The decay of massive strings

Bachelors thesis in Physics and Astronomy

Barry Ruijter (5862752)

June 9, 2010

Supervisor: Dr. M. Taylor
Second supervisor: Dr. K. Skenderis

Extent 12 EC. Project was carried out between February 15, 2010 and May 31, 2010.

Abstract

This paper describes particle decay as the splitting of massive strings. The motion of a classical string is analyzed with the use of Lagrangian mechanics. The formulation of the action integral is then extended to the case of relativistic point particles and the case of relativistic strings. It follows that relativistic strings obey a two-dimensional wave equation. A specific solution to this equation is a closed rotating string with maximum angular momentum. At a certain instant of time, the string splits into two. The outgoing string solutions are given by Fourier expansions. Continuity is demanded in the spatial coordinates and their time derivatives during the splitting process. The decay rates can be described naturally with a semiclassical formula. The estimated masses of the decay products agree very precisely with the masses that follow from quantum calculations. These results confirm that the semiclassical description of the decay process is quite accurate.
Nederlandstalige samenvatting

Er zijn slechts vier fundamentele krachten nodig om alle natuurverschijnselen te verklaren: elektromagnetische krachten, zwakke en sterke kernkrachten en de zwaartekracht. Theoretisch natuurkundigen zoeken al eeuwen naar een theorie die elk van deze vier krachten beschrijft. Een kandidaat is de snaartheorie. Snaartheorie beschrijft elementaire deeltjes als snaren, een soort trillende elastieken. Verschillende trillingswijzen corresponderen met verschillende deeltjes. Vergelijk dit met de snaar van een viool: elke trillingswijze levert een andere (boven)toon.

Elementaire deeltjes kunnen overgaan in andere elementaire deeltjes. Zo vallen vrije neutronen uiteen in een proton, een elektron en een zogenaamd antineutrino. In snaartheorie betekent dit dat een zware snaar splitst in lichtere snaren. Om dit microscopische proces te beschrijven is kwantummechanica nodig. Dit zorgt voor zeer ingewikkelde berekeningen. In dit project proberen we de kwantummechanica daarom te omzeilen. We stellen voor dat de snaar, die er uit ziet als een klein elastiekje, volledig is gestrekt en om zijn middelpunt rondjes draait in de ruimte. De twee helften van de snaar zitten dicht tegen elkaar aan. Op een gegeven moment fuseren de helften op één plaats. Daarna splitst de snaar en ontstaan twee kleinere snaren.

Nu berekenen we de kans dat de splitsing een snaar spontaan plaats vindt. Dit vertelt ons wat de verwachte levensduur is van het deeltje dat de snaar representeert. Verder gaan we na hoe zwaar de snaren zijn die bij het splitsingsproces ontstaan. Al eerder zijn deze berekeningen verricht met behulp van kwantummechanica. Wanneer de snaren relatief zwaar zijn, komen de resultaten van beide methoden nauwkeurig overeen. In dat geval leidt onze eenvoudige beschrijving van deeltjesverval dus tot betrouwbare resultaten.
Contents

1 The classical string 4
  1.1 A Newtonian approach ........................................... 4
  1.2 The classical string in Lagrangian mechanics ................. 5

2 The relativistic point particle 8
  2.1 A Lorentz invariant action ...................................... 8
  2.2 Reparameterization invariance .................................. 9
  2.3 Equations of motion for a relativistic point particle ........ 10

3 The relativistic string action 12
  3.1 The area functional ............................................. 12
  3.2 Reparameterization invariance of the area functional ........ 12
  3.3 Area functional for spacetime surfaces ......................... 14
  3.4 The Nambu-Goto string action .................................. 15
  3.5 Equations of motion for the relativistic string ................ 16
  3.6 The static gauge ................................................. 17
  3.7 The equivalence of tension and mass of a string ............... 18
  3.8 Transverse velocity of the string ................................ 19
  3.9 Motion of string endpoints ...................................... 20

4 Motion of relativistic strings 22
  4.1 Reformulation of the relativistic point particle action ........ 22
  4.2 Reformulation of the relativistic string action ................. 24
  4.3 Symmetries in the string action ................................ 24
  4.4 Deriving the equation of motion ................................ 25
  4.5 Solutions to the two-dimensional wave equation ............... 26
  4.6 Solutions in terms of light-cone coordinates ................... 28
  4.7 Constraints to the string motion ................................ 28
  4.8 Noether’s theorem and conserved quantities .................... 30

5 Semiclassical splitting of a closed string 32
  5.1 A closed rotating string solution with maximum angular momentum .... 32
  5.2 The splitting process ........................................... 33
  5.3 The constants of motion ......................................... 34
  5.4 Solutions for the outgoing strings in terms of Fourier series .... 36
  5.5 Finding a closed-form solution with direct matching .......... 39

6 Properties of the decay products 41
  6.1 Motion of the outgoing strings .................................. 41
  6.2 Decay rate and mass of decay products .......................... 42

7 Discussion and conclusions 45

A Solutions to selected problems from A first course in string theory 47

B Popular scientific Dutch abstract 56
Introduction

For many years, physicists have been searching for a unified theory to describe all four fundamental forces in nature: the electromagnetic force, the weak and strong nuclear forces and gravity. One can state that the development of physics has been marked by unifications. In 1865, Maxwell was the first to present a consistent theory of electromagnetism, which unified magnetic and electric forces. More than hundred years later, in the late 1960’s, Salam, Glashow and Weinberg came up with a theory that connected electromagnetic force to the weak nuclear force, the electroweak theory. This theory was also consistent with quantum mechanics, the correct framework for describing microscopic phenomena. On the other hand, quantization of the strong force resulted in a theory called Quantum Chromodynamics (QCD). Electroweak theory and QCD together form the standard model of physics. A more complete version of this model incorporates supersymmetry, which relates bosons to fermions. Despite the fact that the standard model is a powerful model that can be used for many purposes, it has an important shortcoming: it does not include gravity. The standard model is a quantum theory, while Einstein’s general relativity is a classical theory. It seems impossible to find a theory of quantum gravity that fits into in the standard model.

A candidate for a unification theory that overcomes this problem is string theory. It is a quantum theory that includes gravity. When supersymmetry is included, one speaks of superstring theory. In string theory, particles are not represented by points (which are zero-dimensional), but by one-dimensional objects called strings. Different vibrational modes of the string correspond to different particles. Splitting of strings into two corresponds to heavy particles decaying into two less energetic particles. Describing this process correctly in superstring theory requires complicated quantum field computations. The aim of this project is to describe the splitting process and to make estimates of the decaying rates in a semiclassical way. This approach will significantly simplify the calculations. The accuracy of these calculations will be tested by a comparison with a superstring quantum calculation.

The project can be divided into two parts. The first part discusses general properties of relativistic strings and provides tools for describing the decay process. It is mainly based on the book *A first course in string theory* by Barton Zwiebach [1]. The first chapter describes the motion of a nonrelativistic string with the use of Lagrangian mechanics. This discussion is extended to relativistic point particles and relativistic strings in sections two and three, respectively. The Nambu-Goto action is introduced as the action of the relativistic string. Some important properties, like the equivalence between the string tension and its rest mass will be considered there. In section number four, the Nambu-Goto action is reformulated in a more conventional way. The equation of motion for a two-dimensional relativistic string will turn out to be a wave equation. At the end of this section, the most general solution to this equation that fits the boundary conditions is presented for a closed string.

The second part of the project focuses on the splitting of closed strings in a semiclassical way. The article *Semiclassical decay of strings with maximum angular momentum* by R. Iengo and J.G. Russo [2] is used as a guidance. A rotating closed string solution with maximum angular momentum is considered. It is assumed that, at a certain instant of time, the string splits into two. The outgoing string solutions are described in terms of Fourier modes. The conserved quantities follow naturally from Noether’s theorem. Finally, these results are compared to the quantum field calculations in superstring theory.
1 The classical string

This section focuses on the motion of a nonrelativistic string. One can derive its equations of motion in an analytic way, using Newton’s second law. This will be done in the first subsection. The same result can be obtained with the aid of Lagrangian mechanics, as will be shown in the second subsection. The equations of motion then follow from the action integral.

1.1 A Newtonian approach

Consider a small piece of string in the \((x, y)\)-plane at a fixed time \(t\), as shown in figure 1. The string has tension \(T_0\) and mass density \(\mu_0\). Assume that the vertical displacement of the string is small compared to the total length: \(\left| \frac{\partial y}{\partial x} \right| \ll 1\). This implies that the length of the string and its tension \(T_0\) remain constant.

The different slopes at the endpoints of the string piece cause a net vertical force. The horizontal force is negligible.

At \((x, y)\), the force is the product of \(T_0\) and the slope \(\frac{\partial y}{\partial x}(t, x)\) and is pointing down. Similarly, the force at the point \((x + dx, y + dy)\) is given by the product of \(T_0\) and \(\frac{\partial y}{\partial x}(t, x + dx)\). At this position however, the force is pointing up. The net force on this piece of string is thus

\[
\begin{align*}
\text{d}F &= T_0 \frac{\partial y}{\partial x}(t, x + dx) - \frac{\partial y}{\partial x}(t, x) \text{d}x \approx T_0 \frac{\partial^2 y}{\partial x^2} \text{d}x.
\end{align*}
\]

(1.1)

Newton’s second law states that the net vertical force equals mass times acceleration. The mass of the infinitesimal piece of string is the product \(\mu_0\text{d}x\), and therefore

\[
T_0 \frac{\partial^2 y}{\partial x^2} \text{d}x = \mu_0 \text{d}x \frac{\partial^2 y}{\partial t^2}.
\]

(1.2)

Canceling the \(\text{d}x\) on each side, one can rewrite this expression to the form

\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2} \frac{\partial^2 y}{\partial t^2} = 0,
\]

(1.3)

where \(v_0 = \sqrt{T_0/\mu_0}\). This is just a wave equation, as we could have expected. There are two kinds of boundary conditions that can be imposed on this differential equation. Dirichlet boundary conditions fix the endpoints of the string. For endpoints at \(x = 0\) and \(x = a\), this means

\[
\frac{\partial y}{\partial t}(t, x = 0) = \frac{\partial y}{\partial t}(t, x = a) = 0.
\]

(1.4)

\footnote{This section is based on chapter 4: The nonrelativistic string from B. Zwiebach, A first course in string theory. [1]}
In the case of Neumann boundary conditions, the endpoints can be imagined as infinitesimal, massless loops that slide frictionless over vertical lines. These loops must remain horizontal. Otherwise, the slope of the string would cause a vertical force. Such a force would give the massless loop an infinite acceleration. Therefore, the value of the derivative with respect to $x$ at the endpoints must vanish:

$$\frac{\partial y}{\partial x}(t, x = 0) = \frac{\partial y}{\partial x}(t, x = a) = 0.$$  

The allowed frequencies of oscillation are the same for Neumann and Dirichlet boundary conditions. For example, on substituting the solution $y(t, x) = y(x) \sin(\omega t + \phi)$ into the wave equation, one finds

$$\frac{d^2 y(x)}{dx^2} + \omega^2 \frac{\mu_0}{T_0} y(x) = 0.$$  

This differential equation in $y(x)$ can be solved by using trigonometric functions. The general solution is

$$y_n = A_n \sin\left(\frac{n\pi x}{a}\right) + B_n \cos\left(\frac{n\pi x}{a}\right), \quad \text{where} \quad n = 1, 2, 3 \ldots (1.7)$$

Now imposing Dirichlet boundary conditions (1.4) gives

$$y_n = A_n \sin\left(\frac{n\pi x}{a}\right),$$  

while Neumann boundary conditions (1.5) imply that

$$y_n = B_n \cos\left(\frac{n\pi x}{a}\right).$$  

In both cases, one finds the same spectrum of allowed frequencies

$$\omega_n = \sqrt{\frac{T_0}{\mu_0} \frac{n\pi}{a}}.$$  

1.2 The classical string in Lagrangian mechanics

The wave equation (1.3) can also be found using Lagrangian mechanics. The Lagrangian $L$ of a system is defined by

$$L = T - V,$$  

where $T$ is the kinetic energy and $V$ the potential energy. Further, the action $S$ is defined as

$$S = \int_{\mathcal{P}} L(t) dt,$$  

where $\mathcal{P}$ is a path between an initial position $x_i$ with a corresponding time $t_i$ and a final position $x_f$ at time $t_f$. In general, one can write $L$ as a function of a dynamical variable $q(t)$ and its time derivative $\dot{q}(t)$:

$$S = \int dt L(q(t), \dot{q}(t); t).$$  

The action is not an ordinary function: in that case, there should be a number as input and another number as output. Actually, the action is a functional, which takes a function as input and gives a number as output. The functional can be considered as a function with an infinite number of variables. Now Hamilton’s principle states that on the path which a system actually takes, the action is stationary. Often this is called the principle of least action, which is not formally correct: the definition also allows the action to take a maximum or saddle point value on the physical path. The fact that the action in stationary means that under a small variation of the path, with the endpoints fixed, the change in action is zero.
Formally stated, the transformation \( q \rightarrow q(t) + \delta q(t), \dot{q}(t) \rightarrow \dot{q} + \delta \dot{q} \) should give \( \delta S = 0 \). This principle is demonstrated for a particle in free-fall in figure 2. In order to perform the variation, a first order Taylor expansion is used to get

\[
\delta S = \int dt \left[ \frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right] = 0. \tag{1.14}
\]

To get rid of the derivative in the second term, this equation is integrated by parts:

\[
\delta S = \int dt \left[ \frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}(t)} \delta q(t) \right) - \left( \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}(t)} \right) \delta q(t) \right] = \int dt \left[ \frac{\partial L}{\partial q(t)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}(t)} \right] \delta q(t) + \left[ \frac{\partial L}{\partial \dot{q}(t)} \delta q(t) \right]_{t_f}^{t_i}. \tag{1.15}
\]

The last term equals zero because the variation \( \delta q(t) \) vanishes at the endpoints of the path, by definition. Equating the variation in the action zero for every variation \( \delta q(t) \) implies that the other term should also be zero. Because \( \delta q(t) \) is arbitrary, the term in brackets must vanish. This requirement gives the equation of motion

\[
\frac{\partial L}{\partial q(t)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}(t)} = 0. \tag{1.16}
\]

This principle can be used to find the equations of motion for the classical string. The kinetic energy \( T \) is the sum of the kinetic energies for all infinitesimal pieces of string between \( x = 0 \) and \( x = a \). Formally,

\[
T = \int_0^a \frac{1}{2} (\mu_0 dx) \left( \frac{\partial y}{\partial t} \right)^2. \tag{1.17}
\]

The potential energy equals the work necessary to stretch the string, given by the tension \( T_0 \) (the required force) times the displacement \( \Delta l \). As can be seen from figure 1, this displacement is

\[
\Delta l = \sqrt{(dx)^2 + (dy)^2} - dx = dx \left( \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} - 1 \right) \approx dx \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2, \tag{1.18}
\]

using the assumption \( \frac{\partial y}{\partial x} \ll 1 \). One can multiply this expression by \( T_0 \) and integrate it over the length of the string to find the total potential energy:

\[
V = \int_0^a \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 dx. \tag{1.19}
\]
With the use of (1.17) and (1.19), the action integral becomes

\[ S = \int_{t_i}^{t_f} dt \int_0^a dx \left[ \frac{1}{2} \mu_0 \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T_0 \frac{\partial^2 y}{\partial x^2} \right] \]  

(1.20)

Hamilton’s principle can now be used to variate the path. This leads to

\[ \delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[ \mu_0 \frac{\partial y}{\partial t} \frac{\partial (\delta y)}{\partial t} - T_0 \frac{\partial y}{\partial x} \frac{\partial (\delta y)}{\partial x} \right] \]  

(1.21)

In order to get an expression without derivatives working on variations, one rewrites the two terms in the brackets as a full derivative minus a derivative which does not act on the variation:

\[ \delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[ \mu_0 \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x^2} \right] \]  

(1.22)

Each of the terms in the last expression must vanish separately. This happens for the first term because the variation \( \delta y \) is zero at the endpoints of the path. Note that the variation \( \delta y \) in the last term is arbitrary. Therefore, the expression in brackets is zero. This gives exactly the wave equation (1.3) that was found with a totally different approach! The vanishing of the second term is less obvious. It can be made clear by writing it out explicitly as

\[ \int_{t_0}^{t_f} \left[ -T_0 \frac{\partial y}{\partial x}(t, a)\delta y(t, a) + -T_0 \frac{\partial y}{\partial x}(t, 0)\delta y(t, 0) \right] dt. \]  

(1.23)

This is where the boundary conditions come into play. By imposing Neumann boundary conditions, one must have \( \frac{\partial y}{\partial x}(t, 0) = \frac{\partial y}{\partial x}(t, a) = 0 \). Dirichlet boundary conditions, on the other hand, require that the endpoints are fixed, or formally: \( \frac{\partial y}{\partial t}(t, 0) = \frac{\partial y}{\partial t}(t, a) = 0 \). This would imply that \( \delta y(t, 0) = \delta y(t, a) = 0 \). This confirms that both types of boundary conditions are sufficient.
2 The relativistic point particle

The strings considered in this project are free to move at relativistic speeds. In that case, the classical form of the action is no longer valid. Consider for example the motion of a free particle. The classical action in this specific case is

\[ S = \int L dt = \int \frac{1}{2} m \vec{v}^2(t) dt. \]  

(2.1)

By using the variational principle, it follows that \( \frac{dv}{dt} = 0 \). From a classical viewpoint, this result seems physical. In the relativistic case it’s not correct, though. In principle, the equation allows motion with any velocity: it does not prevent motion that exceeds the speed of light \( c \). The aim of this section is to develop an action that is valid for relativistic point particles. In the next chapter, this result will be extended to a form that is suitable for relativistic strings.

2.1 A Lorentz invariant action

Any physical action must yield Lorentz invariant equations of motion. This means that if a particular Lorentz observer tells that a particle is performing physical motion, any other Lorentz observer must do so. This criterion can be satisfied by requiring the action to be a Lorentz invariant scalar. This can be seen from the following. Suppose that a Lorentz observer states that the action is stationary for a given world line. With a Lorentz invariant action, observers in any other Lorentz frame will agree on this statement. According to Hamilton’s principle, the equations of motion are satisfied on all world lines where the action is stationary. Therefore, all Lorentz observers conclude that the equations of motion are satisfied on the given world line. This is precisely what we need.

Because the action must be Lorentz invariant, it should depend on a quantity that all Lorentz observers agree on. In fact, this is the proper time on the particle’s world line. To find an expression for the proper time, one recalls the invariant interval \( ds^2 \) from special relativity:

\[ -ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \]  

(2.2)

The infinitesimal displacement is zero for an observer who moves with the particle. The infinitesimal time he measures is the infinitesimal proper time: \( d\tau = \frac{dt}{c} \). The proper time is found by integrating this expression over the path that the particle follows. Since the Lagrangian has units of energy, the action has units of energy times time. Therefore, the proper time needs to be multiplied by a Lorentz invariant factor with units of energy. This factor is \( mc^2 \). It turns out that we also need an additional minus sign. The relativistic expression for the action then becomes

\[ S = -mc \int ds. \]  

(2.3)

This integral over \( ds \) can be rewritten to an integral over time, using equation (2.2),

\[ ds = c dt \sqrt{1 - \frac{v^2}{c^2}}. \]  

(2.4)

Then, the action integral becomes

\[ S = -mc^2 \int \sqrt{1 - \frac{v^2}{c^2}} dt. \]  

(2.5)

This equation more obviously shows that the speed of the particle is limited: because the expression under the square root should be positive, the particle’s speed can never exceed the speed of light. In the low speed limit \( v \ll c \), the Lagrangian of (2.5) reduces to

\[ L \approx -mc^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right) = -mc^2 + \frac{1}{2} mv^2, \]  

(2.6)

which is precisely the nonrelativistic Lagrangian. This limit also shows why the additional minus sign in (2.3) is necessary.

\footnote{This section is based on chapter 5: The relativistic point particle from B. Zwiebach, A first course in string theory. [1]}
2.2 Reparameterization invariance

Calculating the action for a specific world line is done by parameterizing the path. Equation (2.3) shows that the value of the action does not depend on the specific parameterization. In fact, the integral is just the sum of $mc \cdot ds$ over all infinitesimal pieces of the path. By parameterizing the path, all coordinates are expressed as functions of a single parameter $\tau$:

$$ x^\mu = x^\mu(\tau), \quad (2.7) $$

as is demonstrated in figure 3.

