-) \frac{1}{(2\pi)^3} \int dz_+ dz_- dz_t \int dk_+ dk_- dk_t \]

\[ \left[ (\alpha_t k_t + \beta m + \frac{1}{\sqrt{2}}(1 - \alpha_z)k_- + \frac{1}{\sqrt{2}}(1 + \alpha_z)k_+) \right] \cdot \]

\[ \exp \left(-i(k_+(x - z)_- + k_-(x - z)_+ - k_t(x - z)_t) \right) \]

\[ \frac{-2k_+ k_- + k_t^2 + m^2}{\exp \left(i(q_t z_t - q_+ z_- - q_- z_+) \right)} \]

\[ \left( \exp(-i\Lambda) - 1 \right)^{i\sqrt{2}(1 - \alpha_z)u(q)} \exp \left( i(q_t - k_t)z_t \right) \]

(B.25)

The \( 1 + \alpha_z \) part dies immediately. Further most integrals are rather trivial to evaluate due to the \( \delta(z_-) \) function. After integrating over \( z_- , z_+ \) and \( k_- \) we obtain:

\[ -\theta(x_-) \frac{1}{(2\pi)^3} \int dz_t \int dk_+ dk_t \]

\[ \left[ (\alpha_t k_t + \beta m - \frac{1}{\sqrt{2}}(1 - \alpha_z)q_-) \exp \left(-i(k_+(x)_- + q_-(x)_+ - k_t(x - z)_t) \right) \right] \cdot \]

\[ \frac{-2k_+ q_- + k_t^2 + m^2}{\exp(-i\Lambda) - 1} \frac{i}{\sqrt{2}(1 - \alpha_z)u(q)} \exp \left( i(q_t - k_t)z_t \right) \]

(B.26)

The last integral over \( k_+ \) (which is due to the Greens function) requires subtlety: we have to apply the boundary condition that the Greens function vanishes for \( x_- < 0 \) if we shift
the pole below the real axis this boundary condition will be fulfilled. We give the result below:

$$\int dk^+ \frac{\exp(-ik_+x_-)}{k_+ + (k_t^2 + q_-/2) + i\epsilon} = -2\pi i \exp\left(i x_- (k_t^2 + m^2)/2q_+\right)$$  \hspace{1cm} (B.27)

This result is very important for another reason: from the integral it follows that $k_+ = [k_t^2 + m^2]/2q_-$. It follows that

$$k_- = q_-$$  \hspace{1cm} (B.28)

We'll make use of this identity at further steps in our derivations. Rewriting some terms, we obtain as final result in the singular gauge:

$$\psi(x) = \theta(-x_-) \exp(-iq \cdot x) u(q) - \theta(x_-) \frac{1}{(2\pi)^2} \int dz_t dk_t \left[ \frac{1}{\sqrt{2}} (\alpha z_t + \beta m) (1 - \alpha z_t) - (1 - \alpha z_t) q_+ \right] \exp\left(i(k_t x_t - k_t^2 + m^2)x_-/2q_- - q_- x_+\right) \exp\left(i z_t(q_t - k_t)\right) \exp\left(i\Lambda(z_t)\right) u(q)$$  \hspace{1cm} (B.29)

Although the derivation of the wavefunction is preferably performed in the singular gauge, for future calculations we prefer to have the wavefunction in the light cone gauge. The wavefunctions are conveniently simple related by just a gauge transformation $\exp\left(i\Lambda(x_t)\right)$.

For our final result we will need to rewrite some terms still. Observe we can rewrite the Dirac equation (in the light cone gauge) as

$$\left(P_t\alpha_t + \beta m - \sqrt{2} p_+ P_+ - \sqrt{2} p_- P_-\right)\psi$$  \hspace{1cm} (B.30)

Using eq. B.28 we can rewrite $p^+$

$$p_+ = \frac{p_t^2 + m^2}{2q_-}$$  \hspace{1cm} (B.31)

Further we can rewrite the spinorial part of eq. B.29. After multiplying with the gauge transformation we obtain (for $x_- > 0$):

$$\psi(x) = \frac{1}{\sqrt{2}} \exp\left(i\Lambda(x_t)\right) \int \frac{dk_t}{(2\pi)^2} dz_t \exp\left(-i\Lambda(z_t)\right) \exp\left(i(z_t q_t - k_t)\right) \exp\left(i(k_t x_t - q_- x_+ - k_+ x_-)\right) \left[1 + \frac{\alpha_t k_t + \beta m}{\sqrt{2}q_-}\right] \alpha_- u(q)$$  \hspace{1cm} (B.32)

In the main text a gauge transformation is associated with $U(x_t)$ and our final solution therefore is:  

