Rascal Tooling for Datatype Defining Rewrite Systems

Wijnand K. van Woerkom
10808981

Bachelor thesis honours extension
Credits: 6 EC

Bachelor Opleiding Kunstmatige Intelligentie

University of Amsterdam
Faculty of Science
Science Park 904
1098 XH Amsterdam

Supervisor
dr. A. Ponse

Informatics Institute
Introduction Faculty of Science
University of Amsterdam
1090 GH Amsterdam

July 30th, 2017
Abstract

It has been argued in [7] that using automated provers such as AProVE [4] and CSI [3] can benefit research projects on the topic of datatype defining rewrite systems (DDRSs), as the size of these DDRSs make manual completeness proofs long and error-prone. The authors of [7] note that no similar software exists for the ground-confluence property, and in [1] it is argued that such software should be developed.

The present work details an implementation of such a prover, developed in Rascal [2], which operates according to the ground-confluence proof methodology used in [1]. Alongside this prover other Rascal tools are detailed that serve practical purposes related to the use of provers. The proof methodology is first shown to be correct, and the tools are then detailed through code fragments and demonstrations of their workings. It is argued that these tools may accelerate DDRS research and help prevent errors. A demonstration of the ground-confluence prover reveals an oversight made in previous research, thus further confirming the need for such tools.

Contents

1 Introduction 3
  1.1 Preliminaries 3
  1.2 Motivation 5
  1.3 Ground-confluence proof methodology 6

2 Auxiliary tools 7
  2.1 The Rascal programming language 8
  2.2 Parsing .recipe files with parse_recipe.rsc 11
  2.3 Functions defined by cases on ADTs 13
  2.4 Automatically proving ground-confluence with GCprover.rsc 14

3 Conclusion 16

A parse_recipe.rsc examples 19
  A.1 A DDRS for terms in decimal tree notation 19
  A.2 The .trs file for Zdt 21

B GCprover.rsc examples 25
  B.1 The naturals in unary notation 25
  B.2 The integers in decimal tree notation 26
1 Introduction

Term rewriting is an area of mathematics that offers a formal framework for the process of deriving new formulas from old ones in a step-wise fashion according to a set of rules that match against the structure of formulas. One application of this theory is that of a datatype defining rewrite system (DDRS), introduced in [1], which serves to yield a sound, ground-complete term rewriting system (TRS) when its equations are interpreted from left to right as rewrite rules.

Other notable results on term rewriting are the development of the automated theorem provers (subsequently, provers) AProVE (presented in [4]) and CSI (presented in [3]) for automatically proving strong-termination and confluence of a given TRS, respectively. The former property is a requirement for DDRSs and it is argued in [7] that the use of these provers accelerates research on the subject, in particular when large DDRSs are the subject of investigation.

The present work is dedicated to detailing a number of additional tools developed in Rascal [2], that serve to further aid in research on DDRSs. Primarily we will consider two pieces of software titled parse_recipe.rsc, and GCprover.rsc. The former can translate a DDRS from succinct notation to formats for other tools such as AProVE and CSI, and the latter can do a ground-confluence proof attempt according to the methodology used in [1] and produce \texttt{\LaTeX} code to report on its findings. Rascal was chosen as the language in which to develop these tools as it offers many functionalities uniquely suited for this purpose.

As follows an overview of the contents of this work. Section 1.1 briefly explains the term rewrite related notations and concepts that will be used. In addition two DDRSs are reviewed: \texttt{ZS}, that will serve as a running example for this work, and \texttt{Zdt}, that will serve as an example for the GCprover.rsc tool. In Section 1.2 the practical benefits of using tools is explained, and in Section 1.3 we formally justify the technique underlying the GCprover.rsc tool. In Section 2 we turn to reviewing the software. In 2.1 the aforementioned Rascal functionalities are further explained and exemplified for the running example. The subsequent sections details the workings of the tools and give examples of their source code. The work ends with a conclusion in Section 3 and some examples of the tools in the Appendix. Appendix A exemplifies the input and output of parse_recipe.rsc; in A.1 the DDRS \texttt{Zdt} is listed along with the corresponding .recipe and in A.3 the corresponding output produced by \texttt{recipe.parser.rsc}. Appendix B exemplifies the output of GCprover.rsc; Appendix B.1 consists of the output for the running example \texttt{ZS} and Appendix B.2 of the output for the DDRS presented in [6]. The latter reveals an oversight in its ground-confluence proof, given in in [6], which further confirms the need for tools.

Note. The tools presented in this work were developed as part of a second year BSc AI honours project at the University of Amsterdam. A report was written on this work and as a result the present work will overlap with said report in certain areas; in particular the examples of the code may be similar or identical.

1.1 Preliminaries

This section will be spend on explaining terminology and notation, and exemplifying these through a DDRS that will serve as a running example throughout this work. The reason for choosing this DDRS as opposed to the others is that it contains few rules and symbols, which
significantly shortens the discussions while still being able to exemplify the essence of a DDRS and the corresponding aspects of the software. To exemplify the tools themselves we will use the DDRS \( Z_{dt} \) listed in Table 2 which is explained at the end of this section.

In general we will follow \([3]\) and \([1]\) in terminology and notation. We briefly repeat some of the concepts as specified in Section 2.1 of \([3]\), as they form an integral part of this work. We will assume some familiarity with set theory, logic, and the notion of a proof by structural induction.

A TRS is a pair consisting of a signature, commonly denoted as \( \Sigma \), and a ruleset, commonly denoted as \( R \). The signature is a set consisting of \( n \)-ary function symbols. The arity is allowed to be 0, in which case the function is usually referred to as a constant, and written without brackets. The signature, along with a set of variables (commonly \( x, y, z, \ldots \)), induce a term set \( \text{Term}_\Sigma(\Sigma) \) consisting of the variables and for every \( n \)-ary function symbol \( F \) and \( t_1, \ldots, t_n \in \text{Term}_\Sigma(\Sigma) \), \( F(t_1, \ldots, t_n) \in \text{Term}_\Sigma(\Sigma) \). The term set induced by \( V = \emptyset \) is called the set of ground terms. The elements of \( R \) are pairs of terms \((t, r)\) such that the variables occurring in \( r \) are a subset of those in \( t \). They are denoted as \( l \to r \), and form a set of rules for rewriting terms by using pattern matching, and may be applied in context. The rules thus define a relation \( \to \) on the set of terms and we denote as \( \to^* \) the transitive reflexive closure of this relation. By associating the terms and the function symbols with a domain we can assign to each term \( t \) its meaning \( \lbrack t \rbrack \).

Lastly some terminology. We say a term is in normal form, or irreducible, if it can not be rewritten by any of the rules. For a TRS \( R = (\Sigma, R) \) its set of normal forms will be denoted as \( \text{NF}(R) \) and similarly \( \text{NF}_\varnothing(R) := \text{NF}(R) \cap \text{Term}_\varnothing(\Sigma) \). The symbol \( N \) will be reserved for referring to the intended set of ground normal forms of DDRSs. A TRS is strongly terminating (henceforth, terminating) if there is no rewrite sequence of infinite length. This implies that if \( t \to r \) for \( t, r \in \text{Term}_\varnothing(\Sigma) \), then there exists \( s \in N \) with \( r \to^* s \). We say a TRS is confluent if \( t, r \in \text{Term}_\varnothing(\Sigma) \) with \( t \to^* r, t \to^* u \) implies there exists \( s \in \text{Term}_\varnothing(\Sigma) \) s.t. \( r \to^* s \) and \( u \to^* s \), or ground-confluent if this holds only for terms in \( \text{Term}_\varnothing(\Sigma) \). If a system is both terminating and (ground-)confluent we say it is (ground-)complete.

Next we give an example of these notions by looking at a DDRS \( Z_S \) of which the rules are listed in Table 1. The signature \( \Sigma_S \) is given by \( \{0, -, S, +, \cdot\} \) with domain \( Z \), and semantics

\[
\lbrack -x \rbrack = -\lbrack x \rbrack, \quad \lbrack x + y \rbrack = \lbrack x \rbrack + \lbrack y \rbrack, \\
\lbrack 0 \rbrack = 0, \quad \lbrack x \cdot y \rbrack = \lbrack x \rbrack \cdot \lbrack y \rbrack, \\
\lbrack S(x) \rbrack = \lbrack x \rbrack + 1,
\]

It is easily verified that the rules in Table 1 are sound with respect to this definition. The set of intended ground normal forms of this system can be defined as

\[
N_S^0 = \{0\} \cup N_S^+ \cup N_S^-,
\]

\[
N_S^+ = \{S(0)\} \cup \{S(x) \mid x \in N_S^+\},
\]

\[
N_S^- = \{-x \mid x \in N_S^+\},
\]

and a ground-completeness proof can be found in \([1]\).

Due to its size \( Z_S \) is a most useful running example for demonstrating the workings of DDRSs and the Rascal functionalities related to the tools presented here, but it simultaneously makes it
a poor example for demonstrating the tools themselves as their usefulness increases with the size of the specifications. For this reason we will use as a second example the DDREs $Z_{dt}$ presented in [6] and [1], which is repeated for convenience in Table 2.

To construct ground normal forms this DDRE uses the constants $0, \ldots, 9$ and a decimal tree function $\hat{d}$ with semantics $[x] \hat{d} y = 10 \cdot [x] + [y]$. Some arbitrary examples of normal forms are $1 \hat{d} 3$, $(2 \hat{d} 4) \hat{d} 0$, and $-(1 \hat{d} 1)$ representing the digits 13, 240, and -11 (in ordinary notation), respectively. See Appendix B.2 for an exact specification of the intended set of ground normal forms.

This DDRE is a good fit for size; the specification contains 277 rules and thus is big enough to exemplify the usefulness of the tools, while still producing a somewhat manageable ground-confluence proof (cf. Appendix B.2, which comprises roughly 20 pages). Additionally it is specified using several meta-notations, the use of which necessitated the development of the parse_recipe.rsc tool, as is further explained in the next section.

1.2 Motivation

Usage of AProVE and CSI may accelerate the process of proving or disproving completeness considerably, but requires the TRS which is under investigation to be specified in a syntax that is accepted by the tools. Commonly the .trs format is used (see e.g. http://aprove.informatik.rwth-aachen.de/help_new/trs.html) for this purpose. However, in order to succinctly specify the DDREs in [1] “digit counters” (see Section 1.1 in [1] and the example in the next paragraph) and other notations are used which are not part of the signature $\Sigma$ and can not be expressed in this format. This notation is used only to succinctly specify the DDREs will be referred to as meta-notation. The use of this notation makes it so that in order to use tools like AProVE and CSI it must first converted to .trs format, which results in a lot of tedious manual labor as DDREs may be comprised of hundreds of rewrite rules (whereas using the meta-notation they generally comprise no more than 30 lines). A concrete example of what this entails is having to manually produce Appendix A.2 based on the equations in Table 2.

A set of equations that exemplifies this issue is the following (from Table 8 in [1]), using constants $0, \ldots, 9$ and the decimal append functions $\hat{d}0, \ldots, \hat{d}9$ with $[x] \hat{d} i = 10 \cdot [x] + i$:

$[d10.i]_{i,j=0}^9 (x;_d i) + (y;_d j) = S^{i}((x + y);_d i)$.

Table 1: A simple DDRE for integer arithmetic, denoted by $Z_S$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[S1]</td>
<td>$-(x) \rightarrow x$</td>
</tr>
<tr>
<td>[S2]</td>
<td>$0 \rightarrow 0$</td>
</tr>
<tr>
<td>[S3]</td>
<td>$S(-S(x)) \rightarrow -x$</td>
</tr>
<tr>
<td>[S4]</td>
<td>$x + (-y) \rightarrow (-(x) + y)$</td>
</tr>
<tr>
<td>[S5]</td>
<td>$x + 0 \rightarrow x$</td>
</tr>
<tr>
<td>[S6]</td>
<td>$x + S(y) \rightarrow S(x + y)$</td>
</tr>
<tr>
<td>[S7]</td>
<td>$x \cdot (-y) \rightarrow -(x \cdot y)$</td>
</tr>
<tr>
<td>[S8]</td>
<td>$x \cdot 0 \rightarrow 0$</td>
</tr>
<tr>
<td>[S9]</td>
<td>$x \cdot S(y) \rightarrow (x \cdot y) + x$</td>
</tr>
</tbody>
</table>
This line represents the set of rules that results from instantiating the occurrences of \(i\) and \(j\) for each pair \((i, j) \in \{0, \ldots, 9\}^2\); i.e. the 100 rewrite rules

\[
\begin{align*}
(x;d_0) + (y;d_0) & \rightarrow S^0((x+y);d_0), \\
\vdots \\
(x;d_9) + (y;d_0) & \rightarrow S^0((x+y);d_9), \\
\vdots \\
(x;d_9) + (y;d_9) & \rightarrow S^0((x+y);d_9).
\end{align*}
\]

The superscript \(i\) of \(S\) is notation for \(i\) applications of \(S\), i.e. \(S^0(t) = t\) and \(S^{n+1} = S(S^n(t))\).

Rather than manually typing the resulting 100 rules we would like to denote \([d10.i.j]_{i,j=0}^{10}\) as a single line of plain text, e.g.:

\[[d10.1.j]\{0..9\} \quad \text{plus}(\text{dai}(x), \text{daj}(y)) = S\_j\{\text{dai}+\text{plus}(x, y)\}\]

Some other meta-notations used in tandem with the digit counters are \(', '\), and \('\), that serve to map constants to other constants according to the equations

\[
i' = i + 1, \quad i'' = i - 1, \quad i^* = 10 - i.
\]

These are not functions in the signature \(\Sigma\) and so they should be spelled out when specifying a DDRS in the .trs format. Making it possible to generate a .trs file according to a specification that uses the meta-notation is the first reason for developing the tools.

The second reason is due to the need for ground-confluence proofs. Ground-confluence is an essential property of DDRSs and while termination and confluence can be proven automatically by AP\textsc{rove} and CSI respectively, no prover exists (to the authors knowledge) that employs the methodology used in [I]. It has been noted in [I] that these proofs should be automated as their substantial size may lead to errors and oversights. Furthermore the process seems well suited to automation as it is algorithmic in nature.

Lastly there is the simple practical issue of having to manually perform rewrite sequences on paper, as the sequences can become quite long. Depending on the DDRS in question it may even be cumbersome to translate ordinary notation to the representation at hand. Being able to easily implement a TRS on a computer speeds up such activities considerably.

### 1.3 Ground-confluence proof methodology

A commonly used technique for determining confluence is to resolve each critical pair of the TRS, that is, find the counter examples to the property and add rules that resolve them. This method can be used for ground-confluence as well (see e.g. [S]) but the nature of DDRSs gives rise to an alternative method based on the following result.

**Theorem 1.** If \(R = (\Sigma, R)\) is a sound, terminating TRS, and \(\text{NF}_{\Sigma}(R)\) satisfies for any two of its members \(t, r\) that \([t] = [r] \rightarrow t \equiv r\), then \(R\) is ground-confluent.

**Proof.** Consider \(t, u, v \in \text{Ter}_\Sigma\) s.t. \(t \rightarrow^* u, t \rightarrow^* v\). Since \(R\) is sound this implies \([t] = [u] = [v]\). Additionally \(R\) is terminating and so there exist \(u', v' \in \text{NF}_{\Sigma}(R)\) with \(u \rightarrow^* u', v \rightarrow^* v'\). Applying soundness again we find \([u'] = [v'], [v] = [v']\), and so \([u'] = [v']\). Hence our assumptions imply \(u' \equiv v'\) which means \(u'\) (or \(v'\)) is a witness to our claim. \(\square\)
The aim of DDRSs is to produce a set of ground normal forms \( \mathcal{N} \) of which the members are canonical representatives of their semantic equivalence classes, and so \( \mathcal{N} \) would satisfy the property mentioned in Theorem 1. Consider for example \( \mathcal{N}_S \) in which each term has unique meaning. This uniqueness is due to the constructor functions, and in a sense is what we require of number representations.

This is what gives rise to the proof technique used in \([1]\): if \( \mathcal{N} \) is the desired set of unique ground normal forms of a DDRS, show that \( \mathcal{N} = \text{NF}_\varnothing(\mathcal{R}) \) for then ground-confluence follows from Theorem \([1]\). This is done by method of structural induction on the complexity of ground terms. In \([1]\) it is first shown separately that \( \mathcal{N} \subseteq \text{NF}_\varnothing(\mathcal{R}) \) because then as

\[
(t \in \text{NF}_\varnothing(\mathcal{R}) \rightarrow t \in \mathcal{N}) \iff (t \notin \text{NF}_\varnothing(\mathcal{R}) \lor t \in \mathcal{N}) \iff (t \text{ can be rewritten } \lor t \in \mathcal{N}),
\]

the latter can be used as induction proposition. The tool presented in Section \([2.4]\) tries to do the proof directly and as such instead uses \( t \in \text{NF}_\varnothing(\mathcal{R}) \iff t \in \mathcal{N} \) as the proposition. The choice between these propositions matters little for the algorithm behind the proof procedure, which is explained in more detail in the next paragraph.

Next we review the specifics of the procedure in some more detail as they underlie the workings of the GCprover.rsc tool detailed in Section \([2.4]\). The base case of the proof treats the constants and the induction case(s) treat for each \( n \)-ary function symbol \( F \) in the signature the term \( F(t_1, \ldots, t_n) \), where we may assume the induction hypothesis for \( t_1, \ldots, t_n \). Furthermore we may assume without loss of generality that \( t_1, \ldots, t_n \in \mathcal{N} \) (we will see why in the next paragraph). For each such case it is tested whether the proposition holds. If it does not, case distinction may be applied to any one of \( t_1, \ldots, t_n \). The proof fails if case distinction has been applied to all of \( t_1, \ldots, t_n \) but a term was found that does not satisfy the proposition. Note that this scenario does not imply the system in question is not ground-confluent.