Figure 3: A world line is fully parameterized by $\tau$. All spacetime coordinates can be expressed as functions of this single parameter. (Adapted from [1])

The parameterization is also valid at the endpoints of the path. The invariant interval (2.2) can be rewritten as

$$ -ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (2.8) $$

with the diagonal Minkowski metric

$$ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.9) $$

Equation (2.8) is just a special form of the relativistic scalar product

$$ a \cdot b \equiv a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \quad (2.10) $$

All these equations are equivalent and will be used interchangeably throughout this paper. The Minkowski metric and its inverse $\eta^{\mu\nu}$ can be used for lowering and raising indices, respectively:

$$ b_\mu \equiv \eta_{\mu\nu} b^\nu, \quad b^\mu \equiv \eta^{\mu\nu} b_\nu. \quad (2.11) $$

One can use (2.8) to express the integrand $ds$ in (2.3) as a function of $\tau$:

$$ ds^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2, \quad (2.12) $$

and so the action takes the form

$$ S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau. \quad (2.13) $$
It is easy to demonstrate that this expression does not depend on the specific parameterization. Choose for example a new parameter \( \tau' \). Then follows by the chain rule that \( \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'} \frac{d\tau'}{d\tau} \). On substituting this into (2.13), one finds

\[
S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau'} \frac{dx^\nu}{d\tau'} \frac{d\tau'}{d\tau}} d\tau = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau'} \frac{dx^\nu}{d\tau'}} d\tau'.
\]  

(2.14)

This has exactly the same form as equation (2.13). We call the action \textit{reparameterization invariant}.

### 2.3 Equations of motion for a relativistic point particle

The action integral can now be varied to obtain the equations of motion. The variation of (2.3) is given by

\[
\delta S = -mc \int \delta (ds). \tag{2.15}
\]

An explicit expression for \( \delta (ds) \) is found by varying both sides of (2.12). The result is

\[
2 ds \delta (ds) = -2 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2, \tag{2.16}
\]

where the factor of 2 arises from the fact that the variations in \( \frac{dx^\mu}{d\tau} \) and \( \frac{dx^\nu}{d\tau} \) give the same contribution. Solving this equation for \( \delta (ds) \) gives

\[
\delta (ds) = -\eta_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{d\tau} d\tau = -\frac{d(\delta x^\mu)}{d\tau} \frac{d}{ds} p_\mu d\tau, \tag{2.17}
\]

where \( \delta (dx^\mu/d\tau) = d(\delta x^\mu)/d\tau \) and where \( \eta_{\mu\nu} \) was used to lower the index of \( dx^\nu \) in the second step. Substituting this expression into (2.15) gives

\[
\delta S = mc \int_{\tau_i}^{\tau_f} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\mu}{d\tau} d\tau = \int_{\tau_i}^{\tau_f} \frac{d(\delta x^\mu)}{d\tau} p_\mu d\tau, \tag{2.18}
\]

where we defined the quantity

\[
p_\mu \equiv mc \frac{dx^\mu}{ds}. \tag{2.19}
\]

To get rid of the derivative working on \( \delta x^\mu \), one integrates this result by parts to get

\[
\delta S = \int_{\tau_i}^{\tau_f} d\tau \left( \frac{d}{d\tau} (\delta x^\mu) p_\mu \right) - \int_{\tau_i}^{\tau_f} d\tau \delta x^\mu (\tau) \frac{dp_\mu}{d\tau}, \tag{2.20}
\]

The first integral of equation (2.20) gives the product of \( \delta x^\mu (\tau) \) with \( p_\mu \) at the endpoints, and is zero because the variation is fixed there. The second term must vanish for arbitrary \( \delta x^\mu \), and thus

\[
\frac{dp_\mu}{d\tau} = \frac{dp^\mu}{d\tau} = 0. \tag{2.21}
\]

This equation of motion states that the momentum of a free relativistic particle is constant along its world line. This is true for arbitrary parameterization \( \tau \), as can be seen from the chain rule:

\[
\frac{dp_\mu}{d\tau} = \frac{dp_\mu}{d\tau'} \frac{d\tau'}{d\tau} = 0. \tag{2.22}
\]

By assumption, \( \frac{d\tau'}{d\tau} \neq 0 \) and therefore \( \frac{dp_\mu}{d\tau} = 0 \). Using \( p_\mu = mc \frac{dx^\mu}{ds} \), one can rewrite this equation of motion as a derivative of the position with respect to the proper time:

\[
\frac{d^2 x^\mu}{ds^2} = 0. \tag{2.23}
\]
It should be noted that this equation does not hold for an arbitrary parameter \( s \). An arbitrary parameter would imply that the distance between two successive marks on the path is arbitrary, which is nonsense. It is, however, possible to write a version of (2.23) which is parameterization invariant. This can be done by setting the variation of the action integral (2.13) equal to zero:

\[
\delta S = \int_{t_i}^{t_f} \left[ \frac{\partial L}{\partial \dot{x}^\rho} \delta x^\rho + \frac{\partial L}{\partial \dot{\dot{x}}^\rho} \delta \dot{x}^\rho \right] d\tau = 0,
\]

(2.24)

where

\[
L = -mc \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau}.
\]

(2.25)

The first term within the brackets is zero because \( L \) only depends on derivatives of \( x^\rho \), so

\[
\delta S = \eta_{\mu\rho} mc \int_{t_i}^{t_f} \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\nu}} \delta x^\rho d\tau = 0.
\]

(2.26)

This expression is integrated by parts to get rid of the derivatives working on the variation \( \delta x^\rho \):

\[
\delta S = \eta_{\mu\rho} mc \int_{t_i}^{t_f} \frac{1}{d\tau} \left[ \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\nu}} \delta x^\rho \right] d\tau - \eta_{\mu\rho} mc \int_{t_i}^{t_f} \left[ \frac{1}{d\tau} \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\nu}} \delta x^\rho \right] d\tau = 0.
\]

(2.27)

The first term vanishes because the variation \( \delta x^\rho \) is zero at the endpoints of the path. In order to make the second integral zero, the term in brackets of the second integral must also vanish. From this statement follows the equation of motion

\[
\frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\nu}} = 0.
\]

(2.28)

Comparing (2.28) to (2.21), one concludes that

\[
p^\mu = \frac{m \dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\nu}},
\]

(2.29)

which is just the relativistic momentum of a point particle: \( \vec{p} = \gamma m \vec{v} \).


3 The relativistic string action

This chapter extends the discussion of sections 1 and 2 to the case of relativistic strings. The path that a relativistic particle traces out is one-dimensional and is called a world line. A string, which has one dimension instead of zero, sweeps out a surface in spacetime. This surface is called the world sheet. For the point particle, we call the product of $c$ and the proper time the proper length of the world line. The equivalent for the string is the proper area. The action of the relativistic string turns out to be proportional to this proper area.

3.1 The area functional

In describing a surface, two kinds of space can be distinguished. The target space is the actual space where the two-dimensional surface lives; for example, this is our four-dimensional spacetime. The parameter space is the space on the surface, which consists of two dimensions. This space is described with the parameters $\xi^1$ and $\xi^2$. Now first, an area functional will be derived for a target space of three (spatial) dimensions. The physical surface is described by the vector

$$\vec{x}(\xi^1, \xi^2) = (x^1(\xi^1, \xi^2), x^2(\xi^1, \xi^2), x^3(\xi^1, \xi^2)).$$

(3.1)

On the parameter space, an infinitesimal area element is simply the product $\xi^1 \xi^2$. In general, this infinitesimal area element maps to target space as a parallelogram. This parallelogram is spanned by the vectors

$$d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1, \quad d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2.$$

(3.2)

The area of the parallelogram is given by the absolute value of the outer product of these vectors. An infinitesimal area element in target space is thus

$$dA = |d\vec{v}_1 \times d\vec{v}_2| = |d\vec{v}_1||d\vec{v}_2| \sin \theta = \sqrt{|d\vec{v}_1|^2|d\vec{v}_2|^2 - |d\vec{v}_1|^2|d\vec{v}_2|^2 \cos^2 \theta},$$

(3.3)

where $\theta$ is the smallest angle between $d\vec{v}_1$ and $d\vec{v}_2$. Rewriting this in terms of spatial dot products gives

$$dA = \sqrt{(d\vec{v}_1 \cdot d\vec{v}_1)(d\vec{v}_2 \cdot d\vec{v}_2) - (d\vec{v}_1 \cdot d\vec{v}_2)^2}.$$

(3.4)

Using equations (3.2) and integrating, one finds an expression for the area functional:

$$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1} \right) \left(\frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \right) - \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \right)^2}.$$

(3.5)

3.2 Reparameterization invariance of the area functional

As in the case of the point particle, one can show that equation (3.5) for the area functional is reparameterization invariant. The resulting equation also turns out to be more compact and elegant. The complicating difference with the point particle is that the area functional depends on two parameters. Consider a new area functional, depending on two parameters $\tilde{\xi}^1(\xi^1, \xi^2)$ and $\tilde{\xi}^2(\xi^1, \xi^2)$. Using the change-of-variable theorem from calculus, one gets

$$d\xi^1 d\xi^2 = \left| \det \left( \frac{\partial \tilde{\xi}^i}{\partial \xi^j} \right) \right| d\tilde{\xi}^1 d\tilde{\xi}^2 = | \det M | d\tilde{\xi}^1 d\tilde{\xi}^2,$$

(3.6)

and similarly

$$d\xi^1 d\xi^2 = \left| \det \left( \frac{\partial \tilde{\xi}^i}{\partial \xi^j} \right) \right| d\tilde{\xi}^1 d\tilde{\xi}^2 = | \det \tilde{M} | d\tilde{\xi}^1 d\tilde{\xi}^2.$$

(3.7)

---

1This section is based on chapter 6: Relativistic strings from B. Zwiebach, A first course in string theory. [1]
In these expressions, \( M = [M_{ij}] = \partial \xi^i / \partial \tilde{\xi}^j \) and \( \tilde{M} = [\tilde{M}_{ij}] = \partial \tilde{\xi}^i / \partial \xi^j \). Combining equations (3.2) and (3.8) gives
\[
|\text{det} M| \cdot |\text{det} \tilde{M}| = 1. \tag{3.8}
\]

Now consider a surface \( S \) in target space. This surface can be described by the vector \( \tilde{x}(\xi^1, \xi^2) \). An infinitesimal tangent vector is then given by \( d\tilde{x} \). By denoting its length as \( ds \), one can write
\[
| ds |^2 = d\tilde{x} \cdot d\tilde{x}. \tag{3.9}
\]
Reexpressing \( d\tilde{x} \) in terms of parameters leads to
\[
d\tilde{x} = \frac{\partial \tilde{x}}{\partial \xi^1} d\xi^1 + \frac{\partial \tilde{x}}{\partial \xi^2} d\xi^2 = \frac{\partial \tilde{x}}{\partial \xi^i} d\xi^i, \tag{3.10}
\]
where \( i \in \{1, 2\} \). Substituting this expression back into (3.9) gives
\[
| ds |^2 = g_{ij}(\xi) d\xi^i d\xi^j, \tag{3.11}
\]
where
\[
g_{ij} = \left( \frac{\partial \tilde{x}}{\partial \xi^1} \cdot \frac{\partial \tilde{x}}{\partial \xi^1}, \frac{\partial \tilde{x}}{\partial \xi^1} \cdot \frac{\partial \tilde{x}}{\partial \xi^2}, \frac{\partial \tilde{x}}{\partial \xi^2} \cdot \frac{\partial \tilde{x}}{\partial \xi^1}, \frac{\partial \tilde{x}}{\partial \xi^2} \cdot \frac{\partial \tilde{x}}{\partial \xi^2} \right). \tag{3.12}
\]
The quantity under the square root in the area functional (3.5) is precisely the determinant of this matrix. The functional can therefore be rewritten into the much simpler form
\[
S = \int d\xi^1 d\xi^2 \sqrt{g}, \tag{3.13}
\]
where \( g = \text{det} g_{ij} \). Note that equation (3.11) defines a geometrical property that should not depend on the particular parameterization. Then the following statement holds:
\[
g_{ij}(\xi) d\xi^i d\xi^j = \tilde{g}_{pq}(\tilde{\xi}) d\tilde{\xi}^p d\tilde{\xi}^q. \tag{3.14}
\]
From using the chain rule on the right-hand-side follows
\[
g_{ij}(\xi) d\xi^i d\xi^j = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j} d\xi^i d\xi^j, \tag{3.15}
\]
and thus
\[
g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j}. \tag{3.16}
\]
One can make use of the previously defined matrix \( \tilde{M} \) to rewrite this equation as
\[
g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi}) \tilde{M}_{pi} \tilde{M}_{qj} = (\tilde{M}^T)_{ip} \tilde{g}_{pq} \tilde{M}_{qj}. \tag{3.17}
\]
This can be considered as being a matrix multiplication. Taking determinants on both sides gives
\[
g = (\text{det} \tilde{M})^2 \tilde{g}(\text{det} \tilde{M}) = \tilde{g}(\text{det} \tilde{M})^2. \tag{3.18}
\]
The last expression can be substituted into (3.13). Thereby making use of equations (3.2) and (3.8) leads to
\[
S = \int d\xi^1 d\xi^2 \sqrt{g} = \int d\tilde{\xi}^1 d\tilde{\xi}^2 |\text{det} M| \sqrt{\tilde{g}} |\text{det} \tilde{M}| = \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\tilde{g}}. \tag{3.19}
\]
Because the choice of parameters \( \tilde{\xi}^1 \) and \( \tilde{\xi}^2 \) is arbitrary, this proves that the area functional is reparameterization invariant.
3.3 Area functional for spacetime surfaces

The area functional can be used to describe the world sheet that a string sweeps out in spacetime. There are two distinct kinds of string: open strings and closed strings. Open strings have endpoints that are free to move in spacetime. Closed strings are topologically circles and can be constructed by connecting the endpoints of an open string. A possible world sheet for both kinds of strings is drawn in figure 4. All world sheet coordinates can be expressed as functions of two parameters. In this particular case, they are named \( \tau \) and \( \sigma \) instead of \( \xi^1 \) and \( \xi^2 \). The parameter \( \tau \) is related to time, while \( \sigma \) refers to positions on the string. The spacetime coordinates are then given by the mapping functions \( x^\mu(\tau, \sigma) \). The convention in string theory is to write this mapping functions as \( X^\mu(\tau, \sigma) \).

This convention is used to distinguish the mapping functions from the spacetime coordinates \( x^\mu \), which we write down as lowercase symbols.

![Figure 4](image)

Figure 4: The picture on the left is the world sheet for an open string. The world sheet for a closed string is a cylinder, as shown on the right. (Adapted from [1])

The \( X^\mu \) are called the string coordinates. These functions map an infinitesimal rectangular surface element in parameter space \((\tau, \sigma)\) to an infinitesimal parallelogram in target spacetime \(x^\mu\). The vectors spanning this parallelogram are given by

\[
dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau, \quad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma. \tag{3.21}
\]

Analogously to the reasoning in subsection 3.1, one can state that

\[
dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}. \tag{3.22}
\]

It turns out that this statement is not true. In fact, an additional minus sign under the square root is necessary. Applying this minus sign and using (3.21) gives

\[
A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \tau}\right)^2}, \tag{3.23}
\]

or in dot product notation

\[
A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \tau}\right)^2}. \tag{3.24}
\]

It will now be demonstrated why the quantity under the square root is positive. It should first be made clear that both spacelike and timelike tangent vectors are needed for each point on a physical world sheet.
Finding a spacelike tangent vector is always possible. For example, any tangent vector along the string at a fixed time is spacelike. To understand the need for a timelike tangent vector, recall the case of a point particle. At any point of its world line, the tangent vector is timelike, because the velocity of a particle is always smaller than c. When considering a string on two closely separated times, one cannot tell how individual points on it move. The exception is that one can keep track of the endpoints. On the other hand, for each point P on the final string, one must be able to find a point P' on the initial string that could have reached P with a speed less than or at most equal to c. A vector that connect these two points is always timelike. Up to a constant scaling factor, the complete set of tangent vectors at a point P can be described by

$$v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}, \quad (3.25)$$

where \(\lambda \in (-\infty, \infty)\). The vector \(\frac{\partial X}{\partial \sigma}\) is reached in the limit \(\lambda \to \infty\). The square of this equation is

$$v^2(\lambda) = v^\mu(\lambda) v^\nu(\lambda) = \lambda^2 \left(\frac{\partial X^\mu}{\partial \tau}\right)^2 + 2\lambda \left(\frac{\partial X^\mu}{\partial \tau} \cdot \frac{\partial X^\nu}{\partial \sigma}\right) + \left(\frac{\partial X^\nu}{\partial \sigma}\right)^2. \quad (3.26)$$

Because \(v^2 = \eta_{\mu\nu} v^\mu v^\nu = -(v_0)^2 + (v_1)^2 + (v_2)^2 + (v_3)^2\), \(v^2 < 0\) means timelike, \(v^2 = 0\) means null and \(v^2 > 0\) means spacelike. To have both timelike and spacelike vectors, the quadratic equation \(v^2(\lambda) = 0\) must take both positive and negative values. Therefore, it has two real roots and its discriminant is positive. This last requirement translates to

$$\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right) - \left(\frac{\partial X}{\partial \sigma}\right)^2 > 0. \quad (3.27)$$

This is precisely what was set out to prove.

