Entanglement entropy and gravitational anomalies
Computing entanglement entropy in different limits of asymptotically AdS$_3$ spacetimes.

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Abstract

Entanglement entropy is a fundamental quantity in Quantum Field Theory, characterizing the amount to which a system is entangled with its environment. For Conformal Field Theories (CFTs) with a holographic dual which is described by Einstein gravity, the entanglement entropy can be calculated using the celebrated Ryu-Takayanagi (RT) formula. However, the dual theory to a CFT with a gravitational anomaly is instead a higher derivative theory of gravity, known as Topologically Massive Gravity (TMG), so the RT procedure does not apply. An extension of the RT formula to TMG has recently been developed in [1] and there it was found that the entanglement entropy can be calculated using solutions of the so-called MPD equations, which describe spinning particles in classical General Relativity. We review this procedure and study the properties of the MPD equations in asymptotically AdS$_3$ spacetimes. Perturbative solutions can be constructed in the limit when the spin $s$ is small, but this perturbation breaks down when the interior of the spacetime is probed. We therefore examine the opposite limit: that of small mass $m$. We will look for perturbative solutions in this regime and outline a procedure to match the different solutions arising from these two limiting cases.
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1 Introduction

This thesis is devoted to computing entanglement entropy holographically in AdS$_3$/CFT$_2$. Before diving into specifics, an introduction into these subjects is appropriate. We will first motivate the choice for three-dimensional gravity and then provide an introduction into the holographic principle and to the role played by entanglement entropy in this framework.

1.1 Three-dimensional gravity

The theory of gravity simplifies markedly when the number of dimensions is reduced to three. This is not just because there are fewer equations to solve, but there are deeper reasons behind this simplification. Pure gravity in three dimensions is described by the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} (R - 2\Lambda),$$  

(1.1)

where $G_3$ is the three-dimensional Newton’s constant and $\Lambda$ is the cosmological constant. The equations of motion are the familiar Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$  

(1.2)

In any number of dimensions, the Riemann tensor $R^{\rho\nu}_{\mu\rho\sigma}$ can be decomposed into the symmetric Ricci tensor $R_{\mu\nu}$ and the antisymmetric Weyl tensor $C_{\mu\nu\rho\sigma}$. In three dimensions, the Ricci and Riemann tensor have the same number of degrees of freedom, which implies the vanishing of the Weyl tensor. The Ricci tensor is related to the energy momentum tensor through Einstein’s equations and this in turn means that the curvature is completely determined by the matter content of the theory [2]. Physically, this has the effect that local propagating degrees of freedom like gravitational waves do not occur in three-dimensional gravity, making it a much simpler model to study conceptual issues in general relativity. As a related fact, the Ricci tensor is completely determined by the metric, when the equations of motion are satisfied: $R_{\mu\nu} = \Lambda g_{\mu\nu}$. This means that vacuum solutions to three-dimensional gravity have constant curvature and can locally be classified by the sign of the cosmological constant $\Lambda$. Solutions with positive cosmological constant have de Sitter (dS) geometry, negative cosmological constant solutions are of anti de Sitter (AdS) type and $\Lambda = 0$ implies that the solution is just Minkowski spacetime. This means that there is for example no Schwarzschild-type geometry in three dimensions. This is however not the full story: solutions can contain more structure than these maximally symmetric spacetimes and the reason for this lies in the meaning of 'locally’.

Three-dimensional gravity may be locally trivial, but there can still be non-trivial global effects. These global effects even give rise to veritable black hole solutions [3], which will be discussed in section 3. The existence of just global degrees of freedom is the reason that three-dimensional gravity is referred to as a topological theory. This fact is made more clear when 3D gravity is written in the first order formalism, where the basic variables are the vielbein $e^\mu_a$ and the spin connection $\omega_{ab}^\mu$. It was first described in [4] that in terms of these variables, the Einstein-Hilbert action (1.1) describing 3D Ads takes the same form of the action of a Chern-Simons gauge theory:

$$I_{EH} = I_{CS}[A] - I_{CS}[\bar{A}],$$  

(1.3)

where

$$I_{CS}[A] = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad A = \omega + \frac{1}{L} e, \quad \bar{A} = \omega - \frac{1}{L} e.$$  

(1.4)
Here we wrote the action in form notation to reduce the amount of indices, $L$ is the AdS radius and $k$ is constant known as the level of the theory. Chern-Simons theories are topological field theories (TFTs) which play a role in many areas of physics [5]. Although the Chern-Simons formulation of 3D gravity can be very useful, we will stick to the metric formulation in this thesis.

For the purpose of this thesis, three-dimensional gravity is useful because entanglement entropy is most easily computable in 3D. Both in three-dimensional AdS and in the dual two-dimensional CFT, exact expressions for entanglement entropy can be found in different setups. We will cover these expressions in the following chapters.

1.2 Holography and entanglement entropy

The holographic principle states that a theory of (quantum) gravity in $d+1$ dimensions is equivalent to a quantum theory without gravity in $d$ dimensions. This is a very non-trivial and in many ways unexpected statement, which has revolutionized many different areas of theoretical physics. The holographic principle was first developed by Susskind [6], who used previous ideas of ’t Hooft [7] and was inspired by black hole thermodynamics. Bekenstein and Hawking had for example discovered that the entropy of a black hole is proportional to the area of its event horizon [8, 9]:

$$S_{BH} = \frac{A}{4G_N}. \quad (1.5)$$

This suggests that at least a black hole in $d+1$ dimensions can be described by data on a $d$-dimensional manifold. More indications that holography could be a central concept in theoretical physics had already been given prior to Susskind by Brown and Henneaux, who proved that the asymptotic symmetry group of three-dimensional AdS spacetime is isomorphic to the symmetry group of two-dimensional conformal field theory [10]. The major advancement however came when Maldacena proposed that type IIB string theory on $AdS_5 \times S^5$ is equivalent to four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [11]. This idea was then generalized to other examples involving AdS-spacetimes and conformal field theories and became known as the AdS/CFT-correspondence. Since its discovery, AdS/CFT has been one of the most researched topics in theoretical physics and many aspects of the duality have been found, together known as the AdS/CFT dictionary. We will not provide an introduction to this dictionary in this thesis, but refer the reader to reviews such as [12, 13, 14, 15].

The AdS/CFT dictionary links observables in the CFT to those in AdS and vice versa. We will in this thesis mainly be occupied with one of these observables: entanglement entropy (EE). The entanglement entropy of some subsystem $A$ in a quantum theory provides a measure for the degree to which $A$ is entangled with its environment. The gravitational dual to entanglement entropy was discovered by Ryu and Takayanagi [16], who found that:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}. \quad (1.6)$$

Here $\gamma_A$ is a minimal surface extending into the bulk, whose boundary coincides with the boundary of $A$ and $G_N$ is the Newton constant of the bulk spacetime. The Ryu-Takayanagi (RT) formula is an example of the power of holography: it relates a complicated non-local field-theoretical quantity like entanglement entropy to a purely geometric and relatively ‘clean’ quantity like geodesic length. The RT procedure provides us with a way to compute entanglement entropy holographically and indirectly learn more about the field theory.

The RT formula applies to static configurations in a conformal field theory with an Einstein gravity dual. It is natural to ask whether it can be extended to more general situations. This thesis is centered around one of these generalizations: we will consider CFTs which suffer from
a gravitational anomaly. In [1] a prescription was developed to compute entanglement entropy holographically for these theories. The authors found that the EE is given not by a solution to the geodesic equation (i.e. a minimal surface) but by a solution to the Mathisson-Papapetrou-Dixon (MPD) equations, which describe spinning particles in classical General Relativity. The anomaly has the effect of introducing extra data in the form of normal vectors which have to be tracked along the geodesic, effectively broadening it into a ribbon.

The method introduced in [1] was later applied in [17] to a class of theories with asymptotically AdS boundary conditions. There the EE is computed explicitly in the limit of the particles having small spin but the validity of this expansion is limited. In this thesis we will review the arguments of [1] and [17] and aim to further the understanding of holographic EE by looking for solutions in different limits and examining how these solutions can be matched.

1.3 Outline

This thesis is organised as follows. We start in section 2 by reviewing the definition of entanglement entropy and the methods to calculate the EE in two-dimensional conformal field theories. AdS/CFT tells us that the dual to these 2D CFTs is three-dimensional Anti de Sitter space and we will examine methods of computing entanglement entropy in AdS$_3$ in section 3. We want to study how entanglement entropy can be computed for anomalous 2D CFTs and in section 4 we turn our attention to the way in which anomalies are introduced in field theories and their effect on the entanglement entropy. In section 5 we focus on the procedure outlined in [1] to compute entanglement entropy holographically in these anomalous theories. We will review their method and see how the MPD equations emerge from the minimization of a new entanglement entropy functional. In section 6 we then consider a more general class of backgrounds, which have asymptotically AdS$_3$ boundary conditions. The authors of [17] managed to construct perturbative solutions to the MPD equations in these backgrounds and we will review their methods. Furthermore, we will consider a different perturbative limit than that of [17] and study how these different perturbations can be related to each other.
2 Entanglement entropy in Conformal Field Theories

Entanglement is a property of quantum mechanics which has no analogue in classical mechanics. It is central in discussions about the interpretation of quantum mechanics [18] and its properties are still the subject of experimental research [19]. EE provides a measure for the entanglement of degrees of freedom in a system. It is used in areas of science as far apart as condensed matter research and information theory. We will look at entanglement entropy in the context of high energy physics and we will see that it captures phenomena which are hard to see in other ways.

We will begin this section by recalling the definition of EE in quantum mechanics and going over some of its well-known properties. We will then look at entanglement entropy in the context of Quantum Field Theory and use methods presented in [20, 21] to find an expression for entanglement entropy in QFT. We will then specialise to two-dimensional conformal field theory, where it is actually possible to evaluate the expression for the entanglement entropy. We will finish by covering several examples.

2.1 Basics of entanglement entropy

We can define a density matrix for a general quantum system in a state $|\psi_i\rangle$ as:

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|.$$ (2.1)

Here $p_i$ is the probability of finding the system in the state $|\psi_i\rangle$. The state $|\psi_i\rangle$ can either be a pure state or a mixed state. If $|\psi_i\rangle$ is a pure state, all except one of the $p_i$ are zero, the non-zero one automatically being unity. Its density matrix is then:

$$\rho = |\Psi\rangle \langle \Psi|.$$ (2.2)

An example of a mixed state is a thermal state at temperature $\beta^{-1}$. The probability of finding the system is such a state is given by the Boltzmann factor, so the density matrix is:

$$\rho_{\text{thermal}} = \sum_i e^{-\beta E_i} |\psi_i\rangle \langle \psi_i|.$$ (2.3)

To define the entanglement entropy, we then assume that the Hilbert space can be written as a direct product of two subsystems $A$ and $B$: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. A general entangled state $|\Psi\rangle_{AB}$ is then given by the Schmidt decomposition:

$$|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |\phi_i\rangle_A \otimes |\chi_i\rangle_B,$$ (2.4)

here $|\phi_i\rangle_A$ and $|\chi_i\rangle_B$ are elements of a complete orthonormal basis for the subsystems $A$ and $B$ respectively. We can now define the reduced density matrix for system $A$ by performing a partial trace over system $B$:

$$\rho_A = \text{Tr}_B \rho.$$ (2.5)

The entanglement entropy of system $A$ with system $B$ is then defined as the von Neumann entropy of the reduced density matrix $\rho_B$:

$$S_A \equiv -\text{Tr}_A \rho_A \log \rho_A.$$ (2.6)
As an easy example of an entangled state, we consider the first of the Bell states, which is given by:

\[ |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B). \]  

(2.7)

The reduced density matrix \( \rho_A \) for this state is:

\[ \rho_A = \frac{1}{2} (|0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A). \]  

(2.8)

From which it follows that the entanglement entropy of this Bell state is:

\[ S_A = -\text{Tr} \left( \frac{1}{2} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right) \log \left( \frac{1}{2} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right) = \log 2, \]  

(2.9)

where we adopted a natural orthonormal 2x2 basis.

### 2.1.1 Properties of entanglement entropy

We can now list some universal properties of entanglement entropy:

- **Positivity**: the entanglement entropy is a manifestly positive quantity. In terms of the probabilities \( p_i \) associated to a state \( |\phi_i\rangle \) we can write:

\[ S_A = \sum_i p_i \log p_i. \]  

(2.10)

From \( 0 \leq p_i \leq 1 \), it follows that \( S_A \geq 0 \).

- **\( S_B = S_A \) for a pure state.** From (2.4) we can see that for a pure state \( |\Psi\rangle_{AB} \), the eigenvalues of both the reduced density matrices \( \rho_A \) and \( \rho_B \) are \( \sqrt{\lambda_i} \). This means that the entanglement entropy for both subsystems is equal:

\[ S_A = S_B. \]  

(2.11)

This equality does not hold for a mixed quantum state, such as a state at finite temperature.

- **Strong subadditivity**: if we divide the system into three non-intersecting subsystems \( A \), \( B \) and \( C \), the entanglement entropy obeys [22]:

\[ S_{A+C} + S_{B+C} \geq S_{A+B+C} + S_B. \]  

(2.12)

This is the most stringent inequalities known so far concerning entanglement entropy [23]. When one of the systems is empty, the subadditivity constraint holds, which is also valid for classical entropy:

\[ S_A + S_B \geq S_{A+B}. \]  

(2.13)

Subadditiviy is satisfied in the case of ordinary thermodynamic entropy, but this is not generally the case for quantum systems. Consider for example a pure state, for which the von Neumann entropy \( S_{A+B} \) vanishes, while the individual entanglement entropies \( S_A \) and \( S_B \) are non-zero. This pure state will however satisfy the constraints imposed by strong subadditivity. The fact that the entropies \( S_A \) and \( S_B \) are non-zero, while \( S_{A+B} \) vanishes, shows that entanglement entropy is a non-local quantity. It represents information encoded in non-local correlations between the subsystems \( A \) and \( B \).
Divergence in the continuum. Entanglement entropy can still be defined for a continuous system, such as a quantum field theory defined on a continuous manifold, but it will generally be divergent. This is easily understood, since there will be degrees of freedom arbitrarily close to the boundary between \(A\) and \(B\) on both sides. These UV degrees of freedom will therefore be arbitrarily strongly entangled, causing the entanglement entropy to diverge. The divergence can be regulated by introducing a UV cutoff \(\epsilon\). A way to circumvent these divergences is to consider the mutual information \(I(A,B)\), defined as

\[
I(A,B) = S_A + S_B - S_{A+B}. \tag{2.14}
\]

See [25] for more on different methods of regularizing entanglement entropy in field theories.

### 2.2 Entanglement entropy in Quantum Field Theory

Now that we have reviewed the basics of entanglement entropy in quantum mechanics, we can look at how it is defined in quantum field theory. This sections closely follows the reasoning presented in [20, 21]. Consider a system in a pure quantum state \(|\Psi\rangle\), so that its density matrix is defined to be \(\rho = |\Psi\rangle \langle \Psi|\). If the Hilbert space \(\mathcal{H}\) can be written as a direct product \(\mathcal{H}_A \otimes \mathcal{H}_B\), we can again define a reduced density matrix for subsystem \(A\) by tracing over subsystem \(B\):

\[
\rho_A = \text{Tr}_B \rho. \tag{2.15}
\]

The Rényi entropies are then defined as:

\[
S_A^{(n)} = \frac{1}{1-n} \ln \text{Tr} \rho_A^n. \tag{2.16}
\]

To relate the entanglement entropy and the Rényi entropies, we take the limit of \(n\) going to 1 and use l'Hôpital’s rule:

\[
\lim_{n \to 1} S_A^{(n)} = - \lim_{n \to 1} \frac{1}{\text{Tr} \rho_A} \cdot \text{Tr} (\rho_A^n \log \rho_A).
\]

In the last line, we used that \(\text{Tr} \rho_A = 1\). The Rényi entropy in this limit is of course equal to the von Neumann entropy and can be taken as an equivalent definition of entanglement entropy.

To compute the entanglement entropy for a given subsystem \(A\) in a quantum field theory, one would normally have to calculate the sum \(S_A = \sum \lambda_i \log \lambda_i\), where \(\lambda_i\) are the eigenvalues of the reduced density matrix. However, calculating these eigenvalues for a generic interacting quantum field theory is often impossible, even when using numerical methods. Here we will follow the reasoning presented in [20] and use the so-called replica trick. Before fully introducing this formalism, let us first make the observation that the eigenvalues \(\lambda_i\) of the reduced density matrix all lie in the interval \([0,1]\) and hence the sum \(\text{Tr} \rho_A = \sum \lambda_i\) is absolutely convergent. This means that the sum is analytic and in particular its derivative is as well, so we can analytically continue \(n\) to an arbitrary complex value. We then use that \(S_A\) can be written as:

\[
S_A = - \lim_{n \to 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n. \tag{2.17}
\]

which we have also used in deriving (2.16). Hence, if we have an expression for \(\rho_A^n\), we can compute the entanglement entropy of region \(A\). In principle, this does not look like a simplification, but it turns out that in 2D conformal field theories, there exists a procedure to relatively easily calculate \(\rho_A^n\).
2.2.1 The replica trick: path integrals and Riemann surfaces

We consider a (1 + 1)-dimensional theory with a discrete spatial variable \( x \), while the time variable \( t \) is continuous. We denote a complete set of commuting observables by \( \{ \phi_x \} \) and their eigenvalues and eigenstates by \( \{ \phi_x \} \) respectively. The states \( \otimes \{ \phi_x \} = \prod \{ \phi_x \} \) then form a basis for the Hilbert space of the theory. We can formally define the elements of a density matrix for this theory as:

\[
\rho(\{ \phi_x \} | \{ \phi'_x \}) = \prod_x \{ \phi_x \} | \rho | \prod_{x'} \{ \phi'_x \} = Z(\beta)^{-1} \prod_x \{ \phi_x \} | e^{-\beta H} | \prod_{x'} \{ \phi'_x \}.
\]  

(2.18)

Here \( Z(\beta) = \text{Tr} e^{-\beta H} \) is the partition function for the theory and serves to normalize the density matrix. We can look at this object differently by defining the field theoretical wave functional \( \Psi \) for the theory at temperature \( \beta \) as a Euclidean path integral:

\[
\Psi(\phi(t = 0)) = Z(\beta)^{-1} \int_{t=-\infty}^{t=0} D\phi e^{-S_E} \prod_x \delta(\phi(x,0) - \phi(x,\beta)).
\]  

(2.19)

The density matrix can then be expressed as:

\[
\rho(\{ \phi_x \} | \{ \phi'_x \}) = Z^{-1} \int [d\phi(y,\tau)] \prod_x \delta(\phi(y,0) - \phi'_x) \prod_x \delta(\phi(y,\beta) - \phi_x) e^{-S_E}.
\]  

(2.20)

The \( \delta \)-functions impose the boundary conditions for the end points of the time interval. \( S_E \) is the Euclidean action for this theory, naturally defined as \( \int_0^L \text{d}t L \tau \), where \( L \) is the Euclidean Lagrangian. Taking the trace of (2.18) amounts to setting \( \{ \phi_x \} = \{ \phi'_x \} \) and doing this, we see why the factor of \( Z(\beta)^{-1} \) is included, since this ensures that \( \text{Tr} \rho = 1 \). Taking the trace has the physical implication of identifying \( \tau = 0 \) and \( \tau = \beta \), thereby putting the theory on a cylinder, which is also equivalent to considering the system at finite temperature.

Now let \( A \) be a subsystem consisting of the points \( x \) in the disjoint intervals \( (u_1, v_1),...,(u_N, v_N) \). To find the reduced density matrix \( \rho_A \), we have to trace over its complement. This can be done by sewing together only those points \( x \) which are not in \( A \), which leaves an open cut for each interval \( (u_i, v_i) \) along the line \( \tau = 0 \). We can then compute \( \rho_A^n \) by making \( n \) copies of this cut manifold and sewing them together in a particular way. Labelling each copy by an integer \( j \) with \( 1 \leq j \leq n \), we impose the condition that the observables \( \phi_i \) on each sheet respect:

\[
\phi_j(x, \tau = \beta^-) = \phi_{j+1}(x, \tau = 0^+), \quad \phi_n(x, \tau = \beta^-) = \phi_1(x, \tau = 0^+) \quad x \in A.
\]  

(2.21)

This results in an \( n \)-sheeted structure which naturally possesses a global \( Z_n \)-symmetry. If we denote the partition function on this \( n \)-sheeted lattice-like structure as \( Z_n(A) \), we have:

\[
\text{Tr} \rho_A^n = \frac{Z_n(A)}{Z_1^n}.
\]  

(2.22)

\( Z_1 \) is the partition function on the original manifold and again serves to normalize the result. We can now express the Rényi entropy (2.15) in terms of the partition function on this \( n \)-sheeted lattice:

\[
S_n = \frac{1}{1-n} (\log Z_n - n \log Z_1).
\]  

(2.23)

As in (2.17), the entanglement entropy can be computed by differentiating (2.22) with respect to \( n \):

\[
S_A = -\lim_{n \to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}.
\]  

(2.24)
This derivative again requires an analytic continuation of the integer \( n \) to arbitrary complex values. If we now take the continuum limit for this theory, which amounts to taking the lattice spacing \( a \to 0 \) while keeping other lengths fixed, the path integral is over an \( n \)-sheeted Riemann surface \( \mathcal{R} \) with branch cuts along the intervals \((u_i, v_i)\). The fact that \( \mathcal{R} \) still possesses a global \( \mathbb{Z}_n \)-symmetry means that we can view the theory as an orbifold theory on \( \mathbb{C}^n/\mathbb{Z}_n \). This orbifold is everywhere regular, except at the endpoints of \( A \). We can relate the entanglement entropy of the orbifold theory to that of \( \mathcal{R} \) by:

\[
S[Z_n] = nS[\hat{Z}_n],
\]

where \( \hat{Z}_n \) is the partition function of the theory defined on the orbifold \( \mathbb{C}^n/\mathbb{Z}_n \).

### 2.2.2 Twist fields

Due to the complicated topology of this sheeted Riemann surface, it is almost always impossible to directly calculate the partition function on such a surface. Following the arguments presented in [21], we can simplify the problem by moving the complicated topology of the base space \( \mathcal{R} \) to the target space of the fields. Instead of taking \( n \) copies of the manifold along with branch cut, we consider fields \( \phi_i, i \leq n \), all living on \( \mathbb{C} \). We of course have to account for the singular boundary points of the entangling interval, which we can do by imposing boundary conditions on the fields:

\[
\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-), \quad x \in [u_i, v_i] \quad i = 1, \ldots n \quad n + i \equiv i,
\]

where \( 0^\pm \) signifies approaching zero from above and below respectively. The path integral can then be written as:

\[
Z_\mathcal{R} = \int_{\mathcal{C}_{u_1, v_1}} [d\phi_1 \ldots d\phi_n] \exp \left[ - \int dxd\tau \mathcal{L}^{(n)}[\phi_1 \ldots \phi_n](x, \tau) \right].
\]

Here we follow the notation of [21] and define \( \int_{\mathcal{C}_{u_1, v_1}} \) as the path integral restricted by the boundary conditions (2.26) and the Lagrangian \( \mathcal{L}^{(n)} \) as the Lagrangian of the multi-copy model, which is just the sum of the individual Lagrangians:

\[
\mathcal{L}^{(n)}[\phi_1 \ldots \phi_n](x, \tau) = \mathcal{L}^{(1)}[\phi_1](x, \tau) + \ldots + \mathcal{L}^{(1)}[\phi_n](x, \tau).
\]

Now we note that the position of the branch cut is arbitrary. If we move the cut, such that some points are now above it which were below it before, we can always use the cyclical \( \mathbb{Z}_n \)-symmetry to map the system back to its original setup. We can view this situation as a theory with operator insertions at the endpoints of the cut, that is, the fixed points of the \( \mathbb{Z}_n \)-symmetry. A field \( \phi_i \) is transformed into \( \phi_{i+1} \) when taken counterclockwise around the endpoint \( u_1 \) and into \( \phi_{i-1} \) when taken counterclockwise around the endpoint \( v_1 \). These insertions can be viewed as local fields \( \Phi_n \) and \( \Phi_{-n} \), which are known as twist fields. They are associated to the global \( \mathbb{Z}_n \)-symmetry in the theory. These fields are not dynamical and are not integrated over in the Lagrangian: they are just a manifestation of the boundary conditions associated to the branch points. For \( x \in [u_i, v_i] \), the twist fields connect consecutive copies \( \phi_i \) and \( \phi_{i+1} \) through \( \tau = 0 \), for \( x \) outside this interval, copies are connected to themselves through \( \tau = 0 \).

Having defined the twist fields, we can observe that the partition function of the theory must be proportional to their correlation function:

\[
Z_\mathcal{R} \propto \langle \Phi_n(u_1, 0)\Phi_{-n}(v_1, 0)\ldots\Phi_n(u_N, 0)\Phi_{-n}(v_N, 0) \rangle_{\mathcal{L}^{(n)}},\nonumber
\]

(2.29)
Here we indicated that the dynamics of the fields are governed by the Lagrangian $\mathcal{L}^{(n)}$ and that the theory lives on $\mathbb{C}$. Having written down the partition function of the theory in terms of the twist fields, we can also relate correlation functions of other operators in the model $\mathcal{L}$ on $\mathbb{R}$ to correlation functions in the model $\mathcal{L}^{(n)}$ on $\mathbb{C}$:

$$
\langle \mathcal{O}_i(x, \tau) \rangle_{\mathcal{L}, \mathbb{R}} = \frac{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) \mathcal{O}_i(x, \tau) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}}{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}}. \tag{2.30}
$$

We considered a single interval for simplicity. $\mathcal{O}_i$ is an operator on the $i^{th}$ sheet of $\mathbb{R}$, or equivalently, it is an operator in the model $\mathcal{L}^{(n)}$ coming from the $i^{th}$ copy of $\mathcal{L}$. The denominator in the above expression ensures normalization of the correlation function.

### 2.3 Entanglement entropy in conformal field theory

We first consider the simplest configuration: a single interval $[u, v]$ of length $l$ in an infinitely long 1D quantum system at zero temperature. Everything stated in this section up to this point has been valid for general QFT’s, but from now on we will assume that the theory under consideration is conformally invariant. This means that we have the ability to perform conformal transformations on the manifold, with the guarantee that the theory itself stays invariant. We can define complex coordinates $w \equiv x + i\tau$ and $\bar{w} \equiv x - i\tau$.