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\[
\psi(x) = \frac{1}{\sqrt{2}} U(x_t) \int \frac{d^2p_t}{(2\pi)^2} d^2z_t U^\dagger(z_t) \exp(ip_t \cdot x_t - iq_- x_+) \exp(iz_t(q_t - p_t))
\]
\[
\exp(-i\frac{(p_0^2 + M^2 - \lambda)}{2q_-^2} x_-) \left[ 1 + \frac{\alpha_t p_t + \beta m}{\sqrt{2q_-}} \right] \alpha_- u(q)
\]

(B.33)

**Appendix C  The Fermionic Propagator in the Classical non-Abelian Gauge Field**

In this section we derive the propagator \( S_A(x, y) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} \psi(x) \bar{\psi}(y) \) for the fermions in the classical non-Abelian background field. Starting point is the form of the spinor as derived in appendix B.

We start by reorganizing some terms: we denote \( \psi(x) = \theta(-x_-)Au(x) + \theta(x_-)Bu(x) \). We directly obtain that \( \bar{\psi}(x) = \theta(-x_-)u^\dagger A^\dagger \gamma_0 + \theta(x_-)u^\dagger B^\dagger \gamma^0 \). Clearly we see we need to compute \( B(x)^\dagger \) and commute \( \gamma^0 \) through the expression.

\[
B(x) = \frac{1}{\sqrt{(2)}} \int \frac{d^2p_t}{(2\pi)^2} d^2z_t \left( U(x_t) U^\dagger(z_t) \right) \exp\left(i(p_t x_t - q_- x_+)\right) \exp\left(iz_t(q_t - p_t)\right)
\]
\[
\exp\left(-i\frac{(p_0^2 + m^2)}{2q_-} x_-\right) \left( 1 + \frac{\alpha_t p_t + \beta m}{\sqrt{(2)}q_-} \right) \alpha_- u(x)
\]

(C.1)

The hermitian conjugate of the phase transformations is trivial, and since the \( U(x_t) U^\dagger(z_t) \) is the only term that acts on the color space of the spinor its conjugate is simply \( U(z_t) U^\dagger(x_t) \). Therefore we focus from here on the spinorial part of the equation. We obtain:

\[
u^\dagger(x) \alpha^\dagger \left( 1 + \frac{\alpha_t p_t + \beta m}{2q_-} \right)^\dagger \gamma_0 \quad (C.2)
\]

Since \( \gamma^\dagger_0 = \gamma_0 \) and \( \gamma^\dagger_i = \gamma_0 \gamma_i \gamma_0 \), we easily obtain \( \alpha^\dagger_0 = \alpha_- \) and \( \alpha^\dagger_+ = \alpha_+ \). We therefore obtain:

\[
u^\dagger(x) \alpha_- \left( 1 + \frac{\alpha_t p_t + \beta m}{2q_-} \right) \gamma_0 \quad (C.3)
\]

Commuting \( \gamma_0 \) easily gives for the spinorial part:

\[
nu \alpha_+ \left( 1 + \frac{\beta m - \alpha_t p_t}{\sqrt{(2)}q_-} \right) \quad (C.4)
\]

We start to investigate the various terms arising in the propagator. For the \( \theta(-x_-) \theta(-y_-) \) part we get an easy result (the free field propagator):

\[
\theta(-x_-) \theta(-y_-) \int \frac{d^4q}{(2\pi)^4} \exp\left(iq(x-y)\right) u \bar{u} \equiv \theta(-x_-) \theta(-y_-) S_0(x-y) \quad (C.5)
\]
For the $\theta(-x_-)\theta(y_-)$ part we have to carefully evaluate the exponents. We obtain:

$$u(x)\bar{u}(y)\theta(-x_-)\theta(y_-) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} \exp \left( i q \cdot x \right) \exp \left( -i p_t y_t + i q \cdot y_+ \right) \cdot \exp \left( -i z_t (q_t - p_t) \right) \exp \left( i \left( \frac{p_t^2 + m^2}{2q_-} \right) \alpha_+ \left( 1 + \frac{\beta m + \alpha_t p_t}{\sqrt{2}q_-} \right) \right)\] \quad \text{(C.6)}$$