We conclude by showing why we may make the aforementioned assumption that \( t_1, \ldots, t_n \in \mathcal{N} \). It is not difficult to see why we may make this assumption when using the approach taken in \([1]\), for if any of them were not then ‘\( F(t_1, \ldots, t_n) \) can be rewritten’ (in context, to be precise). Using the \( t \in \text{NF}_\varnothing(\mathcal{R}) \iff t \in \mathcal{N} \) proposition the same assumption may be made (again w.l.o.g.) but we have to make the additional assumption that for \( G(r_1, \ldots, r_m) \in \mathcal{N} \) we have that \( r_1, \ldots, r_m \in \mathcal{N} \). This is not an unreasonable assumption as \( \mathcal{N} \) is intended to be the exact set of ground normal forms, which is possible only if this assumption holds. We begin by noting that \( t_1, \ldots, t_n \in \mathcal{N} \lor (\exists i \in \{1, \ldots, n\})(t_i \notin \text{NF}_\varnothing(\mathcal{R})) \). We assume the latter to show the induction proposition then holds for \( F(t_1, \ldots, t_n) \). Rewrite rules apply in context so from \( t_i \notin \text{NF}_\varnothing(\mathcal{R}) \) we deduce that that \( F(t_1, \ldots, t_n) \notin \text{NF}_\varnothing(\mathcal{R}) \). However since \( t_i \) satisfies the induction proposition we know that \( t_i \notin \mathcal{N} \) and so by the contrapositive of our additional assumption that \( F(t_1, \ldots, t_n) \notin \mathcal{N} \).

\[2\] Auxiliary tools

The following subsections detail the workings of the parse_recipe.rsc and GCprover.rsc tools, and demonstrate how they can be used to solve the issues described in Section \([1.2]\). Section \([2.1]\) serves to give an introduction to the language; some of the characteristic Rascal functionalities are explained and exemplified for the running example \( \mathcal{Z}_S \), as these functionalities underlie the
implementation of the aforementioned tools. In Section 2.2 it is shown how the functionalities related to parsing according to custom syntax can be used to solve the first issue; translating a succinct description of the rules to other formats to be used by tools such as AProVE and CSI. In Section 2.4 the second issue is dealt with: automatically doing ground-confluence proofs according to the methodology proposed in [1]. Sections 2.1 and 2.3 share a purpose, and jointly show how the third issue can be solved: quickly rewriting and translating terms.

2.1 The Rascal programming language

The tools presented in this work were developed using the Rascal programming language [2]. The reason for choosing Rascal is threefold. The first is that Rascal was designed to facilitate the analysis and transformation of source code written in a user-defined syntax. This made it possible to implement the desired syntax discussed in Section 1.2 and conveniently convert it to different formats. Secondly Rascal offers its user the option to define abstract data types (ADTs). This provides an out-of-the-box mechanism for implementing a term rewriting system and defining custom functions on terms. Lastly, Rascal offers all the usual programming language features such as the common datatypes, loops, and list comprehension. These more typical features made it possible to combine the aforementioned functionalities into a ground-confluence prover.

In the remainder of this section we examine the features that are related to the subsequent discussion of the tools, and exemplify them for the DDRS ZS. These examples are indicated by a vertical line and an indent, and consist both of code fragments as well as the input and output of the interpreter based on expressions. Code fragments were generated using Rascal’s in-build experimental (at the time of writing) feature for producing \LaTeX{} source code based on .rsc files.

We begin by looking at the functionalities related to term rewriting, as underlying most of the tools presented here is the use of algebraic datatypes (ADTs). Rascal makes it easy to define a custom ADT along with functions over them, which behave like rewrite rules. In addition it is possible to do sophisticated pattern matching on instances of the datatype. For example we can define the signature $\Sigma_S$ and term set $\text{Ter}_2(\Sigma_S)$ via

```rascal
data Term = Zero() | neg(Term t) | S(Term t) | plus(Term t, Term r) | mult(Term t, Term r);
```

To incorporate a non-empty $V$ we can add constants that represent variables; in the actual implementation constants like $u()$ and $v()$ are added for this purpose and they will appear in later code examples. The $:\Rightarrow$ operator can be used to perform pattern matching on these terms, returning true or false depending on the outcome of the match. Useful variations are provided such as deep pattern matching (which returns true if the match occurs against a subterm of the expression), and matching against variables, which allows the user to save the result for use in subsequent code. Below are some examples; the first line shows an ordinary match, the second a deep pattern match, and the third a match using variables.

8
Instances of datatypes are represented as trees and a mechanism called visit exists to traverse such trees. Within the visit block cases can be defined for matching against the current node, based on which information about the tree can be recorded or modifications to the tree can be made. The => operator provides a succinct syntax for making such modifications. Below is an example.

```
rascal> visit(plus(S(Zero()), S(S(Zero())))) {
   >>>>>>> case S(x) => x
   >>>>>>>}
   Term: plus(
       Zero(),
       Zero())
```

This functionality gives us an easy method of replacing occurrences of terms within other terms, which will be useful for doing case distinction in the ground-confluence proofs (cf. Section 2.4):

```
// Replace occurrences of i in h with j.
Term replace(Term i, Term h, Term j) {
   return visit(h) {
      case i => j
   }
}
```

Functions can be defined by cases over these custom datatypes. They act like rewrite rules in the sense that they are applied whenever a term that matches them is constructed, and the object that initiated the sequence is replaced with the final result. Additionally this will occur in context. Special syntax was introduced for defining functions to reflect this behaviour; rather than using accolades and the return keyword we can directly assign a return value using =, as demonstrated below.

```
Term plus(x, Zero()) = x;
Term plus(x, S(y)) = S(plus(x, y));
Term mult(x, Zero()) = Zero();
Term mult(x, S(y)) = plus(mult(x, y), x);
```

After importing a file containing these rules in the interpreter we can compute expressions:
The rewrite-esque behavior of functions is not always desirable; for instance the fragment exemplifying the visit statement shown earlier would return Zero() instead of its listed response if these rules were present in the same file. To avoid this they may be wrapped in a step function (as advised in the Rascal documentation), and indeed this is the approach taken in Section 2.4.

Lastly we briefly examine Rascal's syntax parsing functionalities. One of the declarations in Rascal is the 'SyntaxDefinition', which allows users to specify a syntax and use Rascal as a parser for this language. The most important aspect of this functionality is given by the lexical and syntax keywords, which respectively correspond to terminals and non-terminals. Below is an example of a simple general syntax for terms.

```rascal
lexical Constant = [0-9];
lexical Variable = [a-z];
lexical Name = [A-Z]+;

syntax Term = Constant
| Variable
| Name "(" Argument ")";
syntax Argument = Term
| Argument "," Argument;
```

After having imported this code in the interpreter strings can be parsed according to the syntax. The result of the parse is a syntax object which is displayed using the syntax (Nonterminal) 'sentence'. Similarly to how pattern matching can be used on ADTs it can be applied to these objects. Below are examples of parsing a string according to the grammar listed above, and of pattern matching on the result of the parses (note that _ can be the name of a variable and is used here to indicate an anonymous variable, the value of which is irrelevant).

```rascal
rascal>[Term] "S(0)";
Term: (Term) 'S(0)'
rascal>(Term)'<Name _> (<Argument _>)' := [Term] "S(0)"
bool: true
rascal>if((Term)'<Name n> (<Argument a>)' := [Term] "PLUS(S(0), MULT(0, 0))") {

ok

S(0), MULT(0, 0)
ok
```
ADTs and syntax objects are similar to one another; their definitions follow a similar structure, and pattern matching may be used on them in similar ways. Moreover, like we could define functions that act like rewrite rules for ADTs we can define them for syntax objects. We will conclude this section with a demonstration of how this may be used to automatically spell out the superscript i notation used in Section 1.2.

First we expand the syntax Term definition to contain the following line

```
| A: Name "_" Constant "(" Term ")"
```

The prefix A: names this case A, which is added here solely to be able to define a rule for it (the name A was chosen arbitrarily). This rule automatically applies i applications of the function to the input term:

```
Term A(Name n, Constant c, Term t) {
    int i = toInt("<c>");
    if(i == 0) return t;
    else if(i > 0) return [Term] "<n>(<n>_<i-1>(<t>))";
}
```

This rule is activated as soon as a parse is done resulting in a Term of type A, and its output replaces the initial parsed object. The angled brackets that may occur within a string cause the contents between them to be evaluated as an expression, after which its return value is translated to the str datatype and placed back in that position in the string. This is a most convenient feature that will see many applications in the rest of the fragments. As follows some examples.

```
rascal>"<5 + 5> - 3 = <10 - 3>";
str: "10 - 3 = 7"
rascal>[Term] "S.0(x)";
Term: (Term) 'x'
rascal>[Term] "S.1(x)";
Term: (Term) 'S(x)'
rascal>[Term] "S.4(x)";
Term: (Term) 'S(S(S(S(x))))'
rascal>[Term] "PLUS(P.2(x), P.0(y))";
Term: (Term) 'PLUS(P(P(x)), y)'
```

### 2.2 Parsing .recipe files with parse_recipe.rsc

Using the capabilities described in the previous section it was possible to create a syntax for the format described in 1.2. The resulting piece of software can read a file containing plain text that follows this syntax, and transform it to (amongst others) the .trs format. We will henceforth refer to these input files as .recipe files. In this section we review the implementation in more detail and give an example of code that transforms the .recipe file to a different format.

First we extend the example syntax from the last section to one that specifies the format described in the introduction. In particular we need to be able to parse the labels and equations.
Below is a simplified variant of the syntax used in the actual implementation. The lexical definitions have been omitted.

```
syntax Term = Constant | Variable | Name "(" Argument ");
syntax Argument = Term | Argument ";" Argument;
syntax Rule = Label Equation;
syntax Label = "[" Name "]" | "[" Name ItVars "]{" Constant "." Constant "}";
syntax ItVars = "." Variable | ItVars ItVars;
syntax Equation = Term "=" Term;
```

This grammar is used by a `to_rules` function that takes as input the location of the `.recipe` file and gives as output a list of `Rule` objects. The code that does this is somewhat convoluted (presumably this is due to the author, as Rascal offers very sophisticated features for exactly this purpose), but can be summarized as follows: do a first pass to fill in the specific values that the digit counters take, and then do a second pass to translate the different kinds of notation used (superscripts, apostrophes, etc). E.g., a rule that contains a digit counter \(i\) and some term with \(S_i(x)\) will have \(i\) instantiated in the first pass to produce rules that contain \(S_0(x), S_1(x), \ldots\), which will be expanded to \(x, S(x), \ldots\) in the second pass. The resulting software is able to parse all the notations used in [1], as is demonstrated in Appendix A.

We conclude with giving an example of a function that makes use of `to_rules`:

```
void to_trs(loc path) {
    list[Rule] rules = to_rules(path);
    set[str] vars = "{<v> | /(Variable)'<Variable v>' := rules};
    println("(VAR <intercalate(" ", [*vars])>)");
    println("(RULES")
    visit(rules) {
        case (Rule)'[<Name _>] <Term t> = <Term r>': {
            println("<t> \rightarrow <r>");
        }
    }
    println(")");
}
```

The second line of the function scans the rules for variables and stores them in a set (to avoid
duplicates), after which the intercalate function concatenates them (separated by spaces) to form a string that makes up the header of the file. Afterwards the rules are printed in a straightforward fashion. An example of the input is listed in Table 3, the corresponding output can be found in Appendix A.2.

2.3 Functions defined by cases on ADTs

Defining functions case by case for ADTs is most useful, and in particularly it allows us to easily translate instances of the ADT to different datatypes. Consider for example the following fragment for $\mathbb{Z}$ which bears close resemblance to the definition of its semantics.

```
int to_int(neg(x)) = - to_int(x);
int to_int(Zero()) = 0;
int to_int(S(x)) = to_int(x) + 1;
int to_int(plus(x, y)) = to_int(x) + to_int(y);
int to_int(mult(x, y)) = to_int(x) * to_int(y);
```

Additionally we may use this functionality to produce LaTeX source code for a given term, which is what the $\textsc{Gprover.rsc}$ tool uses for generating its output:

```
str to_latex(neg(x)) = "-(<to_latex(x)>);"
str to_latex(Zero()) = "0";
str to_latex(S(x)) = "S(<to_latex(x>)";
str to_latex(plus(x, y)) = "(<to_latex(x>) + (<to_latex(y>)";
str to_latex(mult(x, y)) = "(<to_latex(x>) \cdot (<to_latex(y)>);"
```

The actual implementation uses more cases than this in order to specify when to place brackets, because placing too many can make it difficult to read terms. For instance the example code listed above would map $\text{to_latex}(\text{plus}(\text{S(Zero())}, \text{S(Zero())}))$ to $(S(0)) + (S(0))$, whereas normally this would just be denoted as $S(0)+S(0)$. See Appendix B for examples of how brackets are placed by the tools.

A last application that is useful for our purposes is to define a Boolean function $\text{NF}$ that checks whether a given term is in the intended set of ground normal forms. A definition for our example set $\mathcal{N}_S$ would be as follows.

```
bool NF(Zero()) = true;
default bool NF(x) = NFp(x) || NFm(x);

bool NFp(S(Zero())) = true;
bool NFp(S(S(x))) = NFp(S(x));
default bool NFp(x) = false;

bool NFm(neg(x)) = NFp(x);
default bool NFm(x) = false;
```

The default keyword indicates that a definition should only be tried after the others have failed.
This definition works on ground terms but during the structural induction proof we must perform slightly more complicated membership checks as they deal with terms that contain arguments assumed to be in $\mathcal{N}$, which means that this definition should be expanded to account for this. Consider for instance the term $t = S(u)$ with $u \in \mathcal{N}_S$. We can not tell whether $t \in \mathcal{N}_S$ based on this information, but if in addition we knew that $u = S(u')$ then $u \in \mathcal{N}_S^+$ and so $t \in \mathcal{N}_S$. However if we were to use the above definition the term would evaluate to $NFp(u')$ and return \textit{false}. To implement the correct behaviour in the software we add definitions akin to:

\begin{verbatim}
bool NFp(S(u())) = true;
bool NFm(neg(u())) = true;
\end{verbatim}

\subsection{2.4 Automatically proving ground-confluence with GCprover.rsc}

In this section we present the implementation of a ground-confluence prover which uses the technique described in Section 1.3 (and as such, is reliant on soundness, termination, and semantically unique ground normal forms). Specifically we will summarize its workings and demonstrate parts of its code. Examples of the output can be found in Appendix B.

The workings of this file can be summarized as follows. An outer level function \texttt{check\_gc} is called that iterates over the induction cases and applies a \texttt{handle\_case} function to each. This function respectively checks if the case: is in $\mathcal{N}$, can be rewritten, or can be further expanded by applying case distinction. In the latter scenario the function applies itself to these resulting cases. During this process the software generates the latex source and keeps track of counterexamples, which are presented at the end of the output if there are any. It should be noted that, due to the proof methodology, the terms that are referred to as counterexamples are not counterexamples to the ground-confluence property but counterexamples to the induction proof. Furthermore the base case of the proof is skipped as this can be easily manually verified.

The parts of the GCprover.rsc code that are specific to the DDRS in question are therefore: the check whether a term is in $\mathcal{N}$, the check of whether a term can be rewritten, the induction cases, and the case distinction shapes. Before we continue discussing the DDRS independent code we exemplify the DDRS dependent code for $\mathcal{Z}_S$.

The Boolean $NF$ function has been shown in the previous section, and so we need to define only the induction cases, case distinction, and rewrite rules. For case distinction we can use the \texttt{replace} function defined in Section 2.1 as follows

\begin{verbatim}
list[Term] case\_distinction(Term t, Term v) {
    list[Term] shapes = [
        Zero(),
        S(p(v)),
        neg(p(v))
    ];
    return [replace(v, t, s) | s <- shapes];
}
\end{verbatim}

The function $p$ is added to the \texttt{data Term} definition and is an auxiliary function used to represent a superscript prime (and is translated as such by the \texttt{to\_latex} code). The $\leftarrow$ operator is used
here which essentially means ‘for all ...in’. Similarly to the case distinction we can specify the induction cases simply as a list of terms:

\[
\text{list}[\text{Term}] \ ts = [
\begin{align*}
& S(u()), \\
& \text{neg}(u()), \\
& \text{plus}(u(), v()), \\
& \text{mult}(u(), v())
\end{align*}
\];

Lastly we can use a function (called to.stepfunction) in parse_recipe.rsc to generate the following code, which applies rewrite rules and records the corresponding labels:

\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{neg}(\text{neg}(x))) = <\text{\texttt{S1}}, x>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{neg}(\text{Zero}())) = <\text{\texttt{S2}}, \text{Zero}()>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{S}(\text{neg}(\text{S}(x)))) = <\text{\texttt{S3}}, \text{neg}(x)>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{plus}(x, \text{neg}(y))) = <\text{\texttt{S4}}, \text{neg(plus(neg(x), y))}>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{plus}(x, \text{Zero}())) = <\text{\texttt{S5}}, x>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{plus}(x, \text{S}(y))) = <\text{\texttt{S6}}, \text{S(plus(x, y))}>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{mult}(x, \text{neg}(y))) = <\text{\texttt{S7}}, \text{neg(mult(x, y))}>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{mult}(x, \text{Zero}())) = <\text{\texttt{S8}}, \text{Zero}()>;
\]
\[
\text{tuple}[\text{str}, \text{Term}] \ \text{step}(\text{mult}(x, \text{S}(y))) = <\text{\texttt{S9}}, \text{plus(mult(x, y), x)}>;
\]
\[
\text{default tuple}[\text{str}, \text{Term}] \ \text{step}(x) = <\text{NA}, x>;
\]

The remainder of this section is spent reviewing the DDRS independent code. Amongst this is code for generating the \LaTeX{} source but we will omit it as it is mostly very straightforward. An example of such a line is as follows (here tl is the to_latex function):

\[
\text{println}(\text{"then }\text{tl}(t)\text{ }\to\text{tl}(r)\text{ by }<\text{s}>\text{, and }t \nin \mathcal{N}.\text{"});
\]

First we consider the outermost function check_gc. It starts with the aforementioned code for the list of induction cases ts, and subsequently iterates over them to do the proof:

\[
\begin{align*}
\text{for}(\text{Term } t & \leftarrow \ts) \{ \\
& \text{list}[\text{Term}] \ args = [t[i] \mid \text{int } i \leftarrow [0..\text{arity}(t)]]; \\
& \text{counterexamples } += \text{handle_case}(t, \text{args}); \\
\}
\end{align*}
\]

The list args contains the arguments of t, to which case distinction may be applied. This list is filled using a simple application of list comprehension and the fact that for an ADT instance t the (zero-index) i-th argument can be selected using t[i]. The actual implementation contains a segment after the for loop listed above that prints a list of counterexamples (if there are any) after the proof attempt is completed (cf. the end of Appendix B.3). We omit this code here as it contains just print statements.

