Fitzpatrick-Pejsachowicz degree theory for nonlinear Fredholm mappings of index 0

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Abstract

We extend the well known Leray-Schauder degree to the Fredholm mappings of index 0 and define a degree theory of these mappings. We first define parity, a homotopy invariant of a path of linear Fredholm operators and then we prove properties of parity which is required for developing degree. As a next step, we define a degree theory based on Fitzpatrick and Pejsachowicz work in 1992 [8]. And finally, we apply this degree to nonlinear elliptic partial differential equations.

Details

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Chapter 0

Introduction

The Topological Degree Theory has been widely used in the theory of differential equations and Dynamical systems. The Topological Degree Theory is a generalization of the winding number of a curve in the complex plane, where winding number is the total number of times that a closed curve travels counterclockwise around the origin. L.E.J. Brouwer defined the degree for a map for the first time in 1911.\[3\]

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$, and

$$f : \overline{\Omega} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

be a $C^1$-mapping. $p$ is a regular value of $f$ if $J_f(x) \neq 0$ for all $x \in f^{-1}(p)$. Then for a regular value $p \notin f(\partial \Omega)$, the $C^1$-\textbf{mapping degree} is defined by

$$\text{deg}(f, \Omega, p) := \sum_{x \in f^{-1}(p)} \text{sign} (J_f(x))$$

which takes values in $\mathbb{Z}$. Note that

$$\text{sign} (J_f(x)) = (-1)^\mu,$$

where $\mu$ is the number of negative eigenvalues of $J_f(x)$. Differentiability of $f$ is strongly used in the definition of $C^1$ mapping degree. By using the fact that $C^1$-functions can be approximated by $C^0$-functions and also the homotopy invariance of $C^1$-mapping degree we can define Brouwer degree.

Let $f \in C^0(\overline{\Omega})$, where $C^0(\overline{\Omega})$ is the space of all continuous mappings from $\overline{\Omega} \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$, and let $p \notin f(\partial \Omega)$. The \textbf{Brouwer degree} is defined by

$$\text{deg}_B(f, \Omega, p) := \lim_{k \to +\infty} \text{deg}(f^k, \Omega, p)$$
for any sequence \( f^k \in C^1(\Omega) \) converging to \( f \) in \( C^0 \).

But for infinite dimensional spaces finding a degree theory is not so straightforward. In 1934, J. Leray and J. Schauder extended the Brouwer degree to mappings on infinite dimensional spaces, where the mappings are perturbations of identity map with a compact operators.[11]

Let \( X \) be a real Banach space, \( \Omega \subseteq X \) be open bounded subset of \( X \) and \( f \) be a continuous map of the form \( f = Id - k \) where \( Id \) is the identity map on \( \Omega \subseteq X \). And let \( p \notin f(\partial \Omega) \). Moreover, let \( k^\varepsilon \) be a finite rank perturbation with \( \|k - k^\varepsilon\| < \varepsilon \) and \( k^\varepsilon(\Omega) \subset Y^\varepsilon \subset X \). Then for any finite dimensional subspace \( X^\varepsilon \) containing both \( Y^\varepsilon \) and \( p \), define the Leray-Schauder degree as

\[
\text{deg}_{LS}(f, \Omega, p) := \text{deg}_{B}(f^\varepsilon, \Omega \cap X^\varepsilon, p),
\]

where \( f^\varepsilon = Id - k^\varepsilon \).

We can also define the Leray-Schauder degree in the \( C^1 \)-case, likewise the Brouwer degree. Let \( p \notin f(\partial \Omega) \) be a regular value, then by Inverse Function Theorem there exists an open neighborhood \( V \) of \( f^{-1}(p) \) in \( X \) such that there is only one \( x \in V \) with \( f(x) = p \). In the other words, the set \( f^{-1}(p) \) consisted of isolated points. Let \( x_n \in f^{-1}(p) \), then \( x_n = p + k(x_n) \). Since \( k \) is a compact operator, \( x_n \) has a convergent subsequence, and hence \( f^{-1}(p) \) is compact. Compactness and isolation yields that \( f^{-1}(p) \) is a finite set. Then

\[
\text{deg}_{LS}(f, \Omega, p) = \sum_j \text{deg}_{LS}(f, B_\varepsilon(x_j), p),
\]

where \( x_j \in f^{-1}(p) \). Moreover,

\[
\text{deg}_{LS}(f, B_\varepsilon(x_j), p) = (-1)^\mu,
\]

where \( \mu \) is the sum of multiplicities of negative eigenvalues of \( f'(x_j) \). Leray and Schauder proved that the number of negative eigenvalues of \( f'(x_j) \) counted with multiplicity is finite [11]. Thus \( \mu < \infty \).

Although the Leray-Schauder degree has been widely used in the theory of nonlinear differential equations, it still has some disadvantages. Namely, it can not be defined for general mappings whose the domain and the range of the mappings are different Banach spaces, since Leray-Schauder degree is just defined on compact perturbations of the identity map which must have
the same range and domain. Moreover, Leray-Schauder degree cannot be even extended to general mappings with the same domain and range which are not perturbation of the identity map.

To be more precise let $X$ and $Y$ be Banach spaces and $\text{GL}(X, Y)$ the space of all invertible bounded linear operators from $X$ to $Y$, and $\text{GL}(X)$ the space of all invertible bounded linear operators from $X$ to $X$. If we let $\text{GL}_C(X)$ the subspace of $\text{GL}(X)$ such that all elements of $\text{GL}_C(X)$ are perturbations of the identity map, then we have

**Theorem 0.0.1.** The space of $\text{GL}_C(X)$ has two connected components $\text{GL}^\text{even}_C(X)$ and $\text{GL}^\text{odd}_C(X)$ defined by

$$\text{GL}^\text{even}_C(X) = \{T \in \text{GL}_C(X) | \text{T has even number of negative eigenvalues}\}$$

and

$$\text{GL}^\text{odd}_C(X) = \{T \in \text{GL}_C(X) | \text{T has odd number of negative eigenvalues}\}$$

The idea behind the Leray-Schauder degree is the fact that we can split the space $\text{GL}_C(X)$ in these two connected components. But Kuiper’s Theorem states that the space $\text{GL}(X)$ is contractible [11]. In Topology, a topological space is contractible, if the identity map on $X$ is homotopic to some constant map. i.e. there is a homotopy $H : [0, 1] \times X \rightarrow X$ such that if $x \in X$ then $H(0, x) = x$ and $H(1, x) = c$, where $c \in X$ is a constant. Hence $\text{GL}(X)$ cannot be split into two connected components. To overcome these problems we consider specific class of mappings named Fredholm.

### 0.1 Linear Fredholm Operators

We first introduce some notation. If $X$ and $Y$ are real Banach spaces, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators from $X$ to $Y$ and $\text{GL}(X, Y)$ will denote the invertible members of $\mathcal{L}(X, Y)$. An operator $T \in \mathcal{L}(X, Y)$ is compact if $\overline{T(D)}$ is compact for any bounded subset $D \subset X$. The space of all compact operators from $X$ to $Y$ is denoted by $\mathcal{K}(X, Y)$, if $X = Y$ the spaces are denoted by $\mathcal{L}(X)$, $\text{GL}(X)$, and $\mathcal{K}(X)$ respectively.

Moreover, we let

$$\mathcal{L}_C(X) = \{T \in \mathcal{L}(X) | T = I - K, K \in \mathcal{K}(X)\}$$

the space of all **compact vector fields**, and

$$\text{GL}_C(X) = \mathcal{L}_C(X) \cap \text{GL}(X).$$
The term compact vector field for operators in $\mathcal{L}_C(X)$ is our terminology in this text.[6]

**Remark 0.1.1.** In both $\mathcal{L}_C(X)$ and $\text{GL}_C(X)$ the operators can not be from $X$ to $Y$ because these operators are perturbation of identity map with compact operators. So $\mathcal{L}_C(X,Y)$ and $\text{GL}_C(X,Y)$ are meaningless.

**Definition 0.1.2** (Fredholm operator). An operator $T \in \mathcal{L}_C(X,Y)$ is said to be **Fredholm** if both $\dim \ker T$ and $\dim \text{coker} T$ are finite. Here $\text{coker} T$ means $Y/\text{Range} T$. In which case the index of $T$, $\text{ind} T$, is defined by

$$\text{ind}(T) = \dim \ker T - \dim \text{coker} T$$

$$= \dim \ker T - \dim(Y/\text{Range} T).$$

The Fredholm operators from $X$ to $Y$ will denoted by $\Phi(X,Y)$, and those of index $k \in \mathbb{Z}$ by $\Phi_k(X,Y)$. If $X = Y$ the spaces are denoted by $\Phi_k(X)$ and $\Phi(X)$ respectively.

**Remark 0.1.3.** In some other texts, the definition of Fredholm operator also has the condition that $\text{Range} T$ be closed. But if both $\dim \ker T$ and $\dim \text{coker} T$ be finite, then $\text{Range} T$ is closed (cf. [1], p.156). Therefore this condition is actually redundant.

**Theorem 0.1.4.** If $T \in \mathcal{L}(X,Y)$, then $T \in \Phi_0(X,Y)$ if and only if there exists $S \in \text{GL}(Y,X)$ with $S \circ T$ a compact linear vector field.

**Remark 0.1.5.** One can also prove the general statement for all Fredholm operators as follow:

If $T \in \mathcal{L}(X,Y)$, then $T \in \Phi(X,Y)$ if and only if there exists $S \in \mathcal{L}(Y,X)$ with $S \circ T$ and $T \circ S$ compact vector fields on $X$ and $Y$ respectively.

**Proof of Theorem 0.1.4.** " $\Rightarrow$ " Assume $T \in \Phi_0(X,Y)$, then $\dim \ker T < \infty$ and $\dim \text{coker} T < \infty$, which means $\dim \ker T < \infty$ and $\dim(Y/\text{Range} T) < \infty$. So since $\text{ind}(T) = 0$, we can assume $\dim \ker T = \dim(Y/\text{Range} T) = n$.

Moreover consider decompositions

$$X = X' \oplus \ker T$$

$$Y = \text{Range} T \oplus Y',$$

and block decomposition of

4
\[ T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \]

with respect to above decompositions. Then

\[ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} X' \\ \ker T \end{pmatrix} = \begin{pmatrix} \text{Range} T \\ Y' \end{pmatrix} \]

and \( T_{21} \) must be the 0 map, because if \( T_{21} \neq 0 \) and \( 0 \neq x \in X' \) then

\[ Tx = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11}x \\ T_{21}x \end{pmatrix} \]

which means \( T_{21}x \in Y' \) and since \( T_{21} \neq 0 \), and \( x \notin \ker T \), \( 0 \neq T_{21}x \in Y' \). So \( T_{21}x \in \text{Range} T \cap Y' \) which is a contradiction to \( \text{Range} T \cap Y' = \{0\} \). Hence the block decomposition of \( T \) is

\[ \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \]

where \( T_{11} = T \mid_{X'} : X' \longrightarrow \text{Range} T \) which is invertible.

Furthermore, since both \( Y' \) and \( \ker(T) \) are \( n \)-dimensional spaces we can choose an \( n \times n \) invertible matrix \( A \) such that \( A(Y') = \ker(T) \). Let \( \tilde{S} = T_{11}^{-1} : \text{Range} T \longrightarrow X' \) and

\[ S = \begin{pmatrix} \tilde{S} & 0 \\ 0 & A \end{pmatrix} . \]

\( S \) is invertible since \( \tilde{S} \) and \( A \) both are invertible maps, thus \( S \in \text{GL}(Y, X) \). Furthermore if \( x \in X \) is arbitrary, then \( x = x_1 + x_2 \) where \( x_1 \in X' \), and \( x_2 \in \ker T \). Then we have

\[ (S \circ T)x = S \circ T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tilde{S} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

\[ = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} T_{11}x_1 + T_{12}x_2 \\ T_{22}x_2 \end{pmatrix} \]

\[ = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} T_{11}x_1 \\ 0 \end{pmatrix} \]

\[ = T_{11}^{-1}T_{11}x_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \]

\[ = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \]
Hence $S \circ T = Id - \pi_2(X)$ where \( \pi_2 \) is the projection of \( X \) onto \( \ker T \), which is compact operator. Therefore $S \circ T \in \mathcal{L}_C(X)$.

" \( \Leftarrow \) " Now assume $T \in \mathcal{L}(X, Y)$ and $S \in \text{GL}(Y, X)$ such that $S \circ T \in \mathcal{L}_C(X)$. The classic Fredholm Alternative amounts to the assertion that $\mathcal{L}_C(X) \subseteq \Phi_0(X)$. Hence $S \circ T \in \Phi_0(X)$ and therefore $\dim \ker S \circ T = \dim \text{coker} S \circ T < \infty$.

Moreover $\ker T \subseteq \ker S \circ T$, and if $x \in \ker S \circ T$, then $S \circ T(x) = 0$. Since $S \in \text{GL}(Y, X)$, $(S^{-1} \circ S \circ T)x = S^{-1}0 = 0$, and hence $Tx = 0$ which implies $x \in \ker T$. Thus $\ker S \circ T \subseteq \ker T$, and hence $\ker S \circ T = \ker T$. Therefore $\dim \ker T = \dim \ker S \circ T$.