The conformal mapping

$$
w \to \zeta = \frac{w - u}{w - v}, \tag{2.31}
$$

then maps the branch points $(u, v)$ to $(0, \infty)$. The conformal transformation $\zeta \to z = \zeta^{1/n}$ then maps all sheets of the Riemann surface to $\mathbb{C}$. The fact that the whole theory has now been mapped to $\mathbb{C}$ means that it is also possible to analytically continue $n$ to non-integer values: the theory on $\mathbb{C}$ does not ‘know’ about the $n$-branches of the Riemann surface and the value of $n$ in the definition of $z$ does not have to be integer. We can now look at the stress tensor $T(w)$, see how it transforms under these conformal mappings and relate that to the twist fields correlators through (2.30).

The stress tensor receives an anomalous contribution under a conformal transformation proportional to the Schwarzian derivative [26]:

$$
T(w) = z'^2 T(z) + \frac{c}{12} \frac{z'''z' - \frac{3}{2} z''^2}{z'^2}, \tag{2.32}
$$

where $z'$ denotes $dz/dw$. By translational and rotational invariance, $\langle T(z) \rangle_{\mathbb{C}} = 0$, so the expectation value of $T(w)$ is completely given by the anomalous contribution, which for these particular mapping is given by:

$$
\langle T(w) \rangle_{\mathcal{L}} = \frac{c}{24} \left(1 - \frac{1}{n^2}\right) \frac{(v - u)^2}{(w - u)^2(w - v)^2}. \tag{2.33}
$$

Using (2.30), we can relate this expectation value to the model $\mathcal{L}^{(n)}$ on $\mathbb{C}$:

$$
\langle T(w) \rangle_{\mathcal{L}} = \frac{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) T_i(w) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}}{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}}. \tag{2.34}
$$

This equation is the same for all $i$, so we can just multiply (2.33) by $n$:

$$
\frac{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) T(w) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}}{\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) \rangle_{\mathcal{L}^{(n)}, \mathbb{C}}} = \frac{c}{24n(n^2 - 1)} \frac{(v - u)^2}{(w - u)^2(w - v)^2}. \tag{2.35}
$$
We can then use the conformal Ward identity to express the numerator of (2.35) as:

$$\langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) T(w) \rangle_{L^{(n)}, \mathcal{L}} = \left( \frac{1}{w-u} \frac{\partial}{\partial u} + \frac{h_{\Phi_n}}{(w-u)^2} + \frac{1}{w-v} \frac{\partial}{\partial v} + \frac{h_{\Phi_{-n}}}{(v-u)^2} \right) \langle \Phi_n(u_1, 0) \Phi_{-n}(v_1, 0) \rangle_{L^{(n)}, \mathcal{L}}. $$  \hspace{1cm} (2.36)

Here $h_{\Phi_n}$ and $h_{\Phi_{-n}}$ denote the conformal weights of $\Phi_n$ and $\Phi_{-n}$ respectively. The twist operators are primary operators with scaling dimension $\Delta_n = 2h$. The two-point function of these twist fields can then be written as:

$$\langle \Phi_n(u, 0) \Phi_{-n}(v, 0) \rangle_{L^{(n)}, \mathcal{L}} = \frac{1}{|u-v|^{2h\Delta_n}}. $$ \hspace{1cm} (2.37)

Comparing (2.35) with (2.36), we see that the twist fields have conformal weight:

$$h_{\Phi_n} = h_{\Phi_{-n}} = \frac{c}{24} \left( 1 + \frac{1}{n^2} \right). $$ \hspace{1cm} (2.39)

This puts us in a position to give an explicit expression for the two-point function of $\Phi_n$ and $\Phi_{-n}$ and, according to (2.29) the partition function for the theory:

$$Z_R = \text{Tr} \rho_A^n = c_n \left( \frac{l}{a} \right)^{-\frac{c}{2}(n-1)/n}. $$ \hspace{1cm} (2.40)

Here we reintroduced the lattice spacing $a$ to make the final result dimensionless. In the continuum limit, $a$ can be regarded as a UV-cutoff, $l$ is the length of the subsystem: $l = u - v$. The constants $c_n$ cannot be determined in this way. It is now straightforward to compute the entanglement entropy using (2.16):

$$S_A = \frac{c}{3} \log \left( \frac{l}{a} \right). $$ \hspace{1cm} (2.41)

This result suffers from a UV divergence, since taking the UV cutoff to zero will blow up the logarithm. As we talked about earlier in this section, this divergence comes from correlations arbitrarily close to the partition and is to be expected. Even in the UV though, this expression for the entanglement entropy still useful information. We can obtain the universal part of this equation by taking the logarithmic derivative:

$$\frac{1}{l} \frac{dS_A}{dl} = \frac{c}{3}. $$ \hspace{1cm} (2.42)

This means that the coefficient $c/3$ is universal and that it will appear in every two-dimensional CFT. In 3 dimensions, the leading order term will be proportional to $l$, which for dimensional reasons must come with a power $a^{-1}$, this term is divergent but in a certain combination of derivatives is introduced which will pick out the universal part. In higher even dimensions, the leading order term will always be logarithmic, which will be independent of how the cutoff is chosen. Another way of seeing this, is that the logarithmic terms are related to the Weyl anomaly, which is only non-zero in even dimensions.
2.4 Entanglement entropy at finite size and finite temperature

We can use conformal transformations to investigate the behaviour of the entanglement entropy of the theory on other manifolds. We can for example map the complex $z$-plane to an infinitely long cylinder of circumference $\beta$ using:

$$z \rightarrow w = \frac{\beta}{2\pi} \log z. \quad (2.43)$$

Here the coordinate $w$ is a coordinate on the cylinder and should not be confused with the earlier coordinate $w$ on the Riemann surface $\mathcal{R}$. We see that $\tau = 0$ and $\tau = \beta$ are identified under this map, this procedure therefore corresponds to considering the theory at finite temperature $\beta^{-1}$.

Under a conformal transformation, the two-point function of the twist operator will transform accordingly:

$$\langle \Phi_n(w_1, \bar{w}_1) \Phi_{-n}(w_2, \bar{w}_2) \rangle = (z'(w_1)z'(w_2))^h \langle \Phi_n(z_1, \bar{z}_1) \Phi_{-n}(z_2, \bar{z}_2) \rangle \times \text{anti-hol.} \quad (2.44)$$

Here we only wrote down the holomorphic part of the two-point function, the anti-holomorphic part transforms analogously. Under the transformation (2.43), the two-point function transforms as:

$$\langle \Phi_n(w_1, \bar{w}_1) \Phi_{-n}(w_2, \bar{w}_2) \rangle = \left( \frac{4\pi^2}{\beta^2} e^{\pi(w_1+w_2)} \right)^h \langle \Phi_n(z_1, \bar{z}_1) \Phi_{-n}(z_2, \bar{z}_2) \rangle \times \text{anti-hol.} \quad (2.45)$$

In the second line we used the expression for the twist field two-point function (2.37), the fact that $\Delta_n = h$ (and similarly for the anti-holomorphic part) and finally that we have to take the two-point function in the model $\mathcal{L}$ to the power $n$ to obtain the two-point function in the model $\mathcal{L}^{(n)}$. In the third line we set $w_1 = u$ and $w_2 = v$. To obtain the entanglement entropy, we simply apply (2.16) to this new two-point function:

$$S_A = \frac{c}{3} \log \left( \frac{\frac{\beta}{\pi a} \sinh \pi l}{\beta} \right). \quad (2.46)$$

The above expression applies to a CFT where the temporal coordinate has period $\beta$: $\tau \sim \tau + \beta$. We can also consider a CFT with a compact spatial direction: $x \sim x + R_{\text{cyl}}$, which corresponds to a CFT defined on a cylinder of circumference $R_{\text{cyl}}$. Taking into account the Euclidean signature of the manifolds we are considering, we see that we can switch between these two configurations by identifying $iL \leftrightarrow \beta$. This identification gives the entanglement entropy for a CFT defined on a cylinder:

$$S_A = \frac{c}{3} \log \left( \frac{R_{\text{cyl}}}{\pi a} \frac{\sin \pi l}{R_{\text{cyl}}} \right). \quad (2.47)$$

2.4.1 Conformal field theory on a torus

More generally, we can take both the timelike and spatial coordinate to be compact: this amounts to putting the theory on a torus. The complex coordinate $w$ should then obey $w \sim w + 2\pi$ and $w \sim w + 2\pi\tau$, where $\tau = \tau_1 + i\tau_2$, with $\tau_1$ and $\tau_2$ real parameters. The complex number $\tau$ is
known as the modular parameter of the torus [26]. This torus is schematically depicted in figure 1. The partition function of a CFT on a torus is given by:

$$\int e^{-S} = \text{Tr} e^{2\pi i \tau_1 P} e^{-2\pi \tau_2 H}. \quad (2.48)$$

Note that the momentum operator $P$ generates transformations along the (real) spatial direction while the Hamiltonian operator $H$ generates transformations along the (imaginary) time direction.

![Figure 1: A torus with modular parameter $\tau = \tau_1 + i\tau_2$. Adapted from [26].](image)

The transformations which leave the torus invariant are known as modular transformations and can be written as:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}. \quad (2.49)$$

These transformations form the group $PSL(2,\mathbb{Z})$, also known as the modular group. The group is generated by the transformations $S$, $T$ and $U$:

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1, \quad U : \tau \rightarrow \frac{\tau}{\tau + 1}. \quad (2.50)$$

In fact, only two of these transformations are needed, because of the relation $S = T^{-1}UT^{-1}$. The partition function of a 2D CFT should be invariant under these modular transformations.

One would naively expect that the reasoning we used for a theory on a cylinder would apply to the torus as well, but unlike the cylinder, the torus has non-trivial topology. This also has the effect that there exists no uniformizing transformation to the plane. This means that correlations functions like (2.37) do not just depend on the scaling dimensions of the fields, but on the full operator content of the theory. As a sidenote, this problem also appears when we consider the entanglement entropy of multiple intervals in the field theory: the Riemann surface corresponding to a setup with $n$ intervals will be of genus $n$. For this reason, there are few universal results for toroidal geometries, but there are some approximations we can make.

We can for example consider a very thin torus, with $\tau_1 \gg \tau_2$, this allows us to essentially view the torus as a very long cylinder. According to (2.48), $\tau_1$ is the conjugate variable to the momentum $P$ and can therefore be viewed as an angular potential, similarly $\tau_2$ represents the inverse temperature of the system. We therefore set $\tau_1 = \Omega E$ and $\tau_2 = \beta$ and note that the setup corresponds to the situation where $\beta \ll \Omega E$: the system is at a finite angular potential which
is still much smaller than the temperature. We can now use a conformal transformation which maps the plane to this very thin torus:

\[ z \rightarrow w = \frac{\beta(1 - i\Omega_E)}{2\pi} \log z \]  

(2.51)

This expression is analogous to (2.43). We can again look at how the two-point function transforms to find the entanglement entropy of this region and obtain:

\[ \langle \Phi_n(w_1, \bar{w}_1)\Phi_{-n}(w_2, \bar{w}_2) \rangle = \left[ \frac{\beta(1 + \Omega_E)^2}{\pi^2 \epsilon^2} \sinh \left( \frac{\pi l}{\beta(1 + i\Omega_E)} \right) \sinh \left( \frac{\pi l}{\beta(1 - i\Omega_E)} \right) \right]^{-\frac{c}{36} (\alpha - \frac{1}{4})}. \]  

(2.52)

We explicitly wrote out the anti-holomorphic part, which introduces a minus sign in the argument of the second factor. Also note that we have defined \( l \) in terms of the \( w \)-coordinate:

\[ l = \frac{\beta(1 - i\Omega_E)}{2\pi} \log \frac{u}{v}. \]  

(2.53)

We can analytically continue the angular potential \( \Omega_E \) to Lorentzian signature as by identifying \( \Omega_E = -i\Omega \). This allows us to define effective left and right moving temperatures \( \beta_{\pm} = \beta(1 \pm \Omega) \). The entanglement entropy then becomes:

\[ S_A = \frac{c}{6} \log \left[ \frac{\beta_+ \beta_-}{\pi^2 \epsilon^2} \sinh \left( \frac{\pi l}{\beta_+} \right) \sinh \left( \frac{\pi l}{\beta_-} \right) \right]. \]  

(2.54)

Note that this expression factorizes into left and right moving contributions. If we set \( \Omega = 0 \), we recover the expression for the CFT at a finite temperature (2.47).
3 Holographic entanglement entropy in AdS$_3$

Anti-de Sitter (AdS) spacetime is the solution of the vacuum Einstein equations with a negative cosmological constant. We will focus on the three-dimensional version AdS$_3$. Through the AdS/CFT-correspondence, AdS$_3$ is closely related to the 2D CFT’s we talked about in section 2. In this section we start by reviewing several spacetimes which are locally AdS$_3$. We then turn to the holographic description of entanglement entropy provided by Ryu and Takayanagi in [16, 28]. They conjectured that the quantity dual to entanglement entropy in 2D CFT is the length of particular geodesics. We will go over their argument and explicitly verify their proposal for global AdS$_3$, Poincaré AdS$_3$ and rotating BTZ backgrounds. We will finish this section by going over the proof of the Ryu-Takayanagi (RT) conjecture by Lewkowycz and Maldacena [29].

3.1 Locally AdS$_3$ spacetimes

We first note that AdS$_3$ can be embedded in flat $\mathbb{R}^{2,2}$ [2], which has the natural metric:

$$ds^2 = - (dT^1)^2 - (dT^2)^2 + (dX^1)^2 + (dX^2)^2.$$  \hspace{1cm} (3.1)

AdS$_3$ is then a hyperboloid in this space with radius $L$:

$$-(T^1)^2 - (T^2)^2 + (X^1)^2 + (X^2)^2 = -L^2.$$  \hspace{1cm} (3.2)

From this definition it is immediately clear the isometry group of AdS$_3$ is $SO(2,2)$, which is the same as the conformal group in two dimensions.

3.1.1 Global coordinates

We can use various coordinate systems to describe AdS. Firstly, the so-called global coordinates are given by:

$$T^1 = L \cosh \rho \sin \tau,$$
$$T^2 = L \cosh \rho \cos \tau,$$
$$X^1 = L \sinh \rho \cos \phi,$$
$$X^2 = L \sinh \rho \sin \phi,$$ \hspace{1cm} (3.3)

where $\tau \in [0, 2\pi)$ and $\rho \in [0, \infty)$. In these coordinates, the metric becomes:

$$ds^2 = L^2 \left( - \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \right).$$  \hspace{1cm} (3.4)

$L$ is called the AdS-radius and is the only characteristic length scale in AdS. One might be concerned about the periodic nature of the timelike coordinate $\tau$ and this is a legitimate concern, since the spacetime now allows for closed timelike curves. To avoid these, we extend the range of $\tau$ from $-\infty$ to $\infty$ and obtain what is known as the universal cover of three-dimensional anti-de Sitter space [15]. In these coordinates, global AdS$_3$ has a conformal boundary at $\rho = \infty$, where the metric diverges.

There are two other common forms of representing global coordinates. Firstly, setting $\sinh \rho = \tan \chi$ transforms the metric into:

$$ds^2 = \frac{L^2}{\cos^2 \chi} \left( -d\tau^2 + d\chi^2 + \sin^2 \chi d\phi^2 \right),$$  \hspace{1cm} (3.5)

where $\chi \in [0, \pi/2]$. AdS$_3$ in these coordinates is conformal to a solid cylinder with radius $\chi = \pi/2$. In these coordinates, it is manifest that the conformal boundary of global AdS$_3$ has cylindrical...
topology. This means that the dual field theory should live on the cylinder as well, since the CFT lives on the boundary of the AdS bulk. We conclude that the dual to global AdS$_3$ is a CFT defined on a cylinder.

Lastly, a useful form of AdS$_3$ in global coordinates can be obtained by defining $r = L \sinh \rho$ and $t = L \tau$, which leads to the metric:

$$ds^2 = - \left( 1 + \frac{r^2}{L^2} \right) dt^2 + \left( 1 + \frac{r^2}{L^2} \right)^{-1} dr^2 + r^2 d\phi^2$$  \hspace{1cm} (3.6)

This metric is reminiscent of the BTZ black hole which will be touched upon later.

### 3.1.2 Poincaré coordinates

Another set of often used coordinates, and in fact the coordinates we will use almost exclusively in this thesis, are Poincaré coordinates. Similar to the set of global coordinates, the Poincaré coordinates can be defined in several ways, but we will mainly use the following definition:

$$T^1 = \frac{Lt}{z}, \quad X^1 = \frac{Lx}{z},$$

$$T^2 + X^2 = \frac{L^2}{z},$$

$$T^2 - X^2 = \frac{-t^2 + x^2 + z^2}{z}.$$  \hspace{1cm} (3.7)

This leads to the Poincaré metric:

$$ds^2 = \frac{L^2}{z^2} \left( -dt^2 + dx^2 + dz^2 \right).$$  \hspace{1cm} (3.8)

The Poincaré coordinates only cover part of the spacetime: $z$ divides the hyperboloid into two charts, one for $z > 0$ and one for $z < 0$. We will consider the former region, which is also known as the Poincaré patch. Surfaces of constant $z$ in Poincaré AdS are (conformal to) two-dimensional Minkowski spacetime. At each value of $z$, an observer sees Minkowski spacetime with all lengths rescaled by $z$. The surface bounding this region at $z = 0$ is conformal boundary of Poincaré AdS. Since both coordinates $x$ and $t$ extend from $-\infty$ to $\infty$, we conclude that this boundary has planar topology. This means that the dual to Poincaré AdS is a CFT living on a plane.

### 3.1.3 Geodesics in Poincaré AdS

We will now explicitly solve the geodesic equation in Poincaré AdS, since we will need these geodesics later. We will parameterize the geodesics by their proper length, so we can find the geodesics by solving the Euler-Lagrange equations for the following Lagrangian:

$$\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu.$$  \hspace{1cm} (3.9)

Without loss of generality, we can consider a spacelike geodesic, so we obtain the following Lagrangian:

$$\mathcal{L} = \frac{L^2}{z^2} \left( \dot{x}^2 + \dot{z}^2 \right) = 1.$$  \hspace{1cm} (3.10)

Here an overdot denotes differentiation along the worldline. The Lagrangian is normalized to 1 since we will be concerned with spacelike geodesics later, timelike geodesics can be obtained by
setting $\mathcal{L} = -1$. We immediately see that we have a conserved momentum in the $x$-direction:

$$
\frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{d}{ds} \left( \frac{2L^2}{z^2} \dot{x} \right) = 0, \quad \implies \quad p = \frac{L^2}{z^2} \dot{x}.
$$

By then choosing our affine parameter to be the coordinate $x$, we obtain:

$$
\left( \frac{dz}{dx} \right)^2 = \frac{L^2}{p^2 z^2} - 1.
$$

This is a separable differential equations, which is easily solved

$$
\int \frac{zdz}{\sqrt{L^2/p^2 - z^2}} = \int dx \quad \implies \quad \frac{L^2}{p^2} - z^2 = (x + C)^2.
$$

By applying the boundary condition that $x = \pm \frac{R}{2}$ as $z \to 0$, we can write:

$$
z^2 + x^2 = \frac{R^2}{4},
$$

where we have defined $R = L/2p$. We conclude that geodesics on a constant time slice of Poincaré AdS are semicircles in the $(x, z)$-plane. For future reference, we explicitly compute the tangent vector to these geodesics. By differentiating (3.14) and using the definition of the momentum (3.11) we can write:

$$
z \dot{z} + pxz^2 \frac{L}{z^2} = 0.
$$

This leads to the following expressions for $\dot{z}$ and $\dot{x}$:

$$
\dot{z} = -\frac{2zx}{RL}, \quad \dot{x} = \frac{2z^2}{RL}.
$$

This gives the following tangent vector to the geodesic (using $(x, t, z)$):

$$
v^\mu = \frac{2z}{RL} (z, 0, -x).
$$

### 3.1.4 The BTZ black hole

As was mentioned in the introduction, the only solution to three-dimensional gravity with a negative cosmological constant is three dimensional AdS. Therefore it was long thought that vacuum three-dimensional gravity did not allow for black holes [2]. It therefore came as a surprise when Bañados, Teitelboim and Zanelli (BTZ) showed that there indeed exist 3D black hole solutions with constant negative curvature [3]. This black hole solution shares a lot of characteristics with well known higher-dimensional black holes, such as Schwarzschild or Kerr black holes. BTZ black holes have an event horizon, they have similar thermodynamic properties to other black holes and appear as the final state of collapsing matter [2]. A crucial difference however is of course that the BTZ black hole spacetime has constant negative curvature: it is locally isometric to pure AdS$_3$. It can be viewed as a quotient space of AdS$_3$ when the quotient is taken with a discrete subgroup of the AdS isometry group $SO(2, 2)$. For further details on how to obtain the BTZ geometry by taking quotients of AdS$_3$, see [30] and [2]. The BTZ black hole has the following metric:

$$
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^2 dt + d\phi)^2,
$$
where
\[ N(r)^2 = \left( -8GM + \frac{r^2}{L^2} + \frac{16G^2J^2}{r^2} \right), \quad N^\phi(r) = -\frac{4GJ}{r^2}. \] (3.19)

It is a rotating black hole with an outer and inner horizon \( r_\pm \) given by the zeroes of \( N(r) \):
\[ r_\pm = 4GML^2 \left( 1 \pm \sqrt{1 - \left( \frac{J}{ML} \right)^2} \right). \] (3.20)

The parameters \( M \) and \( J \) correspond to the ADM-mass and angular momentum respectively and can be written in terms of the horizons \( r_\pm \) as:
\[ M = \frac{r_+^2 + r_-^2}{8GL^2}, \quad J = \frac{2(r_+ - r_-)}{8GL}. \] (3.21)

When the black hole is non-rotating, i.e. when \( J = 0 \), the inner horizon vanishes: \( r_- = 0 \). Using the expressions for the mass and angular momentum, we can also write the metric for the rotating black hole in terms of the horizons:
\[ ds^2 = -\frac{(r_-^2 - r_+^2)(r^2 - r_-^2)}{r^2L^2} dt^2 + \frac{L^2r_+^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left( d\phi - \frac{r_+ - r_-}{Lr^2} dt \right)^2. \] (3.22)

The rotating BTZ background is dual to a CFT at finite left- and right-moving temperatures \( \beta_\pm^{-1} \) and finite angular potential \( \Omega \). We covered these CFTs in section 2.4. The temperature and angular potential are related to the horizons through:
\[ \beta_{L,R} = \beta(1 \pm \Omega) = \frac{2\pi R_{\text{cyl}}}{\Delta_\pm}, \quad \Delta_\pm = r_+ \pm r_- \] (3.23)

Note that the BTZ metric reduces to the metric of global AdS in (3.6) for \( r_+^2 = -l^2 \) and \( r_- = 0 \) or equivalently \( J = 0 \) and \( M = -1/8G \). In the words of [30], global AdS appears as a 'bound state' of the BTZ geometry, separated by a mass gap of 1/8G. As was mentioned above, the BTZ black hole can be obtained by making suitable identifications in pure AdS\(^3\). This means that locally, the two spacetimes can be mapped to each other. We will later make use of the explicit mapping between the BTZ metric and Poincaré AdS. The mapping is most easily expressed in light-cone coordinates \( w_{\pm} \), in the metric of Poincaré AdS takes the following form:
\[ ds^2 = \frac{1}{z^2} \left( dz^2 + dw_+ dw_- \right). \] (3.24)

Note that we have normalized the AdS-radius to \( L = 1 \). The mapping is then given as [31]:
\[ w_\pm = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{(\phi \pm t)(r_+ \pm r_-)}, \quad z = \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} e^{r_+ t - tr_-}. \] (3.25)

Note that this mapping does not respect the periodicity \( \phi \sim \phi + 2\pi \). This means that technically, we should view the rotating black hole as an extended black brane, with a non-compact \( \phi \)-coordinate. This does not pose too much of a problem, since the periodicity of the \( \phi \)-coordinate is a global matter, and the mapping is supposed to hold only locally anyway.
3.2 The Ryu-Takayanagi proposal

According to the AdS/CFT correspondence, each observable in the field theory should have a geometric counterpart in the bulk. In 2006, Ryu and Takayanagi proposed that the dual to entanglement entropy in the field theory is given by the area of particular minimal surfaces [16, 28]. Their conjecture can be summarised as:

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}} \]  (3.26)

Here the manifold \( \gamma_A \) is the \( d \)-dimensional static minimal surface in \( AdS^{(d+1)} \) of which the boundary is given by \( \partial A \), which is the boundary of the entangled region in the CFT. In \( AdS_3 \), \( \gamma_A \) is a geodesic, so computing the entanglement entropy in a 2-dimensional CFT comes down to computing the length of a geodesic in 3-dimensional AdS. This geodesic should satisfy the boundary condition that it ends at the boundaries \( \partial A \) of the entangled region in the CFT. The Ryu-Takayanagi (RT) conjecture is an example of the more general nature of the AdS/CFT correspondence: analytical quantities in the CFT like entanglement entropy have a purely geometrical dual in AdS.

Before we can check the RT proposal, we first have to cover two particular entries in de AdS/CFT dictionary, namely the relationship between cutoffs in the different theories and the relationship between the central charge \( c \) and Newton’s constant \( G_N \). We can then outline a general procedure to compute geodesic length in AdS, which we will use in the next section to explicitly show that (3.26) correctly reproduces known results for entanglement entropy.

3.2.1 Cutoffs and the Brown-Henneaux formula

When we computed the entanglement entropy in 2D CFT’s, we noted that it was inherently UV-divergent and to regulate it, we introduced a UV cutoff \( a \). For (3.26) to correctly reproduce the CFT entanglement entropy, it should also be divergent and in fact it is. The minimal surface has to be anchored at the boundary and the boundary in global coordinates is located at \( \rho = \infty \).

Following [16], we therefore introduce an IR cutoff \( \rho_0 \) in global \( AdS_3 \) such that:

\[ e^{\rho_0} \sim \frac{1}{a}. \]  (3.27)

In Poincaré coordinates, the boundary is located at \( z = 0 \), so we need a UV cutoff, which we can just identify with the CFT cutoff \( a \).

