Lepton Flavor Violating Higgs

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Lepton Flavor Violating Higgs

$Z \rightarrow \tau \mu \gamma$ Decay

by

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Abstract

In this thesis, we discuss whether the decay $Z \to \tau\mu\gamma$ can be used to improve bounds on the strength of the lepton flavor violating coupling $H\tau\mu$. The best current bound comes from the measurement of the $H \to \tau\mu$ decay by BaBar. This bound on the coupling strength is $g_{\tau\mu} < 3.1 \times 10^{-3}$. To see if the $Z \to \tau\mu\gamma$ decay can give a better bound, we calculated the branching ratio of $Z \to \tau\mu\gamma$ via a top loop and a Higgs boson, with a lepton flavor violating coupling between the Higgs and the leptons. We used the bound on the coupling $g_{\tau\mu}$ from $H \to \tau\mu$ to get a bound on the size of this branching ratio.

The bound on the branching ratio found is $\text{BR}(Z \to \tau\mu\gamma) < 5.1 \times 10^{-17}$. This makes it unlikely that this decay will be detected in the near future, unless some unexpected progress is made. It is still possible, however, that similar decays where lepton flavor occurs in a different coupling, i.e. not $H\tau\mu$, have larger branching ratios.

Figure: The $Z \to \tau\mu\gamma$ decay discussed in this thesis. It contains the $H\tau\mu$ vertex, marked with a dot. This is a non-Standard Model vertex that violates lepton flavor conservation.
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1. Summary

In the Standard Model of particle physics, the flavor of charged leptons is almost completely conserved; charged leptons can change their flavor only in immeasurably small amounts. This means that if we can measure a lepton changing its flavor, this must be due to physics beyond the Standard Model. This makes it very interesting to look for lepton flavor violation.

Lepton flavor violation occurs naturally in many extensions of the Standard Model, and often it is Higgs interactions that cause the lepton flavor violation. If lepton flavor violation does indeed occur in Higgs interactions, it can be studied by measuring Higgs to lepton decay. This is particularly useful for studying lepton flavor violation between $\tau$- and $\mu$-leptons, since this kind of lepton flavor violation is in general hard to measure. $\tau \to \mu \gamma$ decay, the standard process to measure $\tau \mu$ lepton flavor violation, puts a bound on the coupling strength of $g_{\tau \mu} < 1.0 \times 10^{-1}$. $H \to \tau \gamma$ puts a much stronger bound on the coupling strength of $g_{\tau \mu} < 3.1 \times 10^{-3}$.

It may be possible to get an even better bound from a different process. In this thesis, we studied the decay $Z \to \tau \mu \gamma$ to find out if this process can give a better bound. To do that, we used the limit on the coupling strength from $H \tau \mu$ decay to calculate a limit on the branching ratio of $Z \to \tau \mu \gamma$. In this calculation we did an expansion in the inverse top mass and kept only terms of order $m_t^{-2}$.

The upper limit on the branching ratio we found was $5.1 \times 10^{-17}$. This is very small and that makes it unlikely that this process will be observable in the near future, except if some unexpected progress is made. Therefore, $Z \to \tau \mu \gamma$ cannot be used to

![Figure 1.1:](image)

*Figure 1.1:* The $Z \to \tau \mu \gamma$ decay discussed in this thesis. It contains the $H \tau \mu$ vertex, a non-Standard Model vertex that is lepton flavor violating.
1. Summary

improve upon the bound from $H \to \tau \mu$ decay. However, with lepton flavor violating occurring in interactions with a different particle, it may be possible to make some adjustments to this decay that improve the branching ratio. The resulting decay could be more relevant and may be useful to measure the coupling strength of that different lepton flavor violating interaction.
2. Introduction

2.1. The Standard Model

The Standard Model is the theory that describes the physics of elemental particles. It is a quantum field theory with the symmetries of the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ (Srednicki 2006, p. 527). The general formulation of the Standard Model was complete in the 1970s, and since then it has been a hugely successful theory (Hooft 2007; Oerter 2006). It contains 17 different elementary particles, divided into two categories: fermions and bosons (Griffiths 2008, p. XIII). These are shown in figure 2.1.

The fermions can be considered the more ordinary particles, they can make up matter, while the bosons are involved in interactions between particles (CERN 2012).

The fermions are themselves divided into two more categories, quarks and leptons. There are six of both, and they are divided into three generations. Leptons and quarks of different generations differ in a property called flavor. Because of flavor conservation, particles cannot usually transform into a particle of a different generation (Cottingham et al. 2007, p. 3).

Of the bosons, four, the photon, gluon Z- and W-boson, are carriers of the fundamental forces, electromagnetism and the weak and strong forces. These are called the gauge bosons. The last, the Higgs boson, is rather odd. It does carry a force, like the other bosons, but is necessary to explain some more subtle features of the Standard Model. It gives mass to the other elementary particles and is used to explain why the weak force is short ranged (Carroll 2012). This happens by the spontaneous breaking of the $SU(2)_L \times U(1)_Y$ electro-weak symmetry into the $U(1)_{QED}$ symmetry (Srednicki 2006, p. 543). The subtle role of the Higgs makes it hard to detect, and it was only discovered in 2012. With this discovery of the Higgs, all the particles in the Standard Model now had been found.

2.2. Beyond the Standard Model

But the Standard Model cannot explain everything. Even though the Standard Model successfully describes virtually all experiments on elementary particles, there are some problems. It does not explain gravity or dark matter and seems to be somewhat off in the prediction of $B$-meson decay (Altmannshofer et al. 2017). It cannot explain
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Figure 2.1: The particles of the Standard Model. The twelve leptons are on the left. The purple six at the top are the quarks, the green six at the bottom the leptons. Both quarks and leptons are divided in three generations, each in a column. Particles in different generations are said to have different flavors. The four red particles are the gauge bosons. These are carriers of the fundamental forces. The yellow particle in the top-right is the Higgs boson. Picture by Cush (2017).
the matter-antimatter asymmetry in the universe or the division of fermions in three flavors (Vanhoefer 2017, p. 1). It is clear that there must be something more than the Standard Model. Usually, this is called, physics beyond the Standard Model. The problem is, of course, that we do not know what kind of physics this would be. There are a great deal of beyond the Standard Model theories, and to find out which are true, we have to test their predictions.

Many of these theories predict that there are lepton flavor violating interactions. This contradicts the Standard Model, in which almost all interactions conserve lepton flavor. This means that in the Standard Model, a lepton of one flavor, say an electron, cannot change into a different flavor of lepton, say a muon. If an interaction does change the flavor of a lepton, this conservation law is violated, and that interaction is said to be lepton flavor violating. The only known interactions that are lepton flavor violating, are neutrinos oscillations. In these oscillations, neutral leptons, called neutrinos, can change their flavor freely. This neutral lepton flavor violation necessarily also leads to lepton flavor violation in the other, charged, leptons, but in immeasurably small amounts (Barger et al. 2012, p.33; Bilenky et al. 1977, p. 8). Many beyond the Standard Model theories predict lepton flavor violation in much larger amounts (CMS Collaboration 2015, p. 1). If lepton flavor violation can be observed in such large amounts, this could confirm beyond the Standard Model physics and may help differentiate between theories.

2.3. Lepton Flavor Violating Higgs

One interaction that many theories predict to be lepton flavor violating, is the Higgs decay (Lami et al. 2016, p. 2). In the Standard Model, the Higgs boson can decay into a lepton anti-lepton pair, but because of lepton flavor conservation, these must always have the same flavor. A Higgs can thus decay into a pair of a muon and an anti-muon, but not into a muon and an anti-tauon. If we can observe a reaction in which a Higgs decays into a pair of leptons with different flavor, this reaction cannot be part of the Standard Model, so any signal of such a reaction would indicate new physics, something beyond the Standard Model. This is the reason it is interesting to search for lepton flavor violation.

Lepton flavor violation is possible in three different combinations: muon-electron, tauon-electron and tauon-muon. Of these, muon-electron lepton flavor violation has
been studied the most. This interaction is studied using the muon decay $\mu \rightarrow e\gamma$. If muon electron lepton flavor violation is possible, muons should be able to decay into electrons via some decay diagram, where the difference in energy is carried off by a photon. Unlike charged lepton flavor violation involving the heavier tau lepton, the initial particle of the muon decay is very easy to make. This has made it possible to measure this muon decay very accurately. This has been done in the MEG experiment, yet so far no indication of the $\mu \rightarrow e\gamma$ decay has been found (MEG Collaboration 2016, p. 28). The current bound from MEG on the $\mu \rightarrow e\gamma$ branching ratio is $4.2 \times 10^{-13}$.

Like the muon decay, tauon-electron and tauon-muon lepton flavor violation can be studied using tauon decay. Limits on these experiments come from BaBar, and are far less stringent than the limit on muon decay (BaBar Collaboration et al. 2010, p. 7). The best bound on $\tau \rightarrow e\gamma$ decay is $3.3 \times 10^{-8}$, on $\tau \rightarrow \mu\gamma$ decay it is $4.4 \times 10^{-8}$. This means lepton flavor violation involving tauons could potentially still be quite strong, strong enough that it might be possible to find it in a different way.

In 2015, it appeared that precisely that had happened. The CMS group from LHC seemed to have measured an interaction, different from tauon decay, in which lepton flavor violation had occurred. They published a paper on what seemed to be a resonance in the production of muon-tauon pairs around the mass of the Higgs boson, see figure 2.2 (CMS Collaboration 2015). This sparked interest in the $H \rightarrow \tau\mu$ decay, and made people look into different ways to measure this potential new interaction vertex. Unfortunately, more recent searches by CMS did not show the resonance anymore. Also measurements from ATLAS did not confirm the anomaly (ATLAS Collaboration 2015). Despite this, there remains a possibility that the $H \rightarrow \tau\mu$ is possible, albeit with smaller magnitude (CMS Collaboration 2017b). In this thesis, we will look into a different decay that also contains the $H\tau\mu$ vertex, the $Z \rightarrow \tau\mu\gamma$ decay, to see if this decay can help us find lepton flavor violation.

As mentioned, there are different ways to probe the $H \rightarrow \tau\mu$ interaction. For this thesis, we chose to study $Z \rightarrow \tau\mu\gamma$ decay, mainly because of interest from experimental colleagues. They were looking into $Z \rightarrow \tau\mu$, and noticed they could improve their sensitivity by adding a photon to the final state. This also makes it possible to use a top quark as intermediate particle. This $Z \rightarrow \tau\mu$ decay with a top loop is the subject of this thesis. It is shown in figure 2.4. Using a top in the loop is useful, because the $Z$ boson does not couple directly to the Higgs boson and therefore first has to
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Figure 2.2: Measurement of $\tau\mu$ pair production in the 1 jet $\tau_e$ channel. There is a slight excess visible with energy of about the Higgs mass. This picture shows an older measurement, in newer data no excess is present anymore. Measurement and picture by CMS Collaboration (2015).

![Figure 2.2: Measurement of $\tau\mu$ pair production in the 1 jet $\tau_e$ channel. There is a slight excess visible with energy of about the Higgs mass. This picture shows an older measurement, in newer data no excess is present anymore. Measurement and picture by CMS Collaboration (2015).](image)

Figure 2.3: The Higgs decay $H \rightarrow \tau\mu$ thought to be observed at CMS. This same interaction vertex can also be part of a more complex interaction.

![Figure 2.3: The Higgs decay $H \rightarrow \tau\mu$ thought to be observed at CMS. This same interaction vertex can also be part of a more complex interaction.](image)
2. Introduction

Figure 2.4: The $Z \to \tau \mu \gamma$ decay discussed in this thesis, which contains the $H \tau \mu$ vertex. Important features of this diagram are the top quark loop, which gives a large coupling to the Higgs, and the final state photon, which improves experimental sensitivity.

Figure 2.5: A hypothetical $Z \to \tau \mu$ decay. This decay is not possible, because of spin conservation. The $Z$ is a spin 1 particle, while the $H$ is spinless, so this decay is not possible without adding an extra particle to connect to the loop.

decay into a different, intermediate particle. This intermediate particle than couples to the Higgs. Because of its high mass, the top quark has a strong coupling to the Higgs, potentially making this a significant decay channel. Without photon in the final state, it is not possible to use the same decay diagram with a top quark loop, as shown in figure 2.5, since this violates spin conservation. Different $H \to \tau \mu$ decays are possible, though (Alvarado et al. 2016).

It is important to note that while lepton decay allows us to measure lepton flavor violation regardless of its source, we can use this $Z$ decay only to find lepton flavor violation if it occurs in the interaction between the leptons and the Higgs boson.

In this thesis we will try to find out whether it is useful to study $Z \to \tau \mu \gamma$ decay in order to measure the strength of the $H \to \tau \mu$ interaction. First we will discuss in some more detail why we look at lepton flavor violation to find beyond the Standard Model physics, how the Higgs boson is involved and why we specifically study the
2.3. Lepton Flavor Violating Higgs

$Z \to \tau \mu \gamma$ decay to measure this in chapter 3. Then in chapter 4, we will see how we can use this decay to measure lepton flavor violation. This requires us to do some calculations, which are described in chapter 5. The result of these calculations then are laid out in chapter 6. Finally in chapter 7, we give find a discussion of the results, as well as some recommendations for further research.
3. Background

3.1. Flavor Conservation

In the Standard Model, lepton flavor violation is highly suppressed by the small masses of the neutrinos. While these neutrinos can change their flavor in neutrino oscillations, interactions where a charged lepton changes flavor have immeasurably small branching ratios (Barger et al. 2012, p.33; Cottingham et al. 2007, p. 3). Because of this, any signal we find of lepton flavor violation cannot come from Standard Model interaction, and therefore has to be an indication of new physics.

Many beyond the Standard Model theories predict lepton flavor violation. The law of lepton flavor conservation in the Standard Model is accidental and often naturally violated in Standard Model extensions (CMS Collaboration 2015, p. 1). To understand this, we must consider why lepton flavor is conserved in the Standard Model. This is not immediately obvious; after all, quark flavor is not conserved either.

To find out why lepton flavor is conserved in the Standard Model, we will look at the Higgs mechanism. The Higgs mechanism was originally proposed by Peter Higgs, and others, as a way to give mass to the W and Z boson and consequently limit the range of the weak force that they carry (Higgs 1964; Englert et al. 1964).

3.2. Flavor Violation in Quarks

The reason quark flavor conservation is violated in the Standard Model, is that all quarks have mass (Peskin et al.1995, p. 719). This mass comes from interaction with the Higgs field, and therefore it is the Higgs field that determines their mass eigenstates. The interaction term for down-type quarks with the Higgs field $\phi$ is

$$L_{\text{quark-Higgs}} = -\lambda_{d}^{ij} Q_{L}^{i} \cdot \phi d_{R}^{j} + \text{h.c.}. \quad (3.1)$$

There is a similar one for up-type quarks (Peskin et al. 1995, p. 722). Here a down-type quark $d_{R}^{i}$ couples to the Higgs field $\phi$ and a quark doublet $Q_{L}^{i}$,

$$Q_{L}^{i} = \begin{pmatrix} u^{i} \\ d^{i} \end{pmatrix}_{L}, \quad (3.2)$$
3. Background

where $i$ and $j$ indicate the generation of the quarks. The matrix $\lambda$ can be any complex valued matrix, without restrictions. To simplify this equation, we can rotate $Q_L$ and $d_R$. To do that, we substitute $\lambda_d$ by

$$\lambda_d = U_d D_d W_d^\dagger,$$  \hspace{1cm} (3.3)

where $D_d$ is a diagonal matrix and $U_d$ and $W_d$ are unitary matrices. These unitary matrices can then be absorbed by $Q_L$ and $d_R$, thus rotating these states. Since $D_d$ is diagonal, there is no mixing between the rotated quark states. If we also parametrize the Higgs field $\phi$ using

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix},$$  \hspace{1cm} (3.4)

we can write the interaction term as

$$L_{\text{quark-Higgs}} = -m_d^i \overline{d}_L^i d_R^i \left(1 + \frac{H}{v}\right) + \text{h.c.},$$  \hspace{1cm} (3.5)

where $m_d^i$ is the mass of the down-type quark of generation $i$:

$$m_d^i = \frac{1}{\sqrt{2}} D_{ii} v.$$  \hspace{1cm} (3.6)

This interaction term now actually contains two term. The first term gives mass to the quarks. This means the rotations we did on $d_L$ and $d_R$ converted them to their mass eigenstates. The second term is a Yukawa interaction between the quark and the physical Higgs field $H$. This Yukawa term is also diagonal in the basis of mass eigenstates, so no flavor mixing occurs in the Higgs interaction.

The up-type quarks have a similar interaction term, and it, too, determines their mass eigenstates. These mass eigenstates also diagonalize the Yukawa interaction, so in interactions between up-type quarks, flavor is not mixed either. But the mass eigenstates of these up-type quarks are rotated by different matrices, $U_u$ and $W_u$. These are not equal to $U_d$ and $W_d$, since the original matrix $\lambda_u$, is in general not equal to $\lambda_d$. This gives a mismatch in eigenstates when up-type and down-type quarks are converted into each other: the resulting quark is not in its mass eigenstate, it is in a
linear combination of mass eigenstates. This means that when an interaction converts up-type quarks or down-type quarks into the other, flavor is not conserved.

3.3. Flavor Violation in Leptons

Flavor violation in leptons works virtually the same. The term in the Lagrangian for charged lepton is

\[
L_{\text{lepton-Higgs}} = -\lambda_{ij} \ell_i^L \cdot \phi e^j_R + \text{h.c.},
\]

and the neutrinos have a similar one. Again this causes the eigenstates to be mismatched, and this allows leptons to violate flavor conservation. But just as quarks only change flavor when up- and down-type quarks are converted into each other, leptons only change flavor when charged leptons and neutrinos are converted. This has an important consequence; since neutrinos are always involved in Standard Model lepton flavor violation, these interactions are suppressed by the interaction strength of these neutrinos. In order to change the flavor of a charged lepton there even have to be two neutrino interactions, giving a very large suppression. For example, the \(\tau \to \mu \gamma\) decay depicted in figure 3.1 has a branching ratio of order \(\text{BR} \approx 10^{-54}\) (Bilenky et al. 1977).

Still, in beyond the Standard Model theories, it is possible to introduce lepton flavor violation. Therefore, there is effectively no charged lepton flavor violation in the Standard Model. In beyond the Standard Model theories, however, it is possible to introduce lepton flavor violation in greater amounts. One way in which it may enter,

\[\text{Figure 3.1: } \tau \to \mu \gamma \text{ decay through neutrino oscillations. The cross indicates that the neutrino changes flavor. This is a Standard Model decay that allows charged leptons to change flavor, but its branching ratio is immeasurably small, only about } \text{BR} \approx 10^{-54} \text{ (Bilenky et al. 1977).}\]
3. Background

is in through interaction with the Higgs field. The interaction term between leptons and the Higgs field can be written as

$$L_{\text{lepton-Higgs}} = -m_i^j \bar{e}_L^j \left( 1 + \frac{H}{v} \right) e_R^j + \text{h.c.},$$

(3.8)

analogous to equation 3.5. Here, too, the mass term and the Yukawa interaction term are diagonal in the same basis, that of mass eigenstates. However, it is possible to add additional terms in which the leptons interact with the Higgs field $H$. These may cause the mass matrix and the Yukawa interaction matrix in different ways, so that they cannot be diagonalized simultaneously anymore (Vanhoefer 2017, p. 20). In other words, these extra interaction terms allow lepton flavor mixing, when leptons interact with a Higgs. Lepton flavor violating Higgs interactions arise in many models, such as Multiple Higgs, SuSy and Composite Higgs (Lindner et al. 2016).

The specifics of this lepton flavor violating Higgs interaction are model dependent, but it is possible to make a model agnostic effective Lagrangian. Such an effective Lagrangian, describes the interaction in a general way, regardless of what model exact model is used. In general such a Lagrangian can have a scalar interaction part and a pseudo-scalar interaction part each with independent coefficients. Also the coefficients might have different values when the incoming and outgoing particles are interchanged, $g_{ij} \neq g_{ji}$. This gives 12 different coefficients in total.

$$L_{\text{effective}} = g_{ij}^{\text{scalar}} \phi \bar{\ell}_i \ell_j + g_{ij}^{\text{pseudo-scalar}} \phi \bar{\ell}_i \gamma_5 \ell_j$$

(3.9)

In this thesis, we will only consider the cases where the leptons involved are a $\mu$ and a $\tau$, and we will assume that their coefficient stays the same when the particles are interchanged, $g_{\tau \mu} = g_{\mu \tau}$. Also, we will assume that the coefficient of the scalar and

$$\mu \quad H \quad \tau = -i g_{\tau \mu} (1 - \gamma_5)$$

Figure 3.2: Feynman rule for the $H\tau\mu$ interaction vertex that follows from the Lagrangian in equation 3.10.
pseudo-scalar parts are equal, $g^{ij}_{\text{scalar}} = g^{ij}_{\text{pseudo-scalar}}$. That leaves us with only one coefficient, $g_{\tau\mu}$, and the following Lagrangian:

$$\mathcal{L}_{\text{effective}} = g_{\tau\mu} \phi (1 + \gamma_5) \mu + g_{\tau\mu} \phi (1 + \gamma_5) \tau. \quad (3.10)$$

The Feynman rule that follows from this Lagrangian is shown in figure 3.2.
4. Method

4.1. Measuring the Coupling Strength

We want to find out if we can use the $Z \rightarrow \tau \mu \gamma$ decay to measure the coupling strength of the $H \tau \mu$ interaction vertex. To do this, we will calculate the branching ratio as a function of this coupling strength, $g_{\tau \mu}$. Then we will calculate an upper bound on this coupling strength from other decays, $H \rightarrow \tau \mu$ and $\tau \rightarrow \mu \gamma$. We will then use this upper bound on the coupling strength to find an upper bound on the branching ratio of the $Z \rightarrow \tau \mu \gamma$ decay and consider whether this branching ratio is large enough to be measured. If it is large enough, $Z \rightarrow \tau \mu \gamma$ decay can be used to improve the bounds on the coupling strength, $g_{\tau \mu}$. If the branching ratio is too small to be measured, this decay provides no additional value to existing results from the other decays.