Furthermore assume $y \in Y/\text{Range} T$, then $y = y_1 + Tx_1$ so $Sy = Sy_1 + (S \circ T)x_1$. Since $Sy_1 \in X$ and $(S \circ T)x_1 \in \text{Range}(S \circ T)$, $Sy \in X/\text{Range}(S \circ T)$. Hence $Sy = k_1x_1 + \ldots + k_nx_n$ where \( \{x_1, \ldots, x_n\} \) is the basis of $X/\text{Range}(S \circ T)$. Then $y = S^{-1}(k_1x_1 + \ldots + k_nx_n) = k_1S^{-1}x_1 + \ldots + k_nS^{-1}x_n$. But \( \{S^{-1}x_1, \ldots, S^{-1}x_n\} \) are $n$ independent vectors in $Y/\text{Range} T$, because if a linear combination of these vectors $k_1S^{-1}x_1 + \ldots + k_nS^{-1}x_n$ be zero, then $S(k_1S^{-1}x_1 + \ldots + k_nS^{-1}x_n) = S(0) = 0$, thus $k_1x_1 + \ldots + k_nx_n = 0$. Since \( \{x_1, \ldots, x_n\} \) are linearly independent, $k_1 = \ldots = k_n = 0$. Therefore $\text{dim coker} T = \text{dim coker} S \circ T$. Hence $\dim \ker T = \text{dim coker} T < \infty$, and $T \in \Phi_0(X, Y)$. \( \square \)

To close this section, we list some properties of linear Fredholm operators:[2]

- Every linear operator $T \in \text{GL}(X, Y)$ is Fredholm of index 0.
- If $T \in \Phi(X, Y)$ and $k \in \mathcal{K}(X, Y)$, then $T + k \in \Phi(X, Y)$ and $\text{ind}(T + k) = \text{ind} T$.
- Let $H : [0, 1] \times X \longrightarrow Y$ be a homotopy such that $H(t, .) \in \mathcal{L}(X, Y)$ for every $t \in [0, 1]$. If $H(0, .) \in \Phi_k(X, Y)$, then $H(1, .) \in \Phi_k(X, Y)$ for any $k \in \mathbb{N}$.
- If $T \in \Phi(X, Y)$ and $S \in \Phi(Y, Z)$, then $S \circ T \in \Phi(X, Z)$ and $\text{ind}(S \circ T) = \text{ind} T + \text{ind} S$.

### 0.2 Nonlinear Fredholm Mappings

For definition of nonlinear Fredholm mapping first we need the notion of differentiability at a point for an operator.
Definition 0.2.1. A continuous nonlinear mapping $f$ is called Fréchet differentiable at $x_0$ if there is a linear operator $A \in \mathcal{L}(X,Y)$ such that in a neighborhood $U$ of $x_0$,

$$
\| f(x) - f(x_0) - A(x-x_0) \| = o(\|x-x_0\|).
$$

In this case we say $A = f'(x_0)$, and $f'(x_0)$ is Fréchet derivative of $f$ at $x_0$. Partial derivatives of a nonlinear mapping $f : U \rightarrow Y$ where $U = \prod_{i=1}^N U_i$ and each $U_i$ is an open subset of a Banach space $X_i$ can be defined. If $x = (x_1, ..., x_N) \in U$ with $x_i \in U_i$, the Fréchet partial derivative of $f$ with respect to $x_i$, $D_i f(x)$, is defined by

$$
f(x_1, ..., x_i + h, ..., x_N) - f(x_1, ..., x_i, ..., x_N) = \beta(h) + o(\|h\|),
$$
in which $D_i f(x) = \beta(h) \in \mathcal{L}(X_i,Y)$. Moreover, as in standard calculus textbooks, we can prove that if $D_i f(x) \in \mathcal{L}(X_i,Y)$, for all $1 \leq i \leq N$,

$$
f'(x)h = \sum_{i=1}^N D_i f(x)h_i.
$$

If the mapping $Df : X \rightarrow \mathcal{L}(X,Y)$ is continuous, $f$ is called $C^1$. Note that continuity of $Df$ is respect to convergence in norm, where the space $\mathcal{L}(X,Y)$ is equipped with the norm $\|T\|_{\mathcal{L}(X,Y)} = \inf_{\|x\|=1} \|Tx\|_Y$ in which $T \in \mathcal{L}(X,Y)$.

Definition 0.2.2. Let $X$ and $Y$ be Banach spaces. A $C^1$ mapping $f : X \rightarrow Y$ is called a nonlinear Fredholm mapping if the Fréchet derivative of $f$, $f'(x)$ is a linear Fredholm operator for each $x \in X$. In this case, the index of $f$, $\text{ind} f$, is defined by setting $\text{ind} f(x) = \text{ind} f'(x)$, for $x \in X$.

The above definition is well defined due to the fact that $\text{ind} f(x)$ is independent of $x \in X$, because, if $x, y \in X$, let $H(t, x) = tf'(x) + (1 - t)f'(y)$ where $t \in [0,1]$. Then $H : [0,1] \times X \rightarrow Y$ is a homotopy such that $H(0,.) = f'(x)$ and $H(1,.) = f'(y)$. Third aforementioned property of linear Fredholm operators implies that $\text{ind} f'(x) = \text{ind} f'(y)$. Thus, $\text{ind} f(x)$ is independent of $x \in X$.

In Chapter 1, we begin by definition for the parity of a path of operators $\alpha : [a,b] \rightarrow \Phi_0(X,Y)$ having invertible endpoints, i.e. $\alpha(a), \alpha(b) \in \text{GL}(X,Y)$; it is denoted by $\sigma(\alpha, [a,b])$. We then prove some properties of parity, and finally we finish the chapter by equivalent definitions of parity.
In Chapter 2, by use of the notion of parity we define a degree theory for perturbations of isomorphisms with compact operators. We will use Fitzpatrick-Pejachowicz degree theory in our work and will prove homotopy invariance of our degree.

And finally, Chapter 3 is about some applications of our degree. First we apply our degree theory to prove the existence of solution for elliptic equation $-\Delta u = g(x, u, \nabla u)$, although it has been done using Leray-Schauder degree. We do this to show how our method works in more simple problem, and then we apply our method to prove the existence of solution for the equation $-\Delta_p u = g(x, u, \nabla u)$ where is more involved to apply Leray-Schauder degree theory.
Chapter 1

Parity

Recall that when \( X \) is a separable Hilbert space and \( \text{GL}(X) \) is the space of all invertible linear operators on \( X \) Kuiper’s Theorem \cite{10} asserts that \( \text{GL}(X) \) is contractible. \cite{6} Therefore we can not split the space \( \text{GL}(X) \) into two connected components. But if we let \( \mathcal{D} \) the set of all paths \( \alpha : [a, b] \rightarrow \Phi_0(X, Y) \) with invertible endpoints, then we can split the set into two connected components by using the concept of parity.

In first section we give the definition of Parity for a path of Linear Fredholm operators. Section 2 is devoted to some properties of Parity. And finally in section 3 we give another way of defining Parity by use of Leray-Schauder degree and also Spectral Flow.

1.1 The parity as an intersection index

Let \( X \) and \( Y \) be real Banach spaces, \( \Phi_0(X, Y) \) be the set of operators in \( \mathcal{L}(X, Y) \) which are Fredholm of index 0, and \( \text{GL}(X, Y) \) be the set of invertible operators. Then, \( S(X, Y) = \Phi_0(X, Y) \setminus \text{GL}(X, Y) \) is called the set of singular Fredholm operators. If \( X = Y \) the space is denoted by \( S(X) \).

Furthermore, let \( n \in \mathbb{N} \) and \( S_n(X, Y) = \{ T \in \Phi_0(X, Y) | \dim \ker T = n \} \), \( \mathcal{D}_{\text{reg}} \) is the set of all paths \( \alpha : [a, b] \rightarrow \Phi_0(X, Y) \) with invertible endpoints. \( \mathcal{D}_{\text{reg}} \) is the set of all generic paths, where generic paths are the paths in \( \mathcal{D} \) which are \( C^1 \) paths \( \alpha : [a, b] \rightarrow \Phi_0(X, Y) \) such that \( \alpha \) intersects \( S(X, Y) \) at \( \lambda \) which \( \alpha(\lambda) \in S_1(X, Y) \) and \( \alpha \) is transverse to \( S_1(X, Y) \). A path \( \alpha \) is transverse to \( S(X, Y) \) at \( \lambda \) when \( \alpha'(\lambda) \neq 0 \).

Remark 1.1.1. Note that intersection points of generic paths with \( S(X, Y) \)
are finite. Because if $\lambda_i$ be an intersection point of a generic path $\alpha$ with $S(X,Y)$, i.e. $\alpha(\lambda_i) \in S_1(X,Y)$. Definition of generic path yields that $\alpha'(\lambda_i) \neq 0$, hence there is a neighborhood of $\lambda_i, (\lambda_i - \varepsilon, \lambda_i + \varepsilon)$, such that $\alpha(\lambda) \not\in S_1(X,Y)$, for every $\lambda \in (\lambda_i - \varepsilon, \lambda_i + \varepsilon)$. Now, let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence of intersection points of $\alpha$ with $S(X,Y)$ such that $\lambda_n \to \lambda_0$, then $\alpha(\lambda_n) \in S_1(X,Y)$ for every $n \in \mathbb{N}$ and thus $\lim_{n \to \infty} \alpha(\lambda_n) = \lim_{n \to \infty} \alpha(\lambda_0)$. Hence $\alpha(\lambda_0) \not\in S_1(X,Y)$, which contradicts the existence of a neighborhood of each intersection point such that $\alpha(\lambda) \not\in S_1(X,Y)$ for every $\lambda$ in that neighborhood. Therefore there is no convergent sequence of intersection points. Thus $[a,b] \subseteq \bigcup_{i=1}^{k} (\lambda_i - \varepsilon, \lambda_i + \varepsilon)$, and since $[a,b]$ is compact, there is a $k \in \mathbb{N}$ such that $[a,b] \subseteq \bigcup_{i=1}^{k} (\lambda_i - \varepsilon, \lambda_i + \varepsilon)$. Thus intersection points $\lambda$ are finite.

Recall that given topological pairs $(A,A')$ and $(B,B')$ (i.e. $A' \subseteq A$ and $B' \subseteq B$),

$$\varphi : (A,A') \longrightarrow (B,B')$$

denotes a continuous map $\varphi : A \longrightarrow B$ with $\varphi(A') \subseteq B'$.

### 1.1.1 Parity for generic paths and some properties

First we will give the definition of Parity for generic path and later on we will extend the definition to general paths.

**Definition 1.1.2** (Parity of generic paths). Let $[a,b] = I \subseteq \mathbb{R}$ and consider the path $\alpha : (I, \partial I) \longrightarrow (\Phi_0(X,Y), \text{GL}(X,Y))$, we define the parity of the path $\alpha$, denoted by $\sigma(\alpha, I)$ by

$$\sigma(\alpha, I) := (-1)^k$$

where $k$ is the number of intersection points $\lambda_i \in (a,b)$, $1 \leq i \leq k$ of $\alpha$ with $S(X,Y)$, at which $\alpha(\lambda_i) \in S_1(X,Y)$ and $\alpha$ is transverse to $S_1(X,Y)$.

Before proving some properties of parity for generic paths let us mention a general criteria for paths of Fredholm operators of index 0. By Theorem 0.1.4 we know that if $T \in \mathcal{L}(X,Y)$, then $T \in \Phi_0(X,Y)$ if and only if there exists $S \in \text{GL}(Y,X)$ with

$$ST \in \mathcal{L}_C(X). \quad (1.1.1)$$

In the theory of integral equations, an operator $S \in \text{GL}(Y,X)$ which satisfies (1.1.1) is called an equivalent regularizer of $T$. We will define a parametrix to be a map $S \in \text{GL}(Y,X)$ which satisfies (1.1.1).
Definition 1.1.3. Let $\Lambda$ be a topological space and let 
$$\alpha : \Lambda \rightarrow \Phi_0(X,Y)$$ 
be continuous. A continuous map 
$$\eta : \Lambda \rightarrow \text{GL}(Y,X)$$ 
is called a parametrix for $\alpha$ if 
$$\eta(\lambda) \circ \alpha(\lambda) \in \mathcal{L}_C(X), \text{ for } \lambda \in \Lambda.$$ 
The question of the existence of a parametrix is central to our present purpose. The answer depends on the topology of $\Lambda$. The following result is well-known (cf. [4]).

Theorem 1.1.4. Let $\Lambda$ be compact and contractible. Then each continuous path 
$$\alpha : \Lambda \rightarrow \Phi_0(X,Y)$$ 
has a parametrix.

Since $[a,b] = I \subseteq \mathbb{R}$ is compact and contractible by the above theorem every $\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y))$ has a parametrix like $\eta : I \rightarrow \text{GL}(Y,X)$. Now we will prove the following properties of parity for generic paths.

Lemma 1.1.5. Let $[a,b] = I \subseteq \mathbb{R}$ and $\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y))$ be a generic path. Moreover let $\eta : I \rightarrow \text{GL}(Y,X)$ be a parametrix of $\alpha$. If the endpoints of the paths $\eta \circ \alpha : (I, \partial I) \rightarrow (\mathcal{L}_C(X), \text{GL}_C(X))$ lay in the same connected component of $\text{GL}_C(X)$, then $\sigma(\alpha, I) = 1$.

Proof. Let $\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y))$ be a generic path and $\eta : I \rightarrow \text{GL}(Y,X)$ be a parametrix of $\alpha$. If $\alpha$ intersects $S_1(X,Y)$ transversally at $\lambda_* \in [a,b]$, then $(\eta \circ \alpha)(\lambda_*) \in S(X)$. Moreover $(\eta \circ \alpha)(\lambda_*) \in S_1(X)$, because since $\eta$ is invertible, $\ker(\eta \circ \alpha) = \ker \alpha$ and hence $\dim \ker(\eta \circ \alpha) = 1$. Furthermore, if $\eta \circ \alpha$ intersects $S_1(X)$ transversally at $\lambda_* \in [a,b]$, by equivalence of [1.1.10] and transversality there exists some $c > 0$ and $\delta > 0$ with 
$$\|(\eta \circ \alpha)(\lambda)(x)\| \geq c|\lambda - \lambda_*||x|| \text{ for } |\lambda - \lambda_*| < \delta \text{ and } x \in X.$$
Corollary 1.1.6. Let $|\alpha(I)| = |\alpha(a)| = |\alpha(b)|$. Then $\sigma(\alpha, I) = 1$.