### 3.4 The Nambu-Goto string action

Just like the proper time is used to define an action for the relativistic point particle, the proper area can be used to define the action for a relativistic string. The proper area (3.24) has units of length-squared. This is because all factors of \(X^\mu\) under the square root have units of length and the units of \(\tau\) and \(\sigma\) cancel each other. As always, the action must have units of energy times time: \([S] = [ML^2/T]\). The proper area should therefore be multiplied by a quantity of units \([M/T]\). The correct factor is \(T_0/c\). It has indeed the right units, because \([T_0] = [F] = [ML/T^2]\) and \([c] = [L/T]\). With an additional minus sign, the string action becomes

$$S = -\frac{T_0}{c} \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (X')^2}, \quad (3.28)$$

where \(\dot{X} = \frac{\partial X}{\partial \tau}\) and \(X' = \frac{\partial X}{\partial \sigma}\). This action is called the Nambu-Goto action. Later, it will become clear why the use of an extra minus sign is correct. To rewrite equation (3.28) in a reparameterization invariant form, note that

$$-d\tilde{s}^2 = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} d\xi^a d\xi^b. \quad (3.29)$$

This is a similar equation as (3.11). As before, one can define an induced metric

$$\gamma_{ab} \equiv \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} = \frac{\partial X}{\partial \xi^a} \cdot \frac{\partial X}{\partial \xi^b}. \quad (3.30)$$

The Nambu-Goto action in a reparameterization invariant form is then

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{-\gamma}, \quad (3.31)$$

where \(\gamma \equiv \det \gamma_{ab}\).
3.5 Equations of motion for the relativistic string

To derive equations of motion for the relativistic string, one writes the Nambu-Goto action (3.28) as a double integral over a Lagrangian density $\mathcal{L}$:

$$S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \mathcal{L}(\dot{X}^\mu, X^\mu).$$  \hfill (3.32)

where $\mathcal{L}$ is given by

$$\mathcal{L}(\dot{X}^\mu, X^\mu) = -\frac{T_0}{c} \sqrt{(\dot{X} \cdot \dot{X})^2 - (\ddot{X} \cdot \dot{X})^2}.$$  \hfill (3.33)

The variation of this action integral is

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial (\delta X^\mu)}{\partial \dot{\tau}} + \frac{\partial \mathcal{L}}{\partial X^\mu} \frac{\partial (\delta X^\mu)}{\partial \dot{\sigma}} \right].$$  \hfill (3.34)

Working out the partial derivatives of $\mathcal{L}$ leads to the quantities

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{\dot{X} \cdot \ddot{X}}{(\dot{X} \cdot \dot{X})^2 - (\ddot{X} \cdot \dot{X})^2},$$  \hfill (3.35)

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial {X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot \ddot{X}) - \dot{X} \cdot \dddot{X}}{(\dot{X} \cdot \dot{X})^2 - (\ddot{X} \cdot \dot{X})^2}.$$  \hfill (3.36)

Substituting these expressions back into (3.34) gives

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial (\delta X^\mu)}{\partial \dot{\tau}} + \frac{\partial \mathcal{L}}{\partial {X}^\mu} \frac{\partial (\delta X^\mu)}{\partial \dot{\sigma}} \right].$$  \hfill (3.37)

The first term in brackets is a full derivative in $\tau$. Because the endpoints of the path are stationary, $\delta X^\mu(\tau, f) = \delta X^\mu(\tau, 0) = 0$, and therefore this term vanishes. The remaining expression is

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[ \delta X^\mu \mathcal{P}_\mu \right]_0^{\sigma_f} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \delta X^\mu \left[ \frac{\partial \mathcal{P}_\mu}{\partial \dot{\tau}} + \frac{\partial \mathcal{P}_\mu}{\partial \dot{\sigma}} \right].$$  \hfill (3.38)

Because the variation $\delta X^\mu$ is arbitrary, the expression in brackets of the second term equals zero. This provides the equation of motion for the relativistic string:

$$\frac{\partial \mathcal{P}_\mu}{\partial \dot{\tau}} + \frac{\partial \mathcal{P}_\mu}{\partial \dot{\sigma}} = 0.$$  \hfill (3.39)

The first term in (3.38) depends on the endpoints of the string. For every value of $X^\mu$, one must define a condition to let the term vanish. More explicitly, this means (for $\mu = 1, 2, \ldots, d$)

$$\delta X^0(\tau, \sigma_f) \mathcal{P}^0_0(\tau, \sigma_f) - \delta X^0(\tau, 0) \mathcal{P}^0_0(\tau, 0) = 0,$$

$$\delta X^1(\tau, \sigma_f) \mathcal{P}^1_0(\tau, \sigma_f) - \delta X^1(\tau, 0) \mathcal{P}^1_0(\tau, 0) = 0,$$

$$\vdots$$

$$\delta X^d(\tau, \sigma_f) \mathcal{P}^d_0(\tau, \sigma_f) - \delta X^d(\tau, 0) \mathcal{P}^d_0(\tau, 0) = 0.$$  \hfill (3.40)

There are now two kinds of boundary conditions that can be implied: Dirichlet boundary conditions or free endpoint conditions (which are in fact Neumann boundary conditions). For Dirichlet boundary conditions, the endpoints of the string remain fixed throughout time. Introducing $\sigma_f$ to denote an endpoint (which can take values 0 and $\sigma_f$), this implies

$$\frac{\partial X^\mu}{\partial \dot{\tau}}(\tau, \sigma_f) = 0 \text{ for } \mu \neq 0.$$  \hfill (3.41)
The case \( X^0 \) is excluded because time and \( \tau \) are never independent. Instead of claiming that \( x^\mu \) does not depend on \( \tau \) and therefore is constant in time, one can also assign a constant value to \( X^\mu(\tau, \sigma^*) \). This guarantees that all variations \( \delta X^\mu(\tau, \sigma^*) \) vanish, and therefore that all equations in (3.40) are satisfied. The second possible boundary condition is the free endpoint condition, that is defined by

\[
P_\mu(\tau, \sigma^*) = 0.
\]  

(3.42)

This condition must also apply for \( \mu = 0 \). It is called a free endpoint condition because the variation \( \delta X^0(\tau, \sigma^*) \) is not constrained.

Dirichlet boundary conditions arise if string endpoints are attached to some physical object. These objects are called D-branes (where D stands for Dirichlet) and they are characterized by their dimensionality. When a string endpoint is attached to a D0-brane, it means it is fixed at a (zero-dimensional) point. If the endpoint is free to move over a line, we speak of a D1-brane. In general, a D\( p \)-brane is an object with \( p \) spatial dimensions. When the open string endpoints have free endpoint boundary conditions along all spatial dimensions, we speak of a space-filling D-brane.

### 3.6 The static gauge

In subsection 3.4, it was shown that the Nambu-Goto action is reparameterization invariant. This is analogous to a principle in electrodynamics called gauge invariance. According to this principle, one can choose different potentials \( A_\mu \) to represent the same electromagnetic fields \( \vec{E} \) and \( \vec{B} \). A good choice for a potential makes the calculations of the fields a lot easier. In a similar way, a suitable choice of parameters for the world sheet of a relativistic string makes the equations of motion simpler. This subsection discusses a partial parameterization of the string in which the parameter \( \tau \) is related to the time coordinate \( X^0 \). The principle is demonstrated for an open string in figure 5.

![Figure 5: Left: the parameter space for an open string. Right: the open string world sheet, intersected by the plane \( t = t_0 \). In the static gauge, the string at time \( t = t_0 \) is the image of the line \( AB \), which represents \( \tau = t_0 \) in parameter space. (Adapted from [1])](image)

Imagine the plane \( t = t_0 \) in target space. The intersection of this plane with the world sheet represents the string at the fixed time \( t_0 \), as seen by an observer in our Lorentz frame. Now it is declared that the intersection represents a line of constant \( \tau \). This definition is extended to all times \( t \) and all points \( P \) on the world sheet by

\[
\tau(P) = t(P).
\]  

(3.43)

This is what is called the static gauge, because it represents static strings the in reference frame of our Lorentz observer. The coordinate \( \sigma \) is not parameterized very precisely; one border of the world sheet is denoted as \( \sigma = 0 \) and the other border as \( \sigma = \sigma_1 \). In between, the lines of constant \( \sigma \) are quite arbitrary. The only demand is that they are smooth, non-intersecting and consistent with the values at the borders.
Closed string world sheets require a different parameterization. Because closed strings are topologically circles, their world sheets are cylindrical. Therefore, there must be an identification in the \( \sigma \)-coordinate. This means that when the circumference of the string is \( \sigma_c \), the coordinates \((\tau, \sigma)\) and \((\tau, \sigma + \sigma_c)\) represent the same point. Formally, one writes this as

\[(\tau, \sigma) \sim (\tau, \sigma + \sigma_c).\]  

(3.44)

The closed interval in which \( \sigma \) runs is therefore

\[\sigma \in [0, \sigma_1]\] for an open string,  

(3.45)

\[\sigma \in [0, \sigma_c]\] for a closed string.  

(3.46)

From (3.43) can be concluded that \( X^0(\tau, \sigma) = c\tau, \sigma = c\tau \), which implies \( t = \tau \). The string coordinates can thus be rewritten as

\[X^\mu(\tau, \sigma) = X^\mu(t, \sigma) = \{ct, \vec{X}(t, \sigma)\},\]  

(3.47)

where \( \vec{X} \) represents the spatial coordinates. The derivatives for these coordinates are

\[\frac{\partial X^\mu}{\partial \sigma} = \left(\frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma}\right) = (0, \frac{\partial \vec{X}}{\partial \sigma}),\]  

(3.48)

\[\frac{\partial X^\mu}{\partial \tau} = \left(\frac{\partial X^0}{\partial t}, \frac{\partial \vec{X}}{\partial t}\right) = (c, \frac{\partial \vec{X}}{\partial t}).\]  

(3.49)

These derivatives will be used in section 3.8.

### 3.7 The equivalence of tension and mass of a string

Consider a string with endpoints attached to \( X^1 = 0 \) and \( X^1 = a \), respectively. All other coordinate values are set to zero: \( X^2 = X^3 = ... = X^d = 0 \). The string is described in the static gauge. This means that \( X^0 = c\tau \) and thus that the string is static in our reference frame. The spatial coordinate \( X^1 \) is parameterized by

\[X^1(t, \sigma) = f(\sigma), \quad \text{where} \quad f(0) = 0 \quad \text{and} \quad f(\sigma_1) = a.\]  

(3.50)

One can check that this configuration satisfies the equation of motion (3.39). Between the endpoints, \( f(\sigma) \) must be a strictly increasing and continuous function to assure that every point on the string has an unique value of \( \sigma \). The derivatives of the spacetime coordinates are

\[\dot{X}^\mu = (c, 0, \vec{0}), \quad X''^\mu = (0, f', \vec{0}),\]  

(3.51)

and thus

\[(\dot{X})^2 = -c^2, \quad (X')^2 = (f')^2, \quad \dot{X} \cdot X' = 0.\]  

(3.52)

One can substitute these expressions into the Nambu-Goto action (3.28) to get

\[S = -\frac{T_0}{c} \int_{t_1}^{t_1} dt \int_{\sigma_1}^{\sigma_1} d\sigma \sqrt{0 - (-c^2)(f')^2} = -T_0 \int_{t_1}^{t_1} dt \int_{0}^{\sigma_1} df,\]  

(3.53)

and thus

\[S = -T_0 \int_{t_1}^{t_1} dt (f(\sigma_1) - f(0)) = -T_0 \int_{t_1}^{t_1} dt (-Ta).\]  

(3.54)

Two conclusions can be drawn from this equation. First note that the value of the action is independent of the function \( f(\sigma) \). This confirms the parameterization invariance. Then recall that the action is a path integral over the Lagrangian \( L \). Because the string has no kinetic energy, we have \( L = -V \) and thus

\[S = \int_{t_1}^{t_1} dt(-V).\]  

(3.55)
Comparing this result to equation (3.54) leads to the conclusion

\[ V = T_0 a. \]  

(3.56)

The potential energy of the string is thus proportional to its tension and its length. One can therefore imagine a string as being created from an infinitesimal piece, which is stretched against a force \( T_0 \) over a distance \( a \). The energy created in this way is equivalent to a certain rest mass. The rest mass per unit string length \( \mu_0 \) is

\[ \mu_0 c^2 = V a = T_0, \quad \text{and thus} \quad \mu_0 = \frac{T_0}{c^2}. \]  

(3.57)

This expression shows that rest mass is created because the string has tension. In fact, tension and mass are equivalent. Equation (3.56) further shows why the minus sign in the definition of the Nambu-Goto string action is correct: otherwise, the potential energy would turn out to be negative.

### 3.8 Transverse velocity of the string

In subsection 3.6, the particular parameterization \( X^0 = ct = \sigma \) was used for the relativistic string. With this parameterization, a line of constant \( \tau \) on the string corresponds to the string as observed in the chosen Lorentz frame at time \( t \). Defining a velocity for this string is not as easy as it seems. One could easily think of \( \partial \vec{X}/\partial \tau \) as being the right velocity, but this value depends on the choice of \( \sigma \). From the principle of parameterization invariance follows that this choice is arbitrary and so would be the velocity.

When considering a string at two nearby times, one cannot tell which point from the initial string moved to a point on the final string. This suggests that defining a velocity tangent to the string has no physical meaning. For this reason, the transverse velocity is the only relevant velocity of the string. The transverse velocity is a vector perpendicular to the string and tangent to its world sheet. To define the transverse velocity, one takes the spatial derivative \( \partial \vec{X}/\partial s \) and subtracts the component tangent to the string. In order to do this, one needs to define a unit vector along the string. The length of an infinitesimal vector \( ds \) along the string is given by

\[ ds = |d\vec{X}| = \left| \frac{\partial \vec{X}}{\partial \sigma} \right| d\sigma. \]  

(3.58)

Now consider the quantity \( \partial \vec{X}/\partial s \). Using (3.58), it follows that this is a unit vector

\[ \frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma} \frac{d\sigma}{ds} = 1. \]  

(3.59)

The derivative \( \partial \vec{X}/\partial \sigma \) lies along a line of constant \( \tau \). In the static gauge, lines of constant \( \tau \) are precisely the strings. Therefore,

\[ \frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma} \frac{d\sigma}{ds} \]  

(3.60)

is also tangent to the string. This is precisely the unit vector that had to be found. The transverse velocity \( \vec{v}_\perp \) can now be defined as

\[ \vec{v}_\perp = \frac{\partial \vec{X}}{\partial t} - \frac{\partial \vec{X}}{\partial t} \frac{\partial \vec{X}}{\partial s} \frac{\partial \vec{X}}{\partial s}. \]  

(3.61)

In the definition of the action, the square of this expression is needed:

\[ \vec{v}_\perp^2 = \left( \frac{\partial \vec{X}}{\partial t} \right)^2 - 2 \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2 + \left( \frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2 \]  

(3.62)

One can use equation (3.59) to simplify this expression to

\[ \vec{v}_\perp^2 = \left( \frac{\partial \vec{X}}{\partial t} \right)^2 - \left( \frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2. \]  

(3.63)
Now recall the Nambu-Goto action
\[
S = -\frac{T_0}{c} \int_0^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}
\]  
(3.64)

The quantity under the square root can be reexpressed in terms of \( \vec{v}_\perp \). First, the derivative expressions (3.48) and (3.49) from the static gauge are used to write
\[
(\dot{X})^2 = -c^2 + \left( \frac{\partial \vec{X}}{\partial \tau} \right)^2, \quad (X')^2 = \left( \frac{\partial \vec{X}}{\partial \sigma} \right)^2, \quad \dot{X} \cdot X' = \frac{\partial \vec{X}}{\partial \tau} \cdot \frac{\partial \vec{X}}{\partial \sigma}.
\]  
(3.65)

The argument of the square root can now be rewritten as follows:
\[
(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2 = \left( \frac{d\vec{s}}{d\sigma} \cdot \frac{\partial \vec{X}}{\partial \tau} \right)^2 \left( \frac{d\vec{s}}{d\sigma} \cdot \frac{\partial \vec{X}}{\partial \tau} \right)^2 + c^2 \left( \frac{\partial \vec{X}}{\partial \tau} \right)^2 - \left( \frac{\partial \vec{X}}{\partial \tau} \right)^2 \left( \frac{\partial \vec{X}}{\partial \sigma} \right)^2\sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\]  
(3.66)

The light speed \( c \) is factored out to get
\[
\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2} = c \frac{d\vec{s}}{d\sigma} \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\]  
(3.67)

Substituting this expression into the string action (3.64) gives the Nambu-Goto action in terms of transverse velocity:
\[
S = -T_0 \int dt \int_0^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \frac{d\vec{s}}{d\sigma} \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\]  
(3.68)

In this definition, the Lagrangian is
\[
L = -T_0 \int ds \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\]  
(3.69)

As was demonstrated in subsection 3.7, the rest mass of an infinitesimal piece of string is \( T_0 ds \). Therefore, the Lagrangian is just the rest mass multiplied by a relativistic factor for each infinitesimal piece. This definition is similar to the case of a relativistic point particle (2.5), for which the Lagrangian is given by
\[
L = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\]  
(3.70)

Equation (3.69) can be considered as the generalization of this equation.

### 3.9 Motion of string endpoints

This last subsection discusses two remarkable properties of the motion of string endpoints. Before doing so, the expressions (3.35) and (3.36) for \( P_{\tau \mu} \) and \( P_{\sigma \mu} \), respectively, will be simplified. By using (3.65) and (3.67), it follows that
\[
\begin{align*}
P_{\tau \mu} &= -\frac{T_0}{c} \left( \frac{\partial \vec{X}}{\partial \tau} \cdot \frac{\partial \vec{X}}{\partial \mu} \right) X^\mu - (-c^2 + \left( \frac{\partial \vec{X}}{\partial \tau} \right)^2)X^\mu \frac{d\vec{s}}{d\sigma} \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}} \\
&= -\frac{T_0}{c^2} \left( \frac{\partial \vec{X}}{\partial \tau} \cdot \frac{\partial \vec{X}}{\partial \mu} \right) X^\mu - (-c^2 + \left( \frac{\partial \vec{X}}{\partial \tau} \right)^2) \frac{\partial X^\mu}{\partial \tau} \frac{d\vec{s}}{d\sigma} \sqrt{1 - \frac{\vec{v}_\perp^2}{c^2}}.
\end{align*}
\]  
(3.71)
For the case $\mu = 0$ one has $X^0 = c$ and $\partial X^0 / \partial s = c \partial t / \partial s = 0$. Therefore, the above expression reduces to

$$P^{\sigma \mu} = -\frac{T_0}{c} \left( \frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{\partial \vec{X}}{\partial t} \right) \sqrt{1 - \frac{v^2}{c^2}}. \quad (3.72)$$

A similar calculation for $P^{\tau \mu}$ leads to the following simplified expressions:

$$P^{\tau 0} = \frac{T_0}{c} \frac{ds}{d\sigma} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \vec{P}^{\tau} = \frac{T_0}{c^2} \frac{ds}{d\sigma} \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.73)$$

As discussed before, the motion of individual points on the string has no physical meaning. The endpoints are exceptions to this. Free endpoint conditions require that $P^{\sigma \mu}$ vanishes for every $\mu$. A quick look at (3.72) leads then to the conclusion that

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} = 0 \quad (3.74)$$

at the string endpoints. Because $\frac{\partial \vec{X}}{\partial s}$ is the unit vector tangent to the string, $\frac{\partial \vec{X}}{\partial t}$ is perpendicular to this vector. This implies that the string endpoints move transversely to the string.

Moreover, substituting (3.74) into (3.71) gives

$$P^{\sigma \mu} = -T_0 \sqrt{1 - \frac{v^2}{c^2}} \frac{\partial X^\mu}{\partial s} = 0 \quad (3.75)$$

at the endpoints. If only the spatial coordinates are considered, this expression reduces to

$$\vec{P}^{\sigma} = -T_0 \sqrt{1 - \frac{v^2}{c^2}} \frac{\partial \vec{X}}{\partial s} = 0. \quad (3.76)$$

Because $\frac{\partial \vec{X}}{\partial s} \neq 0$, the only valid conclusion is that

$$v^2 = c^2. \quad (3.77)$$

The string endpoints thus move with the speed of light!
4 Motion of relativistic strings

The aim of this section is to derive the most general solution to the equations of motion for a string with one time dimension and one spatial dimension. In the first two subsections, a slightly different formulation of the Nambu-Goto action will be introduced. The new notation does not contain a square root, which makes it more easy to work with, particularly for the purpose of quantization (although quantization is not relevant in this discussion). After the reformulation of the action, its symmetry properties will be discussed. Thereafter, the equations of motion are derived. The section ends with the most general solution to these equations.