Next we turn to the last and arguably the most important part of the file; the handle_case
function. Below is the code that makes up the essence of the function. Note that $\iff$ is the equivalence operator on Boolean values.

```rascal
list[Term] handle_case(Term t, list[Term] args) {
    list[Term] counterexamples = [];
    <s, _> = step(t);
    if (s == "NA" $\iff$ NF(t)) return [];
    else if (args != []) {
        Term r = args[-1];
        for (Term s <- case_distinction(t, r)) {
            counterexamples += handle_case(r, args - r);
        }
    } else counterexamples += t;
    return counterexamples;
}
```

The elements of `args` respect the order of their occurrence in `t`. This means that because we are selecting an element for applying case distinction via the expression `args[i]`, `i` is a variable of the proof methodology that influences strategy. In this fragment we use $i = -1$ which causes $t$’s arguments to be selected from right to left. This choice is beneficial as the DDRSs may define functions by recursion on the rightmost argument, see for instance \[S_4\]–\[S_9\]. This means that if we were to use `args[0]` and instantiate in the opposite order, we would unnecessarily add many induction cases which causes the output to grow; e.g. the proof in Appendix \[S.2\] comprises 99 pages when generated using $i = 0$.

## 3 Conclusion

In this work two Rascal tools have been presented and detailed, `parse_recipe.rsc` and `GCprover.rsc`, designed to accelerate research on DDRSs and to help avoid errors. Specifically the motivation for their development was threefold. DDRSs are succinctly specified using notation that is not a part of the `.trs` file format, and so converting this notation to use for tools such as AProVE or CSI is cumbersome. Secondly it has been argued that doing ground-confluence proofs using the method in \[1\] should be automated as this work is error-prone. Lastly it is argued in the present work that being able to quickly implement a TRS can accelerate work on DDRSs.

The tools presented here have been shown to provide solutions to these issues. In particular we have shown that by using them the DDRSs can be specified in a more succinct notation to use for other tools like AProVE and CSI, and that the ground-confluence prover is capable of generating valid proofs and relieving a lot of the work involved. The notion that these proofs should be automated was reaffirmed by the tools discovery of an oversight in the ground-confluence proof presented in \[6\]. Furthermore it was argued that Rascal is well suited for the development of such tools, by means of demonstrating brief but expressive code fragments.

Despite these positive results there are some aspects of the tools which could be improved upon. Most notable is the issue that quite some ‘maintenance’ is required before being able
to use them; if the tools are to be applied to a DDRS that they were not configured for then several of the components should be updated to account for this. E.g., for parse_recipe.rsc the syntax should be expanded, and for Gcprover.rsc the induction cases, case distinction shapes, rewrite rules, normal form definition, and to_latex definitions. After having done this a few times it becomes a routine but it is nevertheless cumbersome. Ideally such software would have a GUI so that functions like to_latex could be generated on the basis of a settings screen for determining precedence.

A second issue is that of the size of the generated ground confluence proofs. While a system like the one listed in Table 2 is large in part due to the size of the specification and the number of case distinction shapes, it could easily be manually made shorter by parameterizing arguments using i and j (as in the succinct descriptions of the rules). Examples of this practise can be found in the ground-confluence proofs in [1]. Ideally the software would be capable of performing this same shortening.

Aside from fixing deficiencies the tools’ capabilities could also be further expanded. Having at our disposal the to_latex function it may for instance be possible to expand parse_recipe.rsc to generate a \LaTeX{} Table on the basis of a .recipe file.

Furthermore the syntax described in Section 2.2 could be expanded to allow for infix and postfix notation. After this modification the rules could be specified almost exactly as they are in the tables. Currently this has been done only partially, consider for instance the syntax definition for equations: syntax Equation = Term "=" Term; . Incorporating this for functions like plus and the append functions we could specify \[
[d10.i,j]_{i,j=0}^{9} (x :d i) + (y :d j) = S_{j}((x + y) :d i) \]

A last useful expansion would be to add a check whether all the rules that occur in the system are used during the ground-confluence proof. If the proof succeeds but there are rules that were not used then they can be removed while preserving soundness and ground-completeness. This is because termination and soundness are preserved under the removal of rules, and evidently they were not required for the ground-confluence proof.
References


A parse_recipe.rsc examples

In this appendix we provide examples for the parse_recipe.rsc. In particular we show in Appendix A.1 the DDRS $Z_{dt}$ (Table 2) and its corresponding .recipe file (Table 2). The output of parse_recipe.rsc's function to_trs based on the input in Table 3 is shown in Appendix A.2

A.1 A DDRS for terms in decimal tree notation

The DDRS $Z_{dt}$ is listed in Table 2 and the corresponding .recipe file in Table 3

| [dt1] | 0 $\hat{\alpha} x = x$ | [dt12] | $-0 = 0$ |
| [dt2] | $x \hat{\alpha} (y \hat{\alpha} z) = (x + y) \hat{\alpha} z$ | [dt13] | $-(-x) = x$ |
| [dt3.$i$]$_{i=0}^8$ | $S(i) = i'$ | [dt14] | $P(0) = -1$ |
| [dt4] | $S(9) = 1 \hat{\alpha} 0$ | [dt15.$i$]$_{i=0}^8$ | $P(i') = i$ |
| [dt5.$i$]$_{i=0}^8$ | $S(x \hat{\alpha} i) = x \hat{\alpha} i'$ | [dt16] | $P(x \hat{\alpha} 0) = P(x) \hat{\alpha} 9$ |
| [dt6] | $S(x \hat{\alpha} 9) = S(x) \hat{\alpha} 0$ | [dt17.$i$]$_{i=0}^8$ | $P(x \hat{\alpha} i') = x \hat{\alpha} i$ |
| [dt7.$i$]$_{i=0}^9$ | $x + i = S'(x)$ | [dt18] | $P(-x) = -S(x)$ |
| [dt8.$i$]$_{i=0}^9$ | $x + (y \hat{\alpha} i) = S'(y \hat{\alpha} x)$ | [dt19.$i$]$_{i=0}^8$ | $S(-i') = -i$ |
| [dt9] | $x \cdot 0 = 0$ | [dt20] | $S(-x \hat{\alpha} 0) = -(P(x) \hat{\alpha} 9)$ |
| [dt10.$i$]$_{i=1}^9$ | $x \cdot i = \sum^i x$ | [dt21.$i$]$_{i=0}^8$ | $S(-x \hat{\alpha} i') = -(x \hat{\alpha} i)$ |
| [dt11.$i$]$_{i=0}^9$ | $x \cdot (y \hat{\alpha} i) = ((x \cdot y) \hat{\alpha} 0) + (x \cdot i)$ | [dt22] | $(-x) \hat{\alpha} y = -(x \hat{\alpha} (-y))$ |
| [dt23.$i$.j]$_{i,j=1}^9$ | $i \hat{\alpha} (-j) = i'' \hat{\alpha} j$ | [dt24.$i$.j]$_{i,j=1}^9$ | $(x \hat{\alpha} i) \hat{\alpha} (-j) = (x \hat{\alpha} i'' \hat{\alpha} j$ |
| [dt25] | $x \hat{\alpha} (-y \hat{\alpha} z) = -((y + (-x)) \hat{\alpha} z)$ | [dt26.$i$]$_{i=1}^9$ | $x + (-i) = P^i(x)$ |
| [dt27] | $x + (-y \hat{\alpha} z) = -(y \hat{\alpha} (z + (-x)))$ | [dt28] | $x \cdot (-y) = -(x \cdot y)$ |

Table 2: The DDRS $Z_{dt}$ proposed in [6], using $i'$, $i''$, and $i'''$ as described in Section 1.2 and $\sum^i$ as described in [1]
\[
\begin{align*}
\text{[dt1]} & \quad \text{dt}(0, x) = x \\
\text{[dt2]} & \quad \text{dt}(x, \text{dt}(y, z)) = \text{dt}(\text{plus}(x, y), z) \\
\text{[dt3]} & \quad \text{S}(0) = i' \\
\text{[dt4]} & \quad \text{S}(9) = \text{dt}(1, 0) \\
\text{[dt5]} & \quad \text{S}(\text{dt}(x, i)) = \text{dt}(x, i') \\
\text{[dt6]} & \quad \text{S}(\text{dt}(x, 9)) = \text{dt}(\text{S}(x), 0) \\
\text{[dt7]} & \quad \text{plus}(x, i) = \text{S}.i(x) \\
\text{[dt8]} & \quad \text{plus}(x, \text{dt}(y, i)) = \text{S}.i(\text{dt}(y, x)) \\
\text{[dt9]} & \quad \text{mult}(x, 0) = 0 \\
\text{[dt10]} & \quad \text{mult}(x, i) = \text{sum}^i(x) \\
\text{[dt11]} & \quad \text{mult}(x, \text{dt}(y, i)) = \text{plus}(\text{dt}(\text{mult}(x, y), 0), \text{mult}(x, i)) \\
\text{[dt12]} & \quad \text{neg}(0) = 0 \\
\text{[dt13]} & \quad \text{neg}(\text{neg}(x)) = x \\
\text{[dt14]} & \quad \text{P}(0) = \text{neg}(1) \\
\text{[dt15]} & \quad \text{P}(i') = i \\
\text{[dt16]} & \quad \text{P}(\text{dt}(x, 0)) = \text{dt}(\text{P}(x), 9) \\
\text{[dt17]} & \quad \text{P}(\text{dt}(x, i')) = \text{dt}(x, i) \\
\text{[dt18]} & \quad \text{P}(\text{neg}(x)) = \text{neg}(\text{S}(x)) \\
\text{[dt19]} & \quad \text{S}(\text{neg}(i')) = \text{neg}(i) \\
\text{[dt20]} & \quad \text{S}(\text{neg}(\text{dt}(x, 0))) = \text{neg}(\text{dt}(\text{P}(x), 9)) \\
\text{[dt21]} & \quad \text{S}(\text{neg}(\text{dt}(x, i'))) = \text{neg}(\text{dt}(x, i)) \\
\text{[dt22]} & \quad \text{dt}(\text{neg}(x), y) = \text{neg}(\text{dt}(x, \text{neg}(y))) \\
\text{[dt23]} & \quad \text{dt}(i, \text{neg}(j)) = \text{dt}(i'', j*) \\
\text{[dt24]} & \quad \text{dt}(\text{dt}(x, i), \text{neg}(j)) = \text{dt}(\text{dt}(x, i''), j*) \\
\text{[dt25]} & \quad \text{dt}(x, \text{neg}(\text{dt}(y, z))) = \text{neg}(\text{dt}(\text{plus}(y, \text{neg}(x)), z)) \\
\text{[dt26]} & \quad \text{plus}(x, \text{neg}(i)) = \text{P}.i(x) \\
\text{[dt27]} & \quad \text{plus}(x, \text{neg}(\text{dt}(y, z))) = \text{neg}(\text{dt}(y, \text{plus}(z, \text{neg}(x)))) \\
\text{[dt28]} & \quad \text{mult}(x, \text{neg}(y)) = \text{neg}(\text{mult}(x, y))
\end{align*}
\]

Table 3: Example .recipe file for the DDRS listed in Table 2
A.2 The .trs file for $\mathbb{Z}_{dt}$

The remainder of this section consists of the output of parse_recipe.rsc's to_trs function, wrapped in a verbatim environment, based on the input listed in Table 3.

(VAR x y z)

(RULES
  dt(0, x) -> x
  dt(x, dt(y, z)) -> dt(plus(x, y), z)
  S(0) -> 1
  S(1) -> 2
  S(2) -> 3
  S(3) -> 4
  S(4) -> 5
  S(5) -> 6
  S(6) -> 7
  S(7) -> 8
  S(8) -> 9
  S(9) -> dt(1, 0)
  S(dt(x, 0)) -> dt(x, 1)
  S(dt(x, 1)) -> dt(x, 2)
  S(dt(x, 2)) -> dt(x, 3)
  S(dt(x, 3)) -> dt(x, 4)
  S(dt(x, 4)) -> dt(x, 5)
  S(dt(x, 5)) -> dt(x, 6)
  S(dt(x, 6)) -> dt(x, 7)
  S(dt(x, 7)) -> dt(x, 8)
  S(dt(x, 8)) -> dt(x, 9)
  S(dt(x, 9)) -> dt(S(x), 0)
  plus(x, 0) -> x
  plus(x, 1) -> S(x)
  plus(x, 2) -> S(S(x))
  plus(x, 3) -> S(S(S(x)))
  plus(x, 4) -> S(S(S(S(x))))
  plus(x, 5) -> S(S(S(S(S(x)))))
  plus(x, 6) -> S(S(S(S(S(S(x)))))
  plus(x, 7) -> S(S(S(S(S(S(S(x)))))
  plus(x, 8) -> S(S(S(S(S(S(S(S(x)))))
  plus(x, 9) -> S(S(S(S(S(S(S(S(S(x)))))
  dt(0, x) -> dt(y, x)
  dt(0, dt(y, x)) -> S(dt(y, x))
  dt(0, dt(y, 1)) -> S(dt(y, x))
  dt(0, dt(y, 2)) -> S(dt(y, x))
  dt(0, dt(y, 3)) -> S(dt(y, x))
  dt(0, dt(y, 4)) -> S(dt(y, x))
  dt(0, dt(y, 5)) -> S(dt(y, x))
  dt(0, dt(y, 6)) -> S(dt(y, x))
  dt(0, dt(y, 7)) -> S(dt(y, x))
  dt(0, dt(y, 8)) -> S(dt(y, x))
  dt(0, dt(y, 9)) -> S(dt(y, x))
  dt(0, 0) -> 0
  mult(x, 0) -> 0
  mult(x, 1) -> x
  mult(x, 2) -> plus(x, x)
  mult(x, 3) -> plus(x, x)
  mult(x, 4) -> plus(x, x)
  mult(x, 5) -> plus(x, x)
  mult(x, 6) -> plus(x, x)
  mult(x, 7) -> plus(x, x)
  mult(x, 8) -> plus(x, x)
  mult(x, 9) -> plus(x, x)
  mult(x, dt(y, 0)) -> plus(dt(mult(x, y), 0), mult(x, 0))
  mult(x, dt(y, 1)) -> plus(dt(mult(x, y), 0), mult(x, 1))
  mult(x, dt(y, 2)) -> plus(dt(mult(x, y), 0), mult(x, 2))
  mult(x, dt(y, 3)) -> plus(dt(mult(x, y), 0), mult(x, 3))
  mult(x, dt(y, 4)) -> plus(dt(mult(x, y), 0), mult(x, 4))
  mult(x, dt(y, 5)) -> plus(dt(mult(x, y), 0), mult(x, 5))