Proposition 1.1.7 (concatenation of generic paths). Let $[a, b] = I \subseteq \mathbb{R}$ and $\alpha : (I, \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y))$ be a generic path, where $a < c < b$. Then

$$\sigma(\alpha, [a, b]) = \sigma(\alpha, [a, c]) \cdot \sigma(\alpha, [c, b])$$
Proof. Since $\alpha(c) \in GL_{C}(X,Y)$, $\alpha : [a, c] \longrightarrow \Phi_{0}(X,Y)$ and $\alpha : [c, b] \longrightarrow \Phi_{0}(X,Y)$ are generic. Assume $\alpha([a, c]) \cap S_{1}(X,Y) = \{\lambda_{1}, \ldots, \lambda_{k}\}$ and $\alpha([a, c]) \cap S_{1}(X,Y) = \{\lambda_{k+1}, \ldots, \lambda_{k+j}\}$. Then $\alpha(I) \cap S_{1}(X,Y) = \{\lambda_{1}, \ldots, \lambda_{k}, \lambda_{k+1}, \ldots, \lambda_{k+j}\}$. Thus by definition of parity we have

$$\sigma(\alpha, I) = (-1)^{k+j} = (-1)^{k}(-1)^{j} = \sigma(\alpha, [a, c])\sigma(\alpha, [c, b])$$

\[\Box\]

**Proposition 1.1.8.** Let $[a, b] = I \subseteq \mathbb{R}$, $\alpha$ and $\alpha'$ two generic paths on $I$ with $\alpha(a) = \alpha'(a)$ and $\alpha(b) = \alpha'(b)$. Then

$$\sigma(\alpha, I) = \sigma(\alpha', I)$$

**Proof.** Suppose $\alpha(I) \cap S(X,Y) = \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\} \subseteq S_{1}(X,Y)$ and $\alpha'(I) \cap S(X,Y) = \{\lambda'_{1}, \lambda'_{2}, \ldots, \lambda'_{j}\} \subseteq S_{1}(X,Y)$. Now let

$$\beta([a, b]) = \begin{cases} \alpha(2t - a) & t \in [a, \frac{a+b}{2}] \\ \alpha'(-2t + a + 2b) & t \in [\frac{a+b}{2}, b] \end{cases}$$

Then $\beta$ is $C^{1}$, since $\alpha(b) = \alpha'(b)$ and $\beta(I) \cap S(X,Y) = \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}, \lambda'_{1}, \lambda'_{2}, \ldots, \lambda'_{j}\} \subseteq S_{1}(X,Y)$ and $\beta$ is transverse to $S_{1}(X,Y)$ at each $\lambda_{i}$ and $\lambda'_{j}$. But,

$$\beta(a) = \alpha(a) = \alpha'(a) = \beta(b).$$

Then by Corollary 1.1.6 $\sigma(\beta, I) = 1$. Furthermore, $\beta$ is a concatenation path of $\alpha$ and $\alpha'$. Hence by Proposition 1.1.7 $\sigma(\beta, I) = \sigma(\alpha, I)\sigma(\alpha', I)$ which implies that $\sigma(\alpha, I)\sigma(\alpha', I) = 1$, and thus $\sigma(\alpha, I) = \sigma(\alpha', I)$.

\[\Box\]

### 1.1.2 Existence of a generic path in $\varepsilon$-neighborhood\(^{3}\) of any path

We need another step to be able to define parity for general paths. To do so, we will prove that for every general path like $\alpha$ there exists a generic path $\hat{\alpha}$ which agree with $\alpha$ on the endpoints:

\(^{3}\)The topology that we are using here is the topology of uniform convergence. If we consider $\mathcal{D}$ the set of all paths $\alpha : [a, b] \longrightarrow \Phi_{0}(X,Y)$ with invertible endpoints. The uniform convergent topology on this set is the open balls defined by supremum norm. In other words if we pick an element $\alpha \in \mathcal{D}$ then the open ball of radius $\varepsilon$ centered at $\alpha$ or $\varepsilon$-neighborhood of $\alpha$ is $B_{\varepsilon}(\alpha) = \{\beta \in \mathcal{D} : \|\alpha - \beta\|_{D} < \varepsilon\}$, where $\|\alpha\|_{D} = \sup_{a \leq \lambda \leq b}\|\alpha(\lambda)\|_{\Phi_{0}(X,Y)}$ and $\|\alpha(\lambda)\|_{\Phi_{0}(X,Y)} = \inf_{\|x\|_{X} = 1}\|\alpha(\lambda)x\|_{Y}$. 

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Proposition 1.1.9. Let $n \in \mathbb{N}$ then

$$S_n(X,Y) = \{ T \in \Phi_0(X,Y) \mid \dim \ker T = n \}$$

is a submanifold of $\Phi_0(X,Y)$ of codimension $n^2$. Moreover, a $C^1$ path $\alpha : [a,b] \to \Phi_0(X,Y)$, with $\alpha(\lambda_0) \in S_1(X,Y)$, is transverse to $S_1(X,Y)$ at $\lambda_0$ if and only if

$$\alpha'(\lambda_0)(\ker(\alpha(\lambda_0))) \oplus \alpha(\lambda_0)(X) = Y.$$  \hfill (1.1.2)

Proof. Let $n \in \mathbb{N}$ and let $T \in S_n(X,Y)$. Choose $P \in \mathcal{L}(X,X)$ to be a projection onto $\ker T$ and $Q \in \mathcal{L}(Y,Y)$ to be a projection onto $\text{Range} T$. Then each $L \in \mathcal{L}(X,Y)$ may be represented as a 2 by 2 matrix of operators

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the decompositions

$$X = (I - P)(X) \oplus P(X), \quad Y = Q(Y) \oplus (I - Q)(Y).$$

Since $T_{11} \in \text{GL}((I - P)(X), Q(Y))$ we may select $\varepsilon > 0$ so that $L_{11} \in \text{GL}((I - P)(X), Q(Y))$ if $L \in \mathcal{L}(X,Y)$ and $\|L - T\| < \varepsilon$. In particular, $L \in \Phi_0(X,Y)$ if $\|L - T\| < \varepsilon$.

Let $L \in \mathcal{L}(X,Y)$ with $\|L - T\| < \varepsilon$. Let $x \in X$. Then

$$L(x) = 0$$ \hfill (1.1.3)

if and only if

$$\begin{cases} L_{11}(I - P)(x) + L_{12}P(x) = 0 \\ L_{21}(I - P)(x) + L_{22}P(x) = 0 \end{cases}$$ \hfill (1.1.4)

Where the first equation of (1.1.4) implies

$$(I - P)(x) = -(L_{11})^{-1}L_{12}P(x).$$ \hfill (1.1.5)

Since $L_{11} \in \text{GL}((I - P)(X), Q(Y))$. Then replacing (1.1.5) in the second equation of (1.1.4) we get

$$-L_{21}(L_{11})^{-1}L_{12}P(x) + L_{22}P(x) = 0$$

Therefore $L(x) = 0$ if and only if

$$(L_{21}(L_{11})^{-1}L_{12} - L_{22})P(x) = 0.$$ \hfill (1.1.6)
Now define $\psi : N_\varepsilon(T) \to \mathcal{L}(P(X), (I - Q)(Y))$ by
\[
\psi(L) = L_{21}(L_{11})^{-1}L_{12} - L_{22}
\] for $L \in \mathcal{L}(X, Y)$ with $\|L - T\| < \varepsilon$.

Now, if $\|L - T\| < \varepsilon$ and $L(x) = 0$, it follows that $x \neq 0$ if and only if $P(x) \neq 0$. From this and the equivalence of (1.1.3) and (1.1.6), it follows that
\[
\text{if } \|L - T\| < \varepsilon, \text{ then } L \in S_n(X, Y) \text{ if and only if } \psi(L) = 0 \quad (1.1.7)
\]

It is also clear that $\psi : N_\varepsilon(T) \subseteq \Phi_0(X, Y) \to \mathcal{L}(P(X), (I - Q)(Y))$ is $C^1$.

Let us compute $D\psi(T)$. Indeed, let $L \in \mathcal{L}(X, Y)$. But note that the domain of $T_{12}$ and $T_{22}$ is $P(X)$ which is ker $T$ and hence is mapped to 0. Furthermore, $T_{21}$ maps $(I - P)(X)$ to $(I - Q)(Y)$ which the latter is not in Range $T$ and hence $T_{21}$ has to map everything in $(I - P)(X)$ to 0. Therefore $T_{12} = T_{22} = T_{21} = 0$, it follows that
\[
D\psi(T)(L) = \lim_{t \to 0} \frac{\psi(T + tL) - \psi(T)}{t} = -(I - Q)L\big|_{P(X)},
\]
and hence $D\psi(T)$ is surjective. Since $\mathcal{L}(P(X), (I - Q)(Y))$ is an Euclidean space of dimension $n^2$, from (1.1.7) and the surjectivity of $D\psi(T)$, $S_n(X, Y)$ is a smooth submanifold of $\Phi_0(X, Y)$ of codimension $n^2$.

Now suppose that $n = 1$ and $\alpha : [a, b] \to \Phi_0(X, Y)$ is a $C^1$ curve with $\alpha(\lambda_0) = T$. Then $\alpha : [a, b] \to \Phi_0(X, Y)$ is transverse to $S_1(X, Y)$ at $\lambda_0$ if and only if
\[
D\psi(T)(\alpha'(\lambda_0)) \neq 0;
\]
i.e.
\[
(I - Q)\alpha'(\lambda_0)\big|_{\ker(\alpha(\lambda_0))} \neq 0. \quad (1.1.8)
\]
Since $\alpha(\lambda_0) \in S_1(X, Y)$, $\ker(\alpha(\lambda_0))$ has dimension 1. Hence (1.1.8) means that
\[
\alpha'(\lambda_0)(\ker(\alpha(\lambda_0))) \cap \text{Range}(\alpha(\lambda_0)) = 0. \quad (1.1.9)
\]
Since Range$\alpha(\lambda_0)$ has codimension 1, (1.1.9) holds if and only if
\[
\alpha'(\lambda_0)(\ker(\alpha(\lambda_0))) \oplus \text{Range}(\alpha(\lambda_0)) = Y.
\]
Remark 1.1.10. In [9] it was observed that \( \text{(1.1.2)} \) holds if and only if there exists some \( c > 0 \) and \( \delta > 0 \) with

\[
\|\alpha(\lambda)(x)\| \geq c|\lambda - \lambda_0\|\|x\| \quad \text{for} \quad |\lambda - \lambda_0| < \delta \quad \text{and} \quad x \in X. 
\]  

The foregoing discussion, when \( X = Y = V \) is a \( k \)-dimensional real Euclidean space, leads to the decomposition

\[
\mathcal{L}(V) \setminus \text{GL}(V) = S(V) = \bigcup_{j=1}^{k} S_j(V)
\]

with \( S_j(V) \) a submanifold of \( \mathcal{L}(V) \) of codimension \( j^2 \), if \( 1 \leq j \leq k \).

Lemma 1.1.11. Let \( V \) be a finite-dimensional real vector space. Given \( \alpha : (I, \partial I) \to (\mathcal{L}(V), \text{GL}(V)) \) and \( \varepsilon > 0 \), there is a \( C^1 \) curve \( \hat{\alpha} : I \to \mathcal{L}(V) \) which agrees with \( \alpha \) on \( \partial I \), is a uniform \( \varepsilon \)-approximation of \( \alpha \) and, moreover, \( \hat{\alpha}(I) \cap S(V) = \{\lambda_1, ..., \lambda_j\} \subseteq S_1(V) \) and \( \hat{\alpha} \) is transverse to \( S_1(V) \) at each \( \lambda_i \).

\[\text{Proof.}\] It follows from the Thom Transversality Theorem that the set of all \( C^1 \) curves \( \eta : I \to \mathcal{L}(V) \) which are transverse to all of the \( S_i(V) \)'s is residual in \( C^1(I, \mathcal{L}(V)) \). Accordingly, if we choose \( \delta > 0 \) with \( \delta \leq \max\{\varepsilon, \|\alpha(b) - \alpha(a)\|\} \), such that \( B(\alpha(a), 2\delta) \subseteq \text{GL}(V) \) and \( B(\alpha(b), 2\delta) \subseteq \text{GL}(V) \), we may choose \( \hat{\alpha} : I \to \mathcal{L}(V) \) to be \( C^1 \), transverse to each \( S_j(V) \) and such that

\[
\sup_{\lambda \in I} \|\alpha(\lambda) - \hat{\alpha}(\lambda)\| < \delta.
\]

Let \( \mathcal{D}_1 = B(\alpha(a), \delta), \mathcal{D}_2 = \mathcal{L}(V) \setminus \{\bar{B}(\alpha(a), \delta/2) \cup \bar{B}(\alpha(b), \delta/2)\} \), and \( \mathcal{D}_3 = \bar{B}(\alpha(b), \delta) \). Then \( \{\mathcal{D}_i\}_{i=1}^{3} \) is an open cover of \( \mathcal{L}(V) \), subordinate to which we may choose a \( C^1 \) partition of unity \( \{\gamma_i\}_{i=1}^{3} \). Define

\[
\hat{\alpha}(\lambda) = \gamma_1(\lambda)\alpha(a) + \gamma_2(\lambda)\hat{\alpha}(\lambda) + \gamma_3(\lambda)\alpha(b) \quad \text{for all} \quad \lambda \in I.
\]

Then \( \hat{\alpha} : I \to \mathcal{L}(V) \) is \( C^1 \) and \( \sup_{\lambda \in I} \|\alpha(\lambda) - \hat{\alpha}(\lambda)\| < \varepsilon \). Moreover, one easily sees that \( \hat{\alpha}(\lambda) = \hat{\alpha}(\lambda) \) unless \( \hat{\alpha}(\lambda) \in \text{GL}(V) \), so that \( \hat{\alpha} \) is transverse to each \( S_i(V) \). But each \( S_i(V) \) has codimension greater than \( 1 \) if \( i > 1 \), and so \( \hat{\alpha}(I) \cap S(V) \subseteq S_1(V) \). \( \square \)

Theorem 1.1.12. Let \( X \) and \( Y \) be real Banach spaces, given \( \alpha : (I, \partial I) \to (\Phi_0(X,Y), \text{GL}(X,Y)) \), and \( \varepsilon > 0 \), there is a \( C^1 \) path \( \hat{\alpha} : I \to \Phi_0(X,Y) \) which agrees with \( \alpha \) on \( \partial I \), is a uniform \( \varepsilon \)-approximation of \( \alpha \), and, moreover, \( \hat{\alpha}(I) \cap S(X,Y) = \{\lambda_1, \lambda_2, ..., \lambda_k\} \subseteq S_1(X,Y) \) and \( \hat{\alpha} \) is transverse to \( S_1(X,Y) \) at each \( \lambda_i \).
Proof. By uniformly approximating $\alpha : [a, b] \rightarrow \Phi_0(X, Y)$ by a piecewise linear path and then invoke the Stone-Weierstrass theorem, it is clear that we may, without loss of generality, suppose that $\alpha : [a, b] \rightarrow \Phi_0(X, Y)$ is analytic. If $\alpha(a, b) \cap S(X, Y) = \emptyset$ the proof is complete. Otherwise, choose $\lambda_* \in [a, b]$ with $\alpha(\lambda_*) \in S(X, Y)$.