4.1 Reformulation of the relativistic point particle action

Before reformulating the action for a relativistic string, first consider the simple case of a relativistic point particle. Most textbooks use a notation in which \( \hbar = c = 1 \). The action of a relativistic point particle in this notation is

\[
S = -m \int ds,
\]

where the line element \( ds \) of the world line is given by

\[
ds^2 = -g_{\mu\nu}(X) dX^\mu dX^\nu.
\]

The metric \( g_{\mu\nu}(X) \) is the generalization of the Minkowski metric for curved backgrounds. On substituting this expression, the action (4.1) takes the form

\[
S = -m \int d\tau \sqrt{-g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu}.
\]

Now consider the equivalent expression

\[
\tilde{S} = \frac{1}{2} \int d\tau (e^{-1}\dot{X}^2 - m^2 e),
\]

where \( \dot{X}^2 = g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \). The newly introduced quantity \( e(\tau) \) is called the auxiliary field. Note that this formula for the action lacks a square root and is suitable for massless particles. One can check that it leads to the same equations of motion as (4.3). By setting the variational derivative with respect to \( e(\tau) \) equal to zero, one gets

\[
\delta \tilde{S} = \frac{1}{2} \int d\tau (-e^{-2}\dot{X}^2 - m^2) \delta e = 0,
\]

which implies

\[
\dot{X}^2 + e^2 m^2 = 0.
\]

The original action is recovered by solving this equation for \( e(\tau) \) and substituting it back into (4.4):

\[
\tilde{S} = \frac{1}{2} \int d\tau \left( \frac{\dot{X}^2}{\sqrt{-\dot{X}^2/m}} - m^2 (\sqrt{-\dot{X}^2/m}) \right)
\]

\[
= \frac{m}{2} \int d\tau \left( \frac{(\sqrt{-\dot{X}^2})^2}{\sqrt{-\dot{X}^2}} - \sqrt{-\dot{X}^2} \right)
\]

\[
= -m \int d\tau \sqrt{-\dot{X}^2}
\]

\[
= -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} = S.
\]

The new expression for the action should also be reparameterization invariant. To check this, consider an infinitesimal change of reparameterization

\[
t \rightarrow \tau' = t - \xi(t).
\]

\[\text{This subsection is based on chapter 2: The bosonic string from J. Schwarz et al, String theory and M-theory. [4]}\]
The field $X^\mu$ that describes the particle’s trajectory transforms like a world line scalar. This means that $X^\mu(\tau) = X^\mu(\tau')$. Note the difference between $X^\mu$ and the derivative $X^\mu' = dX^\mu/d\tau$. The variation in $X^\mu$ to first order is then

$$
\delta X^\mu = X^\mu(\tau) - X^\mu(\tau)
= X^\mu(\tau + \xi(\tau)) - X^\mu(\tau)
\approx X^\mu(\tau) + \frac{dX^\mu}{d\tau}(\tau) - X^\mu(\tau)
= \xi(\tau)\delta X^\mu(\tau). \quad (4.9)
$$

The auxiliary field $e(\tau)$ transforms at the same time according to $e'(\tau')d\tau' = e(\tau)d\tau$. This equation can be rewritten to obtain an expression for $e'(\tau)$:

$$
e'(\tau')d\tau' = e(\tau)d\tau
\leftrightarrow e'(\tau') = e(\tau)\left(\frac{d\tau'}{d\tau}\right)^{-1}
\leftrightarrow e'(\tau) = \frac{e(\tau + \xi(\tau))}{1 - \xi(\tau)}. \quad (4.10)
$$

The variation of the auxiliary field is

$$
\delta e = e'(\tau) - e(\tau)
= \frac{e(\tau + \xi(\tau))}{1 - \xi(\tau)} - \frac{e(\tau)(1 - \xi(\tau))}{1 - \xi(\tau)}
\approx \frac{e(\tau) + \dot{e}(\tau)\xi(\tau) - e(\tau) + e(\tau)\ddot{\xi}(\tau)}{1 - \xi(\tau)}
= \frac{e(\tau)\ddot{\xi}(\tau) + e(\tau)\ddot{\xi}(\tau)}{1 - \ddot{\xi}(\tau)}
= \frac{\frac{d}{d\tau}(\xi e)}{\dot{e}} = \frac{d}{d\tau}(\xi e). \quad (4.11)
$$

Because the reparameterization (4.8) is to first order, this is equivalent to $\delta e = \frac{d}{d\tau}(\xi e)$. Now consider a flat spacetime, in which $g_{\mu\nu}(X) = \eta_{\mu\nu}$. The final result, however, is true for arbitrary $g_{\mu\nu}$. The variation in $\tilde{S}$, as given by (4.4), is

$$
\delta \tilde{S} = \frac{1}{2} \int d\tau \left( 2e^{-1}(\xi_{\mu\nu}\dot{X}^\mu)\delta X^\nu + (-e^{-2}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - m^2)\delta e \right)
= \frac{1}{2} \int d\tau \left( \frac{2\dot{X}^\mu\dot{X}_\mu}{e} - \frac{\dot{X}^\mu\dot{X}_\mu}{e^2} - \delta e - m^2 \delta e \right). \quad (4.12)
$$

Note that $\delta X_\mu = \frac{d}{d\tau}\delta X_\mu$. One can use equations (4.9) and (4.11) to rewrite the above expression as

$$
\delta \tilde{S} = \frac{1}{2} \int d\tau \left( \frac{2\dot{X}^\mu}{e}(\xi\dot{X}_\mu + \dot{\xi}\dot{X}_\mu) - \frac{\dot{X}^\mu\dot{X}_\mu}{e^2} (\dot{\xi} + \dot{e}) - m^2 \frac{d(\xi e)}{d\tau} \right). \quad (4.13)
$$

Terms with a total derivative vanish because the variation at the endpoints is zero. Therefore, the third term drops out. The remaining terms can be taken together as another total derivative

$$
\delta \tilde{S} = \frac{1}{2} \int d\tau \frac{d}{d\tau}(\xi\dot{X}_\mu), \quad (4.14)
$$

and therefore $\delta \tilde{S} = 0$. This confirms that the action $\tilde{S}$ is invariant under reparameterizations.
4.2 Reformulation of the relativistic string action

In section 3, the Nambu-Goto action was used as the action for a relativistic string, propagating through a D-dimensional Minkowski-space. With the convention $c = 1$ and without explicitly writing it as a double integral, the Nambu-Goto action is

$$S = -T_0 \int d\sigma d\tau \sqrt{(\dot{X} \cdot \dot{X})^2 - (\dot{X})^2 (X')^2}. \quad (4.15)$$

Just as for the case of the point particle, it is possible to write an equivalent form of this action. This action is called the string sigma model action and is defined as

$$S = -\frac{T_0}{2} \int d\tau d\sigma \sqrt{-hh^{ab}(\sigma)(X)\partial_{\sigma^a}X^\mu \partial_{\sigma^b}X^\mu}. \quad (4.16)$$

In this definition, the auxiliary world sheet metric $h_{ab}(\sigma, \tau)$ plays the same role as the auxiliary field $e(\tau)$ in the case of the point particle. Further,

$$h = \det h_{ab}, \quad h^{ab} = (h^{-1})_{ab}. \quad (4.17)$$

This action is equivalent to the Nambu-Goto action and is simpler to work with, because it has no square root over the derivatives.

4.3 Symmetries in the string action

In this section, it will be demonstrated why string theory uses strings (and no higher dimensional objects) to replace point particles. One can do this with the use of symmetries. A symmetry of the system means that the action can be reformulated in such a way that it still represents the same physical situation. An extended object with $n$ dimensions sweeps out an $(n+1)$-dimensional volume in spacetime. For such an object, one can generalize (4.16) for flat spacetime to

$$S = -\frac{T_0}{2} \int d\sigma^{n+1} \sqrt{-hh^{\alpha\beta}(\sigma)(X)\partial_{\alpha}X^\mu \partial_{\beta}X^\mu}, \quad (4.18)$$

where $\sigma^0 = \tau$ and the spatial coordinates $\sigma^i (i = 1, 2, \ldots, n)$ describe the $n$-dimensional object. The matrix $h_{ab}$ is a symmetric $(d \times d)$-matrix, which has $\frac{1}{2}d(d+1)$ independent components. Because $d = n+1$, the metric $h_{ab}$ has $\frac{1}{2}(n+1)(n+2)$ independent components.

One can distinguish global and local symmetries of the string sigma model action. Global symmetries represent symmetries of the background, while local symmetries reflect symmetries of the string itself. The local symmetries can be used to choose a gauge, especially one that makes the calculations particulary simple. There are two kinds of local symmetries:

1) Reparameterizations

A change in parameters, describing the world sheet, does not change the action. This was already demonstrated in section 3. Formally stated, the transformations

$$\sigma^\alpha \to \sigma'^\alpha = f^\alpha (\sigma), \quad h_{ab}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^a} \frac{\partial f^\delta}{\partial \sigma^b} h_{\gamma\delta}(\sigma'), \quad (4.19)$$

leave the action invariant. For all $n+1$ dimensions of the physical spacetime, there is an independent reparameterization gauge invariance.

---

1This subsection is based on [4].
2This subsection is mainly based on chapter 2: Free bosonic strings from E. Witten et al, Superstring theory, volume 1. [5]
2) Weyl transformations

The auxiliary metric can be rescaled in the following way:

$$ h_{\alpha\beta} \rightarrow \Lambda(\sigma) h_{\alpha\beta}, \quad \delta X^\mu = 0, \quad (4.20) $$

without changing the action. This invariance applies only for the case of strings, as will be shown later on.

Each of the parameterization gauge invariances can be used to eliminate an independent component from the metric $h_{\alpha\beta}$. After they are used, $\frac{1}{2} n(n+1)$ components remain. This is why it is possible to totally eliminate $h_{\alpha\beta}$ (corresponding to $e$) in the case of the point particle, where $n = 0$. For higher dimensional objects, this possibility does not exist. In the case of a string however, one can use the Weyl-scaling to eliminate the last remaining component. Weyl-scaling implies a transformation

$$ \sqrt{-h^{\alpha\beta}} \rightarrow \Lambda^{\frac{1}{2}(n+1)-1} \sqrt{-h^{\alpha\beta}}. \quad (4.21) $$

To see this, first note that

$$ \sqrt{-h^{\alpha\beta}} = \sqrt{-\det h_{\alpha\beta}(h_{\alpha\beta})^{-1}}. \quad (4.22) $$

The matrix $h_{\alpha\beta}$ is a square $(n+1) \times (n+1)$-matrix, where $n$ stands for the dimension of the spatial object considered. For $a \in \mathbb{R}$, such a matrix has the property $\det(a h_{\alpha\beta}) = a^{n+1} \det h_{\alpha\beta}$. One can use this property to write the transformation as

$$ \sqrt{-\det h_{\alpha\beta}(h_{\alpha\beta})^{-1}} \rightarrow \sqrt{-\det(\Lambda h_{\alpha\beta})(\Lambda h_{\alpha\beta})^{-1}} $$

$$ = \sqrt{-\Lambda^{n+1} \det h_{\alpha\beta} \Lambda^{-1} h^{\alpha\beta}} $$

$$ = \Lambda^{\frac{1}{2}(n+1)-1} \sqrt{-h^{\alpha\beta}}. \quad (4.23) $$

The only possibility that leaves the original expression invariant is $n = 1$: the case of the string. The vanishing of $h_{\alpha\beta}$ enormously simplifies further calculations. Arguments from quantum field theory confirm that membranes ($n = 2$) or higher-dimensional objects are not suitable for quantization of the system. These arguments will not be discussed here because they reach beyond the scope of this project.

### 4.4 Deriving the equation of motion

The three local symmetries for a string can be used to choose all the three independent elements of $h_{\alpha\beta}$. With the following choice of gauge

$$ h^{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.24) $$

the string sigma model action (4.16) simplifies to

$$ S = -\frac{T_0}{2} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\mu $$

$$ = -\frac{T_0}{2} \int d\sigma d\tau \left[ \left( \frac{\partial X^\mu}{\partial \tau} \right)^2 + \left( \frac{\partial X^\mu}{\partial \sigma} \right)^2 \right]. \quad (4.25) $$

The equation of motion is derived by taking the variation. Thereby using the notation $\partial X^\mu / \partial \tau = \dot{X}^\mu$ and $\partial X^\mu / \partial \sigma = X^{\mu'}$, it follows that

$$ \delta S = \int d\tau d\sigma \left[ \dot{X}^\nu \delta X^\mu - X^{\mu'} \delta X^{\mu'} \right]. \quad (4.26) $$

---

This subsection is based on [5].
Integrating by parts gives
\[
\delta S = T_0 \int d\tau d\sigma \left[ \frac{d}{d\tau} (X^\mu \delta X^\nu) - X^\mu \delta X^\nu \right] + \left( - \frac{d}{d\sigma} (X^\mu \delta X^\nu) + X^\nu \delta X^\mu \right)
\]
\[
= T_0 \int d\tau d\sigma [ (X^\mu \delta X^\nu) \bigg|_{\tau=T} - (X^\mu \delta X^\nu) \bigg|_{\tau=T_0} ] + T_0 \int d\sigma [ \delta X^\mu (X^\nu \delta X^\mu) \bigg|_{\sigma=0} ] - T_0 \int d\tau [ X^\nu \delta X^\mu \bigg|_{\nu=0, \sigma=0} ].
\] (4.27)

All the three terms must vanish separately. Setting the first term zero gives the equation of motion:
\[
\Box X^\mu \equiv (\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}) X^\mu = 0.
\] (4.28)

This is an ordinary wave equation, just like the that was found for the classical string in section 1. The second term vanishes because the variation at the endpoints is always zero. Finally, equating the third term zero defines a boundary condition for an open string. The third term automatically vanishes for a closed string, because then \( \sigma = 0 \) and \( \sigma = \pi \) represent the same point.

### 4.5 Solutions to the two-dimensional wave equation

To solve the wave equation one must implement the right boundary conditions. There are two types of boundary conditions to consider: one for open strings and one for closed strings.

In the closed string case, there is a periodicity \( X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + a\pi) \). Because the wave equation is a partial differential equation of second order, one tries a solution which is the product of two functions that each depend on one variable: \( X^\mu(\tau, \sigma) = f(\tau) g(\sigma) \). Substituting this solution into (4.28) gives
\[
\frac{1}{g(\sigma)} \frac{\partial^2 g(\sigma)}{\partial\sigma^2} = \frac{1}{f(\tau)} \frac{\partial^2 f(\tau)}{\partial\tau^2}.
\] (4.29)

The left side of the equation only depends on \( \sigma \), while the right side only depends on \( \tau \). Both sides must therefore be equal to a constant \( k \). The separate equations for \( f(\tau) \) and \( g(\sigma) \) are
\[
\frac{\partial^2 g(\sigma)}{\partial\sigma^2} - k g(\sigma) = 0,
\] (4.30)
\[
\frac{\partial^2 f(\tau)}{\partial\tau^2} - k f(\tau) = 0.
\] (4.31)

There are now three possibilities to consider: \( k > 0, \ k = 0 \) and \( k < 0 \). By choosing the sign to be positive, \( k = \tilde{n}^2 \), one gets the following general solution:
\[
X^\mu(\tau, \sigma) = f(\tau) g(\sigma) = (A e^{i\tilde{n}\tau} + B e^{-i\tilde{n}\tau})(C e^{i\tilde{n}\sigma} + D e^{-i\tilde{n}\sigma}).
\] (4.32)

This solution, however, is not consistent with the periodicity condition \( X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + a\pi) \). For both very small and very large values of \( \sigma \), it diverges to infinity. In the case of a negative separation constant, \( k = -\tilde{n}^2 \), the general solution is trigonometric. One can also express such a solution in terms of complex powers of \( e^A + B e^{-A} \):
\[
X^\mu(\tau, \sigma) = f(\tau) g(\sigma) = (A e^{i\tilde{n}\tau} + B e^{-i\tilde{n}\tau})(C e^{i\tilde{n}\sigma} + D e^{-i\tilde{n}\sigma})
\]
\[
= A C e^{i\tilde{n}(\tau + \sigma)} + A D e^{i\tilde{n}(\tau - \sigma)} + B C e^{-i\tilde{n}(\tau + \sigma)} + B D e^{-i\tilde{n}(\tau - \sigma)}
\]
\[
= F e^{-i\tilde{n}(\tau + \sigma)} + G e^{-i\tilde{n}(\tau - \sigma)}
\]
\[
\tilde{n} \in \mathbb{N}, \quad \tilde{n} \in \mathbb{N}, \quad \tilde{n} \in \mathbb{Z} \setminus 0,
\] (4.33)

where all the capital letters represent arbitrary constants. The periodicity condition now requires
\[
X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + a\pi)
\]
\[
\Leftrightarrow e^{-i\tilde{n}(\tau + \sigma)} = e^{-i\tilde{n}(\tau + \sigma + a\pi)}
\]
\[
\Leftrightarrow \cos \tilde{n}(\tau \pm \sigma) - i \sin \tilde{n}(\tau \pm \sigma) = \cos \tilde{n}(\tau \pm \sigma + a\pi) - i \sin \tilde{n}(\tau \pm \sigma + a\pi).
\] (4.34)
The last equation implies that $\tilde{n} = \frac{2n}{a}$ with $n \in \mathbb{Z} \setminus 0$ and so it follows that
\[
X^\mu(\tau, \sigma) = Fe^{-\frac{2\pi}{a}(\tau+\sigma)} + Ge^{-\frac{2\pi}{a}(\tau-\sigma)} \quad n \in \mathbb{Z} \setminus 0. \tag{4.35}
\]
In the special case $k = 0$, the function $X^\mu$ has the following general form:
\[
X^\mu(\tau, \sigma) = A\tau + B\sigma + C\tau\sigma + D, \tag{4.36}
\]
with $A,B,C$ and $D$ arbitrary constants. This solution should also be consistent with the periodicity requirement. This can only be the case if $B = -C\tau$. Therefore,
\[
X^\mu(\tau, \sigma) = A\tau + D. \tag{4.37}
\]
The solutions (4.35) and (4.37) are normal modes of the string. Any possible oscillation must be a linear combination of these modes. One can write the most general solution to the wave equation as a Fourier series
\[
X^\mu = x_0^\mu + \frac{p^\mu}{\pi a} \tau + i \sum_{n \neq 0} \left( x_n^\mu e^{-\frac{2\pi}{a}(\tau-\sigma)} + \bar{x}_n^\mu e^{-\frac{2\pi}{a}(\tau+\sigma)} \right), \tag{4.38}
\]
where $x_n^\mu$ and $\bar{x}_n^\mu$ are the Fourier coefficients and $A = \frac{p^\mu}{\pi a}$ and $D = x_0^\mu$ are chosen for later convenience. The factor $x_0^\mu$ turns out to be the center of mass coordinate of the string, while $p^\mu$ is the center of mass momentum.

In the case of open strings, there is no identification $\sigma \sim \sigma + a\pi$. Instead of that, Neumann boundary condition are implemented at the endpoints. Remember that this boundary condition prevents momentum from flowing out of the string. This is formally stated by
\[
\frac{\partial X^\mu}{\partial \sigma} = 0 \quad \text{for} \quad \sigma = 0, a\pi. \tag{4.39}
\]
Because the general solution is written as a product $X^\mu(\tau, \sigma) = f(\tau)g(\sigma)$, this translates to the requirement
\[
\frac{dg(\sigma)}{d\sigma} = 0 \quad \text{for} \quad \sigma = 0, a\pi. \tag{4.40}
\]
Recall the separation constant $k$ in equations (4.30) and (4.31). Consider the case of $k = \tilde{n}^2$, for which the general solution is (4.32). The derivative of $g$ with respect to $\sigma$ gives
\[
\frac{dg(\sigma)}{d\sigma} = \tilde{n}(Ce^{i\sigma} - De^{-i\sigma}). \tag{4.41}
\]
The condition for $\sigma = 0$ requires
\[
\frac{dg(0)}{d\sigma} = \tilde{n}(C - D) = 0, \tag{4.42}
\]
which is only satisfied when $C = D$. Applying this result to the boundary condition at $a\pi$ gives the requirement
\[
\frac{dg(a\pi)}{d\sigma} = C\tilde{n}(e^{i\pi a\pi} - e^{-i\pi a\pi}) = 0, \tag{4.43}
\]
which is only satisfied if $\tilde{n} = 0$. This brings us to the special case $k = 0$, in which the solution has the same form as for the closed string case: $X^\mu = A\tau + D$. The final possibility is $k = -\tilde{n}^2$. The derivative of $g(\sigma)$ in the general solution (4.33) is
\[
\frac{dg(\sigma)}{d\sigma} = i\tilde{n}(Ce^{i\sigma} - De^{-i\sigma}). \tag{4.44}
\]
For the first endpoint, $\sigma = 0$, this gives $\frac{dg(0)}{d\sigma} = i\tilde{n}(C - D)$=0, which implies that $C = D$. Substituting this result into the boundary condition for $\sigma = a\pi$ gives
\[
\frac{dg(a\pi)}{d\sigma} = 0 \iff i\tilde{n}C(e^{i\pi a\pi} - e^{-i\pi a\pi}) = 0 \iff e^{i\pi a\pi} - e^{-i\pi a\pi} = 0 \iff \cos(\tilde{n}a\pi) + i \sin(\tilde{n}a\pi) - \cos(\tilde{n}a\pi) - i \sin(\tilde{n}a\pi) = 0 \iff 2i \sin(\tilde{n}a\pi) = 0. \tag{4.45}
\]
From the last equation follows that \( n = \frac{a}{a} \) with \( n \in \mathbb{Z} \setminus 0 \). The general solution to the wave equation for an open string is then

\[
X^\mu = x^\mu_0 + \frac{p^\mu}{\pi T} \tau + i \sum_{n \neq 0} \left( x^\mu_n e^{-i \frac{n a}{2} (\tau - \sigma)} + \tilde{x}^\mu_n e^{-i \frac{n a}{2} (\tau + \sigma)} \right).
\]

(4.46)

### 4.6 Solutions in terms of light-cone coordinates

Each solution of the wave equation can be written as the sum of a left-moving mode and a right-moving mode by using the light-cone coordinates

\[
\sigma^- = \tau - \sigma, \quad \sigma^+ = \tau + \sigma,
\]