The EE in 2D CFT’s depends on the central charge \( c \), while (3.26) implies that in AdS it depends on Newton’s constant \( G_N \). We therefore need a way to relate these quantities to each other and it is provided by the AdS/CFT dictionary. Already in 1986 Brown and Henneaux considered the asymptotic symmetries of \( AdS_3 \) at the boundary [10] (for a more accessible review of their argument, see [32]). They found that the asymptotic symmetry algebra factorizes into two copies of the Virasoro algebra, which is of course also the local algebra of symmetries of two dimensional CFT’s. The algebras are isomorphic if:

\[ c = \frac{3L}{2G_N}. \]  (3.28)

This formula is known as the Brown-Henneaux formula.
3.2.2 Geodesic length in AdS$_3$

In section 3.1.3 we found an expression for geodesics in Poincaré AdS$_3$. In principle this could of course also be done for other locally AdS$_3$ spacetimes, but solving the geodesic equation in these spacetimes is rather cumbersome. However, to check the RT proposal we don’t need the full expression of the geodesic, we only need its length. There is an easier method to compute the geodesic length, which makes use of the embedding (3.1) of AdS$_3$ in $\mathbb{R}^{2,2}$ [15]. As was noted in (3.2), AdS$_3$ is given by a hyperboloid in $\mathbb{R}^{2,2}$:
\[ X^A X_A = -L^2. \] (3.29)

Geodesics in AdS$_3$ will thus be ‘straight lines’ in $\mathbb{R}^{2,2}$, with the added condition that they should lie on the hyperbola (3.29). They should therefore extremize the action functional with the aforementioned condition imposed by a Lagrange multiplier:
\[ S = \int ds \left[ \ddot{X}^A \dot{X}_A - \lambda (X^A X_A + L^2) \right]. \] (3.30)

Here an overdot indicates differentiation along the worldline. Minimizing the action with respect to $\lambda$ obviously gives back the constraint (3.29) Taking a derivative along the worldline on both sides, we obtain:
\[ X^A \dot{X}_A = 0. \] (3.31)

Now the Euler-Lagrange equations for the action (3.30) are:
\[ \ddot{X}^A = -2\lambda X^A. \] (3.32)

We can contract with $X_A$ to obtain:
\[ \dddot{X}^A X_A = -2\lambda X^2. \] (3.33)

We can then integrate the left hand side by parts, where (3.31) causes the boundary term to vanish. This allows us to fix the value of the Lagrange multiplier:
\[ \lambda = -\frac{\ddot{X}^2}{2L^2}. \] (3.34)

Hence geodesics obey the simple equation:
\[ L^2 \ddot{X}^A = \dddot{X}^2 X_A. \] (3.35)

We can specialize to spacelike geodesics and normalize them by setting $\dot{X}^2 = 1$. The most general solution to (3.35) is then:
\[ X^\mu(s) = a^\mu e^{s/L} + b^\mu e^{-s/L}. \] (3.36)

Here $a^\mu$ and $b^\mu$ are constant vectors, satisfying:
\[ a^2 = b^2 = 0 \quad 2a \cdot b = -L^2. \] (3.37)

Finally, we can express the geodesic length as:
\[ \text{Length}(\gamma_A) = \int_{s_1}^{s_2} ds = \Delta s. \] (3.38)
Denoting the endpoints of $\gamma_A$ by the points $X^A(s_1)$ and $X^A(s_2)$ on the hyperboloid, we arrive at:

$$X^A(s_1)X_A(s_2) = a \cdot b \left( e^{(s_1-s_2)/L} + e^{-(s_1-s_2)L} \right) = -L^2 \cosh(\Delta s/L). \quad (3.39)$$

Inverting this relationship, we have:

$$\Delta s = L \cosh^{-1} \left[ -\frac{1}{L^2} X^A(s_1)X_A(s_2) \right], \quad (3.40)$$

where again $s_1$ and $s_2$ are the endpoints of the geodesic. If we can construct vectors evaluated at these endpoints, (3.40) will give the geodesic length.

### 3.3 Checking Ryu-Takayanagi

Before checking whether the RT conjecture correctly reproduces the CFT entanglement entropy that we computed in section 2, we can first check whether the formula actually captures the universal properties of entanglement entropy covered in section 2.1.

Firstly, the requirement that $S_A = S_B$ for a pure state implies that $\text{Area}(\gamma_A)$ should be equal to $\text{Area}(\gamma_B)$. Since the boundary $\partial A$ of the CFT entangling region is automatically equal to the boundary of its compliment $\partial B$, we have $\gamma_A = \gamma_B$. Therefore, this property is automatically captured by the RT formula. Secondly, we expect the entanglement entropy to diverge in the CFT, so the dual AdS-entropy should also be divergent. As we noted earlier, this is case. The minimal surface $\gamma_A$ is anchored at the boundary, which is at $r = \infty$ in global coordinates.

Finally, we can check whether the RT formula obeyes strong subadditivity (SSA), given by the inequalities (2.12). The proof for this is given in [23], from which figure 2 is also taken. The regions $A$, $B$ and $C$ in the boundary CFT are displayed and the orange and blue curves represent several minimal surfaces. Noting that these are the bulk surfaces of minimal area connecting the edges of the various boundary intervals, it is immediately clear that the equalities are satisfied: the orange and blue segments always have higher area on the left hand side of the inequality than on the right hand side.

Now that we have checked that the RT formula captures some key universal properties of entanglement entropy, we can check whether it actually reproduces the right expressions for the entanglement entropy in the various set-ups calculated in section 2.

#### 3.3.1 Geodesic length in global coordinates

We consider global AdS in the $(\tau, \phi, \rho)$ coordinates defined in (3.4). As endpoints for $\gamma_A$, we pick the points $(\tau, \phi, \rho) = (0, 0, \rho_0)$ and $(\tau, \phi, \rho) = (0, 2\pi R/R_{cyl}, \rho_0)$, where $R$ is the size of the entangling region, $R_{cyl}$ is the radius of the cylindrical boundary and $\rho_0$ is the IR-cutoff. Plugging these points into the inner product (3.39), we obtain:

$$X^A(s_1)X_A(s_2) = -L^2 \left( \cosh^2 \rho_0 + \sinh^2 \rho_0 \cos \frac{2\pi R}{R_{cyl}} \right), \quad (3.41)$$

$$= -L^2 \left( 1 + 2 \sinh^2 \rho_0 \sin^2 \frac{\pi R}{R_{cyl}} \right). \quad (3.42)$$

So for the geodesic length, we obtain:

$$\Delta s = L \cosh^{-1} \left[ 1 + 2 \sinh^2 \rho_0 \sin^2 \frac{\pi R}{R_{cyl}} \right]. \quad (3.43)$$
Figure 2: Graphic representation of the holographic proof for strong subadditivity. Taken from [23], note that there the plus signs in the graphic were incorrectly swapped for equal signs.

Taking the IR-cutoff $\rho_0$ very large, we can write this as:

$$\Delta s \approx L \log \left[ e^{2\rho_0 \sin^2 \frac{\pi R}{R_{cyl}}} \right]. \quad (3.44)$$

According to the Ryu-Takayanagi proposal, the holographic entanglement entropy is:

$$S_A = L \frac{2G}{\log} \left[ e^{\rho_0 \sin \frac{\pi R}{R_{cyl}}} \right]. \quad (3.45)$$

Using the Brown-Henneaux formula (3.28) and matching the IR-cutoff $\rho_0$ with the inverse of the UV-cutoff $a$ in the CFT, we finally find:

$$S_A = \frac{c}{3} \log \left( \frac{R_{cyl}}{\pi a} \sin \frac{\pi R}{R_{cyl}} \right). \quad (3.46)$$

which exactly matches the CFT-result (2.47).

### 3.3.2 Geodesic length in Poincaré coordinates

We can apply the same procedure of finding the geodesic length to a curve in Poincaré coordinates with endpoints: $(t, x, z) = (0, -l/2, a)$ and $(t, x, z) = (0, l/2, a)$, with $a$ a UV-cutoff. Applying the general procedure and computing the inner product (3.39) for the given coordinates, we obtain:

$$X^A(s_1)X_A(s_2) = -L^2 \left( 1 + \frac{l^2}{2a^2} \right) \implies \Delta s = L \cosh^{-1} \left( 1 + \frac{l^2}{2a^2} \right). \quad (3.47)$$

Expanding this for small UV-cutoff $a$, we obtain:

$$\Delta s = 2L \log \left( \frac{1}{a} \right). \quad (3.48)$$
Which implies:

\[ S_A = \frac{c}{3} \log \left( \frac{1}{a} \right). \]  

(3.49)

This indeed gives the correct entanglement entropy for a CFT on a plane.

Note that we already computed the explicit form of the geodesics in section 3.1.3 and the geodesic length could just as well be directly computed from there. Recall that we found that the geodesics are semicircles, given by:

\[ z^2 + x^2 = \frac{l^2}{4}. \]  

(3.50)

We can parameterize this path as:

\[ (x, z) = \frac{l}{2} (\cos s, \sin s), \quad \left( \frac{2a}{T} \leq s \leq \pi - \frac{2a}{T} \right), \]  

(3.51)

where \( a \) is again the UV cutoff. We can then calculate the geodesic length as:

\[ \Delta s = \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} ds = L \int_{2a/T}^{\pi} \frac{1}{\sin s} ds = L \left[ \log \tan \left( \frac{s}{2} \right) \right]_{2a/T}. \]  

(3.52)

This leads to a geodesic length of:

\[ \Delta s = 2L \log \left( \frac{1}{a} \right). \]  

(3.53)

Which after applying the Brown-Henneaux formula is the same expression as (3.48).

3.3.3 Geodesic length for the rotating BTZ black hole

Finally we compute the geodesic length for the rotating BTZ black hole and compare the result to the entanglement for a CFT at finite temperature \( \beta \) and angular potential \( \Omega \). We will rely heavily on the local mapping between from Poincaré AdS\(_3\) to the BTZ black hole given in (3.25).

Consider an interval on the boundary of the BTZ spacetime extending from \( \phi_1 \) to \( \phi_2 \), where \( \phi \) is the angular coordinate defined in (3.22). Under (3.25), this interval maps to an interval in the Poincaré spacetime \( \Delta x \) given by:

\[ (\Delta x)^2 = \Delta w_+ \Delta w_- = (e^{\Delta_+ \phi_1} - e^{\Delta_+ \phi_2}) (e^{\Delta_- \phi_1} - e^{\Delta_- \phi_2}), \]  

(3.54)

where \( \Delta_\pm = r_+ \pm r_- \). We have to include an IR-cutoff \( r_\infty \) in the BTZ spacetime. This cutoff can be mapped to the UV-cutoffs \( \epsilon_{1,2} \) in the Poincaré spacetime by using the mapping of the \( z \)-coordinate in (3.25):

\[ \epsilon_{1,2} = \sqrt{r_+^2 - r_-^2} e^{r_\infty \phi_{1,2}}. \]  

(3.55)

As was discussed in section 3.2.1, we can relate the BTZ cutoff \( r_\infty \) to the CFT cutoff \( a \) by \( r_\infty = 1/a \). We can now write down the entanglement entropy in the Poincaré spacetime as:

\[ S_A = \frac{c}{6} \log \left( \frac{(\Delta x)^2}{a_1 a_2} \right), \]

\[ = \frac{c}{6} \log \left[ \frac{\beta R \beta L}{4 \pi^2 a^2} \left( e^{r_+(x_1-x_2)} - e^{r_-(x_1-x_2)} - e^{r_-(x_2-x_1)} + e^{r_+(x_2-x_1)} \right) \right], \]  

(3.56)

where we used (3.23). This is again the same expression as we obtained for the CFT at finite temperature and angular potential.
3.4 Proving Ryu-Takayanagi: the Lewkowycz-Maldacena procedure

While Ryu and Takayanagi were able to reproduce correct expressions for entanglement entropy using their formula, they did not provide a proof. For theories which allow a Euclidean continuation, the Ryu-Takayanagi formula (3.26) was proved by Lewkowycz and Maldacena in 2013 [29]. Their approach was reviewed in [33] and we will follow the argument presented there. We first recall (2.23): we can write the Rényi entropies $S_n$ for a quantum field theory as:

$$S_n = \frac{1}{1-n} (\log Z_n - n \log Z_1). \quad (3.57)$$

Here $Z_n$ is the partition function for the theory on the manifold $M_n$, which is obtained by taking $n$ copies of a manifold $M_1$ with a cut along some spatial region $A$ and sewing them together along the cuts, this procedure was covered in detail in section 2. As was also covered in that section, this procedures introduces conical defects on the manifold at the boundary of $A$, which is denoted by $\partial A$. If we locally define an angle $\tau$ around $\partial A$, its range is extended from $2\pi$ to $2\pi n$.

If the field theory has a holographic dual, this dual should be defined on some manifold $B_n$ which has $M_n$ as its boundary. The AdS/CFT dictionary teaches us that we can identify the Euclidean partition function on $M_n$ with the on-shell gravitational action of $B_n$ in the following way [12]:

$$Z_n \equiv Z[M_n] = e^{-S[B_n]} + \ldots \quad (3.58)$$

The dots denote corrections due to two different effects. Firstly, (3.58) only holds in the large-$N$ limit (or in the language of 2D CFT’s: at large central charge). Furthermore, it might be the case that there are several bulk manifolds with $M_n$ as boundary, but we take the one with the lowest action. Hence (3.58) might receive corrections due to both $1/N$ effects and subdominant saddles points.

In section 2 we ensued to map $Z_n$ to $\mathbb{C}$ using conformal transformations and were able to use the transformation properties of the stress tensor to compute the correlation function of the twist fields. These methods arise from the high degree of symmetry in 2D conformal field theories and we do not have these at our disposal in general. That poses a problem, since we need to analytically continue $n$ to non-integer values and it is not clear what effect that would have on (3.58). [29] propose that we should look at this analytic continuation from a bulk perspective.

Remember that $M_n$ has a $Z_n$-symmetry and can be regarded as an orbifold theory, which is regular everywhere, except for at $\partial A$. If we extend the symmetry into the bulk, we can define a similar orbifold of $B_n$:

$$\hat{B}_n = B_n/Z_n.$$

Note that the boundary of the orbifold is $M_n/Z = M_1$, which is precisely the original manifold on which the field theory was defined. This means that the boundary manifold is not susceptible to varying $n$ in the orbifold theory, which hints at the possibility of analytically continuing $n$ without affecting the boundary manifold.

Extending the symmetry into the bulk also entails extending the conical singularities located at $\partial A$: they form a curve into the bulk denoted by $C_n$, which must end on $\partial A$. Therefore $\hat{B}_n$ must have a conical defect along $C_n$ with conical deficit $2\pi (1 - 1/n)$. This conical defect should in the end reduce to the Ryu-Takayanagi minimal surface when the limit of $n \to 1$ is taken. Analogous to (2.25) we can relate the Rényi entropy of $B_n$ to that of $\hat{B}_n$:

$$S[B_n] = n S[\hat{B}_n]. \quad (3.60)$$

Plugging this into (3.57) results in:

$$S_n = \frac{n}{n-1} \left( S[\hat{B}_n] - S[\hat{B}_1] \right). \quad (3.61)$$

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3.4.1 Computing the cone action

We are now left with the question how to compute the gravitational action of the manifold \( \hat{B}_n \) with a conical defect \( C_n \) in it. We now specialize to three dimensions both for notation simplicity and because in this thesis we will almost exclusively be concerned with three-dimensional gravity, but the following argument can easily be extended to higher dimensions. If we view \( C_n \) as the wordline of the tip of a cone, we can write down a metric which is valid near \( C_n \):

\[
ds^2 = \rho^{-2\epsilon}(\rho^2d\tau^2 + (g_{yy} + 2K_{\alpha}x^\alpha)dy^2 + \ldots)
\]

(3.62)

where \( a, b \) label the coordinates \( \rho \) and \( \tau \). We immediately observe that the \( \tau\tau \)-component of the metric is \( \rho^{2-2\epsilon} \), which means that this metric has a conical deficit at \( \rho = 0 \) of \( 2\pi\epsilon \). We can thus relate \( \epsilon \) to \( n \) through:

\[
\epsilon = 1 - \frac{1}{n}.
\]

(3.63)

\( K_{\alpha} \) are the extrinsic curvatures of \( C_n \), which are given by \( \frac{1}{2}\partial_\alpha g_{yy} \). The dots denote higher powers of \( \rho \) which are subleading near \( C_n \). We can now compute the curvature to first order in \( \epsilon \). To do so, it turns out that it is convenient to define a complex coordinate \( z = \rho e^{i\tau} \). To first order in \( \epsilon \) we then find for the \( zz \)-component of the Ricci tensor:

\[
R_{zz} = 2K_z \epsilon z + \ldots
\]

(3.64)

which diverges as \( \rho \to 0 \), the dots denote less divergent terms. This divergence must vanish because the stress tensor of the matter in the spacetime is not expected to diverge, since the unorbifolded solution \( B_n \) is regular. We conclude that \( K_z = 0 \). The fact that the extrinsic curvature vanishes precisely means that the surface has minimal area, which means in three dimensions that the curve is a geodesic.

Now that we know that the extension \( C_n \) of the boundary conical defect is a bulk geodesic, the final step needed to prove the Ryu-Takayanagi conjecture is to show that the variation of the action \( S[\hat{B}_n] \) is actually equal to the entanglement entropy \( S_{EE} \) in the limit \( n \to 1 \). Using (3.61) and remembering (2.17), we see that we have to compute:

\[
S_{EE} = \partial_n S[\hat{B}_n] \bigg|_{n=1}.
\]

(3.65)

The crucial step then is to realise that the manifold \( B_n \) is perfectly regular everywhere in its interior, it is only after orbifolding by \( \mathbb{Z}_n \) that the conical singularities are introduced. If (3.60) is to hold, then the action \( S[\hat{B}_n] \) of the orbifolded manifold should not receive any contribution from the conical defect. As in [33], the action may therefore be defined by excising a small region of radius \( a \) around \( C_n \), calculate the action of this manifold and at the end take the limit of \( a \). The excision of the region around \( C_n \) creates a boundary in the spacetime and when computing the gravitational action of this spacetime extra boundary terms will appear. These boundary terms are computed in [29] and they are precisely equal the area of the minimal surface in the limit of \( a \to 0 \). This proves the Ryu-Takayanagi conjecture for theories which allow a Euclidean continuation.

3.4.2 Extensions of the Ryu-Takayanagi conjecture

The RT formula, as well as the proof by Lewkowycz and Maldacena, only applies to static spacetimes in Einstein gravity. Since the original paper by Ryu and Takayanagi, people have sought to extend their prescription to a broader class of spacetimes.
Hubeny, Rangamani and Takayanagi (HRT) have proposed an extension to the RT formula for non-static spacetimes \cite{31} corresponding to time dependent CFT entanglement entropies. A problem with these spacetimes is that there will be no timelike Killing vector field in the bulk and hence no preferred foliation of the spacetime. This has the effect that the notion of a minimal surface becomes ambiguous, since surfaces can now ‘wiggle’ arbitrarily in the time direction to diminish their covariant area. The relevant surface will in this case thus not be a minimal surface, but rather an extremal surface. The authors of \cite{31} provide a prescription to uniquely choose such a co-dimension one surface \( Y \), which turns out to be a surface where the expansion of null geodesics vanishes. On this surface \( Y \), a unique minimal surface of co-dimension two can be found, the covariant area of which is proportional to the entanglement entropy. The HRT formula correctly reproduces the RT formula in the case of static bulk spacetimes, but it has not been proven as of yet.

Another generalisation is the extension of the RT formula to higher derivative gravity. These theories have actions which contain higher powers of the Riemann tensor and its various contractions. Including higher derivative terms can holographically be motivated by not taking the strict \( N = \infty \) limit in the dual field theory, but including \( 1/N \)-effects. The gravity theory is then not just classical GR but also includes stringy effects: the higher derivative terms are suppressed by powers of the string length \( l_s \). The extension of the RT formula to these higher derivative theories has been made by Camps \cite{34} and later Dong \cite{33}. Dong provides a formula which gives the holographic EE for a higher derivative theory of gravity in which the Lagrangian is formed out of some combination of the Riemann tensor and its contractions and derivatives, evaluated on some surface \( C \). His expression involves the extrinsic curvature of \( C \) and derivatives of the Lagrangian with respect to the Riemann tensor. It correctly reproduces the EE in a number of examples of higher derivative gravity where the EE has already been computed using other methods.

In this thesis, we will also consider higher derivative corrections to classical gravity, but they will not be formed from contractions of the Riemann tensor. They instead involve gravitational Chern-Simons terms, which are formed from the Christoffel symbols \cite{35}. These theories are dual to CFT’s with a gravitational anomaly and it will turn out that the entanglement entropy in these theories possesses extra structure on top of the geodesic length \cite{1}. Before we can see this though, we first have to introduce anomalies into our field theory, which will be the subject of the next section.
Gravitational anomalies in holography

Anomalies occur in a field theory when a symmetry of the classical theory does not remain a symmetry after quantisation. When quantising a theory, the Lagrangian will always remain invariant under the symmetry, but this is not necessarily the case for the measure, which might pick up quantum loop corrections. Well known examples are the axial anomaly in quantum electrodynamics and the trace anomaly in 2-dimensional CFT’s. In this thesis, we will consider a different type of anomaly: the gravitational anomaly.

In this section we will first look at a mathematical framework which describes anomalies in a more general way. We will then focus on gravitational anomalies, which occur when a chiral fermion couples to gravity. We will first look at this anomaly from a field theoretical point of view and compute entanglement entropy in anomalous 2D CFTs. Finally we will use holography to consider the anomaly from a bulk perspective.

4.1 Mathematical framework

Consider a manifold $\mathcal{M}$ covered by a finite number of coordinate patches $U_i$, along with its principal gauge bundle $P \rightarrow \mathcal{M}$ with associated gauge group $G$. On this bundle we can look at transition functions (gauge transformations) $g$ and define a connection one-form $A$ which will transform as $A \rightarrow g^{-1}(A + d)g$. Using this connection, we can define the curvature (or field strength) two-form $F$:

$$F \equiv dA + A \wedge A,$$  

(4.1)

which transforms under a gauge transformation as $F \rightarrow gFg^{-1}$. From the curvature form we can construct so called characteristic classes. A characteristic class $P_m$ is a $2m$-form constructed out of traces of the curvature two-form. We know that the trace is cyclic, which means that:

$$\text{Tr}(\eta_p \zeta_q) = (-1)^{pq} \text{Tr}(\zeta_q \eta_p),$$

(4.2)

for matrix valued $p$- and $q$-forms $\eta_p$ and $\zeta_q$ where wedge products are implied. If we now take $\eta_p$ and $\zeta_q$ to be products of the curvature form $F$ and use the cyclic property, we immediately see that $P_m$ is gauge invariant:

$$\delta_g P_m = 0.$$  

(4.3)

Using the cyclicity along with the Bianchi identity $dF = FA - AF$, we can also show that the characteristic classes $P_m$ are closed:

$$dP_m \sim d\text{Tr}F^m = m\text{Tr}(dFF^{m-1}) = m\text{Tr}(AF - FA)F^{m-1} = 0.$$  

(4.4)

Since the $P_m$ are closed, it follows from the Poincaré lemma that they are locally exact, meaning that we can locally write them as:

$$P_m(F) = dQ_{2m-1}(A, F).$$  

(4.5)

where the subscript $i$ indicates that the equality only holds on local coordinate patches $U_i$: the equality does not generally hold globally. $Q_{2m-1}$ is called the Chern-Simons form, and can be shown to have the general form [36]:

$$Q_{2m-1}(A, F) = m \int_0^1 dt t^{m-1} \text{Tr} \left( A(F + (t - 1)A^2) \right).$$  

(4.6)
We will in particular be concerned with \( Q_3 \), which can be shown to be:

\[
Q_3 = \text{Tr}(dA + \frac{1}{3} A^3) = \text{Tr} \left( AF - \frac{1}{3} A^3 \right),
\]

(4.7)

The Chern-Simons forms \( Q_{2m-1} \) are not gauge invariant, but using the gauge invariance of \( P_m \), we can characterize its gauge variation \( \delta Q_{2m-1} \).

\[
\delta P_m = 0 \implies d\delta Q_{2m-1} = 0.
\]

(4.8)

So the gauge variation of \( Q_{2m-1} \) is a closed form, which again according to Poincaré’s lemma is locally exact and can, up to both closed and gauge invariant forms, be written as:

\[
\delta Q_{2m-1} = d\tilde{Q}_{2m-2} \quad \text{locally on each } U_i.
\]

(4.9)

The fact that the variation of the Chern-Simons term under a gauge transformation is a total derivative will prove to be crucial. We can explicitly calculate \( \tilde{Q}_2 \) from \( Q_3 \). We first note that under an infinitesimal gauge transformation \( g = 1 + v \), \( A \) and \( F \) transform as:

\[
\delta A = dv + [A, v] \quad \delta F = [F, v].
\]

(4.10)

It is then an easy exercise to compute \( \delta Q_3 \):

\[
\delta Q_3 = \delta \text{Tr} \left( AF - \frac{1}{3} A^3 \right) = \text{Tr}(dvF - dvA^2) = d\text{Tr}(vdA).
\]

(4.11)

We conclude that

\[
\tilde{Q}_2 = \text{Tr}(vdA).
\]

(4.12)

The relationships between \( P_m, Q_{2m-1} \) and \( Q_{2m-2} \) are known as the descent equations:

\[
P_m = dQ_{2m-1}, \quad \delta Q_{2m-1} = d\tilde{Q}_{2m-2}.
\]

(4.13)

This relates to field theories when we recall that we can also construct action from contractions and traces of the field strength \( F \). More specifically, the anomaly will appear from the gauge variation of the quantum effective action \( \Gamma \) for the theory \([37]\). This gauge variation can be written as an integral over the a form closely related to \( d\tilde{Q}_{2m-2} \), defined in (4.13). This shows the connection between characteristic classes and anomalies. Much more can be said about this interplay where geometry, topology and field theory come together, but this will not be needed for this thesis.