Since $\alpha$ is analytic we may choose $\varepsilon > 0$ such that $\alpha(\lambda) \in \text{GL}(X, Y)$ if $0 < |\lambda - \lambda_*| < \varepsilon$. Choose $P \in L(X)$ and $Q \in L(Y)$ projections onto ker $\alpha(\lambda_*)$ and Range $\alpha(\lambda_*)$, respectively. Then choose

$$A \in \text{GL}(P(X), (I - Q)(Y)).$$

Observe that $\alpha(\lambda_*) + AP \in \text{GL}(X, Y)$, since $(\alpha(\lambda_*) + AP)(X) = \alpha(\lambda_*)X + APX = \alpha(\lambda_*)X + (I - Q)Y = QY + Y - QY = Y$; hence we may choose $0 < \varepsilon_* < \varepsilon$ with $\alpha(\lambda) + AP \in \text{GL}(X, Y)$ if $|\lambda - \lambda_*| \leq \varepsilon_*$. Observe that if $|\lambda - \lambda_*| \leq \varepsilon_*$, then

$$\alpha(\lambda) = \begin{pmatrix} I & 0 \\ \beta(\lambda) & \eta(\lambda) \end{pmatrix},$$

and we may represent $\psi(\lambda)$ with respect to the decomposition $Y = Q(Y) \oplus (I - Q)(Y)$ as

$$\begin{pmatrix} I & 0 \\ \beta(\lambda) & \eta(\lambda) \end{pmatrix}.$$

Choose a $C^1$ curve $\gamma : [\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*] \rightarrow L((I - Q)(Y))$ as in Lemma 1.1.11 such that

$$\sup \|\gamma(\lambda) - \eta(\lambda)\| < \varepsilon' = \varepsilon[\sup\{||(\alpha(\lambda) + AP)^{-1}||\lambda \in [\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*]\}]^{-1}.$$  

Then $\gamma(\lambda)$ is invertible except at a finite number of points in $(\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*)$, where it crosses $S(\Phi_0((I - Q)(Y)))$ transversally. Set

$$\hat{\alpha}(\lambda) = \hat{\psi}(\lambda)[\alpha(\lambda) + AP],$$

where $\hat{\psi}(\lambda)$ is represented by

$$\begin{pmatrix} I & 0 \\ \beta(\lambda) & \gamma(\lambda) \end{pmatrix}.$$ 

Then $\hat{\alpha} : [\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*] \rightarrow \Phi_0(X, Y)$ is $C^1$, and from the choice of $\varepsilon'$ it follows that it is a uniform $\varepsilon$-approximation of $\alpha$. Moreover, one sees that if $x \neq 0$, then

$$\hat{\alpha}(\lambda)(x) = 0$$

if and only if $\gamma(\lambda)(I - Q)(\alpha(\lambda) + AP)(x) = 0$ and $x \in P(X)$.
It follows that $\hat{\alpha}(\lambda) \in S(X,Y)$ if and only if $\gamma(\lambda) \in S((I - Q)(Y))$ and at each such $\lambda$, $\hat{\alpha}(\lambda) \in S_1(X,Y)$. It remains to check transversality. However, in view of the equivalence of (1.1.10) and transversality, it is clear that $\hat{\alpha}$ inherits transversality from $\gamma$.

We have found that $\hat{\alpha} : [\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*] \to \Phi_0(X,Y)$, which is an $\varepsilon$-approximation of $\alpha : [\lambda_* - \varepsilon_*, \lambda_* + \varepsilon_*] \to \Phi_0(X,Y)$, is $C^1$ and transverse to $S(X,Y)$. Moreover, $\hat{\alpha}$ coincides with $\alpha$ at $\lambda_* \pm \varepsilon_*$, since $\gamma$ coincides with $\eta$ at $\lambda_* \pm \varepsilon_*$ by Lemma 1.1.11.

Since the singular points of $\alpha : [a,b] \to \Phi_0(X,Y)$ are isolated, we may patch the above constructions at each singular point to find $\hat{\alpha} : [a,b] \to \Phi_0(X,Y)$, which is a uniform $\varepsilon$-approximation of $\alpha : [a,b] \to \Phi_0(X,Y)$ and is transverse to $S_1(X,Y)$.

1.1.3 Parity for general path of linear Fredholm operators

Now by Theorem 1.1.12 we can give this definition:

**Definition 1.1.13.** Given $[a,b] = I \subseteq \mathbb{R}$ and $\alpha : (I, \partial I) \to (\Phi_0(X,Y), \text{GL}(X,Y))$.

Theorem 1.1.12 implies that for $\varepsilon > 0$ there is a $C^1$ path $\hat{\alpha} : I \to \Phi_0(X,Y)$ which agrees with $\alpha$ on $\partial I$, is a uniform $\varepsilon$-approximation of $\alpha$, and, moreover, $\hat{\alpha}(I) \cap S(X,Y) = \{\lambda_1, \lambda_2, ..., \lambda_k\} \subseteq S_1(X,Y)$ and $\hat{\alpha}$ is transverse to $S_1(X,Y)$ at each $\lambda_i$. Then we define the parity of $\alpha$ on $I$, denoted by $\sigma(\alpha, I)$ as follows

$$\sigma(\alpha, I) := \sigma(\hat{\alpha}, I) = (-1)^k.$$

To show that the above definition does not depend on the choice of generic path $\hat{\alpha}$ we need the following proposition.

**Proposition 1.1.14.** Let $[a,b] = I \subseteq \mathbb{R}$ and $\alpha : I \to \Phi_0(X,Y)$ be a path with invertible endpoints, $\hat{\alpha} : I \to \Phi_0(X,Y)$ and $\tilde{\alpha} : I \to \Phi_0(X,Y)$ two generic paths and uniform $\varepsilon$-approximations of $\alpha$ which agree with $\alpha$ on $\partial I$. Then $\sigma(\hat{\alpha}, I) = \sigma(\tilde{\alpha}, I)$.

**Proof.** Assume $\hat{\alpha} : I \to \Phi_0(X,Y)$ and $\tilde{\alpha} : I \to \Phi_0(X,Y)$ two generic paths and uniform $\varepsilon$-approximations of $\alpha$ which agree with $\alpha$ on $\partial I$. Then $\hat{\alpha}'(b) = \tilde{\alpha}'(b)$, hence by Proposition 1.1.8 $\sigma(\hat{\alpha}, I) = \sigma(\tilde{\alpha}, I)$.

Thus the above definition is well defined.
1.2 Properties of The Parity

We will prove the following properties of parity for further references:
(I) Homotopy invariance,
(II) Multiplicativity under Concatenation of paths,
(III) Multiplicativity under composition,
(IV) Multiplicativity under direct sum.

To prove Homotopy invariance of parity we need following Theorem:

Theorem 1.2.1. Let \([a, b] = I \subseteq \mathbb{R}\) and consider

\[\alpha : (I, \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y)).\]

The following are equivalent:
(i) \(\sigma(\alpha, I) = 1\).
(ii) there is a homotopy \(\alpha : ([0, 1] \times I, \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y))\) with

\[\alpha_0 = \alpha \text{ and } \alpha_1(I) \subseteq GL(X, Y).\]

Proof. First suppose that (i) holds, By Theorem 1.1.12 there is a generic path \(\hat{\alpha} : I \rightarrow \Phi_0(X, Y)\) which agree with \(\alpha\) on \(\partial I\), and \(\hat{\alpha}(I) \cap S(X, Y) = \{\lambda_1, \lambda_2, ..., \lambda_k\} \subseteq S_1(X, Y)\) and \(a < \lambda_1 < \lambda_2 < ... < \lambda_k < b\). Since \(\sigma(\alpha, I) = 1\), \(k\) must be an even number. Since \(\alpha\) is transversal to \(S_1(X, Y)\) at each \(\lambda_i\), there is \(\varepsilon > 0\) such that \(\alpha(\lambda_i) \pm \varepsilon \in GL(X, Y)\) for \(i = 1, 2, ..., j\). Then by Kuiper’s Theorem we can find paths \(\beta_i : [\lambda_{2i-1} - \varepsilon, \lambda_{2i} + \varepsilon] \rightarrow GL(X, Y)\), \(i = 1, 2, ..., j\), where \(j = \frac{k}{2}\), such that \(\beta_i(\lambda_{2i-1} - \varepsilon) = \alpha(\lambda_{2i-1} - \varepsilon)\) and \(\beta_i(\lambda_{2i} + \varepsilon) = \alpha(\lambda_{2i} + \varepsilon)\) for \(i = 1, 2, ..., j\). Define the homotopy \(\alpha : ([0, 1] \times I, [0, 1] \times \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y))\), by

\[
\alpha(t, \lambda) = \begin{cases} 
\alpha(\lambda) & (t, \lambda) \in [0, 1] \times [a, \lambda_1 - \varepsilon] \\
(1 - t)\alpha(\lambda) + t\beta_1(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_1 - \varepsilon, \lambda_2 + \varepsilon] \\
\alpha(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_2 + \varepsilon, \lambda_3 - \varepsilon] \\
(1 - t)\alpha(\lambda) + t\beta_2(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_3 - \varepsilon, \lambda_4 + \varepsilon] \\
\vdots & \vdots \\
\alpha(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_{k-2} + \varepsilon, \lambda_{k-1} - \varepsilon] \\
(1 - t)\alpha(\lambda) + t\beta_j(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_{k-1} - \varepsilon, \lambda_k + \varepsilon] \\
\alpha(\lambda) & (t, \lambda) \in [0, 1] \times [\lambda_k + \varepsilon, b] 
\end{cases}
\]

Then \(\alpha_0 = \alpha\), and \(\alpha_1 \subseteq GL(X, Y)\).
Now suppose that (ii) holds. Let \( \eta : [0,1] \times I \to GL(Y,X) \) be the parametrix of the \( \alpha_t \). Since \( \alpha_t(a) \in GL(X,Y) \) for all \( t \in [0,1] \), \( \eta_t(a) \circ \alpha_t(a) \in GL_C(X) \) and since \( \eta_t(a) \circ \alpha_t(a) \) is a path in \( GL_C(X) \) so \( \eta_0(a) \circ \alpha_0(a) \) and \( \eta_1(a) \circ \alpha_1(a) \) are in the same connected component of \( GL_C(X) \). (I)

Similarly, \( \eta_0(b) \circ \alpha_0(b) \) and \( \eta_1(b) \circ \alpha_1(b) \) are in the same connected component of \( GL_C(X) \). (II)

Moreover, \( \alpha_1(I) \subseteq GL(X,Y) \), hence \( \eta_1(I) \circ \alpha_1(I) \subseteq GL_C(X) \) and thus \( \eta_1(a) \circ \alpha_1(a) \) and \( \eta_1(b) \circ \alpha_1(b) \) are in the same connected component of \( GL_C(X) \). (III)

Therefore by (I),(II), and (III) \( \eta_0(a) \circ \alpha_0(a) \) and \( \eta_0(b) \circ \alpha_0(b) \) are in the same connected component of \( GL_C(X) \). Thus by Lemma 1.1.5 \( \sigma(\alpha, I) = 1 \). \( \square \)

**Corollary 1.2.2** (Homotopy invariance of Parity). Let \( [a,b] = I \subseteq \mathbb{R} \) and consider the homotopy

\[
\alpha : ([0,1] \times I, [0,1] \times \partial I) \to (\Phi_0(X,Y), GL(X,Y)).
\]

Then

\[
\sigma(\alpha_0, I) = \sigma(\alpha_1, I).
\]

**Proof.** First assume \( \sigma(\alpha_0, I) = 1 \) then by preceding Theorem there is a homotopy

\[
\tilde{\alpha} : ([0,1] \times I, [0,1] \times \partial I) \to (\Phi_0(X,Y), GL(X,Y)) \text{ with }
\]

\[
\tilde{\alpha}_0 = \alpha_0 \text{ and } \tilde{\alpha}_1(I) \subseteq GL(X,Y).
\]

Then consider the homotopy

\[
G(t, \lambda) = \begin{cases} 
\alpha(1 - 2t, \lambda) & (t, \lambda) \in [0, \frac{1}{2}] \times I \\
\tilde{\alpha}(2t - 1, \lambda) & (t, \lambda) \in [\frac{1}{2}, 1] \times I 
\end{cases}
\]

which is continuous, since \( \tilde{\alpha}_0 = \alpha_0 \). And hence \( G \) is in the form

\[
G : ([0,1] \times I, [0,1] \times \partial I) \to (\Phi_0(X,Y), GL(X,Y))
\]

Moreover \( G_0 = \alpha_1 \), and \( G_1(I) = \tilde{\alpha}_1(I) \subseteq GL(X,Y) \), then by using preceding theorem we get \( \sigma(\alpha_1, I) = 1 \).