The first “trick” is to make a shift in the integration variable $p_t \rightarrow p_t + q_t$. Note this shift also affects the $\alpha_t p_t$ term! Also we multiply with a resolution of the identity $\exp \left( i q_+ y_- \right) \exp \left( -i q_+ y_- \right)$ to be able to split out a term $\exp \left( -i q \cdot y \right)$. We obtain:

$$\frac{1}{\sqrt{(2)}} \theta(-x_-)\theta(y_-) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} \exp \left( i q \cdot (x - y) \right) \exp \left( i p_t (z_t - y_t) \right) \cdot \exp \left( i \left( \frac{p_t^2 + m^2 - q_t^2}{2q_-} \right) (m - q) \alpha_+ \left( 1 + \frac{\beta m + \alpha_t (p_t - q_t)}{\sqrt{2}q_-} \right) \right) \quad \text{(C.7)}$$

The last term in the equation demands special attention, we want to write some terms as $m - q$. We start to write the unit term as $\frac{1}{\sqrt{(2)}}q_- \sqrt{(2)}q_-$ and add it to the numerator. Observe that $\alpha_+ = \alpha_+^2 = \gamma_+ - \gamma_+$. Contract $\gamma_+$ with $q_-$. The only term missing before we can rewrite the expression as $q - m$ is $q_+$. We investigate the effect of adding $\alpha_+ q_+$ to the equation. Since the term is multiplied by $\alpha_+$ we can add this “for free”. We rewrite it as:

$$\alpha_+ \alpha_- q_+ = \alpha_+ \gamma_0 q^t_+ = \gamma_- q^t_+ \quad \text{(C.8)}$$

Since we have all components of $q$ contracted we can write the equation as:

$$\theta(-x_-)\theta(y_-) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} \exp \left( i q \cdot (x - y) \right) \exp \left( i p_t (z_t - y_t) \right) \cdot \exp \left( i \left( \frac{p_t^2 + m^2 - q_t^2}{2q_-} \right) (m - q) \gamma_-(m - q - \eta_t) \left( \frac{2q_-}{(2q_-)} \right) \left( U(z_t)U^\dagger(y_t) \right) \right) \quad \text{(C.9)}$$

For the $\theta(x_-)\theta(-y_-)$ part we obtain by an analogous calculation:

$$\theta(x_-)\theta(-y_-) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon} \exp \left( i q \cdot (x - y) \right) \exp \left( i p_t (x_t - z_t) \right) \cdot \exp \left( i \left( \frac{p_t^2 + m^2 - q_t^2}{2q_-} \right) (m - q - \eta_t) \gamma_-(m - q - \eta_t) \left( \frac{2q_-}{(2q_-)} \right) \left( U(x_t)U^\dagger(z_t) \right) \right) \quad \text{(C.10)}$$

The remaining part gives:

$$\theta(x_-)\theta(y_-) \left( V(x_t)V^\dagger(y_t)S_0(x - y) \right) \quad \text{(C.11)}$$

This calculation is left as an exercise. When we apply the propagator in further calculations only the terms $\theta(\pm x_-)\theta(\mp y_-)$ will give non-zero results.
The last step is to show we can write the \( \theta(x_-)\theta(-y_-) \) term in the propagator as:

\[
S(x, y) = -i \int d^4z V(x_t) S_0(x, z) \gamma_- \delta(z_-) S_0(z, y) V^t(z_t)
\]  
(C.12)

(for \( \theta(-x_-)\theta(y_-) \) we’ll naturally obtain an analogous expression). An identity we’ll use is \( \gamma_- \gamma_- = \gamma_0^2 - \{\gamma_0, \gamma_3\} + \gamma_3^2 = 0 \). For notational convenience we’ll leave the \( U \) matrices out of the calculation. Written out eq. C.12 becomes (we’ll subsequently work out parts of this reference equation):

\[
S(x, y) = -i \int dz_+dz_-dp_+dp_-dp_td^4q \exp(iq \cdot x) \exp\left(i(q_+z_- + q_-z_+ + q_tz_t)\right) \exp\left(i(p_+z_- + p_-z_+ + ptz_t)\right) \cdot \exp\left(i(p_+y_- + p_-y_+ + py_t)\right) \frac{m - \frac{q}{q^2 + m^2 - i\epsilon}}{2p_+p_- + p^2_t + m^2 - i\epsilon} \gamma_- \delta(z_-) \frac{m - \gamma_-p_- - \gamma_-p_+ - \gamma_tpt}{2p_+p_- + p^2_t + m^2 - i\epsilon}
\]  
(C.13)

The \( dz_- \) integral puts all \( z_- \) to zero. We collect all terms in \( z_+ \). Upon integration we get:

\[
\int \frac{dz_+}{2\pi} \exp\left(-i(p_- - q_-)z_+\right) = \delta(p_- - q_-)
\]  
(C.14)