### 4.2 Gravitational anomalies

The consequences of a gravitational anomaly are most clear when a chiral fermion is coupled to gravity. Original work on this subject includes \([38, 39]\), see \([37]\) for a more pedagogical introduction. Since gravity is in many ways analogous to gauge theory, we can rephrase the results of section 4.1 for gravity. Fermions do not transform in a fundamental representation of \( SO(d) \), but instead in a spin representation with generators \( T^{ab} = \frac{i}{2} \gamma^{ab} \). In this thesis we will however almost exclusively work in the metric formalism, so we will present the anomaly in a metric framework, remembering that to explicitly calculate the anomaly, we will need the first order formalism. We will thus work in a coordinate basis and define the covariant derivative as \( D \equiv d + [\Gamma, \cdot] \), where the connection one-form takes values in the Lie algebra of \( SO(d-1,1) \) (or...
SO(d) in Euclidean signature) and is defined as: \( \Gamma^\nu_\rho = \Gamma^\nu_{\rho\sigma}dx^\sigma \). The curvature two form is then defined as:

\[
R^\mu_\nu = \Gamma^\mu_\rho + \Gamma^\rho_\mu \Gamma^\nu_\rho - \Gamma^\rho_\sigma \Gamma^\nu_{\rho\sigma}dx^\rho.
\] (4.14)

Note that in this differential form notation, \( R^\mu_\nu \) is in fact the Riemann tensor and not the Ricci tensor. The gauge transformations parameterised by \( \xi \) are now infinitesimal diffeomorphisms \( x^\mu \rightarrow x^\mu - \xi^\mu(x) \) under which the metric transforms as:

\[
\delta \xi g_{\mu\nu} = v_{\mu\nu} + v_{\nu\mu}, \quad v_{\alpha\beta} = \frac{\partial \xi^\alpha}{\partial x^\beta}.
\] (4.15)

The transformation properties of the connection and curvature are entirely analogous to (4.10).

We are now in a position to compute the gravitational anomaly. For a spin-1/2-fermion of positive chirality, the relevant characteristic class is the Dirac genus \( \hat{A}(M) \):

\[
\hat{A}(M) \equiv 1 + \frac{1}{12} \frac{1}{(4\pi)^2} \left[ \frac{1}{360} Tr R^4 + \frac{1}{288} (Tr R^2)^2 \right] + \mathcal{O}(R^6).
\] (4.16)

Here traces are taken in the fundamental representation of \( SO(d-1,1) \). Note that the Dirac genus only involves forms of degrees 4\( k \), where \( k \) is an integer. Using the descent equations (4.13), we can relate the anomaly in \( d = 2n \) dimensions to a term of a characteristic polynomial of degree 2\( n \) + 2. Since \( \hat{A}(M) \) only consists of 4\( k \)-forms, gravitational anomalies can only exist when 2\( n \) + 2 = 4(\( k \) + 1) or equivalently \( d = 4k + 2 \), i.e. gravitational anomalies only occur in \( d = 2, 6, 10, \ldots \) dimensions. We will now specialise to the anomaly in two dimensions, which means that we need only consider the \( R^2 \)-term in (4.16). It turns out that:

\[
P_4 = 2\pi \left[ \hat{A}(M_4) \right]_4.
\] (4.17)

The extra factor of 2\( \pi \) is related to the index of the Dirac operator on a 4-dimensional manifold. The associated Chern-Simons form and its gauge variation are:

\[
P_4 = \frac{1}{96\pi} Tr R^2, \quad Q_3 = \frac{1}{96\pi} Tr \left( \Gamma d\Gamma + \frac{2}{3} \Gamma^3 \right), \quad \tilde{Q}_2 = \frac{1}{96\pi} Tr d\Gamma.
\] (4.18)

We conclude that the anomalous variation of the action under a gauge transformation with parameter \( \xi \) is given by:

\[
\delta S = \frac{1}{96\pi} \int Tr d\Gamma.
\] (4.19)

More explicitly, under an infinitesimal diffeomorphism, the action transforms as:

\[
\delta \xi S = \frac{1}{96\pi} \int \xi^\alpha \partial_\alpha \Gamma^\beta_{\alpha\nu} dx^\nu \wedge dx^\alpha,
\] (4.20)

\[
= -\frac{1}{96\pi} \int \xi^\alpha \epsilon^{\alpha\mu\nu} \partial_\mu \partial_\nu \Gamma^\beta_{\alpha\nu} d^2x.
\]

The anomaly has the effect that the action is no longer diffeomorphism invariant. Using the definition of the stress tensor, we can in general write a variation of the action due to an infinitesimal diffeomorphism as:

\[
\delta \xi S = \frac{1}{2} \int d^2x \sqrt{g} T^\nu_\mu \delta g_{\mu\nu} = \int d^2x \sqrt{g} T^\nu_\mu \nabla_\mu \xi_\nu = -\int d^2x \sqrt{g} \nabla_\nu T^\mu_\nu \xi_\mu.
\] (4.21)
Comparing (4.20) and (4.21), we see that:

\[ \nabla_\mu T^{\mu\nu} = \frac{1}{96\pi} g^{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha \partial_\rho \Gamma^\rho_{\nu\beta}. \]  

(4.22)

That is, in the presence of a gravitational anomaly, the stress tensor is no longer conserved: its derivative is now proportional to the second derivative of the Christoffel symbol. In Minkowski spacetime, the Christoffel symbols vanish, but a coordinate transformation might render the stress tensor non-conserved. This anomalous divergence could of course be kept track of and it turns out that in the first order formalism, the anomaly is actually proportional to the curvature tensor, which vanishes in flat space for all coordinate systems. In flat space therefore, the effect of the anomaly is not that dramatic, but has now become impossible to dynamically couple the theory to gravity. If the metric is promoted to a dynamical field, energy would constantly leak out', depending on the geometry.

We computed the anomaly for a spin-$\frac{1}{2}$ fermion of positive chirality. If we had considered a fermion of negative chirality instead, the anomalous divergence of the stress tensor would have picked up an extra minus sign. In the framework of 2D-CFT’s, the central charge can be viewed as representing the number of degrees of freedom in a theory and chiral fermions have $c = \pm \frac{1}{2}$, depending on the chirality. If we specialise to 2D CFTs, this motivates us to write the anomalous divergence of the stress tensor as [35]:

\[ \nabla_\mu T^{\mu\nu} = \frac{c_L - c_R}{96\pi} g^{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha \partial_\rho \Gamma^\rho_{\nu\beta}. \]  

(4.23)

It is immediately clear that the anomaly vanishes when the theory has equal left and right moving central charges $c_{L,R}$. There is however nothing that forces the central charges to be equal and we will compute the entanglement entropy in theories with unequal central charges in the next section.

As mentioned earlier, instead of working in a coordinate basis where we use the metric and the Christoffel connection as the fundamental degrees of freedom, we could also have worked in a non-coordinate basis. Here the geometry is represented by the vielbein one-forms $e^a = e^a_\mu dx^\mu$ and the connection is given by the spin connection $\omega^a_{\mu} = \omega^a_{\mu\nu} dx^\nu$. The action would then not be invariant under local Lorentz transformations $\delta e^a = -\alpha^a e^\nu$, rather than under diffeomorphisms. The calculation of the anomaly goes analogously, but one finds that the stress tensor now remains conserved, yet it is not symmetric anymore:

\[ T_{ab} - T_{ba} = 2\beta^* R_{ab}, \]  

(4.24)

where $^* R_{ab}$ is the Hodge dual of the curvature two-form. In [35], it is shown that one can always go back and forth between the two pictures by adding a local counterterm to the action. We will from now on work in the metric formalism, where the stress tensor is not conserved.

### 4.3 Gravitational anomalies and entanglement entropy

In section 2, we presented a method to compute entanglement entropy of intervals in 2D conformal field theories. This method almost entirely carries over to theories with a gravitational anomaly: the central charges being different will not have much effect, since the calculation factorizes holomorphically. The conformal dimension $\Delta = h_L + h_R$ of the twist fields for example remains the same as in (2.39). There are however some differences to the non-anomalous case: the twist fields for example now acquire a non-zero spin:

\[ \Delta = \frac{c_L + c_R}{24} \left( 1 + \frac{1}{n} \right), \quad s = \frac{c_L - c_R}{24} \left( 1 + \frac{1}{n} \right). \]  

(4.25)
This non-zero spin is not just there for the twist fields, the theory itself also acquires a non-zero ‘Casimir momentum’ in addition to the usual Casimir energy $E_0$:

$$J_0 = \frac{c_L - c_R}{24}$$

(4.26)

Using the replica trick for a region of length $l$ in the field theory and taking the limit of $n \rightarrow 1$, this reproduces the same entanglement entropy as in (2.41):

$$S_{EE} = \frac{c_L}{6} \log \left( \frac{l}{a} \right) + \frac{c_R}{6} \log \left( \frac{\bar{l}}{\bar{a}} \right).$$

(4.27)

However, if we now consider a boosted complex interval $[0, z]$ with $z = le^{i\theta}$, note that in Euclidean signature, a boost is just a rotation, (2.41) becomes:

$$S_{EE} = \frac{c_L}{6} \log \left( \frac{z}{a} \right) + \frac{c_R}{6} \log \left( \frac{\bar{z}}{\bar{a}} \right),$$

(4.28)

$$= \frac{c_L + c_R}{6} \log \left( \frac{l}{a} \right) + \frac{c_R - c_L}{6} i\theta.$$}

If we analytically continue (4.28) to Lorentzian signature, the angle $\theta$ is related to a boost parameter $\kappa$ as $\theta = i\kappa$ and we obtain:

$$S_{EE} = \frac{c_L + c_R}{6} \log \left( \frac{l}{a} \right) - \frac{c_L - c_R}{6} \kappa.$$

(4.29)

This shows explicitly that the entanglement entropy in a theory with a gravitational anomaly depends on the choice of coordinates used to define a constant time slice. A nice physical argument for why boosting the theory would alter the entanglement entropy can be found in [40]. For theories at finite size and finite temperature, we obtain similar results. The expressions for 2D CFTs at finite temperature (2.46) and on a manifold of finite size (2.47) only depend on the sum of the central charges, so there will be no anomalous contribution. Just as in the case of the CFT on a plane however, these expressions will still acquire anomalous contributions under a boost. Finally we can consider a CFT at non-zero temperature $\beta^{-1}$ and angular potential $\Omega_E$. In section 2.4 we already noted that this expression factorizes in left and right moving contributions. This fact immediately implies that these CFTs will receive an anomalous correction if $c_L \neq c_R$.

### 4.4 Holographic gravitational anomalies: Topologically Massive Gravity

The AdS-CFT correspondence teaches us that there should be a dual gravitational theory to a two dimensional conformal field theory with a gravitational anomaly. However, diffeomorphism invariance cannot be violated inside the bulk, since without it, it would be impossible to have a well-defined theory of gravity. We thus need a mechanism to break diffeomorphism on the boundary, yet preserve it in the bulk. The solution is provided by the Chern-Simons term in (4.18). As mentioned earlier, the variation of this term under a gauge transformation is precisely a total derivative. Adding the three-dimensional Chern-Simons term to the action will produce precisely the non-covariant term on the boundary we want.

The theory of three-dimensional Einstein gravity with a Chern-Simons term added to it, is known as Topologically Massive Gravity (TMG) and it was first described in [41, 42]. It is a
higher derivative theory of gravity and has the following action in the presence of a negative cosmological constant:

$$S_{TMG} = S_{EH} + \frac{1}{32\pi G\mu} \int d^3x \sqrt{-g} \mu^{\mu\nu\lambda} \Gamma^\rho_{\lambda\sigma} \left( \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right),$$  \hspace{1cm} (4.30)

where $S_{EH}$ is the usual Einstein-Hilbert action and $\mu$ is a real coupling constant. Varying this action with respect to the metric will yield the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = -\frac{1}{\mu} C_{\mu\nu}.  \hspace{1cm} (4.31)$$

Here $C_{\mu\nu}$ is the Cotton tensor:

$$C_{\mu\nu} = \epsilon^{\alpha\beta\mu} \nabla_\alpha \left( R_{\beta\nu} - \frac{1}{4} R \right).  \hspace{1cm} (4.32)$$

Using the Bianchi identities, it is easy to see that the Cotton tensor is symmetric, transverse and traceless:

$$\epsilon^{\alpha\mu\nu} C_{\mu\nu} = \nabla_{\mu} C_{\mu\nu} = C_{\mu\mu} = 0.  \hspace{1cm} (4.33)$$

Taking the trace of the equations of motion then yields $R = -6/l^2$ and we can write:

$$R_{\mu\nu} + \frac{2}{l^2} g_{\mu\nu} = -\frac{1}{\mu} C_{\mu\nu}.  \hspace{1cm} (4.34)$$

Solutions of the three-dimensional vacuum Einstein equations with negative cosmological constant are all conformally flat and conformally flat manifolds also have vanishing Cotton tensor. Hence all solutions of pure 3D-gravity are also solutions of TMG. TMG however admits a wider class, such as warped AdS$_3$ spacetimes, which were for example studied in [43].

Applying the Brown-Henneaux method to a TMG background, one can find the algebra of asymptotic symmetries. This was for example done in [44] and [45] and there it was found that, as in ordinary AdS$_3$, the algebra of asymptotic charges is organized in two copies of the Virasoro algebra, but now with different central charges:

$$c_L = \frac{3l}{2G_3} \left( 1 - \frac{1}{\mu l} \right), \hspace{1cm} c_R = \frac{3l}{2G_3} \left( 1 + \frac{1}{\mu l} \right).  \hspace{1cm} (4.35)$$

The thermodynamic properties of the spacetime are different when the Chern-Simons term is added to the action. Using the conventional ADM-procedure, one can compute for global AdS$_3$:

$$M_{AdS} = -c_L + c_R \frac{24l}{24}, \hspace{1cm} J_{AdS} = \frac{c_L - c_R}{24}.  \hspace{1cm} (4.36)$$

Note that TMG has the effect of introducing a non-zero angular momentum in global AdS$_3$. This matches the Casimir momentum (4.26) of the boundary field theory.

Because of the epsilon-tensor, the Chern-Simons term is odd under time reversal $t \rightarrow -t$. In the next section we will need the Euclidean version of TMG. If we perform a Wick rotation $t \rightarrow t_E = it$, then the second term in the action picks up an $i$:

$$S_E = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) + i \frac{l}{32\pi G\mu} \int d^3x \sqrt{-g} \mu^{\mu\nu\lambda} \Gamma^\rho_{\lambda\sigma} \left( \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right).  \hspace{1cm} (4.37)$$

The equations of motion then become:

$$R_{\mu\nu} + \frac{2}{l^2} g_{\mu\nu} = -\frac{i}{\mu} C_{\mu\nu}.  \hspace{1cm} (4.38)$$
This shows that real metrics which are solutions of Euclidean TMG also necessarily have to be solutions of Einstein gravity, since the Einstein tensor and the Cotton tensor have to vanish independently.

As mentioned earlier, three-dimensional gravity can be written as a Chern-Simons theory. Perhaps rather confusingly, adding a Chern-Simons term to the action actually has the effect that TMG is not a Chern-Simons theory anymore. It can however be cast in a Chern-Simons formulation, with an extra constraint added to the action. In this thesis we will only use the metric formulation of TMG, but for details on the Chern-Simons formulation, see Appendix E of [1].

As mentioned earlier, the Ryu-Takayanagi formula (3.26) only applies to pure GR, so we cannot use it to holographically compute the entanglement entropy in 2D CFT’s with a gravitational anomaly. Extending the Ryu-Takayanagi procedure to TMG was the main goal of [1] and we will review their results in the next section.
5 Holographic entanglement entropy in TMG

We have established that the dual theory to a 2D CFT with a gravitational anomaly is Topologically Massive Gravity. As mentioned before, this means that we cannot use the Ryu-Takayanagi formula to compute the entanglement entropy holographically. We can however use the line of reasoning of the proof of the RT formula in [29] to find a modification of this formula for these theories. This was the aim of [1] and in this section we will cover their methods and results.

We expect the procedure to somehow depend on the coordinate frame which is chosen in the boundary CFT, since diffeomorphism invariance is broken there. It will turn out that this is indeed the case: we will have to track the normal vectors to the RT worldline from one endpoint, through the bulk, to the other endpoint. This has the effect of 'broadening' the worldline into a ribbon and the extra contribution to the entanglement entropy due to the anomaly will be given by the twist of this ribbon. We will see that this twist can be expressed as:

\[ S_{\text{anom}} = \frac{1}{4G_3} \int ds(\tilde{n} \cdot \nabla n). \]  \hspace{1cm} (5.1)

We will begin this section by applying the reasoning presented in section 3.4 to TMG, again finding the action of an extended conical defect, now for both the Einstein-Hilbert term and the Chern-Simons term in (4.37). We will find that we have to add (5.1) to the usual geodesic action. We will then minimize this action and find that this minimization yields the so-called Mathisson-Papapetrou-Dixon (MPD) equations. These equations have been known for a long time and describe the motion of particles with a continuously tunable spin in classical GR, original papers on the subject are [46, 47, 48]. The fact that we have to consider spinful particles is a reflection of the fact that the twist fields in the boundary CFT have non-zero spin when a gravitational anomaly is present. We will end the section by explaining how to use this prescription to actually compute entanglement entropy and checking that it reproduces the correct results from anomalous CFTs, computed in section 4.3.

5.1 Cones in TMG

In section 3.4 we discussed the proof of the Ryu-Takayanagi conjecture provided by [29]. By evaluating the action of a spacetime with a conical defect cut out of it, it was shown that the proper length of this conical defect is equal to \( 4G_3 \) times the entanglement entropy of an interval in the boundary field theory, when the limit of \( n \to 1 \) was taken. In this section, we will follow the reasoning in [1] to apply this procedure to a conical defect in TMG. For the geometry we will use a regularized version of the conical metric given in (3.62):

\[ ds^2 = e^{\phi(\sigma)} \delta_{ab} d\sigma^a d\sigma^b + (g_{yy} + K_a \sigma^a) dy^2 + e^{\phi(\sigma)} U_a(\sigma, y) d\sigma^a dy. \]  \hspace{1cm} (5.2)

Here, again \( y \) is a coordinate parameterising the tip of the cone and \( \sigma^a, a = 1, 2 \), parameterise the plane transverse to the tip. \( \phi(\sigma) \) is a function regulating the tip of the cone, it is defined to fall off exponentially outside the core region of the cone which has radius \( a \). If we define complex coordinates \((z, \bar{z}) = \sigma^1 \pm i \sigma^2\), it is shown in [1] that \( \phi \) takes the explicit form:

\[ \phi(z, \bar{z}) = -\exp\left(-\frac{|z|}{a}\right) \log(|z|). \]  \hspace{1cm} (5.3)

The cone has opening angle \( 2\pi n \), so we will compute its action to first order in \( \epsilon \equiv n - 1 \). Firstly we note that the Einstein-Hilbert part of the TMG action (4.37) will just give the same results as
in the non-anomalous pure gravity. The Einstein-Hilbert part of the action evaluated around the conical defect to first order in $\epsilon$ is thus just geodesic length:

$$S_{EH} = -\frac{\epsilon}{4G_3} \int_C dy \sqrt{g_{yy}} + \mathcal{O}(\epsilon^2).$$

(5.4)

If we parameterise the worldline of the cone by $X^\mu(s)$, we can write the familiar expression:

$$S_{EH} = -\frac{\epsilon}{4G_3} \int_C ds \sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + \mathcal{O}(\epsilon^2).$$

(5.5)

We now have to do the same calculation for the Chern-Simons part of (4.30). This is done explicitly in Appendix B of [1], there it is found that to first order in $\epsilon$ the Chern-Simons part of the action is:

$$S_{CS} = -\frac{i\epsilon}{16\mu G_3} \int dy \left( \epsilon^{ab} \partial_a U_b \right),$$

(5.6)

where we used the coordinates defined in (5.2). The explicit appearance of the functions $U_a$ means that this expression is not coordinate independent, as expected. In writing down the metric (5.2) a choice has to be made how to define the coordinates $\sigma^a$ transverse to the tip of the cone and the functions $U_a$ reflect this choice. In fact, we can perform an infinitesimal local rotation on the $\sigma$-coordinates:

$$\delta \sigma^a = -\theta(y) \epsilon_a^b \sigma^b.$$

(5.7)

The $d\sigma^2$-component of the metric (5.2) then picks up a term of the form (there will be additional terms, but they will be higher order in $\epsilon$):

$$d\sigma^a d\sigma^b \rightarrow d\sigma^a d\sigma^b - 2\theta'(y) \epsilon_c^a \sigma^b d\sigma^c dy.$$

(5.8)

This means that the curl of the vector field $U_a$ transforms as:

$$\delta \left( \epsilon^{ab} \partial_a U_b \right) = 4 \partial_y \theta.$$

(5.9)

If we vary the action under such an infinitesimal rotation, it will pick up a boundary term:

$$\delta S_{CS} = -\frac{i\epsilon}{4\mu G_3} (\theta(y_f) - \theta(y_i)).$$

(5.10)

This shows that we have to track the twisting of the coordinate frame parameterised by the $\sigma^a$-coordinates. To do so, we define a normal frame $n_1$ and $n_2$ with respect to the coordinates $\sigma^a$:

$$n_1 = \frac{\partial}{\partial \sigma^1}, \quad n_2 = \frac{\partial}{\partial \sigma^2}.$$

(5.11)

To first order in $\epsilon$, these vectors have unit norm and are orthonormal to the tangent vector $v$ along the curve. Looking at the metric (5.2), we see that the Chern-Simons part of the cone action (5.6) can be written as one Christoffel symbol:

$$S_{CS} = -\frac{i\epsilon}{16\mu G_3} \int dy \left( \epsilon^{ab} \partial_a U_b \right) = -\frac{i\epsilon}{4\mu G_3} \int_C ds \Gamma^2_1 y = -\frac{i\epsilon}{4\mu G_3} \int_C ds n_2 \cdot \nabla n_1.$$

(5.12)

Here $\nabla$ denotes differentiation along the wordline:

$$\nabla V^\mu = \frac{dV^\mu}{ds} + \Gamma^\mu_{\rho\sigma} \frac{dX^\rho}{ds} V^\sigma.$$

(5.13)
(5.12) is not a covariant expression, since it depends on the choice of the normal vectors \( n_1 \) and \( n_2 \). We will later see that the values of these normal vectors at the boundary are closely related to the anomalous contribution to the entanglement entropy. The only thing we have to now do to find the entanglement entropy for the Euclidean theory, is taking the \( \epsilon \) derivative of (5.5) and (5.12):

\[
S_{EE} = \frac{1}{4G_3} \int_C ds \left( \sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + \frac{i}{\mu} n_2 \cdot \nabla n_1 \right)_E.
\]  

(5.14)

We can perform a Wick rotation to obtain this expression in Lorentzian signature by analytically continuing the complex coordinates \( z \) and \( \bar{z} \) via \( z = x - t \), \( \bar{z} = x + t \). The normal vector then change accordingly:

\[
n \equiv in_1 = \partial_t, \quad \tilde{n} \equiv n_2 = \partial_x.
\]  

(5.15)

In terms of \( n \) and \( \tilde{n} \), the entanglement entropy is:

\[
S_{EE} = \frac{1}{4G_3} \int_C ds \left( \sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + \frac{1}{\mu} \tilde{n} \cdot \nabla n \right).
\]  

(5.16)

This expression should be evaluated on the worldline \( C \) which is traced out by the tip of the cone. In section 3.4 we went over the argument presented in [29] that in the case of Einstein gravity, the wordline is actually a geodesic. This was shown using the requirement that the coefficients of certain divergent terms in the Einstein equations vanish. A geodesic is of course also an extremum of the entanglement functional itself. It is not immediately clear that the functional (5.16) should also be evaluated on an extremum. In the next section we will show that if we do extremize (5.16), we obtain the so called MPD equations. In Appendix D of [1], it is shown that the MPD equations can be expressed in the metric functions defined in (5.2). In a similar way to the geodesic case, it turns out that the TMG equations of motion (4.38) contain divergences which will vanish precisely when the MPD equations are satisfied. This shows that the action (5.16) should be evaluated on a curve which is an extremum of this action.

Finally, as is noted in [1], it is not entirely clear whether the analytic continuation from (5.14) to (5.16) is always possible. The Euclidean action (5.14) is always well defined but there is no mathematically rigorous way of proving that the Lorentzian version (5.16) exists. We will nevertheless use it to compute entanglement entropy in TMG and find results which match the CFT results calculated in section 4.3.

5.2 The MPD equations: spinning particles and ribbons

We can write the new entanglement functional (5.16) as:

\[
S_{EE} = \int_C d\tau \left( m \sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + s \tilde{n} \cdot \nabla n \right),
\]  

(5.17)

where

\[
m = \frac{1}{4G_3}, \quad s = \frac{1}{4G_3 \mu},
\]  

(5.18)

and \( \tau \) is a parameter along the wordline.

It seems at first sight that by introducing \( n \) and \( \tilde{n} \) into the action, we introduced six new degrees of freedom, but we know that we have to impose constraints on \( n \) and \( \tilde{n} \). They have to be normalized to +1 and −1 respectively and they have to be mutually orthonogal and perpendicular to the wordline:

\[
n \cdot v = 0, \quad \tilde{n} \cdot v = 0, \quad n \cdot \tilde{n} = 0, \quad n^2 = -1, \quad \tilde{n}^2 = 1.
\]  

(5.19)
Here the velocity vector $v^\mu$ is defined by differentiation with respect to an affine parameter $s$, which we will choose to be the proper length along the spacelike path:

$$v^\mu \equiv \frac{dX^\mu}{ds} = \dot{X}^\mu, \quad v^2 = 1. \quad (5.20)$$

We can enforce these constraints by introducing Lagrange multipliers $\lambda_i$:

$$S_C = \int_C ds \left[ \lambda_1 n \cdot \tilde{n} + \lambda_2 n \cdot v + \lambda_3 \tilde{n} \cdot v + \lambda_4 (n^2 + 1) + \lambda_5 (\tilde{n}^2 - 1) \right]. \quad (5.21)$$

The full entanglement functional is then the sum of (5.17) and (5.21):

$$S = \int_C ds \left( m \sqrt{g} \dot{X}^\mu \dot{X}_\mu + s \tilde{n} \cdot \nabla n \right) + S_C. \quad (5.22)$$

This is the full action for the spinning particle, which should be varied with respect to $X^\mu$, $n$ and $\tilde{n}$. It appears as though we still have one dynamical degree of freedom along the wordline left: $n$ and $\tilde{n}$ have three components each, the five constraints fix five of these six degrees of freedom, leaving one. That this is in fact not the case can be seen by varying the action with respect to $n$ and $\tilde{n}$ and contracting both equations with $n^\mu$, $\tilde{n}^\mu$ and $v^\mu$, which yields the following six equations for the $\lambda_i$:

$$\lambda_1 = s \tilde{n} \cdot \nabla \tilde{n}, \quad \lambda_2 = sv \cdot \nabla \tilde{n}, \quad 2\lambda_4 = -sn \cdot \nabla n. \quad (5.23)$$

$$\lambda_1 = sn \cdot \nabla n, \quad \lambda_3 = sv \cdot \nabla n, \quad 2\lambda_5 = -s\tilde{n} \cdot \nabla \tilde{n}. \quad (5.24)$$

We observe that this fixes all multipliers except for $\lambda_1$, which appears in both sets of equations and seems to give a relation between $n$ and $\tilde{n}$:

$$\tilde{n} \cdot \nabla \tilde{n} = n \cdot \nabla n. \quad (5.25)$$

This is however an identity, since both sides of the equation are identically zero because $\tilde{n}^2 = -n^2 = 1$. We thus conclude that $(n, \tilde{n})$ do not have a dynamical equation of motion: $S$ is insensitive to small variations of the normal frame along the worldline. The boundary values of $n$ and $\tilde{n}$ can however not be fixed by choosing the $\lambda_i$ and these boundary values will turn out to be crucial in computing the entanglement entropy.