Now assume \( \sigma(\alpha_0, I) = -1 \) and \( \sigma(\alpha_1, I) = 1 \), by applying the above argument for \( \alpha_1 \) we get \( \sigma(\alpha_0, I) = 1 \) which is a contradiction. So this situation does not happen and \( \sigma(\alpha_0, I) = \sigma(\alpha_1, I) = -1 \). \( \square \)
Theorem 1.2.3 (The parity of concatenation of paths). Let \([a, b] = I \subseteq \mathbb{R}\) and \(\alpha : (I, \{a, b, c\}) \rightarrow (\Phi_0(X, Y), GL(X, Y))\), where \(a < c < b\) be a path of Fredholm operators with index zero. Then

\[
\sigma(\alpha, [a, b]) = \sigma(\alpha, [a, c]) \cdot \sigma(\alpha, [c, b]).
\]

Proof. Since \(\alpha(c) \in GL(X, Y)\), By Theorem 1.1.12 for given \(\varepsilon > 0\) there are generic paths \(\hat{\alpha} : [a, c] \rightarrow \Phi_0(X, Y)\) and \(\hat{\alpha} : [c, b] \rightarrow \Phi_0(X, Y)\) \(\varepsilon\)-approximations of \(\alpha([a, c])\) and \(\alpha([c, b])\) respectively, which agree with \(\alpha\) on \([a, b, c]\). Moreover, \(\hat{\alpha} : [a, b] \rightarrow \Phi_0(X, Y)\) which is a concatenation of above generic paths is \(\varepsilon\)-approximation of \(\alpha([a, b])\). Then by definition of parity and Proposition 1.1.7 we get

\[
\sigma(\alpha, [a, b]) = \sigma(\hat{\alpha}, [a, b]) = \sigma(\hat{\alpha}, [a, c]) \cdot \sigma(\hat{\alpha}, [c, b]) = \sigma(\alpha, [a, c]) \cdot \sigma(\alpha, [c, b]).
\]

\(\square\)

Theorem 1.2.4 (Multiplicativity under composition). Let \([a, b] = I \subseteq \mathbb{R}\). If \(\alpha : (I, \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y))\) and \(\beta : (I, \partial I) \rightarrow (\Phi_0(Y, Z), GL(Y, Z))\) are two paths and if \(\gamma : (I, \partial I) \rightarrow (\Phi_0(X, Z), GL(X, Z))\) is the pointwise composition of \(\alpha\) with \(\beta\), then

\[
\sigma(\gamma, I) = \sigma(\alpha, I) \cdot \sigma(\beta, I).
\]

Proof. By Theorem 1.1.12 for given \(\varepsilon > 0\) there are generic paths \(\hat{\alpha} : I \rightarrow \Phi_0(X, Y)\) and \(\hat{\beta} : I \rightarrow \Phi_0(Y, Z)\) \(\varepsilon\)-approximations of \(\alpha\) and \(\beta\) respectively, which agree with \(\alpha\) and \(\beta\) on \(\partial I\). Let \(\hat{\alpha}(I) \cap S(X, Y) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subseteq S_1(X, Y)\) and \(\hat{\beta}(I) \cap S(Y, Z) = \{\mu_1, \mu_2, \ldots, \mu_j\} \subseteq S_1(Y, Z)\) with \(\lambda_i \neq \mu_l\) for all \(1 \leq i \leq k\) and \(1 \leq l \leq j\). Since \(\hat{\alpha}(\lambda_i) \in S_1(X, Y)\), and \(\hat{\beta}(\mu_i) \in GL(Y, Z)\) for all \(1 \leq i \leq k\), \(\hat{\beta}(\lambda_i) \circ \hat{\alpha}(\lambda_i) \in S_1(X, Z)\). Because since \(\hat{\beta}(\lambda_i)\) is invertible, \(\ker(\hat{\beta}(\lambda_i) \circ \hat{\alpha}(\lambda_i)) = \ker \hat{\alpha}(\lambda_i)\) and hence \(\dim(\ker(\hat{\beta}(\lambda_i) \circ \hat{\alpha}(\lambda_i))) = 1\). Moreover, Since \(\hat{\beta}(\mu_i) \in S_1(Y, Z)\), and \(\hat{\alpha}(\mu_i) \in GL(X, Y)\) for all \(1 \leq i \leq j\), \(\hat{\beta}(\mu_i) \circ \hat{\alpha}(\mu_i) \in S_1(X, Z)\). Because since \(\hat{\alpha}(\mu_i)\) is invertible, \(\ker(\hat{\beta}(\mu_i) \circ \hat{\alpha}(\mu_i)) = \ker \hat{\beta}(\mu_i)\) and hence \(\dim(\ker(\hat{\beta}(\mu_i) \circ \hat{\alpha}(\mu_i))) = 1\). Let \(\hat{\gamma} = \beta \circ \hat{\alpha}\), then \(\hat{\gamma}\) is \(\varepsilon\)-approximation of \(\gamma\) and \(\hat{\gamma}(I) \cap S(X, Z) = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_j\} \subseteq S_1(X, Z)\). Therefore we have

\[
\sigma(\gamma, I) = (-1)^{k+j} = (-1)^k \cdot (-1)^j = \sigma(\alpha, I) \cdot \sigma(\beta, I).
\]

\(\square\)

Theorem 1.2.5 (Multiplicativity under direct sum). Let \([a, b] = I \subseteq \mathbb{R}\), \(X = X_1 \oplus X_2\), and \(Y = Y_1 \oplus Y_2\). If \(\alpha : (I, \partial I) \rightarrow (\Phi_0(X_1, Y_1), GL(X_1, Y_1))\) and \(\beta : (I, \partial I) \rightarrow (\Phi_0(X_2, Y_2), GL(X_2, Y_2))\) are two paths and if \(\gamma : (I, \partial I) \rightarrow (\Phi_0(X, Y), GL(X, Y))\) is the pointwise direct sum of \(\alpha\) with \(\beta\), then

\[
\sigma(\gamma, I) = \sigma(\alpha, I) \cdot \sigma(\beta, I).
\]
Proof. By Theorem 1.1.12 for given $\varepsilon > 0$ there are generic paths $\hat{\alpha} : I \to \Phi_0(X_1,Y_1)$ and $\hat{\beta} : I \to \Phi_0(X_2,Y_2)$ $\varepsilon$-approximations of $\alpha$ and $\beta$ respectively, which agree with $\alpha$ and $\beta$ on $\partial I$. Let $\hat{\alpha}(I) \cap S(X_1,Y_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subseteq S_1(X_1,Y_1)$ and $\hat{\beta}(I) \cap S(X_2,Y_2) = \{\mu_1, \mu_2, \ldots, \mu_j\} \subseteq S_1(X_2,Y_2)$ with $\lambda_i \neq \mu_i$ for all $1 \leq i \leq k$ and $1 \leq l \leq j$. Since $\hat{\alpha}(\lambda_i) \in S_1(X_1,Y_1)$, and $\hat{\beta}(\lambda_i) \in \text{GL}(X_2,Y_2)$ for all $1 \leq i \leq k$, $\hat{\alpha}(\lambda_i) \oplus \hat{\beta}(\lambda_i) \in S_1(X,Z)$. Because $\ker(\hat{\alpha}(\lambda_i) \oplus \hat{\beta}(\lambda_i)) = \ker \hat{\alpha}(\lambda_i) \cup \ker \hat{\beta}(\lambda_i)$ and since $\hat{\beta}(\lambda_i)$ is invertible, $\ker \hat{\beta}(\lambda_i) = \{0\}$, thus $\ker(\hat{\alpha}(\lambda_i) \oplus \hat{\beta}(\lambda_i)) = \ker \hat{\alpha}(\lambda_i)$ and hence $\dim \ker(\hat{\alpha}(\lambda_i) \oplus \hat{\beta}(\lambda_i)) = 1$. Moreover, since $\hat{\beta}(\mu_i) \in S_1(X_2,Y_2)$, and $\hat{\alpha}(\mu_i) \in \text{GL}(X_1,Y_1)$ for all $1 \leq i \leq j$, $\hat{\alpha}(\mu_i) \oplus \hat{\beta}(\mu_i) \in S_1(X,Y)$. Because since $\hat{\alpha}(\mu_i)$ is invertible, $\ker \hat{\alpha}(\mu_i) = \{0\}$, thus $\ker(\hat{\alpha}(\mu_i) \oplus \hat{\beta}(\mu_i)) = \ker \hat{\beta}(\mu_i)$ and hence $\dim \ker(\hat{\alpha}(\lambda_i) \oplus \hat{\beta}(\lambda_i)) = 1$. Let $\hat{\gamma} = \hat{\alpha} \oplus \hat{\beta}$, then $\hat{\gamma}$ is $\varepsilon$-approximation of $\gamma$ and $\hat{\gamma}(I) \cap S(X,Y) = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_j\} \subseteq S_1(X,Y)$. Therefore we have

$$\sigma(\gamma, I) = (-1)^{k+j} = (-1)^k \cdot (-1)^j = \sigma(\alpha, I) \cdot \sigma(\beta, I)$$

$$\square$$

Corollary 1.2.6. Let $[a,b] = I \subseteq \mathbb{R}$ and $\alpha : (I, \partial I) \to (\Phi_0(X,Y), \text{GL}(X,Y))$ a continuous path. Suppose that there are decompositions $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ with $X_k$ (respectively $Y_k$) $k = 1, 2$, closed subspaces of $X$ (respectively $Y$) relative to which $\alpha(\lambda)$ has the block decomposition

$$\alpha(\lambda) = \begin{pmatrix} \alpha_{11}(\lambda) & \alpha_{12}(\lambda) \\ 0 & \alpha_{22}(\lambda) \end{pmatrix} \quad (1.2.1)$$

for every $\lambda \in [a,b]$. Suppose further that $\alpha_{22}(\lambda) \in \text{GL}(X_2,Y_2)$ for every $\lambda \in [a,b]$. Then, $\alpha_{11} : (I, \partial I) \to (\Phi_0(X,Y), \text{GL}(X,Y))$ is continuous, and one has

$$\sigma(\alpha, I) = \sigma(\alpha_{11}, I).$$

Proof. Let $Q_1$ and $Q_2$ denote the embeddings of $X_1$ and $X_2$ in $X$, and $P_1$ and $P_2$ denote the projections of $Y$ to $Y_1$ and $Y_2$. Then $\alpha_{ij}(\lambda) = P_i \alpha(\lambda) Q_j$, $i,j = 1, 2$. Thus $\alpha_{11}(a) = P_1 \alpha(a) Q_1$, and since $\alpha(a) \in \text{GL}(X,Y)$, $\alpha_{11}(a) \in \text{GL}(X_1,Y_1)$. Similarly, $\alpha_{11}(b) \in \text{GL}(X_1,Y_1)$. Now, set

$$\alpha_1(\lambda) = \begin{pmatrix} \alpha_{11}(\lambda) & 0 \\ 0 & I_{X_2} \end{pmatrix}, \quad \alpha_2(\lambda) = \begin{pmatrix} I_{X_2} & \alpha_{12}(\lambda) \\ 0 & \alpha_{22}(\lambda) \end{pmatrix},$$

which $\alpha_2 : I \to \text{GL}(Y_1 \times X_2,Y)$ and $\alpha_1 : I \to \Phi_0(X,Y_1 \times X_2)$. Then we have

$$\alpha_2(\lambda) \alpha_1(\lambda) = \begin{pmatrix} I_{X_2} & \alpha_{12}(\lambda) \\ 0 & \alpha_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \alpha_{11}(\lambda) & 0 \\ 0 & I_{X_2} \end{pmatrix} = \begin{pmatrix} \alpha_{11}(\lambda) & \alpha_{12}(\lambda) \\ 0 & \alpha_{11}(\lambda) \end{pmatrix} = \alpha(\lambda).$$
Theorem 1.2.4 (Multiplicativity under composition) implies that $\sigma(\alpha, I) = \sigma(\alpha_1, I) \cdot \sigma(\alpha_2, I)$. But since $\alpha_2 \subseteq GL(Y_1 \times X_2, Y)$, by Theorem 1.2.1 $\sigma(\alpha, I) = 1$, and hence $\sigma(\alpha, I) = \sigma(\alpha_1, I)$. Moreover, Theorem 1.2.5 implies that $\sigma(\alpha, I) = \sigma(\alpha_{11}, I) \cdot \sigma(\lambda_{X_2}, I)$ which with the same argument $\sigma(\lambda_{X_2}, I) = 1$, and thus $\sigma(\alpha, I) = \sigma(\alpha_{11}, I)$. 

When we say that $\alpha(\lambda)$ has the block decomposition (1.2.1) we mean $\alpha(\lambda)X_1 \subseteq Y_1$ for $\lambda \in [a, b]$. We will generalize the above corollary by replacing the condition $\alpha(\lambda) X_1 \subseteq Y_1$ by a weaker one, where now the space $X_1$ may depend on $\lambda$.

**Theorem 1.2.7.** [Main Preliminary lemma] Let $[a, b] = I \subseteq \mathbb{R}$ and $\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), GL(X,Y))$ a continuous path. Suppose that there is a subspace $Y_1$ of $Y$ with finite dimension such that $\text{Range}\alpha(\lambda) + Y_1 = Y$ for every $\lambda \in [a, b]$ and set

$$E' = \{ (\lambda, x) \in [a, b] ; \alpha(\lambda)x \in Y_1 \} \subset [a, b] \times X.$$

Then:

(i) $E'$ is a trivial finite dimensional subbundle of $[a, b] \times X$ with fibre isomorphic to $Y_1$. In other words, there is a finite dimensional space $X' \approx Y_1$ and a continuous mapping (trivialization) $\psi' : [a, b] \rightarrow \mathcal{L}(X', X)$, such that $\psi'(\lambda) \in GL(X', E'_\lambda)$, $\forall \lambda \in [a, b]$, where

$$E'_\lambda = \{ x \in X ; (\lambda, x) \in E' \} = \alpha(\lambda)^{-1}Y_1.$$

(ii) given any trivialization $\psi' : [a, b] \rightarrow \mathcal{L}(X', X)(X' \approx Y_1)$, one has $\alpha(a)\psi'(a), \alpha(b)\psi'(b) \in GL(X', Y_1)$ and

$$\sigma(\alpha, [a, b]) = \text{sgn}(det(\alpha)\psi'(a)) \cdot \text{sgn}(det(\alpha)\psi'(b)),$$

(Note that $\alpha(\lambda)\psi'(\lambda) \in \mathcal{L}(X', Y_1)$ for $\lambda \in [a, b]$).

A subset $E' \subset [a, b] \times X$ is called a subbundle of $[a, b] \times X$ if the following two conditions hold:

1. $E' = \bigcup_{\lambda \in [a, b]} \{ \lambda \} \times E'_\lambda$, where $E'_\lambda$ is a closed subspace of $X$,

2. for every $\lambda_0 \in [a, b]$, there is a neighborhood $U$ of $\lambda_0$, a Banach space $X'$ and a mapping $\psi' : U \rightarrow \mathcal{L}(X', X)$ such that $\psi'(\lambda) \in GL(X', E'_\lambda)$, $\forall \lambda \in U$ (local triviality).

To be able to prove the above theorem we will give two lemmas. For proof of these lemmas you can see [S].

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Lemma 1.2.8. Suppose that $E'$ is a finite dimensional subbundle of $[a, b] \times X$ (i.e. $\dim E'_\lambda < \infty, \forall \lambda \in [a, b]$). Then, $E'$ can be complemented.