21
\[
mult(x, \ dt(y, 6)) \rightarrow \ plus(\mult(x, y), 0), \mult(x, 6))
\]
\[
mult(x, \ dt(y, 7)) \rightarrow \ plus(\mult(x, y), 0), \mult(x, 7))
\]
\[
mult(x, \ dt(y, 8)) \rightarrow \ plus(\mult(x, y), 0), \mult(x, 8))
\]
\[
mult(x, \ dt(y, 9)) \rightarrow \ plus(\mult(x, y), 0), \mult(x, 9))
\]
\[
\neg(0) \rightarrow 0
\]
\[
\neg(\neg(x)) \rightarrow x
\]
\[
P(0) \rightarrow \neg(1)
\]
\[
P(1) \rightarrow 0
\]
\[
P(2) \rightarrow 1
\]
\[
P(3) \rightarrow 2
\]
\[
P(4) \rightarrow 3
\]
\[
P(5) \rightarrow 4
\]
\[
P(6) \rightarrow 5
\]
\[
P(7) \rightarrow 6
\]
\[
P(8) \rightarrow 7
\]
\[
P(9) \rightarrow 8
\]
\[
P(\dt(x, 0)) \rightarrow \dt(P(x), 9)
\]
\[
P(\dt(x, 1)) \rightarrow \dt(x, 0)
\]
\[
P(\dt(x, 2)) \rightarrow \dt(x, 1)
\]
\[
P(\dt(x, 3)) \rightarrow \dt(x, 2)
\]
\[
P(\dt(x, 4)) \rightarrow \dt(x, 3)
\]
\[
P(\dt(x, 5)) \rightarrow \dt(x, 4)
\]
\[
P(\dt(x, 6)) \rightarrow \dt(x, 5)
\]
\[
P(\dt(x, 7)) \rightarrow \dt(x, 6)
\]
\[
P(\dt(x, 8)) \rightarrow \dt(x, 7)
\]
\[
P(\dt(x, 9)) \rightarrow \dt(x, 8)
\]
\[
P(\neg(x)) \rightarrow \neg(5(x))
\]
\[
S(\neg(1)) \rightarrow \neg(0)
\]
\[
S(\neg(2)) \rightarrow \neg(1)
\]
\[
S(\neg(3)) \rightarrow \neg(2)
\]
\[
S(\neg(4)) \rightarrow \neg(3)
\]
\[
S(\neg(5)) \rightarrow \neg(4)
\]
\[
S(\neg(6)) \rightarrow \neg(5)
\]
\[
S(\neg(7)) \rightarrow \neg(6)
\]
\[
S(\neg(8)) \rightarrow \neg(7)
\]
\[
S(\neg(9)) \rightarrow \neg(8)
\]
\[
S(\neg(\dt(x, 0))) \rightarrow \neg(\dt(P(x), 9))
\]
\[
S(\neg(\dt(x, 1))) \rightarrow \neg(\dt(x, 0))
\]
\[
S(\neg(\dt(x, 2))) \rightarrow \neg(\dt(x, 1))
\]
\[
S(\neg(\dt(x, 3))) \rightarrow \neg(\dt(x, 2))
\]
\[
S(\neg(\dt(x, 4))) \rightarrow \neg(\dt(x, 3))
\]
\[
S(\neg(\dt(x, 5))) \rightarrow \neg(\dt(x, 4))
\]
\[
S(\neg(\dt(x, 6))) \rightarrow \neg(\dt(x, 5))
\]
\[
S(\neg(\dt(x, 7))) \rightarrow \neg(\dt(x, 6))
\]
\[
S(\neg(\dt(x, 8))) \rightarrow \neg(\dt(x, 7))
\]
\[
S(\neg(\dt(x, 9))) \rightarrow \neg(\dt(x, 8))
\]
\[
dt(1, \neg(1)) \rightarrow \dt(0, 9)
\]
\[
dt(1, \neg(2)) \rightarrow \dt(0, 8)
\]
\[
dt(1, \neg(3)) \rightarrow \dt(0, 7)
\]
\[
dt(1, \neg(4)) \rightarrow \dt(0, 6)
\]
\[
dt(1, \neg(5)) \rightarrow \dt(0, 5)
\]
\[
dt(1, \neg(6)) \rightarrow \dt(0, 4)
\]
\[
dt(1, \neg(7)) \rightarrow \dt(0, 3)
\]
\[
dt(1, \neg(8)) \rightarrow \dt(0, 2)
\]
\[
dt(1, \neg(9)) \rightarrow \dt(0, 1)
\]
\[
dt(2, \neg(1)) \rightarrow \dt(1, 9)
\]
\[
dt(2, \neg(2)) \rightarrow \dt(1, 8)
\]
\[
dt(2, \neg(3)) \rightarrow \dt(1, 7)
\]
\[
dt(2, \neg(4)) \rightarrow \dt(1, 6)
\]
\[
dt(2, \neg(5)) \rightarrow \dt(1, 5)
\]
\[
dt(2, \neg(6)) \rightarrow \dt(1, 4)
\]
\[
dt(2, \neg(7)) \rightarrow \dt(1, 3)
\]
\[
dt(2, \neg(8)) \rightarrow \dt(1, 2)
\]
\[
dt(2, \neg(9)) \rightarrow \dt(1, 1)
\]
\[
dt(3, \neg(1)) \rightarrow \dt(2, 9)
\]
dt(3, neg(2)) -> dt(2, 8)
dt(3, neg(3)) -> dt(2, 7)
dt(3, neg(4)) -> dt(2, 6)
dt(3, neg(5)) -> dt(2, 5)
dt(3, neg(6)) -> dt(2, 4)
dt(3, neg(7)) -> dt(2, 3)
dt(3, neg(8)) -> dt(2, 2)
dt(3, neg(9)) -> dt(2, 1)
dt(4, neg(1)) -> dt(3, 9)
dt(4, neg(2)) -> dt(3, 8)
dt(4, neg(3)) -> dt(3, 7)
dt(4, neg(4)) -> dt(3, 6)
dt(4, neg(5)) -> dt(3, 5)
dt(4, neg(6)) -> dt(3, 4)
dt(4, neg(7)) -> dt(3, 3)
dt(4, neg(8)) -> dt(3, 2)
dt(4, neg(9)) -> dt(3, 1)
dt(5, neg(1)) -> dt(4, 9)
dt(5, neg(2)) -> dt(4, 8)
dt(5, neg(3)) -> dt(4, 7)
dt(5, neg(4)) -> dt(4, 6)
dt(5, neg(5)) -> dt(4, 5)
dt(5, neg(6)) -> dt(4, 4)
dt(5, neg(7)) -> dt(4, 3)
dt(5, neg(8)) -> dt(4, 2)
dt(5, neg(9)) -> dt(4, 1)
dt(6, neg(1)) -> dt(5, 9)
dt(6, neg(2)) -> dt(5, 8)
dt(6, neg(3)) -> dt(5, 7)
dt(6, neg(4)) -> dt(5, 6)
dt(6, neg(5)) -> dt(5, 5)
dt(6, neg(6)) -> dt(5, 4)
dt(6, neg(7)) -> dt(5, 3)
dt(6, neg(8)) -> dt(5, 2)
dt(6, neg(9)) -> dt(5, 1)
dt(7, neg(1)) -> dt(6, 9)
dt(7, neg(2)) -> dt(6, 8)
dt(7, neg(3)) -> dt(6, 7)
dt(7, neg(4)) -> dt(6, 6)
dt(7, neg(5)) -> dt(6, 5)
dt(7, neg(6)) -> dt(6, 4)
dt(7, neg(7)) -> dt(6, 3)
dt(7, neg(8)) -> dt(6, 2)
dt(7, neg(9)) -> dt(6, 1)
dt(8, neg(1)) -> dt(7, 9)
dt(8, neg(2)) -> dt(7, 8)
dt(8, neg(3)) -> dt(7, 7)
dt(8, neg(4)) -> dt(7, 6)
dt(8, neg(5)) -> dt(7, 5)
dt(8, neg(6)) -> dt(7, 4)
dt(8, neg(7)) -> dt(7, 3)
dt(8, neg(8)) -> dt(7, 2)
dt(8, neg(9)) -> dt(7, 1)
dt(9, neg(1)) -> dt(8, 9)
dt(9, neg(2)) -> dt(8, 8)
dt(9, neg(3)) -> dt(8, 7)
dt(9, neg(4)) -> dt(8, 6)
dt(9, neg(5)) -> dt(8, 5)
dt(9, neg(6)) -> dt(8, 4)
dt(9, neg(7)) -> dt(8, 3)
dt(9, neg(8)) -> dt(8, 2)
dt(9, neg(9)) -> dt(8, 1)
dt(dt(x, 1), neg(1)) -> dt(dt(x, 0), 9)
dt(dt(x, 1), neg(2)) -> dt(dt(x, 0), 8)
dt(dt(x, 1), neg(3)) -> dt(dt(x, 0), 7)
dt(dt(x, 1), neg(4)) -> dt(dt(x, 0), 6)
dt(dt(x, 1), neg(5)) -> dt(dt(x, 0), 5)
dt(dt(x, 1), neg(6)) -> dt(dt(x, 0), 4)
dt(dt(x, 1), neg(7)) -> dt(dt(x, 0), 3)
dt(dt(x, 1), neg(8)) -> dt(dt(x, 0), 2)
dt(dt(x, 1), neg(9)) -> dt(dt(x, 0), 1)
dt(dt(x, 2), neg(1)) -> dt(dt(x, 1), 9)
dt(dt(x, 2), neg(2)) -> dt(dt(x, 1), 8)
dt(dt(x, 2), neg(3)) -> dt(dt(x, 1), 7)
dt(dt(x, 2), neg(4)) -> dt(dt(x, 1), 6)
dt(dt(x, 2), neg(5)) -> dt(dt(x, 1), 5)
dt(dt(x, 2), neg(6)) -> dt(dt(x, 1), 4)
dt(dt(x, 2), neg(7)) -> dt(dt(x, 1), 3)
dt(dt(x, 2), neg(8)) -> dt(dt(x, 1), 2)
dt(dt(x, 2), neg(9)) -> dt(dt(x, 1), 1)
dt(dt(x, 3), neg(1)) -> dt(dt(x, 2), 9)
dt(dt(x, 3), neg(2)) -> dt(dt(x, 2), 8)
dt(dt(x, 3), neg(3)) -> dt(dt(x, 2), 7)
dt(dt(x, 3), neg(4)) -> dt(dt(x, 2), 6)
dt(dt(x, 3), neg(5)) -> dt(dt(x, 2), 5)
dt(dt(x, 3), neg(6)) -> dt(dt(x, 2), 4)
dt(dt(x, 3), neg(7)) -> dt(dt(x, 2), 3)
dt(dt(x, 3), neg(8)) -> dt(dt(x, 2), 2)
dt(dt(x, 3), neg(9)) -> dt(dt(x, 2), 1)
dt(dt(x, 4), neg(1)) -> dt(dt(x, 3), 9)
dt(dt(x, 4), neg(2)) -> dt(dt(x, 3), 8)
dt(dt(x, 4), neg(3)) -> dt(dt(x, 3), 7)
dt(dt(x, 4), neg(4)) -> dt(dt(x, 3), 6)
dt(dt(x, 4), neg(5)) -> dt(dt(x, 3), 5)
dt(dt(x, 4), neg(6)) -> dt(dt(x, 3), 4)
dt(dt(x, 4), neg(7)) -> dt(dt(x, 3), 3)
dt(dt(x, 4), neg(8)) -> dt(dt(x, 3), 2)
dt(dt(x, 4), neg(9)) -> dt(dt(x, 3), 1)
dt(dt(x, 5), neg(1)) -> dt(dt(x, 4), 9)
dt(dt(x, 5), neg(2)) -> dt(dt(x, 4), 8)
dt(dt(x, 5), neg(3)) -> dt(dt(x, 4), 7)
dt(dt(x, 5), neg(4)) -> dt(dt(x, 4), 6)
dt(dt(x, 5), neg(5)) -> dt(dt(x, 4), 5)
dt(dt(x, 5), neg(6)) -> dt(dt(x, 4), 4)
dt(dt(x, 5), neg(7)) -> dt(dt(x, 4), 3)
dt(dt(x, 5), neg(8)) -> dt(dt(x, 4), 2)
dt(dt(x, 5), neg(9)) -> dt(dt(x, 4), 1)
dt(dt(x, 6), neg(1)) -> dt(dt(x, 5), 9)
dt(dt(x, 6), neg(2)) -> dt(dt(x, 5), 8)
dt(dt(x, 6), neg(3)) -> dt(dt(x, 5), 7)
dt(dt(x, 6), neg(4)) -> dt(dt(x, 5), 6)
dt(dt(x, 6), neg(5)) -> dt(dt(x, 5), 5)
dt(dt(x, 6), neg(6)) -> dt(dt(x, 5), 4)
dt(dt(x, 6), neg(7)) -> dt(dt(x, 5), 3)
dt(dt(x, 6), neg(8)) -> dt(dt(x, 5), 2)
dt(dt(x, 6), neg(9)) -> dt(dt(x, 5), 1)
dt(dt(x, 7), neg(1)) -> dt(dt(x, 6), 9)
dt(dt(x, 7), neg(2)) -> dt(dt(x, 6), 8)
dt(dt(x, 7), neg(3)) -> dt(dt(x, 6), 7)
dt(dt(x, 7), neg(4)) -> dt(dt(x, 6), 6)
dt(dt(x, 7), neg(5)) -> dt(dt(x, 6), 5)
dt(dt(x, 7), neg(6)) -> dt(dt(x, 6), 4)
dt(dt(x, 7), neg(7)) -> dt(dt(x, 6), 3)
dt(dt(x, 7), neg(8)) -> dt(dt(x, 6), 2)
dt(dt(x, 7), neg(9)) -> dt(dt(x, 6), 1)
dt(dt(x, 8), neg(1)) -> dt(dt(x, 7), 9)
dt(dt(x, 8), neg(2)) -> dt(dt(x, 7), 8)
dt(dt(x, 8), neg(3)) -> dt(dt(x, 7), 7)
dt(dt(x, 8), neg(4)) -> dt(dt(x, 7), 6)
dt(dt(x, 8), neg(5)) -> dt(dt(x, 7), 5)
dt(dt(x, 8), neg(6)) -> dt(dt(x, 7), 4)
dt(dt(x, 8), neg(7)) -> dt(dt(x, 7), 3)
B GCprover.rsc examples

In this section we exemplify the output of GCprover.rsc by doing proof attempts for the DDRSs listed in Tables [1] and [2]. The first (in Appendix [B.1]) succeeds and the second (in Appendix [B.2]) fails. As such the latter demonstrates the tools behaviour in this case: printing a list of counterexamples.

B.1 The naturals in unary notation

The LATEX code for this section past this paragraph was generated by the GCprover.rsc tool, after being configured for the DDRS $Z_S$ using $N_S$ and the rules listed in Table [1].

(1) $t = S(u)$: apply case distinction on $u$.

$u = 0$: then, $S(0)$ is irreducible and $t \in N$.
$u = S(u')$: then, $S(S(u'))$ is irreducible and $t \in N$.
$u = -S(u')$: then, $S(-S(u')) \rightarrow -u'$ by [S3], and $t \notin N$.

(2) $t = -u$: apply case distinction on $u$.

$u = 0$: then, $-0 \rightarrow 0$ by [S2], and $t \notin N$.
$u = S(u')$: then, $-S(u')$ is irreducible and $t \in N$.
$u = -S(u')$: then, $-(-S(u')) \rightarrow S(u')$ by [S1], and $t \notin N$.

(3) $t = u + v$: apply case distinction on $v$. 

This completes the proof.

B.2 The integers in decimal tree notation

Denoting as D the set of constants \( \{0, 1, \ldots, 9\} \) we can define the desired set of ground normal forms \( \mathcal{N}_{dt} \) for the DDRS listed in Table 2 by the equations

\[
\mathcal{N}_{dt} = \{0\} \cup \mathcal{N}_{G} \cup \mathcal{N}_{S},
\]

\[
\mathcal{N}_{G} = \{D - \{0\}\} \cup \{x \quad | \quad x \in \mathcal{N}_{dt}, i \in D\},
\]

\[
\mathcal{N}_{S} = \{-x \quad | \quad x \in \mathcal{N}_{dt}\}.
\]

The \LaTeX{} code for this section past this paragraph was generated by the \texttt{GCproever.rsc} tool, after being configured for \( \mathcal{Z}_{dt} \), using \( \mathcal{N}_{dt} \) and the rules listed in Table 2. The list of counterexamples for this configuration can be found at the end of the proof.

(1) \( t = S(u) \): apply case distinction on \( u \).

\( u = 0 \): then, \( S(0) \rightarrow 1 \) by [dt3.0], and \( t \notin \mathcal{N} \).

\( u = 1 \): then, \( S(1) \rightarrow 2 \) by [dt3.1], and \( t \notin \mathcal{N} \).

\( u = 2 \): then, \( S(2) \rightarrow 3 \) by [dt3.2], and \( t \notin \mathcal{N} \).

\( u = 3 \): then, \( S(3) \rightarrow 4 \) by [dt3.3], and \( t \notin \mathcal{N} \).

\( u = 4 \): then, \( S(4) \rightarrow 5 \) by [dt3.4], and \( t \notin \mathcal{N} \).

\( u = 5 \): then, \( S(5) \rightarrow 6 \) by [dt3.5], and \( t \notin \mathcal{N} \).

\( u = 6 \): then, \( S(6) \rightarrow 7 \) by [dt3.6], and \( t \notin \mathcal{N} \).

\( u = 7 \): then, \( S(7) \rightarrow 8 \) by [dt3.7], and \( t \notin \mathcal{N} \).

\( u = 8 \): then, \( S(8) \rightarrow 9 \) by [dt3.8], and \( t \notin \mathcal{N} \).

\( u = 9 \): then, \( S(9) \rightarrow 1 \) by [dt4.0], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 0 \): then, \( S(u' \quad \hat{a} \quad 0) \rightarrow u' \quad \hat{a} \quad 1 \) by [dt5.0], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 1 \): then, \( S(u' \quad \hat{a} \quad 1) \rightarrow u' \quad \hat{a} \quad 2 \) by [dt5.1], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 2 \): then, \( S(u' \quad \hat{a} \quad 2) \rightarrow u' \quad \hat{a} \quad 3 \) by [dt5.2], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 3 \): then, \( S(u' \quad \hat{a} \quad 3) \rightarrow u' \quad \hat{a} \quad 4 \) by [dt5.3], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 4 \): then, \( S(u' \quad \hat{a} \quad 4) \rightarrow u' \quad \hat{a} \quad 5 \) by [dt5.4], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 5 \): then, \( S(u' \quad \hat{a} \quad 5) \rightarrow u' \quad \hat{a} \quad 6 \) by [dt5.5], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 6 \): then, \( S(u' \quad \hat{a} \quad 6) \rightarrow u' \quad \hat{a} \quad 7 \) by [dt5.6], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 7 \): then, \( S(u' \quad \hat{a} \quad 7) \rightarrow u' \quad \hat{a} \quad 8 \) by [dt5.7], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 8 \): then, \( S(u' \quad \hat{a} \quad 8) \rightarrow u' \quad \hat{a} \quad 9 \) by [dt5.8], and \( t \notin \mathcal{N} \).

\( u = u' \quad \hat{a} \quad 9 \): then, \( S(u' \quad \hat{a} \quad 9) \rightarrow S(u') \quad \hat{a} \quad 0 \) by [dt6.0], and \( t \notin \mathcal{N} \).

u = -1: then, \( S(-1) \rightarrow -0 \) by [dt19.0], and \( t \notin \mathcal{N} \).
\( u = -2: \) then, \( S(-2) \to -1 \) by \([dt19.1]\), and \( t \notin \mathcal{N} \).

\( u = -3: \) then, \( S(-3) \to -2 \) by \([dt19.2]\), and \( t \notin \mathcal{N} \).

\( u = -4: \) then, \( S(-4) \to -3 \) by \([dt19.3]\), and \( t \notin \mathcal{N} \).

\( u = -5: \) then, \( S(-5) \to -4 \) by \([dt19.4]\), and \( t \notin \mathcal{N} \).

\( u = -6: \) then, \( S(-6) \to -5 \) by \([dt19.5]\), and \( t \notin \mathcal{N} \).

\( u = -7: \) then, \( S(-7) \to -6 \) by \([dt19.6]\), and \( t \notin \mathcal{N} \).

\( u = -8: \) then, \( S(-8) \to -7 \) by \([dt19.7]\), and \( t \notin \mathcal{N} \).

\( u = -9: \) then, \( S(-9) \to -8 \) by \([dt19.8]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 0): \) then, \( S(-(u' ȧ 0)) \to -(P(u') ȧ 9) \) by \([dt20]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 1): \) then, \( S(-(u' ȧ 1)) \to -(u' ȧ 0) \) by \([dt21.0]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 2): \) then, \( S(-(u' ȧ 2)) \to -(u' ȧ 1) \) by \([dt21.1]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 3): \) then, \( S(-(u' ȧ 3)) \to -(u' ȧ 2) \) by \([dt21.2]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 4): \) then, \( S(-(u' ȧ 4)) \to -(u' ȧ 3) \) by \([dt21.3]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 5): \) then, \( S(-(u' ȧ 5)) \to -(u' ȧ 4) \) by \([dt21.4]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 6): \) then, \( S(-(u' ȧ 6)) \to -(u' ȧ 5) \) by \([dt21.5]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 7): \) then, \( S(-(u' ȧ 7)) \to -(u' ȧ 6) \) by \([dt21.6]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 8): \) then, \( S(-(u' ȧ 8)) \to -(u' ȧ 7) \) by \([dt21.7]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 9): \) then, \( S(-(u' ȧ 9)) \to -(u' ȧ 8) \) by \([dt21.8]\), and \( t \notin \mathcal{N} \).

\( (2) \ t = P(u): \) apply case distinction on \( u \).

\( u = 0: \) then, \( P(0) \to -1 \) by \([dt14]\), and \( t \notin \mathcal{N} \).

\( u = 1: \) then, \( P(1) \to 0 \) by \([dt15.0]\), and \( t \notin \mathcal{N} \).

\( u = 2: \) then, \( P(2) \to 1 \) by \([dt15.1]\), and \( t \notin \mathcal{N} \).

\( u = 3: \) then, \( P(3) \to 2 \) by \([dt15.2]\), and \( t \notin \mathcal{N} \).