The integral over \( p_- \) becomes trivial, but note we do get \( q_- \) in the last propagator. However, since \( \gamma^2 = 0 \) the \( q_- \) part in the numerator of the propagator conveniently drops out. Next we turn to the integral over \( p_+ \). The terms containing \( p_+ \) are:

\[
\int \frac{dp_+}{2\pi} \frac{\exp(-ip_+y_-)}{2q_-(p_+ + \frac{p^2_t + m^2 - i\epsilon}{2q_-})}
\]  
(C.15)

The pole is located in the left upper plane. We can close the contour in the upper hemisphere since the integrand then vanishes. So we obtain just \( 2\pi i \) times the residue and the remainder is:

\[
\frac{i \exp\left(1\left(\frac{p^2_t + m^2}{2q_-}\right)y_-\right)}{2q_-}
\]  
(C.16)

Note the similarity between this pole and the pole we already encountered in deriving eq. B.28. We give the intermittent integral below.

\[
S(x, y) = \int dz_t dp_td^4q \exp(iq \cdot x) \exp\left(i(pt - qt)z_t\right) \exp\left(i(q_-y_+ + py_t)\right) \cdot \exp\left(i\left(\frac{p^2_t + m^2}{2q_-}\right)y_-\right) \frac{m - \frac{q}{q^2 + m^2 - i\epsilon}}{2q_-(q^2 + m^2 - i\epsilon)} \gamma_- \left(m - \gamma_-p_- - \gamma_tpt\right)
\]  
(C.17)

In order to get back our original form, we need to obtain a term \( \exp\left(iq(x - y)\right) \). We insert a resolution of the identity of the form \( \exp(iq_+y_-) \exp(-i(q_+y_-)) \). As well we make it...
a shift in the integration variable \( p_t \rightarrow p_t + q_t \). At the end add “for free” \( \gamma_+ q_+ \) in the numerator of the remainder of the last propagator. Collecting terms we find eq. C.9. In future calculations we will make use of the propagator as given in eq. C.9.

We can however further elaborate the propagator which yields new insight. We define a gauge transformation matrix \( G(x_t, x_-) = \theta(-x_-) + \theta(x_-) V(x) \). It can be shown that the propagator can be written as:

\[
S(x, y) = G(x) S_0(x - y) G\dagger(y) - i \int d^4z G(x) \left( \theta(x_-) \theta(-y_-) (V\dagger(z_t) - 1) - \theta(-x_-) \theta(y_-) (V(z_t) - 1) \right) G\dagger(y) S_0(x - z) \gamma_- \delta(z_-) S_0(z - y)
\]

We observe that the propagator takes the form of a gauge transformation of the free propagator and an additional term. Therefore, in calculations yielding an explicit gauge invariant result, we can use a particular simple form of this propagator:

\[
S(x, y) = S_0(x - y) - i \int d^4z \left( \theta(x_-) \theta(-y_-) (V\dagger(z_t) - 1) - \theta(-x_-) \theta(y_-) (V(z_t) - 1) \right) S_0(x - z) \gamma_- \delta(z_-) S_0(z - y)
\]

We see now the propagator becomes the free propagator plus one extra terms; the extra terms sums up all insertions on the propagator due to the classical field.

**Appendix D  The Calculation of the Hadronic Tensor \( W^{\mu \nu} \)**

The starting point of the calculation of the hadronic tensor is:

\[
W^{\mu \nu}(q^2, P \cdot q) = \frac{1}{2\pi} \sigma \frac{P^+}{m} \text{Im} \int d^4x dX_- \exp(\imath q \cdot x) \cdot \text{Tr}(\gamma^\mu S_A(X_- + x/2, X_- - x/2) \gamma^\nu S_A(X_- - x/2, X_- + x/2))
\]

We will start with using the form of the propagator as given in eq. C.5, C.11 and C.12. We will prove that only the \( \theta(\pm x_-) \theta(\mp y_-) \) part is non-vanishing; for notational convenience we will for now suppress the argument of the exponents.

We introduce the following variables:

\[
k = X_- + \frac{z}{2} \quad l = X_- - \frac{z}{2}
\]
Then the tensor becomes:

\[
W_{\mu\nu} = \frac{\sigma P^+}{\pi^2 m} \text{Im} \int dX_\cdot d^4 z d^4 s d^4 t \exp(i q \cdot x) \text{Tr}\{ \left[ \theta(-k_-)\theta(-l_-)S_0(k-l) +
\theta(l_-)\theta(k_-)V(k_t)S_0(k-l)V^\dagger(l_t) + \theta(-l_-)\theta(-k_-)S_0(k-l)\gamma_\mu S_0(k-s)\gamma_ \delta S_0(s-l)V^\dagger(s_t) +
\theta(-k_-)\theta(l_-)V(l_t)S_0(l-s)\gamma_\mu S_0(s-k)V^\dagger(s_t) \right] \gamma_\nu \}
\]