To get a bound on the coupling strength $g_{\tau \mu}$, we need an interaction that contains the $H \tau \mu$ coupling. Any interaction that contains this coupling can be used to find the coupling strength, at least in principle. However, some will work better than others. Whether a decay is useful to us, depends largely on three things:

1. How easy the initial particle is to produce (Prevalence)
2. How often that initial particle decays into the desired products (Branching Ratio)
3. How easily we can detect the products of the decay (Sensitivity)

4.2. $H \rightarrow \tau \mu$

The most straightforward process is simply the Higgs decay, shown in figure 4.1. In this decay the only interaction is the lepton flavor violating $H \tau \mu$ coupling. This diagram contains no elements that suppress the decay, like loops. This allows the branching ratio to be relatively large. However, the initial particle, the Higgs, is hard to produce, the cross section of the most important production channel is only $4.858 \times 10^{-2}$ nb (Anastasiou et al. 2016, p. 1). Also the experimental sensitivity for the $\tau \mu$ final state is low, resulting in a current experimental upper limit on the branching ratio of $7.55 \times 10^{-3}$ (Patrignani et al. 2017a, p. 6).
4. Method

Figure 4.1: $H \rightarrow \tau \mu$ decay. This is the simplest decay involving the $H\tau\mu$ vertex. Since it contains no elements that suppress it, it can have a relatively large branching ratio, up to $7.55 \times 10^{-3}$.

4.3. $\tau \rightarrow \mu\gamma$

There are other diagrams that contain the $H\tau\mu$ coupling, with different initial particles that may be easier to make, or a final state that can be measured to higher sensitivity. We might therefore be able to get a more precise bound on the coupling strength from these, even if their branching ratio is more suppressed and thus smaller.

One such other diagram is the $\tau$ decay diagram in figure 4.2. In this diagram the lepton interacts twice with the Higgs here, but only one of these is the lepton flavor violating coupling. The other is just a Standard Model coupling. Depending on which is which, the intermediate particle is either a $\mu$ or a $\tau$. Unlike the $H \rightarrow \tau \mu$ decay, the $\tau \rightarrow \mu\gamma$ decay has a greatly suppressed branching ratio. The number of $\tau$s available is larger, however, the relevant production cross section is $\sigma_{e^+e^- \rightarrow \tau^+\tau^-} = 9.19 \times 10^{-1}$ nb (BaBar Collaboration et al. 2010, p. 4). Also, the final state can be measured more accurately, which leads to a current experimental upper limit on the branching ratio of $4.4 \times 10^{-8}$ (Lindner et al. 2016, p. 13). Still, put together, the Higgs decay puts a stronger bound on the coupling strength $g_{H\tau\mu}$ than the $\tau$ decay, because the $\tau$’s decay rate is suppressed so much.

4.4. $Z \rightarrow \tau\mu\gamma$

This $Z$ decay is the decay from which we try to get a new bound on the Higgs coupling, see figure 4.3. The $Z$ boson is easier to produce than the Higgs, the production cross section is $\sigma_{pp \rightarrow ZZ} = 1.72 \times 10^{-2}$ nb (CMS Collaboration 2017a, p. 19). Also, because of the extra final state photon, the experimental sensitivity of this decay could be significantly higher than for a decay with only a $\mu$ and a $\tau$ in the final state (Personal communication with Olga Igonkina, September 19, 2017). This means that
Figure 4.2: $\tau \rightarrow \mu \gamma$ decay, via a Higgs loop. In this decay the lepton interacts with the Higgs twice. The lepton flavor violation can happen in either of these couplings, the other one is just the Standard Model coupling. Its branching ratio is $4.4 \times 10^{-8}$.

Figure 4.3: The $Z \rightarrow \tau \mu \gamma$ decay that is the subject of this thesis. This decay has some elements that suppress it, such as the loop and the three-particle phase space of the final state, but the extra photon makes it possible to measure this decay with great sensitivity and the heavy top quark has a strong coupling to the Higgs boson. This decay can also be useful for finding a bound on the coupling strength, provided its branching ratio is large enough.

We will now try to find out how large the branching ratio of $Z \rightarrow \tau \mu \gamma$ can still be, given the bounds the other decays already put on $g_{\tau \mu}$. To do that we will calculate the branching ratio of the $Z$ decay as a function of $g_{\tau \mu}$, calculate the bounds on $g_{\tau \mu}$ and plug them into the branching ratio of the $Z \rightarrow \tau \mu \gamma$ decay. This will give an upper bound on this branching ratio. If this upper bound large enough, we may be able to use it to set a new bound on $g_{\tau \mu}$ in the future.
5. Branching Ratio Calculations

In this chapter, we will calculate the branching ratios of the three processes mentioned in chapter 4: $H \rightarrow \tau \mu$ decay, $\tau \rightarrow \mu \gamma$ decay and $Z \rightarrow \tau \mu \gamma$ decay. We will also calculate the branching ratio of the decay $H \rightarrow \gamma \gamma$. This decay is similar to the $Z \rightarrow \tau \mu \gamma$ decay and we can use it to verify its result.

The branching ratio of the other three processes all depend on the lepton flavor violating coupling strength $g_{\tau \mu}$. For the first two, $H \rightarrow \tau \mu$ and $\tau \rightarrow \mu \gamma$, we will compare the calculated branching ratio to experimental bounds and use these to extract upper bounds on $g_{\tau \mu}$. We will then insert these upper bounds in the formula for the branching ratio of $Z \rightarrow \tau \mu \gamma$ and estimate an upper bound on this branching ratio.
5. Branching Ratio Calculations

5.1. $H \to \tau \mu$

In this section, we will calculate the branching ratio of the $H \to \tau \mu$ decay. This branching ratio will depend on $g_{\tau \mu}$, the coupling strength of the lepton flavor violating $H \tau \mu$ interaction. We will later use this dependence to calculate a bound on $g_{\tau \mu}$, using experimental bounds on the branching ratio.

$H \to \tau \mu$ is the simplest decay with the lepton flavor violating interaction $H \tau \mu$. To calculate its branching ratio, we first have to calculate the matrix element of the decay $\mathcal{M}$. We then have to take the square of $\mathcal{M}$ and use that to calculate the decay rate. Finally, we will divide the decay rate by the total decay rate of $H$, which will yield the branching ratio.

To get the matrix element $\mathcal{M}$, we have to apply the Feynman rules in Appendix A to the decay diagram, shown in figure 5.1. Doing this gives

$$
\mathcal{M} = \bar{u}_\tau(p_2) \left( -ig_{\tau \mu} \left( 1 + \gamma_5 \right) \right) u_\mu(p_3).
$$

We now need to square the matrix element $\mathcal{M}$. We will also sum over all possible spins of the $\tau$ and $\mu$, since we cannot know what their spins are. We can then replace the sum over spins by a trace and evaluate this trace to find the value of $|\mathcal{M}|^2$.

$$
\sum_{\text{spins}} |\mathcal{M}|^2 = \sum_{\text{spins}} \left[ \bar{u}_\tau(p_2) \left( -ig_{\tau \mu} \left( 1 + \gamma_5 \right) \right) u_\mu(p_3) \right]^* (\text{Griffiths 2008, p. 251})
$$

$$
= g_{\tau \mu}^2 \text{Tr} \left[ (1 + \gamma_5) (p_3 + m_3) (1 - \gamma_5) (p_2 + m_2) \right]
$$

$$
= 8g_{\tau \mu}^2 p_2 \cdot p_3
$$

Figure 5.1: The Feynman diagram of the $H \to \tau \mu$ decay. This is the simplest decay containing the lepton flavor violating coupling.
The square of the matrix element now still depends on $p_2$ and $p_3$, the momenta of the outgoing particles. These are, neglecting the mass of the $\tau$ and $\mu$

$$p_1 = \left( \frac{m_H}{0} \right), \quad p_2 = \left( \frac{1}{2} m_H \right), \quad p_3 = \left( \frac{1}{2} m_H \hat{z} \right).$$

We can then put the squared matrix element into the formula for the decay rate of a two-particle decay. This gives

$$\Gamma(H \rightarrow \tau \mu) = \frac{|p_\tau|}{8\pi m_H^2} \sum_{\text{spins}} |\mathcal{M}|^2$$

(Griffiths 2008, p. 429) (5.6)

$$= g_{\tau\mu}^2 \frac{m_H}{4\pi}$$

(5.7)

$$= 9.95 \text{ GeV} g_{\tau\mu}^2.$$  

(5.8)

We divide this decay rate by the total decay rate of $H$

$$\Gamma_{\text{Total}}(H) < 0.013 \text{ GeV},$$  

(Patrignani et al. 2017a, p. 5) (5.9)

to get the branching ratio

$$\text{BR}(H \rightarrow \tau \mu) = \frac{\Gamma(H \rightarrow \tau \mu)}{\Gamma_{\text{Total}}(H)} > g_{\tau\mu}^2 \frac{9.95 \text{ GeV}}{0.013 \text{ GeV}} = 7.7 \times 10^2 g_{\tau\mu}^2.$$  

(5.10)
5. Branching Ratio Calculations

5.2. $\tau \rightarrow \mu \gamma$

In this section, we will calculate the branching ratio of $\tau \rightarrow \mu \gamma$ decay with a top quark loop. Like that of the $H \rightarrow \tau \mu$ decay, this branching ratio will depend on $g_{\tau \mu}$, the coupling strength of the lepton flavor violating $H \tau \mu$ interaction. We will later use this dependence to calculate a bound on $g_{\tau \mu}$, using experimental bounds on the branching ratio.

In this $\tau \rightarrow \mu \gamma$ decay, the $\tau$ decays with a $H$ loop to a $\mu$ and a $\gamma$. Because of the loop, this calculation is more complicated than that of the $H \rightarrow \tau \mu$ decay. The method of the calculation we will use is based on Peskin et al. (1995, p. 184 - 196). Again, to find the branching ratio, we first have to calculate the matrix element of the decay $\mathcal{M}$, square it to calculate the decay rate and then divide the decay rate by the total decay rate to get the branching ratio.

A more detailed overview of this calculation can be found in Appendix C.

To get the matrix element $\mathcal{M}$, we have to apply the Feynman rules in Appendix A to the decay diagram, shown in figure 5.2. Doing this gives

$$\mathcal{M} = \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2_H} \frac{i}{(p_2 - k)^2 - m^2_\tau} \frac{i}{(p_1 - k)^2 - m^2_\tau} \epsilon_\mu(p_3)$$

$$\cdot \bar{u}_\tau(p_2)(ig_{\tau \mu})(p_2 - k + m_\tau)(-ie\gamma^\mu)(p_1 - p_3 + m_\tau)(ig_{H \tau})u_\mu(p_1)$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{N^\mu}{D'} \epsilon_\mu(p_3).$$

(5.11)

(5.12)

We have to integrate over all possible momenta $k$ of the particle in the loop, the $H$. We will use a standard integral to do that, but before we can apply the standard integral, we have to rewrite $\mathcal{M}$ in the right form. To do that, we will introduce another integral and a variable shift in the momentum $k$. The first step of the rewriting is to split the matrix element into two parts, the “numerator” $N^\mu$ and the “denominator” $D'$

$$N^\mu = -(e g_{\tau \mu} g_{H \tau})\bar{u}_\mu(p_2)(m_\mu - k + m_\tau)(\gamma^\mu)(2m_\tau - k)u_\tau(p_1),$$

(5.13)

$$D' = [k^2 - m^2_H] [k^2 - m^2_\tau] [(p_2 - k)^2 - m^2_\tau] [(p_1 - k)^2 - m^2_\tau],$$

(5.14)
5.2. \( \tau \rightarrow \mu \gamma \)

Figure 5.2: The Feynman diagram of \( \tau \rightarrow \mu \gamma \) decay, with an intermediate \( H \). This is a more complex diagram with the lepton flavor violating \( H \) coupling. In this diagram, a \( \tau \) first couples to a \( H \) via a Standard Model coupling. The \( \tau \) then emits a \( \gamma \) and recombines with the \( H \), this time using the lepton flavor violating vertex, so that it becomes a \( \mu \). The Standard Model and lepton flavor violating vertices could be interchanged, but this would mean that the Standard Model coupling would involve a \( \mu \) instead of a \( \tau \), which does not couple as strongly to the \( H \).

and use Feynman parameters to replace \( D' \) by a new “denominator” \( D \):

\[
\frac{1}{D'} = \int_{0}^{1} dx \ dy \ dz \ \delta(x + y + z - 1) \ \frac{2}{D^3} , \quad \text{(Peskin et al. 1995, p. 190)} \quad (5.15)
\]
\[
D = x[k^2 - m_{H}^2] + y[(p_2 - k)^2 - m_{\tau}^2] + z[(p_1 - k)^2 - m_{\tau}^2] . \quad (5.16)
\]

The whole matrix element has now become:

\[
\mathcal{M} = 2 \int_{0}^{1} dx \ dy \ dz \ \delta(x + y + z - 1) \ \int \frac{d^d k}{(2\pi)^d} \frac{N_\mu}{D^3} (p_3)
\]

(5.17)

To solve this integral, we will rewrite \( D \) in the form \( D = \ell^2 - \Delta \), with \( \ell = (k - \text{constant}) \) and \( \Delta \) constant in \( \ell \). The new variable \( \ell \) is only shifted by a constant, so \( d\ell = dk \). Since the numerator \( N_\mu \) contains \( k \)s as well, we must rewrite it too before we can carry out the integral over \( k \) (\( \ell \) by then).

We perform the substitution \( k \rightarrow \ell = (k - \text{constant}) \) in the denominator:

\[
D = \ell^2 - m_{\tau}^2(z^2 + yz + y) + p_3^2yz - xm_{H}^2 + m_{\mu}^2(y - y^2 - yz)
\]

\[
= \ell^2 - \Delta ,
\]

(5.18)
5. Branching Ratio Calculations

and also in the numerator:

\[ N^\mu = - \left[ e g_{\tau \mu} g_{HT} \right] \pi_\mu \left( p_2 \right) \left[ m_\mu + m_\tau - \ell - p_2 y - p_1 z \right] \gamma^\mu \]  
\[ \cdot \left[ 2m_\tau - \ell - p_2 y - p_1 z \right] u_\tau(p_1). \]  

(5.20)

(5.21)

We now rewrite the numerator in such a way, that it consists of a part proportional to \( \sigma^{\mu\nu} p_{3\nu} \) and a part proportional to \( \gamma^\mu \). We can then drop the part that is proportional to \( \gamma^\mu \), because it will not contribute to the decay rate.

\[ N^\mu = i e g_{\tau \mu} g_{HT} \pi_\mu \left[ \sigma^{\mu\nu} p_{3\nu} m_\tau \left( z^2 - 2z + yz - y \right) \right] u_\tau + O(\gamma^\mu) \]  

(5.22)

The whole matrix element is now

\[ M = 2 \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{N^\mu}{D^3} \epsilon_\mu(p_3) \]  
\[ = M_\sigma + O(\gamma^\mu). \]  

(5.23)

(5.24)

We drop the part with \( \gamma^\mu \) and proceed with the part containing \( \sigma^{\mu\nu} p_{3\nu}, M_\sigma \):

\[ M_\sigma = 2 i e g_{\tau \mu} g_{HT} \pi_\mu \int_0^1 dx dy dz \delta(x + y + z - 1) \sigma^{\mu\nu} p_{3\nu} \]  
\[ \cdot m_\tau \left( z^2 - 2z + yz - y \right) u_\tau \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \epsilon_\mu(p_3). \]  

(5.25)

The matrix element is now in the right form, so we can use the standard integral to solve the integral over the momentum \( \ell \) of \( H \). The standard integral is

\[ \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^3} = -i \frac{1}{2 \cdot (4\pi)^2} \frac{1}{\Delta}. \]  

(Peskin et al. 1995, p. 193)  

(5.26)

Applying it to \( M_\sigma \) this gives

\[ M_\sigma = -e g_{\tau \mu} g_{HT} m_\tau \pi_\mu \sigma^{\mu\nu} p_{3\nu} u_\tau \epsilon_\mu(p_3) \]  
\[ \cdot \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{z^2 - 2z + yz - y}{m_\tau^2 (z^2 + yz + y) + x m_H^2}. \]  

(5.27)
We have solved the momentum integral, but are still left with the integral over $x$, $y$ and $z$. We call this integral $I_{\mu, 1}^{++}$, analogous to Lindner et al. (2016, p. 20). It evaluates to

$$I_{\mu, 1}^{++} = \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \frac{z^2 - 2z + yz - y}{m_T^2(z^2 + yz + y) + \tau m_H^2}$$

$$= -\frac{1}{m_T^2} \left[ \frac{1}{6} - \left( \frac{3}{2} \log(\lambda^2) \right) \right],$$

where $\lambda \equiv \frac{m_T}{m_H}$. We plug this back in $M_\sigma$, which now becomes

$$M_\sigma = \frac{e \bar{\sigma}_T \gamma_T m_T}{(4\pi)^2} \frac{1}{m_T^2} \bar{u}_\mu \sigma^{\mu\nu} p_{3\nu} \left[ \frac{1}{6} - \left( \frac{3}{2} \log(\lambda^2) \right) \right] u_\tau \epsilon_\mu(p_3).$$

Now we square the matrix element and average over all possible spins of the $\tau$ and sum over the spins of the $\mu$ and the polarizations of the $\gamma$. This allows us to evaluate the polarization vectors and the lepton spinors, which gives

$$\frac{1}{2} \sum_{\text{spins, polarizations}} |M_\sigma|^2 = 4\pi \alpha_{em} m_T^6 \left( \frac{\bar{\sigma}_T \gamma_T m_T}{(4\pi)^2} \right)^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{2}{m_H^4}. \tag{5.31}$$

Here we have used that the momenta of the particles are

$$p_1 = \left( m_T, 0 \right), \quad p_2 = \left( \frac{1}{2} m_T, -\frac{1}{2} m_T \hat{z} \right), \quad p_3 = \left( \frac{1}{2} m_T, \frac{1}{2} m_T \hat{z} \right). \tag{5.32}$$

We can then put the squared matrix element into the formula for the decay rate of a two-particle decay. This gives the decay rate

$$\Gamma(\tau \to \mu\gamma) = \frac{|p_\mu|}{8\pi m_T^2} \frac{1}{2} \sum_{\text{spins}} |M|^2 \tag{5.33}$$

$$= \frac{1}{2} \alpha_{em} m_T^5 \left( \frac{\bar{\sigma}_T \gamma_T m_T}{(4\pi)^2} \right)^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{2}{m_H^4}, \tag{5.34}$$

$$= 9.40 \times 10^{-9} \text{eV} \bar{\sigma}_T. \tag{5.35}$$
5. Branching Ratio Calculations

We divide this decay rate by the total decay rate of $\tau$

$$\Gamma_{\text{Total}}(\tau) = 0.0023 \text{ eV}, \quad \text{(Patrignani et al. 2017c, p. 2)} \quad (5.36)$$

to get the branching ratio

$$\text{BR}(\tau \rightarrow \mu \gamma) = \frac{\Gamma(\tau \rightarrow \mu \gamma)}{\Gamma_{\text{Total}}(\tau)} = \frac{9.4 \times 10^{-9} \text{ eV}}{0.0023 \text{ eV}} g_{\tau\mu}^2$$

$$= 4.1 \times 10^{-6} g_{\tau\mu}^2. \quad (5.37)$$
5.3. $H \rightarrow \gamma\gamma$

In this section we will calculate the branching ratio of $H \rightarrow \gamma\gamma$ with a top quark loop. This is a Standard Model process that does not involve lepton flavor violation, and is not dependent on $g_{\tau\mu}$. We do this calculation because this decay’s diagram is very similar to that of $Z \rightarrow \tau\mu\gamma$’s. That means we can verify the result of the latter by checking the result of this calculation.

The diagram of this decay consists of a Higgs particle that decays via a top loop into two photons. Because this decay also has one loop, the calculation of this decay has many similarities that that of $\tau \rightarrow \mu\gamma$ and the method of calculation is again analogous to Peskin et al. (1995, p. 184 - 196). The basic outline is that we first calculate the matrix element $M$ of the decay, then square it and plug it into the formula for the decay rate, and finally divide the decay rate by the total decay rate of $H$ the get the branching ratio.

During this calculation of the $H \rightarrow \gamma\gamma$ decay, we will heavily rely on Mathematica and Package X Patel 2016. The Mathematica code used in this calculation can be found in section C.

We get the matrix element $M$ by applying the Feynman rules in Appendix A to the decay diagram, shown in figure 5.3. In this diagram, the direction of the top quark’s momentum is clockwise, but in reality it can also be counterclockwise. This means we have to evaluate two different diagrams. The matrix element is the sum of the matrix elements of the two diagrams, $M = M_1 + M_2$. The color factor $N_C$ accounts for the different colors the top quark can have.