\( u = 4: \) then, \( P(4) \to 3 \) by \([dt15.3]\), and \( t \notin \mathcal{N} \).

\( u = 5: \) then, \( P(5) \to 4 \) by \([dt15.4]\), and \( t \notin \mathcal{N} \).

\( u = 6: \) then, \( P(6) \to 5 \) by \([dt15.5]\), and \( t \notin \mathcal{N} \).

\( u = 7: \) then, \( P(7) \to 6 \) by \([dt15.6]\), and \( t \notin \mathcal{N} \).

\( u = 8: \) then, \( P(8) \to 7 \) by \([dt15.7]\), and \( t \notin \mathcal{N} \).

\( u = 9: \) then, \( P(9) \to 8 \) by \([dt15.8]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 0: \) then, \( P(u' ȧ 0) \to P(u') ȧ 9 \) by \([dt16]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 1: \) then, \( P(u' ȧ 1) \to u' ȧ 0 \) by \([dt17.0]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 2: \) then, \( P(u' ȧ 2) \to u' ȧ 1 \) by \([dt17.1]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 3: \) then, \( P(u' ȧ 3) \to u' ȧ 2 \) by \([dt17.2]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 4: \) then, \( P(u' ȧ 4) \to u' ȧ 3 \) by \([dt17.3]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 5: \) then, \( P(u' ȧ 5) \to u' ȧ 4 \) by \([dt17.4]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 6: \) then, \( P(u' ȧ 6) \to u' ȧ 5 \) by \([dt17.5]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 7: \) then, \( P(u' ȧ 7) \to u' ȧ 6 \) by \([dt17.6]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 8: \) then, \( P(u' ȧ 8) \to u' ȧ 7 \) by \([dt17.7]\), and \( t \notin \mathcal{N} \).

\( u = u' ȧ 9: \) then, \( P(u' ȧ 9) \to u' ȧ 8 \) by \([dt17.8]\), and \( t \notin \mathcal{N} \).

\( u = 1: \) then, \( P(-1) \to -S(1) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 2: \) then, \( P(-2) \to -S(2) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 3: \) then, \( P(-3) \to -S(3) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 4: \) then, \( P(-4) \to -S(4) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 5: \) then, \( P(-5) \to -S(5) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 6: \) then, \( P(-6) \to -S(6) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 7: \) then, \( P(-7) \to -S(7) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 8: \) then, \( P(-8) \to -S(8) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = 9: \) then, \( P(-9) \to -S(9) \) by \([dt18]\), and \( t \notin \mathcal{N} \).

\( u = -(u' ȧ 0): \) then, \( P(-(u' ȧ 0)) \to -S(u' ȧ 0) \) by \([dt18]\), and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 1) \): then, \( P(-(u' \hat{a} 1)) = -S(u' \hat{a} 1) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = (u' \hat{a} 2) \): then, \( P(-(u' \hat{a} 2)) = -S(u' \hat{a} 2) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 3) \): then, \( P(-(u' \hat{a} 3)) = -S(u' \hat{a} 3) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 4) \): then, \( P(-(u' \hat{a} 4)) = -S(u' \hat{a} 4) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 5) \): then, \( P(-(u' \hat{a} 5)) = -S(u' \hat{a} 5) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 6) \): then, \( P(-(u' \hat{a} 6)) = -S(u' \hat{a} 6) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 7) \): then, \( P(-(u' \hat{a} 7)) = -S(u' \hat{a} 7) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 8) \): then, \( P(-(u' \hat{a} 8)) = -S(u' \hat{a} 8) \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = -(u' \hat{a} 9) \): then, \( P(-(u' \hat{a} 9)) = -S(u' \hat{a} 9) \) by [dt18], and \( t \notin \mathcal{N} \).

(3) \( t = u \hat{a} v \): apply case distinction on \( v \).
\( v = 0 \): apply case distinction on \( u \).
\( u = 0 \): then, \( 0 \hat{a} 0 \rightarrow 0 \) by [dt18], and \( t \notin \mathcal{N} \).
\( u = 1 \): then, \( 1 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 2 \): then, \( 2 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 3 \): then, \( 3 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 4 \): then, \( 4 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 5 \): then, \( 5 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 6 \): then, \( 6 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 7 \): then, \( 7 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 8 \): then, \( 8 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = 9 \): then, \( 9 \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 0) \): then, \( (u' \hat{a} 0) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 1) \): then, \( (u' \hat{a} 1) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 2) \): then, \( (u' \hat{a} 2) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 3) \): then, \( (u' \hat{a} 3) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 4) \): then, \( (u' \hat{a} 4) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 5) \): then, \( (u' \hat{a} 5) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 6) \): then, \( (u' \hat{a} 6) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 7) \): then, \( (u' \hat{a} 7) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = (u' \hat{a} 8) \): then, \( (u' \hat{a} 8) \hat{a} 0 \) is irreducible and \( t \in \mathcal{N} \).
\( u = -(u' \cdot \hat{q} \cdot 9) \): then, \(-(u' \cdot \hat{q} \cdot 9)) \cdot \hat{q} \cdot 0 \rightarrow -(u' \cdot \hat{q} \cdot 9) \cdot \hat{q} (-0) \) by \([dt22]\), and \( t \in \mathcal{N} \).

\( v = 1 \): apply case distinction on \( u \).

- \( u = 0 \): then, \( 0 \cdot \hat{q} \cdot 1 \rightarrow 1 \) by \([dt1]\), and \( t \in \mathcal{N} \).
- \( u = 1 \): then, \( 1 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 2 \): then, \( 2 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 3 \): then, \( 3 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 4 \): then, \( 4 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 5 \): then, \( 5 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 6 \): then, \( 6 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 7 \): then, \( 7 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 8 \): then, \( 8 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 9 \): then, \( 9 \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).

- \( u = u' \cdot \hat{q} \cdot 0 \): then, \((u' \cdot \hat{q} \cdot 0) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 1 \): then, \((u' \cdot \hat{q} \cdot 1) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 2 \): then, \((u' \cdot \hat{q} \cdot 2) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 3 \): then, \((u' \cdot \hat{q} \cdot 3) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 4 \): then, \((u' \cdot \hat{q} \cdot 4) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 5 \): then, \((u' \cdot \hat{q} \cdot 5) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 6 \): then, \((u' \cdot \hat{q} \cdot 6) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 7 \): then, \((u' \cdot \hat{q} \cdot 7) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 8 \): then, \((u' \cdot \hat{q} \cdot 8) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = u' \cdot \hat{q} \cdot 9 \): then, \((u' \cdot \hat{q} \cdot 9) \cdot \hat{q} \cdot 1 \) is irreducible and \( t \in \mathcal{N} \).

\( v = 2 \): apply case distinction on \( u \).

- \( u = 0 \): then, \( 0 \cdot \hat{q} \cdot 2 \rightarrow 2 \) by \([dt1]\), and \( t \in \mathcal{N} \).
- \( u = 1 \): then, \( 1 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 2 \): then, \( 2 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 3 \): then, \( 3 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 4 \): then, \( 4 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 5 \): then, \( 5 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
- \( u = 6 \): then, \( 6 \cdot \hat{q} \cdot 2 \) is irreducible and \( t \in \mathcal{N} \).
u = 7: then, $7 \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = 8: then, $8 \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = 9: then, $9 \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 0$: then, $(u' \mathring{a} 0) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 1$: then, $(u' \mathring{a} 1) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 2$: then, $(u' \mathring{a} 2) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 3$: then, $(u' \mathring{a} 3) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 4$: then, $(u' \mathring{a} 4) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 5$: then, $(u' \mathring{a} 5) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 6$: then, $(u' \mathring{a} 6) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 7$: then, $(u' \mathring{a} 7) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 8$: then, $(u' \mathring{a} 8) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 9$: then, $(u' \mathring{a} 9) \mathring{a} 2$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 0$: then, $-((u' \mathring{a} 0) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 1$: then, $-((u' \mathring{a} 1) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 2$: then, $-((u' \mathring{a} 2) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 3$: then, $-((u' \mathring{a} 3) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 4$: then, $-((u' \mathring{a} 4) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 5$: then, $-((u' \mathring{a} 5) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 6$: then, $-((u' \mathring{a} 6) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 7$: then, $-((u' \mathring{a} 7) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 8$: then, $-((u' \mathring{a} 8) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 9$: then, $-((u' \mathring{a} 9) \mathring{a} 2)$ is irreducible and $t \in \mathbb{N}$.

v = 3: apply case distinction on $u$.

u = 0: then, $0 \mathring{a} 3 \rightarrow 3$ by [dt1], and $t \notin \mathbb{N}$.

u = 1: then, $1 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 2: then, $2 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 3: then, $3 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 4: then, $4 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 5: then, $5 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 6: then, $6 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 7: then, $7 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 8: then, $8 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = 9: then, $9 \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 0$: then, $(u' \mathring{a} 0) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 1$: then, $(u' \mathring{a} 1) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 2$: then, $(u' \mathring{a} 2) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 3$: then, $(u' \mathring{a} 3) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 4$: then, $(u' \mathring{a} 4) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.

u = $u' \mathring{a} 5$: then, $(u' \mathring{a} 5) \mathring{a} 3$ is irreducible and $t \in \mathbb{N}$.
\[ u = u' \hat{a} 6: \] then, \((u' \hat{a} 6) \hat{a} 3\) is irreducible and \(t \in \mathcal{N}\).

\[ u = u' \hat{a} 7: \] then, \((u' \hat{a} 7) \hat{a} 3\) is irreducible and \(t \in \mathcal{N}\).

\[ u = u' \hat{a} 8: \] then, \((u' \hat{a} 8) \hat{a} 3\) is irreducible and \(t \in \mathcal{N}\).

\[ u = u' \hat{a} 9: \] then, \((u' \hat{a} 9) \hat{a} 3\) is irreducible and \(t \in \mathcal{N}\).

\[ u = -1: \] then, \((-1) \hat{a} 3 \rightarrow -(1 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -2: \] then, \((-2) \hat{a} 3 \rightarrow -(2 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -3: \] then, \((-3) \hat{a} 3 \rightarrow -(3 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -4: \] then, \((-4) \hat{a} 3 \rightarrow -(4 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -5: \] then, \((-5) \hat{a} 3 \rightarrow -(5 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -6: \] then, \((-6) \hat{a} 3 \rightarrow -(6 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -7: \] then, \((-7) \hat{a} 3 \rightarrow -(7 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -8: \] then, \((-8) \hat{a} 3 \rightarrow -(8 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -9: \] then, \((-9) \hat{a} 3 \rightarrow -(9 \hat{a} (3))\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 0: \] then, \(-(u' \hat{a} 0) \hat{a} 3 \rightarrow -(u' \hat{a} 0) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 1: \] then, \(-(u' \hat{a} 1) \hat{a} 3 \rightarrow -(u' \hat{a} 1) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 2: \] then, \(-(u' \hat{a} 2) \hat{a} 3 \rightarrow -(u' \hat{a} 2) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 3: \] then, \(-(u' \hat{a} 3) \hat{a} 3 \rightarrow -(u' \hat{a} 3) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 4: \] then, \(-(u' \hat{a} 4) \hat{a} 3 \rightarrow -(u' \hat{a} 4) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 5: \] then, \(-(u' \hat{a} 5) \hat{a} 3 \rightarrow -(u' \hat{a} 5) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 6: \] then, \(-(u' \hat{a} 6) \hat{a} 3 \rightarrow -(u' \hat{a} 6) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 7: \] then, \(-(u' \hat{a} 7) \hat{a} 3 \rightarrow -(u' \hat{a} 7) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 8: \] then, \(-(u' \hat{a} 8) \hat{a} 3 \rightarrow -(u' \hat{a} 8) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).

\[ u = -(u' \hat{a}) 9: \] then, \(-(u' \hat{a} 9) \hat{a} 3 \rightarrow -(u' \hat{a} 9) \hat{a}(3)\) by \([dt22]\), and \(t \in \mathcal{N}\).
$u = -6$: then, $(-6) \not{\rightarrow} (6 \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -7$: then, $(-7) \not{\rightarrow} (7 \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -8$: then, $(-8) \not{\rightarrow} (8 \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -9$: then, $(-9) \not{\rightarrow} (9 \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 0)$: then, $((-u' \not{\rightarrow} 0)) \not{\rightarrow} (-(u' \not{\rightarrow} 0) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 1)$: then, $((-u' \not{\rightarrow} 1)) \not{\rightarrow} (-(u' \not{\rightarrow} 1) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 2)$: then, $((-u' \not{\rightarrow} 2)) \not{\rightarrow} (-(u' \not{\rightarrow} 2) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 3)$: then, $((-u' \not{\rightarrow} 3)) \not{\rightarrow} (-(u' \not{\rightarrow} 3) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 4)$: then, $((-u' \not{\rightarrow} 4)) \not{\rightarrow} (-(u' \not{\rightarrow} 4) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 5)$: then, $((-u' \not{\rightarrow} 5)) \not{\rightarrow} (-(u' \not{\rightarrow} 5) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 6)$: then, $((-u' \not{\rightarrow} 6)) \not{\rightarrow} (-(u' \not{\rightarrow} 6) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 7)$: then, $((-u' \not{\rightarrow} 7)) \not{\rightarrow} (-(u' \not{\rightarrow} 7) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 8)$: then, $((-u' \not{\rightarrow} 8)) \not{\rightarrow} (-(u' \not{\rightarrow} 8) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 9)$: then, $((-u' \not{\rightarrow} 9)) \not{\rightarrow} (-(u' \not{\rightarrow} 9) \not{\rightarrow} (-4))$ by $[dt22]$, and $t \in \mathcal{N}$.

**v = 5**

apply case distinction on $u$.

$u = 0$: then, 0 $\not{\rightarrow} 5 \rightarrow 5$ by $[dt1]$, and $t \in \mathcal{N}$.
$u = 1$: then, 1 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 2$: then, 2 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 3$: then, 3 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 4$: then, 4 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 5$: then, 5 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 6$: then, 6 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 7$: then, 7 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 8$: then, 8 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = 9$: then, 9 $\not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.

$u = u' \not{\rightarrow} 0$ then, $(u' \not{\rightarrow} 0) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 1$ then, $(u' \not{\rightarrow} 1) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 2$ then, $(u' \not{\rightarrow} 2) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 3$ then, $(u' \not{\rightarrow} 3) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 4$ then, $(u' \not{\rightarrow} 4) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 5$ then, $(u' \not{\rightarrow} 5) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 6$ then, $(u' \not{\rightarrow} 6) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 7$ then, $(u' \not{\rightarrow} 7) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 8$ then, $(u' \not{\rightarrow} 8) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.
$u = u' \not{\rightarrow} 9$ then, $(u' \not{\rightarrow} 9) \not{\rightarrow} 5$ is irreducible and $t \in \mathcal{N}$.

$u = -1$: then, $(-1) \not{\rightarrow} 5 \rightarrow (1 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -2$: then, $(-2) \not{\rightarrow} 5 \rightarrow (2 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -3$: then, $(-3) \not{\rightarrow} 5 \rightarrow (3 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -4$: then, $(-4) \not{\rightarrow} 5 \rightarrow (4 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -5$: then, $(-5) \not{\rightarrow} 5 \rightarrow (5 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -6$: then, $(-6) \not{\rightarrow} 5 \rightarrow (6 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -7$: then, $(-7) \not{\rightarrow} 5 \rightarrow (7 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -8$: then, $(-8) \not{\rightarrow} 5 \rightarrow (8 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -9$: then, $(-9) \not{\rightarrow} 5 \rightarrow (9 \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.

$u = -(u' \not{\rightarrow} 0)$ then, $(-(u' \not{\rightarrow} 0)) \not{\rightarrow} 5 \rightarrow (-(u' \not{\rightarrow} 0) \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 1)$ then, $(-(u' \not{\rightarrow} 1)) \not{\rightarrow} 5 \rightarrow (-(u' \not{\rightarrow} 1) \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 2)$ then, $(-(u' \not{\rightarrow} 2)) \not{\rightarrow} 5 \rightarrow (-(u' \not{\rightarrow} 2) \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 3)$ then, $(-(u' \not{\rightarrow} 3)) \not{\rightarrow} 5 \rightarrow (-(u' \not{\rightarrow} 3) \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$.
$u = -(u' \not{\rightarrow} 4)$ then, $(-(u' \not{\rightarrow} 4)) \not{\rightarrow} 5 \rightarrow (-(u' \not{\rightarrow} 4) \not{\rightarrow} (-5))$ by $[dt22]$, and $t \in \mathcal{N}$. 

32
\[ u = -(u' \hat{y} 5): \text{ then, } ((u' \hat{y} 5) \hat{y} 5) = -((u' \hat{y} 5) \hat{y} (-5)) \text{ by } [dt22], \text{ and } t \in \mathcal{N}. \]
\[ u = -(u' \hat{y} 6): \text{ then, } ((u' \hat{y} 6) \hat{y} 5) = -((u' \hat{y} 6) \hat{y} (-5)) \text{ by } [dt22], \text{ and } t \in \mathcal{N}. \]
\[ u = -(u' \hat{y} 7): \text{ then, } ((u' \hat{y} 7) \hat{y} 5) = -((u' \hat{y} 7) \hat{y} (-5)) \text{ by } [dt22], \text{ and } t \in \mathcal{N}. \]
\[ u = -(u' \hat{y} 8): \text{ then, } ((u' \hat{y} 8) \hat{y} 5) = -((u' \hat{y} 8) \hat{y} (-5)) \text{ by } [dt22], \text{ and } t \in \mathcal{N}. \]
\[ u = -(u' \hat{y} 9): \text{ then, } ((u' \hat{y} 9) \hat{y} 5) = -((u' \hat{y} 9) \hat{y} (-5)) \text{ by } [dt22], \text{ and } t \in \mathcal{N}. \]

\[ v = 6: \text{ apply case distinction on } u. \]
\[ u = 0: \text{ then, } 0 \hat{y} 6 \to 6 \text{ by } [dt11], \text{ and } t \in \mathcal{N}. \]
\[ u = 1: \text{ then, } 1 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 2: \text{ then, } 2 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 3: \text{ then, } 3 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 4: \text{ then, } 4 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 5: \text{ then, } 5 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 6: \text{ then, } 6 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 7: \text{ then, } 7 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 8: \text{ then, } 8 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 9: \text{ then, } 9 \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 0: \text{ then, } (u' \hat{y} 0) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 1: \text{ then, } (u' \hat{y} 1) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 2: \text{ then, } (u' \hat{y} 2) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 3: \text{ then, } (u' \hat{y} 3) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 4: \text{ then, } (u' \hat{y} 4) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 5: \text{ then, } (u' \hat{y} 5) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 6: \text{ then, } (u' \hat{y} 6) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 7: \text{ then, } (u' \hat{y} 7) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 8: \text{ then, } (u' \hat{y} 8) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = u' \hat{y} 9: \text{ then, } (u' \hat{y} 9) \hat{y} 6 \text{ is irreducible and } t \in \mathcal{N}. \]

\[ v = 7: \text{ apply case distinction on } u. \]
\[ u = 0: \text{ then, } 0 \hat{y} 7 \to 7 \text{ by } [dt11], \text{ and } t \in \mathcal{N}. \]
\[ u = 1: \text{ then, } 1 \hat{y} 7 \text{ is irreducible and } t \in \mathcal{N}. \]
\[ u = 2: \text{ then, } 2 \hat{y} 7 \text{ is irreducible and } t \in \mathcal{N}. \]
$u = 3$: then, $3 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 4$: then, $4 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 5$: then, $5 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 6$: then, $6 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 7$: then, $7 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 8$: then, $8 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = 9$: then, $9 \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 0$: then, $(u' \vartriangle 0) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 1$: then, $(u' \vartriangle 1) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 2$: then, $(u' \vartriangle 2) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 3$: then, $(u' \vartriangle 3) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 4$: then, $(u' \vartriangle 4) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 5$: then, $(u' \vartriangle 5) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 6$: then, $(u' \vartriangle 6) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 7$: then, $(u' \vartriangle 7) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 8$: then, $(u' \vartriangle 8) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 9$: then, $(u' \vartriangle 9) \vartriangle 7$ is irreducible and $t \in \mathcal{N}$.