Lemma 1.2.9. Every subbundle $E'$ of $[a, b] \times X$ is trivial.

Proof of Theorem 1.2.7. From Lemma 1.2.8 above, we do not need to prove triviality of $E'$. Initially, we show that $\dim E'_\lambda = \dim Y_1 = n$. Let $\tilde{\alpha}(\lambda) \in \mathcal{L}(X \times Y_1, Y)$ with $\tilde{\alpha}(\lambda)(x, y_1) = \alpha(\lambda)x - y_1$. Since $\alpha(\lambda) \in \Phi_0(X, Y)$ and $Y_1$ is $n$ dimensional, $\tilde{\alpha}(\lambda) \in \Phi_n(X \times Y_1, Y)$. Furthermore, $\tilde{\alpha}(\lambda)$ is onto because $Y = \text{Range} \alpha(\lambda) + Y_1$, which implies that $\text{dim} \ker \tilde{\alpha}(\lambda) = n$. But $(x, y_1) \in \ker \tilde{\alpha}(\lambda)$ if and only if $\alpha(\lambda)x - y_1 = 0$, and thus $(x, y_1) \in \ker \tilde{\alpha}(\lambda)$ if and only if $x \in E'_{\lambda}$. Hence the mapping $\nu : \ker \tilde{\alpha}(\lambda) \rightarrow E'_{\lambda}$, $(x, y_1) \mapsto x$ is a linear bijection. Therefore, $\dim E'_\lambda = \dim Y_1 = n$, and hence fibre is isomorphic to $Y_1$.

Let $\lambda_0 \in [a, b]$ be given and choose a closed subspace $Y_2$ of $Y$ such that $Y = Y_1 \oplus Y_2$. Call $Q_1$ and $Q_2$ projections of $Y$ onto $Y_1$ and $Y_2$, and choose a closed subspace $E''_{\lambda_0}$ of $X$ such that $X = E''_{\lambda_0} \oplus E'_{\lambda_0}$. Now suppose $x'' \in E''_{\lambda_0}$ and $Q_2 \alpha(\lambda_0)x'' = 0$. Then since $Q_2$ is a projection onto $Y_2$, $\alpha(\lambda_0)x'' \in Y_1$. Thus, $x'' \in E'_{\lambda_0}$ and therefore $x'' = 0$. Hence $Q_2 \alpha(\lambda_0)$ is one to one. On the other hand, given $y_2 \in Y_2$, there is $x \in X$, $y_1 \in Y_1$ such that $y_2 = \alpha(\lambda_0)x + y_1$, because $Y = \alpha(\lambda_0)X + Y_1$. Thus, $y_2 = Q_2y_2 = Q_2(\alpha(\lambda_0)x + y_1) = Q_2 \alpha(\lambda_0)x$. Then, since $X = E'_{\lambda_0} \oplus E''_{\lambda_0}$, $x = x' + x''$ where $x' \in E'_{\lambda_0}$ and $x'' \in E''_{\lambda_0}$. Hence $Q_2 \alpha(\lambda_0)x'' = y_2$, because by definition of $E''_{\lambda_0}$, $\alpha(\lambda_0)x' \in Y_1$ and $Q_2$ is a projection onto $Y_2$. Thus $Q_2 \alpha(\lambda_0)$ is onto and therefore $Q_2 \alpha(\lambda_0)|_{E''_{\lambda_0}} \in \text{GL}(E''_{\lambda_0}, Y_2)$.

Moreover from the above and continuity of $\alpha$, there is an open neighborhood $U$ of $\lambda_0$ such that $Q_2 \alpha(\lambda)|_{E''_{\lambda_0}} \in \text{GL}(E''_{\lambda_0}, Y_2)$, $\forall \lambda \in U$. Set

$$P''(\lambda) = (Q_2 \alpha(\lambda)|_{E''_{\lambda_0}})^{-1}Q_2 \alpha(\lambda) \in \mathcal{L}(X), \quad \forall \lambda \in U.$$ 

If $x' \in E'_{\lambda}$, Since $\alpha(\lambda)x' \in Y_1$ and $Q_2$ is a projection onto $Y_2$, $Q_2 \alpha(\lambda)x' = 0$, and hence, $P''(\lambda)x' = 0$. Moreover, if $x'' \in E''_{\lambda_0}$,

$$P''(\lambda)x'' = (Q_2 \alpha(\lambda)|_{E''_{\lambda_0}})^{-1}Q_2 \alpha(\lambda)x'' = x''.$$ 

Thus $P''(\lambda)$ is a continuous projection onto $E''_{\lambda_0}$ with $\ker P''(\lambda) = E'_{\lambda}$. Continuity is due to the fact that all the operators are continuous with respect to topology of $X$. Furthermore, since $\alpha$ is continuous in topology of $\mathbb{R}$ and $Q_2$ does not depend on $\lambda$, the map $\kappa'' : U \rightarrow \mathcal{L}(X), \lambda \mapsto P''(\lambda)$ is continuous.
Therefore, $P'(\lambda) = I - P''(\lambda) \in \mathcal{L}(X)$ is a continuous projection onto $E'_\lambda$, and $\kappa': U \to \mathcal{L}(X)$, $\lambda \mapsto P'(\lambda)$ is continuous.

Now let $\psi'(\lambda) = P'(\lambda)|_{E'_\lambda}$. Then $P'(\lambda)|_{E'_\lambda} \in \text{GL}(E'_{\lambda_0}, E'_\lambda)$ for $\lambda \in U$, because $E'_{\lambda_0}$ and $E'_\lambda$ have the same dimension $n$, and $P'(\lambda)$ is a projection onto $E'_\lambda$. And also if $x' \in E'_{\lambda_0}$ is such that $P'(\lambda)x' = 0$, then $x' \in E''_{\lambda_0}$ and hence $x' = 0$. Therefore, if we let $X' = E'_{\lambda_0}$, then $\psi'$ satisfies the condition required in part 2 of the definition of subbundles of $[a, b] \times X$. Hence, $E'$ is a trivial finite dimensional subbundle of $[a, b] \times X$.

To prove the second part of Theorem 1.2.7, we let $\psi : [a, b] \to \mathcal{L}(X', X)$ be a trivialization of $E'$. First note that since $\psi' \in \text{GL}(X', E'_{\lambda})$, $\forall \lambda \in [a, b]$, $\psi(a) \in \text{GL}(X', E'_a), \psi(b) \in \text{GL}(X', E'_b)$. Moreover, since $\alpha(a), \alpha(b) \in \text{GL}(X, Y), \alpha(a) \in \text{GL}(E'_a, Y_1), \alpha(b) \in \text{GL}(E'_b, Y_1)$, and thus
\[
\alpha(a)\psi'(a), \alpha(b)\psi'(b) \in \text{GL}(X', Y_1).
\]

Now Lemma 1.2.8 implies that there exists a closed subspace $E''$ of $X$ such that $X = E' \oplus E''$. Then by Lemma 1.2.9 $E''$ is also a trivial subbundle of $[a, b] \times X$. Let $\psi'' : [a, b] \to \mathcal{L}(X'', X)$ be a trivialization corresponding to $E''$. Now, define $\psi : [a, b] \to \mathcal{L}(X' \times X'', X)$ by
\[
\psi(\lambda)(x', x'') = \psi'(\lambda)x' + \psi''(\lambda)x'', \quad \forall \lambda \in [a, b].
\]

Note that $X'$ and $X''$ are Banach spaces isomorphic to the fibres of $E'$ and $E''$, respectively. And since $X = E'_\lambda \oplus E''_{\lambda}$, for every $\lambda \in [a, b]$, $\psi(\lambda) \in \text{GL}(X' \times X'', X)$, for every $\lambda \in [a, b]$. Then using Theorem 1.2.4 (Multiplicativity under composition) we get $\sigma(\alpha \psi, [a, b]) = \sigma(\alpha, [a, b], \sigma(\psi, [a, b]))$, where $\sigma(\alpha \psi)(\lambda) = \alpha(\lambda)\psi(\lambda)$. Moreover, $\psi \subseteq \text{GL}(X' \times X'', X)$ together with the Theorem 1.2.1 implies $\sigma(\psi, [a, b]) = 1$. Thus
\[
\sigma(\alpha, [a, b]) = \sigma(\alpha \psi, [a, b]).
\]

Let $Y_2$ be any closed subspace of $Y$ such that $Y = Y_1 \oplus Y_2$ and once again $Q_1$ and $Q_2$ projections of $Y$ onto $Y_1$ and $Y_2$, respectively. Relative to the decompositions
\[
X' \times X'' = X' \times \{0\} \oplus \{0\} \times X'',
\]
\[
Y = Y_1 \oplus Y_2,
\]
and upon identifying $X' \times \{0\}$ and $\{0\} \times X''$ with $X'$ and $X''$, respectively, $(\alpha \psi)(\lambda), \lambda \in [a, b], \text{possesses the block decomposition}
\[
(\alpha \psi)(\lambda) = \begin{pmatrix}
(Q_1 \alpha \psi)(\lambda) & (Q_1 \alpha \psi''(\lambda)) \\
(Q_2 \alpha \psi)(\lambda) & (Q_2 \alpha \psi''(\lambda))
\end{pmatrix}.
\]
But, since \( \alpha(\lambda)\psi'(\lambda)x' \in Y_1 \), for \( x' \in X' \), \( Q_1\alpha\psi' = \alpha\psi' \) and \( Q_2\alpha\psi' = 0 \). Therefore, we find

\[
(\alpha\psi)(\lambda) = \begin{pmatrix} (\alpha\psi')(\lambda) & (Q_1\alpha\psi'')(\lambda) \\ 0 & (Q_2\alpha\psi'')(\lambda) \end{pmatrix}.
\]

Observe that \( (Q_2\alpha\psi'')(\lambda) \in \text{GL}(X'', Y_2) \), because \( \psi''(\lambda)\text{GL}(X'', E_\lambda) \) (\( \psi'' \) is a trivialization of \( E'' \)) and \( Q_2\alpha(\lambda) \in \text{GL}(E_\lambda, Y_2) \) (was proved earlier). Now, we are able to apply Corollary 1.2.6 and we get

\[
\sigma(\alpha\psi, [a, b]) = \sigma(\alpha\psi', [a, b]).
\]

But \( \alpha(\lambda)\psi'(\lambda) \in \mathcal{L}(X', Y_1) \) are \( n \times n \) matrices for every \( \lambda \in [a, b] \). Therefore, if we let \( \mu_a \) and \( \mu_b \) be the number of negative eigenvalues of \( \alpha(a)\psi'(a) \) and \( \alpha(b)\psi'(b) \), respectively, then the number of intersection points of the path \( \alpha\psi' \) with \( S(X', Y_1) \) is the number of negative eigenvalues of \( \alpha(\lambda)\psi'(\lambda) \) which connected to a positive eigenvalue of \( \alpha(b)\psi'(b) \) through the path \( \alpha\psi' \). Thus, the number of intersection points \( \alpha\psi' \) with \( S(X', Y_1) \) is equal to \( \mu_a - \mu_b \), and hence

\[
\sigma(\alpha\psi', [a, b]) = (-1)^{\mu_a - \mu_b} = (-1)^{\mu_a}(-1)^{\mu_b} = \text{sgn}(\det(\alpha(a)\psi'(a)))\text{sgn}(\det(\alpha(b)\psi'(b))).
\]

Therefore we have

\[
\sigma(\alpha, [a, b]) = \text{sgn}(\det(\alpha(a)\psi'(a)))\text{sgn}(\det(\alpha(b)\psi'(b))).
\]

\( \square \)

Now we would like to mention another property of the parity as follow:

**Proposition 1.2.10.** Let \( [a, b] = I \subseteq \mathbb{R} \), \( \alpha : (I, \partial I) \rightarrow (\Phi_0(X, Y), \text{GL}(X, Y)) \) and \( \alpha' : (I, \partial I) \rightarrow (\Phi_0(X, Y), \text{GL}(X, Y)) \) with \( \alpha(a) = \alpha'(a) \) and \( \alpha(b) = \alpha'(b) \). Then

\[
\sigma(\alpha, I) = \sigma(\alpha', I)
\]

**Proof.** By Theorem 1.1.12 for given \( \varepsilon > 0 \) there are generic paths \( \hat{\alpha} : I \rightarrow \Phi_0(X, Y) \) and \( \hat{\alpha}' : I \rightarrow \Phi_0(X, Y) \) \( \varepsilon \)-approximations of \( \alpha \) and \( \alpha' \) respectively, which agree with \( \alpha \) and \( \alpha' \) on \( \partial I \). Then \( \hat{\alpha}(a) = \alpha(a) = \alpha'(a) = \hat{\alpha}'(a) \) and \( \hat{\alpha}(b) = \alpha(b) = \alpha'(b) = \hat{\alpha}'(b) \). Thus by Proposition 1.1.8 \( \sigma(\hat{\alpha}, I) = \sigma(\hat{\alpha}', I) \). And therefore

\[
\sigma(\alpha, I) = \sigma(\alpha', I)
\]

\( \square \)

**Remark 1.2.11.** The above proposition states that the parity depends only on the endpoints of a path.
1.3 Alternative definitions of the Parity and connections with our definition

In some other texts parity has been defined in other way. In this section we give two different definitions of parity and prove the equivalence of them with our definition. The first one is a definition which is used by Leray-Schauder degree \[6, 7, 8\], and another one is using spectral flow.

1.3.1 parity and Leray-Schauder degree

Let \([a, b] = I \subseteq \mathbb{R}\) and \(\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y))\). Then Theorem 1.1.4 implies that there exists a path \(\eta : I \rightarrow \text{GL}(Y,X)\) such that \(\eta(\lambda)\alpha(\lambda) \in L_C(X)\), for every \(\lambda \in [a, b]\). We denote by \(\text{deg}_{LS}(T)\) the Leray-Schauder degree of \(T\) with respect to a ball about the origin. Now we can define Fitzpatrick-Pejsachowicz parity of a path as follows:

**Definition 1.3.1.** Given \([a, b] = I \subseteq \mathbb{R}\) and

\[\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y)),\]

Fitzpatrick-Pejsachowicz parity of \(\alpha\) on \(I\) denoted by \(\sigma_{\text{FP}}(\alpha, I)\), is defined by

\[\sigma_{\text{FP}}(\alpha, I) = \text{deg}_{LS}(\eta(a)\alpha(a)) \text{deg}_{LS}(\eta(b)\alpha(b))\]

where \(\eta : I \rightarrow \text{GL}(Y,X)\) is any parametri for \(\alpha\).