The matrix element of the first diagram is

$$M_1 = - \int \frac{d^4k}{(2\pi)^d} e_v(p_{3})e_{\rho}(p_{2})(-ig_{Ht})i(i g_{\gamma t})i(i g_{\gamma t})i N_C \cdot \text{Tr} \left[ \left( \frac{k - p_{3} + m_t}{(k - p_{3})^2 - m_t^2} \right) \gamma^\nu \left[ \left( \frac{k + m_t}{(k + m_t)^2 - m_t^2} \right) \gamma^\rho \left( \frac{k + p_{2} + m_t}{(k + p_{2})^2 - m_t^2} \right) \right] \right] .$$

(5.39)
5. Branching Ratio Calculations

\[ k + p_2 \]
\[ H \rightarrow \gamma \gamma \]
\[ k - p_3 \]
\[ t \]
\[ \rho_2 \]
\[ \rho_1 \]
\[ \rho_3 \]

Figure 5.3: The Feynman diagram of the \( H \rightarrow \gamma \gamma \) decay, with a top loop. This is a Standard Model decay, without any lepton flavor violation. This decay diagram is very similar that of the \( Z \rightarrow \tau \mu \gamma \) decay studied in this thesis and we will use it for cross checking.

We will call the trace the “numerator” \( N_1 \)

\[ N_1 = \text{Tr} \left[ \left( k - p_3 + m_t \right) \gamma^\nu \left( k + m_t \right) \gamma^\rho \left( k + p_2 + m_t \right) \right], \quad (5.40) \]

which, as we find with Package X, equals

\[ N_1 = 16m_t k^\nu k^\rho + 8m_t k^\nu p_2^\rho - 8m_t k^\rho p_3^\nu - 4k^2 m_t g^{\nu \rho} \]
\[ + 4m_t^2 g^{\nu \rho} - 4m_t p_2^\rho p_3^\nu + 4m_t p_2^\nu p_3^\rho - 4m_t p_2 \cdot p_3 g^{\nu \rho}. \quad (5.41) \]

From the last row of equation 5.39 we call this the “denominator” \( D' \):

\[ D' = \left[ (k - p_3)^2 - m_t^2 \right] \left[ k^2 - m_t^2 \right] \left[ (k + p_2)^2 - m_t^2 \right]. \quad (5.43) \]

The matrix element of the second diagram, where the direction of the top quark’s momentum is reversed, is equal to that of the first, except for the “numerator”:

\[ M_2 = - \int \frac{d^d k}{(2\pi)^d} e_\nu(p_3) e_\rho(p_2)(-ig_{H\bar{t}}i(i g_{\bar{\gamma} t})i(i g_{\gamma t})iNC \]
\[ \cdot \text{Tr} \left[ \left( - k - p_2 + m_t \right) \gamma^\rho \left( - k + m_t \right) \gamma^\nu \left( - k + p_3 + m_t \right) \right] \]
\[ \cdot \frac{1}{(k - p_3)^2 - m_t^2} \frac{1}{k^2 - m_t^2} \frac{1}{(k + p_2)^2 - m_t^2}. \quad (5.44) \]
5.3. $H \to \gamma \gamma$

The “numerator” $N_2$ is here

\[
N_2 = 16m_t k^\nu k^\rho + 8m_t k^\nu p_2^\rho - 8m_t k^\rho p_3^\nu - 4k^2 m_t g^{\nu\rho} \\
+ 4m_t^3 g^{\nu\rho} - 4m_t p_2^\rho p_3^\nu + 4m_t p_2^\nu p_3^\rho - 4m_t p_2 \cdot p_3 g^{\nu\rho}. \tag{5.45}
\]

The total matrix element is thus

\[
\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \tag{5.47}
\]

\[
= - g_{H\gamma\gamma} N C e_\nu(p_3) e_\rho(p_2) \int \frac{d^d k}{(2\pi)^d} \frac{N_1 + N_2}{D'} \tag{5.48}
\]

We have to integrate over all possible momenta $k$ of the particle in the loop, the $t$. We will use a standard integral to do that, but before we can apply the standard integral we have to rewrite $\mathcal{M}$ in the right form. To do that, we will introduce another integral and a variable shift in the momentum $k$.

The calculation can be simplified by replacing momenta by masses whenever possible, using on-shell relations. To do that, we need to know the momenta of the particles, which are

\[
p_1 = \begin{pmatrix} m_H \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} \frac{1}{2} m_H \\ \frac{1}{2} m_H \hat{z} \end{pmatrix}, \quad p_3 = \begin{pmatrix} \frac{1}{2} m_H \\ -\frac{1}{2} m_H \hat{z} \end{pmatrix}. \tag{5.49}
\]

Also, we can use the fact that photons cannot have transverse polarization, since they travel at the speed of light. This means we can drop all terms with $p_2^\rho$ and $p_3^\nu$, as

\[
p_2^\rho e_\rho = 0, \quad p_3^\nu e_\nu = 0. \tag{5.50}
\]

The next step is to rewrite the “denominator” $D'$ into the form $(k^2 - \Delta)^3$, where $\Delta$ is independent of $k$. We use the “Feynman trick” to get $D'$ into the form $(k^2 + ak + b)^3$ and then we perform a shift in $k$ to lose the part linear in $k$.

\[
\frac{1}{D'} = \frac{1}{(k - p_3)^2 - m_t^2} \frac{1}{k^2 - m_t^2} \frac{1}{((k + p_2)^2 - m_t^2)} \tag{5.51}
\]

\[
= \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{D^3} \tag{5.52}
\]
5. Branching Ratio Calculations

with $D = k^2 - \Delta, \Delta = m_t^2 - m_H^2 yz$. The shift in $k$ we used here is

$$k \rightarrow k - p_2 y + p_3 z.$$  \hfill (5.53)

The matrix element $M$ is now

$$M = -g_{Ht} g_{\gamma l} g_{\gamma l} N_C e_{\nu} (p_3) e_{\bar{\nu}} (p_2) \cdot \int_0^1 dxdydz \delta (x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{2(N_1 + N_2)}{D^3},$$

where the momentum $k$ in the “numerators” $N_1$ and $N_2$ is still the old, unshifted $k$, and the $k$ in the “denominator” $D$ is the new, shifted $k$.

In the numerator we have to perform the same shift in $k$. We can drop any terms that are linear in $k$ after the shift, because these will vanish when we integrate over $k$ (since the denominator is even in $k$). When we do the shift and also apply the on shell conditions, wherever possible, we end up with

$$N = N_1 + N_2$$

$$= 32 m_t k^0 k^0 - 8 k^2 m_t g^{\nu \rho} + 8 m_t^2 m_H y z g^{\nu \rho} - 4 m_H^2 m_t g^{\nu \rho}$$

$$+ 8 m_t g^{\nu \rho} + 8 m_t p_2 p_3^\rho - 32 m_t y z p_2 p_3^\rho.$$  \hfill (5.55)

$$N = N_{k^0} + N_{k^2}$$

$$N_{k^0} = 8 m_t^2 m_H y z g^{\nu \rho} - 4 m_H^2 m_t g^{\nu \rho} + 8 m_t^3 g^{\nu \rho} + 8 m_t p_2 p_3^\rho - 32 m_t y z p_2 p_3^\rho$$

$$N_{k^2} = 32 m_t k^0 k^0 - 8 k^2 m_t g^{\nu \rho}$$

We split $N$ in two parts, one constant in $k$ and one quadratic in $k$, because we will need to use different standard integral for each part.

$$N = N_{k^0} + N_{k^2}$$

$$N_{k^0}$$

$$N_{k^2}$$

The matrix element has become

$$M = -g_{Ht} g_{\gamma l} g_{\gamma l} N_C e_{\nu} (p_3) e_{\bar{\nu}} (p_2) \cdot \int_0^1 dxdydz \delta (x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{2(N_{k^0} + N_{k^2})}{D^3}.$$  \hfill (5.60)
The matrix element is now in the right form, so we can do the integral over the top quark momentum $k$ using the standard integrals:

\[
\int \frac{d^d k}{2 \pi^d (k^2 - \Delta)^n} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n i}{(4 \pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta} n^{-d/2}, \quad \text{(Peskin et al. 1995, p. 807) (5.61)}
\]

\[
\int \frac{d^d k}{2 \pi^d (k^2 - \Delta)^n} \frac{k^2}{(k^2 - \Delta)^n} = \frac{(-1)^{n-1} d}{2 (4 \pi)^{d/2}} \frac{\Gamma(n - d - 1)}{\Gamma(n)} \frac{1}{\Delta} n^{-d/2-1}.
\]

To use the second standard integral, we first need to replace $k^\mu k^n$ using

\[
k^\mu k^n \rightarrow \frac{1}{d} k^2 g^{\mu \nu}, \quad \text{(Peskin et al. 1995, p. 807) (5.63)}
\]

\[
r \cdot k k^n = r^\mu k^\mu k^\nu \rightarrow r^\mu \frac{1}{d} k^2 g^{\mu \nu} = r^\nu \frac{1}{d} k^2.
\]

Now Mathematica can do the integral:

\[
\int \frac{d^d k}{(2 \pi)^d} \frac{2(N_{k^0} + N_{k^2})}{D^3} = -\frac{im_t (4yz - 1) (m_H^2 g^{\nu \rho} - 2p_2^\nu p_3^\rho)}{4\pi^2 (m_t^2 - m_H^2 yz)}.
\]

The full matrix element is now

\[
\mathcal{M} = -g_H g_{\gamma\gamma} g_{\gamma t} N_C e_\nu(p_3) e_\rho(p_2) \cdot \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{-im_t (4yz - 1) (m_H^2 g^{\nu \rho} - 2p_2^\nu p_3^\rho)}{4\pi^2 (m_t^2 - m_H^2 yz)}.
\]

We also have to solve the integrals over $x, y$ and $z$. Mathematica can do those too:

\[
\int_0^1 dx dy dz \delta(x + y + z - 1) \frac{-im_t (4yz - 1) (m_H^2 g^{\nu \rho} - 2p_2^\nu p_3^\rho)}{4\pi^2 (m_t^2 - m_H^2 yz)} = im_t \left( \frac{m_H^2 g^{\nu \rho} - 2p_2^\nu p_3^\rho}{4\pi^2 m_I^4} \right) \left[ 2m_I^2 + \left( m_I^2 - 4m_t^2 \right) \right]
\]

\[
\cdot \left( \text{Li}_2 \left( \frac{-2im_H}{\sqrt{4m_t^2 - m_I^2} - im_H} \right) + \text{Li}_2 \left( \frac{2im_H}{im_H + \sqrt{4m_t^2 - m_I^2}} \right) \right).
\]
This gives the matrix element

\[
\mathcal{M} = -g_h g_\gamma^2 g_\gamma^2 N_C e_{\nu} (p_3) e_\nu (p_2) \\
\cdot \frac{m_i^2 g^{\mu \nu} - 2 p_i^\mu p_j^\nu}{4 \pi^2 m_i^4} \left[ 2 m_i^2 + (m_i^2 - 4 m_t^2) \right] \\
\cdot \left( \text{Li}_2 \left( \frac{-2 i m_i}{\sqrt{4 m_i^2 - m_i^2}} \right) + \text{Li}_2 \left( \frac{2 i m_i}{m_i + \sqrt{4 m_i^2 - m_i^2}} \right) \right) .
\]

\[(5.68)\]

Now we square the matrix element and sum over all possible spins of the photons. This allows us to evaluate the polarization vectors, so that we get \(|\mathcal{M}|^2:\n\]

\[
\sum_{\text{polarizations}} |\mathcal{M}|^2 = (g_h g_\gamma^2 g_\gamma^2 N_C)^2 \frac{m_i^2}{8 m_i^4 \pi^4} \\
\cdot \left[ (m_i^2 - 4 m_t^2) \text{Li}_2 \left( \frac{m_H \left( m_H + i \sqrt{4 m_i^2 - m_i^2} \right)}{2 m_i^2} \right) \right] \\
+ (m_i^2 - 4 m_t^2) \text{Li}_2 \left( \frac{m_H \left( m_H - i \sqrt{4 m_i^2 - m_i^2} \right)}{2 m_i^2} \right) + 2 m_i^2 \gamma^2 .
\]

\[(5.69)\]

We put this into the formula for the decay rate:

\[
\Gamma (H \rightarrow \gamma \gamma) = \frac{1/2}{16 \pi m_H} \sum_{\text{polarizations}} |\mathcal{M}|^2 \quad \text{(Griffiths 2008, p. 429)} \quad (5.70)
\]

\[
= (g_h g_\gamma^2 g_\gamma^2 N_C)^2 \frac{m_i^2}{256 m_i^4 \pi^5} \\
\cdot \left[ (m_i^2 - 4 m_t^2) \text{Li}_2 \left( \frac{m_H \left( m_H + i \sqrt{4 m_i^2 - m_i^2} \right)}{2 m_i^2} \right) \right] \\
+ (m_i^2 - 4 m_t^2) \text{Li}_2 \left( \frac{m_H \left( m_H - i \sqrt{4 m_i^2 - m_i^2} \right)}{2 m_i^2} \right) + 2 m_i^2 \gamma^2 .
\]

\[(5.71)\]
Here the factor $1/2$ accounts for the two interchangeable final state particles.

We divide this decay rate by the total decay rate of $H$

$$\Gamma_{\text{Total}}(H) < 0.013 \times 10^9 \text{ eV} ,$$

(Patrignani et al. 2017a, p. 5) (5.73)

to get the branching ratio

$$\text{BR}(H \to \gamma\gamma) = \frac{\Gamma(H \to \gamma\gamma)}{\Gamma_{\text{Total}}(H)}$$

(5.74)

$$\frac{7.27 \times 10^2 \text{ eV}}{0.013 \times 10^9 \text{ eV}} = 5.6 \times 10^{-5} .$$

(5.75)
5. Branching Ratio Calculations

5.4. $Z \rightarrow \tau \mu \gamma$

In this section we will calculate the branching ratio of $Z \rightarrow \tau \mu \gamma$ decay with a top quark loop and an intermediate $H$. This branching ratio will depend on $g_{\tau \mu}$, the coupling strength of the $H \tau \mu$ interaction. We will later use bounds on $g_{\tau \mu}$ from the decays in section 5.1 and section 5.2 to calculate a bound on the branching ratio of this decay.

The diagram of this decay consists of a $Z$ boson that decays via a top loop into two photons. The calculation of this decay is very similar to the calculation of $H \rightarrow \gamma \gamma$. The main difference between the two is that with $Z \rightarrow \tau \mu \gamma$, one of the resulting particles decays one more time, leading to a 3-particle final state. The method of calculation is again analogous to Peskin et al. (1995, p. 184 - 196). The basic outline is that we first calculate the matrix element $\mathcal{M}$ of the decay, then square it and plug it into the formula for the decay rate, and finally divide the decay rate by the total decay rate of $Z$ to get the branching ratio.

Like with the calculation of $H \rightarrow \gamma \gamma$, we will use Mathematica and Package X Patel 2016. The code for the $Z \rightarrow \tau \mu \gamma$ calculation is mostly identical to the code for $H \rightarrow \gamma \gamma$ and can be found in section C.

We get the matrix element $\mathcal{M}$ by applying the Feynman rules in Appendix A to the decay diagram, shown in figure 5.4. In this diagram, the direction of the top quark’s momentum is clockwise, but in reality it can also be counterclockwise. This means we

![Figure 5.4: The Feynman diagram of $Z \rightarrow \tau \mu \gamma$ decay, with a top quark loop and an intermediate $H$. This diagram contains the lepton flavor violating $H$ coupling and has a three-particle final state, but is otherwise very similar to the $H \rightarrow \gamma \gamma$ decay in section 5.3.](image-url)
have to evaluate two different diagrams. The matrix element is the sum of the matrix elements of the two diagrams, \( M = M_1 + M_2 \). The color factor \( N_C \) accounts for the different colors the top quark can have. We neglect the width of the Higgs boson, as its momentum \( p_2 \) is much smaller than its mass \( m_H \), so it cannot be on-shell.

The matrix element of the first diagram is

\[
M_1 = -\int \frac{d^4k}{(2\pi)^4} \epsilon_\mu(p_1)\epsilon_\nu(p_3) \left( \frac{-igZt}{2} \right) i(ig\gamma_t)i(-igH_t)iN_C
\]

\[
\cdot \text{Tr} \left[ \gamma_\mu \left( c^f_V - c^f_A \gamma^5 \right) \left[ (k - p_3 + m_t)\gamma^\nu \left[ (k + m_t) \right]\left[ (k + p_2 + m_t) \right] \right] \right]
\]

\[
\cdot \frac{1}{(k - p_3)^2 - m_t^2} \frac{1}{k^2 - m_t^2} \frac{1}{(k + p_2)^2 - m_t^2}
\]

\[
\cdot \frac{1}{p_2^2 - m_H^2} (-ig\gamma_\mu)u_\mu(p_4)v_\nu(p_5)
\]

We will call the trace the “numerator” \( N_1 \)

\[
N_1 = \text{Tr} \left[ \gamma_\mu \left( c^f_V - c^f_A \gamma^5 \right) \left[ (k - p_3 + m_t)\gamma^\nu \left[ (k + m_t) \right]\left[ (k + p_2 + m_t) \right] \right] \right],
\]

which, as we find with Package X, equals

\[
N_1 = 8ic_A m_t \epsilon_{\mu,\nu \nu,\nu(p_3)} + 4ic_A m_t \epsilon_{\mu,\nu \nu,\nu(p_2),\nu(p_3)} - 4c_V k \cdot km_t g_{\mu\nu}
\]

\[
+ 8c_V m_t k \cdot p_3 g_{\mu\nu} + 4c_V m_t^3 g_{\mu\nu} + 4c_V m_t p_2 \cdot p_3 g_{\mu\nu}
\]

\[
+ 16c_V m_t k \cdot k V + 8c_V m_t k \cdot p_2 + 8c_V m_t k \cdot p_3 - 8c_V m_t k \cdot p_3
\]

\[
- 8c_V m_t k \cdot p_3 - 4c_V m_t p_3 \cdot p_3 - 4c_V m_t p_2 \cdot p_3
\]

From the next to last row of equation 5.76 we call this the “denominator” \( D' \):

\[
D' = \left[ (k - p_3)^2 - m_t^2 \right] \left[ k^2 - m_t^2 \right] \left[ (k + p_2)^2 - m_t^2 \right].
\]

The last row of equation 5.76 contains the propagator of the Higgs boson and the lepton spinors, which we call \( L \):

\[
L = \frac{i}{p_2^2 - m_H^2} (-ig\gamma_\mu)u_\mu(p_4)v_\nu(p_5)
\]
5. Branching Ratio Calculations

The matrix element of the second diagram, where the direction of the top quark’s momentum is reversed, is equal to that of the first, except for the “numerator”:

\[
M_2 = - \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu(p_1) \epsilon_\nu(p_3) \left( \frac{-igZt}{2} \right) i(i\gamma_\nu)i(-igHt)iN_C
\]

\[
\cdot \text{Tr} \left[ \gamma_\mu \left( c_v^f - c_A^f \gamma^5 \right) \left[ (-k - p_2 + m_t) \left[ (-k + m_t) \right] \right] \right.
\]

\[
\cdot \gamma_\nu [(-k + p_3 + m_t)]
\]

\[
\cdot \frac{1}{(k - p_3)^2 - m_t^2} \frac{1}{k^2 - m_t^2} \frac{1}{(k + p_2)^2 - m_t^2}
\]

\[
\cdot \frac{i}{p_2^2 - m_t^2} (-ig_\tau\mu) u_\mu(p_4) v_\tau(p_5)
\]

The “numerator” \( N_2 \) is here

\[
N_2 = -8ic_A m_t \epsilon_{\mu,\nu,\{k\},\{p_3\}} - 4ic_A m_t \epsilon_{\mu,\nu,\{p_2\},\{p_3\}} - 4c_V k \cdot km_t g_{\mu\nu}
\]

\[
+ 8c_V m_t k \cdot p_3 g_{\mu\nu} + 4c_V m_t^3 g_{\mu\nu} + 4c_V m_t p_2 \cdot p_3 g_{\mu\nu}
\]

\[
+ 16c_V m_t k_\mu k_\nu + 8c_V m_t k_\mu p_2_\mu - 8c_V m_t k_\nu p_3_\mu
\]

\[
- 8c_V m_t k_\mu p_3_\nu - 4c_V m_t p_2_\nu p_3_\mu - 4c_V m_t p_2_\mu p_3_\nu
\]

The total matrix element is now

\[
M = M_1 + M_2
\]

\[
= \frac{1}{2} g Z t g_\gamma t g H t \epsilon_\mu(p_1) \epsilon_\nu(p_3) N_C \cdot L \int \frac{d^4k}{(2\pi)^4} \frac{N_1 + N_2}{D'}
\]

We have to integrate over all possible momenta \( k \) of the particle in the loop, the \( t \). We will use a standard integral to do that, but before we can apply the standard integral we have to rewrite \( M \) in the right form. To do that, we will introduce another integral and a variable shift in the momentum \( k \).