$v = 8$: apply case distinction on $u$.
$v = 0$: then, $0 \vartriangle 8 \to 8$ by $[dt1]$, and $t \in \mathcal{N}$.
$v = 1$: then, $1 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 2$: then, $2 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 3$: then, $3 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 4$: then, $4 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 5$: then, $5 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 6$: then, $6 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 7$: then, $7 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 8$: then, $8 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$v = 9$: then, $9 \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 0$: then, $(u' \vartriangle 0) \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
$u = u' \vartriangle 1$: then, $(u' \vartriangle 1) \vartriangle 8$ is irreducible and $t \in \mathcal{N}$.
\[u = u' \tilde{a} 2:\] then, \((u' \tilde{a} 2) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 3:\] then, \((u' \tilde{a} 3) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 4:\] then, \((u' \tilde{a} 4) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 5:\] then, \((u' \tilde{a} 5) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 6:\] then, \((u' \tilde{a} 6) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 7:\] then, \((u' \tilde{a} 7) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 8:\] then, \((u' \tilde{a} 8) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = u' \tilde{a} 9:\] then, \((u' \tilde{a} 9) \tilde{a} 8\) is irreducible and \(t \in \mathcal{N}\).
\[u = -1:\] then, \((-1) \tilde{a} 8 \to -(-1) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -2:\] then, \((-2) \tilde{a} 8 \to -(2 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -3:\] then, \((-3) \tilde{a} 8 \to -(3 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -4:\] then, \((-4) \tilde{a} 8 \to -(4 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -5:\] then, \((-5) \tilde{a} 8 \to -(5 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -6:\] then, \((-6) \tilde{a} 8 \to -(6 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -7:\] then, \((-7) \tilde{a} 8 \to -(7 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -8:\] then, \((-8) \tilde{a} 8 \to -(8 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -9:\] then, \((-9) \tilde{a} 8 \to -(9 \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 0):\] then, \(-(u' \tilde{a} 0) \tilde{a} 8 \to -((u' \tilde{a} 0) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 1):\] then, \(-(u' \tilde{a} 1) \tilde{a} 8 \to -((u' \tilde{a} 1) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 2):\] then, \(-(u' \tilde{a} 2) \tilde{a} 8 \to -((u' \tilde{a} 2) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 3):\] then, \(-(u' \tilde{a} 3) \tilde{a} 8 \to -((u' \tilde{a} 3) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 4):\] then, \(-(u' \tilde{a} 4) \tilde{a} 8 \to -((u' \tilde{a} 4) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 5):\] then, \(-(u' \tilde{a} 5) \tilde{a} 8 \to -((u' \tilde{a} 5) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 6):\] then, \(-(u' \tilde{a} 6) \tilde{a} 8 \to -((u' \tilde{a} 6) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 7):\] then, \(-(u' \tilde{a} 7) \tilde{a} 8 \to -((u' \tilde{a} 7) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 8):\] then, \(-(u' \tilde{a} 8) \tilde{a} 8 \to -((u' \tilde{a} 8) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\[u = -(u' \tilde{a} 9):\] then, \(-(u' \tilde{a} 9) \tilde{a} 8 \to -((u' \tilde{a} 9) \tilde{d} -8)\) by \([dt22]\), and \(t \notin \mathcal{N}\).

\(v = 9:\) apply case distinction on \(u\).

\(u = 0:\) then, \(0 \tilde{a} 9 \to 9\) by \([dt1]\), and \(t \notin \mathcal{N}\).
\(u = 1:\) then, \(1 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 2:\) then, \(2 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 3:\) then, \(3 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 4:\) then, \(4 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 5:\) then, \(5 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 6:\) then, \(6 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 7:\) then, \(7 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 8:\) then, \(8 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = 9:\) then, \(9 \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 0:\) then, \((u' \tilde{a} 0) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 1:\) then, \((u' \tilde{a} 1) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 2:\) then, \((u' \tilde{a} 2) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 3:\) then, \((u' \tilde{a} 3) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 4:\) then, \((u' \tilde{a} 4) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 5:\) then, \((u' \tilde{a} 5) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 6:\) then, \((u' \tilde{a} 6) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 7:\) then, \((u' \tilde{a} 7) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 8:\) then, \((u' \tilde{a} 8) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = u' \tilde{a} 9:\) then, \((u' \tilde{a} 9) \tilde{a} 9\) is irreducible and \(t \in \mathcal{N}\).
\(u = -1:\) then, \((-1) \tilde{a} 9 \to -(1 \tilde{d} -9)\) by \([dt22]\), and \(t \notin \mathcal{N}\).
\( u = -2: \) then, \((-2) \quad 9 \rightarrow (-2) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -3: \) then, \((-3) \quad 9 \rightarrow (-3) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -4: \) then, \((-4) \quad 9 \rightarrow (-4) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -5: \) then, \((-5) \quad 9 \rightarrow (-5) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -6: \) then, \((-6) \quad 9 \rightarrow (-6) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -7: \) then, \((-7) \quad 9 \rightarrow (-7) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -8: \) then, \((-8) \quad 9 \rightarrow (-8) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -9: \) then, \((-9) \quad 9 \rightarrow (-9) \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 0): \) then, \((-u' \quad 0) \rightarrow (-u' \quad 0) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 1): \) then, \((-u' \quad 1) \rightarrow (-u' \quad 1) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 2): \) then, \((-u' \quad 2) \rightarrow (-u' \quad 2) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 3): \) then, \((-u' \quad 3) \rightarrow (-u' \quad 3) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 4): \) then, \((-u' \quad 4) \rightarrow (-u' \quad 4) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 5): \) then, \((-u' \quad 5) \rightarrow (-u' \quad 5) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 6): \) then, \((-u' \quad 6) \rightarrow (-u' \quad 6) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 7): \) then, \((-u' \quad 7) \rightarrow (-u' \quad 7) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 8): \) then, \((-u' \quad 8) \rightarrow (-u' \quad 8) \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( u = -(u' \quad 9): \) then, \((-u' \quad 9) \rightarrow (-u' \quad 9) \) by \([dt22]\), and \( t \notin \mathcal{N}\).

\( v = v' \quad 0: \) then, \(u \rightarrow (u + v') \quad 0 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 1: \) then, \(u \rightarrow (v' \quad 1) \rightarrow (u + v') \quad 1 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 2: \) then, \(u \rightarrow (v' \quad 2) \rightarrow (u + v') \quad 2 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 3: \) then, \(u \rightarrow (v' \quad 3) \rightarrow (u + v') \quad 3 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 4: \) then, \(u \rightarrow (v' \quad 4) \rightarrow (u + v') \quad 4 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 5: \) then, \(u \rightarrow (v' \quad 5) \rightarrow (u + v') \quad 5 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 6: \) then, \(u \rightarrow (v' \quad 6) \rightarrow (u + v') \quad 6 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 7: \) then, \(u \rightarrow (v' \quad 7) \rightarrow (u + v') \quad 7 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 8: \) then, \(u \rightarrow (v' \quad 8) \rightarrow (u + v') \quad 8 \) by \([dt22]\), and \( t \notin \mathcal{N}\).
\( v = v' \quad 9: \) then, \(u \rightarrow (v' \quad 9) \rightarrow (u + v') \quad 9 \) by \([dt22]\), and \( t \notin \mathcal{N}\).

\( v = -1: \) apply case distinction on \( u \).
\( u = 0: \) then, \(0 \rightarrow -1 \) by \([dt1]\), and \( t \notin \mathcal{N}\).
\( u = 1: \) then, \(1 \rightarrow 0 \) by \([dt23.1.1]\), and \( t \notin \mathcal{N}\).
\( u = 2: \) then, \(2 \rightarrow 1 \) by \([dt23.2.1]\), and \( t \notin \mathcal{N}\).
\( u = 3: \) then, \(3 \rightarrow 2 \) by \([dt23.3.1]\), and \( t \notin \mathcal{N}\).
\( u = 4: \) then, \(4 \rightarrow 3 \) by \([dt23.4.1]\), and \( t \notin \mathcal{N}\).
\( u = 5: \) then, \(5 \rightarrow 4 \) by \([dt23.5.1]\), and \( t \notin \mathcal{N}\).
\( u = 6: \) then, \(6 \rightarrow 5 \) by \([dt23.6.1]\), and \( t \notin \mathcal{N}\).
\( u = 7: \) then, \(7 \rightarrow 6 \) by \([dt23.7.1]\), and \( t \notin \mathcal{N}\).
\( u = 8: \) then, \(8 \rightarrow 7 \) by \([dt23.8.1]\), and \( t \notin \mathcal{N}\).
\( u = 9: \) then, \(9 \rightarrow 8 \) by \([dt23.9.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 0: \) \((u' \quad 0) \rightarrow (u' \quad 0) \) could not be resolved.
\( u = u' \quad 1: \) then, \((u' \quad 1) \rightarrow (u' \quad 1) \) by \([dt24.1.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 2: \) then, \((u' \quad 2) \rightarrow (u' \quad 1) \) by \([dt24.2.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 3: \) then, \((u' \quad 3) \rightarrow (u' \quad 2) \) by \([dt24.3.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 4: \) then, \((u' \quad 4) \rightarrow (u' \quad 3) \) by \([dt24.4.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 5: \) then, \((u' \quad 5) \rightarrow (u' \quad 4) \) by \([dt24.5.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 6: \) then, \((u' \quad 6) \rightarrow (u' \quad 5) \) by \([dt24.6.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 7: \) then, \((u' \quad 7) \rightarrow (u' \quad 6) \) by \([dt24.7.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 8: \) then, \((u' \quad 8) \rightarrow (u' \quad 7) \) by \([dt24.8.1]\), and \( t \notin \mathcal{N}\).
\( u = u' \quad 9: \) then, \((u' \quad 9) \rightarrow (u' \quad 8) \) by \([dt24.9.1]\), and \( t \notin \mathcal{N}\).
\[
\begin{align*}
u = -1: & \quad \text{then, } (-1, \tilde{a} \rightarrow (-1, \tilde{a} \rightarrow (-1, \tilde{a} \rightarrow (-1))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -2: & \quad \text{then, } (-2, \tilde{a} \rightarrow (-2, \tilde{a} \rightarrow (-2, \tilde{a} \rightarrow (-2))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -3: & \quad \text{then, } (-3, \tilde{a} \rightarrow (-3, \tilde{a} \rightarrow (-3, \tilde{a} \rightarrow (-3))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -4: & \quad \text{then, } (-4, \tilde{a} \rightarrow (-4, \tilde{a} \rightarrow (-4, \tilde{a} \rightarrow (-4))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -5: & \quad \text{then, } (-5, \tilde{a} \rightarrow (-5, \tilde{a} \rightarrow (-5, \tilde{a} \rightarrow (-5))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -6: & \quad \text{then, } (-6, \tilde{a} \rightarrow (-6, \tilde{a} \rightarrow (-6, \tilde{a} \rightarrow (-6))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -7: & \quad \text{then, } (-7, \tilde{a} \rightarrow (-7, \tilde{a} \rightarrow (-7, \tilde{a} \rightarrow (-7))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -8: & \quad \text{then, } (-8, \tilde{a} \rightarrow (-8, \tilde{a} \rightarrow (-8, \tilde{a} \rightarrow (-8))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -9: & \quad \text{then, } (-9, \tilde{a} \rightarrow (-9, \tilde{a} \rightarrow (-9, \tilde{a} \rightarrow (-9))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -10: & \quad \text{then, } (-10, \tilde{a} \rightarrow (-10, \tilde{a} \rightarrow (-10, \tilde{a} \rightarrow (-10))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -11: & \quad \text{then, } (-11, \tilde{a} \rightarrow (-11, \tilde{a} \rightarrow (-11, \tilde{a} \rightarrow (-11))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
\[
\begin{align*}
u = -12: & \quad \text{then, } (-12, \tilde{a} \rightarrow (-12, \tilde{a} \rightarrow (-12, \tilde{a} \rightarrow (-12))) \text{ by } [dt22], \text{ and } t \notin \mathcal{N}\).
\]
$v = -3$: apply case distinction on $u$. 

$u = 0$: then, $0 \hat{a} (-3) \rightarrow -3$ by $[dt1]$, and $t \notin \mathcal{N}$. 

$u = 1$: then, $1 \hat{a} (-3) \rightarrow 0 \hat{a} 7$ by $[dt23.1.3]$, and $t \notin \mathcal{N}$. 

$u = 2$: then, $2 \hat{a} (-3) \rightarrow 1 \hat{a} 7$ by $[dt23.2.3]$, and $t \notin \mathcal{N}$. 

$u = 3$: then, $3 \hat{a} (-3) \rightarrow 2 \hat{a} 7$ by $[dt23.3.3]$, and $t \notin \mathcal{N}$. 

$u = 4$: then, $4 \hat{a} (-3) \rightarrow 3 \hat{a} 7$ by $[dt23.4.3]$, and $t \notin \mathcal{N}$. 

$u = 5$: then, $5 \hat{a} (-3) \rightarrow 4 \hat{a} 7$ by $[dt23.5.3]$, and $t \notin \mathcal{N}$. 

$u = 6$: then, $6 \hat{a} (-3) \rightarrow 5 \hat{a} 7$ by $[dt23.6.3]$, and $t \notin \mathcal{N}$. 

$u = 7$: then, $7 \hat{a} (-3) \rightarrow 6 \hat{a} 7$ by $[dt23.7.3]$, and $t \notin \mathcal{N}$. 

$u = 8$: then, $8 \hat{a} (-3) \rightarrow 7 \hat{a} 7$ by $[dt23.8.3]$, and $t \notin \mathcal{N}$. 

$u = 9$: then, $9 \hat{a} (-3) \rightarrow 8 \hat{a} 7$ by $[dt23.9.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 0$: $(u' \hat{a} 0) \hat{a} (-3)$ could not be resolved. 

$u = u' \hat{a} 1$: then, $(u' \hat{a} 1) \hat{a} (-3) \rightarrow (u' \hat{a} 0) \hat{a} 7$ by $[dt24.1.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 2$: then, $(u' \hat{a} 2) \hat{a} (-3) \rightarrow (u' \hat{a} 1) \hat{a} 7$ by $[dt24.2.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 3$: then, $(u' \hat{a} 3) \hat{a} (-3) \rightarrow (u' \hat{a} 2) \hat{a} 7$ by $[dt24.3.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 4$: then, $(u' \hat{a} 4) \hat{a} (-3) \rightarrow (u' \hat{a} 3) \hat{a} 7$ by $[dt24.4.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 5$: then, $(u' \hat{a} 5) \hat{a} (-3) \rightarrow (u' \hat{a} 4) \hat{a} 7$ by $[dt24.5.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 6$: then, $(u' \hat{a} 6) \hat{a} (-3) \rightarrow (u' \hat{a} 5) \hat{a} 7$ by $[dt24.6.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 7$: then, $(u' \hat{a} 7) \hat{a} (-3) \rightarrow (u' \hat{a} 6) \hat{a} 7$ by $[dt24.7.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 8$: then, $(u' \hat{a} 8) \hat{a} (-3) \rightarrow (u' \hat{a} 7) \hat{a} 7$ by $[dt24.8.3]$, and $t \notin \mathcal{N}$. 

$u = u' \hat{a} 9$: then, $(u' \hat{a} 9) \hat{a} (-3) \rightarrow (u' \hat{a} 8) \hat{a} 7$ by $[dt24.9.3]$, and $t \notin \mathcal{N}$. 

$u = -1$: then, $(-1) \hat{a} (-3) \rightarrow (1 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -2$: then, $(-2) \hat{a} (-3) \rightarrow (2 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -3$: then, $(-3) \hat{a} (-3) \rightarrow (3 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -4$: then, $(-4) \hat{a} (-3) \rightarrow (4 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -5$: then, $(-5) \hat{a} (-3) \rightarrow (5 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -6$: then, $(-6) \hat{a} (-3) \rightarrow (6 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -7$: then, $(-7) \hat{a} (-3) \rightarrow (7 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -8$: then, $(-8) \hat{a} (-3) \rightarrow (8 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$. 

$u = -9$: then, $(-9) \hat{a} (-3) \rightarrow (9 \hat{a} (-3))$ by $[dt22]$, and $t \notin \mathcal{N}$.
\[ u = -(u' \hat{a} 9) : \text{ then, } -(u' \hat{a} 9) \hat{a} (-3) \rightarrow -((u' \hat{a} 9) \hat{a} (-3))) \] by [dt22], and \( t \notin \mathcal{N} \).

\( v = -4: \) apply case distinction on \( u \).

\( u = 0: \) then, 0 \( \hat{a} (-4) \rightarrow -4 \) by [dt1], and \( t \notin \mathcal{N} \).