(4.47)

Their name comes from the fact that the axes, in these coordinates, represent light paths through the origin. This can easily be seen from the following. The \( \sigma^+ \)-axis (where \( \sigma^- = 0 \)) represents the line \( \tau = \sigma \). Similarly, \( \tau = -\sigma \) on the \( \sigma^- \)-axis. The speed in both cases is \( |\vec{v}| = \frac{\sigma}{\tau} = 1 \). Because \( c = 1 \) in these coordinates, the axes represent light paths and together thus form the light cone. With the use of light-cone coordinates, the general closed string solution (4.38) can be rewritten as a sum

\[
X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-),
\]

(4.48)

where

\[
X^\mu_L(\sigma^+) = \frac{1}{2} x^\mu_0 + \frac{p^\mu}{2\pi T} \tau^+ + i \sum_{n \neq 0} x^\mu_n e^{-i \frac{n a}{2} \sigma^+},
\]

(4.49)

\[
X^\mu_R(\sigma^-) = \frac{1}{2} x^\mu_0 + \frac{p^\mu}{2\pi T} \tau^- + i \sum_{n \neq 0} x^\mu_n e^{-i \frac{n a}{2} \sigma^-}.
\]

(4.50)

Note that \( X^\mu_L \) and \( X^\mu_R \) are only functions of \( \sigma^- \) and \( \sigma^+ \), respectively. The corresponding derivatives in these coordinates are

\[
\partial_\pm = \frac{1}{2} (\partial_\tau \pm \partial_\sigma).
\]

(4.51)

### 4.7 Constraints to the string motion

An important constraint on the string motion that needs to be considered follows from the energy-momentum tensor. This tensor is a general concept in physics and is defined as

\[
T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h_{\alpha\beta}},
\]

(4.52)

where \( \frac{\delta S}{\delta h^{\alpha\beta}} \) is the variational derivative of the action with respect to the two dimensional metric \( h_{\alpha\beta} \). The \( T \) in this definition has nothing to do with the string tension. In order to perform the variation, one uses the following formula:

\[
\delta h = -hh_{\alpha\beta} \delta h^{\alpha\beta},
\]

(4.53)

which implies

\[
\delta \sqrt{-h} = \frac{d}{dh} \sqrt{-h} \delta h = -\frac{1}{2} \sqrt{-h} h_{\alpha\beta} \delta h^{\alpha\beta}.
\]

(4.54)

Now recall the relativistic string action as defined in the previous section

\[
S = -\frac{T_0}{2} \int d\tau d\sigma \sqrt{-h} \delta h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu.
\]

(4.55)

\(^1\)This subsection is based on [5].
Its variation is given by
\[
\delta S = -\frac{T_0}{2} \int d\tau d\sigma \left( \sqrt{-h} \partial_\alpha X^\mu \partial_\beta X_\mu \delta h^{\alpha\beta} + h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \delta \sqrt{-h} \right)
\]
\[
= -\frac{T_0}{2} \int d\tau d\sigma \left( \sqrt{-h} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \right) \delta h^{\alpha\beta}.
\] (4.56)

Now using the formula for the energy-momentum tensor (4.52) gives
\[
T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\mu\nu} \partial_\mu X^\nu \partial_\beta X_\mu.
\] (4.57)

In addition, the Euler-Lagrange equation is
\[
\frac{\partial S}{\partial h_{\alpha\beta}} - \frac{d}{d\mu} \left( \frac{\partial S}{\partial \partial_\mu h_{\alpha\beta}} \right) = 0
\]
\[
\Leftrightarrow \frac{\partial S}{\partial h_{\alpha\beta}} = 0.
\] (4.58)

Together with (4.52), this equation implies that \(T_{\alpha\beta} = 0\). This constraint can be expressed in terms of the coordinates \(\tau\) and \(\sigma\) but also in light-cone coordinates. Consider the first case. Remember that the symmetries in the sigma string model action allowed for choosing \(h_{\alpha\beta} = \eta_{\alpha\beta}\). Then
\[
T_{00} = \dot{X}^2 - \frac{1}{2}(\dot{X}^2 - X^2) = \frac{1}{2}(X^2 + X^2),
\] (4.59)
\[
T_{01} = X' \cdot X',
\] (4.60)
\[
T_{0\tau} = X' \cdot X,
\] (4.61)
\[
T_{11} = X'^2 + \frac{1}{2}(\dot{X}^2 - X^2) = \frac{1}{2}(\dot{X}^2 + X^2).
\] (4.62)

The vanishing of \(T\) gives the following constraints:
\[
X \cdot X' = 0,
\] (4.63)
\[
\dot{X}' + X'^2 = 0.
\] (4.64)

Now consider the case of light-cone coordinates. The metric can be rewritten as follows:
\[
h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu = \partial_\alpha X^\mu \partial_\alpha X^\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu
\]
\[
= \partial_\alpha X^\mu \partial_\alpha X^\mu + \partial_\alpha X^\mu \partial_\alpha X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu
\]
\[
= \left( \partial_\alpha X^\mu \partial_\alpha X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu \right)
\]
\[
+ \left( \partial_\alpha X^\mu \partial_\alpha X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu \right)
\]
\[
= \partial_\alpha X^\mu \partial_\alpha X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu.
\] (4.65)

The inverse of the auxiliary metric in light-cone coordinates is thus
\[
h^{\alpha\beta} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.
\] (4.66)

Using this result and the vanishing of the energy-momentum tensor gives
\[
T_{++} = \partial_+ X^\mu \partial_+ X_\mu = X^2_L = 0,
\] (4.67)
\[
T_{--} = \partial_- X^\mu \partial_- X_\mu = X^2_R = 0.
\] (4.68)

To find the other matrix elements, consider the following product:
\[
h^{\alpha\beta} T_{\alpha\beta} = h^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X_\mu) - \frac{1}{2} (h^{\alpha\beta} h_{\alpha\beta}) (h^{\mu\nu} \partial_\alpha X^\mu \partial_\beta X_\nu)
\]
\[
= h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu
\]
\[
= 0.
\] (4.69)

where was used that \(h^{\alpha\beta} h_{\alpha\beta} = \delta^2_{\mu} = 2\) (this is a two-dimensional delta function). Because the elements \(h^{++}\) and \(h^{--}\) are nonzero, this implies that \(T_{++} = T_{--} = 0\).
4.8 Noether’s theorem and conserved quantities

Subsection 4.3 discussed the symmetries of the relativistic string action. These symmetries can be used to find the constants of motion. If the Lagrangian of a system does not explicitly contain a particular coordinate of displacement, then a corresponding canonical momentum is conserved. The absence of explicit dependence means that the Lagrangian is unaffected by a transformation that alters the value of that coordinate. One calls the Lagrangian invariant or symmetric under the transformation. Invariance of the Lagrangian under time displacement, for example, implies conservation of energy. Invariance under a linear transformation implies conservation of linear momentum. If the Lagrangian is invariant under a rotation, the angular momentum is conserved. The formal description of the relationship between invariance properties and conserved quantities is given by Noether’s theorem.

With Noether’s method, one constructs a conserved current \( J_\alpha \) that is associated with a global symmetry transformation \( \phi(\sigma) \rightarrow \phi(\sigma) + \epsilon \delta \phi(\sigma) \), where \( \phi(\sigma) \) is any field in the theory and \( \epsilon \) is an infinitesimal parameter. Now consider the more general transformation

\[
\phi(\sigma) \rightarrow \phi(\sigma) + \epsilon(\sigma) \delta \phi(\sigma).
\]

(4.70)

In this case, \( \epsilon \) is not constant but an arbitrary function of \( \sigma \). The action is generally not invariant under such a transformation, because it is only invariant for constant \( \epsilon \). Now assume that \( \epsilon \) is a linear function of \( \sigma \). The action would then be invariant if its variation is proportional to the first derivative of \( \epsilon \). It can be written in the following form:

\[
\delta S = \int d\sigma d\tau J_\alpha \partial^\alpha \epsilon.
\]

(4.71)

In this expression, \( J_\alpha \) is the conserved ‘current’. One can see this from rewriting the above equation as follows:

\[
\delta S = \int d\sigma d\tau J_\alpha \partial^\alpha \epsilon = \int d\sigma d\tau (\partial_\alpha J_\alpha \epsilon) - \epsilon \partial_\alpha J_\alpha = 0.
\]

(4.72)

The integral over the first term in brackets is zero because it is a total derivative. Note that \( \epsilon \) in the second term is arbitrary. The variation is therefore only zero if \( \partial_\alpha J_\alpha = 0 \). Equation (4.71) also holds for \( \epsilon(\sigma) \) that are higher order functions of \( \sigma \) (for which \( \partial^\alpha \epsilon \) obviously is not constant). Using integration by parts, an integral over a term of order \( n \) can be rewritten as

\[
\delta S = \int d\sigma d\tau J_\alpha (\partial^\alpha \epsilon)^n = \int d\sigma d\tau \left[ \partial_\alpha (J_\alpha (\partial^\alpha \epsilon)^{n-1} \epsilon) - \partial^\alpha J_\alpha (\partial^\alpha \epsilon)^{n-1} \epsilon \right] = -\int d\sigma d\tau \partial^\alpha J_\alpha (\partial^\alpha \epsilon)^{n-1} \epsilon.
\]

(4.73)

This procedure can be repeated until the partial derivative is of first order. The more general form of (4.71) with higher order terms is

\[
\delta S = \int d\sigma d\tau \left( J_\alpha \partial^\alpha \epsilon + J_{\beta\alpha} \partial^{\beta} \epsilon + J_{\gamma\beta\alpha} \partial^{\gamma\beta} \epsilon + \ldots \right),
\]

(4.74)

where \( \partial^\alpha \partial^\beta = \partial^{\alpha\beta} \) and \( \partial^\alpha \partial^\beta \partial^\gamma = \partial^{\alpha\beta\gamma} \). The result of repeatedly integrating this equation by parts is

\[
\delta S = \int d\sigma d\tau \left( J_\alpha - \partial^\alpha J_{\beta\alpha} + \partial^{\beta\gamma} J_{\gamma\beta\alpha} + \ldots \right) \partial^\alpha \epsilon \equiv \int d\sigma d\tau J_\alpha \partial^\alpha \epsilon,
\]

(4.75)

which has exactly the same form as (4.71). Now recall the relativistic string action

\[
S = -\frac{T_0}{2} \int d\sigma d\tau \eta^{\mu\nu} \partial_\mu X^\nu \partial_\nu X_\mu,
\]

(4.76)

and consider a linear transformation

\[
X_\mu \rightarrow X_\mu + \epsilon_\mu(\sigma).
\]

(4.77)

\(^{1}\)This subsection is mainly based on [5] and [6].
The variation of the action is
\[
\delta S = -T_0 \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \delta(\partial_\beta X_\mu)
\]
\[
= -T_0 \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu
\]
\[
= -T_0 \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta \epsilon_\mu
\]
\[
= -T_0 \int d\sigma d\tau \partial_\alpha X^\mu \partial^\alpha \epsilon_\mu.
\] (4.78)

The last expression has the form (4.71), and thus
\[
J_\alpha = -T_0 \partial_\alpha X^\mu.
\] (4.79)

This is the current associated with translation invariance. The linear momentum (the conserved quantity) is found by integrating this expression over \(\sigma\) at \(\tau = 0\):
\[
P^\mu = T_0 \int_0^\pi d\sigma \left( \frac{\partial X^\mu}{\partial \sigma} + \frac{\partial X^\mu}{\partial \tau} \right) = T_0 \int_0^\pi d\sigma X^\mu.
\] (4.80)

It is possible to do a similar calculation for rotations. These are characterized by the transformation
\[
X \rightarrow X + \epsilon Y \equiv X',
\]
\[
Y \rightarrow Y - \epsilon X \equiv Y'.
\] (4.81)

(4.82)

One can use the metric \(\eta^{\alpha\beta}\) to change the partial derivative from \(\partial_\beta\) to \(\partial^\alpha\):
\[
S = -\frac{T_0}{2} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \partial^\beta X_\mu = -T_0 \int d\sigma d\tau \partial^\alpha X_\mu \partial_\alpha X^\mu.
\] (4.83)

Note that a rotation only alters the value of the coordinates \(X\) and \(Y\). The upper and lower index values for these coordinates are the same. The variation of the action is then given by
\[
\delta S = -T_0 \int d\sigma d\tau \left( \partial_\alpha X \delta(\partial^\alpha X) + \partial_\alpha Y \delta(\partial^\alpha Y) \right)
\]
\[
= -T_0 \int d\sigma d\tau \left( \partial_\alpha X \partial^\alpha \delta(X) + \partial_\alpha Y \partial^\alpha \delta(Y) \right)
\]
\[
= -T_0 \int d\sigma d\tau \left( \partial_\alpha X (Y \partial^\alpha \epsilon + \epsilon \partial^\alpha Y) + \partial_\alpha Y (-X \partial^\alpha \epsilon - \epsilon \partial^\alpha X) \right)
\]
\[
= -T_0 \int d\sigma d\tau \left( Y \partial_\alpha X - X \partial_\alpha Y \right) \partial^\alpha \epsilon.
\] (4.84)

The conserved current for rotations is thus
\[
J_\alpha = T_0 (X \partial_\alpha Y - Y \partial_\alpha X).
\] (4.85)

The angular momentum is found by integrating this expression over \(\sigma\) at \(\tau = 0\):
\[
J = T_0 \int_0^\pi d\sigma \left( \frac{\partial Y}{\partial \tau} + X \frac{\partial Y}{\partial \sigma} - Y \frac{\partial X}{\partial \tau} - Y \frac{\partial X}{\partial \sigma} \right) = T_0 \int_0^\pi d\sigma \left( XY - YX \right).
\] (4.86)
5 Semiclassical splitting of a closed string

The preceding sections provide all the necessary tools for a semiclassical description of the particle decay. This process is represented by the splitting of a closed string. The first subsection discusses a particular solution to the equation of motion. Then, the description of the splitting process is given and there will be a discussion about the conserved quantities. Finally, the outgoing string solutions are expressed in terms of Fourier series.

5.1 A closed rotating string solution with maximum angular momentum

Consider a solution to the wave equation (4.28) for a closed rotating string. It is parameterized as

\[ X = L \cos(2\sigma) \cos(2\tau), \quad Y = L \cos(2\sigma) \sin(2\tau), \quad X^0 = 2L\tau, \]

where \( \sigma \in [0, \pi) \). This solution represents the state of maximum angular momentum. This statement will not be proved here, but it can be made clear intuitively. A point on the string of constant \( \sigma \) moves on a circle of radius \( L \cos(2\sigma) \) around the origin. All points have the same time dependence. Therefore, the collection of points on the string can be imagined as a straight line rotating around the origin. The circular string is exactly folded on this line. Because the string is stretched as far as possible, the angular momentum has the maximum possible value. The world sheet of this string for a short time interval is shown in figure 6.

One can check that the constraints (4.63) and (4.64) are satisfied for this solution. With

\[ X = (X^0, X, Y) = (2L\tau, L \cos(2\sigma) \cos(2\tau), L \cos(2\sigma) \sin(2\tau)), \]

the derivatives are

\[ \dot{X} = (2L, -2L \cos(2\sigma) \sin(2\tau), 2L \cos(2\sigma) \cos(2\tau)), \]

\[ X' = (0, -2L \sin(2\sigma) \cos(2\tau), -2L \sin(2\sigma) \sin(2\tau)). \]

Figure 6: World sheet of the folded rotating string. For a fixed instant of time, the string is a straight line of length \( 2L \). It is represented by any of the horizontal lines.

One section is based on the article Semiclassical decay of strings with maximum angular momentum from R. Iengo, J. Russo. [2]
Substituting these expressions into the constraints gives

\[ X \cdot X + X' \cdot X' = 4L^2(-1 + \cos^2(2\tau) \sin^2(2\tau) + \cos^2(2\tau) \cos^2(2\tau)) \]
\[ + 4L^2(\sin^2(2\tau) \cos^2(2\tau) + \sin^2(2\tau) \sin^2(2\tau)) \]
\[ = 4L^2(-1 + \cos^2(2\tau) + \sin^2(2\tau)) \]
\[ = 4L^2(-1 + 1) \]
\[ = 0, \quad (5.5) \]

and

\[ X \cdot X' = 4L^2(\cos(2\tau) \sin(2\tau) \sin(2\tau) \cos(2\tau) - \cos(2\tau) \cos(2\tau) \sin(2\tau)) \]
\[ = 0. \quad (5.6) \]

Because the string is folded to a line, one can divide it into two segments that overlap: the ‘upper’ segment is described by \( 0 < \sigma < \frac{\pi}{2} \) and the ‘lower’ segment by \( \frac{\pi}{2} < \sigma < \pi \). At \( \tau = 0 \), these segments are stretched between \( X = -L \) and \( X = L \). The total length of the string is therefore \( 4L \).

The formulas from section 4.8 will now be used to calculate the constants of motion. The energy and linear momenta are components of the momentum vector \( P = (p^0, p^1, p^2) = (E, p^\tau, p^y) \). Using formula (4.80) gives

\[ E = T_0 \int_0^{\pi} d\sigma \dot{X}^0 = T_0 \int_0^{\pi} 2Ld\sigma = 2\pi T_0 L, \quad (5.7) \]
\[ p^\tau = T_0 \int_0^{\pi} d\sigma \dot{X} = -2LT_0 \sin(2\tau) \int_0^{\pi} \cos(2\sigma)d\sigma = 0, \quad (5.8) \]
\[ p^y = T_0 \int_0^{\pi} d\sigma \dot{Y} = 2LT_0 \cos(2\tau) \int_0^{\pi} \cos(2\sigma)d\sigma = 0. \quad (5.9) \]

The integrals for \( p^\tau \) and \( p^y \) are zero because the integrated function is periodic. The angular momentum follows from (4.86):

\[ J = T_0 \int_0^{\pi} (XY - \dot{X}Y)d\sigma \]
\[ = 2T_0L^2 \int_0^{\pi} (\cos^2(2\sigma) \cos^2(2\tau) + \cos^2(2\sigma) \sin^2(2\tau))d\sigma \]
\[ = 2T_0L^2 \int_0^{\pi} \cos^2(2\sigma)d\sigma \]
\[ = T_0L^2 \int_0^{\pi} (1 + \cos(4\sigma))d\sigma \]
\[ = T_0L^2 \left[ \sigma + \frac{\sin(2\sigma)}{2} \right]_0^{\pi} \]
\[ = \pi T_0 L^2. \quad (5.10) \]

5.2 The splitting process

Assume that at \( \tau = 0 \) the string splits into two pieces of length \( 4L_I \) and \( 4L_{II} \), respectively, where \( L_I + L_{II} = L \). The new coordinates become \( X^\mu(\sigma) \rightarrow \{ X^\mu_I(\sigma), X^\mu_{II}(\sigma) \} \). The splitting occurs at \( \sigma = \frac{\pi}{2} \) for the upper segment and at \( \sigma = \pi - \frac{\pi}{2} \) for the lower segment, where \( a \) is defined as

\[ \cos(a\pi) \equiv -\frac{L_I - L_{II}}{L}, \quad 0 < a < 1. \quad (5.11) \]

This situation is shown in figure 7.
The initial conditions for both strings at $\tau = 0$ are determined by the string solution (5.1). Continuity is required in the spatial coordinates and their time derivatives. This means for string I

$$X_I^\mu(\sigma,0) = X^\mu(\sigma,0), \quad \dot{X}_I^\mu(\sigma,0) = \dot{X}_I^\mu(\sigma,0),$$

(5.12)

where $0 < \sigma < \frac{a\pi}{2}$ represents the upper segment and $\pi > \sigma > \pi - \frac{a\pi}{2}$ the lower segment. String II has similar equations:

$$X_{III}^\mu(\sigma,0) = X^\mu(\sigma,0), \quad \dot{X}_{II}^\mu(\sigma,0) = \dot{X}_I^\mu(\sigma,0),$$

(5.13)

for $\frac{a\pi}{2} < \sigma < \frac{\pi}{2}$ (upper segment) and $\frac{\pi}{2} < \sigma < \pi - \frac{a\pi}{2}$ (lower segment). These boundary conditions uniquely determine the solution describing both outgoing strings.