The calculation can be simplified by replacing momenta by masses whenever possible, using on-shell relations. To do that, we need to know the momenta of the particles,
which are:

\[
p_1 = \left( \frac{m_Z}{0} \right), \quad p_2 = \left( \frac{m_Z - E_\gamma}{E_\gamma z} \right), \quad p_3 = \left( \frac{E_\gamma}{-E_\gamma z} \right),
\]

\[
p_4 = \left( \begin{array}{c} E_\tau \\ 0 \\ a \\ b \end{array} \right), \quad p_5 = \left( \begin{array}{c} \sqrt{m^2_\mu + (a + b - E_\gamma)^2} \\ 0 \\ -a \\ E_\gamma - b \end{array} \right),
\]

with

\[
a = \sqrt{-\frac{(2E_\gamma (E_\tau - m_Z) - 2E_\tau m_Z - m^2_\mu + m^2_\tau + m^2_Z)^2}{4E^2_\gamma} + E^2_\gamma - m^2_\tau},
\]

\[
b = -\frac{2E_\gamma E_\tau - 2E_\gamma m_Z - 2E_\tau m_Z - m^2_\mu + m^2_\tau + m^2_Z}{2E_\gamma}.
\]

Also, we can use the fact that photon cannot have transverse polarization, since it travels at the speed of light. This means we can drop all terms with \(p_3^\nu\), as

\[
p_3^\nu \epsilon_\nu = 0.
\]

The next step is to rewrite the “denominator” \(D'\) into the form \((k^2 - \Delta)^3\), where \(\Delta\) is independent of \(k\). We do this with the “Feynman trick” to get it into the form \((k^2 + ak + b)^3\) and then we perform a shift in \(k\) to lose the part linear in \(k\).

\[
\frac{1}{D'} = \frac{1}{[(k - p_3)^2 - m^2_\tau]} \frac{1}{[k^2 - m^2_\tau]} \frac{1}{[(k + p_2)^2 - m^2_\tau]} \frac{1}{D^3}
\]

\[
= \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{D^3}
\]

with \(D = k^2 - \Delta, \Delta = m^2_\tau - m_H^2 yz\).

The shift in \(k\) we used here is

\[
k \rightarrow k - p_2 y + p_3 z.
\]
5. Branching Ratio Calculations

The matrix element \( M \) is now
\[
M = \frac{1}{2} g Z \gamma g H \mathbf{N} \epsilon_{
u}(p_1) \epsilon_{\nu}(p_3) \cdot L
\]
\[
\cdot \int_0^1 \! dx dy dz \delta(x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{2(N_1 + N_2)}{D^3},
\]
where the momentum \( k \) in the “numerator” \( N_1 \) and \( N_2 \) is still the old, unshifted \( k \), and the \( k \) in the “denominator” \( D \) is the new, shifted \( k \).

In the numerator we have to perform the same shift in \( k \). We can drop any terms that are linear in \( k \) after the shift, because these will vanish when we integrate over \( k \) (since the denominator is even in \( k \)). When we do the shift and also apply the on shell conditions, wherever possible, we end up with
\[
N = N_1 + N_2
\]
\[
= -16c_V m_1 y p_{2\mu} p_{2\nu} + 32c_V m_1 y^2 p_{2\mu} p_{2\nu} - 8c_V m_1 p_{2\nu} p_{3\mu} + 16c_V m_1 y p_{2\nu} p_{3\mu} - 32c_V m_1 y p_{2\nu} p_{3\mu} + 8c_V m_1^3 g_{\mu\nu}
\]
\[
+ 8c_V E_\gamma m_1 m Z g_{\mu\nu} - 16c_V E_\gamma m_1 m Z y g_{\mu\nu} + 16c_V E_\gamma m_1 m Z y^2 g_{\mu\nu}
\]
\[
- 8c_V m_1 m_2 y^2 g_{\mu\nu} + 16c_V E_\gamma m_1 m Z y g_{\mu\nu} + 32c_V m_1 k_{\mu} k_{\nu}
\]
\[
- 8c_V m_1 k^2 g_{\mu\nu},
\]

We split \( N \) in two parts, one constant in \( k \) and one quadratic in \( k \), because we will need to use different standard integral for each part.
\[
N = N_{k_0} + N_{k_2}
\]
\[
N_{k_0} = 8c_V E_\gamma m_1 m Z g_{\mu\nu} + 16c_V E_\gamma m_1 m Z y^2 g_{\mu\nu} - 16c_V E_\gamma m_1 m Z y g_{\mu\nu}
\]
\[
+ 16c_V E_\gamma m_1 m Z y g_{\mu\nu} + 8c_V m_1^3 g_{\mu\nu} - 8c_V m_1 m Z^2 y^2 g_{\mu\nu}
\]
\[
- 8c_V m_1 p_{2\mu} p_{3\mu} + 16c_V m_1 y p_{2\nu} p_{3\mu} - 32c_V m_1 y p_{2\nu} p_{3\mu}
\]
\[
+ 32c_V m_1 y^2 p_{2\mu} p_{2\nu} - 16c_V m_1 y p_{2\mu} p_{2\nu}
\]
\[
N_{k_2} = 32c_V m_1 k_{\mu} k_{\nu} - 8c_V k \cdot k m_1 g_{\mu\nu}
\]
The matrix element has become

\[ \mathcal{M} = \frac{1}{2} g_\gamma g_\gamma t g_H t N C e_\mu(p_1) e_\nu(p_3) \cdot L \]

\[ \cdot \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{2(N_k^0 + N_k^2)}{D^3} . \]  

The matrix element is now in the right form, so we can do the integral over the top quark momentum \( k \) using the standard integrals:

\[ \int \frac{d^d k}{2\pi^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left( \frac{1}{\Delta} \right)^{n-d/2} , \text{ (Peskin et al. 1995, p. 807)} \]  

\[ \int \frac{d^d k}{2\pi^d} \frac{k^2}{(k^2 - \Delta)^n} = \frac{(-1)^{n-1} i d}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 1)}{2 \Gamma(n)} \left( \frac{1}{\Delta} \right)^{n-d/2-1} . \]

To use the second standard integral, we first need to replace \( k^\mu k^\nu \) in the matrix element, using

\[ k^\mu k^\nu \rightarrow \frac{1}{d} k^2 g^{\mu\nu} , \text{ Peskin et al. 1995, p. 807} \]

\[ r \cdot k k^\nu = r_\mu k^\mu k^\nu \rightarrow r_\mu \frac{1}{d} k^2 g^{\mu\nu} = r^\nu \frac{1}{d} k^2 . \]

Then Mathematica can do the integral:

\[ \int \frac{d^d k}{(2\pi)^d} \frac{2(N_k^0 + N_k^2)}{D^3} \]  

\[ = - \frac{ic\gamma m_t}{2\pi^2} \left( -2E_\gamma m_Z y^2 - 2E_\gamma m_Z y z + 2E_\gamma m_Z y + m_t^2 + m_Z^2 y^2 - m_Z^2 y \right) \]

\[ \cdot \left( 4E_\gamma m_Z y^2 g^{\mu\nu} + 4E_\gamma m_Z y z g^{\mu\nu} - 4E_\gamma m_Z y g^{\mu\nu} + E_\gamma m_Z g^{\mu\nu} \right. \]

\[ - 2m_Z^2 y^2 g^{\mu\nu} + 2y p_2^\nu p_3^\mu + 2yp_2^\nu p_3^\mu \]

\[ - 4yp_2^\nu p_3^\mu + 4y^2 p_2^\mu p_2^\nu - 2yp_2^\nu p_2^\nu \right) . \]
5. Branching Ratio Calculations

The full matrix element is now

\[
\mathcal{M} = \frac{1}{2} g_Z g_\gamma g_{hl} N_C e_\mu(p_1) e_\nu(p_3) \cdot L \int_0^1 \, dx dy dz \delta(x + y + z - 1) \quad (5.106)
\]

\[
\cdot \frac{i c_V}{m_t} \cdot 2 \pi^2 (-2E_\gamma m_Z y^2 - 2E_\gamma m_Z y z + 2E_\gamma m_Z y + m_t^2 + m_Z^2 y^2 - m_Z^2 y)
\]

\[
\cdot \left(4E_\gamma m_Z y^2 g^{\mu \nu} + 4E_\gamma m_Z y z g^{\mu \nu} - 4E_\gamma m_Z y g^{\mu \nu} + E_\gamma m_Z g^{\mu \nu} - 2m_Z^2 y^2 g^{\mu \nu}
\right.
\]

\[
+ m_Z^2 y g^{\mu \nu} - p_2^\nu p_3^\mu + 2y p_2^\nu p_3^\mu - 4yz p_2^\nu p_3^\mu + 4y^2 p_2^\mu p_2^\nu - 2yp_2^\mu p_2^\nu \right)
\]

\[
\int_0^1 \, dx dy dz \delta(x + y + z - 1) \quad (5.107)
\]

\[
= \frac{i c_V p_2^\nu p_3^\mu}{6 \pi^2 m_t} - \frac{i c_V E_\gamma m_Z g^{\mu \nu}}{6 \pi^2 m_t} + O \left( \frac{1}{m_t^2} \right).
\]

To see that this expansion is valid, we can keep terms up to order \(1/m_t^2\). This changes the result only by a few percent. For more information, see chapter 7.

The total matrix element is now

\[
\mathcal{M} = \frac{1}{2} g_Z g_\gamma g_{hl} N_C e_\mu(p_1) e_\nu(p_3) \cdot L \cdot \left( \frac{i c_V p_2^\nu p_3^\mu}{6 \pi^2 m_t} - \frac{i c_V E_\gamma m_Z g^{\mu \nu}}{6 \pi^2 m_t} \right) \quad (5.108)
\]

\[
+ O \left( \frac{1}{m_t^2} \right).
\]
The next step is to square the matrix element. First we consider only the part of the matrix element $L$.

$$M = M_{\text{Loop}} \cdot L$$  \hspace{1cm} (5.109)

We can square $L$ independently from the rest of the element:

$$|M|^2 = |M_{\text{Loop}}|^2 \cdot |L|^2$$  \hspace{1cm} (5.110)

We will also sum over all possible spins of the $\tau$ and $\mu$, since we cannot know what their spins are. We can then replace the sum over spins by a trace and evaluate this trace to find the value of $|L|^2$.

$$\sum_{\text{spins}} |L|^2 = \sum_{\text{spins}} \frac{i}{p_2^2 - m_H^2} \left| (-i g_{\tau\mu}) u_\mu (p_4) v_\tau (p_5) \right|^2$$  \hspace{1cm} (5.111)

$$= 2 g_{\tau\mu}^2 \frac{m_Z (-2 E_\gamma + m_Z)}{(m_H^2 + 2 E_\gamma m_Z - m_Z^2)^2}$$  \hspace{1cm} (5.112)

Now we square the rest of the matrix element and average over all possible polarizations of the $Z$ and sum over the polarizations of the $\gamma$. This allows us to evaluate the polarization vectors, using

$$\frac{1}{3} \sum_{\text{spins}} \epsilon_\xi (p_1) \epsilon_\mu (p_1) \rightarrow - \left( \epsilon_{\mu\xi} - \frac{p_1 \epsilon p_1 \mu}{m_Z^2} \right), \quad \text{(Peskin et al. 1995, p. 149)}$$  \hspace{1cm} (5.113)

$$\sum_{\text{polarizations}} \epsilon_\eta^*(p_3) \epsilon_\nu (p_3) \rightarrow - g_{\eta\nu}.$$  \hspace{1cm} (5.114)

That gives us the squared matrix element

$$\frac{1}{3} \sum_{\k, \nu} |M|^2 = \left( \frac{g_{Z \gamma} g_{\gamma H} N_{\text{CC}} \epsilon_\nu}{12 \pi m_1 m_Z} \right)^2 2 g_{\tau\mu} m_Z (-2 E_\gamma + m_Z)$$  \hspace{1cm} (5.115)

$$\cdot \left( 4 E_\gamma^2 m_Z^4 - E_\gamma^2 m_Z^2 p_1 \cdot p_1 
- 2 E_\gamma m_Z^3 p_2 \cdot p_3 + 2 E_\gamma m_Z p_1 \cdot p_2 p_1 \cdot p_3 
+ m_Z^2 p_2 \cdot p_2 p_3 \cdot p_3 - p_2 \cdot p_2 (p_1 \cdot p_3)^2 \right).$$
5. Branching Ratio Calculations

Now we will insert this into the formula for the decay rate. Since this decay has a three-particle final state, this formula is different from the formula used in the previous sections (Savage 1995, p. 19). It involves doing two integrals, over the energy of the $\tau$ and the $\gamma$. Mathematica can do both these integrals and gives us the decay rate:

$$\Gamma(Z \rightarrow \tau\mu\gamma) = \frac{1}{m_Z^2 2^6 \pi^3} \int_0^{m_Z/2} dE_\gamma \int_{m_Z/2}^{m_Z/2} dE_{\tau} \frac{1}{3} \sum_{s,p} |M|^2$$

\[ (5.116) \]

$$= - \frac{c_V^2 s_\gamma^2 s_{Ht}^2 s_{\tau\mu}^2 s_{Zt}^2 N_C^2}{331776 \pi^7 m_H^2 m_t^2 m_Z^3} \cdot \left( 24 m_H^6 m_Z^2 - 42 m_H^4 m_Z^4 - 18 m_H^4 m_\mu^2 m_Z^2 
- 18 m_H^4 m_\tau^2 m_Z^2 + 17 m_H^2 m_Z^6 + 27 m_H^2 m_\mu^2 m_Z^4 
+ 27 m_H^2 m_\tau^2 m_Z^4 - 6 m_\mu^2 m_Z^6 - 6 m_\tau^2 m_Z^6 \right)$$

\[ (5.117) \]

$$- \frac{c_V^2 s_\gamma^2 s_{Ht}^2 s_{\tau\mu}^2 s_{Zt}^2}{331776 \pi^7 m_H^2 m_t^2 m_Z^3} \log \left( 1 - \frac{m_Z^2}{m_H^2} \right) \cdot \left( 24 m_H^8 - 18 m_H^6 m_\mu^2 - 18 m_H^6 m_\tau^2 - 54 m_H^6 m_Z^2 
+ 36 m_H^4 m_\mu^2 m_Z^2 + 36 m_H^4 m_\tau^2 m_Z^2 + 36 m_H^4 m_\mu^2 m_Z^4 
- 6 m_H^2 m_Z^6 - 18 m_H^2 m_\mu^2 m_Z^4 - 18 m_H^2 m_\tau^2 m_Z^4 \right)$$

\[ (5.118) \]

We divide this decay rate by the total decay rate of Z

$$\Gamma_{\text{Total}}(Z) = 2.5 \text{ GeV} ,$$

\[ (5.119) \]

to get the branching ratio

$$\text{BR}(Z \rightarrow \tau\mu\gamma) = \frac{\Gamma(Z \rightarrow \tau\mu\gamma)}{\Gamma_{\text{Total}}(Z)}$$

\[ (5.120) \]

$$= \frac{1.30 \times 10^{-2} \text{ eV}}{2.5 \text{ GeV}} s_{\tau\mu}^2 \cdot$$

\[ (5.121) \]

$$= 5.2 \times 10^{-12} s_{\tau\mu}^2 .$$

\[ (5.122) \]
6. Resulting Bounds on $g_{\tau\mu}$ and $Z \rightarrow \tau\mu\gamma$

In this chapter, we will use bounds on the branching ratios of $H \rightarrow \tau\mu$ and $\tau \rightarrow \mu\gamma$ to find a bound on $g_{\tau\mu}$, the coupling strength of the lepton flavor violating interaction $H\tau\mu$. When we have calculated that bound, we will insert it into the formula for the $Z \rightarrow \tau\mu\gamma$ decay, to get a bound on this branching ratio.

The results we found in chapter 5 were:

<table>
<thead>
<tr>
<th>Decay</th>
<th>$\Gamma$</th>
<th>BR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H \rightarrow \tau\mu$</td>
<td>$10 \times 10^9$ eV $\cdot g_{\tau\mu}^2$</td>
<td>$7.7 \times 10^2 \cdot g_{\tau\mu}^2$</td>
</tr>
<tr>
<td>$\tau \rightarrow \mu\gamma$</td>
<td>$9.4 \times 10^{-9}$ eV $\cdot g_{\tau\mu}^2$</td>
<td>$4.1 \times 10^{-6} \cdot g_{\tau\mu}^2$</td>
</tr>
<tr>
<td>$H \rightarrow \gamma\gamma$</td>
<td>$7.3 \times 10^2$ eV</td>
<td>$5.6 \times 10^{-5}$</td>
</tr>
<tr>
<td>$Z \rightarrow \tau\mu\gamma$</td>
<td>$1.3 \times 10^{-2}$ eV $\cdot g_{\tau\mu}^2$</td>
<td>$5.2 \times 10^{-12} \cdot g_{\tau\mu}^2$</td>
</tr>
</tbody>
</table>

First we will use the branching ratio of the $H \rightarrow \tau\mu$ decay to calculate a bound on $g_{\tau\mu}$. The branching ratio we calculated is

$$\text{BR}(H \rightarrow \tau\mu) > 7.7 \times 10^2 g_{\tau\mu}^2.$$  \hspace{5em} (6.1)

This branching ratio is plotted in figure 6.1.

The experimental bound on this branching ratio is

$$\text{BR}(H \rightarrow \tau\mu) + \text{BR}(H \rightarrow \tau\mu) < 0.015, \quad (\text{Patrignani et al. 2017a, p. 6})$$  \hspace{5em} (6.2)

$$\text{BR}(H \rightarrow \tau\mu) < \frac{0.015}{2} = 7.6 \times 10^{-3}.$$  \hspace{5em} (6.3)

Combining these gives

$$g_{\tau\mu} < 3.1 \times 10^{-3}.$$  \hspace{5em} (6.4)

The bound on the coupling strength set by $H \rightarrow \tau\mu$ decay is $g_{\tau\mu} < 3.1 \times 10^{-3}$.

Now we will calculate the bound on $g_{\tau\mu}$ from the $\tau \rightarrow \mu\gamma$ decay. The branching
6. Resulting Bounds on $g_{\tau\mu}$ and $Z \to \tau\mu\gamma$

The branching ratio we calculated is

$$BR(\tau \to \mu\gamma) = 4.1 \times 10^{-6} g_{\tau\mu}^2$$  \hspace{1cm} (6.5)

This branching ratio is plotted in figure 6.2.

The experimental bound on this branching ratio is

$$BR(\tau \to \mu\gamma) < 4.4 \times 10^{-8}$$ \hspace{1cm} (Lindner et al. 2016, p. 12) \hspace{1cm} (6.6)

Combining these gives

$$g_{\tau\mu} < 1.0 \times 10^{-1}$$ \hspace{1cm} (6.7)

The bound on the coupling strength set by $\tau \to \mu\gamma$ decay is $g_{\tau\mu} < 1.0 \times 10^{-1}$.

The bound set by $H \to \tau\mu$ is thus much stronger than the bound set by $\tau \to \mu\gamma$ decay. This is why we will use this value, $g_{\tau\mu} < 3.1 \times 10^{-3}$, to calculate the bound on the branching ratio of $Z \to \tau\mu\gamma$. We insert this value into the branching ratio

$$BR(Z \to \tau\mu\gamma) = 5.2 \times 10^{-12} \cdot g_{\tau\mu}^2,$$  \hspace{1cm} (6.8)

and get the bound

$$BR(Z \to \tau\mu\gamma) < 5.1 \times 10^{-17}.$$  \hspace{1cm} (6.9)

So, given the bound on the branching ratio of $H \to \tau\mu$ decay, the branching ratio of $Z \to \tau\mu\gamma$ can be at most $5.1 \times 10^{-17}$. This branching ratio is plotted in figure 6.3, with indications of the bounds.

In figure 6.4, the branching ratio of $Z \to \tau\mu\gamma$ is plotted against the branching ratio of $H \to \tau\mu$. This plot shows how the magnitude of these branching ratios change together as a function of $g_{\tau\mu}$. 
Figure 6.1: The branching ratio of $H \rightarrow \tau\mu$ plotted against the coupling strength $g_{\tau\mu}$. The experimental upper bound on the branching ratio is indicated on the $y$ axis, as well as the resulting upper bound on $g_{\tau\mu}$. The region excluded by these bounds is marked with red stripes.

Figure 6.2: The branching ratio of $\tau \rightarrow \mu\gamma$ plotted against the coupling strength $g_{\tau\mu}$. The experimental upper bound on the branching ratio is indicated on the $y$ axis, as well as the resulting upper bound on $g_{\tau\mu}$. The region excluded by these bounds is marked with red stripes.
6. Resulting Bounds on $g_{\tau\mu}$ and $Z \to \tau\mu\gamma$

**Figure 6.3:** The branching ratio of $Z \to \tau\mu\gamma$, plotted against $g_{\tau\mu}$. The upper bound on $g_{\tau\mu}$ from the $H \to \tau\mu$ decay is indicated, as is the resulting upper bound on the branching ratio of $Z \to \tau\mu\gamma$. The region excluded by these bounds is marked with red stripes.

**Figure 6.4:** The branching ratio of $Z \to \tau\mu\gamma$, plotted against the branching ratio of $H \to \tau\mu$. The value of $g_{\tau\mu}$ is indicated by the color. Given the sensitivity with which one of these decays can be measured, this plot shows how sensitive the measurement of the other has to be, to probe $g_{\tau\mu}$ with equal precision. The upper bounds of these branching ratio are indicated. These correspond to a value of $g_{\tau\mu} = 3.1 \times 10^{-3}$. The region excluded by the bounds is marked with red stripes.
7. Discussion and Conclusion

7.1. Reliability of the Results

The upper bound we have found on the branching ratio of $Z \rightarrow \tau \mu \gamma$ is $5.1 \times 10^{-17}$. It is important to ask to what extend this number is correct. We will assess this in two ways: we will cross-check with existing results and we will look at the expansion we did in $1/m_t$ to see if it holds at higher orders.