(D.3)

(S\(_0\) has the usual meaning of the free fermionic propagator). We see we get, in principle, 16 terms. However most die due to the \(\theta\) functions. We start to separate the multiplication, since the first two \(\theta\) functions have an equal sign in their argument and don’t have overlap with the last two terms with mixed signs in these functions. We start to focus on the \(\theta\) functions with even signs. We obtain:

\[
\frac{\sigma P^+}{\pi^2 m} \text{Im} \int dX_\cdot d^4 z d^4 s d^4 t \exp(i q \cdot x) \text{Tr}\{ \left[ \theta(-k_-)\theta(-l_-)S_0(k-l)\gamma_\mu S_0(l-k)\gamma_\nu \right] +
\left[ \theta(l_-)\theta(k_-)V(k_t)S_0(k-l)V^\dagger(l_t)\gamma_\nu \theta(l_-)\theta(k_-)V(l_t)S_0(l-k)V^\dagger(k_t) \right] \gamma_\mu \}
\]

(D.4)

We note that in the second term all the gauge matrices \(V\) cancel (remember these are unitary matrices). So the above term is fully independent of \(V\). Although various terms in eq. D.4 are divergent, we note that the terms are tadpoles without an imaginary part, so we conclude this term is zero. In the multiplication of the odd terms also just two from the four terms contribute due to the \(\theta\) functions.

Basically we can think of the hadronic tensor as consisting of a product of three factors: a spinorial part (due to the fermionic propagators and the \(\gamma_-\) matrices), an exponential part (due to the exponents of the fermionic propagator and the overall exponent) and the \(\theta\) functions and \(U\) matrices. We have contained the complete spinorial part of the hadronic tensor in \(M_{\mu\nu}(p,l,p',l')\) below since in a fair part of the calculation below we won’t have to deal with it.

\[
M_{\mu\nu}(p,l,p',l') = \text{Tr}\left\{ \frac{(M - p')\gamma_-(M - l')\gamma_\mu(M - p)\gamma_-(M - l)\gamma_\nu}{(p^2 + M^2 - i\epsilon)(l^2 + M^2 - i\epsilon)(l'^2 + M^2 - i\epsilon)(p'^2 + M^2 - i\epsilon)} \right\}
\]

(D.5)

We focus next on the \(\theta\) functions and \(V\) matrices of the two remaining terms. We obtain:

\[
\text{Tr}\left\{ \theta(k_-)\theta(-l_-)V(k_t)V^\dagger(s_t)V(t_t)V^\dagger(k_t) + \theta(-k_-)\theta(l_-)V(s_t)V^\dagger(l_t)V(t_t)V^\dagger(t_t) \right\}
\]

(D.6)
We rewrite this equation using the definition of the $\gamma$ function from eq. 4.14. We average $\frac{1}{N_c}V(x)V^\dagger(y)$ over the colors of partons at higher rapidities. We will show later for a particular distribution how to relate $\gamma$ to the gluon density. Using the unitarity of $V$, the cyclicity of the trace and the fact that the correlation functions are rotationally, translationally and parity invariant (i.e. $\gamma(x-y) = \gamma(y-x)$) we rewrite eq. D.6 as:

$$\left(\theta(k_-)\theta(-l_-) + \theta(-k_-)\theta(l_-)\right)N_c\gamma(k-s)$$  \hfill (D.7)

Inserting the identities we obtain:

$$W_{\mu\nu} = \frac{N_c\sigma P^+}{\pi^2m} \text{Im} \int dX_- d^4z d^4s d^4t d^4p d^4l d^4l' \exp(iq \cdot x) \left[ \theta(z_+ + X_-)\theta(z_- - X_-) + \theta(X_+ - z_+)\theta(-z_- + X_-) \right] \gamma(kt - st) \exp\left(i(X_+ + z/2 - s) \cdot p\right) \exp\left(i(s + z/2 - X_-) \cdot l\right) \exp\left(i(X_+ + z/2 - t) \cdot l'\right) \exp\left(i(t + z/2 - X_-) \cdot p'\right) \delta(s) \delta(t) M_{\mu\nu}(p, l, p', l') \right]$$  \hfill (D.8)

The integral can be simplified considerably, but it is necessary to work in the proper order. We start with collecting the terms in $X_-$ and do the corresponding integral. The terms containing $X_-$ are:

$$\int dX_- \exp\left(iX_- (p_+ - p'_+ + l_+ - l'_+)\right)$$  \hfill (D.9)

We obtain for the integral:

$$\epsilon(z_-) \frac{2\sin\left(\frac{z_-}{2}(p_+ - p'_+ + l_+ - l'_+)\right)}{p_+ - p'_+ + l_+ - l'_+}$$  \hfill (D.10)

in which $\epsilon(x) = \theta(x) - \theta(-x)$.