### 5.3 The constants of motion

With the information from subsection 5.2 and the formulas following from Noether’s theorem, it is possible to calculate the constants of motion for the outgoing strings. First consider string I. Its momentum vector components at $\tau = 0$ are

$$E_I = 2T_0 \int_0^{\frac{a\pi}{2}} d\sigma \dot{X}_I^0(\sigma,0) = 2T_0 \int_0^{\frac{a\pi}{2}} d\sigma 2L = 2\pi a L T_0,$$

(5.14)

$$p_{xI} = 2T_0 \int_0^{\frac{a\pi}{2}} d\sigma \dot{X}_I(\sigma,0) = 0,$$

(5.15)

$$p_{yI} = 2T_0 \int_0^{\frac{a\pi}{2}} d\sigma \dot{Y}_I(\sigma,0) = 4L T_0 \int_0^{\frac{a\pi}{2}} d\sigma \cos(2\sigma) = 4LT_0 \left[ \frac{\sin(2\pi)}{2} \right]_0^{\frac{a\pi}{2}} = 2LT_0 \sin(\pi a).$$

(5.16)
The angular momentum is given by
\[ J_I = 2T_0 \int_{0}^{\pi} \left( X_I \dot{Y}_I - \dot{X}_I X_I \right) d\sigma \]
\[ = 2T_0 \int_{0}^{\pi} \left( 2L^2 \cos^2(2\sigma) + 2L^2 \cos^2(2\tau) \right) d\sigma \]
\[ = 4L^2 T_0 \int_{0}^{\pi} \cos^2(2\sigma) d\sigma \]
\[ = 2L^2 T_0 \int_{0}^{\pi} [1 + \cos(4\sigma)] d\sigma \]
\[ = \pi a L^2 T_0 + \frac{L^2 T_0}{2} \sin(2\pi a). \] (5.17)

As usual for the angular momentum, one can split it up into a spin component \( S_I \) and an orbital component \( l_I \). First consider the orbital angular momentum. In classical mechanics, it is given by
\[ \vec{I}_I = \vec{r} \times \vec{p} = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ x_0 & y_0 & 0 \\ p_{I}^x & p_{I}^y & 0 \end{vmatrix} = (x_0 p_{I}^y - y_0 p_{I}^x) \hat{z}, \] (5.18)
where \( \vec{r} \) is the center of mass coordinate vector. Because \( p_{I}^z = 0 \), the absolute value of the angular momentum is \( l_I = x_0 p_{I}^y \). The center of mass coordinate \( x_0 \) can be found by using the classical formula
\[ x_0 = \frac{1}{M} \int_{0}^{K} d\sigma X(\sigma, 0) d\sigma, \] (5.19)
where \( M \) is the total mass of the string, \( k \) its length in parameter space and \( \mu_0 \) the mass density per unit length. Assuming the mass density to be constant, \( \mu_0 = 1 \), one finds
\[ x_0 = \frac{1}{K} \int_{0}^{K} d\sigma X(\sigma, 0) = \frac{2L}{\pi a} \int_{0}^{\pi} d\sigma \cos(2\sigma) = \frac{2L}{\pi a} \left[ \frac{\sin(2\sigma)}{2} \right]_0^\pi = \frac{L \sin(\pi a)}{\pi a} = \frac{p_{I}^y}{2\pi a T_0}. \] (5.20)

This expression can then be substituted into the orbital angular momentum formula:
\[ l_I = x_0 p_{I}^y = \frac{(p_{I}^y)^2}{2\pi a T_0} = \frac{2L^2 T_0 \sin^2(\pi a)}{\pi a}. \] (5.21)

Because \( J_I = l_I + S_I \), the spin component is
\[ S_I = L^2 T_0 \left( \pi a + \frac{\sin(2\pi a)}{2} - \frac{2 \sin^2(\pi a)}{\pi a} \right). \] (5.22)

Finally, it is possible to calculate the mass of string I. The relativistic energy-momentum equation is \( E^2 - (pc)^2 = (mc^2)^2 \). Because we have set \( c = 1 \), it follows that
\[ m^2_I = E_I^2 - p_I^2 = E_I^2 - (p_I^y)^2 = 4L^2 T_0^2 \left( a^2 \pi^2 - \sin^2((\pi a)) \right). \] (5.23)

Now consider the conserved quantities for string II at \( \tau = 0 \). The momentum vector is given by
\[ E_{II} = 2T_0 \int_{0}^{\pi} d\sigma X_{II}^0(\sigma, 0) = 2T_0 \int_{0}^{\pi} d\sigma 2L = 2\pi(1-a)LT_0, \]
\[ p_{II}^x = 2T_0 \int_{0}^{\pi} d\sigma X_{II}(\sigma, 0) = 0, \] (5.24)
\[ p_{II}^y = 2T_0 \int_{0}^{\pi} d\sigma Y_{II}(\sigma, 0) = -4L T_0 \int_{0}^{\pi} d\sigma \cos(2\sigma) = -2L T_0 \sin(\pi a). \] (5.25)
The angular momentum for string II is
\[
J_{II} = 2T \int_{0}^{\frac{2\pi}{aT}} (X_{II} \dot{Y}_{II} - \dot{X}_{II} Y_{II}) d\sigma
\]
\[
= 2T \int_{0}^{\frac{2\pi}{aT}} (2L^2 \cos^2(2\sigma) \cos^2(\tau) + 2L^2 \cos^2(2\sigma) \sin^2(2\tau)) d\sigma
\]
\[
= 4L^2 T \int_{0}^{\frac{2\pi}{aT}} \cos^2(2\sigma) d\sigma
\]
\[
= 2L^2 T \int_{0}^{\frac{2\pi}{aT}} [1 + \cos(4\sigma)] d\sigma
\]
\[
= \pi(1-a)L^2 T_0 + \frac{L^2 T_0}{2} \sin(2\pi(1-a))
\]
\[
= \pi(1-a)L^2 T_0 - \frac{L^2 T_0}{2} \sin(2\pi a).
\] (5.26)

As before, the angular momentum can be decomposed into an orbital component and a spin component:
\[
l_{II} = \frac{(p_{II})^2}{2\pi a T_0} = \frac{2L^2 T_0 \sin^2(\pi a)}{\pi(1-a)}
\]
\[
S_{II} = L^2 T_0 (\pi(1-a) + \frac{\sin(2\pi a)}{2} - \frac{2 \sin^2(\pi a)}{\pi(1-a)}).
\] (5.27)

Finally, the mass of string II is
\[
m_{II}^2 = E_{II}^2 - p_{II}^2 = E_{II}^2 - (p_{II})^2 = 4L^2 T_0^2 ((1-a)^2 \pi^2 - \sin^2(\pi a)).
\] (5.28)

One can check that energy, linear momentum and angular momentum are conserved in the process of splitting:
\[
E_I + E_{II} = 2\pi a L T_0 + 2\pi(1-a)L T_0 = 2\pi LT_0 = E,
\] (5.29)
\[
p_I^2 + p_{II}^2 = 2LT_0 \sin(\pi a) - 2LT_0 \sin(\pi a) = 0 = p,
\] (5.30)
\[
J_I + J_{II} = \pi a L^2 T_0 + \frac{LT_0}{2} \sin(2\pi a) + \pi(1-a)L^2 T_0 - \frac{LT_0}{2} \sin(2\pi a) = \pi L^2 T_0 = J.
\] (5.31)

## 5.4 Solutions for the outgoing strings in terms of Fourier series

All coordinates of the outgoing strings should separately obey the two-dimensional wave equation (4.28). First consider string I. Using the most general solution (4.38) for the spatial coordinates gives
\[
X_I(\sigma, \tau) = x_{0I} + \frac{p_{I}}{a \pi T_0} \tau + i \sum_{n \neq 0} (x_n e^{-\frac{2\pi}{a}(\tau - \sigma)} + \bar{x}_n e^{-\frac{2\pi}{a}(\tau + \sigma)}).
\] (5.32)
\[
Y_I(\sigma, \tau) = y_{0I} + \frac{p_{I}}{a \pi T_0} \tau + i \sum_{n \neq 0} (y_n e^{-\frac{2\pi}{a}(\tau - \sigma)} + \bar{y}_n e^{-\frac{2\pi}{a}(\tau + \sigma)}).
\] (5.33)
\[
X_{0I}(\sigma, \tau) = 2L \tau.
\] (5.34)

The constraints \(X \cdot \dot{X} + X' \cdot X' = 0\) and \(Y \cdot Y' = 0\) are automatically satisfied at \(\tau = 0\) because the solutions \(X_I^{n}, X_{0I}^{n}\) then coincide with the original string solution \(X^0\). Moreover, they are satisfied for all \(\tau\), and therefore they are a constant of motion. One can see this from the following. Recall that \(T_{++} = \frac{1}{2}(X \cdot X') + \frac{1}{2}(\dot{X} \cdot \dot{X} + X' \cdot X') = 0\) and \(T_{--} = \frac{1}{2}X_{+}X_{-} = 0\). Because \(T_{++} = T_{++}(\sigma^+)\) and \(T_{--} = T_{--}(\sigma^-)\), these expressions are independent and therefore they must be constant in time. This implies that they vanish for any \(\tau\).

The outgoing string solutions (5.32) and (5.33) are still quite general. The boundary conditions can be used to determine the Fourier coefficients. First consider the time derivative of (5.32):
\[
\dot{X}_I(\sigma, \tau) = \frac{p_{I}}{\pi a T_0} + 2 \sum_{n \neq 0} n(x_n e^{-\frac{2\pi}{a}(\tau - \sigma)} + \bar{x}_n e^{-\frac{2\pi}{a}(\tau + \sigma)}).
\] (5.35)
It was already demonstrated that $p_f^2 = 0$. The boundary condition $X_f(\sigma, 0) = X(\sigma, 0) = 0$ now becomes

$$X_f(\sigma, 0) = \frac{2}{a} \sum_{n \neq 0} n(x_n e^{\frac{2\pi in\sigma}{a}} + \bar{x}_n e^{-\frac{2\pi in\sigma}{a}}) = 0. \quad (5.36)$$

This statement can only be true if each term with positive $n$ is canceled by a corresponding term with negative $n$. Formally stated,

$$x_n e^{\frac{2\pi in\sigma}{a}} + \bar{x}_n e^{-\frac{2\pi in\sigma}{a}} = x_{-n} e^{\frac{2\pi in\sigma}{a}} + \bar{x}_{-n} e^{-\frac{2\pi in\sigma}{a}}. \quad (5.37)$$

This implies

$$x_n = \bar{x}_{-n}. \quad (5.38)$$

It is possible to derive a similar statement for the Fourier coefficients of $Y_f$. The center of mass coordinate $y_{0f}$ is zero, because $Y(\sigma, 0) = 0$ for all values of $\sigma$. The boundary condition $Y_f(\sigma, 0) = Y(\sigma, 0) = 0$ then implies

$$Y_f(\sigma, 0) = i \sum_{n \neq 0} (y_n e^{\frac{2\pi in\sigma}{a}} + \bar{y}_n e^{-\frac{2\pi in\sigma}{a}}) = 0. \quad (5.39)$$

This requirement translates to $y_n e^{\frac{2\pi in\sigma}{a}} + \bar{y}_n e^{-\frac{2\pi in\sigma}{a}} = -(y_{-n} e^{-\frac{2\pi in\sigma}{a}} + \bar{y}_{-n} e^{\frac{2\pi in\sigma}{a}})$ and thus

$$y_n = -\bar{y}_{-n}. \quad (5.40)$$

Before explicitly calculating the Fourier coefficients, recall the general formula for expanding a periodic function $f(x)$ with period $2L$ into a Fourier series. According to Kreyszig [7], these formulas are

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{2\pi inx}{L}} \, dx. \quad (5.41)$$

For our specific case, one can rewrite these expressions into the following form:

$$X_f(\sigma) = \hat{x}_0 + \sum_{n \neq 0} \hat{x}_n e^{\frac{2\pi in\sigma}{a}}, \quad \hat{x}_n = \frac{1}{\pi a} \int_{\frac{\pi}{a}}^{\frac{3\pi}{a}} X_f(\sigma) e^{-\frac{2\pi in\sigma}{a}} \, d\sigma. \quad (5.42)$$

The solution (5.32) has a slightly different normalization. This can be seen from rewriting the solution at $\tau = 0$ with the use of (5.38):

$$X_f(\sigma, 0) = x_{0f} + i \sum_{n \neq 0} (x_n e^{\frac{2\pi in\sigma}{a}} + \bar{x}_n e^{-\frac{2\pi in\sigma}{a}})$$

$$= x_{0f} + i \sum_{n \neq 0} (x_n e^{\frac{2\pi in\sigma}{a}} + x_{-n} e^{\frac{2\pi i(n+1)\sigma}{a}})$$

$$= x_{0f} + 2i \sum_{n \neq 0} x_n e^{\frac{2\pi in\sigma}{a}}. \quad (5.43)$$

A comparison with (5.42) shows that $x_n = -\frac{i}{2} \hat{x}_n$. Therefore,

$$x_n = -\frac{i}{2\pi a} \int_{\frac{\pi}{a}}^{\frac{3\pi}{a}} X_f(\sigma) e^{-\frac{2\pi in\sigma}{a}} \, d\sigma. \quad (5.44)$$
The two remaining boundary conditions can be used for actually calculating the Fourier coefficients \( x_n \) and \( y_n \). The condition \( X_f(\sigma, 0) = X(\sigma, 0) = L \cos 2\sigma \) gives

\[
\begin{align*}
x_n &= -iL \int_{\frac{-\pi}{\alpha}}^{\frac{\pi}{\alpha}} \cos(2\sigma) e^{-\frac{2in\sigma}{\alpha}} \, d\sigma \\
&= -iL \frac{2\alpha}{\pi a} \int_{\frac{-\pi}{\alpha}}^{\frac{\pi}{\alpha}} \cos(2\sigma) \cos\left(\frac{2n\sigma}{a}\right) - i \cos(2\sigma) \sin\left(\frac{2n\sigma}{a}\right) \, d\sigma \\
&= -iL \frac{2\alpha}{\pi a} \int_{\frac{-\pi}{\alpha}}^{\frac{\pi}{\alpha}} \cos(2\sigma) \cos\left(\frac{2n\sigma}{a}\right) \, d\sigma \\
&= -iL \frac{2\alpha}{\pi a} \int_{\frac{-\pi}{\alpha}}^{\frac{\pi}{\alpha}} \cos(2\sigma) \cos\left(\frac{2n\sigma}{a} + 2\sigma + \frac{2n\sigma}{a}\right) \, d\sigma \\
&= -iL \frac{2\alpha}{\pi a} \int_{\frac{-\pi}{\alpha}}^{\frac{\pi}{\alpha}} \cos(2\sigma - \frac{2n\sigma}{a}) + \cos(2\sigma + \frac{2n\sigma}{a}) \, d\sigma \\
&= -iL \left[ \frac{\sin(2\sigma - \frac{2n\sigma}{a})}{a - n} + \frac{\sin(2\sigma + \frac{2n\sigma}{a})}{a + n} \right]_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}}. \quad (5.45)
\end{align*}
\]

Because \( n \) is always an integer, \( \sin(n\pi) = 0 \) and \( \cos(n\pi) = (-1)^n \). Therefore,

\[
x_n = \frac{-iL \sin(an)}{2\alpha a} \left( -1 \right)^n. \quad (5.46)
\]

Note that \( x_n = x_{-n} \). The final solution is found by substituting these results into the general solution (5.32), thereby using equations (5.15) and (5.38):

\[
X_f(\sigma, \tau) = x_0 + \frac{\beta_1 T}{a\pi} \tau + i \sum_{n \neq 0} \left( x_n e^{-\frac{2in\tau}{\alpha}} + \bar{x}_n e^{-\frac{2in\tau}{\alpha}} \right)
= x_0 + i \sum_{n \neq 0} x_n \left( e^{-\frac{2in\tau}{\alpha}} + e^{-\frac{2in\tau}{\alpha}} \right) + \bar{x}_n e^{-\frac{2in\tau}{\alpha}} \tau + i \sum_{n \neq 0} \bar{x}_n \left( e^{-\frac{2in\tau}{\alpha}} + e^{-\frac{2in\tau}{\alpha}} \right)
= x_0 + 2i \sum_{n=1}^{\infty} x_n \left( \cos\left( \frac{2n\tau}{a} (\tau - \sigma) \right) + \cos\left( \frac{2n\tau}{a} (\tau + \sigma) \right) \right)
= x_0 + 2i \sum_{n=1}^{\infty} x_n \left( \frac{2n\tau}{a} \cos\left( \frac{2n\tau}{a} \right) \right)
= \frac{L \sin(\pi a)}{\pi a} \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1 - \frac{a^2}{\pi^2}} \cos\left( \frac{2n\tau}{a} \right) \cos\left( \frac{2n\sigma}{a} \right) \right]. \quad (5.47)
\]

A similar calculation for \( Y_f \) gives

\[
Y_f(\sigma, \tau) = \frac{L \sin(\pi a)}{\pi a} \left[ 2\tau + 2a \sum_{n=1}^{\infty} \frac{(-1)^n}{n(1 - \frac{a^2}{\pi^2})} \sin\left( \frac{2n\tau}{a} \right) \cos\left( \frac{2n\sigma}{a} \right) \right]. \quad (5.48)
\]

Deriving the solutions for string II from the solution for string I is easy. This can be done by making the substitutions \( a \to (1 - a) \) and \( \sigma \to (\sigma - \frac{\pi}{\alpha}) \). Note that the initial solution (5.1) then changes to

\[
X = -L \cos(2\sigma) \cos(2\tau), \quad Y = -L \cos(2\sigma) \sin(2\tau), \quad X^0 = 2L\tau. \quad (5.49)
\]
It is therefore essential to multiply \( X_I \) and \( Y_I \) with an additional minus sign. Using this fact and substituting \((1 - a)\) for \( a \) into equations (5.47) and (5.48) gives

\[
X_{II}(\sigma, \tau) = \frac{L \sin(\pi a)}{\pi(1 - a)} \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1 - \frac{n^2}{(1 - a)^2}} \cos\left(\frac{2n\tau}{1 - a}\right) \cos\left(\frac{2n\sigma}{(1 - a)}\right) \right] \quad (5.50)
\]

\[
Y_{II}(\sigma, \tau) = \frac{L \sin(\pi a)}{\pi(1 - a)} \left[ 2\tau + 2(1 - a) \sum_{n=1}^{\infty} \frac{(-1)^n}{n(1 - \frac{n^2}{(1 - a)^2})} \sin\left(\frac{2n\tau}{1 - a}\right) \cos\left(\frac{2n\sigma}{(1 - a)}\right) \right] \quad (5.51)
\]

\[
X_0^{\prime}(\sigma, \tau) = 2L\tau. \quad (5.52)
\]

Note that there is made use of the fact that \( \sin(\pi - \pi a) = -\sin(-\pi a) = \sin(\pi a) \). In this case, \( -\frac{\pi(1-a)}{2} \leq \sigma < \frac{\pi(1-a)}{2} \).

5.5 Finding a closed-form solution with direct matching

Iengo and Russo use a second method to derive solutions for the outgoing strings in closed form. In order to do so, they write the initial string solution (5.1) in terms of light-cone coordinates \( \sigma^\pm = \tau \pm \sigma \). The rotating string solution in these coordinates is

\[
X = \frac{L}{2} \cos(2\sigma^+) + \frac{L}{2} \cos(2\sigma^-), \quad (5.53)
\]

\[
Y = \frac{L}{2} \sin(2\sigma^+) + \frac{L}{2} \sin(2\sigma^-), \quad (5.54)
\]

\[
X^0 = L\sigma^+ + L\sigma^- \quad (5.55)
\]

All these expressions have the form \( X^\mu(\sigma, \tau) = X_0^\mu(\sigma^+) + X_0^\mu(\sigma^-) \). Then, they require continuity in the spatial coordinates and their time derivatives during the splitting process. Further, the world sheet parameters are rescaled in such a way that both outgoing strings have periodicity \( \Delta \sigma = \pi \). This requirement is shown in figure 8. String I gets the rescaling \( \sigma^+ \to a\sigma^+ \), while string II is rescaled as \( \sigma^+ \to (1 - a)\sigma^+ \).