To cross-check, we can compare the result of the calculation of $H \rightarrow \gamma \gamma$, equation 5.71, with existing results. This calculation is done using the same Mathematica code as the calculation of $Z \rightarrow \tau \mu \gamma$, so by verifying the result of the former, we can justify the result of the latter.

In Ilisie (2011, p. 10), the decay rate of $H \rightarrow gg$, the decay of a Higgs to two gluons, with an intermediate top quark loop is calculated. The decay rate of this process is equal to the $H \rightarrow \gamma \gamma$ decay, up to a constant factor. By comparing equation 5.71 to equation 1.52 from Ilisie (2011, p. 10), we can see that this indeed is what we calculated.

We can also compare the result numerically against Abdullayev et al. (2017, p. 336), equation 39:

$$\Gamma(H \rightarrow \gamma \gamma) = \frac{G_F a^2 m_H^3}{8\sqrt{2\pi}^3} N_C q_t^4 2 \frac{m_H^2}{4m_t^2} \left[ 1 + \left(1 - \frac{1}{m_H^4 m_t^4}\right) \arcsin^2 \sqrt{\frac{m_H^2}{4m_t^2}} \right]^{-1}$$

$$= 7.27 \times 10^2 \text{ eV.}$$

This matches the result we found of $7.27 \times 10^2 \text{ eV.}$

Since these results both agree, we can be confident that the Mathematica code is correct.

To see if the expansion we did in $1/m_t$ is allowed, we will include higher order terms. In the calculation of this branching ratio, we kept only terms of order up to $1/m_t^4$, which resulted in a branching ratio of $5.1 \times 10^{-17}$. When we keep terms of up to
7. Discussion and Conclusion

order $1/m_t^6$, the branching ratio becomes $\text{BR}(Z \to \tau\mu\gamma) = 5.5 \times 10^{-17}$. This is a difference of only 6.1%, confirming that the expansion is indeed justified.

7.2. Usefulness of the Results

The calculated upper bound on the $Z \to \tau\mu\gamma$ branching ratio of $5.1 \times 10^{-17}$ is very small. It is unlikely that this can be measured anytime soon. In comparison, the upper limit on the branching ratio of $Z \to \tau\mu$ at ATLAS is $1.69 \times 10^{-5}$ (ATLAS Collaboration et al. 2017, p. 23). Even with a significantly higher sensitivity because of the final state photon, this difference of almost 12 orders of magnitude seems insurmountable.

The upper bound on the branching ratio of $H \to \tau\mu$, the decay that gives the best current bound, is $1.5 \times 10^{-2}$, almost 15 orders of magnitude larger than the branching ratio of $Z \to \tau\mu\gamma$. The main reasons for this difference are the three-particle phase-space of the final state, the suppression from the quark loop, the higher number of couplings involved and the size of the total decay rate of the $Z$.

Relative to the two-particle phase space of $H \to \tau\mu$, the three particle phase space gives an extra suppression by a factor of $2^4\pi^3$, which is about $5 \times 10^2$. This follows from the formula for the decay rate (Griffiths 2008, p. 429):

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_1} \left( \frac{d^3p_2}{(2\pi)^32E_2} \right) \left( \frac{d^3p_3}{(2\pi)^32E_3} \right) \cdots \left( \frac{d^3p_n}{(2\pi)^32E_n} \right) \cdot (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \ldots - p_n),$$

for each particle added, there is an extra factor of $\frac{1}{(2\pi)^2}$.

Integrating over the loop momentum of the quark loop gives a suppression of $(2\pi)^4$, about $2 \times 10^3$ (Peskin et al. 1995, p. 189).

The couplings between the top quark and the other particles are all smaller than 1. This gives another suppression of $(\frac{g_{Zt}}{2}g_{\gamma t}g_{Ht})^2$, about $9 \times 10^3$.

Another suppression comes from the larger total decay rate of the $Z$. The total decay rate of the $Z$ is 2.5 GeV, compared to less than $1.3 \times 10^{-2}$ GeV for the $H$. This causes a difference of a factor of $2 \times 10^2$. 

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These different reasons together suppress the branching ratio of the $Z$ decay by a factor of $1 \times 10^{12}$, compared to the $H$ decay. All of these reasons are unavoidable. The larger total decay rate of the $Z$ is inherent to any $Z$ decay. There exist no tree-level diagrams of this decay, so the loop is also necessary. This loop then introduces more coupling vertices. The extra final photon is also necessary, because the $Z$ has spin 1 and the $H$ has spin 0. That means there has to be another particle to carry off the spin.

In a $Z \to \tau\mu$ decay, the suppression might be a lot smaller. However, for the reasons listed above, this decay is not possible when a $H$ boson is involved, unless extra loops are introduced. The $Z \to \tau\mu$ decay could only happen at zero- or one-loop level with a different lepton flavor violating vertex.

Another interesting cause for the low branching ratio seems more accidental. In the decay rate, two terms of almost equal magnitude, but opposite sign, are added. These are

$$
\log \left(1 - \frac{m_Z^2}{m_H^2}\right) \cdot \left(24m_H^8 - 18m_H^6m_H^2 - 18m_H^6m_Z^2 - 54m_H^6m_Z^4 + 36m_H^4m_Z^4 - 36m_H^4m_Z^2 + 36m_H^2m_Z^2 - 6m_H^2m_Z^6 - 18m_H^2m_Z^4 - 18m_H^2m_Z^4 - 18m_H^2m_Z^2 + 17m_H^2m_Z^6 + 27m_H^2m_Z^2 + 27m_H^2m_Z^4 - 6m_H^2m_Z^6 - 6m_H^2m_Z^6\right)
$$

$$= 2.0761 \times 10^{17} \text{ GeV}^6$$

(7.4)

and

$$
\left(24m_H^6m_Z^2 - 42m_H^4m_Z^4 - 18m_H^4m_H^2m_Z^2 - 18m_H^4m_H^2m_Z^2 + 17m_H^2m_Z^6 + 27m_H^2m_H^2m_Z^4 + 27m_H^2m_H^2m_Z^4 - 6m_H^2m_Z^6 - 6m_H^2m_Z^6\right)
$$

$$= -2.0644 \times 10^{17} \text{ GeV}^6.$$

(7.5)

These terms cancel out for the most part:

$$2.0761 \times 10^{17} \text{ GeV}^6 - 2.0644 \times 10^{17} \text{ GeV}^6 = 1.17 \times 10^{15} \text{ GeV}^6.$$

(7.6)

This means that while each of these terms themselves are of order $10^{17} \text{ GeV}^6$, their
7. Discussion and Conclusion

The sum is only of order $10^{15}$ GeV. This is a suppression by a factor of $\frac{2.1 \times 10^{17} \text{ GeV}^6}{1.2 \times 10^{18} \text{ GeV}^6} = 1.8 \times 10^2$.

It seems this cancellation is just an unfortunate accident. If, for example, the $Z$ boson would have had a mass of 120 GeV, instead of its actual mass of 91.2 GeV, these terms would not be of similar magnitude, and the resulting upper limit on the branching ratio would be $5.7 \times 10^{-16}$, an improvement by a factor of 11.

7.3. Conclusion

We estimated an upper limit coupling strength of the $H \tau \mu$ vertex using $H \rightarrow \tau \mu$ and $\tau \rightarrow \mu \gamma$ decays. We used this coupling strength to calculate an upper bound on branching ratio of the $Z \rightarrow \tau \mu \gamma$ decay to find out if we can use this decay to probe the $H \tau \mu$ vertex more accurately. The upper limit on the $Z \rightarrow \tau \mu \gamma$ branching ratio we found is $5.1 \times 10^{-17}$. This is very small and that makes it unlikely that this process will be observable in the near future, unless there is some unexpected, very significant progress. For now, the two mentioned decays can measure the interaction much more accurately.

The $Z \rightarrow \tau \mu \gamma$ decay seems no improvement upon the $Z \rightarrow \tau \mu$ decay, despite possible higher experimental sensitivity, because of a huge suppression by a number of factors. For lepton flavor violating couplings with particles different from the Higgs, however, it might be a different story. If there is lepton flavor violation in interactions with some other particle, it may be possible to make similar diagrams that are much more relevant. By replacing some particles in the decay, it could be possible to get diagrams with larger branching ratios. In this way, some $Z \rightarrow \tau \mu \gamma$ decay could provide insights in $\tau \mu$ lepton flavor violating interactions.
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A. Feynman Rules

External Legs

\[ Z(q) = e^\mu(q), \quad \gamma(q) = e_\mu(q) \]

\[ \mu(p') = \pi_m(p'), \quad e(p) = u_e(p) \]

\[ H(p) = 1 \]

Vertices

\[ f \quad H \quad f = -i g_{Hf} \]

\[ q \quad \gamma \quad q = i g_{\gamma q} \gamma^\mu \]

\[ Z \quad H \quad Z = i g_{Hf} \hat{g}^{\mu\nu} \]

\[ g_{\gamma q} = \frac{-q}{e} \sqrt{4\pi\alpha_{em}} \]  

(Griffiths 2008, p. 244) (A.1)

\[ g_{Hf} = \frac{-m_f}{VeV_H} \]

(Peskin et al. 1995, p. 716) (A.2)
A. Feynman Rules

\( g_{HZ} = 2 \frac{m_Z^2}{\text{VeV}_H} \)  \hspace{1cm} (Peskin et al. 1995, p. 716)  \hspace{1cm} (A.3) \\

\( g_{Zf} = \frac{g_{\gamma e}}{\sin \theta_W \cos \theta_W} \)  \hspace{1cm} (Griffiths 2008, p. 332)  \hspace{1cm} (A.4) \\

Propagators

\[
\begin{align*}
H(k) &= \frac{i}{k^2 - m_H^2} \\
Z(q) &= \frac{-i(g_{\mu\nu} - q\mu q\nu / m_Z^2)}{q^2 - m_Z^2}
\end{align*}
\]

\[
\begin{align*}
\quad = \quad & i(p - k + m_f) \\
\quad \Rightarrow \quad & (p - k)^2 - m_f^2
\end{align*}
\]

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B. Constants

\[ m_H = 125.09(32) \text{ GeV} \quad \text{(Patrignani et al. 2017a, p. 1)} \]
\[ m_\mu = 0.1056583745(24) \text{ GeV} \quad \text{(Patrignani et al. 2017b, p. 1)} \]
\[ m_\tau = 1.77686(12) \text{ GeV} \quad \text{(Patrignani et al. 2017c, p. 1)} \]
\[ m_t = 173.1(6) \text{ GeV} \quad \text{(Patrignani et al. 2017d, p. 1)} \]
\[ m_Z = 91.1876(21) \quad \text{(Patrignani et al. 2017e, p. 1)} \]
\[ \alpha_{\text{em}} = \frac{1}{137.036} \quad \text{(Griffiths 2008, p. 10)} \]
\[ v_{eH} = 246 \text{ GeV} \quad \text{(Griffiths 2008, p. 403)} \]
\[ \theta_W = 0.5018 \quad \text{(Griffiths 2008, p. 332)} \]
\[ c_V = \frac{1}{2} - \frac{4}{3} \sin^2 \theta_W \quad \text{(for interactions with top quarks) (Griffiths 2008, p. 332)} \]
\[ c_A = \frac{1}{2} \quad \text{(for interactions with top quarks) (Griffiths 2008, p. 332)} \]
\[ N_C = 3 \quad \text{(for quarks) (Abdullayev et al. 2017, p. 330)} \]
C. Calculation of $\tau \rightarrow \mu\gamma$

The following sections are based on Peskin et al. (1995, p. 184 - 196).

When we apply the Feynman rules to the diagram this yields the matrix element

$$iM = \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m_{H}^2} \frac{i}{(p_2 - k)^2 - m_{\tau}^2} \frac{i}{(p_1 - k)^2 - m_{\tau}^2} \epsilon_\mu(p_3)$$

$$\cdot \tau_\tau(p_2)(i\gamma_\mu)(p_2 - k + m_\tau)(-ie\gamma^\mu)(p_1 - p_3 + m_\tau)(i\gamma_{H\tau})u_\mu(p_1)$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{N_\mu}{D'} \epsilon_\mu(p_3).$$

Here we split the matrix elements into two parts, the “numerator” $N_\mu$ and the “denominator” $D'$

$$N_\mu = - (e g_{\tau\mu} g_{H\tau})\tau_\mu(p_2)(p_2 - k + m_\tau)(\gamma^\mu)(p_1 - k + m_\tau)u_\mu(p_1)$$

$$= - (e g_{\tau\mu} g_{H\tau})\tau_\mu(p_2)(m_\mu - k + m_\tau)(\gamma^\mu)(2m_\tau - k)u_\mu(p_1),$$

$$D' = \left[k^2 - m_{H}^2\right] \left[(p_2 - k)^2 - m_{\tau}^2\right] \left[(p_1 - k)^2 - m_{\tau}^2\right],$$

and use Feynman parameters to replace $D'$ by a new “denominator” $D$:

$$\frac{1}{D'} = \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{D^3}$$

(Peskin et al. 1995, p. 190),

$$D = x[k^2 - m_{H}^2] + y[(p_2 - k)^2 - m_{\tau}^2] + z[(p_1 - k)^2 - m_{\tau}^2].$$
C. Calculation of $\tau \rightarrow \mu \gamma$

The whole matrix element has now become:

$$iM = 2 \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \int \frac{d^4k}{(2\pi)^d} \frac{N^\mu}{D^3} \epsilon^\mu(p_3)$$  \hspace{1cm} (C.8)

To solve the integral over $k$, we will rewrite $D$ in the form

$$D = \ell^2 - \Delta,$$  \hspace{1cm} (C.9)

with $\ell = (k - constant)$ and $\Delta$ constant in $\ell$. The new variable $\ell$ is only shifted by a constant, so $d\ell = dk$. Since the numerator $N^\mu$ contains $k$ as well, we must rewrite it too before we can carry out the integral over $k$ ($\ell$ by then).

First we get $D$ in the form

$$D = x[k^2 - m_H^2] + y[(p_2 - k)^2 - m_2^2] + z[(p_1 - k)^2 - m_1^2]$$  \hspace{1cm} (C.10)

$$= (x + y + z)k^2 + yp_2^2 + zp_1^2 - 2k(p_2y + zp_1) - xm_H^2 - (y + z)m_2^2$$  \hspace{1cm} (C.11)

$$= k^2 - 2k(p_2y + p_1z) + p_2^2y + p_1^2z - xm_H^2 - (y + z)m_2^2.$$  \hspace{1cm} (C.12)

At this point we do the substitution $k \rightarrow \ell = (k - constant)$ so that that the part linear in $k$ vanishes:

$$\ell \equiv k - (p_2y + p_1z), \quad k = \ell + (p_2y + p_1z),$$  \hspace{1cm} (C.13)

$$\ell^2 = k^2 - 2k(p_2y + p_1z) + (p_2y)^2 + (p_1z)^2 + 2p_2p_1yz,$$  \hspace{1cm} (C.14)

$$D = \ell^2 - p_2^2y^2 + p_1^2z^2 + p_2^2z - 2p_2p_1yz - xm_H^2 - (y + z)m_2^2$$  \hspace{1cm} (C.15)

$$= \ell^2 + m_H^2(z - z - z^2 - yz - y) + p_3^2yz - xm_H^2 + m_2^2(y - y^2 - yz)$$  \hspace{1cm} (C.16)

$$= \ell^2 - m_H^2(z^2 + yz + y) + p_3^2yz - xm_H^2 + m_2^2(y - y^2 - yz)$$  \hspace{1cm} (C.17)

$$= \ell^2 - \Delta,$$  \hspace{1cm} (C.18)

$$\Delta = -m_H^2(z^2 + yz + y) + p_3^2yz - xm_H^2 + m_2^2(y - y^2 - yz).$$  \hspace{1cm} (C.19)
Substituting $\ell$ for $k$:

$$-k = -\ell - (p_2y + p_1z), \quad (C.20)$$

$$N^\mu = -(e^\ast g_{\tau\mu} g_{H\tau}) u_\mu(p_2)(m_\mu - k + m_\tau)(\gamma^\mu)(2m_\tau - k)u_\tau(p_1) \quad (C.21)$$

$$\quad = -[e^\ast g_{\tau\mu} g_{H\tau}] u_\mu(p_2)[m_\mu + m_\tau - \ell - p_2y - p_1z] \gamma^\mu \quad (C.22)$$

$$\quad \cdot [2m_\tau - \ell - p_2y - p_1z]u_\tau(p_1). \quad (C.23)$$

The momenta $p$ and $p_2$ are on-shell, so their operators $/p_1$ and $/p_2$ can be replaced by the appropriate particle mass when they work on their eigenstates:

$$p_1 u_\tau(p_1) = m_\tau u_\tau(p_1), \quad (C.24)$$

$$\pi_\mu(p_2) p_2 = \pi_\mu(p_2)m_\mu. \quad (C.25)$$

To get the $p_1$ operators next to their eigenstates, you need to get them across the $\gamma^\mu$'s with the identities

$$p_1 \gamma^\mu = -\gamma^\mu p_1 + 2p_1^\mu \quad \text{and} \quad (C.26)$$

$$\gamma^\mu p_1 = -p_1 \gamma^\mu + 2p_1^\mu \quad (C.27)$$

Putting these all together yields the identities

$$p_1 \gamma^\mu u_\tau(p_1) = (-m_\tau \gamma^\mu + 2p_1^\mu)u_\tau(p_1) \quad \text{and} \quad (C.28)$$

$$\pi_\mu(p_2) \gamma^\mu p_2 = \pi_\mu(p_2)(-\gamma^\mu m_\mu + 2p_2^\mu). \quad (C.29)$$

Finally, we can swap the $p_1$'s with each other with

$$p_1 p_2 = -p_2 p_1 + 2p_1^\alpha p_2^\alpha \quad (C.30)$$

so that

$$p_1 \gamma^\mu p_2 = p_1[-p_2 \gamma^\mu + 2p_2^\mu] = (-2p_1^\alpha p_2^\alpha + p_2 p_1) \gamma^\mu + 2p_1 p_2^\mu \quad (C.31)$$

$$\quad = -2p_1^\alpha p_2^\alpha \gamma^\mu - p_2 \gamma^\mu p_1 + 2p_2 p_1^\mu + 2p_1 p_2^\mu. \quad (C.32)$$
Now $N^\mu$ becomes, using the shorthands

$$G = -[e g_{\tau\mu} g_{\tau\tau}],$$

(C.33)

$$M_1 = (m_\tau + m_\mu (1 - y)) \text{ and}$$

(C.34)

$$M_2 = (m_\tau (2 - z)), \tag{C.35}$$

$$N^\mu = -[e g_{\tau\mu} g_{\tau\tau}] \bar{\pi}_\mu [m_\mu + m_\tau - \ell - p_2 y - p_1 z] \gamma^\mu$$

\[\cdot [2m_\tau - \ell - p_2 y - p_1 z] u_\tau \tag{C.36}\]

$$= G \bar{\pi}_\mu [m_\mu + m_\tau - \ell - m_\mu y - p_1 z] \gamma^\mu [2m_\tau - \ell - p_2 y - m_\tau z] u_\tau \tag{C.37}$$

$$= G \bar{\pi}_\mu [M_1 - \ell - p_1 z] \gamma^\mu [M_2 - \ell - p_2 y] u_\tau \tag{C.38}$$

$$= G \bar{\pi}_\mu [\ell \gamma^\mu \ell + M_1 \gamma^\mu M_2 + y M_1 m_\mu \gamma^\mu - 2 y M_1 p_2^\mu$$

\[+ z M_2 m_\tau \gamma^\mu - 2 z M_2 p_1^\mu + y z p_1 \gamma^\mu p_2^\mu] u_\tau \tag{C.39}\]

$$= G \bar{\pi}_\mu [\ell \gamma^\mu \ell + M_1 \gamma^\mu M_2 + y M_1 m_\mu \gamma^\mu - 2 y M_1 p_2^\mu$$

\[+ z M_2 m_\tau \gamma^\mu - 2 z M_2 p_1^\mu - y z m_\tau ^2 \gamma^\mu - y z m_\mu ^2 \gamma^\mu$$

\[- y z m_\mu \gamma^\mu m_\tau + 2 y z m_\mu p_1^\mu + 2 y z m_\tau p_1^\mu] u_\tau \tag{C.40}$$

$$= G \bar{\pi}_\mu [\ell \gamma^\mu \ell] u_\tau$$

(C.41)

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [2z M_2 p_1^\mu + 2 y z m_\mu p_1^\mu - 2 y M_1 p_2^\mu + 2 y z m_\tau p_2^\mu] u_\tau$$

(C.42)

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [M_1 M_2 - y z m_\tau ^2 + z M_2 m_\tau - y z m_\mu m_\tau + y M_1 m_\mu - y z m_\mu ^2] u_\tau$$

$$= G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [- (1/2) \ell^2] u_\tau$$

(C.43)

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [M_1 M_2 - y z m_\tau ^2 + z M_2 m_\tau - y z m_\mu m_\tau + y M_1 m_\mu - y z m_\mu ^2] u_\tau$$

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [- (1/2) \ell^2] u_\tau$$

(C.44)