\( u = 1: \) then, 1 \( \hat{a} (-4) \rightarrow 0 \hat{a} 6 \) by [dt23.1.4], and \( t \notin \mathcal{N} \).

\( u = 2: \) then, 2 \( \hat{a} (-4) \rightarrow 1 \hat{a} 6 \) by [dt23.2.4], and \( t \notin \mathcal{N} \).

\( u = 3: \) then, 3 \( \hat{a} (-4) \rightarrow 2 \hat{a} 6 \) by [dt23.3.4], and \( t \notin \mathcal{N} \).

\( u = 4: \) then, 4 \( \hat{a} (-4) \rightarrow 3 \hat{a} 6 \) by [dt23.4.4], and \( t \notin \mathcal{N} \).

\( u = 5: \) then, 5 \( \hat{a} (-4) \rightarrow 4 \hat{a} 6 \) by [dt23.5.4], and \( t \notin \mathcal{N} \).

\( u = 6: \) then, 6 \( \hat{a} (-4) \rightarrow 5 \hat{a} 6 \) by [dt23.6.4], and \( t \notin \mathcal{N} \).

\( u = 7: \) then, 7 \( \hat{a} (-4) \rightarrow 6 \hat{a} 6 \) by [dt23.7.4], and \( t \notin \mathcal{N} \).

\( u = 8: \) then, 8 \( \hat{a} (-4) \rightarrow 7 \hat{a} 6 \) by [dt23.8.4], and \( t \notin \mathcal{N} \).

\( u = 9: \) then, 9 \( \hat{a} (-4) \rightarrow 8 \hat{a} 6 \) by [dt23.9.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 0: \) (\( u' \hat{a} 0 \) \( \hat{a} (-4) \)) could not be resolved.

\( u = u' \hat{a} 1: \) then, \( u' \hat{a} 1 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 0) \hat{a} 6 \) by [dt24.1.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 2: \) then, \( u' \hat{a} 2 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 1) \hat{a} 6 \) by [dt24.2.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 3: \) then, \( u' \hat{a} 3 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 2) \hat{a} 6 \) by [dt24.3.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 4: \) then, \( u' \hat{a} 4 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 3) \hat{a} 6 \) by [dt24.4.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 5: \) then, \( u' \hat{a} 5 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 4) \hat{a} 6 \) by [dt24.5.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 6: \) then, \( u' \hat{a} 6 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 5) \hat{a} 6 \) by [dt24.6.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 7: \) then, \( u' \hat{a} 7 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 6) \hat{a} 6 \) by [dt24.7.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 8: \) then, \( u' \hat{a} 8 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 7) \hat{a} 6 \) by [dt24.8.4], and \( t \notin \mathcal{N} \).

\( u = u' \hat{a} 9: \) then, \( u' \hat{a} 9 \) \( \hat{a} (-4) \rightarrow (u' \hat{a} 8) \hat{a} 6 \) by [dt24.9.4], and \( t \notin \mathcal{N} \).

\( v = -1: \) then, \( -(1) \hat{a} (-4) \rightarrow -(1 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -2: \) then, \( -(2) \hat{a} (-4) \rightarrow -(2 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -3: \) then, \( -(3) \hat{a} (-4) \rightarrow -(3 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -4: \) then, \( -(4) \hat{a} (-4) \rightarrow -(4 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -5: \) then, \( -(5) \hat{a} (-4) \rightarrow -(5 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -6: \) then, \( -(6) \hat{a} (-4) \rightarrow -(6 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -7: \) then, \( -(7) \hat{a} (-4) \rightarrow -(7 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -8: \) then, \( -(8) \hat{a} (-4) \rightarrow -(8 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -9: \) then, \( -(9) \hat{a} (-4) \rightarrow -(9 \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 0: \) then, \( -(u' \hat{a} 0) \hat{a} (-4) \rightarrow -((u' \hat{a} 0) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 1: \) then, \( -(u' \hat{a} 1) \hat{a} (-4) \rightarrow -((u' \hat{a} 1) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 2: \) then, \( -(u' \hat{a} 2) \hat{a} (-4) \rightarrow -((u' \hat{a} 2) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 3: \) then, \( -(u' \hat{a} 3) \hat{a} (-4) \rightarrow -((u' \hat{a} 3) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 4: \) then, \( -(u' \hat{a} 4) \hat{a} (-4) \rightarrow -((u' \hat{a} 4) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 5: \) then, \( -(u' \hat{a} 5) \hat{a} (-4) \rightarrow -((u' \hat{a} 5) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 6: \) then, \( -(u' \hat{a} 6) \hat{a} (-4) \rightarrow -((u' \hat{a} 6) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 7: \) then, \( -(u' \hat{a} 7) \hat{a} (-4) \rightarrow -((u' \hat{a} 7) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 8: \) then, \( -(u' \hat{a} 8) \hat{a} (-4) \rightarrow -((u' \hat{a} 8) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -(u' \hat{a}) 9: \) then, \( -(u' \hat{a} 9) \hat{a} (-4) \rightarrow -((u' \hat{a} 9) \hat{a} (-4))) \) by [dt22], and \( t \notin \mathcal{N} \).
$u = 7$: then, $7 \not\in (5) \rightarrow 6 \not\in 5$ by $[dt23.7.5]$, and $t \notin N$.

$u = 8$: then, $8 \not\in (5) \rightarrow 7 \not\in 5$ by $[dt23.8.5]$, and $t \notin N$.

$u = 9$: then, $9 \not\in (5) \rightarrow 8 \not\in 5$ by $[dt23.9.5]$, and $t \notin N$.

$u = u' \not\in 0$: $(u' \not\in 0) \not\in (5)$ could not be resolved.

$u = u' \not\in 1$: then, $(u' \not\in 1) \not\in (5) \rightarrow (u' \not\in 0) \not\in 5$ by $[dt24.1.5]$, and $t \notin N$.

$u = u' \not\in 2$: then, $(u' \not\in 2) \not\in (5) \rightarrow (u' \not\in 1) \not\in 5$ by $[dt24.2.5]$, and $t \notin N$.

$u = u' \not\in 3$: then, $(u' \not\in 3) \not\in (5) \rightarrow (u' \not\in 2) \not\in 5$ by $[dt24.3.5]$, and $t \notin N$.

$u = u' \not\in 4$: then, $(u' \not\in 4) \not\in (5) \rightarrow (u' \not\in 3) \not\in 5$ by $[dt24.4.5]$, and $t \notin N$.

$u = u' \not\in 5$: then, $(u' \not\in 5) \not\in (5) \rightarrow (u' \not\in 4) \not\in 5$ by $[dt24.5.5]$, and $t \notin N$.

$u = u' \not\in 6$: then, $(u' \not\in 6) \not\in (5) \rightarrow (u' \not\in 5) \not\in 5$ by $[dt24.6.5]$, and $t \notin N$.

$u = u' \not\in 7$: then, $(u' \not\in 7) \not\in (5) \rightarrow (u' \not\in 6) \not\in 5$ by $[dt24.7.5]$, and $t \notin N$.

$u = u' \not\in 8$: then, $(u' \not\in 8) \not\in (5) \rightarrow (u' \not\in 7) \not\in 5$ by $[dt24.8.5]$, and $t \notin N$.

$u = u' \not\in 9$: then, $(u' \not\in 9) \not\in (5) \rightarrow (u' \not\in 8) \not\in 5$ by $[dt24.9.5]$, and $t \notin N$.

$u = -1$: then, $-1 \not\in (5) \rightarrow -(1 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -2$: then, $-2 \not\in (5) \rightarrow -(2 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -3$: then, $-3 \not\in (5) \rightarrow -(3 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -4$: then, $-4 \not\in (5) \rightarrow -(4 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -5$: then, $-5 \not\in (5) \rightarrow -(5 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -6$: then, $-6 \not\in (5) \rightarrow -(6 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -7$: then, $-7 \not\in (5) \rightarrow -(7 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -8$: then, $-8 \not\in (5) \rightarrow -(8 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -9$: then, $-9 \not\in (5) \rightarrow -(9 \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 0)$: then, $-(u' \not\in 0) \not\in (5) \rightarrow -((u' \not\in 0) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 1)$: then, $-(u' \not\in 1) \not\in (5) \rightarrow -((u' \not\in 1) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 2)$: then, $-(u' \not\in 2) \not\in (5) \rightarrow -((u' \not\in 2) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 3)$: then, $-(u' \not\in 3) \not\in (5) \rightarrow -((u' \not\in 3) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 4)$: then, $-(u' \not\in 4) \not\in (5) \rightarrow -((u' \not\in 4) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 5)$: then, $-(u' \not\in 5) \not\in (5) \rightarrow -((u' \not\in 5) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 6)$: then, $-(u' \not\in 6) \not\in (5) \rightarrow -((u' \not\in 6) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 7)$: then, $-(u' \not\in 7) \not\in (5) \rightarrow -((u' \not\in 7) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 8)$: then, $-(u' \not\in 8) \not\in (5) \rightarrow -((u' \not\in 8) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -(u' \not\in 9)$: then, $-(u' \not\in 9) \not\in (5) \rightarrow -((u' \not\in 9) \not\in (5))$ by $[dt22]$, and $t \notin N$.

$u = -6$: apply case distinction on $u$.

$u = 0$: then, $0 \not\in (6) \rightarrow -6$ by $[dt1]$, and $t \notin N$.

$u = 1$: then, $1 \not\in (6) \rightarrow 0 \not\in 4$ by $[dt23.1.6]$, and $t \notin N$.

$u = 2$: then, $2 \not\in (6) \rightarrow 1 \not\in 4$ by $[dt23.2.6]$, and $t \notin N$.

$u = 3$: then, $3 \not\in (6) \rightarrow 2 \not\in 4$ by $[dt23.3.6]$, and $t \notin N$.

$u = 4$: then, $4 \not\in (6) \rightarrow 3 \not\in 4$ by $[dt23.4.6]$, and $t \notin N$.

$u = 5$: then, $5 \not\in (6) \rightarrow 4 \not\in 4$ by $[dt23.5.6]$, and $t \notin N$.

$u = 6$: then, $6 \not\in (6) \rightarrow 5 \not\in 4$ by $[dt23.6.6]$, and $t \notin N$.

$u = 7$: then, $7 \not\in (6) \rightarrow 6 \not\in 4$ by $[dt23.7.6]$, and $t \notin N$.

$u = 8$: then, $8 \not\in (6) \rightarrow 7 \not\in 4$ by $[dt23.8.6]$, and $t \notin N$.

$u = 9$: then, $9 \not\in (6) \rightarrow 8 \not\in 4$ by $[dt23.9.6]$, and $t \notin N$.

$u = u' \not\in 0$: $(u' \not\in 0) \not\in (6)$ could not be resolved.

$u = u' \not\in 1$: then, $(u' \not\in 1) \not\in (6) \rightarrow (u' \not\in 0) \not\in 4$ by $[dt24.1.6]$, and $t \notin N$.

$u = u' \not\in 2$: then, $(u' \not\in 2) \not\in (6) \rightarrow (u' \not\in 1) \not\in 4$ by $[dt24.2.6]$, and $t \notin N$.

$u = u' \not\in 3$: then, $(u' \not\in 3) \not\in (6) \rightarrow (u' \not\in 2) \not\in 4$ by $[dt24.3.6]$, and $t \notin N$.

$u = u' \not\in 4$: then, $(u' \not\in 4) \not\in (6) \rightarrow (u' \not\in 3) \not\in 4$ by $[dt24.4.6]$, and $t \notin N$.

$u = u' \not\in 5$: then, $(u' \not\in 5) \not\in (6) \rightarrow (u' \not\in 4) \not\in 4$ by $[dt24.5.6]$, and $t \notin N$. 

\( u = u' \partial 6 \): then, \((u' \partial 6) \partial (-6) \to (u' \partial 5) \partial 4 \) by [dt24.6.6], and \( t \notin \mathcal{N} \).

\( u = u' \partial 7 \): then, \((u' \partial 7) \partial (-6) \to (u' \partial 6) \partial 4 \) by [dt24.7.6], and \( t \notin \mathcal{N} \).

\( u = u' \partial 8 \): then, \((u' \partial 8) \partial (-6) \to (u' \partial 7) \partial 4 \) by [dt24.8.6], and \( t \notin \mathcal{N} \).

\( u = u' \partial 9 \): then, \((u' \partial 9) \partial (-6) \to (u' \partial 8) \partial 4 \) by [dt24.9.6], and \( t \notin \mathcal{N} \).

\( u = -1 \): then, \((-1) \partial (-6) \to (-1 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -2 \): then, \((-2) \partial (-6) \to (-2 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -3 \): then, \((-3) \partial (-6) \to (-3 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -4 \): then, \((-4) \partial (-6) \to (-4 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -5 \): then, \((-5) \partial (-6) \to (-5 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -6 \): then, \((-6) \partial (-6) \to (-6 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -7 \): then, \((-7) \partial (-6) \to (-7 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -8 \): then, \((-8) \partial (-6) \to (-8 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -9 \): then, \((-9) \partial (-6) \to (-9 \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 0) \): then, \(-(u' \partial 0) \partial (-6) \to -((u' \partial 0) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 1) \): then, \(-(u' \partial 1) \partial (-6) \to -((u' \partial 1) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 2) \): then, \(-(u' \partial 2) \partial (-6) \to -((u' \partial 2) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 3) \): then, \(-(u' \partial 3) \partial (-6) \to -((u' \partial 3) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 4) \): then, \(-(u' \partial 4) \partial (-6) \to -((u' \partial 4) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 5) \): then, \(-(u' \partial 5) \partial (-6) \to -((u' \partial 5) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 6) \): then, \(-(u' \partial 6) \partial (-6) \to -((u' \partial 6) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 7) \): then, \(-(u' \partial 7) \partial (-6) \to -((u' \partial 7) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 8) \): then, \(-(u' \partial 8) \partial (-6) \to -((u' \partial 8) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -(u' \partial 9) \): then, \(-(u' \partial 9) \partial (-6) \to -((u' \partial 9) \partial (-6))\) by [dt22], and \( t \notin \mathcal{N} \).

\( v = -7 \): apply case distinction on \( u \).

\( v = 0 \): then, \( 0 \partial (-7) \to 7 \) by [dt1], and \( t \notin \mathcal{N} \).

\( v = 1 \): then, \( 1 \partial (-7) \to 0 \partial 3 \) by [dt23.1.7], and \( t \notin \mathcal{N} \).

\( v = 2 \): then, \( 2 \partial (-7) \to 1 \partial 3 \) by [dt23.2.7], and \( t \notin \mathcal{N} \).

\( v = 3 \): then, \( 3 \partial (-7) \to 2 \partial 3 \) by [dt23.3.7], and \( t \notin \mathcal{N} \).

\( v = 4 \): then, \( 4 \partial (-7) \to 3 \partial 3 \) by [dt23.4.7], and \( t \notin \mathcal{N} \).

\( v = 5 \): then, \( 5 \partial (-7) \to 4 \partial 3 \) by [dt23.5.7], and \( t \notin \mathcal{N} \).

\( v = 6 \): then, \( 6 \partial (-7) \to 5 \partial 3 \) by [dt23.6.7], and \( t \notin \mathcal{N} \).

\( v = 7 \): then, \( 7 \partial (-7) \to 6 \partial 3 \) by [dt23.7.7], and \( t \notin \mathcal{N} \).

\( v = 8 \): then, \( 8 \partial (-7) \to 7 \partial 3 \) by [dt23.8.7], and \( t \notin \mathcal{N} \).

\( v = 9 \): then, \( 9 \partial (-7) \to 8 \partial 3 \) by [dt23.9.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 0 \): \( (u' \partial 0) \partial (-7) \) could not be resolved.

\( u = u' \partial 1 \): then, \((u' \partial 1) \partial (-7) \to (u' \partial 0) \partial 3 \) by [dt24.1.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 2 \): then, \((u' \partial 2) \partial (-7) \to (u' \partial 1) \partial 3 \) by [dt24.2.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 3 \): then, \((u' \partial 3) \partial (-7) \to (u' \partial 2) \partial 3 \) by [dt24.3.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 4 \): then, \((u' \partial 4) \partial (-7) \to (u' \partial 3) \partial 3 \) by [dt24.4.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 5 \): then, \((u' \partial 5) \partial (-7) \to (u' \partial 4) \partial 3 \) by [dt24.5.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 6 \): then, \((u' \partial 6) \partial (-7) \to (u' \partial 5) \partial 3 \) by [dt24.6.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 7 \): then, \((u' \partial 7) \partial (-7) \to (u' \partial 6) \partial 3 \) by [dt24.7.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 8 \): then, \((u' \partial 8) \partial (-7) \to (u' \partial 7) \partial 3 \) by [dt24.8.7], and \( t \notin \mathcal{N} \).

\( u = u' \partial 9 \): then, \((u' \partial 9) \partial (-7) \to (u' \partial 8) \partial 3 \) by [dt24.9.7], and \( t \notin \mathcal{N} \).

\( u = -1 \): then, \((-1) \partial (-7) \to -(-1 \partial (-7))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -2 \): then, \((-2) \partial (-7) \to -(2 \partial (-7))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -3 \): then, \((-3) \partial (-7) \to -(3 \partial (-7))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -4 \): then, \((-4) \partial (-7) \to -(4 \partial (-7))\) by [dt22], and \( t \notin \mathcal{N} \).