Figure 8: This figure shows the effect of implying a new periodicity to the outgoing strings. Picture (1) shows the old parameter values and the splitting points. Picture (2) demonstrates how the splitting points in the upper and lower segments of the initial strings become connected and how the new parameter values are chosen. In fact, both the initial as the outgoing strings remain folded on a straight line. They are just unfolded in this figure to give a clearer view of the situation.
The resulting solutions are as follows. String I is split into an upper and a lower part. The solution for the upper part \((0 \leq \sigma^+ < \frac{\pi}{2})\) becomes

\[
X_i^0(\sigma^+) = L \sigma^+, \tag{5.56}
\]

\[
X_i^{\pm}(\sigma^+) = \frac{L}{2} \cos(2a\sigma^+) , \tag{5.57}
\]

\[
Y_i^{\pm}(\sigma^+) = \frac{L}{2} \sin(2a\sigma^+) , \tag{5.58}
\]

while the result for the lower part \((\frac{\pi}{2} \leq \sigma^+ < \pi)\) is

\[
X_i^0(\sigma^+) = L \sigma^+ , \tag{5.59}
\]

\[
X_i^{\pm}(\sigma^+) = \frac{L}{2} \cos(2a\sigma^+ - 2\pi) \cos(2a\sigma^+) , \tag{5.60}
\]

\[
Y_i^{\pm}(\sigma^+) = \frac{L}{2} \sin(2a\sigma^+ - 2\pi). \tag{5.61}
\]

The solution for string II is

\[
X_{II}^0(\sigma^+) = L(1-a)\sigma^+ , \tag{5.62}
\]

\[
X_{II}^{\pm}(\sigma^+) = \frac{L}{2} \cos(2(1-a)\sigma^+ + a\pi) , \tag{5.63}
\]

\[
Y_{II}^{\pm}(\sigma^+) = \frac{L}{2} \sin(2(1-a)\sigma^+ + a\pi) , \tag{5.64}
\]

where \(0 \leq \sigma^+ < \pi\).

The authors claim that these solutions are equivalent to the previously derived Fourier series. However, this is not true. The Fourier solutions (5.48) and (5.52) have a term linear in \(\tau\), associated with the linear momenta (5.16) and (5.25), respectively. The solutions for \(Y_i\) and \(Y_{II}\) above certainly lack such a linear term. Moreover, their linear momenta are not conserved. Recalling the formula for the linear momentum (4.80), one finds for string I

\[
P_Y = T \int_0^{\infty} d\sigma Y_I^\mu
\]

\[
= T \int_0^{\frac{\pi}{2}} d\sigma [2aL \cos(2a\tau) \cos(2a\sigma) + T \int_{\frac{\pi}{2}}^{\infty} d\sigma [2aL \cos(2a\tau) \cos(2a\sigma - 2a\pi)]
\]

\[
= 2aLT \cos(2a\tau) \int_0^{\frac{\pi}{2}} d\sigma \cos(2a\sigma) + \int_{\frac{\pi}{2}}^{\infty} d\sigma \cos(2a\sigma - 2a\pi)
\]

\[
= 2aLT \cos(2a\tau) \sin(2a\sigma) \bigg|_0^{\frac{\pi}{2}} + \sin(2a\sigma - 2a\pi) \bigg|_{\frac{\pi}{2}}
\]

\[
= LT \cos(2a\tau) \sin(2a\pi). \tag{5.65}
\]

This gives the correct value for \(\tau = 0\) (compare with (5.16)), but for general \(\tau\) is is time-dependent. This is not only inconsistent with the Fourier solution, but it also harms the general principle of momentum conservation. There must be a conceptual problem, occurring in the matching procedure. For these reasons, all further calculations will be done with the Fourier solutions.
6 Properties of the decay products

This last section concerns the properties of the outgoings string solutions. Their motion is analyzed by parametric plotting of the Fourier solutions. Thereafter, the mass of the strings and the decay rates of the splitting process are discussed. In a previous article of R. Iengo and J. Russo [8], the full quantum calculation of the string decay was done in ten-dimensional type II superstring theory. A detailed discussion of these computations reaches beyond the scope of this paper. Only the global idea and the most important results will be discussed and compared to the results of the semiclassical calculations.

6.1 Motion of the outgoing strings

The initial string solution (5.1) and the Fourier series for the outgoing strings are used to visualize the splitting process. This is done for the case $a = 0.4$ in figure 9. The picture on the left shows the world sheet of the strings during the splitting process. The graph on the right represents both outgoing strings at three successive instants of time after the splitting. Note that the outgoing strings remain folded and that they move away from each other as time goes by. The last observation agrees with the fact that both string have a nonzero momentum in the $y$-direction, as follows from (5.16) and (5.25). The larger string, which is more massive, has a smaller linear velocity. Another interesting feature is the fact that both outgoing strings have an angular bending that moves back and forth over the string as time goes by. These ‘kinks’ are the result of a discontinuity in $dY/dX$ at the splitting points.

![World sheet of splitting process](image1)

![The outgoing strings at three successive instants of time](image2)

Figure 9: The splitting string for $a = 0.4$. On the left, we see the surface that the strings sweep out during the splitting process. The figure on the right shows the both outgoing strings at three instances of time, shortly after the splitting.

The same plots are drawn for the case $a = 0.2$ in figure 10. Note that the higher linear velocity in the $y$-direction for the smaller outgoing string is more obvious in this case. The pictures of the string at fixed instants of time suggest that the angles under which string I and string II bend sum up to $180^\circ$. Iengo and Russo [2] claim that the respective angles are given by

$$\theta_1 = a\pi, \quad \theta_1 = (1 - a)\pi.$$  \hspace{1cm} (6.1)

This agrees with the results in figures 9 and 10. However, these formulas are proven with the use of the incorrect closed form solution from subsection 5.5. A remaining problem is thus to prove this angle formulas for the Fourier solutions.
6.2 Decay rate and mass of decay products

First, a global idea will be given of how to come to a formula for quantum particle decay rates, based on [9]. Consider a process in which an unstable particle 1 of mass $M$ decays into other particles, denoted by the numbers 2, . . . , n. The decay rate is given by Fermi’s golden rule

$$\text{Decay rate} = \frac{2\pi}{\hbar} |M|^2 \times \text{(phase space)}. \quad (6.2)$$

The first important factor in this formula is the quantum mechanical amplitude $M$, which is the inner product between the initial quantum state $|1\rangle$ and the final quantum state $|2,\ldots,n\rangle$. It can be interpreted as the overlap between these states. The phase space is also called the density of final states and depends on the masses, energies and momenta of the particles. It describes the number of quantum states at each energy level that is available to be occupied. In order to calculate the total decay rate of a process, one calculates the matrix element $M$ and then integrates the absolute value squared over the available phase space. The decay rate is then derived by summing over all possible final states:

$$\Gamma = \frac{2\pi}{\hbar} \sum_{\text{final states}} \int d\Pi |M(1 \to 2,\ldots,n)|^2, \quad (6.3)$$

where $d\Pi$ is a short notation for the fact that the integration is done over all possible final states. The sum over the squared matrix elements is related to the imaginary part of the mass squared of unstable particle by a certain optical theorem:

$$\text{Im}(\Delta M^2) = \sum_{\text{final states}} |M|^2 \times \text{(phase space)}. \quad (6.4)$$

This theorem follows from unitarity (the fact that the sum of probabilities of all possible outcomes sums up to 1).
It is analogous to the classical optical theorem which relates the forward scattering amplitude to the cross section of the scatterer. The decay rate to a specific final state with two particles of rest mass \( M_1, M_2 \), respectively, is given by

\[
\Gamma_{1,2} \sim \frac{d^2 \text{Im}(\Delta M^2)}{dM_1^2 dM_2^2}.
\]  

(6.5)

From the full quantum superstring calculation in [8] follows that

\[
\frac{d^2 \text{Im}(\Delta M^2)}{dM_1^2 dM_2^2} \sim g_s^2 M^{-3} \exp(2M^2 S_0(M_1/M, M_2/M)).
\]  

(6.6)

In this formula, \( g_s \) is a coupling constant. The exponent contains a factor \( S_0 \) that is a function of the masses \( M_1, M_2 \) of the decay products. This function is plotted in figure 11. Note that \( S_0 \leq 0 \) and that it has a maximum value \( S_0 = 0 \) at a single line in the top of the diagram. This maximum value gives the largest contribution to the decay rate. It also defines a mass function \( M_1 = M_1^{\text{quantum}}(M_2) \).

![Figure 11: The function \( S_0 \) as a function of the masses \( M_1 \) and \( M_2 \). (Adapted from [8])](image)

Now recall the semiclassical formulas for the outgoing particle masses, given by equations (5.23) and (5.28). Both masses can be expressed as a function of \( a \):

\[
M_1(a) = \frac{L}{a'} \sqrt{a^2 - \frac{\sin^2(\pi a)}{\pi^2}},
\]  

(6.7)

\[
M_2(a) = \frac{L}{a'} \sqrt{(1-a)^2 - \frac{\sin^2(\pi a)}{\pi^2}}.
\]  

(6.8)

One can use these equations to express \( M_1 \) as a function of \( M_2 \). This classical mass function and the quantum mass function are plotted together in figure 12. They almost perfectly coincide.

It will now be explained why the quantum decay rate formula (6.6) makes sense. This is done by finding a relation between quantum mechanics and the action principle. A clear description of this relationship is given by Richard Feynman [10]. One can ask about the variational principle: "How does the particle precisely find the right path, for which the action is extremized? Does it 'smell' the neighboring paths to find out whether or not they have more or less action?". In order to answer this question, first consider the case of light. A ray of light always chooses the path that takes the shortest time to travel. If a photon chooses to take another path, it will take a different amount of time and the photon arrives at the endpoint with a different phase. The total amplitude of light is the sum of the contributions from all different paths that the light can travel. Paths for which the resulting phase is very different from the dominant path contribute almost nothing. The important path becomes the path for which there are many nearby paths that all give the same phase.
Now the laws of quantum mechanics tell that the situation for a particle is exactly similar: the particle does not take just one single path, but it travels every possible path with a certain probability. The probability that a particle at point \( a \) with a corresponding time \( t_i \) will arrive at point \( b \) at time \( t_f \) is the square of a probability amplitude. This amplitude is proportional to
\[
Ce^{iS/h},
\]
where \( C \) is a normalization constant and \( S \) the action corresponding to the path. The factor \( S/h \) can be interpreted as the complex phase angle. The contribution of the different paths to the total amplitude depends on the relative value of \( S \) compared to \( h \). If \( S \) and \( h \) have the same order of magnitude for all paths, a small change in \( S \) for a neighboring path does not have much influence on the phase angle. This implies that many paths can contribute to the total amplitude. Now consider the case that \( S \) is very large compared to \( h \). A small change in \( S \) for a nearby path then gives a completely different phase angle. Therefore, such paths hardly contribute to the total amplitude. The only relevant contribution comes from the region in which \( S \) does not change (in a first order approximation) for neighboring paths. This means that in the classical limit, in which \( h \to 0 \), only the path for which the variation in \( S \) to first order is zero is relevant. This is precisely the relation between quantum mechanics and the principle of least action!

From figure 12 follows that there is a very close agreement between the full quantum superstring calculation and the classical string splitting process. This indicates that the quantum decay of the string is dominated by the classical path. From (6.9) could therefore be expected that the rate of decay should be captured by \( e^{S/c_0}/h \) where \( S/c_0 \) is the action evaluated on the classical solution. In practice this is most conveniently calculated by an analytic continuation of the time coordinate from the Lorentzian regime to the Euclidean, i.e. \( t \to i\tau_E \), resulting in a decay rate which goes as
\[
e^{-S_E/h},
\]
where \( S_E \) is the Euclidean action of the classical solution. This exponential term corresponds to the exponential term in the quantum decay rate (6.6). The peaking of the decay around \( S_0 = 0 \), as demonstrated in figure 11, can therefore be interpreted as a peaking around the classical limit.


7 Discussion and conclusions

This project concerned the splitting process of a rotating closed string. It started with the description of a classical string in section 1, with the use of Lagrangian mechanics. The solution for the classical string was a wave equation, as one could have expected. It was also shown that two kinds of boundary conditions were consistent with the action principle: Neumann boundary conditions, corresponding to fixed endpoints of the string and Dirichlet boundary conditions, that apply to free endpoints.

The classical description of the action, where the Lagrangian is defined as the kinetic energy minus the potential energy, fails at relativistic speeds. By setting the action proportional to the proper time, one finds an action that is applicable to relativistic point particles. The path that the particle actually takes needs to be parameterized in order to perform action integrals. It was demonstrated that the value of the action is independent of the particular parameterization chosen. This property is called reparameterization invariance. A clever choice of parameters makes the derivation of the equations of motion much simpler.

In section 3, the discussion was extended to the case of relativistic strings. Analogously to the world line for a point particle, one can define a world sheet for strings. The string equivalent for proper time is proper area. Just like the proper time can be used to define an action for the relativistic point particle, one can use the proper area to define an action for relativistic strings. This action goes under the name Nambu-Goto action. It can be written in a reparameterization invariant form and can be expressed as a function of the transverse velocity of the string. The static gauge is thereby introduced as a specific choice of parameters for which the string is static in the observer’s Lorentz frame. It was shown that there is an equivalence between the mass and tension of a string and that endpoints of a string move with the speed of light.

The auxiliary world sheet metric was introduced in section 4 in order to obtain a more conventional notation of the Nambu-Goto action. The local symmetries of the string (reparameterizations and Weyl transformations) can then be used to gauge away the auxiliary metric, which makes the calculations much simpler. From the simplified form of the Nambu-Goto action followed that the relativistic string should obey a two-dimensional wave equation. This wave equation was solved in the most general way in the form of a Fourier series. Thereby, it was shown that all solutions to the wave equation can be written as a sum of a left- and a right-moving solution with the use of light-cone coordinates. In the last part of section 4, Noether’s theorem was presented as a method to find all conserved quantities of the motion. This theorem was specifically used to derive explicit formulas for the calculation of the linear and angular momenta.

Section one to four provided all the tools to describe the splitting process of a relativistic string in a semiclassical way. Section 5 started with the parameterized solution of a rotating folded string with maximum angular momentum. It was assumed that, at a certain instant of time, the string splits into two. Thereby, continuity was required in the spatial derivatives and their time derivatives. The equations of motion for the outgoing strings were derived by making a Fourier expansion, using the general solution from section 4. The continuity requirements were used to determine the Fourier coefficients. It was checked that the total energy and momenta are conserved in this process.

The outgoing string solutions have some interesting features. They remain folded and keep rotating like the initial string. Remarkably, there is an angular bending that travels back and forth over the outgoing strings as a function of time. Because the kink is created locally during the splitting process, it must be a generic feature, regardless of the string’s angular momentum. In the last section, the results found with the semiclassical method were compared to the results from the correct quantum computations. The semiclassical method very accurately reproduces the relation between the masses of the decay products that was found in a previous quantum computation. Also, they decay rate that follows from quantum computations can be described by a semiclassical formula. These findings confirm that the description of the splitting process as given in this paper is quite accurate.
References


A Solutions to selected problems from A first course in string theory

This appendix contains solutions to a selection of problems from Barton Zwiebach’s book A first course in string theory [1]. I have made these exercises in order to improve my understanding of the theory. The problem numbers refer to the original numbers used in the book.

Problem 4.4: Evolving an initial open string configuration

A string with tension T_0, mass density μ_0 and wave velocity v_0 = \sqrt{T_0/\mu_0}, is stretched from (x, y) = (0, 0) to (x, y) = (a, 0). The string endpoints are fixed, and the string can vibrate in the y direction.

(a) Write y(t, x) in the form y(t, x) = h_+(x-v_0t)+h_-(x+v_0t), and prove that the above Dirichlet boundary conditions imply

\[ h_+(u) = -h_-(u) \quad \text{and} \quad h_+(u) = h_+(u+2a). \]

Here u ∈ (-∞, ∞) is a dummy variable that stands for the argument of the functions h_+.

The general expression for y(x, t) is

\[ y(x, t) = h_+(x-v_0t) + h_-(x+v_0t). \]

The boundary condition at x = 0 implies

\[ y(t, 0) = h_+(0) + h_-(0) = 0. \]

Using u = -v_0t, one can rewrite this equation as

\[ h_+(u) = -h_-(u). \] (A.1)

Imposing the boundary condition at x = a results in

\[ y(t, a) = h_+(a-v_0t) + h_-(a+v_0t) = 0. \]

⇒ \[ h_+(a-v_0t) = -h_-(a+v_0t). \]

Combining this result with (A.1) gives

\[ h_+(u) = h_+(u+2a). \]

Now consider an initial value problem for this string. At t = 0 the transverse displacement is identically zero, and the velocity is

\[ \frac{\partial y}{\partial t}(0, x) = v_0 \frac{x}{a} \left( 1 - \frac{x}{a} \right), \quad x \in (0, a). \]

(b) Calculate h_+(u) for u ∈ (-a, a). Does this define h_+(u) for all u?

At t = 0, the transverse displacement is identically zero, and thus

\[ y(0, x) = h_+(x) + h_-(x) = 0 \]

⇒ \[ h_+(x) = -h_-(x). \] (A.2)

Note that

\[ \frac{\partial y}{\partial t}(t, x) = v_0 (h_+''(x-v_0t) + h_-'(x+v_0t)). \]

It follows, using equation (A.2), that

\[ \frac{\partial y}{\partial t}(0, x) = v_0 \left( -h_+''(x) + h_-''(x) \right) = -2v_0 h_+''(x) = \frac{v_0}{a} \left( 1 - \frac{x}{a} \right). \]
This leads to the differential equation
\[ \frac{dh_+}{dx} = -\frac{x}{2a} - \frac{x^2}{2a^2}, \]
which can be solved easily:
\[ \int dh_+(x) = \int \left( \frac{-x}{2a} + \frac{x^2}{2a^2} \right) dx. \]
\[ h_+(x) = \frac{-x^2}{4a} + \frac{x^3}{6a^2} + c \quad \text{for} \quad x \in (0, a). \]

From part (a) follows that \( h_+(u) = h_+(u + 2a) \), so \( h_+(a) = h_+(-a) \). This implies that
\[ h_+(x) = \frac{-x^2}{4a} - \frac{x^3}{6a^2} + c \quad \text{for} \quad x \in (-a, 0). \]

This does indeed define \( h_+ \) for all values of \( u \), because \( h_+ \) is periodic with \( 2a \).

(c) Calculate \( y(t, x) \) for \( x \) and \( v_0t \) in the domain \( D \) defined by the two conditions
\[ D = \{(x, v_0t) | 0 \leq x \pm v_0t \leq a\}. \]

Exhibit the domain \( D \) in a plane with axes \( x \) and \( v_0t \).

The domain is drawn in figure 13.

![Figure 13: The domain is the diamond-shaped region, enclosed by the four lines.](image)

Now, take the solution \( h_+ = (x - v_0t) \) for \( x \in (0, a) \) from part (b) and use \( h_+(x) = -h_-(x) \). Then, it follows that
\[ y(t, x) = -\frac{(x - v_0t)^2}{4a} + \frac{(x - v_0t)^3}{6a^2} + \frac{(x + v_0t)^2}{4a} - \frac{(x + v_0t)^3}{6a^2}. \]

(d) At \( t = 0 \) the midpoint \( x = a/2 \) has the largest velocity of all points in the string. Show that the velocity of the midpoint reaches the value of zero at time \( t_0 = a/(2v_0) \) and that \( y(t_0, a/2) = a/12 \). This is the maximum vertical displacement of the string.