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [\{m_\tau + m_\mu (1 - y)\} (m_\tau (2 - z)) + z (m_\tau (2 - z)) m_\tau$$

\[- y z m_\tau ^2 - y z m_\mu m_\tau + y (m_\tau + m_\mu (1 - y)) m_\mu - y z m_\mu ^2] u_\tau$$

$$+ G \bar{\pi}_\mu \gamma^\mu \gamma^\mu [- 2 z (m_\tau (2 - z)) p_1^\mu + 2 y z m_\mu p_1^\mu$$

\[- 2 y (m_\tau + m_\mu (1 - y)) p_2^\mu + 2 y z m_\tau p_2^\mu] u_\tau.$$
$p_1^\mu$ and $p_2^\mu$. It turns out that the part proportional to $\gamma^\mu$ does not affect the decay rate, so we can leave that part for what it is and continue with only the part containing $p_1^\mu$ and $p_2^\mu$. We will rewrite it to the form $A(p_1^\mu + p_2^\mu) + B(p_1^\mu - p_2^\mu) = A(p_1^\mu + p_2^\mu) + Bp_3^\mu$:

$$\begin{align*}
-2z(m_\tau(2-z)p_1^\mu + 2yzp_2^\mu - 2y(m_\tau + m_\mu(1-y))p_2^\mu \\
+ 2yzm_\tau p_2^\mu + O(\gamma^\mu)
\end{align*}$$

$$= m_\tau [2z^2 - 4z] p_1^\mu + m_\mu [2yz] p_1^\mu$$

$$+ m_\tau [2yz - 2y] p_2^\mu + m_\mu [2y^2 - 2y] p_2^\mu + O(\gamma^\mu)$$

(C.44)

(C.45)

To simplify things, we also set $m_\mu = 0$ from now on. Then

$$-2z(m_\tau(2-z))p_1^\mu + 2yzm_\tau p_2^\mu - 2y(m_\tau + m_\mu(1-y))p_2^\mu$$

$$= m_\tau [2z^2 - 2z + yz - y] (p_1^\mu + p_2^\mu) + m_\mu [yz + y^2 - y] (p_1^\mu + p_2^\mu)$$

$$+ m_\tau [z^2 - 2z - yz + y] p_3^\mu + m_\mu [yz - y^2 + y] p_3^\mu + O(\gamma^\mu)$$

(C.46)

We can replace $p_2^\mu + p_1^\mu$ now with $\gamma^\mu(m_\tau + m_\mu) - i\sigma^{\mu\nu}p_{3\nu}$ since

$$\pi_\mu(p_2)\gamma^\mu u_\tau(p_1) = \pi_\mu(p_2) \left[ \frac{p_2^\mu + p_1^\mu}{m_\tau + m_\mu} + \frac{i\sigma^{\mu\nu}p_{3\nu}}{m_\tau + m_\mu} \right] u_\tau(p_1)$$

(C.49)

$$\pi_\mu(p_2) \left[ p_2^\mu + p_1^\mu \right] u_\tau(p_1) = \pi_\mu(p_2) \left[ \gamma^\mu(m_\tau + m_\mu) - i\sigma^{\mu\nu}p_{3\nu} \right] u_\tau(p_1).$$

(C.50)

Because the part proportional to $\gamma^\mu$ does not affect the decay rate, we can replace

$$(p_2^\mu + p_1^\mu)m_\tau \left[ z^2 - 2z + yz - y \right]$$

by

$$-i\sigma^{\mu\nu}p_{3\nu}m_\tau \left[ z^2 - 2z + yz - y \right].$$

(C.51)

(C.52)
C. Calculation of $\tau \rightarrow \mu \gamma$

We are left now with

$$N^\mu = i e g_{\tau \mu} g_\mu H \bar{\pi}_\mu \left[ \sigma^{\mu \nu} p_{3\nu} m_\tau (z^2 - 2z + yz - y) \right] u_\tau + \mathcal{O}(\gamma^\mu) \tag{C.53}$$

Since the part proportional to $\gamma^\mu$ does not matter, we only look at the part containing $\sigma^{\mu \nu} p_{3\nu}$, $\mathcal{M}_\sigma$.

$$\mathcal{M} = 2 \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} N^\mu \mathcal{D} \epsilon_\mu(p_3) \tag{C.54}$$
$$= \mathcal{M}_\sigma + \mathcal{O}(\gamma^\mu) \tag{C.55}$$

$$\mathcal{M}_\sigma = 2 i e g_{\tau \mu} g_\mu H \bar{\pi}_\mu \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \sigma^{\mu \nu} p_{3\nu}$$
$$\cdot m_\tau (z^2 - 2z + yz - y) u_\tau \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^{3/2}} \epsilon_\mu(p_3) \tag{C.56}$$

We can solve the integral over $\ell$ using these standard integrals (Peskin et al. 1995, p. 193):

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^m} = i(-1)^m \frac{1}{(4\pi)^2} \frac{1}{(m - 1)(m - 2)} \frac{1}{\Delta^{m-2}} \tag{C.57}$$

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^3} = -i \frac{1}{2 \cdot (4\pi)^2} \frac{1}{\Delta} \tag{C.58}$$

$$\mathcal{M}_\sigma = -\frac{e g_{\tau \mu} g_\mu H m_\tau}{(4\pi)^2} \bar{\pi}_\mu \sigma^{\mu \nu} p_{3\nu} u_\tau \epsilon_\mu(p_3)$$
$$\cdot \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \frac{z^2 - 2z + yz - y}{m_\tau^2(z^2 + yz + y) + x m_t^2}$$
$$= -\frac{e g_{\tau \mu} g_\mu H m_\tau}{(4\pi)^2} \bar{\pi}_\mu \sigma^{\mu \nu} p_{3\nu} I_{\mu,1}^{++} u_\tau \epsilon_\mu(p_3) \tag{C.59}$$

This still leaves the integral over $x, y, z$; which we call $I_{\mu,1}^{++}$, analogous to Lindner et al. (2016, p. 20).
\[ I_{\mu, 1}^{++} = \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \frac{z^2 - 2z + yz - y}{m_T^2 (z^2 + yz + y) + \tau m_T^2} \]  
(C.61)

\[ = \int_0^1 dx \, dy \, dz \, \delta(x + y + z - 1) \frac{-zx - 1 + x}{m_T^2 (1 - x - zx) + \tau m_T^2} \]  
(C.62)

\[ = \int_0^1 dx \int_0^{1-x} dz \, \frac{-zx - 1 + x}{(1 - x - zx)m_T^2 + \tau m_T^2} \]  
(C.63)

\[ = -\frac{1}{m_T^2} \int_0^1 dx \int_0^{1-x} dz \, \frac{zx + 1 - x}{(1 - x - zx)\lambda^2 + x} \]  
(C.64)

\[ = \int_0^1 dx \int_0^{1-x} dy \, \frac{(1-x)yx + 1 - x}{(1 - x - (1-x)xy)\lambda^2 + x} \]  
(C.65)

\[ = -\frac{1}{m_T^2} \int_0^1 dx \int_0^{1-x} dy \, \frac{x[(1-x)y + 1]}{(1-x)yz + 1} \]  
(C.66)

\[ = -\frac{1}{m_T^2} \int_0^1 dx \int_0^{1-x} dy \, \frac{z(x - xy^2y)\lambda^2 + 1 - x}{(1-x)(1 - xy\lambda^2) + x\lambda^2} \]  
(C.67)

\[ = -\frac{1}{m_T^2} \left[ -\frac{4}{3} - 2\log(\lambda) \right] + \frac{1}{6} - \left( \frac{3}{2} + \log(\lambda^2) \right) \]  
(C.68)

where \( \lambda \equiv \frac{m_T}{m_H} \).

Here the identity \( y + z = 1 - x \) comes from the \( \delta \) function and the substitution \( x \to (1-x) \) is allowed because we integrate \( x \) from 0 to 1.

We plug this back in \( \mathcal{M}_\sigma \), which then becomes

\[ \mathcal{M}_\sigma = \frac{-e S_{\tau\mu} S_{H\tau} m_T}{(4\pi)^2} \frac{\pi \sigma^{\mu\nu} p_3 v}{m_T^2} I_{\mu, 1}^{++} u_\tau e_\mu(p_3) \]  
(C.70)

\[ = \frac{e S_{\tau\mu} S_{H\tau} m_T}{(4\pi)^2} \frac{\pi \sigma^{\mu\nu} p_3 v}{m_T^2} \left[ \frac{1}{6} - \left( \frac{3}{2} + \log(\lambda^2) \right) \right] u_\tau e_\mu(p_3) \]  
(C.71)

\[ = \frac{e S_{\tau\mu} S_{H\tau} m_T}{(4\pi)^2} \frac{1}{m_T^2} \frac{\pi \sigma^{\mu\nu} p_3 v}{m_T^2} \left[ \frac{1}{6} - \left( \frac{3}{2} + \log(\lambda^2) \right) \right] u_\tau e_\mu(p_3) \]  
(C.72)
The square is

\[ |\mathcal{M}|^2 = \frac{4\pi \alpha_{em}}{m_H^4} \left( \frac{g_{\tau\mu} g_{H\tau}}{(4\pi)^2} \right)^2 m_T^2 \left| \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} \left[ \frac{4}{3} + \log(\lambda^2) \right] u_T \epsilon_\mu(p_3) \right|^2 \]  

(C.73)

Now sum over all spins and polarizations.

\[ \sum_{\text{polarizations}} \epsilon_\mu^* \epsilon_\alpha \rightarrow -\epsilon_{\mu\nu} \quad \text{(Peskin et al. 1995, p. 159)} \]  

(C.74)

This allows us to use

\[ \sum_{\text{all spins}} [\bar{u}(a) \Gamma_1 u(b)] [\bar{u}(a) \Gamma_2 u(b)]^* = \text{Tr} \left[ \Gamma_1 (p_b + m_b) \Gamma_2 (p_a + m_a) \right] \quad \text{(Griffiths 2008, p. 251)} \]  

(C.76)

Therefore,

\[ \sum_{s, p} \left| \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} \left[ \frac{4}{3} + \log(\lambda^2) \right] u_T \epsilon_\mu(p_3) \right|^2 \]  

(C.77)

\[ = \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \sum_{s, p} \left( \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} u_T \epsilon_\mu(p_3) \right) \left( \tau_{\mu} \sigma^{\alpha\beta} p_{3\beta} u_T \epsilon_\alpha(p_3) \right)^* \]  

(C.78)

\[ = \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \sum_{s, p} \left( \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} u_T \right) \left( \tau_{\mu} \sigma^{\alpha\beta} p_{3\beta} u_T \right)^* \epsilon_\mu(p_3) \epsilon_\alpha^*(p_3) \]  

(C.79)

\[ = - \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \sum_s \left( \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} u_T \right) \left( \tau_{\mu} \sigma^{\alpha\beta} p_{3\beta} u_T \right)^* \sigma_{\mu\alpha} \]  

(C.80)

\[ = - \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \sum_s \left( \tau_{\mu} \sigma^{\mu\nu} p_{3\nu} u_T \right) \left( \tau_{\mu} \sigma^{\alpha\beta} p_{3\beta} u_T \right)^* \]  

(C.81)

\[ = - \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \text{Tr} \left[ \sigma^{\mu\nu} p_{3\nu} (p_1 + m_T) \sigma_{\mu\beta} p_{3\beta} (p_2 + m_T) \right]. \]  

(C.82)

(Griffiths 2008, p. 309)
\[ \sigma^{HV} = \frac{i}{2} \left[ \gamma^{\mu}, \gamma^{\nu} \right] = \frac{i}{2} \left( \gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu} \right) \quad \text{(Peskin et al. 1995, p. 50)} \]  
\[ \text{(C.83)} \]

\[ \sigma^{HV} p_1 = \frac{i}{2} \left[ \gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu} \right] \gamma^\rho p_\rho \]  
\[ = \frac{i}{2} \left[ \gamma^{\mu} (-\gamma^{\rho} \gamma^{\nu} + 2g^{\rho\nu}) - \gamma^{\nu} (-\gamma^{\rho} \gamma^{\mu} + 2g^{\rho\mu}) \right] p_\rho \]  
\[ = \frac{i}{2} \left[ -\gamma^{\mu} \gamma^{\rho} \gamma^{\nu} + 2\gamma^{\mu} g^{\rho\nu} + \gamma^{\nu} \gamma^{\rho} \gamma^{\mu} - 2\gamma^{\nu} g^{\rho\mu} \right] p_\rho \]  
\[ = \frac{i}{2} \left[ \gamma^{\nu} \gamma^{\mu} \gamma^{\rho} + 2\gamma^{\mu} g^{\rho\nu} - 2\gamma^{\nu} g^{\rho\mu} \right] p_\rho \]  
\[ = \gamma_1 \sigma^{HV} + 2ip_1^\nu \gamma^{\mu} - 2ip_1^\mu \gamma^{\nu} \]  
\[ \text{(C.84)} \]

\[ \sigma^{HV} \gamma_\mu \sigma_\mu^\beta = \left( \frac{i}{2} \right)^2 \left[ \gamma^{\mu}, \gamma^{\nu} \right] \left[ \gamma^{\mu}, \gamma^{\beta} \right] \]  
\[ = -\frac{1}{4} \left( \gamma^{\mu} \gamma^{\nu} \gamma^{\mu} \gamma^{\beta} + \gamma^{\nu} \gamma^{\mu} \gamma^{\beta} \gamma^{\mu} - \gamma^{\nu} \gamma^{\mu} \gamma^{\beta} \gamma^{\mu} - \gamma^{\mu} \gamma^{\nu} \gamma^{\beta} \gamma^{\mu} \right) \]  
\[ = -\frac{1}{4} \left( -2\gamma^{\nu} \gamma^{\beta} - 2\gamma^{\nu} \gamma^{\beta} - \gamma^{\nu} 4\gamma^{\beta} - 4g^{\nu\beta} I_4 \right) \]  
\[ = 2\gamma^{\nu} \gamma^{\beta} + 8^{\nu\beta} I_4 \]  
\[ \text{(C.90)} \]

\[ \gamma^{\mu} \gamma_\mu \beta = \frac{i}{2} \gamma^{\mu} \gamma^{\mu} \gamma^{\beta} - \frac{i}{2} \gamma^{\mu} \gamma^{\beta} \gamma^{\mu} \]  
\[ = 3i\gamma^{\beta} \]  
\[ \text{(C.94)} \]

\[ \sigma_\mu^{\beta} p_\beta = \frac{i}{2} \gamma_\mu p_3 - \frac{i}{2} \gamma^{\beta} \gamma_\mu p_3^\beta \]  
\[ = \frac{i}{2} \gamma_\mu p_3 - \frac{i}{2} \left( -\gamma_\mu \gamma^{\beta} + 2g^{\beta}_\mu I_4 \right) p_3^\beta \]  
\[ = \frac{i}{2} \gamma_\mu p_3 + \frac{i}{2} \gamma_\mu \gamma^{\beta} p_3^\beta - \frac{i}{2} 2g^{\beta}_\mu I_4 p_3^\beta \]  
\[ \text{(C.96)} \]
C. Calculation of $\tau \to \mu\gamma$

\begin{align*}
= i\gamma_\mu p_3 - iI_4 p_{3\mu}
\end{align*}

(C.99)

Using the trace theorems from Griffiths (2008, p. 252-253):

\begin{align*}
\text{Tr} \left[ \sigma^{\mu\nu} p_{3\nu}(p_1 + m_\tau)\sigma_\mu^\beta p_{3\beta}(p_2 + m_\mu) \right] \quad (C.100) \\
= \text{Tr} \left[ \sigma^{\mu\nu} p_{3\nu} \sigma_\mu^\beta p_{3\beta} p_2 \right] + \text{Tr} \left[ \sigma^{\mu\nu} m_\tau \sigma_\mu^\beta p_{3\beta} p_2 \right] \quad (C.101) \\
= \text{Tr} \left[ p_{3\nu} \left( p_1 \sigma^{\mu\nu} + 2ip_1^\nu\gamma^\mu - 2ip_1^\mu\gamma^\nu \right) \sigma_\mu^\beta p_{3\beta} p_2 \right] \quad (C.102) \\
= \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma^\nu \gamma_\mu p_3 p_2 \right] + \text{Tr} \left[ 2ip_{3\nu} p_1^\gamma i\gamma_\mu p_3 p_2 \right] \quad (C.103) \\
&\quad - \text{Tr} \left[ 2ip_{3\nu} p_1^\gamma i\gamma_\mu p_3 p_2 \right] \quad (C.104) \\
= 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] + \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right]
\end{align*}

(C.105)

\begin{align*}
= 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] + 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] - 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right]
\end{align*}

(C.106)

\begin{align*}
= 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] + 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] - 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right]
\end{align*}

(C.107)

\begin{align*}
= 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] + 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right] - 2 \text{Tr} \left[ p_{3\nu} p_1^\gamma \gamma_\mu p_3 p_2 \right]
\end{align*}

(C.108)

\begin{align*}
= -32(p_3 \cdot p_1)(p_3 \cdot p_2) + 8 \left( (p_3 \cdot p_1)(p_3 \cdot p_2) - (p_3 \cdot p_3)(p_1 \cdot p_2) + (p_3 \cdot p_2)(p_1 \cdot p_3) \right)
\end{align*}

(C.109)

\begin{align*}
= -8(p_3 \cdot p_1) \text{Tr} \left[ p_3 p_2 \right] + 2 \text{Tr} \left[ p_3 p_1 p_3 p_2 \right]
\end{align*}

(C.110)

\begin{align*}
= -32(p_3 \cdot p_1)(p_3 \cdot p_2) + 16(p_3 \cdot p_1)(p_3 \cdot p_2)
\end{align*}

(C.111)

\begin{align*}
= -16(p_3 \cdot p_1)(p_3 \cdot p_2)
\end{align*}

(C.112)

\begin{align*}
= -4m_\tau^4
\end{align*}

(C.113)
The square of $\mathcal{M}$ is now:

$$\frac{1}{2} \sum_{s, p} |\mathcal{M}_{sp}|^2 = \frac{4\pi \alpha_{\text{em}}}{m_H^4} \left( \frac{g_{\tau \mu} g_{H\tau}}{(4\pi)^2} \right)^2 m_{\tau}^2$$

(C.114)

$$\cdot \left| \Pi_{\mu} \sigma_{\mu\nu} p_{3\nu} \left[ \frac{4}{3} + \log(\lambda^2) \right] u_{\tau\epsilon}(p_3) \right|^2$$

$$= \frac{4\pi \alpha_{\text{em}}}{m_H^4} \left( \frac{g_{\tau \mu} g_{H\tau}}{(4\pi)^2} \right)^2 m_{\tau}^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{1}{2} \frac{1}{4} m_{\tau}^4$$

(C.115)

$$= \frac{4\pi \alpha_{\text{em}} m_{\tau}^6}{m_H^4} \left( \frac{g_{\tau \mu} g_{H\tau}}{(4\pi)^2} \right)^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{2}{m_{H}^4}$$

(C.116)

The particles have momenta

$$p_1 = \left( \frac{m_{\tau}}{2} \right), \quad p_2 = \left( p_3 - p_{3\hat{z}} \right), \quad p_3 = \left( p_3, p_{3\hat{z}} \right).$$

(C.117)

$$p_1 \cdot p_2 = \frac{1}{2} m_{\tau}^2$$

$$p_1 \cdot p_1 = m_{\tau}^2$$

(C.118)

$$p_1 \cdot p_3 = \frac{1}{2} m_{\tau}^2$$

$$p_2 \cdot p_2 = 0$$

(C.119)

$$p_2 \cdot p_3 = \frac{1}{4} m_{\tau}^2 (1 + z^2) = \frac{1}{2} m_{\tau}^2$$

$$p_3 \cdot p_3 = 0$$

(C.120)

$$\Gamma(\tau \rightarrow \mu\gamma) = \frac{|p_{\mu}|}{8\pi m_{\tau}^2} \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2$$

(Griffiths 2008, p. 429) (C.121)

$$= \frac{m_{\tau}}{8\pi m_{\tau}^2} \frac{4\pi \alpha_{\text{em}} m_{\tau}^6}{m_{H}^4} \left( \frac{g_{\tau \mu} g_{H\tau}}{(4\pi)^2} \right)^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{2}{m_{H}^4}$$

(C.122)

$$= \frac{1}{2} \alpha_{\text{em}} m_{\tau}^5 \left( \frac{g_{\tau \mu} g_{H\tau}}{(4\pi)^2} \right)^2 \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \frac{2}{m_{H}^4}$$

(C.123)

$$= 9.404 \times 10^{-9} g_{\tau \mu}^2.$$
C. Calculation of $\tau \rightarrow \mu \gamma$

Now we can calculate the branching ratio by relating it to the Standard Model process $\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)$,

$$BR(\tau \rightarrow \mu\gamma) = \frac{\Gamma(\tau \rightarrow \mu\gamma)}{\Gamma_{\text{Total}}(\tau)} = \frac{\Gamma(\tau \rightarrow \mu\gamma)}{\Gamma_{\text{Total}}(\tau)} \frac{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)}{BR(\tau \rightarrow \mu\nu\bar{\tau}\mu)},$$

since

$$BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) = \frac{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)}{\Gamma_{\text{Total}}(\tau)}, \quad (C.126)$$

$$\Gamma_{\text{Total}}(\tau) = \frac{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)}{BR(\tau \rightarrow \mu\nu\bar{\tau}\mu)}. \quad (C.127)$$