To derive the equations of motion of the spinning probe, we now need to vary the action (5.21) with respect to $X^\mu$. This is a rather tedious procedure, which is performed in detail in Appendix C of [1]. The geodesic part of (5.21) will of course give rise to the geodesic equation:

$$m \nabla v^\mu = 0. \quad (5.26)$$

The variation of the anomalous part (5.1) of the action (5.21) is more involved and will give rise to non-covariant terms. These are however cancelled by the variation of the constraint action $S_C$. Using the definition and properties of the Riemann tensor we arrive at:

$$\nabla [mv^\mu + v_\rho \nabla s^{\rho\sigma}] = -\frac{1}{2} v^\nu s^{\rho\sigma} R_{\nu\rho\sigma}, \quad (5.27)$$

where the spin tensor $s^{\mu\nu}$ is defined as:

$$s^{\mu\nu} = s(n^\mu \tilde{n}^\nu - \tilde{n}^\mu n^\nu). \quad (5.28)$$
These equations are known as the Mathisson-Papapetrou-Dixon (MPD) equations which were introduced at the beginning of this section. They have been known for a long time but have also attracted more recent interest, for example in the context of torsional extensions of GR known as Riemann-Cartan theory [49, 50]. The equations can in general be obtained by writing down a Lagrangian for a spinning particle and finding the equation of motion from there. In higher than three dimensions, the MPD-equations have to be supplemented with a relation which describes the evolution of the spin tensor. In three dimensions though, the spin tensor does not contain any dynamical degrees of freedom, as we have already seen, so such a relation is not necessary. This is further emphasized by the fact that in three dimensions the spin tensor can be written purely in terms of the velocity:
\[
s_{\mu\nu} = -s\epsilon^{\mu\nu\lambda\rho}v_{\rho}.
\]  
(5.29)

In Appendix D of [1], the consistency of this procedure is checked by considering a spinning probe in a TMG background. There it is shown that it is possible to write down a stress tensor for the probe and that the requirement that this stress tensor is covariantly conserved is equivalent to the MPD equations.

We can summarize this section by noting that the entanglement entropy in topologically massive gravity is found by solving the MPD equations (5.27) and then evaluating the action (5.17) on this solution.

### 5.2.1 The MPD equations in locally AdS$_3$ spacetimes

The MPD equations simplify in spacetimes which are locally AdS$_3$. In such spacetimes, the Riemann tensor takes a simple form:
\[
R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}).
\]  
(5.30)

where $L$ is the AdS radius. It is an easy exercise to check that the contraction with the spin tensor on the right hand side of the MPD equations vanishes in this case. This simplifies the MPD equations greatly: if we write the spin tensor in terms of the velocity as in (5.29), we can write them as:
\[
\nabla [mv_{\mu} - s\epsilon^{\mu\nu\lambda}v_{\nu}v_{\lambda}] = 0.
\]  
(5.31)

It is immediately clear that a geodesic $\nabla v_{\mu} = 0$ is a solution to (5.31). This means that in locally AdS$_3$ spacetimes, the spinning particles will move on geodesics. Note that the MPD equations are third order in derivatives, which means that there might be other solutions. We will return to this possibility in section 6.6, where we will show that the other solutions do not obey the right boundary conditions. We conclude that in locally AdS$_3$ spacetimes, the desired entanglement entropy will be the sum of the length of the geodesic term and the anomalous term in (5.31), evaluated on the geodesic.

As in [1], we can write the anomalous part of the action in terms of a single normal vector $n$ using $\tilde{n}_{\mu} = \epsilon^{\mu\nu\lambda}v_{\nu}n_{\lambda}$:
\[
S_{\text{anom}} = \frac{1}{4G_3\mu} \int_C ds \epsilon_{\mu\nu\lambda}v^\mu n^\nu (\nabla n^\lambda).
\]  
(5.32)

If we define the value of $n(s)$ at the boundary as $n(s_i)$ and $n(s_f)$, the anomalous part of the action is proportional to the twist of $n_f$ relative to $n_i$. To compute this twist, we introduce another normal frame along the curve, made up of two vectors $(q^\mu, \tilde{q}^\mu)$. The fact that the curve is a geodesic allows us to demand that this tangent frame is parallel transported along the
curve: $\nabla q = \nabla \tilde{q} = 0$. $n$ is related to $q$ and $\tilde{q}$ by an $SO(1,1)$-transformation: we can perform an arbitrary local boost on $q$ and $\tilde{q}$ to obtain:

$$n(s) = \cosh(\eta(s))q(s) + \sinh(\eta(s))\tilde{q}(s).$$

(5.33)

Plugging this into the action (5.32), the $q$ and $\tilde{q}$ actually drop out and the action reduces to a total derivative:

$$S_{\text{anom}} = \frac{1}{4G_3\mu} \int_C ds \epsilon_{\mu\nu\lambda} v^\mu (\cosh^2 \eta q^\nu \tilde{q}^\lambda - \sinh^2 \eta q^\nu \tilde{q}^\lambda) \dot{\eta} = \frac{1}{4G_3\mu} \int_C \dot{\eta}(s),$$

(5.34)

where we used that $q \cdot \tilde{q} = 0$ and $\epsilon_{\mu\nu\lambda} v^\mu q^\nu \tilde{q}^\lambda = 1$. We then note that there is a convenient way to write the twist in terms of the boundary values of $q$, $\tilde{q}$ and $n$:

$$S_{\text{anom}} = \frac{1}{4G_3\mu} \log \left( \frac{q(s_f) \cdot n_f - \tilde{q}(s_f) \cdot n_f}{q(s_i) \cdot n_i - \tilde{q}(s_i) \cdot n_i} \right)$$

(5.35)

To see that (5.35) is actually equal to (5.34), we use the definition $q$ and $\tilde{q}$ (5.33):

$$S_{\text{anom}} = \frac{1}{4G_3\mu} \int_C ds \dot{\eta}.$$

(5.36)

Note again that the anomalous contribution to the entanglement entropy only depends on the boundary values of $n$. Finally we need a way to fix the direction of $n_i$ and $n_f$. At the boundary, the coordinate system in the bulk should coincide with the coordinate system parameterizing the CFT. We note that $n$ is a timelike vector so (5.15) instructs us to identify:

$$n_i = n_f = (\partial_t)_{\text{CFT}}.$$

(5.37)

This is a reflection of the fact that the boundary is not diffeomorphism invariant: the entanglement entropy now depends on the choice of time coordinate in the field theory, just as we saw in section 4.3.

### 5.3 Examples

In the previous sections, we discussed the procedure to compute entanglement entropy in TMG. To get more evidence that this procedure is actually correct, we will compute the entanglement entropy in the case of Poincaré AdS and the rotating BTZ black hole. Since we established that geodesics are solutions to the MPD equations in locally AdS$_3$ spacetimes, we can use the geodesics we computed earlier in sections 3.1.3 and 3.3.3.

#### 5.3.1 Poincaré AdS

In section 3.1.3 we computed the explicit form of geodesics in Poincaré AdS$_3$. In particular, we found that the tangent vector to the geodesic had the form:

$$v^\mu = \frac{2z}{RL}(z,0,0),$$

(5.38)

where $R$ is the length of half the entangled interval in the CFT (or equivalently the radius of the semicircle) and $L$ is the AdS radius. Given this tangent vector we can construct a parallel
transported normal frame \((q, \tilde{q})\) by first demanding that \(v \cdot q = v \cdot \tilde{q} = 0\). \(q\) and \(\tilde{q}\) then have the following form:

\[
q^\mu = A(0, z, 0), \quad \tilde{q}^\mu = B(x, 0, z).
\]

The values of \(A\) and \(B\) are fixed by demanding that \(q^2 = -1\) and \(\tilde{q}^2 = 1\):

\[
A = \frac{1}{l}, \quad B = \frac{2z}{RL}.
\]

This gives the following tangent frame:

\[
q^\mu = \frac{1}{l}(0, z, 0), \quad \tilde{q}^\mu = \frac{2z}{RL}(x, 0, z).
\]

For completeness, it should be checked that \(q\) and \(\tilde{q}\) are parallelly transported: \(\nabla q = 0\) and \(\nabla \tilde{q} = 0\). Having constructed a normal frame that is actually parallelly transported, we can go on and compute the entanglement entropy coming from the anomaly by using (5.35). We immediately note that \(q^\mu(s_i) = q^\mu(s_f)\) is a vector which has norm \(-1\) and points in the time-direction. Note that the value of \(q^\mu\) is actually not well defined at the boundary, since the metric diverges at \(z = 0\). We can thus also immediately conclude that \(q(s_f) \cdot n_f = 1\) and \(\tilde{q}(s_f) \cdot n_f = 0\) and similarly for \(s_i\). Hence \(q\) already satisfies the right boundary conditions and \(\eta(s) = 0\): the spin does not contribute to the entanglement entropy. This means that the entanglement entropy is given by the proper distance, which was already computed in (3.49):

\[
S_{EE} = \frac{c_L + c_R}{6} \log \left( \frac{R}{\epsilon} \right).
\]

which of course matches the CFT result (4.27).

### 5.3.2 Poincaré AdS, boosted interval

We again consider Poincaré AdS, but now the entangling interval \((x, t)\) is boosted to \((x', t')\). In a Lorentz invariant theory, this would obviously have no effect, but since the theory suffers from gravitational anomalies, it will turn out that boosting does indeed influence the entanglement entropy. Under a boost:

\[
\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \kappa & \sinh \kappa \\ \sinh \kappa & \cosh \kappa \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix},
\]

the tangent vector \(v^\mu = (\dot{x}, \dot{t}, \dot{z})\) transforms to:

\[
v^\mu = \frac{2z}{RL}(z \cosh \kappa, z \sinh \kappa, -x),
\]

where we have used that \(\dot{t} = 0\) in the original frame. We proceed in a fashion similar to the non-boosted case. We can construct a normal frame \((q, \tilde{q})\) by simply boosting the normal frame given in (5.41), which gives:

\[
q^\mu = \frac{1}{L}(z \sinh \kappa, z \cosh \kappa, 0), \quad \tilde{q}^\mu = \frac{2z}{RL}(x \cosh \kappa, x \sinh \kappa, z).
\]

Using the constraint that \(n_i = n_f = (\partial_t)_{\text{CFT}}\) and again using the expression for the anomalous part of the entanglement entropy (5.35), we can compute the anomalous contribution:

\[
S_{\text{anom}} = \frac{1}{4G_3 \mu} \log \left( \frac{\cosh \kappa + \sinh \kappa}{\cosh \kappa - \sinh \kappa} \right) = \frac{1}{2G_3 \mu} \kappa.
\]
We can relate this to the difference of the central charges of the boundary field theory by noting that:
\[
\frac{1}{2G_{3H}} = 2s = \frac{c_L - c_R}{6},
\]
where we used (5.18) and (4.25). Adding the geodesic length, we find that the total entanglement entropy is:
\[
S_{EE} = \frac{c_L + c_R}{6} \log \left( \frac{R}{\epsilon} \right) - \frac{c_L - c_R}{6} \kappa.
\]
This matches the CFT result (4.28).

5.3.3 Rotating BTZ

Finally, we compute the entanglement entropy for a rotating BTZ geometry in a TMG background. The metric reads:
\[
ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} d\tau^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left( d\phi + \frac{r_+ - r_-}{r^2} d\tau \right)^2.
\]
The boundary of the spacetime lies at \( r = \infty \) in these coordinates. In this limit, the metric becomes:
\[
ds^2_{r \to \infty} = -d\tau^2 + d\phi^2.
\]
Both the \( \tau \) and the \( \phi \) coordinate are periodically identified, so this is also the metric of the spacetime on which the CFT lives. The boundary conditions on the normal vector are defined with respect to the time coordinate of the black hole:
\[
n_i = n_f = (\partial_\tau)_{CFT}.
\]
This normal vector is again a vector with unit length in the \( \tau \)-direction:
\[
n_i = n_f = \frac{1}{\sqrt{-g_{\tau\tau}}}(0, 1, 0, 0) \sim \frac{1}{r}(0, 1, 0),
\]
where we have taking the limit \( r \to \infty \). We can now use the mapping from the BTZ coordinates to Poincaré AdS3, which we covered earlier. We repeat the mapping for convenience:
\[
x = \sqrt{r_2 - r_2^2 r_-^2} e^{2\tau / \beta_{R, L}}, \quad z = \sqrt{r_2 + r_2^2 e^{2\phi / \beta_{R, L}}}.
\]
We can now proceed to view the problem in Poincaré coordinates only to transform back to BTZ coordinates at the final step. We consider a boundary interval of length \( R \) along the \( \phi \)-direction at \( \tau = 0 \), which can be mapped to Poincaré coordinates \( (x, t)_{1, 2} \) with the relation above. We obtain:
\[
\begin{align*}
x_1 &= e^{-\frac{n_x}{2l}} \cosh \left( \frac{R \tau}{2l} \right), & t_1 &= -e^{-\frac{n_x}{2l}} \sinh \left( \frac{R \tau}{2l} \right), \\
x_2 &= e^{\frac{n_x}{2l}} \cosh \left( \frac{R \tau}{2l} \right), & t_2 &= e^{\frac{n_x}{2l}} \sinh \left( \frac{R \tau}{2l} \right).
\end{align*}
\]
Meanwhile, as \( r \to \infty \), the values of \( z \) become:
\[
\begin{align*}
z_1 &= \sqrt{r_2 - r_2^2} e^{-\frac{n_x}{2l}}, & z_2 &= \sqrt{r_2 + r_2^2} e^{\frac{n_x}{2l}}.
\end{align*}
\]
The length of the interval in Poincaré coordinates is:
\[ R_P = \sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2}, \]  
and we can define a vector pointing along this boundary interval:
\[ p^i = \frac{1}{R_P} (x_2 - x_1, t_2 - t_1, 0). \]

Using this information, we can write down the \( q \)'s (again in Poincaré coordinates):
\[ q^\mu = z_l p^t, p^x, 0 \]  
\[ \tilde{q}^\mu = -\frac{2z}{l R_P} (\lambda p^x, \lambda p^t, z), \quad \lambda^2 + z^2 = \frac{R_P}{4}. \]

Here \( \lambda \) parametrizes movement along the curve. Previously we could use the coordinate \( x \) for this, but this doesn’t work anymore, since the boundary is now boosted out of the \((x, t)\)-plane.

The differences in \( x \) and \( t \) can be expressed as follows:
\[ x_2 - x_1 = 2 \sinh \left( Rr + \frac{2l}{R_P} \right) \cosh \left( \frac{Rr}{2l} \right) = \sinh \left( \frac{R\pi}{\beta_R} \right) + \sinh \left( \frac{R\pi}{\beta_L} \right) \]  
\[ t_2 - t_1 = 2 \cosh \left( Rr + \frac{2l}{R_P} \right) \sinh \left( \frac{Rr}{2l} \right) = \sinh \left( \frac{R\pi}{\beta_R} \right) - \sinh \left( \frac{R\pi}{\beta_L} \right). \]

This means that
\[ R_P = 2 \sqrt{\sinh \left( \frac{R\pi}{\beta_R} \right) \sinh \left( \frac{R\pi}{\beta_L} \right)}. \]

To transform \( q \) and \( \tilde{q} \) from Poincaré to BTZ coordinates, we also need to compute the Jacobian. Since \( g_{\tau \phi} \) vanishes as \( r \to \infty \), we only have to compute \( q^\tau \) and \( \tilde{q}^\tau \). Using the definition of the coordinate transformation, we conclude:
\[ q^\tau = \frac{l}{r_- x} q^x + \frac{l}{r_+ t} q^t + \frac{l}{r_- u} q^z. \]

and accordingly for \( \tilde{q} \). We are now in a position to compute \( q^\tau \) and \( \tilde{q}^\tau \):
\[ q^\tau = \frac{u}{R_P} \left( \frac{(t_2 - t_1)}{x r_-} + \frac{(x_2 - x_1)}{t r_+} \right), \]
\[ \tilde{q}^\tau = -\frac{2z}{R_P} \left( \lambda (x_2 - x_1) + \lambda (t_1 - t_2) + \frac{1}{r_-} \right). \]

Putting everything together and going through the algebra, we find that:
\[ S_{\text{anom}} = \frac{1}{4G_3 \mu} \left[ \log \left( \frac{\beta_+}{\pi a} \sinh \left( \frac{\pi R}{\beta_+} \right) \right) - \log \left( \frac{\beta_-}{\pi a} \sinh \left( \frac{\pi R}{\beta_-} \right) \right) \right]. \]

Together with the geodesic contribution computed in (2.54), this precisely accounts for the anomalous entanglement entropy in a 2D CFT with a gravitational anomaly.

In this section we have reviewed how the procedure developed in [1] correctly reproduces the entanglement entropy of anomalous 2D CFTs using holography. We take this as a starting point to consider geometries which are more complicated than locally AdS3 spacetimes, which will be the subject of the next section.
6 The MPD-equations in asymptotically AdS$_3$ backgrounds

In the previous chapter, we covered the prescription given in [1] to compute entanglement entropy in 2D CFT’s with a gravitational anomaly. We found that the entanglement entropy is given by the value of the action (5.17) evaluated at an extremum of this action, which is a solution of the MPD equations (5.27). In spacetimes which are locally AdS$_3$, we saw that geodesics are always a solution to the MPD-equations. In this section, we will consider backgrounds which are not locally AdS$_3$: the spacetimes we will consider only asymptote to AdS at the boundary, but are deformed in the interior by some radial profile $f(z)$. On the field theory side, it is well known that the radial coordinate in AdS corresponds to the energy scale in the CFT [12, 13]. The non-trivial radial profile can then be interpreted as a deformation of the CFT by a relevant operator and moving into the AdS-interior corresponds to renormalization group flow.

In these asymptotically AdS backgrounds, a geodesic is not a solution to the MPD equations. To compute the entanglement entropy, we then have to construct solutions to the full MPD equations and evaluate the action (5.17) on these solutions. Since the MPD equations are coupled, third order, non-linear differential equations, this is very hard to do exactly, hence we will try to construct perturbative solutions in different limits. In [17], this was done in the limit of small $s$, which corresponds to the limit where curves stay close to the boundary. We will go over their arguments and look closely at the limitations of their calculation. These limitations will press us to consider the opposite limit, where the mass $m$ of the particle is small. We will look in detail at possible solutions in this limit as well as commenting on the possibilities for matching the two perturbative solutions together. We finish this section by returning to spacetimes which are locally AdS$_3$ to look for solutions to the MPD equations which are not geodesics.

6.1 RG flow and asymptotically AdS$_3$ backgrounds

Conformal field theories are interesting in their own right, but of course not all field theories are conformal. We can however use a lot of the knowledge from CFT’s when considering non-conformal theories. We can break conformal invariance by adding a local operator $\mathcal{O}$ to the action:

$$S \rightarrow S + h \int d^2 x \mathcal{O}, \quad (6.1)$$

with some coupling constant $h$. We will only consider relevant operators $\mathcal{O}$, meaning the perturbation becomes more important at low energies. This constrains the scaling dimension $\Delta$ of $\mathcal{O}$ to be less than 2. The theory can flow under the renormalization group to a theory in the IR. This theory can be a new conformal field theory, but it might also be a gapped theory, where confinement has occurred at low energies and a mass gap has formed. For 2D CFT’s, Zamolodchikov’s $c$-theorem states how the central charge $c = (c_L + c_R) / 2$ behaves under RG flow [51]. The theorem states that there exists a function $c(h)$ of the coupling constant $h$ which continuously decreases under influence of a renormalization group transformation. At fixed points of the RG flow, which are conformal field theories, the value of $c(h)$ coincides with the central charge of the theory.

While $c$ is allowed to flow, this is not the case for $\hat{c} = c_L - c_R$. This can be seen from modular invariance, in particular from the fact that the partition function $Z(\tau)$ should remain invariant under the T-transformation $\tau \rightarrow \tau + 1$. It can then easily be derived that $\hat{c} \in 24\mathbb{Z}$ and therefore $\hat{c}$ cannot continuously vary [26]. This means that if we have a theory with a gravitational anomaly $c_L \neq c_R$ in the UV, this anomaly will still be present in the IR. This implies in particular that the IR theory cannot be gapped, if that was not the case for the UV theory. Note that for a theory without a gravitational anomaly, the IR theory can actually be trivial, even if the UV theory
is not. The reason for this is that \( \hat{c} \) already vanishes to begin with in a non-anomalous theory. \cite{17} looked in detail at how adding a gravitational anomaly prevents theories from becoming gapped in the IR. The fact that \( \hat{c} \) does not flow is an example of the more general phenomenon that anomalies can constrain the RG flow of a theory. 't Hooft developed an anomaly matching procedure which amounts to anomalies persisting along the RG flow: the UV and IR theories suffer from the same anomaly \cite{52}.

The AdS/CFT dictionary provides a dual description for the energy scale of a theory: it is identified with the radial coordinate \( z \) \cite{12}. In this construction, the boundary \( z = 0 \) of AdS corresponds to the UV of the field theory, while flowing to the IR corresponds to moving into the AdS interior. A relevant deformation in the CFT will have the result that the AdS metric is deformed in the interior by some radial profile \( f(z) \):

\[
\text{ds}^2 = \frac{1}{z^2} \left( \frac{dz^2}{f(z)} + dx^2 - dt^2 \right) \quad (6.2)
\]

Note that we have set the AdS radius to unity: \( \mathcal{L} = 1 \). The field theory is conformal in the UV, so at the boundary \( z = 0 \) this metric should be equal to that of AdS\(_3\): we require that \( f(z) \to 1 \) as \( z \to 0 \). In the infrared limit, \( z \to \infty \) we will in this thesis be concerned with functions of the form \( f(z) \sim az^n \) for \( n \geq 0 \). Sometimes it is more convenient to consider the metric in a different form, where we set \( z = e^{-A(u)} \) and \( \partial_u = f_1^{1/2} \partial_z \):

\[
\text{ds}^2 = g_{\mu\nu} = e^{2A(u)}(du^2 + dx^2 - dt^2) \quad (6.3)
\]

The geometries (6.2) and (6.3) are not in fact solutions to the vacuum Einstein equations, since they are not locally AdS\(_3\) for \( f(z) \neq 1 \). Matter has to be introduced into the spacetime to sustain these asymptotically AdS\(_3\) geometries. It turns out that we can construct the desired geometry by including a minimally coupled scalar field \( \phi \) into our spacetime with potential

\[
V(\phi) \sim \exp(-4\delta\phi). \quad (6.4)
\]

This reproduces the geometry (6.2) with \( f(z) \sim z^n \) if \( n = 2\delta^2 \). In the case of a non-anomalous boundary theory, we can relate the entanglement entropy to geodesics in the background (6.2) using the Ryu-Takayanagi formula, as is done in \cite{27, 53}. In analogy with section 3.1.3, the length of a geodesic in this background is given by:

\[
L = \int \frac{1}{z^2} \left( \dot{x}^2 + \frac{\dot{z}^2}{f(z)} \right) d\lambda = \int_{-R}^R \frac{dx}{z(x)} \sqrt{1 + \frac{z'(x)^2}{f(z)}}. \quad (6.5)
\]

Equivalently, we can express the length of the boundary interval as a function of the turning point of the geodesic \( z_0 \), which is defined by \( z_0 = z(x = 0) \). We use the fact that the background (6.2) has a conserved momentum in the \( x \)-direction to write:

\[
R = 2 \int_{z_0}^{z_0} \frac{dz}{\sqrt{f(z)}} = \frac{z}{\sqrt{z_0^2 - z^2}}. \quad (6.6)
\]

Near the boundary, \( f(z) \sim 1 \) and the integrand describes the semicircular geodesics known from Poincaré AdS. To leading order in \( z_0 \), we have:

\[
R = 2z_0. \quad (6.7)
\]

Moving into the interior, \( f(z) \) changes: \( f(z) = az^n + \mathcal{O}(z^{n-1}) \). The integral (6.6) then gives:

\[
R = z_0^{-n/2} \sqrt{\frac{\pi}{a} \Gamma \left( 1 - \frac{n}{4} \right) \frac{1}{\Gamma \left( \frac{3}{2} - \frac{n}{4} \right)}} + \mathcal{O}(z_0^{-n/2}). \quad (6.8)
\]
We can immediately see that for \( n > 2 \), \( R \) goes to zero both as \( z_0 \) becomes arbitrarily small and arbitrarily large. This shows that for values \( R \) larger than some \( R_{\text{max}} \), the geodesics become disconnected: they just drop down into the interior and end on the singularity at \( z = \infty \). The fact that \( S_{EE} \) does not depend on \( R \) above some value \( R_{\text{max}} \) indicates that the dual theory is gapped in the IR. The maximum radius can be viewed as a minimal energy below which no excitations are possible.

This argument is repeated in [17] to show that the IR theory is gapped for \( n > 2 \) in Einstein gravity. There this is also shown by looking at the Schrödinger equation of a probe scalar on the background (6.2). However, as we have seen, the dual theory to a CFT with a gravitational anomaly is not Einstein gravity, but rather TMG, and as discussed above, we do not expect the IR theory to be gapped there. This is explicitly shown in [17] using two methods. Firstly they considered the linearized spectrum of the metric \( g_{\mu \nu} \) and the scalar field \( \phi \) on the background (6.2). They found that in TMG, indeed all vacuum solutions have a continuous spectrum in the IR. They further corroborated this fact by computing the entanglement entropy for these backgrounds in TMG. This is done using the prescription presented in section 5 and it is the subject of the next section.