The integral over $t_-$ and $s_-$ is trivial and multiplies some + components of momenta with 0. However note we have also + components of momenta that get multiplied by $z_-$. Next we collect the terms in $t_+$ and $s_+$ perform the integral:

$$\int ds_+ dt_+ \exp\left(i(p_- - l_-)s_+ + i(l'_- - p'_-)t_+\right) = (2\pi)^2 \delta(p_- - l_-)\delta(l'_- - p'_-)$$  \hfill (D.11)

We collect all terms in $s_t$ and $t_t$, these are:

$$\int dt_t ds_t \exp\left(is_t(l_t - p_t)\right) \exp\left(it_t(p'_t - l'_t)\right) \gamma(s_t - t_t)$$  \hfill (D.12)

We have to make to shifts now: first shift $s_t \rightarrow s_t + t_t$ (then we have just the momentum variable $s_t$ in the $\gamma$ function). Next shift $t_t \rightarrow t_t - s_t/2$, such that we obtain again two exponents which each just contain $t_t$ or $s_t$. We obtain:
\[
\int dt_{i}ds_{i}\exp\left(\frac{is_{i}}{2}(t_{i}-p_{t}-p'_{t}+l'_{i})\right)\exp\left(\frac{-it_{i}}{2}(t_{i}-p_{t}-p'_{t}+l'_{i})\right)\gamma(s_{i})
\]
\[
= (2\pi)^4\delta(p'_{t} - l'_{t} + l_{t} - p_{t}) \int ds_{t}\exp\left(\frac{is_{t}}{2}(t_{t}-p_{t}-p'_{t}+l'_{t})\right)\gamma(s_{t})
\]  

(D.13)

Note that, due to the antisymmetry of the integral, we can subtract a constant term “for free” from the gamma function. Thus we obtain the Fourier transform \(\tilde{\gamma}\) of the \(\gamma\) function (see eq. 4.14). We summarize the result:

\[
\tilde{\gamma}(\frac{l_{t}}{2} - p_{t} - p'_{t} + l'_{t}) \exp\left(\frac{iz_{t}}{2}(p_{t} + l_{t} - p'_{t} - l'_{t})\right) \exp\left(\frac{iz_{t}}{2}(p_{t} - p'_{t})\right).
\]  

(D.14)

We first integrate over \(l'_{t}\). Next we do the \(l'_{t}\) integral; this changes a.o. the argument of the \(\tilde{\gamma}\) function to \(l_{t} - p_{t}\). The remaining integral over \(z_{t}\) gives \(\delta(p_{t} - p'_{t} + q_{t})\).

We first collect all terms in \(z_{\pm}\) to compute the following integral:

\[
\int dz_{\mp}\epsilon(z_{\mp})\exp\left(\frac{iq_{+}z_{\mp}}{2}\right)\left[\exp\left(\frac{-iz_{\mp}(p_{+} - p'_{+})}{2}\right) - \exp\left(\frac{-iz_{\mp}(l_{+} - l'_{+})}{2}\right)\right]
\]

(D.15)

In principle the integral is not well defined due to the exponent. However we can add a small imaginary part \(\imath\epsilon\) to \(q_{+}\). Then we’re left with two simple integrals (note we have to pick the sign of \(\imath\epsilon\) careful per case). We obtain:

\[
i\left[\frac{l'_{+} - l_{+} - p'_{+} + p_{+}}{(q_{+} - p_{+} + p'_{+} + \imath\epsilon)(q_{+} - l_{+} + l'_{+} + \imath\epsilon)} + (\imath\epsilon \to -\imath\epsilon)\right]
\]

(D.16)

The numerator just cancels a similar term in the tensor. We can now apply a contour integral over \(p'_{+}\) and \(l'_{+}\). The residue of the integral is unitary and we obtain the constraints \(p'_{+} = p_{+} - q_{+}\) and \(l'_{+} = l_{+} - q_{+}\). We apply all of the above obtained relations to the tensor. We obtain

\[
W_{\mu\nu} = \frac{N_{c}\sigma P^{+}}{\pi^{2}m} \text{Im} \int \frac{d^{4}p}{(2\pi)^4} \frac{d^{2}k_{t}}{(2\pi)^2} \frac{dk_{+}}{2\pi} \tilde{\gamma}(k_{t})M_{\mu\nu}
\]

(D.17)

in which \(p' = p - q\), \(l' = l - q\) and \(l = p - k\) (which follows from all the constraints derived above).