To calculate the branching ratio, we need to insert the following two decay rates:

$$\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu) = \frac{G_F^2 m_\tau^5}{3(4\pi)^3}, \quad (Griffiths 2008, p. 313) \quad (C.128)$$

$$\Gamma(\tau \rightarrow \mu\gamma) = \frac{|p_\mu|}{8\pi m_\tau^2} \frac{1}{2} \sum_{\text{spins}} |M|^2. \quad (Griffiths 2008, p. 429) \quad (C.129)$$

$$BR(\tau \rightarrow \mu\gamma) = \frac{\Gamma(\tau \rightarrow \mu\gamma)}{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)} \frac{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)}{\Gamma_{\text{Total}}(\tau)} \frac{\Gamma(\tau \rightarrow \mu\nu\bar{\tau}\mu)}{BR(\tau \rightarrow \mu\nu\bar{\tau}\mu)} \quad (C.130)$$

$$= \frac{1}{16\pi m_\tau^2} \frac{3(4\pi)^3}{G_F^2 m_\tau^6} \sum_{\text{spins}} |M|^2 \frac{3(4\pi)^3}{4G_F^2} \frac{4\pi m_\tau^6}{4\pi m_\tau^6} \lambda_{em} \left( \frac{S_{\tau\mu} S_{H\tau}}{(4\pi)^2} \right) \left( \frac{4}{3} + \log(\lambda^2) \right) \frac{2}{m_H^4} \cdot BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) \quad (C.131)$$

$$= \frac{3(4\pi)^3}{4G_F^2} \lambda_{em} \left( \frac{S_{\tau\mu} S_{H\tau}}{(4\pi)^2} \right) \left( \frac{4}{3} + \log(\lambda^2) \right) \frac{2}{m_H^4} \cdot BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) \quad \cdot BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) \quad (C.132)$$

$$= \frac{3(4\pi)^3}{4G_F^2} \lambda_{em} \left( \frac{S_{\tau\mu} S_{H\tau}}{(4\pi)^2} \right) \left( \frac{4}{3} + \log(\lambda^2) \right) \frac{2}{m_H^4} \cdot BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) \quad \cdot BR(\tau \rightarrow \mu\nu\bar{\tau}\mu) \quad (C.133)$$

(Lindner et al. 2016, p. 17; Dauncey 2009, p. 3; Harnik et al. 2013, p. 7)
And similarly:

\[
\text{BR}(\mu \rightarrow e\gamma) = \frac{3(4\pi)^3}{4G_F^2}\alpha_{\text{em}} \left( \frac{\mathcal{G}_{\tau\mu} \mathcal{G}_{H\tau}}{(4\pi)^2} \right)^2 \frac{2}{m_H^4} \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \cdot \text{BR}(\mu \rightarrow e\bar{\nu}_e \nu_e) \tag{C.135}
\]

\[
\text{BR}(\tau \rightarrow e\gamma) = \frac{3(4\pi)^3}{4G_F^2}\alpha_{\text{em}} \left( \frac{\mathcal{G}_{\tau\mu} \mathcal{G}_{H\tau}}{(4\pi)^2} \right)^2 \frac{2}{m_H^4} \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \cdot \text{BR}(\tau \rightarrow e\bar{\nu}_e \nu_e) \tag{C.136}
\]

The branching ratio of $\tau \rightarrow \mu\nu\tau\bar{\nu}_\mu$ is

\[
\text{BR}(\tau \rightarrow \mu\nu\tau\bar{\nu}_\mu) = 17.39\% . \tag{Patrignani et al. 2017c, p. 6) \tag{C.137}
\]

That of $\mu \rightarrow e\nu\bar{\nu}_e$ is

\[
\text{BR}(\mu \rightarrow e\nu\bar{\nu}_e) \approx 100\% , \tag{Patrignani et al. 2017b, p. 4) \tag{C.138}
\]

\[
\text{BR}(\tau \rightarrow \mu\gamma) < 4.4 \times 10^{-8} \tag{C.139}
\]

\[
\text{BR}(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13} \tag{C.140}
\]

\[
\text{BR}(\tau \rightarrow e\gamma) < 3.3 \times 10^{-8} \tag{C.141}
\]

(Lindner et al. 2016, p. 13)

\[
\mathcal{G}_{\tau\mu}^2 = \frac{\text{BR}(\tau \rightarrow \mu\gamma)}{\text{BR}(\tau \rightarrow \mu\nu\tau\bar{\nu}_\mu)} \frac{4G_F^2}{3(4\pi)^3}\alpha_{\text{em}} \left( \frac{(4\pi)^2}{\mathcal{G}_{H\tau}} \right)^2 \frac{2}{m_H^4} \left[ \frac{4}{3} + \log(\lambda^2) \right]^2 \tag{C.142}
\]

\[
\mathcal{G}_{\tau\mu} = \sqrt{\frac{\text{BR}(\tau \rightarrow \mu\gamma)}{\text{BR}(\tau \rightarrow \mu\nu\tau\bar{\nu}_\mu)}} \frac{2G_F}{\sqrt{3}(4\pi)^3}\sqrt{\alpha_{\text{em}} \mathcal{G}_{H\tau}} \frac{(4\pi)^2}{\sqrt{2}} \frac{m_H^2}{\sqrt{2}} \left[ \frac{4}{3} + \log(\lambda^2) \right]^{-1} \tag{C.143}
\]
C. Calculation of $\tau \rightarrow \mu \gamma$

$$\delta_{\mu e} = \sqrt{\frac{\text{BR}(\mu \rightarrow e\gamma)}{\text{BR}(\mu \rightarrow e\mu\bar{\nu}_e)}} \frac{2G_F}{\sqrt{3}(4\pi)^{3/2}} \frac{1}{\sqrt{8em}} \frac{(4\pi)^2 m_H^2}{g_H^H} \left[ \frac{4}{3} + \log(\lambda^2) \right]^{-1}$$ (C.144)

$$\delta_{\tau e} = \sqrt{\frac{\text{BR}(\tau \rightarrow e\gamma)}{\text{BR}(\tau \rightarrow e\nu_\tau\bar{\nu}_e)}} \frac{2G_F}{\sqrt{3}(4\pi)^{3/2}} \frac{1}{\sqrt{8em}} \frac{(4\pi)^2 m_H^2}{g_H^H} \left[ \frac{4}{3} + \log(\lambda^2) \right]^{-1}$$ (C.145)

<table>
<thead>
<tr>
<th>Process</th>
<th>Calculated Bound</th>
<th>Bound from literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu \rightarrow e\gamma$:</td>
<td>$\delta_{s1}^{\mu e} &lt; 1.0 \times 10^{-3}$</td>
<td>$\delta_{s1}^{\mu e} &lt; 3.6 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\tau \rightarrow e\gamma$:</td>
<td>$\delta_{s1}^{\tau e} &lt; 7.4 \times 10^{-2}$</td>
<td>$\delta_{s1}^{\tau e} &lt; 1.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\tau \rightarrow \mu\gamma$:</td>
<td>$\delta_{s1}^{\tau\mu} &lt; 8.5 \times 10^{-2}$</td>
<td>$\delta_{s1}^{\tau\mu} &lt; 1.6 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

(Harnik et al. 2013, p. 6)

Discrepancy because of second order terms. (Harnik et al. 2013, p. 9)
This file calculates the decay rate of the decay \( H \rightarrow \gamma \gamma \), where the \( H \) decays into a top quark loop, which emits two photons.

Two different diagrams contribute to this decay rate, they are identical except for the direction of the momentum in the top loop.

The momenta of the particles are:
- \( p_1: H \)
- \( p_2: \gamma \)
- \( p_3: \gamma \)

Method from Peskin & Schroeder chapter 6

Diagram element

First the elements of the Feynman diagram are defined.

- \( G \) collects constant scalar factors from the vertices and propagators
- \( a, b, c \) are the momenta of the top quark in the loop.
- The propagators of the top quark are split into two variables, one for the numerators and one for the denominators.
- \( \epsilon_\nu \) and \( \epsilon_\rho \) is the polarization of the \( \gamma \)
- NC is the color factor of the top quark in the loop

\[
G := \text{Chop}[\text{NC} (-I gHt) (I) (I g\gamma t) (I) (I g\gamma t) (I)]
\]

\[
a := k + p2
\]
\[
b := k
\]
\[
c := k - p3
\]

\[
t\text{PropLoop1D} := \frac{1}{a.a - m_t^2}
\]
\[
t\text{PropLoop2D} := \frac{1}{b.b - m_t^2}
\]
\[
t\text{PropLoop3D} := \frac{1}{c.c - m_t^2}
\]

\[
denom := t\text{PropLoop1D}^{-1} \ast t\text{PropLoop2D}^{-1} \ast t\text{PropLoop3D}^{-1}
\]
numDiagram1 = Expand@Spur[c, \[Gamma] + 1 mt], \[Gamma] \[Vee] , (b, \[Gamma] + 1 mt), \[Gamma] \[Vee] , (a, \[Gamma] + 1 mt)]
numDiagram2 = Expand@Spur[(-a), \[Gamma] + 1 mt], \[Gamma] \[Vee] , ((-b), \[Gamma] + 1 mt), \[Gamma] \[Vee] , ((-c), \[Gamma] + 1 mt)]

num := numDiagram1 + numDiagram2

16 mt \[Kappa], k, + 8 mt \[Kappa], p, 2, \[Kappa], 2, p, 0, - 8 mt \[Kappa], p, 3, - 4 mt p, 2, p, 3, + 
4 mt p, 2, p, 3, + 4 mt^3 \[Gamma] \[Kappa], \[Kappa], - 4 mt k \[Gamma] \[Kappa], \[Kappa], - 4 mt p, 2, p, 3, \[Gamma] \[Kappa], \[Kappa]

16 mt \[Kappa], k, + 8 mt \[Kappa], p, 2, \[Kappa], 2, p, 0, - 8 mt \[Kappa], p, 3, - 4 mt p, 2, p, 3, + 
4 mt p, 2, p, 3, + 4 mt^3 \[Gamma] \[Kappa], \[Kappa], - 4 mt k \[Gamma] \[Kappa], \[Kappa], - 4 mt p, 2, p, 3, \[Gamma] \[Kappa], \[Kappa]

Coefficient[expression, n_] :=
Coefficient[Expand[Contract @ expression /. k \[Rule] \[Kappa] k, \[Kappa], n]
RemoveSinglek[expression_] :=
Expand[(Contract @ expression - Contract @ Coefficient k[expression, 1])]

OnShellp1p2 = OnShellp3p3
OnShellp1p3 = OnShellp2p2

Assump3 = Element[PolyLog[2, _], Reals];
$Assumptions =
{Element[mH, Reals] Element[mt, Reals], mH < mt, mt > 0, mt > \[Theta], Element[g\[ScriptMu]\[ScriptNu], Reals],
g\[ScriptMu]\[ScriptNu] > 0, Element[g\[ScriptE]t, Reals], Element[g\[ScriptE]t, Reals], Element[g\[ScriptE]t, Reals],
Element[g\[ScriptH]t, Reals], Element[C, Reals], Element[CA, Reals], Element[n, Reals],
(n) > 0, Element[p, 2, p, 2, Reals], Element[p, 2, p, 3, Reals], Element[g\[ScriptC], \[ScriptMu], Reals],
Element[g\[ScriptU] \[ScriptV], Reals], Element[g\[ScriptU] \[ScriptV], Reals], Element[p, 2, p, 3, Reals],
Element[p, 3, \[Kappa], Reals], Assump3, EY > 0, mZ > 0, EY < \[Fraction] mZ^2
2

Kinematics

Transverseys[expression_] := expression /. p, 2, 0, \[Rule] 0 /. p, 3, 0, \[Rule] 0

OnShellp1p1[expression_] := expression /. p, 1, p, 1 \[Rule] mH^2
OnShellp2p2[expression_] := expression /. p, 2, p, 2 \[Rule] 0
OnShellp3p3[expression_] := expression /. p, 3, p, 3 \[Rule] 0

OnShellp1p2[expression_] := expression /. p, 1, p, 2 \[Rule] \[Fraction] mH^2
2

2

2

OnShell[expression_] :=
Transverseys @ OnShellp1p2 @* OnShellp2p3 @* OnShellp1p3 @* OnShellp2p2 @* OnShellp3p3 @* OnShellp1p3 @ expression
Denominator

We simplify the denominator using the ‘Feynman trick’. This prevents terms with higher order than \( k^2 \) from appearing, at the cost of three integrals over Feynman parameters \( x, y, z \).

The integral over the loop momentum \( k \) is easier to perform if the denominator contains not linear part in \( k \). Here we will shift the variable \( k \) so that the denominator contains only a \( k^2 \) and a part constant in \( k \), called \( \Delta \). We have to apply the same shift in \( k \) to the numerator as well.

\[
\text{denominator} := k.k - \Delta \\
\Delta := \Delta_{\text{automatic}} \\
\text{Shiftk}[\text{expression}_\text{\_}] := \text{expression} /. k \rightarrow k + \text{DenomShift}
\]

Automatic Calculation

\[
\text{xyzSimplify}[\text{expression}_\text{\_}] := \text{Simplify}[\text{expression}, \\
\text{Assumptions} \rightarrow \{x + y + z == 1, x \geq 0, x \leq 1, y \geq 0, y \leq 1, z \geq 0, z \leq 1\}] /. -1 + y + z \rightarrow -x
\]

\[
\text{Axyz} := \text{tPropLoop2D}^{-1}; \\
\text{Bxyz} := \text{tPropLoop1D}^{-1}; \\
\text{Cxyz} := \text{tPropLoop3D}^{-1}; \\
\text{Denomxyz} = x \text{Axyz} + y \text{Bxyz} + z \text{Cxyz}
\]

\[
x \left( -m^2 + k.k \right) + y \left( -m^2 + k.k + 2 k.p2 + p2.p2 \right) + z \left( -m^2 + k.k - 2 k.p3 + p3.p3 \right)
\]

\[
\text{DenomxyzLinearink} = \text{Coefficient}[\text{Expand}[\text{Denomxyz}], 1] \\
\text{DenomShift} = \text{Expand}\left[\frac{\text{DenomxyzLinearink}}{2} \right] /. \text{var}_\text{\_} \rightarrow \text{var} \rightarrow k \rightarrow \text{Denomxyz}
\]

\[
2 y k.p2 - 2 z k.p3 \\
- p2 y + p3 z
\]

\[
\text{DenomShifted} = \text{xyzSimplify}@\text{RemoveSinglek}@\text{OnShell}@\text{Shiftk}@\text{Denomxyz}
\]

\[
- m^2 + mH^2 y z + k.k
\]

\[
\Delta_{\text{automatic}} = \text{Expand}\left[ - \left( \text{DenomShifted} - k.k \right) \right]
\]

\[
mt^2 - mH^2 y z
\]

Numerator

The numerator contains integrals over \( x, y, z \). It also contains a delta function \( \delta(x+y+z-1) \), so we can do one of these integrals by simply replacing \( x \rightarrow 1-y-z \).

After that we have to perform the shift in \( k \) we performed to simplify the denominator.

Any parts linear in \( k \) that remain will vanish upon integration, so \( \text{RemoveSinglek} \) gets rid of them straight away.
The numerator is then split into a part quadratic in k and a part constant in k.

\[
\text{Intx[Integrand_]} := \text{Integrand} /. x \rightarrow 1 - y - z
\]

\[
\text{NShiftedIntegratedx} = \text{Expand}[\text{xyzSimplify}@\text{Intx@RemoveSinglek@Contract@Expand@OnShell@Shiftk@num}];
\]

\[
\text{NShiftedk0Integratedx} = \text{Coefficient}[\text{NShiftedIntegratedx}, 0]
\]

\[
\text{NShiftedk2Integratedx} = \text{Coefficient}[\text{NShiftedIntegratedx}, 2]
\]

\[
8 m t p_2, p_3, -32 m t y z p_2, p_3, -4 m H^2 m t g_{\nu, \rho} + 8 m t^3 g_{\nu, \rho} + 8 m H^2 m t y z g_{\nu, \rho}
\]

\[
32 m t k, k, -8 m t k . k
\]

\[
\text{Integrals}
\]

\textbf{Momentum integral}

\(\varepsilon\) tensor with more than 1 k is always zero upon “Contract”ion

\[
\text{Substitutitekk}[\text{expression_}] := \\
\text{expression} /. \{k_\alpha k_\beta \rightarrow \frac{1}{d} k . k g_{\alpha, \beta}\} /. \{1_{-} . k k_\alpha \rightarrow \frac{1}{d} k . k\} /. \\
\{k . p_\varepsilon \varepsilon_{\mu, \nu, \rho, \sigma}(k) \rightarrow \frac{1}{d} k . k \varepsilon_{\mu, \nu, \rho, \sigma}(p)\} /. \{k . p_\varepsilon \varepsilon_{\mu, \nu, \rho, \sigma}(k) \rightarrow \frac{1}{d} k . k \varepsilon_{\mu, \nu, \rho, \sigma}(p)\} /. \\
\{k_\rho \varepsilon_{\mu, \nu, \sigma}(k)(p_3) \rightarrow \frac{1}{d} k . k \varepsilon_{\mu, \nu, \sigma}(p_3)\} /. \{k_\nu \varepsilon_{\mu, \rho, \sigma}(k)(p_2) \rightarrow \frac{1}{d} k . k \varepsilon_{\mu, \rho, \sigma}(p_2)\};
\]

\[
\text{Peskin 807:}
\]

\[
\text{k0IntegrationFactor} := \frac{-1}{2 * (4 \pi)^2} \frac{1}{\text{Intx@}\Delta}
\]

\[
\text{k2IntegrationFactor} := \frac{(-1)^{n-1} d}{(4 \pi)^{d/2}} \frac{1}{\text{Gamma}[n - 1 - \frac{d}{2}]} \left(\frac{1}{\text{Intx@}\Delta}\right)^{n - 1 - \frac{d}{2}} / n \rightarrow 3
\]

\[
\text{MShiftedk0Integrateddxk} = \\
\text{ExpandNumerator}[2 \text{NShiftedk0Integrateddx} \ast \text{k0IntegrationFactor}];
\]

\[
\text{MShiftedk2Integrateddxk} = \text{Simplify@}
\]

\[
\text{LoopRefine}[2 \text{Substitutitekk@NShiftedk2Integrateddx} \ast \text{k2IntegrationFactor}];
\]

\[
\text{MShiftedIntegrateddxk} = \text{Factor@LoopRefine[}
\text{MShiftedk0Integrateddxk + MShiftedk2Integrateddxk]}
\]

\[
\frac{-1}{4 \pi^2} \frac{4 y z}{m t} \left(-2 p_2 \cdot p_3, m H^2 g_{\nu, \rho}\right)
\]

\[
\frac{-1}{4 \pi^2} \frac{4 y z}{m t} \left(-2 y z m H^2 m t g_{\nu, \rho}\right)
\]
Feynman Parameter integral

\[
\begin{align*}
M_{\text{ShiftedIntegrated}}(x_k, z) &= \text{Transverse}\int M_{\text{ShiftedIntegrated}}(x_k, z) \, dz; \\
M_{\text{ShiftedIntegrated}}(x_k, z) &= \text{Factor} \left( \frac{M_{\text{ShiftedIntegrated}}(x_k, z) / (z \to 1-y) - M_{\text{ShiftedIntegrated}}(x_k, z) / (z \to 0)}{z^{1-y} - z^0} \right); \\
M_{\text{ShiftedIntegrated}}(x_k, z) &= \text{Simplify} \left( \text{Transverse}\int M_{\text{ShiftedIntegrated}}(x_k, z) \, dy, \{y, 0, 1\} \right); \\
M &= \varepsilon \cdot \varepsilon_{\rho} \cdot M_{\text{ShiftedIntegrated}}(x_k, z)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{4 \, mH^2 \pi^2} & \int \varepsilon \cdot \varepsilon_{\rho} \left( -2 \, p_2 \cdot p_3 + mH^2 \, g_{\nu, \rho} \right) \left( \begin{array}{c}
2 \, mH^2 + \\
\left( mH^2 - 4 \, m^2 \right) \left( \text{PolyLog}[2, \frac{2 \, mH}{mH + i \sqrt{-mH^2 + 4 \, m^2}}] - \text{PolyLog}[2, \frac{mH + i \sqrt{-mH^2 + 4 \, m^2}}{2 \, m^2}] \right) \right)
\end{array} \right)
\end{align*}
\]

\section*{Sum over Spins and Polarizations}

\begin{align*}
\text{MyConjugate} \left[ x \right] := x \, \text{/.} \left\{ \text{Complex}\left[ \text{part}_R, \text{part}_I \right] \to \text{part}_R - i \, \text{part}_I \right\} \\
\text{ConjugateM} \left[ M \right] := \left( \text{MyConjugate} \left[ M \right] \right) \, \text{/.} \left\{ \varepsilon \to \eta \right\} \, \text{/.} \left\{ \rho \to \theta \right\} \\
\text{SumSpins} \left[ M \right] := \\
\text{Contract} \circ \text{Expand} \left[ M \, \text{/.} \left\{ \varepsilon_{\eta} \to 1 \right\} \, \text{/.} \left\{ \varepsilon_{\theta} \to 1 \right\} \, \text{/.} \left\{ \varepsilon_{\nu} \to -g_{\nu, \eta} \right\} \, \text{/.} \left\{ \varepsilon_{\rho} \to -g_{\rho, \eta} \right\} \right] \\
\text{SquareM} \left[ M \right] := \text{LoopRefine} \circ \text{Expand} \circ \text{SumSpins} \left[ M \, \text{ConjugateM} @ M \right]
\end{align*}
Decay Rate

Griffiths p. 429

{SetsOfIndenticleParticles} := \{1, 2\}

\text{ComputeS}[\text{ParticleSets}_\_] := \frac{1}{\text{Length}[\text{ParticleSets}]} \prod_{j=1}^{\text{Length}[\text{ParticleSets}]} \left\{\text{ParticleSets}[j]\right\}!