### 6.2 MPD in asymptotically AdS\(_3\)

We recall that the holographic entanglement entropy in TMG is given by a solution to the MPD equations, which are obtained by minimizing the functional:

\[
S_{EE}^{(\text{TMG})} = \int d\tau \left( m \sqrt{g_{\mu \nu} v^\mu v^\nu} + s \tilde{n} \cdot \nabla n \right) .
\]  

Here \( \tau \) is an affine parameter along the wordline. As noted earlier, using \( s^{\mu \nu} = -s \epsilon^{\mu \nu \lambda \sigma} v^\lambda v^\sigma \), we can write the MPD equations as:

\[
\nabla \left[ mv^\mu - s v^\nu \epsilon^{\mu \nu \lambda} \nabla v^\lambda \right] = \frac{1}{2} s v^\nu \epsilon^{\rho \sigma \lambda} v^\lambda R^\mu_{\nu \rho \sigma} .
\]  

We adopt the following conventions for the Levi-Civita tensor:

\[
\epsilon_{txu} = \sqrt{-g}, \quad \epsilon^{txu} = -\frac{1}{\sqrt{-g}} .
\]

Note that the right hand side of (6.10) was identically zero in locally AdS\(_3\) spacetimes, but since the spacetime (6.2) is only asymptotically AdS\(_3\), the contraction op the Riemann tensor and the spin tensor does not vanish here.

To simplify the MPD equations, we note that the action (6.9) is invariant under translations in both the \( x \) and \( t \) directions, so we can find conserved quantities associated to these translations. There is however a subtlety regarding the constraints on \( n \) and \( \tilde{n} \): the action (6.9) does not ‘know’ that these normal vectors are orthonormal to each other and the wordline. We have seen that the normal vectors do not carry dynamical degrees of freedom, but this is only the case after these constraints are imposed. We could include these constraints by extremizing the action (6.9) with respect to \( n \) and \( \tilde{n} \) as well and view these as functions of \( v^\mu \), but here we will explicitly include the constraint by considering the complete action \( S + S_{\text{constraints}} \). In particular, the constraint that \( n \) and \( \tilde{n} \) are orthogonal to the wordline depend on \( v^\mu \), so they will contribute to the change of the action under a translation. This means that we have to consider the change in the following action:

\[
S = S_{\text{geo}} + S_{\text{anom}} + S_{\text{constraints}} = \int d\lambda \left( m \sqrt{g_{\mu \nu} v^\mu v^\nu} + s \tilde{n} \cdot \nabla n + \lambda_2 n \cdot v + \lambda_3 \tilde{n} \cdot v \right) .
\]  

(6.12)
We can then for example consider the change in the action (6.9) under an infinitesimal translation in the $x$-direction: $x \to x + \epsilon$. The $x$-component of the velocity will then change accordingly: $v^x \to v^x + \partial \epsilon$. The different terms in the action will then change as:

$$\delta S_{\text{geo}} = m \int \mathrm{d} \tau \left( m c^A(u) v^x \right) \partial \epsilon. \tag{6.13a}$$

$$\delta S_{\text{anom}} = s \int \mathrm{d} \lambda \left( \partial_u A(u) e^{2A(u)} s^u x^2 \right) \partial \epsilon = -s \int \mathrm{d} \tau \left( \partial_u A(u) e^{A(u)} v^t \right) \partial \epsilon. \tag{6.13b}$$

$$\delta S_{\text{constraints}} = s \int \mathrm{d} \tau \left( v^u \nabla_s x^u \right) \partial \epsilon, \tag{6.13c}$$

To arrive at these variations, we used the values of the Lagrange multipliers given in (5.23) and (5.24), the Christoffel symbols in asymptotically AdS$_3$ and the definition of the spin tensor in terms of both the normal vectors and the tangent vector:

$$s^u \nu = s(n^u \tilde{n}^\nu - \tilde{n}^u n^\nu) = -se^{\mu \lambda} v^{\lambda}. \tag{6.14}$$

Lastly, we use that the tangent vector along the wordline is normalized:

$$g_{\mu \nu} v^\mu v^\nu = 1 \implies (v^u)^2 + (v^x)^2 - (v^t)^2 = e^{-2A(u)}. \tag{6.15}$$

We find that we have a conserved quantity in the $x$-direction given by:

$$P_x = mc^{2A(u)} v^x + sc^{3A(u)} \left( v^u \tilde{v}^t - v^t \tilde{v}^u + v^t \partial_u A \left( (v^u)^2 - (v^t)^2 + (v^x)^2 - e^{-2A} \right) \right),$$

$$= mc^{2A(u)} v^x + sc^{3A(u)} (v^u)^2 \partial_t \left( \frac{v^t}{v^u} \right), \tag{6.16}$$

where $P_x$ is a constant.

The derivation of the momentum in the $t$-direction goes entirely analogously. We can thus express the MPD equations as:

$$P_x = mc^{2A(u)} v^x + sc^{3A(u)} (v^u)^2 \partial_t \left( \frac{v^t}{v^u} \right), \tag{6.17}$$

$$P_t = mc^{2A(u)} v^t + sc^{3A(u)} (v^u)^2 \partial_t \left( \frac{v^x}{v^u} \right). \tag{6.18}$$

We can take a derivative of (6.15):

$$v^u \partial_x v^u + v^x \partial_x v^x - v^t \partial_x v^t = -\partial_x A(u) e^{-2A(u)}. \tag{6.19}$$

Using (6.15) and (6.19), we see that the combination $P_x v^t - P_t v^x$ can be written down in a concise way, which will be useful in future sections:

$$P_x v^t - P_t v^x = sc^{3A(u)} (v^u)^2 \left[ v^t \partial_t \left( \frac{v^x}{v^u} \right) - v^x \partial_t \left( \frac{v^t}{v^u} \right) \right],$$

$$= sc^{3A(u)} v^u \left[ v^t \partial_x v^t - v^x \partial_x v^u - v^t \partial_x v^x \right] + sc^A \partial_x v^u,$$

$$= s \partial_t \left( e^A v^u \right). \tag{6.20}$$
To construct a perturbative solution, it will be convenient to adopt a radial gauge, using the $u$-coordinate to parameterize the proper length of the curve. We have:

$$\partial_t = v^u \partial_u, \quad \hat{v}(u) \equiv e^{A(u)} v^u = (1 - (\partial_u t)^2 + (\partial_u x)^2)^{-1/2}. \quad (6.21)$$

In this gauge, we can write the $x$-component of the MPD equations (6.17) as:

$$P_x = me^2 \partial_x x + s \hat{v}^2 e^{A(u)} \partial_t \left(\frac{\partial_t t}{v^u}\right),$$

$$= me^A \hat{v} \partial_u x + s \hat{v}^3 \partial_x^2 t. \quad (6.22)$$

The form of the $t$-component is again analogous, so we can write down the MPD equations in radial gauge as:

$$P_x = me^A \hat{v} \partial_u x + s \hat{v}^3 \partial_x^2 t,$$

$$P_t = me^A \hat{v} \partial_u t + s \hat{v}^3 \partial_x^2 x. \quad (6.23)$$

In this gauge it becomes manifest that we have reduced the order of the MPD equations by using global symmetries. The MPD equations are third order in derivatives, but integrating by parts once to obtain the expressions for $P_x$ and $P_t$ has reduced them to second order ordinary differential equations. This is entirely analogous to the geodesic equation reducing to a first order differential equation when translational invariance was used.

In the next section, we will construct perturbative solutions to the MPD equations (6.23) and (6.24). Note that $u$ is not single valued along the curve, since it has to return to the boundary. To take this into account, we just consider the curve up to its turning point $u_0$, which is defined to be the value where the $u$-derivative diverges: $\partial_u x|_{u=u_0} = \partial_u t|_{u=u_0} \to \infty$. This can of course only be done if the curve is actually symmetric about its turning point, but this is warranted by the symmetry of the geodesics in Poincaré AdS. Since setting the spin to zero reduces the MPD equations to the geodesic equation, the zeroth order solution is just the geodesic, as we will reaffirm in the next section. Because the geodesic is symmetric about $u_0$, the perturbative solutions will be as well.

6.2.1 No static solutions

In the Ryu-Takayanagi procedure, the geodesic must lie on a constant time slice. It is instructive to consider the static situation for the MPD equations as well, since being able to consider static geometries would considerably simplify the problem. Therefore we set:

$$\hat{v}^t = \frac{\partial t}{\partial \tau} = 0. \quad (6.25)$$

In that case the MPD equations reduce to:

$$P_x = me^{2A(u)} v^x, \quad P_t = -s e^{A(u)} v^x \partial_u A(u). \quad (6.26)$$

We assume that both $s \neq 0$ and $m \neq 0$. As is argued in [17], this restricts the form $A(u)$ can take. By substituting $v^x = P_x/me^{2A}$ into the second equation, we obtain:

$$e^{-A(u)} \partial_u A(u) = \text{constant}. \quad (6.27)$$

The solution to this equation is $A(u) = -\log u + C$, or $e^{A(u)} = e^C u^{-1}$, where $C$ is a constant. This choice corresponds to having pure AdS$_3$ as a background. Static solutions to the MPD
equations are therefore only consistent in exactly AdS$_3$. In this background, we can also relate $m$ and $s$:

$$m P_t = s P_x$$  \hspace{1cm} (6.28)$$

As we discussed in section 5.2, these solutions might in principle be different from geodesics, since the MPD equations are third order in derivatives. In section 6.6 however, we will show that these other solutions do not obey the right boundary conditions. Therefore we conclude that solutions with $v^t = 0$ can only exist in pure AdS$_3$, where they will follow a geodesic. We have already discussed these solutions in section 5.3, so they are not of much interest at this moment. There is another solution for the MPD equations with $v^t = 0$ and that is to also set $v^x = 0$, which renders the equations trivial. This situation is also not very interesting, since the curves remain at a given value of $x$, so they will never return to the boundary. We conclude that for a general background, setting $v^t = 0$ in the MPD equations will not allow for connected solutions. This means we have to solve the full MPD equations (6.23) and (6.24).

6.3 Perturbative solution in $s$

The MPD equations (6.23) and (6.24) are two second order, coupled, non-linear differential equations. Solving these equations analytically is a tremendously difficult task, so we will resort to perturbative methods in this section and the next. This section will be devoted to the perturbative expansions in the spin $s$ which is performed in [17]. It will turn out to be helpful to also examine the opposite limit, which is done in the next section.

Considering the limit where $s$ to be small means that we stay close to the boundary: our zeroth order solution will be the UV of the theory and we are perturbing away from that. This can also be seen from the MPD equations themselves: the terms proportional to $\partial^2 u$ will become dominant over the terms proportional to $e^A \partial u$ when we move into the interior of the spacetime. It will therefore be interesting to see how far into the IR the expansion will hold.

We can expand the curve $(x(u), t(u))$ as:

$$x(u) = x_0(u) + \sum_{i=1}^{\infty} s^i x_i(u), \quad t(u) = t_0(u) + \sum_{i=1}^{\infty} s^i t_i(u).$$  \hspace{1cm} (6.29)$$

And similarly for the conserved charges:

$$P_{t,x} = p_{t,x}^0 + \sum_{i=0}^{\infty} s^i p_{t,x}^i.$$  \hspace{1cm} (6.30)$$

We insert these expansions in (6.23) and (6.24) to write down the perturbative form of the MPD-equations:

$$p_0^x + \sum_{i=0}^{\infty} s^i p_i^x = me^A \hat{\nu} \left[ \partial_u x_0(u) + \sum_{i=1}^{\infty} s^i \partial_u x_i(u) \right] + s \hat{\nu} \left[ \partial^2_u t_0(u) + \sum_{i=1}^{\infty} s^i \partial^2_u t_i(u) \right].$$  \hspace{1cm} (6.31)$$

$$p_0^t + \sum_{i=0}^{\infty} s^i p_i^t = me^A \hat{\nu} \left[ \partial_u t_0(u) + \sum_{i=1}^{\infty} s^i \partial_u t_i(u) \right] + s \hat{\nu} \left[ \partial^2_u x_0(u) + \sum_{i=1}^{\infty} s^i \partial^2_u x_i(u) \right].$$  \hspace{1cm} (6.32)$$

Note that on top of the explicit $s$-dependence, these equations also depend on $s$ through $\hat{\nu}$, which is a function of the full $x(u), t(u)$. We are only interested in solutions which are anchored at the boundary and we want the solutions to return to the same value of $t$, since the entanglement
entropy in the field theory is of course measured at one moment in time. We therefore impose
the following boundary conditions:
\[ R \equiv \Delta x = x(u_f) - x(u_i), \quad \Delta t = t(u_f) - t(u_i) = 0, \quad u_i, u_f \to 0. \] (6.33)

**Zeroth order** At order \( s \), the coupling to the Chern-Simons term disappears and we are left
with the regular Ryu-Takayanagi prescription of computing entanglement entropy. The MPD
equations reduce to:
\[ p_0^x = me^A \hat{v}_0 \partial_u x_0(u), \]
\[ p_0^t = me^A \hat{v}_0 \partial_u t_0(u). \] (6.34)

Here \( \hat{v}_0(u) \) is the zeroth order contribution to \( \hat{v}(u) \):
\[ \hat{v}_0(u) = \frac{1}{\sqrt{1 + (\partial_u x_0)^2 - (\partial_u t_0)^2}}. \] (6.35)

Since the spin is now zero, there is no inconsistency in choosing a static geodesic, i.e.:
\[ p_0^t = 0, \quad t_0(u) = 0, \quad \hat{v}_0(u) = \frac{1}{\sqrt{1 + (\partial_u x_0)^2}}. \] (6.36)

To solve the other equation, we reintroduce the coordinate \( z \) through \( z(u) \equiv e^{-A(u)} \) and define
\( z_0 \equiv m/p_0^x \), which is the turning point of the curve in the \( z \)-coordinate.
\[ \frac{1}{z_0} = \frac{1}{z} \frac{\partial_u x_0}{\sqrt{1 - (\partial_u x_0)^2}} \quad \Rightarrow \quad \partial_u x_0 = \frac{z(u)}{\sqrt{z_0^2 - z^2}}. \] (6.37)

Using \( \partial_u = f^{1/2} \partial_z \), we see that this expression reduces to the familiar semicircular geodesics for
\( f(z) = 1 \), while it gets deformed when \( f(z) = az^n \). We can use this expression to express \( \hat{v}_0 \), the
zeroth order contribution to \( \hat{v}(u) \) in terms of \( z \) and \( z_0 \):
\[ \hat{v}_0 = (1 + (\partial_u x_0)^2)^{-1/2} = \frac{1}{z_0} \sqrt{z_0^2 - z^2}. \] (6.38)

For future reference, we compute the first \( u \)-derivative of \( \hat{v}_0 \) as well:
\[ \partial_u \hat{v}_0 = \frac{\partial^3}{z_0^2 \hat{v}_0^2} (\partial_u x_0) (\partial_u^2 x_0) = -\frac{zf^{1/2}}{z_0^2 \hat{v}_0}. \] (6.39)

**First order** We first observe that the \( x \)-component of the MPD equations (6.31) does not receive
a correction due to the spin coupling at linear order, since \( t_0(u) = 0 \). For this reason, we are free
to set \( p_1^x \) to zero and therefore also: \( \partial_u x_1 = 0 \). Looking at the equation for \( p_1^t \), (6.32), we see that
it gives:
\[ p_1^t = me^A \hat{v}_0 \partial_u t_1 + \hat{v}_0^3 \partial_u^2 x_0. \] (6.40)

Recalling the zeroth order MPD-equation (6.34), we can rewrite this as:
\[ \partial_u t_1 = \frac{1}{p_0^t} \left( p_1^t \partial_u x_0 + \hat{v}_0^3 \partial_u^2 x_0 \cdot \partial_u x_0 \right), \] (6.41)

where we have used that there are no first order corrections in \( s \) to \( \hat{v} \). Using the expression for the derivative of \( \hat{v} \), (6.39), we see that we can write the second term as:
\[ \partial_u t_1 = \frac{1}{p_0^t} \left( p_1^t \partial_u x_0 + \partial_u \hat{v}_0 \right). \] (6.42)
Writing this in terms of the \( z \)-coordinate system defined in (6.2), inserting the definition of \( \partial_x x_0 \) and recalling (6.38), we can also write:

\[
\partial_u t_1 = \frac{z}{p_0^2 \sqrt{z_0 - z^2}} \left( p_1 - f(z)^{1/2} \right), \tag{6.43}
\]

Since the conserved charges are constant, we can integrate (6.42) to obtain:

\[
\Delta t_1 = \frac{1}{p_0} \left( p_1' \Delta x_0 + \Delta \hat{v}_0 \right). \tag{6.44}
\]

Since we can use invariance under time translations to set \( t_0 = 0 \), we can use the boundary conditions (6.33) to set \( \Delta t_1 = 0 \). We furthermore observe that \( \Delta \hat{v}_0 = \hat{v}_0(u_f) - \hat{v}_0(u_i) = 2 \) which yields a simple expression for \( p_1' \):

\[
p_1' = \frac{2}{\Delta x_0}. \tag{6.45}
\]

We now recall (6.20) and rewrite it as:

\[
s \partial_u \hat{v} = P_u \partial_u t - P \partial_u x. \tag{6.46}
\]

Here we used the definition of the radial gauge (6.21). We can integrate this equality up to obtain:

\[
s \Delta \hat{v} = P_u \Delta t - P \Delta x. \tag{6.47}
\]

Which when we use the boundary conditions (6.33) gives an expression for \( P_t \):

\[
P_t = \frac{s \Delta \hat{v}}{R}. \tag{6.48}
\]

**Second order** Before we look at expansions of \( t(u) \) and \( x(u) \) to second order, we first look at the order \( s^2 \) contributions to \( \hat{v} \):

\[
\hat{v} = \frac{1}{\sqrt{1 + \left( \partial_u x_0 + s^2 \partial_u x_2 \right)^2} - s^2 \left( \partial_u t_1 \right)^2} + \mathcal{O}(s^4), \tag{6.49}
\]

\[
= \hat{v}_0 + \frac{1}{2} \hat{v}_0^3 \left( \left( \partial_u t_1 \right)^2 - 2 \left( \partial_u x_0 \left( \partial_u x_2 \right) \right) s^2 + \mathcal{O}(s^4). \right.
\]

Since we concluded that \( x_1(u) = 0 \), the second order correction to the temporal constraint only receives contributions from one term and is therefore relatively simple:

\[
p_2' = me^A \hat{v}_0 \partial_u t_2 \implies \partial_u t_2 = \frac{p_2'}{p_0} \partial_u x_0. \tag{6.50}
\]

Integrating up and requiring that \( \Delta t = 0 \), we see that this sets \( p_2' = 0 \). Since \( m, e^A \) or \( \hat{v}_0 \) are never zero, we can set \( t_2 = 0 \) without loss of generality. This alternating behaviour for \( t(u) \) and \( x(u) \) is general: \( t(u) \) will only receive corrections from odd powers of \( s \), while \( x(u) \) will only be corrected with even powers of \( s \). Collecting terms of order \( s^2 \) for the spatial component yields:

\[
p_2^s = me^A \hat{v}_0 \partial_u x_0 + me^A \hat{v}_0 \partial_u x_2 + \hat{v}_0^3 \partial_u^2 t_1. \tag{6.51}
\]

We can rewrite this as:

\[
me^A \partial_u x_2 \left( \hat{v}_0 - \hat{v}_0^3 \left( \partial_u x_0 \right)^2 \right) = p_2^s - \hat{v}_0 \partial_u^2 t_1 - \frac{me^A \hat{v}_0}{2} \left( \partial_u t_1 \right)^2 \left( \partial_u x_0 \right), \tag{6.52}
\]

\[
\frac{m}{\partial_u x_2} \partial_u x_2 = \frac{p_2^s}{\hat{v}_0} - \partial_u^2 t_1 - \frac{p_0^6}{2 \hat{v}_0} \left( \partial_u t_1 \right)^2.
\]
Here we again used that \( z = e^{-\Lambda(u)} \) and we used the expressions for \( x_0(u) \) given in (6.34) and for \( \hat{v}_0 \) given in (6.36). We can write this in terms of the coordinate \( z \) as:

\[
\frac{m}{z(u)} \partial_u x_2 = \frac{1}{2 \hat{v}_0^2 \hat{v}_0} \left[ \frac{z}{z_0} \partial_z f + \left( p_1 - \frac{f^{1/2}}{z_0} \right)^2 \right] + \frac{1}{v_0^2} \left[ \frac{1}{2 \hat{v}_0^2} \left( \frac{f}{z_0} - (p_1)^2 \right) + p_2^2 \right]. \tag{6.53}
\]

We have to integrate this expression to find \( \Delta x_2 \), but since the second term is proportional to \( \hat{v}_0^{-3} \), it will diverge proportional to \( (z_0 - z)^{-1/2} \) near \( z_0 \). We therefore have to fix the value of \( p_2^2 \) to cancel this divergence:

\[
p_2^2 = \frac{1}{2 \hat{v}_0^2} \left( (p_1)^2 - \frac{f(z_0)}{z_0} \right), \tag{6.54}
\]

We then arrive at the following correction to \( \Delta x \):

\[
\Delta x_2 = \frac{z_0}{m^2} \int_0^1 \frac{dq}{f(z_0 q)^{1/2}} \frac{1}{\sqrt{1 - q^2}} \left[ q \partial_q f(z_0 q) + \left( z_0 p_1 - \sqrt{f(z_0 q)} \right)^2 + \frac{f(z_0 q) - f(z_0)}{1 - q^2} \right], \tag{6.55}
\]

where \( q \equiv z/z_0 \).

We can evaluate (6.55) for different \( f(z) = f(z_0 q) \). In the limit of small \( z_0 \), the curve is well approximated by a geodesic. This is because small \( z_0 \) means that the curve remains close to the boundary and hence \( f(z_0 q) \approx 1 \). Inserting this for \( f(z_0 q) \), using (6.45) and noting that for the geodesic we have \( 2z_0 = \Delta x_0 \), we see that \( \Delta x_2 \approx 0 \). We can also consider the regime for large \( z_0 \), where \( f(z) = a(z_0 q)^n \), for which \( \Delta x_0 \) was already computed in (6.8). Computing \( \Delta x_2 \) to leading order in \( z_0 \) by inserting this form of \( f(z) \) into (6.55), we find:

\[
\Delta x_2 \sim \frac{a}{m^2} z_0^{1+n/2} + O(z_0^{n/2}). \tag{6.56}
\]

Collecting the two terms, we write for \( R \):

\[
R \sim 2 \left( \frac{\alpha z_0}{\sqrt{\alpha z_0}} + O(z_0^{-n/2}) \right) + \frac{s^2}{m^2} \left( \beta \sqrt{\alpha} z_0^{1+n/2} + O(z_0^{n/2}) \right) + O(s^4). \tag{6.57}
\]

We can immediately see that the second order correction prevents the system from being gapped: \( R \) can now be arbitrarily large for \( n > 0 \). More importantly for this thesis, we can see that higher power of \( s \) come with higher powers of \( z_0 \). Higher order corrections will contain higher powers of \( f(z) \) and we can write:

\[
\Delta x_{2i} \sim s^{2i} z_0^{(1+ni)/2}. \tag{6.58}
\]

Since the small \( s \) limit is only valid near the boundary, we see that the perturbation will break down at order \( i \) if \( z_0 \sim s^{1/(1+ni)} \). Hence, the corrections do not converge for arbitrarily large \( z_0 \), meaning that this perturbative expansion will break down at some point in the IR. Although it was indeed shown in [17] that connected solutions to the MPD equations exist in asymptotically AdS\(_3\) spacetimes, we cannot construct them deep into the bulk using just this procedure. To obtain more information about the solutions deep in the bulk, we have to consider a different limit, which will be done in the next section.

[17] then goes on to look how the perturbative corrections in \( s \) influence the entanglement entropy of the interval, by using the method outlined in the previous section. Mutually orthogonal normal vectors to the solutions are defined with the demand that they are Fermi-Walker transported along the curves (note that the parallel transport we demanded in the previous section has to be generalized, since the curves are not geodesics). The anomalous twisting term for the curves can
then be computed in a fashion similar to the previous section. In the end, the authors of [17] find corrections to the entanglement entropy which scale with $z_0^{n/2}$ in the large-$z_0$ region. This means that the entanglement entropy is always dependent on the interval size, which provides another indication that the system is not gapped in the IR. In this thesis we are more interested in constructing connected solutions to the MPD equations in these backgrounds and the fact that the perturbative expansion breaks down for large $z_0$ is a big obstruction to that cause. To get more information about the IR aspects of the solutions, we will now consider the opposite limit where $s \gg m$.

6.4 Perturbative solution in $m$

Since the perturbative expansions in the regime $s \ll m$ breaks down if we move deeper into the bulk, we will now discuss the opposite regime: $m \ll s$. To find the zeroth order solution, we set $m = 0$. The MPD-equations then reduce to:

$$P_x = s \hat{v}^3 \partial_u^2 u, \quad P_t = s \hat{v}^3 \partial_u^2 x. \quad (6.59)$$

Before immediately attempting to solve these equations and see if we can construct connected solutions in this limit, let us first see to what this limit actually corresponds physically. The meaning of the expansion in $s$ is intuitively clear: we consider the Chern-Simons term as a small correction to classical GR, which might for example have its origin in string theory [5]. However, when we set $m = 0$, the Einstein-Hilbert term vanishes and with it the conventional theory of classical General Relativity disappears. What we are left with is an action which consists just of the gravitational Chern-Simons term and the vacuum dynamics are fully governed by the vanishing of the Cotton tensor:

$$C_{\mu \nu} = 0. \quad (6.60)$$

We saw earlier that the Cotton tensor vanishes for all conformally flat metrics, which means that this theory is a conformal theory of gravity. This can also be seen by noting that the equations (6.59) are independent of the interior geometry, since the factor of $e^{2A}$ has disappeared. Conformal gravity has been considered in other contexts: one can for example construct actions for four dimensional conformal theories of gravity out of contractions of the Weyl tensor. These are studied as potential explanations for dark matter and other long distance phenomena in GR [54]. From a holographic perspective though, it is less clear what conformal gravity means. If the mass $m$ is set to zero, this means that in this limit $c_R + c_L$ also goes to zero, which is not unexpected, since $c_R + c_L$ is allowed to flow and this just means that the theory becomes trivial in the deep IR. We already saw however, that the difference of the central charges $c_R - c_L$ is not allowed to flow: the anomaly persists along RG flow. This would mean that the left and right moving charges can not both be zero, so one of these will become negative in this limit. This would mean that the dual CFT would be non-unitary at least and other details of the theory are unclear. However, we will just use the results in this limit to learn more about the characteristics of the connected solutions we want to construct and will be less concerned with the properties of the theory by itself.