We apply the Landau Cutkosky rule to the above equation and shift \(p \to p + k\). We
find:

\[ W_{\mu\nu} = \frac{N_c \sigma P^+}{\pi^2 m} \text{Im} \int \frac{d^4 p}{(2\pi)^4} \frac{d^2 k_t}{(2\pi)^2} \frac{dk_+}{2\pi} \gamma(k_t) \cdot \theta(p_+ + k_+) \theta(q_+ - p_+) (2\pi)^2 \delta((p + k)^2 + M^2) \delta((p - q)^2 + M^2). \]

\[ \frac{1}{(p^2 + M^2)} \left( \frac{1}{(p + k - q)^2 + M^2} \right) \theta(p^+) \theta(q^+ - p^+) (2\pi)^2 \delta(p^2 + M^2) \delta((p + k - q)^2 + M^2). \]

\[ \frac{1}{(p + k)^2 + M^2} \left( \frac{1}{(p - q)^2 + M^2} \right) M'_{\mu\nu} \] (D.18)

in which \( M'_{\mu\nu} \) is defined as:

\[ M'_{\mu\nu}(p, l, p', l') = \text{Tr} \{ (M - p - k) \gamma_-(M - p) \gamma_\mu (M - p + q) \gamma_-(M - p - k + q) \gamma_\nu \} \] (D.19)

With an appropriate change of variables we see the second term is the first term with \( \mu \leftrightarrow \nu \). So overall we obtain:

\[ W_{\mu\nu} = \frac{N_c \sigma P^+}{\pi^2 m} \text{Im} \int \frac{d^4 p}{(2\pi)^4} \frac{d^2 k_t}{(2\pi)^2} \frac{dk_+}{2\pi} \gamma(k_t) \left[ \theta(p_+ + k_+) \theta(-p_+) (2\pi)^2 \delta((p + k)^2 + M^2) \delta((p - q)^2 + M^2) \right] \quad (D.20) \]

in which \( M''_{\mu\nu} \) is defined as:

\[ M''_{\mu
u}(p, l, p', l') = \text{Tr} \{ (M - p - k) \gamma_-(M - p) \gamma_\mu (M - p + q) \gamma_-(M - p - k + q) \gamma_\nu + \nu \leftrightarrow \mu \} \] (D.21)

Integrating over the \( \delta \) functions we finally obtain:

\[ W_{\mu\nu} = \frac{N_c \sigma P^+}{16\pi^2 m} \frac{1}{(q^2)^2} \text{Im} \int \frac{d^2 p_t}{(2\pi)^2} \frac{d^2 k_t}{(2\pi)^2} \int_{-\infty}^{-M^2_{p-q}} \frac{dp_+}{2\pi} \gamma(k_t) \cdot \frac{M'_{\mu\nu}}{M_{p+k-q}^2} \cdot I(k_t, p_t, q, p^+) \] (D.22)

in which \( M_{p-q} = (p_t - q_t)^2 + M^2 \). \( I \) is defined as:

\[ I(k_t, p_t, q, p^+) = \frac{1}{p_+ - \frac{(M^2_\mu - M^2_{p-q})}{2q^-}} \cdot \frac{1}{p_+ - \frac{M^2_{p-q}(M^2_{p+k-q} - M^2_{p+k-q})}{2q^- M^2_{p+k-q}}} \] (D.23)
It is argued in the main section the structure functions are given by (see eq. 4.21):

\[
F_1 = \frac{F_2}{2x} + \frac{q^2}{(q^-)^2}W^- \\
F_2 = -\frac{(q^-)^2}{q^2}W^+ 
\]

We can compute the trace \( M_{++} \) explicitly. It is given by:

\[
\frac{1}{16}M_{++} = \frac{1}{2} \left( M_p^2 M_{p+k-q}^2 + M_{p+k}^2 M_{p-q}^2 \right) - \frac{1}{2} q^2 k_t^2 
\]

We can carry out the integral over \( p^+ \), and from the above relations we obtain for \( F_2 \):

\[
F_2 = \frac{\sigma N_c}{2\pi^2} \int \frac{d^2p_t}{(2\pi)^2} \frac{d^2k_t}{(2\pi)^2} \tilde{\gamma}(k_t) \frac{(M_p^2 M_{p+k-q}^2 + M_{p+k}^2 M_{p-q}^2) - \frac{1}{2} q^2 k_t^2}{(M_p^2 M_{p+k-q}^2 - M_{p+k}^2 M_{p-q}^2)} \log \left( \frac{M_{p+k}^2 M_{p}^2}{M_{p-k}^2 M_{p}^2} \right) 
\]