\( u = -5 \): then, \((-5) \partial (-7) \to -(5 \partial (-7))\) by [dt22], and \( t \notin \mathcal{N} \).
\( u = -6: \) then, \((-6) \hat{a}(-7) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -7: \) then, \((-7) \hat{a}(-7) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -8: \) then, \((-8) \hat{a}(-8) \rightarrow \hat{a}(-8)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -9: \) then, \((-9) \hat{a}(-9) \rightarrow \hat{a}(-9)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 0): \) then, \((-u' \hat{a} 0) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 1): \) then, \((-u' \hat{a} 1) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 2): \) then, \((-u' \hat{a} 2) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 3): \) then, \((-u' \hat{a} 3) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 4): \) then, \((-u' \hat{a} 4) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 5): \) then, \((-u' \hat{a} 5) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 6): \) then, \((-u' \hat{a} 6) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 7): \) then, \((-u' \hat{a} 7) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 8): \) then, \((-u' \hat{a} 8) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 9): \) then, \((-u' \hat{a} 9) \rightarrow \hat{a}(-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -8: \) apply case distinction on \( u. \)

\( u = 0: \) then, \(0 \hat{a}(8) \rightarrow \hat{a}(8)) by [dt1], and \( t \in \mathcal{N}. \)

\( u = 1: \) then, \(1 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 2: \) then, \(2 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 3: \) then, \(3 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 4: \) then, \(4 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 5: \) then, \(5 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 6: \) then, \(6 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 7: \) then, \(7 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 8: \) then, \(8 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = 9: \) then, \(9 \hat{a}(8) \rightarrow \hat{a}(8)\) by [dt31.2.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 0: \) \((u' \hat{a} 0) \rightarrow \hat{a}(8)) could not be resolved.\n
\( u = u' \hat{a} 1: \) then, \((u' \hat{a} 1) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 2: \) then, \((u' \hat{a} 2) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 3: \) then, \((u' \hat{a} 3) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 4: \) then, \((u' \hat{a} 4) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 5: \) then, \((u' \hat{a} 5) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 6: \) then, \((u' \hat{a} 6) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 7: \) then, \((u' \hat{a} 7) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 8: \) then, \((u' \hat{a} 8) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = u' \hat{a} 9: \) then, \((u' \hat{a} 9) \rightarrow \hat{a}(8)) by [dt24.1.8], and \( t \in \mathcal{N}. \)

\( u = -1: \) then, \((-1) \rightarrow (-1)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -2: \) then, \((-2) \rightarrow (-2)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -3: \) then, \((-3) \rightarrow (-3)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -4: \) then, \((-4) \rightarrow (-4)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -5: \) then, \((-5) \rightarrow (-5)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -6: \) then, \((-6) \rightarrow (-6)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -7: \) then, \((-7) \rightarrow (-7)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -8: \) then, \((-8) \rightarrow (-8)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -9: \) then, \((-9) \rightarrow (-9)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 0): \) then, \((-u' \hat{a} 0) \rightarrow (-u' \hat{a} 0)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 1): \) then, \((-u' \hat{a} 1) \rightarrow (-u' \hat{a} 1)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 2): \) then, \((-u' \hat{a} 2) \rightarrow (-u' \hat{a} 2)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 3): \) then, \((-u' \hat{a} 3) \rightarrow (-u' \hat{a} 3)) by [dt22], and \( t \in \mathcal{N}. \)

\( u = -(u' \hat{a} 4): \) then, \((-u' \hat{a} 4) \rightarrow (-u' \hat{a} 4)) by [dt22], and \( t \in \mathcal{N}. \)

42
$v = -9$: apply case distinction on $u$.

$u = 0$: then, $0 \ avoids (-9) \to -9$ by [dt1], and $t \notin \mathcal{N}$.

$u = 1$: then, $1 \ avoids (-9) \to 0 \ avoids (-9)$ by [dt23.1.9], and $t \notin \mathcal{N}$.

$u = 2$: then, $2 \ avoids (-9) \to 1 \ avoids (-9)$ by [dt23.2.9], and $t \notin \mathcal{N}$.

$u = 3$: then, $3 \ avoids (-9) \to 2 \ avoids (-9)$ by [dt23.3.9], and $t \notin \mathcal{N}$.

$u = 4$: then, $4 \ avoids (-9) \to 3 \ avoids (-9)$ by [dt23.4.9], and $t \notin \mathcal{N}$.

$u = 5$: then, $5 \ avoids (-9) \to 4 \ avoids (-9)$ by [dt23.5.9], and $t \notin \mathcal{N}$.

$u = 6$: then, $6 \ avoids (-9) \to 5 \ avoids (-9)$ by [dt23.6.9], and $t \notin \mathcal{N}$.

$u = 7$: then, $7 \ avoids (-9) \to 6 \ avoids (-9)$ by [dt23.7.9], and $t \notin \mathcal{N}$.

$u = 8$: then, $8 \ avoids (-9) \to 7 \ avoids (-9)$ by [dt23.8.9], and $t \notin \mathcal{N}$.

$u = 9$: then, $9 \ avoids (-9) \to 8 \ avoids (-9)$ by [dt23.9.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids (-9)$: then, $(u' \ avoids (-9)) \ avoids (-9)$ could not be resolved.

$u = u' \ avoids 0$: then, $(u' \ avoids 0) \ avoids (-9) \to (u' \ avoids 0) \ avoids (-9)$ by [dt24.1.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 2$: then, $(u' \ avoids 2) \ avoids (-9) \to (u' \ avoids 2) \ avoids (-9)$ by [dt24.2.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 3$: then, $(u' \ avoids 3) \ avoids (-9) \to (u' \ avoids 3) \ avoids (-9)$ by [dt24.3.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 4$: then, $(u' \ avoids 4) \ avoids (-9) \to (u' \ avoids 4) \ avoids (-9)$ by [dt24.4.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 5$: then, $(u' \ avoids 5) \ avoids (-9) \to (u' \ avoids 5) \ avoids (-9)$ by [dt24.5.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 6$: then, $(u' \ avoids 6) \ avoids (-9) \to (u' \ avoids 6) \ avoids (-9)$ by [dt24.6.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 7$: then, $(u' \ avoids 7) \ avoids (-9) \to (u' \ avoids 7) \ avoids (-9)$ by [dt24.7.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 8$: then, $(u' \ avoids 8) \ avoids (-9) \to (u' \ avoids 8) \ avoids (-9)$ by [dt24.8.9], and $t \notin \mathcal{N}$.

$u = u' \ avoids 9$: then, $(u' \ avoids 9) \ avoids (-9) \to (u' \ avoids 9) \ avoids (-9)$ by [dt24.9.9], and $t \notin \mathcal{N}$.

$u = -1$: then, $(-1) \ avoids (-9) \to -1 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -2$: then, $(-2) \ avoids (-9) \to -2 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -3$: then, $(-3) \ avoids (-9) \to -3 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -4$: then, $(-4) \ avoids (-9) \to -4 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -5$: then, $(-5) \ avoids (-9) \to -5 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -6$: then, $(-6) \ avoids (-9) \to -6 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -7$: then, $(-7) \ avoids (-9) \to -7 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -8$: then, $(-8) \ avoids (-9) \to -8 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -9$: then, $(-9) \ avoids (-9) \to -9 \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 0)$: then, $-(u' \ avoids 0) \ avoids (-9) \to -(u' \ avoids 0) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 1)$: then, $-(u' \ avoids 1) \ avoids (-9) \to -(u' \ avoids 1) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 2)$: then, $-(u' \ avoids 2) \ avoids (-9) \to -(u' \ avoids 2) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 3)$: then, $-(u' \ avoids 3) \ avoids (-9) \to -(u' \ avoids 3) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 4)$: then, $-(u' \ avoids 4) \ avoids (-9) \to -(u' \ avoids 4) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 5)$: then, $-(u' \ avoids 5) \ avoids (-9) \to -(u' \ avoids 5) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 6)$: then, $-(u' \ avoids 6) \ avoids (-9) \to -(u' \ avoids 6) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 7)$: then, $-(u' \ avoids 7) \ avoids (-9) \to -(u' \ avoids 7) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 8)$: then, $-(u' \ avoids 8) \ avoids (-9) \to -(u' \ avoids 8) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.

$u = -(u' \ avoids 9)$: then, $-(u' \ avoids 9) \ avoids (-9) \to -(u' \ avoids 9) \ avoids (-9)$ by [dt22], and $t \notin \mathcal{N}$.
\( v = -(\nu' \dot{a} 4) \): then, \( \dot{u} = -(\nu' \dot{a} 4) \rightarrow -(\nu' + u) \dot{a} 4 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  
\( v = -(\nu' \dot{a} 5) \): then, \( \dot{u} = -(\nu' \dot{a} 5) \rightarrow -(\nu' + u) \dot{a} 5 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  
\( v = -(\nu' \dot{a} 6) \): then, \( \dot{u} = -(\nu' \dot{a} 6) \rightarrow -(\nu' + u) \dot{a} 6 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  
\( v = -(\nu' \dot{a} 7) \): then, \( \dot{u} = -(\nu' \dot{a} 7) \rightarrow -(\nu' + u) \dot{a} 7 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  
\( v = -(\nu' \dot{a} 8) \): then, \( \dot{u} = -(\nu' \dot{a} 8) \rightarrow -(\nu' + u) \dot{a} 8 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  
\( v = -(\nu' \dot{a} 9) \): then, \( \dot{u} = -(\nu' \dot{a} 9) \rightarrow -(\nu' + u) \dot{a} 9 \) by \([dt25]\), and \( t \notin \mathcal{N} \).  

(4) \( t = u + v \): apply case distinction on \( v \).  
\( v = 0 \): then, \( u + 0 \rightarrow u \) by \([dt7.0]\), and \( t \notin \mathcal{N} \).  
\( v = 1 \): then, \( u + 1 \rightarrow S(u) \) by \([dt7.1]\), and \( t \notin \mathcal{N} \).  
\( v = 2 \): then, \( u + 2 \rightarrow S(S(u)) \) by \([dt7.2]\), and \( t \notin \mathcal{N} \).  
\( v = 3 \): then, \( u + 3 \rightarrow S(S(S(u))) \) by \([dt7.3]\), and \( t \notin \mathcal{N} \).  
\( v = 4 \): then, \( u + 4 \rightarrow S(S(S(S(u)))) \) by \([dt7.4]\), and \( t \notin \mathcal{N} \).  
\( v = 5 \): then, \( u + 5 \rightarrow S(S(S(S(S(u)))) \) by \([dt7.5]\), and \( t \notin \mathcal{N} \).  
\( v = 6 \): then, \( u + 6 \rightarrow S(S(S(S(S(S(u)))) \) by \([dt7.6]\), and \( t \notin \mathcal{N} \).  
\( v = 7 \): then, \( u + 7 \rightarrow S(S(S(S(S(S(S(u)))) \) by \([dt7.7]\), and \( t \notin \mathcal{N} \).  
\( v = 8 \): then, \( u + 8 \rightarrow S(S(S(S(S(S(S(S(u)))) \) by \([dt7.8]\), and \( t \notin \mathcal{N} \).  
\( v = 9 \): then, \( u + 9 \rightarrow S(S(S(S(S(S(S(S(S(S(u)))) \) by \([dt7.9]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 0 \): then, \( u + (\nu' \dot{a} 0) \rightarrow (\nu' \dot{a} u) \) by \([dt8.0]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 1 \): then, \( u + (\nu' \dot{a} 1) \rightarrow S((\nu' \dot{a} u)) \) by \([dt8.1]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 2 \): then, \( u + (\nu' \dot{a} 2) \rightarrow S(S((\nu' \dot{a} u))) \) by \([dt8.2]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 3 \): then, \( u + (\nu' \dot{a} 3) \rightarrow S(S(S((\nu' \dot{a} u))) \) by \([dt8.3]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 4 \): then, \( u + (\nu' \dot{a} 4) \rightarrow S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.4]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 5 \): then, \( u + (\nu' \dot{a} 5) \rightarrow S(S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.5]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 6 \): then, \( u + (\nu' \dot{a} 6) \rightarrow S(S(S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.6]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 7 \): then, \( u + (\nu' \dot{a} 7) \rightarrow S(S(S(S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.7]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 8 \): then, \( u + (\nu' \dot{a} 8) \rightarrow S(S(S(S(S(S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.8]\), and \( t \notin \mathcal{N} \).  
\( v = \nu' \dot{a} 9 \): then, \( u + (\nu' \dot{a} 9) \rightarrow S(S(S(S(S(S(S(S(S(S(S((\nu' \dot{a} u))) \) by \([dt8.9]\), and \( t \notin \mathcal{N} \).  

(5) \( t = u + v \): apply case distinction on \( v \).  
\( v = 0 \): then, \( u + 0 \rightarrow u \) by \([dt9.0]\), and \( t \notin \mathcal{N} \).  
\( v = 1 \): then, \( u + 1 \rightarrow u \) by \([dt10.1]\), and \( t \notin \mathcal{N} \).
$v = 2$: then, $u \cdot 2 \to u + u$ by $[dt10.2]$, and $t \notin \mathcal{N}$.
$v = 3$: then, $u \cdot 3 \to (u + u) + u$ by $[dt10.3]$, and $t \notin \mathcal{N}$.
$v = 4$: then, $u \cdot 4 \to ((u + u) + u) + u$ by $[dt10.4]$, and $t \notin \mathcal{N}$.
$v = 5$: then, $u \cdot 5 \to (((u + u) + u) + u) + u$ by $[dt10.5]$, and $t \notin \mathcal{N}$.
$v = 6$: then, $u \cdot 6 \to (((u + u) + u) + u) + u$ by $[dt10.6]$, and $t \notin \mathcal{N}$.
$v = 7$: then, $u \cdot 7 \to (((((u + u) + u) + u) + u) + u)$ by $[dt10.7]$, and $t \notin \mathcal{N}$.
$v = 8$: then, $u \cdot 8 \to (((((u + u) + u) + u) + u) + u)$ by $[dt10.8]$, and $t \notin \mathcal{N}$.
$v = 9$: then, $u \cdot 9 \to (((((u + u) + u) + u) + u) + u)$ by $[dt10.9]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}0$: then, $u \cdot (v'/\hat{a}0) \to ((u \cdot v')\hat{a}0) + (u \cdot 0)$ by $[dt11.0]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}1$: then, $u \cdot (v'/\hat{a}1) \to ((u \cdot v')\hat{a}0) + (u \cdot 1)$ by $[dt11.1]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}2$: then, $u \cdot (v'/\hat{a}2) \to ((u \cdot v')\hat{a}0) + (u \cdot 2)$ by $[dt11.2]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}3$: then, $u \cdot (v'/\hat{a}3) \to ((u \cdot v')\hat{a}0) + (u \cdot 3)$ by $[dt11.3]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}4$: then, $u \cdot (v'/\hat{a}4) \to ((u \cdot v')\hat{a}0) + (u \cdot 4)$ by $[dt11.4]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}5$: then, $u \cdot (v'/\hat{a}5) \to ((u \cdot v')\hat{a}0) + (u \cdot 5)$ by $[dt11.5]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}6$: then, $u \cdot (v'/\hat{a}6) \to ((u \cdot v')\hat{a}0) + (u \cdot 6)$ by $[dt11.6]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}7$: then, $u \cdot (v'/\hat{a}7) \to ((u \cdot v')\hat{a}0) + (u \cdot 7)$ by $[dt11.7]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}8$: then, $u \cdot (v'/\hat{a}8) \to ((u \cdot v')\hat{a}0) + (u \cdot 8)$ by $[dt11.8]$, and $t \notin \mathcal{N}$.
$v = v'/\hat{a}9$: then, $u \cdot (v'/\hat{a}9) \to ((u \cdot v')\hat{a}0) + (u \cdot 9)$ by $[dt11.9]$, and $t \notin \mathcal{N}$.
$v = -1$: then, $u \cdot -1 \to -(u \cdot 1)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -2$: then, $u \cdot -2 \to -(u \cdot 2)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -3$: then, $u \cdot -3 \to -(u \cdot 3)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -4$: then, $u \cdot -4 \to -(u \cdot 4)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -5$: then, $u \cdot -5 \to -(u \cdot 5)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -6$: then, $u \cdot -6 \to -(u \cdot 6)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -7$: then, $u \cdot -7 \to -(u \cdot 7)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -8$: then, $u \cdot -8 \to -(u \cdot 8)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -9$: then, $u \cdot -9 \to -(u \cdot 9)$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}0)$: then, $u \cdot -(v'/\hat{a}0) \to -(u \cdot (v'/\hat{a}0))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}1)$: then, $u \cdot -(v'/\hat{a}1) \to -(u \cdot (v'/\hat{a}1))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}2)$: then, $u \cdot -(v'/\hat{a}2) \to -(u \cdot (v'/\hat{a}2))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}3)$: then, $u \cdot -(v'/\hat{a}3) \to -(u \cdot (v'/\hat{a}3))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}4)$: then, $u \cdot -(v'/\hat{a}4) \to -(u \cdot (v'/\hat{a}4))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}5)$: then, $u \cdot -(v'/\hat{a}5) \to -(u \cdot (v'/\hat{a}5))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}6)$: then, $u \cdot -(v'/\hat{a}6) \to -(u \cdot (v'/\hat{a}6))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}7)$: then, $u \cdot -(v'/\hat{a}7) \to -(u \cdot (v'/\hat{a}7))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}8)$: then, $u \cdot -(v'/\hat{a}8) \to -(u \cdot (v'/\hat{a}8))$ by $[dt28]$, and $t \notin \mathcal{N}$.
$v = -(v'/\hat{a}9)$: then, $u \cdot -(v'/\hat{a}9) \to -(u \cdot (v'/\hat{a}9))$ by $[dt28]$, and $t \notin \mathcal{N}$.

$(6) t = -u$: apply case distinction on $u$.

$u = 0$: then, $-0 \to 0$ by $[dt12]$, and $t \notin \mathcal{N}$.
$u = 1$: then, $-1$ is irreducible and $t \notin \mathcal{N}$.
$u = 2$: then, $-2$ is irreducible and $t \notin \mathcal{N}$.
$u = 3$: then, $-3$ is irreducible and $t \notin \mathcal{N}$.
$u = 4$: then, $-4$ is irreducible and $t \notin \mathcal{N}$.
$u = 5$: then, $-5$ is irreducible and $t \notin \mathcal{N}$.
$u = 6$: then, $-6$ is irreducible and $t \notin \mathcal{N}$.
$u = 7$: then, $-7$ is irreducible and $t \notin \mathcal{N}$.
$u = 8$: then, $-8$ is irreducible and $t \notin \mathcal{N}$.
$u = 9$: then, $-9$ is irreducible and $t \notin \mathcal{N}$.
$u = u'/\hat{a}0$: then, $-(u'/\hat{a}0)$ is irreducible and $t \notin \mathcal{N}$. 

45
The following counterexamples were found:

1. \( (u' \hat{a})^0 \hat{a} \rightarrow -1 \)
2. \( (u' \hat{a})^0 \hat{a} \rightarrow -2 \)
3. \( (u' \hat{a})^0 \hat{a} \rightarrow -3 \)
4. \( (u' \hat{a})^0 \hat{a} \rightarrow -4 \)
5. \( (u' \hat{a})^0 \hat{a} \rightarrow -5 \)
6. \( (u' \hat{a})^0 \hat{a} \rightarrow -6 \)
7. \( (u' \hat{a})^0 \hat{a} \rightarrow -7 \)
8. \( (u' \hat{a})^0 \hat{a} \rightarrow -8 \)
9. \( (u' \hat{a})^0 \hat{a} \rightarrow -9 \)

...