Turning to the computation, the gravity theory being conformal means that the MPD equations in this limit (6.59) are invariant under rescalings of the coordinates. In particular, we can rescale all our coordinates: $u \rightarrow P_x^{-1} u$, $x \rightarrow P_x^{-1} x$, $t \rightarrow P_x^{-1} t$, this causes (6.59) to only depend on the ratio of $P_t$ and $P_x$:

$$s \hat{v}^3 \partial_u^2 t = 1, \quad s \hat{v}^3 \partial_u^2 x = \frac{P_t}{P_x}. \quad (6.61)$$
As is done in [17], we can combine the solutions into:

\[ P_t \partial_t^2 u = P_x \partial_x^2 u. \]  

(6.62)

And taking a derivative of (6.59) to obtain conditions on the third derivatives of \( x \) and \( t \), we can also write:

\[ s^2 \partial_u^3 \partial_u^2 v = P_x^2 - P_t^2. \]  

(6.63)

Note that setting \( \partial_u t = 0 \) will in this regime in fact yield consistent solutions: it will just have the effect of setting \( P_x = 0 \), which is not possible when \( m \) and \( s \) are both non-zero. The solutions have to satisfy the same boundary conditions as before:

\[ \Delta x \equiv x(u_f) - x(u_i) = R \] \[ \Delta t \equiv t(u_f) - t(u_i) = 0. \]  

(6.64)

Note however that to satisfy these boundary conditions it is not necessary that \( \partial_u t = 0 \): solutions which move through time might exist, as long as they return to the same time slice. It is unclear what the most general class of solutions to perturb around is, so we have to investigate both the static and non-static solutions.

### 6.4.1 Non-static solutions

Finding non-static solutions amounts to solving the full MPD equations (6.59) in the conformal gravity background which, as noted above, are equivalent to (6.62) and (6.63). We can use (6.62) to express \( \partial_u t \) in terms of \( \partial_u x \):

\[ \partial_u t = \frac{P_x}{P_t} \partial_u x + c, \quad c \in \mathbb{R}. \]  

(6.65)

We can immediately note that the derivatives of \( x(u) \) and \( t(u) \) only differ by a constant, meaning that their shape will be the same: they can at most be tilted with respect to each other. This spells a problem for possible solutions in this class: the boundary conditions demand the solution for \( t(u) \) to return to the same time slice, while the solution for \( x(u) \) has to return to \( u = 0 \) at some non-zero value \( R \). Nevertheless, we will explore this class of solutions a bit further, to get a feel for the problem and see whether the this intuitive argument holds.

We can then see that (6.63) reduces to a second order non-linear differential equation in \( x(u) \), which is given in (6.62). Note that this equation only depends on derivatives of \( x(u) \), so we can view it as a first order differential equation in \( \partial_u x \) and solve it as such. The solutions will depend on four parameters: the ratio of \( P_x \) and \( P_t \), \( s \), the integration constant \( c \) as defined in (6.65) and another integration constant \( C_1 \) which comes from solving the differential equation. It turns out that this constant has to be chosen purely imaginary for the solutions to be real. We will again be looking for solutions which return to the boundary, so the solutions have to exhibit turning points in both \( x \) and \( t \), which effectively means that their \( u \)-derivatives should diverge at some value of \( u \). Furthermore, the solutions have to return to the same time slice. Note that in this procedure, we assume that \( \partial_u t \) is non-zero. The static situation will therefore not appear as a limit to these solutions and will have to be considered separately.

If we solve this differential equation, we obtain four different branches of solutions, corresponding to two different square roots. The solutions are too involved and long to display explicitly, but we can examine their properties. The four solutions are displayed in figure 3 for a generic choice of the four parameters.

We can immediately note some general properties of the solutions. Firstly, they always appear in pairs, symmetric about a particular value of \( x \):

\[ x_{\text{sym}} = \frac{cP_x P_t}{P_t^2 - P_x^2}. \]  

(6.66)

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Furthermore, there is a symmetry about the \( u \)-axis, it turns the solutions on both sides of the \( u \)-axis swap if the sign of the integration constant \( C_1 \) is changed from plus to minus. The Poincaré metric is only defined for positive values of \( u \) and this symmetry in the solutions means that we are in fact allowed to only consider the solutions for positive \( u \) without loss of generality. Lastly, and most importantly, even for this generic choice of parameters, the solutions for \( \partial_u x \) all diverge at some value of \( u \), which for this choice of parameters turns out to be 

\[
\begin{align*}
\pm \left( 4 - \frac{\sqrt{3}}{13} \right) \approx \pm 3.87.
\end{align*}
\]

This strengthens the belief that there might be connected solutions obeying the right boundary conditions in this regime.

If we now integrate the solutions once, we obtain the actual curves \( x(u) \) and \( t(u) \). These will of course contain another set of integration constants, but because of the translational invariance in both \( x \) and \( t \), we are free to choose these independently for each solution. To get more of a feel for the solutions, we can look at the curves for \( x(u) \) corresponding to the solutions \( \partial_u x \) in figure 3. These curves are displayed in figure 4. From now on we will only consider curves for positive \( u \).

We immediately note that the two branches of solutions link up to form one connected solution which returns to the boundary. This could already have been inferred from figure 3, since the turning points occur at the same value of \( u \) with one solution blowing up to \( \infty \) and one to \(-\infty\). We therefore see that the boundary condition that \( x(u) \) returns to the boundary is already satisfied for a generic choice of parameters. The boundary condition on \( t(u) \) is more stringent: it should not only return to the boundary, but in fact to the same time slice. Let us therefore now look at the solutions for \( t(u) \). These can be obtained by integrating \( \partial_u t = \partial_u x + c \) once and as noted earlier, we can choose the integration constant such that \( t(0) = 0 \) for all solutions. Since the solutions now always intersect at \( u = 0 \), we just have to see whether their turning point matches up. If we set \( P_x = s = 1 \), we can find the values of \( c \) for which the turning points of the two solutions coincide as a function of \( P_t \) and \( C = iC_1 \):

\[
\begin{align*}
c = \pm \sqrt{1 - \frac{1}{P_t^2}} \pm \sqrt{\frac{P_t^2 - 1}{CP_t^4}}.
\end{align*}
\]

(6.67)

Note that we could have equivalently expressed \( P_t \) or \( C \) in terms of the other two parameters. From this expression we see that real solutions exist for \( P_t > 1 \). It turns out that the solutions

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Plot of the four solutions to the \( m = 0 \) MPD equations with parameters \( P_x/P_t = 0.5 \), \( s = 1 \), \( c = 2 \) and \( C_1 = -i \). The four colours depict the different solutions.}
\end{figure}
belonging to the positive inner square root are disconnected, so we consider solutions with:

\[ c = \pm \sqrt{1 - \frac{1}{P_t^2} - \frac{\sqrt{P_t^2 - 1}}{CP_t^4}}. \]  

(6.68)

If we choose values \( P_t = 2 \) and \( C = 1/4 \), we obtain \( c = \pm 1/2 \sqrt{3 - \sqrt{3}} \), the solutions corresponding to these values are plotted in figure 5. We see that these solutions move through time and return to the same time slice. It can be shown that these returning solutions exist for every combination of \( P_t \) and \( C \), with the condition that \( P_t > 1 \).

Figure 5: The two positive \( u \) branches for \( t(u) \). Both plots have \( P_t = 2 \), \( C = 1/4 \), while \( c = 1/2 \sqrt{3 - \sqrt{3}} \) for the plot on the left and \( c = -1/2 \sqrt{3 - \sqrt{3}} \) for the plot on the right.

This shows that we have found solutions \( t(u) \) which obey the correct boundary conditions for specific choices of the parameters \( c \), \( C \) and \( P_t \). We can now look whether the solutions \( x(u) \) also obey the correct boundary conditions for this choice of parameters. Integrating the solutions \( \partial_u x \) and forcing their turning points to coincide, we obtain the plots depicted in figure 6 for the choice that \( c = \pm 1/2 \sqrt{3 - \sqrt{3}} \).

It is immediately clear that the curves belonging to the solutions \( x(u) \) also return to the same
point. This will not happen generically, but for every choice of boundary conditions where \( t(u) \)
returns to the same point, the same will hold for \( x(u) \). We already mentioned this at the start of
this section: the solutions for \( x(u) \) and \( t(u) \) will be tilted with respect to each other, but their
shape will be the same. It is clear that we should look to construct other solutions in the limit
\( m = 0 \), since this class of solutions does not obey the correct boundary conditions. The obvious
choice are the static solutions, which might still exist, since they cannot be formed from the
non-static solutions. To obtain static solutions, we first have to set \( \partial_u t = 0 \) by hand and then
solve the differential equations.

### 6.4.2 Static solutions

Now that we know that the non-static solutions to the MPD equations (6.59) do not obey the
correct boundary conditions, we will consider the class of static solutions, setting \( \partial_u t = P_t = 0 \).
This makes the problem at hand a lot easier to solve: we are only left with the equation for \( P_t \),
which can be written as:

\[
P_t = \frac{s \partial_u^2 x}{(1 + (\partial_u x)^2)^{3/2}}. \tag{6.69}
\]

This differential equation is readily solved for \( \partial_u x \):

\[
\partial_u x = \frac{h(u)}{\sqrt{s^2 - h(u)^2}}, \quad h(u) = P_t u + h_1, \tag{6.70}
\]

with \( h_1 \) a constant. We see that the turning point \( u_0 \) occurs where \( h(u_0) = s \). We can integrate
once more to obtain \( x(u) \):

\[
x(u) = \frac{1}{P_t} \sqrt{s^2 - h(u)^2}. \tag{6.71}
\]

We did not include an extra constant of integration, since we can again use translational invariance
to shift the \( x \)-coordinates of the solution. To compare (6.71) with the geodesics in Poincaré AdS,
we rewrite it as:

\[
x^2 + \left( u + \frac{h_1}{P_t} \right)^2 = \frac{s^2}{P_t^2}. \tag{6.72}
\]

The solutions are semicircles like the geodesics, but they are offset by \( h_1 \) in the \( u \)-direction. Some
solutions are plotted in figure 7: we see that the case \( h_1 = 0 \) corresponds to the geodesic. The
constant \( h_1 \) effectively determines the angle with which the curves intersect the boundary at

\[
\begin{align*}
\text{Figure 6: The two positive } u \text{ branches for } x(u) \text{ with the same choice of parameters } c, P_t \text{ and } C \\
as in figure 5.
\end{align*}
\]
$u = 0$, for the geodesics, this angle was always 90 degrees, but this does not have to be the case anymore. We can read off the interval size, by setting $u = 0$:

$$R = 2\sqrt{s^2 - h_1^2}. \tag{6.73}$$

Some solutions to the static MPD equations in conformal gravity are depicted in figure 7. The turning point occurs at $x(u_0) = 0$ for all solutions for fixed $s$ and $P_1$, but the interval size shrinks as $h_1$ becomes non-zero.

Figure 7: Solutions to the static MPD-equations in conformal gravity with $s = P_1 = 1$. Note that the interval size shrinks when $h_1$ gets larger.

Note that the equation (6.69) allows a rescaling of the u-coordinate $u \rightarrow u - h_1/P_1$. This redefinition removes $h_1$ from the expression for $x(u)$ and effectively shifts the location of the u-axis such that the location of the circle center is at the origin.

We can gain further intuition about this limit by computing the action of the solutions. To do so, we again employ the method introduced in [1] and evaluate the entanglement entropy functional on these solutions. The entanglement functional is given by:

$$S_{EE} = s \int d\tau (\vec{n} \cdot \nabla n). \tag{6.74}$$

In section 5 we found that computing the entanglement entropy amounted to computing:

$$S_{EE} = \frac{1}{4G_3\mu} \log \left( \frac{q(s_f) \cdot n_f - \tilde{q}(s_f) \cdot n_f}{q(s_i) \cdot n_i - \tilde{q}(s_i) \cdot n_i} \right), \tag{6.75}$$

which was found in (5.35). There $(q, \tilde{q})$ were a parallely transported normal frame, satisfying $\nabla q = \nabla \tilde{q} = 0$. In this limit, we do not have geodesic solutions, so the $q, \tilde{q}$ now define a Fermi-Walker transported normal frame, meaning they obey:

$$\nabla q^\mu = -(q_\alpha \nabla v^\alpha)v^\mu, \quad \nabla \tilde{q}^\mu = -(\tilde{q}_\alpha \nabla v^\alpha)v^\mu. \tag{6.76}$$

This transport just takes into account the spinning of the reference frame. The $q$ and $\tilde{q}$ also obey $q^2 = -1$, $\tilde{q}^2 = -1$ and $v \cdot q = v \cdot \tilde{q} = 0$. Finally we remember that $n_i$ and $n_f$ are defined as the Killing vector along the time direction in the CFT: $n_i = n_f = (\partial_t)_{\text{CFT}}$. A normalized tangent vector to the path (6.71) is (using $(x, t, u)$ again):

$$v^\mu = B(u) \left( \frac{h(u)}{\sqrt{s^2 - h(u)^2}}, 0, 1 \right), \quad B(u) = \frac{\sqrt{s^2 - h(u)^2}}{e^{\Delta(u)} s}. \tag{6.77}$$
The normal vectors are then easily constructed:

\[ q^\mu = e^{-A(u)}(0, 1, 0), \quad \bar{q}^\mu = B(u) \left(-1, 0, \frac{h(u)}{\sqrt{s^2 - h(u)^2}}\right). \] (6.78)

It is straightforward to check that these vectors are already Fermi-Walker transported. In particular, \( q_{\mu} = (\partial_{\mu})_{\text{CFT}} \) and \( \bar{q} \cdot n_f = \bar{q} \cdot n_i = 0 \). Evaluating (5.35) on these vectors thus gives us 0. This was to be expected, since we are in the conformal gravity regime: every conformally flat metric is equivalent to the Minkowski metric and there is no twisting of normal vectors in flat space. This means that there is no anomalous contribution to the entanglement entropy for these solutions. However, just as for the geodesic solutions in Poincaré AdS, the entanglement entropy is not invariant under boosts because of the loss of diffeomorphism invariance. This means that the entanglement entropy will receive an anomalous contribution if the interval is boosted. Note that we have a 1-parameter family of geodesic-like solutions to the MPD-equations in this regime: the solutions can be labelled by the value of the constant \( h_1 \) and the associated entanglement entropy receives contributions under boosts which depend on \( h_1 \). This is potentially problematic:

a connected 1-parameter family of saddle points for the action would make the theory unstable. Remember however, that this solution only describes the deep IR of the theory. If a full solution to the MPD equations is constructed, the perturbative solution in the UV should be matched to that in the IR and requiring continuity of the curves should then fix the value of \( h_1 \). We will come back to this in section 6.5.

We can summarize the results of this section by noting that we found that the only class of connected solutions which obey the correct boundary conditions in the \( m = 0 \) limit are the ones where \( \partial_u t \) is set to zero. This is to be contrasted with solutions to the full MPD equations, since we already concluded that these in fact have to move through time. The situation then is as follows: very near to the boundary, the curve should remain on the same time slice, since the geodesics are static. Going further away from the boundary, the curve should start to move through time, since it cannot remain static in a region where the full MPD equations are valid. Then in the very deep IR, the curves should again become static, staying on one time slice.

### 6.4.3 Perturbative expansion

Now that we know that the correct solutions to perturb around are the static ones, we can perform an expansion in \( m \) around these solutions and try to find the solutions moving away from the deep IR. In analogy to the \( s \)-expansion, we can write:

\[ P^{x,t} = \tilde{p}_{0}^{x,t} + \sum_{i=1}^{\infty} m^i \tilde{p}_{i}^{x,t}, \] (6.79)

and:

\[ x(u) = \tilde{x}_{0}(u) + \sum_{i=1}^{\infty} m^i \tilde{x}_{i}(u), \quad t(u) = \tilde{t}_{0}(u) + \sum_{i=1}^{\infty} m^i \tilde{t}_{i}(u). \] (6.80)

We wrote tildes on \( x, t \) and \( p \) to distinguish them from their counterparts in the UV-expansion.

**Zeroth order** We repeat the results for the zeroth order solution from the previous subsection:

\[ \partial_u \tilde{x}_0 = \frac{h(u)}{\sqrt{s^2 - h(u)^2}}, \quad h(u) = \tilde{p}_{0}^{0} u + h_1. \] (6.81)
\( \tilde{p}_0^t \) is related to the interval size:
\[
\tilde{p}_0^t = 2 \sqrt{s^2 - h_1^2} \frac{R}{s}.
\]  
(6.82)

Since we are considering static solutions, we can set \( \partial_u \tilde{t}_0 = 0 \) and using the fact that \( \Delta t = 0 \), we see that it is consistent to set \( \tilde{t}_0(u) = 0 \). We also note that:
\[
\hat{v}_0 = (1 + (\partial_u \tilde{x}_0)^2)^{-1/2} = \sqrt{s^2 - h(u)^2} \frac{s}{s}.
\]  
(6.83)

**First order** To first order in \( m \), the MPD-equations reduce to:
\[
\tilde{p}_1^t = e^A \hat{v} \partial_u \tilde{x}_0 + s \tilde{p}_0^t \partial_u^2 \tilde{t}_1, \quad (6.84)
\]
\[
\tilde{p}_1^x = s \tilde{v}_0^3 \partial_u^2 \tilde{x}_1. \quad (6.85)
\]

Equation (6.85) does not receive any contribution from the geodesic term. We can combine it with the zeroth order equation (6.69) to find:
\[
\partial_u^2 \tilde{x}_1 = \frac{\tilde{p}_1^t}{\tilde{p}_0^t} \tilde{p}_0^t \partial_u^2 \tilde{x}_0. \quad (6.86)
\]

We can integrate this equation twice to obtain:
\[
\Delta \tilde{x}_1 = \frac{\tilde{p}_1^t}{\tilde{p}_0^t} (R + 2au_0). \quad (6.87)
\]

Here \( a \) is an integration constant. Since this term receives no contribution from the geodesic term in the action, we have to set \( \Delta \tilde{x}_1 = 0 \). This can be achieved by setting \( \tilde{p}_1^t = 0 \), as was done in the UV, but in this case we also have the freedom to set \( \Delta \tilde{x}_1 = 0 \) by setting \( a = -\frac{R}{2u_0} \).

Furthermore, we can rewrite (6.84) as:
\[
\partial_u^2 \tilde{t}_1 = \frac{s}{(s^2 - h(u)^2)^{3/2}} \left( s \tilde{p}_1^t - me^A h(u) \right). \quad (6.88)
\]

We have to integrate this expression to obtain conditions on \( \tilde{p}_1^t \) and also because it appears in the higher order terms in the expansion. This is a practical problem with the \( m \)-expansion: the lowest order term at each order in \( m \) is a second derivative which needs to be integrated to obtain terms at the next order. This is to be contrasted to the \( s \)-expansion, where the lowest order term was a first derivative, which then had to be differentiated for the higher order terms.

Before we can integrate (6.88), we have to specify a particular radial profile \( f(z) \), which translates to a particular form of \( e^{A(u)} \). We will choose the simplest non-trivial \( f(z) \):
\[
f(z) = 1 + Bz, \quad (6.89)
\]

Using the expression for the metric in the different coordinates (6.2) and (6.3), this translates to:
\[
A(u) = \log \left( \frac{4B}{B^2 u^2 - 4} \right). \quad (6.90)
\]

We can now integrate (6.88) to obtain \( \partial_u \tilde{t}_1 \). The expression we obtain is rather unwieldy, but it is a combination of polynomial and logarithmic fractions, which can schematically be written as:
\[
\partial_u \tilde{t}_1 = \frac{c_1 + c_2 u + c_3 h(u)}{c_4 \sqrt{s^2 - h(u)^2}} + \frac{f_-(u)}{(B^2 s^2 - (B h_1 - 2 \tilde{p}_0^t)^2)^{3/2}} + \frac{f_+(u)}{(B^2 s^2 - (B h_1 + 2 \tilde{p}_0^t)^2)^{3/2}} + Cu, \quad (6.91)
\]
where
\[ f_{\pm}(u) = B^2(2p_0^\pm + Bh) \left( (\log(2 \mp Bu) \right) \]
\[ \pm \log \left( \mp 2p_0^\prime h(u) \sqrt{(B^2s^2 - (Bh_1 \mp 2p_0^\prime)^2)(s^2 - h(u)^2)} - B(h_1^2 - s^2 + h_1p_0^\prime u) \right). \]  
(6.92)

\( C \) is an integration constant and the \( c_i \) are constants formed from \( s, p_0^\prime, p_1^\prime, B \) and \( h_1 \).

We can now integrate and demand that \( \Delta t_1 = 0 \):
\[ \int_0^{\bar{t}_1} \partial_\nu \bar{t}_1 du = 0. \]  
(6.93)

This should in principle allow us to fix \( \bar{p}_1^\prime \) in terms of the other parameters, which can in practice only be done implicitly, due to the complexity of the solutions.

Even after imposing the boundary conditions \( \Delta x = R \) and \( \Delta t = 0 \), we still have undetermined integration constants \( a \) and \( C \) as defined in (6.87) and (6.91). As we said before, these might be fixed when the UV and IR solutions are glued together. If we would somehow be able to solve the MPD equations analytically in each regime, these constants would not appear, so a consistent matching procedure should fix them at each level. While we cannot exclude the possibility that the constants turn out to be non-zero, the fact that the solutions display alternating behaviour in \( x \) and \( t \) in the UV hints that this will be the same in the IR, which would for example allow us to set the above constant \( a = 0 \). We will talk more about matching procedures in section 6.5.

**Second order** Moving to second order, we find that the MPD equations reduce to:
\[ \bar{p}_2^\prime = e^A \bar{v}_0 \partial_\nu \bar{x}_1 + s \bar{v}_0^\prime \partial_\nu^2 \bar{x}_2, \]  
(6.94a)
\[ \bar{p}_2 = me^A \bar{v}_0 \partial_\nu \bar{l}_1 + s \bar{v}_0^3 \partial_\nu^2 \bar{x}_2 + \frac{3}{2} \bar{v}_0^\prime \partial_\nu^2 \bar{x}_0 (2 \partial_\nu \bar{x}_0 \partial_\nu \bar{x}_2 - (\partial_\nu \bar{l}_1)^2). \]  
(6.94b)

We can again solve (6.94a) by either setting \( \bar{p}_2^\prime = 0 \) or choosing the value of an integration constant to ensure that \( \Delta \bar{l}_2 = 0 \). (6.94b) can be rewritten as:
\[ \bar{p}_2^\prime = \frac{e^A \partial_\nu \bar{l}_1}{s} \sqrt{s^2 - h(u)^2} + \frac{\partial_\nu^2 \bar{x}_2}{s} (s^2 - h(u)^2)^{3/2} + \frac{3}{2} \frac{(s^2 - h(u)^2)p_0^\prime}{s^3} \frac{2h(u)}{\sqrt{s^2 - h(u)^2}} \partial_\nu \bar{x}_2 - (\partial_\nu \bar{l}_1)^2 . \]  
(6.95)

If we take the same radial profile as in (6.90), then we can use (6.91) for \( \partial_\nu \bar{l}_1 \). (6.95) is then a differential equation involving \( \partial_\nu \bar{x}_2 \) and \( \partial_\nu^2 \bar{x}_2 \) which can in principle be solved (it will again introduce an integration constant). This will then give the first non-trivial correction \( \Delta x_2 \) to the interval \( R \). It would be interesting to examine these corrections to the interval size due to \( m \) and see how these relate to the corrections due to \( s \). Even without a way to match the different limits, there could be similarities in the structure of the expansions, which would reveal more about the full solution of the MPD equations. If these corrections are obtained, one would then have to evaluate the entanglement functional on them to find their action. The expressions (6.91) and (6.95) however are quite hard to deal with and we will not attempt to find the corrections here. Instead we now turn to a procedure to glue the two perturbative solutions together, working out the corrections in the \( m \)-expansion could be a direction for future research to head in.
6.5 Matching of limits

Even if we are capable to construct perturbative solutions in two different regimes, this does not mean that connected solutions actually exist. For that there has to be a way to glue the UV solutions to the IR solutions without the need to solve the full MPD equations which describe the intermediate region. To see how such a matching procedure would work, we first consider the simpler situation of just Einstein gravity with a deforming radial profile $f(z)$. If it is possible to glue UV and IR solutions together in this background, we might be able to generalize this to the TMG backgrounds we are interested in. Constructing connected solutions and in that way computing entanglement entropy was the subject of [53]. The authors found a procedure to construct connected solutions for entangling regions of arbitrary dimension. We will first review their method and then comment on its applicability to TMG. We will use the coordinate system parameterized by $(t,x,z)$:

$$ds^2 = \frac{L^2}{z^2} \left( -dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right).$$  \hfill (6.96)

We again impose the condition that the geometry is AdS near the boundary, so:

$$f(z) \to 1, \quad z \to 0.$$  \hfill (6.97)

And we again consider polynomial deformations of the interior:

$$f(z) = az^n + \ldots, \quad a > 0, \quad n > 0.$$  \hfill (6.98)

We will assume that there is some crossover scale $z_{CO}$, such that (6.98) is valid for $z \gg z_{CO}$. The matching procedure will be consistent when the end result does not depend on $z_{CO}$ anymore.