\text{S} = \text{ComputeS}[\text{SetsOfIndenticleParticles}]

\frac{1}{2}

\Gamma = \text{FullSimplify}\@\text{Expand}\left[\frac{\text{S}}{16 \pi mH} \text{G}^2 \text{MSummedSquared}\right]
\[ \Gamma = \frac{1}{256 \frac{m_H^5}{m_t^5}} g_{Ht}^2 g_{\gamma t}^4 m_t^2 N_C^2 \left( 2 m_H^2 + (m_H^2 - 4 m_t^2) \right) \]
\[ = \Phi \left( \text{PolyLog} \left[ 2, \frac{m_H \left( m_H - i \sqrt{-m_H^2 + 4 m_t^2} \right)}{2 m_t^2} \right], \text{PolyLog} \left[ 2, \frac{m_H + i \sqrt{-m_H^2 + 4 m_t^2}}{2 m_t^2} \right] \right)^2 \]

\[
\begin{align*}
m_Z &= 91.1876 \times 10^9; \\
m_W &= 80.385 \times 10^8; \\
m_t &= 1776.82 \times 10^8; \\
m_H &= 125.09 \times 10^9; \\
m_{\mu} &= 1056.583745; \\
\text{vev}_H &= 246.22 \times 10^9; \\
\alpha_{em} &= \frac{1}{137.0359991}; \\
(\ast \text{vev}_H = \frac{1}{\sqrt{2} \alpha_F}; \ast) \\
GF &= \frac{\text{vev}_H^2}{\sqrt{2}}; \\
\theta_W &= \text{ArcCos}\left( \frac{m_W}{m_Z} \right) \\
c_V &= \frac{1}{2} - \frac{4}{3} \sin[\theta_W]^2; \\
c_A &= \frac{1}{2}; \\
ge &= \sqrt{4 \pi \alpha_{em}}; \\
g_{Zt} &= \frac{ge}{\sin[\theta_W] \cos[\theta_W]}; \\
g_{\gamma t} &= -\frac{2}{3} \sqrt{4 \pi \alpha_{em}}; \\
g_{Ht} &= -\frac{m_t}{\text{vev}_H}; \\
\Gamma_{\text{total}} &= 0.013 \times 10^9; \\
N_C &= 3; \\
\end{align*}

\[ \Gamma \]

\[ 726.93 + 0. \, i \]

\[ BR = \frac{\Gamma}{\Gamma_{\text{total}}} \]

\[ 0.0000559177 + 0. \, i \]
This file calculates the decay rate of the decay \( Z \rightarrow \tau \mu \gamma \), where the \( Z \) decays into a top quark loop, which emits a photon and an off-shell Higgs, that then decays further into a \( \tau \) and a \( \mu \).

Two different diagrams contribute to this decay rate, they are identical except for the direction of the momentum in the top loop.

The momenta of the particles are:
- \( p_1: Z \)
- \( p_2: H \)
- \( p_3: \gamma \)
- \( p_4: \tau \)
- \( p_5: \mu \)

Method from Peskin & Schroeder chapter 6

### Diagram element

First the elements of the Feynman diagram are defined.

G collects constant scalar factors from the vertices and propagators

\( a, b, c \) are the momenta of the top quark in the loop

The propagators of the top quark are split into two variables, one for the numerators and one for the denominators.

\( \epsilon_{\mu} \) is the polarization of the \( Z \)

\( \epsilon_{\gamma} \) is the polarization of the \( \gamma \)

\( NC \) is the color factor of the top quark in the loop

\[
G := \text{Chop}\left[ NC \left( -\frac{\text{I gZt}}{2} \right) \left( \text{I gYt} \right) \left( \text{I} \right) \left( -\text{I gHt} \right) \left( \text{I} \right) \right]
\]

\[
a := k + p_2
\]
\[
b := k
\]
\[
c := k - p_3
\]
tPropLoop1D := \frac{1}{a.a - mt^2}
tPropLoop2D := \frac{1}{b.b - mt^2}
tPropLoop3D := \frac{1}{c.c - mt^2}
denom := tPropLoop1D * tPropLoop2D * tPropLoop3D

numDiagram1 = Expand@Spur[DiracMatrix[\gamma_\mu, (cV1 - cA \gamma 5)], (c.\gamma +1 mt), \gamma_\nu, (b.\gamma +1 mt), (a.\gamma +1 mt)]
numDiagram2 = Expand@Spur[DiracMatrix[\gamma_\mu, (cV1 - cA \gamma 5)], ((-a).\gamma +1 mt), ((-b).\gamma +1 mt), \gamma_\nu, ((-c).\gamma +1 mt)]
num := numDiagram1 + numDiagram2

16 cV mt k_\mu, k_\nu + 8 cV mt k, p_2, -8 cV mt k, p_3, -4 cV mt p_2, p_3, -8 cV mt k, p_3, -4 cV mt p_2, p_3,
+ 4 cV mt p_2, p_3, 8 i cA mt \epsilon_{\mu,\nu,\alpha}(k), (p_3, 4 i cA mt \epsilon_{\mu,\nu,\alpha}(p_2, (p_3)

16 cV mt k_\mu, k_\nu + 8 cV mt k, p_2, -8 cV mt k, p_3, -4 cV mt p_2, p_3, -8 cV mt k, p_3, -4 cV mt p_2, p_3,
+ 4 cV mt p_2, p_3, 8 i cA mt \epsilon_{\mu,\nu,\alpha}(k), (p_3, 4 i cA mt \epsilon_{\mu,\nu,\alpha}(p_2, (p_3)

Coefficient[expression\_, n\_] := Coefficient[Expand[Contract@expression /. \alpha \rightarrow k], \alpha, n]
RemoveSinglek[expression\_] := Expand[{(Contract@expression - Contract@Coefficientk[expression, 1])]

LScalarQ[x] = True;
LScalarQ[y] = True;
LScalarQ[z] = True;
LScalarQ[\alpha] = True;

Assump3 = Element[PolyLog[2, _], Reals];
$Assumptions =
{Element[mH, Reals] Element[mt, Reals], mH < mt, mH > 0, mt > 0, Element[gz\mu, Reals],
gz\mu > 0, Element[ge, Reals], Element[gzt, Reals], Element[gzh, Reals],
Element[gh\mu, Reals], Element[cV, Reals], Element[cA, Reals], Element[n, Reals],
(n) \geq 0, Element[p2, p2, Reals], Element[p2, p3, Reals], Element[gz\mu\nu, Reals],
Element[gz\nu, Reals], Element[ge, Reals], Element[gh, Reals], Element[p2, Reals],
Element[p3, Reals], Assump3, EY > 0, mZ > 0, EY < \frac{mZ}{2}, mH^2 \geq mZ^2};

Kinematics

Transversey[expression\_] := expression /. p_3 \rightarrow 0
Denominator

We simplify the denominator using the ‘Feynman trick’. This prevents terms with higher order than \( k^2 \) from appearing, at the cost of three integrals over Feynman parameters \( x, y, z \).

The integral over the loop momentum \( k \) is easier to perform if the denominator contains not linear part in \( k \). Here we will shift the variable \( k \) so that the denominator contains only a \( k^2 \) and a part constant in \( k \), called \( \Delta \). We have to apply the same shift in \( k \) to the numerator as well.

denominator := \( k.k - \Delta \)
\( \Delta := \Delta_{\text{automatic}} \)
Shiftk[expression_] := expression /. k \[RightTostate] k + DenomShift

Automatic Calculation

xyzSimplify[expression_] := \text{Simplify[expression,}
Asumptions \rightarrow \{x + y + z == 1, x \[GreaterEqual] 0, x \[LessEqual] 1, y \[GreaterEqual] 0, y \[LessEqual] 1, z \[GreaterEqual] 0, z \[LessEqual] 1\}] \[RightTostate] \(-1 + y + z \[RightTostate] -x\)

Axyz := tPropLoop2D\(^{-1}\);
Bxyz := tPropLoop1D\(^{-1}\);
Cxyz := tPropLoop3D\(^{-1}\);
Denomxyz = x Axyz + y Bxyz + z Cxyz

\( x \{ -m^2 + k.k \} + y \{ -m^2 + k.k + 2 k.p2 + p2.p2 \} + z \{ -m^2 + k.k - 2 k.p3 + p3.p3 \} \)

\text{DenomxyzLinearink = Coefficient[k[Expand[Denomxyz], 1]}\)
\text{DenomShift = Expand[\frac{\text{DenomxyzLinearink}}{-2}}/. \text{var} _\rightarrow \text{. k \[RightTostate] var}]\)

\text{2 y k.p2 - 2 z k.p3}
- p2 y + p3 z
\[ \text{DenomShifted} = \text{xyzSimplify@RemoveSinglek@OnShell@Shiftk@Denomxyz} \]
\[-m t^2 + m Z y (m Z - 2 E y x - m Z y) + k.k \]
\[ \Delta_{\text{automatic}} = \text{Expand}\left[-\left(\text{DenomShifted} - k.k\right)\right] \]
\[ m t^2 - m Z^2 y + 2 E y m Z x y + m Z^2 y^2 \]

**Numerator**

The numerator contains integrals over x, y, z. It also contains a delta function \(\delta(x+y+z-1)\), so we can do one of these integrals by simply replacing x\(\rightarrow\)1-y-z.

After that we have to perform the shift in k we performed to simplify the denominator.

Any parts linear in k that remain will vanish upon integration, so \text{RemoveSinglek} gets rid of them straight away.

The numerator is then split into a part quadratic in k and a part constant in k.

\[ \text{Int}_x[\text{Integrand}_\_] := \text{Integrand} /. x \rightarrow 1 - y - z \]

\[ \text{NShiftedIntegrated}_x = \text{Expand}[\text{xyzSimplify@Int}_x@\text{RemoveSinglek@Contract@Expand@OnShell@Shiftk@num}] ; \]

\[ \text{NShiftedk}_0\text{Integrated}_x = \text{Coefficient}_k[\text{NShiftedIntegrated}_x, 0] \]
\[ \text{NShiftedk}_2\text{Integrated}_x = \text{Coefficient}_k[\text{NShiftedIntegrated}_x, 2] \]

\[ -16 c V m t y p_2^2 + 32 c V m t y^2 p_2^2 + p_2^4 - 8 c V m t p_2^3 + 16 c V m t y p_2^3 + p_3^2 - 32 c V m t y z p_2^2 + p_3^2 + 8 c V m t m Z y g_{\mu,v} - 16 c V E y m Z y g_{\mu,v} + 16 c V E y m Z^2 y^2 g_{\mu,v} - 8 c V m t m Z^2 y^2 g_{\mu,v} + 16 c V E y m Z y z g_{\mu,v} + 32 c V m t k_\mu k_\nu - 8 c V m t k.k g_{\mu,v} \]

**Integrals**

**Momentum integral**

\(\epsilon\) tensor with more than 1 k is always zero upon "Contract"ion

\[ \text{Substitute}_k[\text{expression}_\_] := \text{expression} /. \{k_{\alpha_k} k_{\beta_k} \rightarrow \frac{1}{d} k.k g_{\alpha_k,\beta_k}\} / . \{l_{\alpha_k}.k k_{\alpha_k} \rightarrow \frac{1}{d} k.k\} ; \]

\[ \text{k}_0\text{IntegrationFactor} := \frac{-1}{2 \times (4 \pi)^2} \frac{1}{\text{Int}_x\Delta} \]
\[ \text{k}_2\text{IntegrationFactor} := \frac{-1}{(4 \pi)^{d/2}} \frac{1}{\text{Gamma}\left[n - 1 - \frac{d}{2}\right]} \frac{1}{\text{Gamma}\left[n\right]} \left(\frac{1}{\text{Int}_x\Delta}\right)^{n - 1 - \frac{d}{2}} / . \ n \rightarrow 3 \]
MShiftedk0Integratedxk =
ExpandNumerator[2 NShiftedk0Integratedx * k0IntegrationFactor];
MShiftedk2Integratedxk = Simplify@
LoopRefine[2 Substitutetekk@NShiftedk2Integratedx * k2IntegrationFactor];
MShiftedIntegratedxk =
Factor@LoopRefine[MShiftedk0Integratedxk + MShiftedk2Integratedxk];

- (i cV mt (-2 p2, p2, + 4 y^2 p2, p2, - p2, p3, + 2 y p2, p3, - 4 y z p2, p3, + E y mZ g_{\mu,\nu} -
4 E y mZ y g_{\mu,\nu} + mZ^2 y g_{\mu,\nu} + 4 E y mZ y z g_{\mu,\nu}) ) /
(2 \pi^2 (m^2 + 2 E y mZ y - mZ^2 y - 2 E y mZ y^2 + mZ^2 y^2 - 2 E y mZ y z ))

\textbf{Feynman Parameter integral}

OrderOfmtExpansion = 1;

MShiftedIntegratedxkExpansionmt =
Expand@Normal[Series[Expand[MShiftedIntegratedxk /. mt \to 1/mtninverse],
{mtninverse, 0, OrderOfmtExpansion}]] /. mtninverse \to 1/mt;

MShiftedIntegratedxkExpansionmtPrimitivez =
Transverse@Integrate[MShiftedIntegratedxkExpansionmt, z];
MShiftedIntegratedxyzkExpansionmt = Transverse@
Integrate[{MShiftedIntegratedxkExpansionmtPrimitivez /. z \to 1 - y} -
{MShiftedIntegratedxyzkExpansionmtPrimitivez /. z \to 0}, {y, 0, 1}]

M = \epsilon_{\mu} \epsilon_{\nu} * MShiftedIntegratedxyzkExpansionmt;

\textbf{Lepton Current}

\nu\text{current} = \langle \mu[p4, ml1], (1 + \gamma 5), \nu[p5, ml2] \rangle;
M_{\mu\text{part}} = \mu\text{current} (-i g_{\tau\mu}) \frac{1}{p2.p2 - mH^2};

\text{SumLeptonSpins[expression_]} := \text{Refine}[\text{expression}, \{g_{\tau\mu} \in \text{Reals}, p2.p2 \in \text{Reals}\} /. \mu\text{current} \to \text{Spur}[\{i + \gamma 5\}, (p4.\gamma + i m\tau), (1 - \gamma 5), (p5.\gamma + i m\mu)]]

LSquared = OnShell@SumLeptonSpins[(-M_{\mu\text{part}}) M_{\mu\text{part}}]
4 g_{\tau\mu}^2 (mZ (-2 E y + mZ) - m_{\mu}^2 - m_{\tau}^2)
(-mH^2 - 2 E y mZ + mZ^2)^2
Sum over Spins and Polarizations

\[
\text{MyConjugate}[x_\_] := x /. \{\text{Complex}[\text{partR\_}, \text{partI\_}] \rightarrow \text{partR} - i \text{partI}\}
\]

\[
\text{ConjugateM}[M\_] := (\text{MyConjugate}[M]) /. \mu \rightarrow \xi / \gamma \rightarrow \eta
\]

\[
\text{SumSpins}[M\_] := \text{Contract} @ \text{Expand}\left[ M / . \{\epsilon_\xi \rightarrow 1\} / . \{\epsilon_\gamma \rightarrow 1\} / . \{\epsilon_\eta \rightarrow 1\} / . \{\epsilon_\mu \rightarrow -\left(\frac{p_1 \mu \cdot \mu_1}{m^2}\right)\} / . \{\epsilon_\nu \rightarrow -g_{\alpha_1,\gamma}\}\right]
\]

\[
\text{SquareM}[M\_] := \text{LoopRefine} @ \text{Expand} @ \text{SumSpins}[M \ast \text{ConjugateM}[M]]
\]

Decay Rate

Griffiths p. 429

\[
\text{SetsOfIndenticleParticles} := \{1, 1, 1, 1\}
\]

\[
\text{ComputeS}[\text{ParticleSets\_}] := \text{Length}[\text{ParticleSets}] / \prod_{j=1}^{\text{Length}[\text{ParticleSets}]} \text{ParticleSets}[[j]]!
\]

\[
S = \text{ComputeS}[\text{SetsOfIndenticleParticles}]
\]

\[
\text{MSummedSquaredWOLEptons} = \text{SquareM}[M]
\]

\[
\text{MSummedSquared} = \text{Expand} @ \text{Cancel} @ \text{OnShell} @ \text{Expand} @ [\text{MSummedSquaredWOLEptons} \ast \text{LSquared}]
\]

\[
\left(\text{c}\text{V}^2 \left(4 E^2 m_Z^4 - E_Y m_Z^2 p_1.p_1 + 2 E_Y m_Z p_1.p_2 p_1.p_3 - \left(p_1.p_3\right)^2 p_2.p_2 - 2 E_Y m_Z^2 p_2.p_3 - m_Z^2 p_2.p_2 p_3.p_3\right)\right) / (36 m_t^2 m_Z^2 \pi^4)
\]

\[
4 \text{c}\text{V}^2 E^2 g_{\gamma\mu}^2 m_Z^3 + 2 \text{c}\text{V}^2 E^2 g_{\gamma\mu}^2 m_Z^4 - 9 m_t^2 (m_H^2 + 2 E_Y m_Z - m_Z^2)^2 \pi^4 + 9 m_t^2 (m_H^2 + 2 E_Y m_Z - m_Z^2)^2 \pi^4
\]

\[
9 m_t^2 (m_H^2 + 2 E_Y m_Z - m_Z^2)^2 \pi^4 - 2 \text{c}\text{V}^2 E^2 g_{\gamma\mu}^2 m_Z^2 m_{\mu}\acute{\mu} - 2 \text{c}\text{V}^2 E^2 g_{\gamma\mu}^2 m_Z^2 m_{\acute{\mu}}^2
\]

\[
9 m_t^2 (m_H^2 + 2 E_Y m_Z - m_Z^2)^2 \pi^4 - 9 m_t^2 (m_H^2 + 2 E_Y m_Z - m_Z^2)^2 \pi^4
\]

\[
\text{MSummedSquaredIntE} = \text{Expand} @ \text{Integrate} @ [\text{MSummedSquared}, \{E_Y, \frac{m_Z}{2} - E_Y, \frac{m_Z}{2}\}];
\]

\[
\text{MSummedSquaredIntE} = \text{Expand} @ \text{Integrate} @ [\text{MSummedSquaredIntE} \ast \{E_Y, 0, \frac{m_Z}{2}\}];
\]

\[
\Gamma = \text{Collect} @ \text{Factor} @ \text{Expand} \left[\frac{1}{(2 \pi)^3} \frac{1}{8 m_Z} \frac{1}{3} \text{MSummedSquaredIntE} \ast \text{Integrate} [\text{MSummedSquaredIntE}, \{E_Y, \frac{m_Z}{2}\}];\right]
\]
\[
\begin{align*}
\Gamma & = - \left( (cV^2 gH^2 gZt^2 g\gamma^2 g\tau^2 g\zeta^2 \mu^2) (24 mH^6 mZ^2 - 42 mH^4 mZ^4 + 17 mH^2 mZ^6 - 18 mH^4 mZ^2 m\mu^2 + 27 mH^2 mZ^4 m\mu^2 - 6 mZ^2 m\mu^2 - 18 mH^4 mZ^2 m^2 + 27 mH^2 mZ^4 m^2 - 6 mZ^2 m^2) NC^2 \right) / (331776 mH^2 mZ^2 m^2) - \\
& \left( cV^2 gH^2 gZt^2 g\gamma^2 g\tau^2 g\zeta^2 \mu^2 \right) (24 mH^8 - 54 mH^6 mZ^2 + 36 mH^4 mZ^4 - 6 mH^2 mZ^6 - 18 mH^4 m\mu^2 + \\
36 mH^2 mZ^2 m\mu^2 - 18 mH^2 mZ^4 m\mu^2 - 18 mH^2 m^2 + 36 mH^2 mZ^2 m^2 - 18 mH^2 mZ^4 m^2) NC^2 \log \left( 1 - \frac{m^2}{mH^2} \right) / (331776 mH^2 mZ^2 m^2) \right)
\end{align*}
\]

\[
mZ = 91.1876 \times 10^9; \\
m\bar{W} = 80.385 \times 10^9; \\
m\tau = 1776.82 \times 10^9; \\
mH = 125.09 \times 10^9; \\
m\mu = 1056583745; \\
vevH = 246.22 \times 10^9; \\
\alphaem := \frac{1}{137.0359991}; \\
(*vevH := \frac{1}{\sqrt{\frac{3}{2} GF}};*)
\]

\[
GF = \frac{vevH^2}{\sqrt{2}};
\]

\[
\thetaW := \text{ArcCos} \left( \frac{m\bar{W}}{mZ} \right)
\]

\[
cV := \frac{1}{2} - \frac{4}{3} \sin[\thetaW]^2; \\
cA := \frac{1}{2}; \\
ge := \sqrt{4 \pi \alphaem}; \\
gZt := \frac{ge}{\sin[\thetaW] \cos[\thetaW]}; \\
g\gamma \tau := \frac{-2}{3} \sqrt{4 \pi \alphaem}; \\
gHt := \frac{-m\tau}{vevH}; \\
\rhoZTotal = 2.4952810 \times 10^9; \\
NC = 3;
\]

\[
0.0130124 g\zeta^2
\]

\[
\text{BR} = \frac{\Gamma}{\rhoZTotal} = 5.21482 \times 10^{-12} g\zeta^2
\]