The above definition is well defined because:

**Lemma 1.3.2.** Let \(I = [a, b] \subseteq \mathbb{R}\),

\[\eta : I \rightarrow \text{GL}(Y,X)\] and \(\tilde{\eta} : I \rightarrow \text{GL}(Y,X)\) be parametrices for

\[\alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y)).\]

Then

\[\text{deg}_{LS}(\eta(a)\alpha(a)) \text{deg}_{LS}(\eta(b)\alpha(b)) = \text{deg}_{LS}(\tilde{\eta}(a)\alpha(a)) \text{deg}_{LS}(\tilde{\eta}(b)\alpha(b)).\]

**Proof.** When \(\alpha \in \Phi_0(X,Y)\) and \(\eta\) and \(\tilde{\eta}\) are parametrices for \(\alpha\), then it follows immediately that

\[\eta(\tilde{\eta}^{-1}) \in L_C(X).\]
Consequently
\[ \eta(\tilde{\eta}^{-1}): [a, b] \to \text{GL}_C(X) \]
is continuous and hence by the invariance of the Leray-Schauder degree
\[ \text{deg}_{\text{LS}}(\eta(a)(\tilde{\eta}(a))^{-1}) = \text{deg}_{\text{LS}}(\eta(b)(\tilde{\eta}(b))^{-1}), \]
and hence
\[ \text{deg}_{\text{LS}}(\eta(a)(\tilde{\eta}(a))^{-1}) \text{deg}_{\text{LS}}(\eta(b)(\tilde{\eta}(b))^{-1}) = 1. \]
This last equality together with the composition property for the Leray-Schauder degree gives:
\[ \text{deg}_{\text{LS}}(\eta(a)(\tilde{\eta}(a))^{-1}) \text{deg}_{\text{LS}}(\eta(b)(\tilde{\eta}(b))^{-1}) = \text{deg}_{\text{LS}}(\eta(a)(\tilde{\eta}(a))^{-1}) \text{deg}_{\text{LS}}(\eta(b)(\tilde{\eta}(b))^{-1}) \]
Observe that for a path of compact vector fields, i.e. \( \alpha : (I, \partial I) \to (\mathcal{L}_C(X), \text{GL}_C(X)) \), one has:
\[ \sigma_{\text{FP}}(\alpha, I) = \text{deg}_{\text{LS}}(\alpha(a)) \text{deg}_{\text{LS}}(\alpha(b)) \]
Now we prove the equivalence of Fitzpatrick-Pejsachowicz parity with our definition of parity.

**Theorem 1.3.3.** Given \([a, b] = I \subseteq \mathbb{R} \) and \( \alpha : (I, \partial I) \to (\Phi_0(X, Y), \text{GL}(X, Y)) \),
\[ \sigma(\alpha, I) = \sigma_{\text{FP}}(\alpha, I). \]

**Proof.** By Theorem 1.1.12 there is a generic path \( \hat{\alpha} : I \to \Phi_0(X, Y) \) which agree with \( \alpha \) on \( \partial I \), and \( \hat{\alpha}(I) \cap S(X, Y) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subseteq S_1(X, Y) \). Let \( \eta : I \to \text{GL}(Y, X) \) be a parametrices for \( \alpha \), then \( \eta \) is also a parametrices for \( \hat{\alpha} \). Now, if \( \eta(a)\hat{\alpha}(a) \) and \( \eta(b)\hat{\alpha}(b) \) lay in the same connected component of \( \text{GL}_C(X) \), then properties of Leray-Schauder degree yields that
\[ \text{deg}_{\text{LS}}(\eta(a)\alpha(a)) = \text{deg}_{\text{LS}}(\eta(b)\alpha(b)). \]
Thus \( \sigma_{\text{FP}}(\alpha, I) = 1. \) On the other hand, Lemma 1.1.5 implies that \( \sigma(\alpha, I) = 1. \) Furthermore, if \( \eta(a)\hat{\alpha}(a) \) and \( \eta(b)\hat{\alpha}(b) \) lay in the different connected component of \( \text{GL}_C(X) \), then \( \sigma(\alpha, I) = \sigma_{\text{FP}}(\alpha, I) = -1. \) Therefore \( \sigma(\alpha, I) = \sigma_{\text{FP}}(\alpha, I). \) □
1.3.2 parity and Spectral Flow

In this section, we consider paths of self adjoint operators which are self adjoint, i.e. \( \alpha : (I, \partial I) \rightarrow (\Phi_0(X,Y), \text{GL}(X,Y)) \) such that \((\alpha(\lambda))^* = \alpha(\lambda)\), for every \( \lambda \in [a,b] \), and \( \alpha^* = \alpha \). These paths are not only generic, but also intersects \( S_1(X,Y) \) with a direction either +1 or -1. The Spectral Flow \( \omega \) of such paths are the algebraic sum of these directions. In some texts, the parity of these paths are defined by

\[ \sigma_{\text{SF}}(\alpha, I) = (-1)^\omega. \]

Note that if we let \( k \) be the number of intersection points of \( \alpha \) with \( S_1(X,Y) \), then \( k = \omega + e \) where \( e \) is the number of intersection points with opposite direction, and hence are in pairs. Therefore \( e \) must be an even number. Hence we find

\[ \sigma(\alpha, I) = (-1)^k = (-1)^{\omega+e} = (-1)^\omega.(-1)^e = (-1)^\omega \]

which the latter equality is due to the fact that \( e \) is an even number and hence \( (-1)^e = 1 \). Therefore, \( \sigma(\alpha, I) = \sigma_{\text{SF}}(\alpha, I) \).
Chapter 2

Fitzpatrick-Pejsachowicz Degree Theory

In this chapter we will use the Fitzpatrick-Pejsachowicz degree theory for special class of Nonlinear Fredholm mappings, $F(x) = Lx + f(x)$ where $L$ is invertible linear Fredholm operator with index 0 and $f$ is a compact operator. P.M. Fitzpatrick, J. Pejsachowicz, and P.J. Rabier defined a degree theory for general proper Fredholm mappings in 1992. But here we adopt their degree to the special case when $F(x) = Lx + f(x)$.

To do this, in Section 1 we will give the definition for regular values of $F$ and a proof of homotopy invariance in following Section. Section 3 is devoted to general definition of degree and homotopy invariance of the degree. And finally we will briefly explain the general Fitzpatrick-Pejsachowicz degree in section 4.

2.1 The degree at regular values

Let $X$ and $Y$ be real Banach spaces and $F : X \rightarrow Y$ be a $C^1$ Fredholm mapping in the form $F(x) = Lx + f(x)$ where $L$ is invertible linear Fredholm operator with index 0 and $f$ is a compact operator. If $\Omega$ is an open subset of $X$. A mapping $f : X \rightarrow Y$ is said to be proper if $f^{-1}(K) = \{ x \in X | f(x) \in K \}$ is compact for any compact set $K \subset X$. Proper mappings are closed, i.e. a mapping $f$ is called a closed mapping if it maps closed sets $A \subset X$ to closed sets $f(A) \subset Y$. Furthermore, recall that $p \in Y$ is said to be a regular value of $F$ if $DF(x) \in GL(X,Y)$ for every $x \in F^{-1}(p)$. 
Remark 2.1.1. Let $\Omega \subseteq X$ be open and bounded and $F : \overline{\Omega} \to Y$ be a $C^1$ Fredholm mapping in the form $F(x) = Lx + f(x)$ where $L$ is invertible linear Fredholm operator with index 0 and $f$ is a compact operator. Then $F$ is proper.

Proof. Let $\Omega \subseteq X$ be open and bounded, if $F : \overline{\Omega} \to Y$ be a $C^1$ closed mapping which is Fredholm of index 0, then it is proper (cf. [15], Lemma 2.1, p.449). Since $L$ is an invertible operator, is closed. Moreover, compactness of $f$ implies closeness. Thus $F$ is a closed map. Therefore $F$ is proper. \qed

Let $\Omega \subseteq X$ be open and bounded, and $p \in Y$ be a regular value of $F$ with $p \notin F(\partial \Omega)$. $\{p\}$ is a compact set in $Y$ and since $F$ is proper (by above Remark), $F^{-1}(p) = \{x \in \overline{\Omega} | F(x) = p\}$ is compact. This together with $p \notin F(\partial \Omega)$ implies that $\overline{\Omega} \cap F^{-1}(p) = \Omega \cap F^{-1}(p)$ is compact. Furthermore, since $p$ is a regular value of $F$, Inverse Mapping Theorem implies that $\Omega \cap F^{-1}(p)$ consists of isolated points. This isolation together with compactness implies that, $\Omega \cap F^{-1}(p)$ is finite. On the other hand, $F^1(x) = L + f'(x)$, i.e. $DF(x) = Lx + Df(x)$, where $Df(x)$ is compact. Then for given $x \in X$ there is a path between $L$ and $L + f'(x)$ by $\alpha : [a,b] \to L + (\frac{t-a}{b-a})f'(x)$. Note that since $(\frac{t-a}{b-a})f'(x) \in K(X,Y)$ for every $t \in [a,b]$, $L + (\frac{t-a}{b-a})f'(x) \in \Phi_0(X,Y)$ for every $t \in [a,b]$. Now we introduce the degree of $F$ on $\Omega$ with respect to $p$, denoted by $\deg(F,\Omega, p)$ by setting

$$\deg(F,\Omega, p) := \sum_{x \in \Omega \cap F^{-1}(p)} \sigma(\alpha, [a,b]), \quad (2.1.1)$$

where $\sigma$ is defined through the notion of parity discussed in Chapter 1, and

$$\alpha : ([a,b], [a,b]) \to (\Phi_0(X,Y), GL(X,Y)), \text{ with } \alpha(a) = L$$

$$\alpha(b) = L + f'(x) = DF(x), \quad \forall x \in \Omega \cap F^{-1}(p),$$

be any curve.

Definition (2.1.1) may be justified as follows: first, $\sigma$ in (2.1.1) is well defined since $\alpha$ is a continuous path in $\Phi_0(X,Y)$ with endpoints $L, DF(x_i) \in GL(X,Y)$, the latter is because of the fact that $p$ is a regular value of $F$. Next, let $\tilde{\alpha}_i$ be another $C^0$ path with $\tilde{\alpha}_i(a_i) = L$ and $\tilde{\alpha}_i(b_i) = DF(x_i)$. Then there is a homotopy $\Gamma_i : ([0,1] \times [a_i, b_i], [0,1] \times \{a_i, b_i\}) \to (\Phi_0(X,Y), GL(X,Y))$ such that $\Gamma_i(0,.) = \alpha_i$ and $\Gamma_i(1,.) = \tilde{\alpha}_i$. Hence by Corollary [1.2.2] (Homotopy invariance of Parity), $\sigma(\alpha_i, [a_i, b_i]) = \sigma(\tilde{\alpha}_i, [a_i, b_i])$. Therefore $\sigma$ in (2.1.1) is independent of the path $\alpha_i$. 31
Remark 2.1.2. If we let $X = Y$ and consider $F(X) = x + f(x)$ where $f$ is a compact operator and also letting $p \in X$ a regular value of $F$, then by definition (2.1.1) we have:

$$\deg(F, \Omega, p) = \sum_{x \in \Omega \cap F^{-1}(p)} \sigma(\alpha, [a, b]),$$

where

$$\alpha : ([a, b], \{a, b\}) \rightarrow (\Phi_0(X), \text{GL}(X)), \text{ with } \alpha(a) = \text{Id} \text{ and } \alpha(b) = DF(x), \forall x \in \Omega \cap F^{-1}(p),$$

Then by Theorem 1.3.3 and (1.3.1) we get:

$$\sigma(\alpha, [a, b]) = \deg_{\text{LS}}(\alpha(a)) \deg_{\text{LS}}(\alpha(b))$$

$$= \deg_{\text{LS}}(\text{Id}) \deg_{\text{LS}}(DF(x)),$$

and since $\deg_{\text{LS}}(\text{Id}) = 1$,

$$\deg(F, \Omega, p) = \sum_{x \in \Omega \cap F^{-1}(p)} \deg_{\text{LS}}(\text{Id}) \deg_{\text{LS}}(DF(x))$$

$$= \sum_{x \in \Omega \cap F^{-1}(p)} \deg_{\text{LS}}(DF(x))$$

$$= \deg_{\text{LS}}(F, \Omega, p).$$

Therefore our degree coincide with the Leray-Schauder degree when we consider the class of compact vector fields.

Remark 2.1.3. Let $\Omega \subseteq X$ be open and bounded, and $p \in Y$, $p \notin F(\partial \Omega)$ be a regular value of $F$. Moreover let $G(x) = F(x) - p$, then $0 \notin G(\partial \Omega)$ and $0$ is a regular value of $G$, because if $x \in G^{-1}(0)$, then $G(x) = 0$, which means $F(x) - p = 0$, and hence $F(x) = p$ and $x \in F^{-1}(p)$. Therefore $G^{-1}(0) = F^{-1}(p)$. But since $p$ is a regular value of $F$, $DF(x) \in \text{GL}(X,Y)$. On the other hand, $DG(x) = DF(x)$, thus $DG(x) \in \text{GL}(X,Y)$. Therefore, $0$ is a regular value of $G$ and if $\alpha$ be a path joining $L$ to $DF(x)$, then by $DG(x) = DF(x)$, $\alpha$ is also a path joining $L$ to $DG(x)$. Thus all in all we get

$$\deg(F, \Omega, p) = \deg(F - p, \Omega, 0).$$
To close this section we will prove the local constancy of the degree at regular values.