The analysis in [53] was done for theories in an arbitrary number of dimensions and their results are most pronounced in that case. We will therefore review their work in arbitrary dimension as well, the simplification to $d = 2$ is readily made. If we again consider a region $A$ in the CFT of which we want to compute the entanglement entropy, its boundary $\partial A$ is in [53] referred to as the entangling region. There, two types of entangling region were considered: regions with the topology of either a sphere or a strip. Since the entangling region in 2D just consists of two disconnected points, this distinction is not important for our purpose, so we focus on the results for a spherical entangling region with metric:

$$d\vec{x}^2 = d\rho^2 + \rho^2 d\Omega^2_{d-2}.$$  \hfill (6.99)

The entanglement entropy for such a region of radius $R$ in the CFT is given by the Ryu-Takayanagi formula and can be written as:

$$S(R) = \frac{L^{d-1}}{4G_N} \omega^{d-2} A.$$  \hfill (6.100)

Here $\omega^{d-2}$ is the area of a $d-2$ dimensional hypersphere and $A$ is the entanglement functional which is obtained by minimizing the surface area. We can parametrize $A$ either by the radial coordinate $\rho$ or by the bulk direction $z$, noting that each polar direction will contribute a factor $\rho/z$:

$$A \equiv \int_0^R d\rho \frac{\rho^{d-2}}{z^{d-1}} \sqrt{1 + \frac{z^2}{f(z)}} = \int_0^{z_0} dz_0 \frac{z_0^{d-2}}{z^{d-1}} \sqrt{\rho^2 + \frac{1}{f(z)}}.$$  \hfill (6.101)

Here $z_0$ is the turning point of the curve. We impose the obvious boundary conditions:

$$\rho(z = 0) = R, \quad \rho(z_0) = 0, \quad \rho'(z_0) = \infty.$$  \hfill (6.102)
In Einstein gravity, the geometry can be gapped, causing the minimal surface to have the topology of a cylinder, rather than a disk, depending on the form of \( f(z) \). It was shown in [17] that this gapping does not occur in TMG, so we will not be concerned with gapped geometries. We can now straightforwardly write down the Euler Lagrange equations for \( \rho \):

\[
(d - 2) \frac{1}{f} + (d - 1) \frac{\rho' z}{f} = \rho \sqrt{\rho'^2 + \frac{1}{f}} \frac{\rho'}{\sqrt{\rho'^2 + \frac{1}{f}}}.
\]

(6.103)

Note that this equation for the geodesics could be reduced to first order in two dimensions, since there we made use of a conserved momentum in the \( x \) direction. Solving the geodesic equation and then evaluating the entanglement functional (6.101) then gives the entanglement entropy. This is however impossible to do analytically, so we solve it perturbatively in different limits.

### 6.5.1 UV expansion

Firstly, we can expand \( \rho(z) \) for small \( z \) as:

\[
\rho(z) = c_0 + c_1 z + c_2 z^2 + \ldots
\]

(6.104)

We impose the additional boundary condition that \( \rho(0) = c_0 = R \). We set \( f(z) = 1 + a z^n \) and then solve the equations of motion order by order which yields the following form of \( \rho(z) \):

\[
\rho(z) = R \left( -\frac{z^2}{2R} - \frac{z^4}{4R^3} + \ldots + c_d(R) z^d + \ldots \right)
\]

(6.105)

All coefficients \( c_i \) with \( i < d \) can be determined locally, while the terms \( c_i \) with \( i > d \) can be determined in terms of \( c_d \). The terms up to \( c_d \) only contain even powers of \( z \), while higher order terms contain integer powers of \( z \) greater than \( d \). We can similarly expand \( f(z) \) for small \( z \):

\[
f(z) = 1 + \sum_{n=0}^{\infty} f_n z^{d+n} + \sum_{m=2}^{\infty} c_n z^{m\alpha}.
\]

(6.106)

Here \( m \) is a mass scale and \( \alpha \) is some positive coefficient, which is related to a source deformation as \( \alpha = d - \Delta \) where \( \Delta \) is the UV-dimension of the leading relevant operator. Since the boundary has a surface of which the area grows as \( z^d \), the first geometric correction to \( f(z) \) occurs at \( O(z^d) \).

We observe that terms in (6.105) of order \( i < d \) involve only odd powers of \( 1/R \), so we could alternatively write down an expansion for \( \hat{\rho}(z) \) in \( 1/R \):

\[
\rho(z) = R \left( \frac{\rho_1(z)}{R} - \frac{\rho_3(z)}{R^3} + \ldots - \frac{\hat{\rho}(z)}{R^\nu} + \ldots \right)
\]

(6.107)

Here the term at order \( \nu \) is the first term which gets affected by the IR behaviour, and it is the first term which has even dependence on \( 1/R \). We can write a differential equation for each term in the expansion (6.107):

\[
\frac{z^{d-1}}{\sqrt{f}} \left( \frac{\sqrt{T}}{z^{d-1}} \rho_i' \right)' = s_i.
\]

(6.108)

The \( s_i \) are source terms from lower order terms and it is crucial that these will be zero for \( \hat{\rho}(z) \), since it is the first term with even dependence on \( 1/R \). \( \hat{\rho}(z) \) is therefore a solution to the homogeneous differential equation and can be written as:

\[
\hat{\rho}(z) = \hat{b} \int_0^z \frac{du}{\sqrt{f(u)}} u^{d-1}.
\]

(6.109)
6.5.2 IR expansion

We now turn to finding a perturbative expansion in the IR, where \( f(z) \sim az^n \). To find such an expansions, we first note that the equations of motion are invariant under scalings of a particular form. Firstly, if \( \bar{\rho}(z) \) is a solution to (6.103) with \( a = 1 \), then:

\[
\rho(z) = \bar{\rho}\left(a^{-\frac{1}{2n}}z\right),
\]

satisfies (6.103) for any \( a \). Here not just the argument of \( \rho(z) \) should be replaced with \( a^{-\frac{1}{2n}}z \), but also the explicit factors of \( z \) in (6.103). Furthermore, the equations of motion are invariant under scalings of the form:

\[
\rho(z) \to \lambda^{2-n} \rho \left(\lambda^{-1}z\right).
\]

Combining these two scalings, we see that we can write \( \rho(z) \) as:

\[
\rho(z) = \bar{\alpha}_0 z^{\frac{2-n}{d-n}} \tilde{\rho}_d(u), \quad u, \equiv \frac{z}{z_0}, \quad \tilde{\rho}_d(u = 1) = 0.
\]

Here \( \tilde{\rho}_d(z) \) satisfies (6.103) with \( a = 1 \). This scaling behaviour is helpful because if we now take \( z_0 \) sufficiently large, \( u \) will be small even for \( z \gg z_{CO} \). We can therefore expand \( \tilde{\rho}_d(u) \) in small \( u \) and solve the equations of motion order by order again. This expansion is rather involved so we will just state the result, for the full derivation, see Appendix C of [27].

\[
\tilde{\rho}_d(u) = \bar{\alpha}_0 + \frac{\alpha_1}{\bar{\alpha}_0} u^{2-n} + \frac{\alpha_2}{\bar{\alpha}_0^2} u^{2(2-n)} + \ldots + \frac{\tilde{h}}{\bar{\alpha}_0^{\frac{d-n}{2}}} u^{d-n/2} + \ldots
\]

Here \( \bar{\alpha}_0 \) and \( \tilde{h} \) are constants which will be fixed upon matching the solutions. \( \eta \) is given by:

\[
\eta = \frac{2 - n}{d - 1}
\]

and the \( \alpha_i \) can be expressed in terms of \( n \) and \( d \), \( \alpha_1 \) can for example be written as:

\[
\alpha_1 = \frac{2(d-2)}{(n-2)(n+2d-4)}
\]

We can then find the expansion for \( \rho(z) \) by using (6.112):

\[
\rho(z) = \rho_0 + \frac{\alpha_1}{\rho_0} z^{2-n} + \ldots + \frac{\tilde{h}}{\bar{\alpha}_0^{\frac{d-n}{2}}} z^{d-n/2} + \ldots
\]

where

\[
\rho_0 = \frac{\bar{\alpha}_0}{\sqrt{a}} z_0^{\frac{2-n}{d-n}}.
\]

6.5.3 Matching

The last thing to do to match the solutions together is evaluate for sufficiently large \( z \) such that \( f(z) \sim az^n \) applies. In particular, we can calculate \( \hat{\rho}(z) \) in this regime by explicitly integrating (6.109):

\[
\hat{\rho}(z) = \frac{\hat{b}}{\sqrt{a} d - n/2} z^{d-n/2}
\]
We can then compare (6.118) with (6.113) and find that to match them, we have to set:

\[ \alpha_0 = R, \quad \hat{b} = -\left( \frac{d}{2} \right) ha^{-\frac{1}{\eta}}, \quad \nu = \frac{2}{\eta}, \]

(6.119)

From (6.117), we see that \( z_0 \) scales with \( R \) as:

\[ z_0 \sim R^{\frac{2}{n+2}}, \]

(6.120)

which reproduces the correct semicircular behaviour for \( n = 0 \). The authors of [53] go on to compute the entanglement entropy for these connected solutions and find that the expressions they find do not depend on the location of the crossover scale \( z_{CO} \), which gives credence to this matching procedure.

For the purposes of this thesis, we are of course not primarily interested in matching just Einstein gravity spacetimes to each other. We want to match the solutions resulting from the expansions in \( s \) and \( m \) and in doing so, also take into account the deformation in the interior which is caused by \( f(z) \). There are two facts are important to take from the matching procedure outlined in this section. Firstly, the procedure simplifies a lot by noting that there were no source terms for \( \hat{\rho}(z) \) in (6.108). This stems from the fact that the UV expansion of \( \rho(z) \) only receives odd contributions in \( 1/R \) close to the boundary. It remains to be seen whether this behaviour is also displayed by the MPD equations close to the boundary. More importantly, it was crucial for the IR expansion to have a small parameter \( u = z/z_0 \) to expand in. This effectively pushes the turning point very far down into the deep IR and creates a matching region far from the turning point.

When we now introduce the spin of the particle and change the Einstein gravity background to a TMG background, we introduce another matching region where the perturbative solutions in both \( s \) and \( m \) have to be matched. We thus have two matching regions superposed on each other and depending on the situation, the location and order of these matching regions will be different. In the extreme situations of this setup, we can find valid perturbative solutions. Close to the boundary for example, when both \( z_0 \) and \( s \) are small, we know the curves have to reduce to geodesics in Poincaré AdS. In [17] it was found that turning on the deformation \( f(z) \) gives corrections to the interval size, but only for small \( z_0 \). Meanwhile, very deep in the IR, when both \( s \) and \( z_0 \) are very large, a perturbative solution in \( m \) can in principle be constructed along the lines of section 6.4. The existence of an additional matching region however, means that it is much harder to come up with a solution in the intermediate matching regions. We cannot use the same procedure of ‘pushing down’ the IR region for one of the variables very deep to obtain a small parameter to expand in, since we would still be in the matching region for the other variable. It is unclear how a consistent matching procedure would have to be set up in this case of interplaying limits.

Note that the perturbative series in [53] is of a different nature from the perturbative series from [17] and the earlier chapters of this thesis. In the perturbations we considered before, the expansion is done in one of the parameters characterizing the system, in this case \( s \). In the type of expansion from [53], the expansion variable is a coordinate, a geometrical quantity. To gain more intuition about the MPD equations using the matching procedure from [53], it would be interesting to expand the MPD equations for small \( z \) and solve the equations order by order again. One could then see whether the structure of such a perturbative expansion corresponds to the perturbative solutions in \( s \) considered earlier. This would give more information about the structure of the MPD equations in the UV limit and perhaps also give an insight in the interplay of the matching regions.
6 THE MPD-EQUATIONS IN ASYMPTOTICALLY ADS3 BACKGROUNDS

6.6 Non-geodesic solutions to the MPD equations in pure AdS

To finish this thesis, we return to the MPD equations in backgrounds which have pure AdS3 geometry, rather than just asymptotically AdS3. In section 5.2 we observed that the MPD equations in these backgrounds reduce to:

$$\nabla [m v^\mu - s \epsilon^{\mu\nu\lambda} v_\nu v_\lambda] = 0. \tag{6.121}$$

It is clear that a geodesic is a solution, but since the MPD equations are third order in derivatives, there might in principle be other solutions. We can just write out the covariant derivatives and see whether we can solve the equations. This approach provides some insight to solutions to the MPD equations in flat space and is applied to global AdS3 in appendix A. The boundary conditions are however more easily applied in Poincaré AdS in the radial gauge we considered before and that is the approach we will take in this section.

In pure AdS3, we have $e^{2\Lambda(u)} = u^{-2}$. This means that the MPD equations can be written in the radial gauge introduced in (6.21) as:

\begin{align*}
P_x &= \frac{m\dot{v}}{u} \partial_u x + s\dot{v}^3 \partial_u^2 t, \tag{6.122a} \\
P_t &= \frac{m\dot{v}}{u} \partial_u t + s\dot{v}^3 \partial_u^2 x. \tag{6.122b}
\end{align*}

In section 6.2, we noted that setting $\partial_u t = 0$ in the MPD equations is only consistent in pure AdS3 spacetimes, which we are now considering. The MPD equations then reduce to:

\begin{align*}
P_x &= \frac{m\dot{v}}{u} \partial_u x, & P_t &= s\dot{v}^3 \partial_u^2 x. \tag{6.123}
\end{align*}

The curve $x(u)$ has to satisfy both relationships. The first equation in (6.123) is just the geodesic equation, to which we know the solution:

$$x(u) = -\sqrt{\left(\frac{R}{2}\right)^2 - u^2}. \tag{6.124}$$

The second equation seems to impose another constraint on the path, but taking derivatives of (6.124), we see that:

$$P_x = \frac{m}{R}, \quad P_t = \frac{s}{R}, \tag{6.125}$$

which is just an instance of the general relationship of $P_t$ and $P_x$, which we already noted in (6.28). We can conclude that the only static solutions to the MPD equations in asymptotically AdS3 spacetimes are geodesics in pure AdS3.

But that does leave room for non-static solutions in pure AdS. This means that we look for a solution to the full MPD equations (6.122). These equations are still impossible to solve analytically, but they can be solved numerically. Integrating by parts to obtain $P_x$ and $P_t$ reduced the equations from third to second order so we want to solve two coupled second order ordinary differential equations. To do so, we will need four boundary conditions to obtain a unique solution. The first two boundary conditions are clear: they are the values of $x$ and $t$ at $u = 0$: $x(u) = R/2$ and $t(0) = 0$. As an aside, note that the numerical solutions should diverge at the boundary, so the boundary value should be represented by a UV cutoff $a$. The values of the derivatives $\partial_u x$ and $\partial_u t$ at $u \to 0$ constitute the other boundary conditions and these are a bit more subtle. We know that the solutions should behave as geodesics as $u \to 0$, which means that their derivatives
should linearly approach zero as \( u \to 0 \). It is too restrictive to just set the derivatives equal to zero, we should instead capture the linear behaviour as \( u \to 0 \). This can be done by imposing:

\[
\frac{\partial u_x}{u} \bigg|_{u \to 0} = \alpha_x, \quad \frac{\partial u_t}{u} \bigg|_{u \to 0} = \alpha_t, \quad (6.126)
\]

where \( \alpha_x \) and \( \alpha_t \) are constants.

We can now try to find numerical solutions. These solutions should of course exhibit a turning point \( u_0 \) and we will only be able to see them up to this turning point. This means that we expect to see \( x(u) \) and \( t(u) \) diverge at some value of \( u \). The numerical solutions to the MPD equations (6.122) are given in figures 8 and 9.

![Figure 8: Numerical solution for \( x(u) \) in (6.122). The parameters have been chosen as: \( s = 1, m = 2, P_x = 3, P_t = 1, \alpha_x = 3.3 \) and \( \alpha_t = 1.0 \).](image1)

Both \( x(u) \) and \( t(u) \) blow up at \( u \approx 0.128 \), making them candidates for a non-geodesic solution to the MPD equations. We can explicitly check whether the solutions are not in fact timelike geodesics by evaluating the geodesic part of the MPD equations on for example the solution for \( x(u) \) depicted in figure 8. If the solution is a geodesic, the geodesic part of the MPD-equations \( P_x = m \hat{v} u \partial_u x \) should of course be constant. In figure 10, \( P_x \) is plotted versus \( u \) and it is clear that the solution is not a geodesic, since \( P_x \) is not constant.

![Figure 9: Numerical solution for \( t(u) \) in (6.122). The values of the parameters are equal to those in figure 8.](image2)

![Figure 10: Plot of the numerical solution for \( x(u) \) given in figure 8, evaluated on the geodesic part of the MPD equations \( P_x = m \hat{v} u \partial_u x \).](image3)
We might be tempted now to conclude that there are non-geodesic solutions to the MPD equations in pure AdS$_3$, since we have solutions with a vertical tangent which are not geodesics. There is however still a problem with these them: they do not return to the same time slice. We are only allowed to reflect the solutions about the turning point $u_0$, which means that the curve will end at some later moment in time. This means that these numerical solutions do not obey the correct boundary conditions and leads us to the conclusion that there are no non-geodesic solutions to the MPD equations which satisfy the appropriate boundary conditions in spacetimes which are locally AdS.

This result could have been expected when we look at it from the CFT point of view. As discussed earlier, the dual to Poincaré AdS is a CFT living on a line. The entanglement entropy in this situation only depends on the two-point function of the twist fields (2.29). Even in the presence of the gravitational anomaly, the entanglement entropy is completely given by this two-point function and was computed in (4.27). This CFT entanglement entropy is already given by the geodesic solution to the MPD equations. An extra solution in pure AdS would at least correspond to corrections to this expression, which is forbidden by conformal symmetry.
7 Conclusion

Let us now summarize this thesis and give some indications for future research on these subjects. We started this thesis in section 2 by reviewing the concept of entanglement entropy. We closely examined the replica method to compute EE in conformal field theories and covered several examples where exact results can be obtained. We then turned to the AdS side of the correspondence in section 3. Firstly, we went over the different coordinates systems with which to describe pure AdS$_3$ and the BTZ black hole. We then introduced the Ryu-Takayanagi conjecture to compute EE holographically, went on to check the conjecture in different cases and reviewed the Lewkowycz-Maldacena procedure to prove the conjecture. In section 4, we then discussed the framework in which anomalies in a field theory can be described and how a gravitational anomaly affects both a 2D CFT and its holographic dual. These chapters all built towards the main focus of this thesis: computing entanglement entropy holographically in TMG backgrounds, which was the subject of sections 5 and 6. In section 5 we reviewed the procedure introduced in [1] to find a new functional to compute EE for these anomalous CFTs, the minimization of which results in the MPD equations. We then went over the examples which provide more evidence for the procedure. In section 6, we applied this procedure to asymptotically AdS$_3$ spacetimes, along the lines of [17].

We reviewed the connected solutions to the MPD equations in the limit of small $s$ which were constructed in [17] and discussed the fact that these solutions break down when the interior of the spacetime is probed too deeply. This prompted the inspection of the opposite limit: that of small $m$. We first focused on the zeroth order solutions, where $m$ is set to zero. In this limit, it is not clear whether the most general solutions move through time or not, so we first closely examined the non-static solutions to the $m = 0$ MPD equations. We found that there exist solutions which obey the correct boundary conditions as far as the time component $t(u)$ is concerned, but it turned out that these solutions do not move in $x$, making them unsuitable for our purposes. This meant that the zeroth order solutions in the $m$ expansion are static, simplifying the problem at hand. We then ensued to perturb around these static solutions in $m$, corrections to the zeroth order solution can in principle be found, but the expressions are complicated enough to make that task non-trivial. Even if these corrections can be found, a matching procedure is needed to glue the two perturbative solutions together and solve the full problem. We reviewed how such a procedure is applied to the problem of having just Einstein gravity in asymptotically AdS$_3$, as was put forward in [27, 53]. This matching procedure was effective in the case of Einstein gravity, but runs into trouble in the TMG case. We are unable to solve the problem exactly with respect to both the spin being turned on and the deformation in the interior becoming more important, so we have to resort to perturbative approaches in both limits. This means that in the TMG case, another matching region appears, where perturbative solutions in $s$ and $m$ have to be glued together. The methods described in [27, 53] are insufficient to deal with this additional matching and it is unclear how this problem should be tackled.

A natural avenue for further research to head into is more investigation in both the open problems of this thesis. Corrections to the $m = 0$ solution to the MPD equations can in principle be found and their form can shed more light on the structure of the solutions to the MPD equations in general. The same goes for the action of these perturbative solutions: it would be interesting to see how much the deep IR solutions contribute to the entanglement entropy. On top of that, the matching procedure is of course important. The precise method of [27, 53] does not apply, but there might be other methods of which the author is unaware to systematically match the different solutions together. More likely, the form of the different perturbative solutions might have properties which makes matching them easier, as exemplified by the Einstein gravity solutions having very specific $1/R$-dependence. These characteristics are hard to guess in advance.
and will appear (if at all) when the solutions themselves are found. A possible route to discovering these simplifying properties is to expand the MPD equations in the limit of small $u$ (or small $z$) and comparing it to both the expansion for small $s$ and the geodesic case. If there are similarities, this too might give more insight into the structure of the full MPD equations.

There are different directions of research which could be pursued regarding the construction of [1] itself. It provides a way to compute entanglement entropy holographically which could be applied to other solutions of TMG and perhaps generalized to other types of anomalous CFTs. A natural avenue of research is to generalize the method to higher dimensions, which has been done in [23] and [55]. Furthermore, considering TMG with asymptotically flat boundary conditions might lead to a flat space generalisation. More generally, the construction has proven its merit by providing a way to compute EE holographically in TMG, but it has itself not been proven yet. Coming up with a proof would lend the method further credence.

Over the past several decades, holography has grown to be one of the most important areas in theoretical physics. On top of the new physics it has brought along, holography has also served to refine our understanding of concepts which were already known. Entanglement entropy is such a concept and the aim of this thesis was to further explore this important quantity. This exploration ends here as far as this thesis is concerned, but will no doubt continue in general.
A  Exact solution to the MPD equations in global AdS$_3$

We want to find non-geodesic solutions for the MPD-equations in pure AdS$_3$:

$$\nabla \left( mv^\mu - se^{\mu\nu\lambda} v_\nu \nabla v_\lambda \right) = 0 \quad (A.1)$$

First of all, it was shown in section 6.6 that static solutions all reduce to the geodesic equation.

Before looking at the MPD equations in AdS, it is instructive to look at flat space first. The Christoffel symbols vanish and we obtain:

$$\dot{v}^\mu - se^{\mu\nu\lambda} v_\nu \dot{v}_\lambda = 0 \quad (A.2)$$

This solution to this equation has the form:

$$v^\mu = \left( \sqrt{1 + \gamma^2}, \gamma \cos(\omega t), \gamma \sin(\omega t) \right) \quad (A.3)$$

where $\omega = \frac{m}{s} \frac{1}{\sqrt{1 + \gamma^2}}$. This solution is both non-static and non-perturbative in $s$, since it contains terms proportional to $e^{i/s}$, which will not be analytic in Euclidean signature. Note that this solution describes an (anyonic) particle moving on a circular path. In polar coordinates $(t, r, \phi)$, the solution has the following form:

$$v^\mu = \left( \sqrt{1 + \gamma^2}, 0, \gamma \omega r \right) \quad (A.4)$$

Motivated by this solution, we look for solutions of the MPD-equations in pure AdS$_3$. We take $v^r = 0$ and using reparameterization invariance, we can always set $v^t = A$, with $A$ some constant. The normalisation constraint $g_{\mu\nu} v^\mu v^\nu = -1$ then implies that $v^\phi$ is also constant.

To make the situation as similar as possible to the flat case, we consider AdS$_3$ in global coordinates:

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\phi^2 \quad (A.5)$$

Written out in full, the MPD-equations look like:

$$\frac{d}{d\tau} \left[ mv^\mu - se^{\mu\nu\lambda} v_\nu \left( \dot{v}_\lambda - \Gamma^\alpha_{\lambda\beta} v_\alpha v^\beta \right) \right] + \Gamma^\mu_{\rho\sigma} \left[ mv^\rho v^\sigma - se^{\mu\nu\lambda} v_\nu v^\sigma \left( \dot{v}_\lambda - \Gamma^\alpha_{\lambda\beta} \right) \right] = 0 \quad (A.6)$$

The $t$- and $\phi$-components of the MPD-equations are trivially satisfied if $v^\phi$ is constant, while the $r$-component reduces to:

$$\left( \frac{(v^t)^2}{L^2} - \frac{(v^\phi)^2}{r^2} \right) \left( m + \frac{s}{r} v^\phi v^t \right) = 0 \quad (A.7)$$

We conclude that:

$$\frac{(v^\phi)^2}{L^2} = \frac{(v^t)^2}{L^2} \quad \text{or} \quad v^\phi = -\frac{mr}{sv^t} \quad (A.8)$$

We can then finally use the constraint that $g_{\mu\nu} v^\mu v^\nu = -1$, which translates to:

$$1 + \frac{r^2}{L^2} \left( v^t \right)^2 + r^2 (v^\phi)^2 = -1 \quad (A.9)$$

\[1\] I thank Nabil Iqbal for providing the flat space solution and putting me on this path.
This means that the first equality in (A.8) is only satisfied when \((v^t)^2 = 1\) and does not depend on the parameters \(m\) and \(s\). The second equality however implies:

\[
v^\phi = \pm \frac{1}{\sqrt{2r}} \sqrt{-1 + \sqrt{1 + \frac{B}{s^2}}}
\]

where

\[
B = 4m^2 r^4 \left(1 + \frac{r^2}{L^2}\right)
\]

This results in the following solution:

\[
v^\mu = (v^t, v^r, v^\phi) = \left(\pm \sqrt{2mr^2} \sqrt{-1 + \sqrt{1 + \frac{B}{s^2}}}, 0, \pm \frac{1}{\sqrt{2r}} \sqrt{-1 + \sqrt{1 + \frac{B}{s^2}}}\right)
\]

This solution doesn’t nearly look as nice as the one in the flat case, but it does imply that a 'constant-radius' solution to MPD exists in AdS, the dependence on the different constants is just uglier. The solution also goes as \(1/\sqrt{s}\) for small \(s\), which would still mean that it would not be perturbatively connected to a geodesic, since terms like \(e^{1/\sqrt{s}}\) would appear in Poincaré coordinates.

This solution does however not obey the correct boundary conditions, since a constant \(v^t\) means that it will not return to the boundary. However, as we noted above, \(v^t\) and \(v^\phi\) will always be constant when \(v^r\) is set to zero. Obtaining a solution which might have a non-constant \(v^t\) would therefore also imply setting \(v^t \neq 0\). This makes the equations a lot more complicated and highly non-linear. We conclude that it is more fruitful to first simplify the equations by using conserved quantities, as is done in section 6, rather than solving the equations directly.
References


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