Differentiating the function \( y(t, x) \) with respect to \( t \) gives
\[ \frac{dy}{dt} = \frac{-v_0(x - v_0t)}{2a} - \frac{v_0(x - v_0t)^2}{2a^2} + \frac{v_0(x + v_0t)}{2a} - \frac{v_0(x + v_0t)^2}{2a^2} = \frac{v_0}{2a} \left[ 2x - \frac{(x - v_0t)^2}{a} - \frac{(x + v_0t)^2}{a} \right]. \]
The vertical velocity at \( x = a/2 \) is then set equal to zero:

\[
\frac{\partial y_t}{\partial t} = \frac{v_0}{2a} \left[ a - \left( \frac{a}{2} - v_0 t_0 \right)^2 - \left( \frac{a}{2} + v_0 t_0 \right)^2 \right] = \frac{v_0}{4} - \frac{v_0^3}{a^2 t_0} = 0.
\]

From this follows \( t_0 = \frac{a}{2v_0} \). One substitutes this value into the expression for \( y(t, x) \) to get

\[
y\left( \frac{a}{2v_0}, \frac{a}{2} \right) = -\frac{(0)^2}{4a} + \frac{(0)^3}{6a^2} + \frac{(a)^2}{4a} - \frac{(a)^3}{6a^2} = \frac{a}{12}.
\]

**Problem 4.8: Deriving Euler-Lagrange equations**

(a) Consider an action for a dynamical variable \( q(t) \):

\[
S = \int dt L(q(t), \dot{q}(t); t).
\]

Calculate the variation \( \delta S \) of the action under a variation \( \delta q(t) \) of the coordinate. Use the condition \( \delta S = 0 \) to find the equation of motion for the coordinate \( q(t) \) (the Euler-Lagrange equation).

The equation of motion for the coordinate \( q(t) \) can be found by calculating the variation \( \delta S \) of the equation above:

\[
\delta S = \int dt \left[ \frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right].
\]

To get rid of the derivative on \( \delta q \) in the second term, it is integrated by parts:

\[
\delta S = \int dt \left[ \frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right] = \int dt \left[ \frac{\partial L}{\partial q(t)} \delta q(t) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}(t)} \right) \delta q(t) \right] = \int \left[ \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right] dt.
\]

The last term equals zero because the variation \( \delta q(t) \) vanishes at the endpoints of the path. For the first term, note that because the action on the physical path is stationary, \( \delta S \) must be zero for every variation \( \delta q(t) \). Therefore, the term in brackets has to be zero. This gives the equation of motion:

\[
\frac{\partial L}{\partial \dot{q}(t)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial q(t)} = 0 \tag{A.3}
\]

(b) Consider an action for a dynamical field variable \( \phi(t, \vec{x}) \). As indicated, the field is a function of space and time, and is briefly written as the spacetime function \( \phi(x) \). The action is obtained by integrating the Lagrangian density \( L \) over spacetime. The Lagrangian density is a function of the field and the spacetime derivatives of the field:

\[
S = \int d^D x L(\phi, \partial_\mu \phi).
\]

Here \( d^D x = dx^1 \ldots dx^d \) and \( \partial_\mu \phi = \partial \phi(\vec{x}) \). Calculate the variation \( \delta S \) of the action under a variation \( \delta \phi(x) \) of the field. Use the condition \( \delta S = 0 \) to find the equation of motion for the field \( \phi(x) \) (the Euler-Lagrange equation).

The expression for \( S \) can be varied as before:

\[
\delta S = \int d^D x \left[ \frac{\partial L}{\partial \phi(x)} \delta \phi(x) + \frac{\partial L}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right].
\]
The second term in brackets is then integrated by parts:

$$\delta S = \int d^Dx \left[ \frac{\partial L}{\partial \phi(x)} \delta \phi(x) + \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \phi(x))} \right) \delta \phi(x) \right] - \left( \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial (\partial_{\mu} \phi(x))} \right) \delta \phi(x) \right] dDx = \int d^Dx \left[ \frac{\partial L}{\partial \phi(x)} \delta \phi(x) + \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \phi(x))} \right) \delta \phi(x) \right]_{\gamma_o = x_i}.$$

In the last term, \(d^Dx = dt dx^1 \ldots dx^{d-1} dx^{d+1} \ldots dx^D\). Because the variation in \(\phi(x)\) is zero at the endpoints of the path, the second term must vanish. Because \(\delta S = 0\) and \(\delta \phi(x)\) is arbitrary, the term in brackets of the first integral must equal zero. This gives the equation of motion for the field \(\phi(x)\):

$$\frac{\partial L}{\partial \phi(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \phi(x))} \right) = 0.$$

**Problem 5.1: Point particle equation of motion and reparameterizations**

If the path of a point particle is parameterized by proper time, the equation of motion is \(\frac{d^2 X^\mu}{ds^2} = 0\). Consider now a new parameter \(\tau = f(s)\). Find the most general function \(f\) for which the equation of motion implies \(\frac{d^2 X^\mu}{d\tau^2} = 0\)

Using the chain rule gives

$$\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau} \frac{d\tau}{ds},$$

and thus

$$\frac{d^2 x^\mu}{ds^2} = \frac{d}{ds} \left( \frac{dx^\mu}{d\tau} \frac{d\tau}{ds} \right) = \frac{d^2 x^\mu}{d\tau^2} \left( \frac{d\tau}{ds} \right)^2 + \frac{dx^\mu}{d\tau} \frac{d^2 \tau}{ds^2} = 0.$$

From the last equation follows that

$$\frac{d^2 x^\mu}{d^2 \tau} = - \left( \frac{d\tau}{ds} \right)^2 \frac{dx^\mu}{d\tau} \frac{d^2 \tau}{d\tau^2}.$$

Since by assumption \(\tau = f(s)\), its derivative with respect to \(s\) cannot be zero. Also, \(\frac{dx^\mu}{d\tau} \neq 0\) because the path must depend on the parameter \(\tau\). Therefore, \(\frac{d^2 x^\mu}{d\tau^2} = 0\) implies

$$\frac{d^2 \tau}{ds^2} = 0,$$

and thus \(\frac{d^2 \tau}{ds^2} = \text{constant}\). The most general function is \(\tau = cs\), with \(c\) an arbitrary constant.

**Problem 5.2: Particle equation of motion with arbitrary parameterization**

Vary the point particle action

$$S = -mc \int_{t_i}^{t_f} \int_{\gamma_i}^{\gamma_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau$$

to find a manifestly reparameterization invariant form of the free particle equation of motion.

The equations of motion can be found in two different ways:

1. It can be done explicitly, by setting the variation of the action integral equal to zero:
The first term within the brackets is zero because $L$ only depends on derivatives of $x^\rho$. So,

$$\delta S = \int_t^f \frac{d}{d\tau} \left[ \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \delta \dot{x}^\rho \right] d\tau = \eta_{\rho\mu} m c \int_t^f \left[ \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right) \delta \dot{x}^\rho \right] d\tau = 0.$$

Because there should be no derivatives working on the variation $\delta x^\rho$, the expression above is integrated by parts to get

$$\delta S = \eta_{\rho\mu} m c \int_t^f \left[ \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right) \delta \dot{x}^\rho \right] d\tau = 0.$$

This equation can be written out more explicitly as

$$\delta S = \eta_{\rho\mu} m c \int_t^f \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \delta \dot{x}^\rho \left|_t^f \right. - \eta_{\rho\mu} m c \int_t^f \left[ \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right) \delta \dot{x}^\rho \right] d\tau = 0.$$

The variation $\delta \dot{x}^\rho$ is zero at the endpoints of the path. Therefore, the first term vanishes. Because $\delta \dot{x}^\rho$ is arbitrary, the term in brackets of the second integral must equal zero. The equation of motion is thus

$$\frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} = 0.$$

(2) The equation of motion can also be derived more directly, by using the Euler Lagrange equations:

$$\frac{\partial L}{\partial \dot{x}^\rho} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}^\rho} = 0.$$

The first term drops because $L$ does not depend on $x^\rho$. Therefore,

$$\frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}^\rho} = 0.$$

Differentiating $L$ gives

$$\frac{\partial L}{\partial \dot{x}^\rho} = \frac{\partial}{\partial \dot{x}^\rho} (-mc \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}) = mc \eta_{\mu\rho} \dot{x}^\mu + \eta_{\rho\nu} \ddot{x}^\nu = mc \eta_{\mu\rho} \dot{x}^\mu \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu},$$

where in the last step was used that $\eta_{\mu\rho} \dot{x}^\mu = \eta_{\rho\mu} \dot{x}^\mu = \eta_{\rho\nu} \ddot{x}^\nu$. This leads to the same equation of motion as was found with the first method:

$$\frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} = 0.$$

**Problem 5.7: Point particle action in curved space**

A curved space described by the metric $g_{\mu\nu}(x)$ has an invariant interval $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. The motion of a point particle of mass $m$ on curved space is studied using the action

$$S = -mc \int ds.$$

Show that the equation of motion obtained by variation of the world-line is

$$\frac{d}{ds} \left[ g_{\mu\nu} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}(x)}{\partial x^\rho} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}.$$

This is called the geodesic equation. When the metric is constant we recover the equation of motion of a free point particle.
The action in the curved background is

$$S = -mc \int ds = -mc \int \sqrt{-g_{\mu \nu}(x)dx^\mu dx^\nu} = -mc \int \sqrt{-g_{\mu \nu}(x)} \frac{dx^\nu}{d\tau} d\tau.$$  

Note that $ds/d\tau = L$, where $L = (-g_{\mu \nu}(x)\partial^\mu \partial^\nu)^{-1/2}$. The equations of motion follow directly from the Euler Lagrange Equations. The required derivatives are

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} = mc(\partial^\mu \mathbf{x}(-g_{\mu \nu}(x)\partial^\mu \partial^\nu)^{-1/2},$$

$$\frac{\partial L}{\partial \mathbf{x}} = 2mc(\partial^\mu (-g_{\mu \nu}(x)\partial^\mu \partial^\nu)^{-1/2}.$$  

Substituting the derivatives of $L$ into the general Euler Lagrange equation (A.3) gives

$$\frac{d}{ds}(g_{\mu \nu} \frac{dx^\mu}{ds} L^{-1}) = \frac{1}{2} \frac{\partial g_{\mu \nu}(x)}{\partial t} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} L^{-1}.$$  

One the cancelation of the $L$'s and $L^{-1}$s one finds the equation of motion:

$$\frac{d}{ds} \left( g_{\mu \nu} \frac{dx^\mu}{ds} \right) = \frac{1}{2} \frac{\partial g_{\mu \nu}(x)}{\partial t} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}.$$  

**Problem 6.6: Planar motion for open string with attached endpoints**

Consider the motion of a relativistic open string on the $(x, y)$ plane. The string endpoints are attached to $(x, y) = (0, 0)$ and $(x, y) = (a, 0)$, where $a > 0$. We want to study motions that can be described using the function $y(t, x)$ which gives the vertical displacement of the string for $x \in [0, a]$. Show that the Lagrangian $L = -T_0 \int ds \sqrt{1 - \frac{y'^2}{c^2}}$ can be written in the form:

$$L = -T_0 \int_x^a dx \sqrt{1 - y'^2} - \frac{\dot{y}^2}{c^2}.$$  

Here $y' = \partial y/\partial x$ and $\dot{y} = \partial y/\partial t$.

The position of the string can be described by the vector $\mathbf{X} = (x, y(x, t))$. An infinitesimal segment (at a fixed time $t$) is then given by:

$$d\mathbf{X} = (dx, \frac{\partial y}{\partial x} dx) = (1, y') dx,$$

with a corresponding length

$$ds = |d\mathbf{X}| = \sqrt{1 + y'^2} dx.$$  

These expressions can be used to note that

$$\frac{\partial \mathbf{X}}{\partial s} = \frac{\partial \mathbf{X}}{\partial x} \frac{\partial x}{\partial s} = (1, y') \frac{1}{\sqrt{1 + y'^2}},$$

and

$$\frac{\partial \mathbf{X}}{\partial t} = (0, \dot{y}), \quad \text{where} \quad \dot{y} = \frac{\partial y}{\partial t}.$$  

52
Substituting these expressions into formula (6.83) from Zwiebach gives

\[ v_\perp^2 = \left( \frac{\partial \vec{X}}{\partial t} \right)^2 - \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2 = \dot{y}^2 - \frac{\dot{y}^2 y'^2}{1 + y'^2} = \frac{\dot{y}^2}{1 + y'^2}. \]

Now, the Lagrangian

\[ L = -T_0 \int ds \sqrt{1 - \frac{v_\perp^2}{c^2}} \]

becomes

\[ L = -T_0 \int \sqrt{1 + y'^2} dx \sqrt{1 - \frac{\dot{y}^2}{c^2(1 + y'^2)}} = -T_0 \int_0^\alpha dx \sqrt{1 - y'^2 - \frac{\dot{y}^2}{c^2}}. \]

### 6.7 Time evolution of a closed circular string

At \( t = 0 \), a closed string forms a circle of radius \( R \) on the \((x, y)\) plane and has zero velocity. The time evolution of this string can be studied using the action in terms of transverse velocity. The string will remain circular, but its radius will be a time-dependent function \( R(t) \). Give the Lagrangian \( L \) as a function of \( R(t) \) and its time derivative. Calculate the radius and velocity as functions of time. Sketch the spacetime surface traced by the string in a three-dimensional plot with \( x, y, \) and \( ct \) axes. [Hint: calculate the Hamiltonian associated with \( L \) and use energy conservation.]

Because the string remains in the \((x, y)\)-plane, \( dz = 0 \). The invariant interval at a fixed time \((dt = 0)\) is therefore

\[ ds^2 = dx^2 + dy^2. \]

Because this problem has circular symmetry, it is better to work in polar coordinates \( x = r \cos \theta, y = r \sin \theta \), where

\[ dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos \theta dr - r \sin \theta d\theta, \]
\[ dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin \theta dr + r \cos \theta d\theta, \]

and thus

\[ ds^2 = dr^2 + r^2 d\theta^2. \]

In these coordinates, the transverse velocity of the string is simply \( v_\perp = \frac{dR(t)}{dt} = \dot{R} \). Therefore, the Lagrangian (6.89 in Zwiebach) becomes

\[ L = -T_0 \int ds \sqrt{1 - \frac{\dot{R}^2}{c^2}} = -2\pi RT_0 \sqrt{1 - \frac{\dot{R}^2}{c^2}}, \]

where the fact was used that \( \int ds = 2\pi R \), which is simply the length of the string. Its derivatives are

\[ \frac{\partial L}{\partial R} = -2\pi T_0 \sqrt{1 - \frac{\dot{R}^2}{c^2}}, \]
\[ \frac{\partial L}{\partial \dot{R}} = \frac{2\pi T_0 R \dot{R}}{c^2 \sqrt{1 - \frac{\dot{R}^2}{c^2}}}. \]

Substituting these expressions into the Euler-Lagrange equation (A.3) gives

\[ \frac{d}{dt} \left( \frac{RR}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} \right) + c^2 \sqrt{1 - \frac{\dot{R}^2}{c^2}} = 0. \]
This equation can be made more explicit by performing the differentiation:

\[
\frac{\dot{R}^2}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} + \frac{RR}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} + \frac{RR^2 \ddot{R}}{c^2(1 - \frac{\dot{R}^2}{c^2})^{3/2}} + c^2 \sqrt{1 - \frac{\dot{R}^2}{c^2}} = 0
\]

\[
\Rightarrow \frac{\dot{R}^2}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} + RR(1 - \frac{\dot{R}^2}{c^2}) + \frac{RR^2 \ddot{R}}{c^2} + c^2(1 - \frac{\dot{R}^2}{c^2})^2 = 0
\]

\[
\Rightarrow -\ddot{R}^2 + RR + c^2 = 0.
\]

Now it is possible to write down the Hamiltonian of the system. Remember that \( L = T - V \) and \( H = T + V \).

Then follows that

\[
H = 2T - L \quad \text{or} \quad H = \vec{p} \cdot \vec{v} - L.
\]

The previous results are helpful in calculating this quantity:

\[
H = \frac{\partial L}{\partial \dot{R}} \frac{dR}{dt} - L = \frac{2\pi T_0 R \dot{R}^2}{c^2 \sqrt{1 - \frac{\dot{R}^2}{c^2}}} + 2\pi T_0 R \sqrt{1 - \frac{\dot{R}^2}{c^2}}
\]

\[
= \frac{2\pi T_0 R \left[R^2 + c^2(1 - \frac{\dot{R}^2}{c^2})\right]}{c^2 \sqrt{1 - \frac{\dot{R}^2}{c^2}}}
\]

\[
= \frac{2\pi T_0 R}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}}.
\]

Because the energy is conserved during the motion, \( H \) must be a constant. At \( t = 0 \), the string radius is \( R_0 \), while the velocity \( \frac{\partial R}{\partial t} \) equals zero. Then it follows that

\[
H(t = 0) = 2\pi T_0 R_0 = \text{constant},
\]

and thus

\[
\frac{R}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} = R_0
\]

\[
\Rightarrow \dot{R}^2 + \left(\frac{c}{R_0}\right)^2 R^2 - c^2 = 0.
\]

This is an equation of harmonic motion, with general solution \( R(t) = A \cos \omega t + B \sin \omega t \). Its time derivative is \( \dot{R}(t) = -A\omega \sin \omega t + B\omega \cos \omega t \). Due to the boundary condition \( \dot{R}(0) = 0 \), \( B \) must equal zero. The other boundary condition requires \( R(0) = R_0 = A \). Substituting

\[
R(t) = R_0 \cos \omega t
\]

into the equation of motion (A.5) gives

\[
R_0^2 \omega^2 \sin^2 \omega t + c^2 \cos^2 \omega t = c^2.
\]

From this expression follows that \( \omega = \frac{c}{R_0} \). The radius as a function of time is then

\[
R(t) = R_0 \cos \left(\frac{ct}{R_0}\right),
\]

and the corresponding velocity is

\[
\dot{R}(t) = -\omega \sin \left(\frac{ct}{R_0}\right).
\]
The evolution of the string radius is shown in figure 14. From $t = 0$, the string starts to shrink and its radius eventually becomes zero. As follows from the formula, this happens when $t = \frac{\pi R_0}{2}$. In an equal amount of time, the radius of the string grows again to the maximum value $R_0$. This cycle repeats itself forever.

Figure 14: This diagram shows the change of the string’s radius as a function of time. In fact, the graph is the world sheet of the string.
Natuurverschijnselen verklaren met splitsende snaren

Bliksem en onweer, het ontstaan van sterren, radioactief verval, de werking van je mobiele telefoon... Wanneer je de wereld om je heen probeert te begrijpen, lijkt deze vaak oneindig ingewikkeld. Toch zijn er maar vier fundamentele krachten nodig om alle natuurverschijnselen te beschrijven: elektromagnetische kracht, zwakke en sterke kernkracht en de zwaartekracht. Theoretisch natuurkundigen zoeken al eeuwen naar een theorie die elk van deze krachten verklaart. Ze noemen dit een unificatie theorie of theorie van alles. Het succesvolste resultaat van hun inspanningen is het standaardmodel van de natuurkunde.


Op basis van hun benaming zou je verwachten dat elementaire deeltjes onverwoestbaar zijn. Toch blijken ze in bepaalde situaties uiteen te vallen in andere elementaire deeltjes. In snaartheorie betekent dit dat een zware snaar splitst in lichtere snaren. Om dit microscopische proces te beschrijven is kwantummechanica nodig. Dit zorgt voor zeer ingewikkelde berekeningen. In dit project proberen we de kwantummechanica daarom te omzeilen. We stellen voor dat de snaar, die er uit ziet als een klein elastiekje, volledig is gestrekt en om zijn middelpunt rondjes draait in de ruimte. De twee helften van de snaar zitten dicht tegen elkaar aan. Op een gegeven moment fuseren de helften op één plaats. Daarna splitst de snaar en ontstaan twee kleinere snaren.

Nu berekenen we de kans dat de splitsing van een snaar spontaan plaats vindt. Dit vertelt ons wat de verwachte levensduur is van het deeltje dat de snaar representeert. Verder gaan we na hoe zwaar de snaren zijn die bij het splitsingsproces ontstaan. Al eerder zijn deze berekeningen verricht met behulp van kwantummechanica. Wanneer de snaren relatief zwaar zijn, komen de resultaten van beide methoden nauwkeurig overeen. In dat geval leidt onze eenvoudige beschrijving van deeltjesverval dus tot betrouwbare resultaten.

Figure 15: Deze afbeelding laat het spoor zien dat de snaren tijdens het splitsingsproces achter laten. Het onderste gedeelte is het spoor van de oorspronkelijke snaar. Deze is volledig gestrekt en draait om zijn middelpunt. Na splitsing ontstaan twee kleinere snaren met een knik, die van elkaar af bewegen.