We like to point out that alternatively we can as well write the tensor as:

\[
W_{++} = -\frac{N_c\sigma D^+}{2\pi^2 m} \frac{1}{(q^-)^2} \int_0^1 dz \int \frac{d^2p_t}{(2\pi)^2} \frac{d^2k_t}{(2\pi)^2} \tilde{\gamma}(k_t) \left( \frac{1}{M_p^2 + z(1-z)q_t^2} \right) \left( M_{p+k}^2 M_{p+k-q}^2 + M_{p+k}^2 M_{p-k}^2 - q_t^2 k_t^2 \right) 
\]

in which \( z = p^-/q^- \).

Plugging in the definition of \( \tilde{\gamma}(k_t) \) we obtain:

\[
F_2 = \frac{\sigma N_c}{2\pi^2} \int \frac{d^2p_t}{(2\pi)^2} \frac{d^2k_t}{(2\pi)^2} \frac{d^2x_t}{(2\pi)^2} \left( \gamma(x_t) - 1 \right) \exp \left( -i(k_t \cdot x_t) \right) \frac{(M_p^2 M_{p+k-q}^2 + M_{p+k}^2 M_{p-q}^2) - \frac{1}{2} q^2 k_t^2}{(M_p^2 M_{p+k-q}^2 - M_{p+k}^2 M_{p-k}^2)} \log \left( \frac{M_{p+k}^2 M_{p}^2}{M_{p-k}^2 M_{p}^2} \right) 
\]

This is the final result of this section.

**Appendix E Conventions, Light Cone Coordinates and Inelastic Variables**

**E.1 Light Cone Coordinates and some Conventions**

In the renormalization chapter we’ll use the metric \( g^\mu\nu = (+, -, -, -) \), while in the other chapters we use \( g^\mu\nu = (-, +, +, +) \).
We always work with Dirac \( \gamma \) matrices in the chiral representation and occasionally use (following conventions in literature on this topic) \( \beta = \gamma^0 \).

In the derivation of the IP-Sat model we work in light cone coordinates. These are defined as:

\[ p^\pm = \frac{1}{\sqrt{2}} (p^0 \pm p^3) \tag{E.1} \]

and \( p^t \) denotes the transversal components. From these relations we obtain:

\[ p^2 = -2p^+p^- + (p^t)^2 \tag{E.2} \]

Note as well in light cone coordinates we have \( g^{+-} = g^{--} = 0 \) and \( g^{-+} = g^{++} = -1 \) and \( g^{t1,t2} = 1 \). From the definitions above it trivially follows that \( \gamma^\pm = \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^3) \). With these definitions we can define projection operators:

\[ \alpha^\pm = \gamma^0 \gamma^\pm \tag{E.3} \]

These have the usual property that

\[ (\alpha^\pm)^2 = \alpha^\pm \]
\[ \alpha^\pm \alpha^\mp = 0 \]
\[ \alpha^\pm + \alpha^\mp = 1 \tag{E.4} \]

Using \( \gamma^\mu_\mu = \gamma^0 \gamma_\mu \gamma^0 \) we can easily verify \( (\alpha^\pm)^\dagger = \alpha^\pm \).

### E.2 Inelastic Coordinates

Inelastic scattering processes of a lepton with momentum \( k \) and hadron with momentum \( P \) are conveniently described in inelastic variables. A constituent (a quark or gluon) hadron carries a fraction \( x \) of the total longitudinal momentum \( P \) of the hadron and \( p = xP \).

It is practical to define variables in the center of mass system of the constituent and the lepton, these are denoted by \( \hat{s}, \hat{t}, \hat{u} \) (these correspond to the normal Mandelstam variables). For massless particles, \( \hat{t} = Q^2 \) and \( \hat{s} = 2p \cdot k = xs \). Also, considering the mass of the scattered parton to be small w.r.t. \( s \) and \( Q^2 \), we have for the parton:

\[ 0 \approx (p + q)^2 = 2p \cdot q + q^2 = 2xP \cdot q - Q^2 \tag{E.5} \]

from which it follows \( x = \frac{Q^2}{2P \cdot q} \).

The other inelastic variable we often encounter (and make use of in this thesis) is \( y \):

\[ y \equiv \frac{2P \cdot q}{2P \cdot k} \tag{E.6} \]

When we consider this (Lorentz invariant) quantity in the frame in which the target is at rest we obtain \( y = \frac{Q^2}{k^2} \) and we see \( y \) is the fractional momentum loss of the lepton. The variables \( x \) and \( y \) have as well the convenient property that they are dimensionless.