**Theorem 2.1.4.** Let $\Omega \subseteq X$ be open and bounded, and $p \in Y$, $p \notin F(\partial \Omega)$ be a regular value of $F$. If $\tilde{p}$ be close enough to $p$, then $\tilde{p} \notin F(\partial \Omega)$, $\tilde{p}$ is a regular value of $F$, and

$$\deg(F, \Omega, p) = \deg(F, \Omega, \tilde{p})$$

**Proof.** Proper mappings are closed, and since $\partial \Omega$ is closed, the set $F(\partial \Omega)$ is closed. Therefore, $\tilde{p} \notin F(\partial \Omega)$ if $\|\tilde{p} - p\|$ be small enough (otherwise it contradicts $Y \setminus F(\partial \Omega)$ being open). By the same argument $\tilde{p} \notin F(\Omega)$ if $p \notin F(\Omega)$ and $\|\tilde{p} - p\|$ is small enough. Hence if $\Omega \cap F^{-1}(p) = \emptyset$ then $\Omega \cap F^{-1}(\tilde{p}) = \emptyset$, and thus

$$\deg(F, \Omega, p) = \deg(F, \Omega, \tilde{p}) = 0.$$ 

Now assume $\Omega \cap F^{-1}(p) = \{x_1, \ldots, x_k\}$ where $1 \leq k < \infty$. Since $p$ is a regular value of $F$ Inverse Mapping theorem implies that there exist Neighborhood around $p$ in $Y$ and a continuously differential map $G : V \to X$ such that $F(G(y)) = y$ for every $y \notin V$. Thus if $\tilde{p}$ be close enough to $p$, then $\Omega \cap F^{-1}(\tilde{p}) \supset \{\tilde{x}_1, \ldots, \tilde{x}_k\}$ where $\tilde{x}_i$ is close to $x_i$, and hence $\tilde{x}_i$s are distinct. Since $\Omega \subseteq X$ is bounded, Remark 2.1.1 implies that $F$ is proper and since $\|\tilde{p} - p\|$ is small enough, no other solution to $F(x) = \tilde{p}$ exists in $\Omega$.

So far, we have $\Omega \cap F^{-1}(\tilde{p}) = \{\tilde{x}_1, \ldots, \tilde{x}_k\}$ where $\tilde{x}_i$ is arbitrarily close to $x_i$, $1 \leq i \leq k$, if $\|\tilde{p} - p\|$ is small enough. Hence, we may choose $\tilde{p}$ so that for $t \in [0, 1]$ with no loss of generality $DF(t\tilde{x}_i + (1 - t)x_i) \in GL(X, Y)$, i.e. $\tilde{p}$ is a regular value of $F$. Let $\tilde{\alpha}_i$ be a continuous curve in $\Phi_0(X, Y)$ joining $L$ to $DF(\tilde{x}_i)$, which is obtained by first joining $L$ to $DF(x_i)$ through a curve $\alpha_i$ and next $DF(x_i)$ to $DF(\tilde{x}_i)$ through the path $\gamma_i(t) = DF(t\tilde{x}_i + (1 - t)x_i)$, $t \in [0, 1]$.

**Theorem 1.2.3** (The parity of concatenation of paths) implies that

$$\sigma_i(\tilde{\alpha}_i, [\tilde{a}_i, \tilde{b}_i]) = \sigma_i(\alpha_i, [a_i, b_i]) \sigma(\gamma_i, [0, 1]).$$

(Note that $\tilde{a}_i = a_i$ and $\tilde{b}_i = 1$.) Moreover by Theorem 1.2.1 $\sigma(\gamma_i, [0, 1]) = 1$, since $DF(t\tilde{x}_i + (1 - t)x_i) \in GL(X, Y)$, $\forall t \in [0, 1]$. Therefore $\tilde{\sigma}_i(\tilde{\alpha}_i, [\tilde{a}_i, \tilde{b}_i]) = \sigma_i(\alpha_i, [a_i, b_i])$, and hence $\deg(F, \Omega, p) = \deg(F, \Omega, \tilde{p})$ from (2.1.1).

**2.2 Homotopy invariance of the degree for regular values**

We will present some results in this section which are fundamental preliminaries for defining the general degree in next section. We will also use these
results for proving homotopy variance of the degree in the following section.

Let \( H : [0, 1] \times X \to Y \) be \( C^1 \) such that \( D_x H(t, x) \in \Phi_0(X, Y) \) for every \((t, x) \in [0, 1] \times X\). We shall say that if for every \( t \in [0, 1], H(t \times X) \) is in the form \( Lx + f(x) \) where \( L \) is invertible linear Fredholm operator with index 0 and \( f \) is a compact operator(\( L \) is fixed but \( f \) differs in \([0, 1] \times X\)), \( H \) is an \textbf{\( L \)-homotopy}. If we let \( \Omega \subset X \) a bounded set, then \( H(t, \cdot) \) is proper(by Remark 2.1.1). And since \([0, 1] \times \Omega \) is also bounded, \( H_{|[0,1] \times \Omega} \) is proper(by Remark 2.1.1).

\textbf{Theorem 2.2.1.} Let \( \Omega \subset X \) be open and bounded, and \( H : [0, 1] \times \overline{\Omega} \to Y \) be an \( L \)-homotopy. Moreover, let \( p \in Y \), \( p \not\in H([0, 1] \times \partial \Omega) \), be a regular value of \( H \) (i.e. \( DH(t, x) \in \mathcal{L}(\mathbb{R} \times X, Y) \) is onto for every \((t, x) \in H^{-1}(y)\)) and also be a regular value of \( H(0, \cdot) \) and \( H(1, \cdot) \). Then

\[
\deg(H(0, \cdot), \Omega, p) = \deg(H(1, \cdot), \Omega, p).
\]

For proving this Theorem, we need a preliminary lemma.

\textbf{Lemma 2.2.2.} If the hypotheses of Theorem 2.2.1 hold and if we let

\[
\lambda \in [a, b] \to (t(\lambda), x(\lambda)) \in [0, 1] \times \Omega
\]

be a \( C^1 \) curve such that \( H(t(\lambda), x(\lambda)) = p, (\dot{t}(\lambda), \dot{x}(\lambda) \neq (0, 0), \forall \lambda \in [a, b]. \)

Furthermore, assume that \( t([a, b]) \subset \{0, 1\} \) (i.e. the mapping \( t(\cdot) \) maps the boundary of \([a, b] \) into the boundary of \([0, 1] \)). Then

\[
\sigma(D_x H(t(.), x(.)), [a, b]) = \left\{ \begin{array}{cl}
-1 & \text{if } t(a) = t(b) \\
1 & \text{if } t(a) \neq t(b)
\end{array} \right.
\]

\textbf{Note.} Since \( p \) is a regular value of \( H(0, \cdot)|\Omega \) and \( H(1, \cdot)|\Omega \), and \( t([a, b]) \subset \{0, 1\} \), the parity on the left-hand side is well defined.

\textbf{Proof.} For \( \lambda \in [a, b] \) set \( \alpha(\lambda) \equiv D_x H(t(\lambda), x(\lambda)) \in \Phi_0(X, Y) \). Since \( \alpha(a), \alpha(b) = D_x H(0, p) \) or \( D_x H(1, p) \)(it depends on \( t(a) \) and \( t(b) \), i.e. \( t(a), t(b) = 0 \) or 1, and since \( p \) is a regular value of \( H(0, \cdot)|\Omega \) and \( H(1, \cdot)|\Omega \), \( \alpha(a), \alpha(b) \in GL(X, Y) \).

Define \( \tilde{\alpha}(\lambda) \in \mathcal{L}(X \times \mathbb{R}, Y) \) by

\[
\tilde{\alpha}(\lambda) = \begin{pmatrix}
\alpha(\lambda) & D_t H(t(\lambda), x(\lambda)) \\
0 & 1
\end{pmatrix}.
\]

(2.2.1)

Since \( \alpha(\lambda) \in \Phi_0(X, Y) \), by Theorem 1.1.4 there exists \( S(\lambda) \in GL(Y, X) \) with \( S(\lambda) \circ \alpha(\lambda) \in \mathcal{L}_C(X) \) for every \( \lambda \in [a, b] \). Let

\[
\tilde{S}(\lambda) = \begin{pmatrix}
S(\lambda) & -(S(\lambda) \circ D_t H(t(\lambda), x(\lambda))) \\
0 & 1
\end{pmatrix}
\]

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(Note that $D_1H(t(\lambda), x(\lambda)) : \mathbb{R} \to Y$ and $S(\lambda) : Y \to X$, hence $S(\lambda) \circ D_1H(t(\lambda), x(\lambda)) : \mathbb{R} \to X$.) Then

$$\tilde{S}(\lambda) \circ \tilde{\alpha}(\lambda) = \begin{pmatrix} S(\lambda) & -S(\lambda) \circ D_1H(t(\lambda), x(\lambda)) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha(\lambda) & D_1H(t(\lambda), x(\lambda)) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} S(\lambda) \circ \alpha(\lambda) & S(\lambda) \circ D_1H(t(\lambda), x(\lambda)) - S(\lambda) \circ D_1H(t(\lambda), x(\lambda)) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} S(\lambda) \circ \alpha(\lambda) & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Hence $\tilde{S}(\lambda) \circ \tilde{\alpha}(\lambda) \in \mathcal{L}_C(X \times \mathbb{R})$, since $S(\lambda) \circ \alpha(\lambda) \in \mathcal{L}_C(X)$. Thus $\tilde{\alpha}(\lambda) \in \Phi_0(X \times \mathbb{R}, Y \times \mathbb{R})$. Furthermore, $\tilde{\alpha}$ is continuous, because $\alpha(\lambda)$ and $D_1H(t(\lambda), x(\lambda))$ are continuous. Moreover, $\tilde{\alpha}(a), \tilde{\alpha}(b) \in \text{GL}(X \times \mathbb{R}, Y \times \mathbb{R})$, since $p$ is a regular value of $H(0, \ldots, \Omega)$ and $H(1, \ldots, \Omega)$ and the same argument above. Therefore by Corollary $\ref{cor:1.2.6}$ $\sigma(a, [a, b]) = \sigma(\tilde{\alpha}, [a, b])$.

On the other hand

$$\tilde{\alpha}(\lambda)(X \times \mathbb{R}) = \begin{pmatrix} \alpha(\lambda) & D_1H(t(\lambda), x(\lambda)) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \mathbb{R} \end{pmatrix} = \begin{pmatrix} \alpha(\lambda)X + D_1H(t(\lambda), x(\lambda))\mathbb{R} \\ \mathbb{R} \end{pmatrix} = \begin{pmatrix} DH(t(\lambda), x(\lambda))(X \times \mathbb{R}) \\ \mathbb{R} \end{pmatrix}.$$ 

And since $H(t(\lambda), x(\lambda)) = p$ for every $\lambda \in [a, b]$ and $p$ is a regular value of $H|_{[0,1] \times \Omega}$, $DH(t(\lambda), x(\lambda)) \in \mathcal{L}(\mathbb{R} \times X, Y)$ is onto for every $\lambda \in [a, b]$. Thus $Y \times \mathbb{R} = \text{Range} \tilde{\alpha}(\lambda) + ([0] \times \mathbb{R})$, for every $\lambda \in [a, b]$. Now, we are able to apply Theorem $\ref{thm:1.2.7}$ if we replace $X, Y, Y_1$, and $\alpha$ by $X \times \mathbb{R}, Y \times \mathbb{R}, \{0\} \times \mathbb{R},$ and $\tilde{\alpha}$, respectively. By part (i) of Theorem $\ref{thm:1.2.7}$ the set

$$E' = \{(x, \theta) \in [a, b] \times (X \times \mathbb{R}); \tilde{\alpha}(\lambda)(x, \theta) \in \{0\} \times \mathbb{R} \}$$

$$= \{(x, \theta) \in [a, b] \times (X \times \mathbb{R}); \left(\alpha(\lambda)x + \theta D_1H(t(\lambda), x(\lambda))\right) \in \{0\} \times \mathbb{R} \}$$

$$= \{(x, \theta) \in [a, b] \times (X \times \mathbb{R}); \alpha(\lambda)x + \theta D_1H(t(\lambda), x(\lambda)) = 0 \}$$

is a one-dimensional trivial subbundle of $[a, b] \times (X \times \mathbb{R})$, because $\{0\} \times \mathbb{R}$ is one-dimensional. Next, part (ii) of the same theorem provides, for any
given trivialization \( \psi' : [a, b] \to \mathcal{L}(\mathbb{R}, X \times \mathbb{R}) \) of \( E' \), \( \tilde{\alpha}(a)\psi'(a), \tilde{\alpha}(b)\psi'(b) \in \text{GL}(\mathbb{R}, \{0\} \times \mathbb{R}) \) and
\[
\sigma(\tilde{\alpha}, [a, b]) = \text{sgn(det}(\tilde{\alpha}(a)\psi'(a))\text{sgn(det}(\tilde{\alpha}(b)\psi'(b)).
\]
But since \( \tilde{\alpha}(a)\psi'(a), \tilde{\alpha}(b)\psi'(b) : \mathbb{R} \to \{0\} \times \mathbb{R} \), and \( \{0\} \times \mathbb{R} \) is one-dimensional, we can omit "det":
\[
\sigma(\tilde{\alpha}, [a, b]) = \text{sgn}(\tilde{\alpha}(a)\psi'(a))\text{sgn}(\tilde{\alpha}(b)\psi'(b)). \tag{2.2.2}
\]
Now, if we differentiate the identity \( H(t(\lambda), x(\lambda)) = y \), we find
\[
\alpha(\lambda)\dot{x} + \dot{t}(\lambda)D_t H(t(\lambda), x(\lambda)) = 0
\]
for \( \lambda \in [a, b] \). Thus \( (\lambda, \dot{x}(\lambda), \dot{t}(\lambda)) \in E' \). Furthermore, since \( E' \) is a one-dimensional subbundle of \( [a, b] \times (X \times \mathbb{R}) \), and \( (\dot{x}(\lambda), \dot{t}(\lambda)) \neq (0, 0) \), we have
\[
E' = \{(\lambda, (\xi\dot{x}(\lambda), \xi\dot{t}(\lambda)) \); \( (\lambda, \xi) \in [a, b] \times \mathbb{R} \}.
\]
Hence the mapping \( \psi' : [a, b] \to \mathcal{L}(\mathbb{R}, X \times \mathbb{R}) \) defined by \( \psi'(\lambda)\xi = \xi(\dot{x}(\lambda), \dot{t}(\lambda)) \) is a trivialization of \( E' \), because of the above characterization.

Moreover, by (2.2.1) we have
\[
\left(\tilde{\alpha}(\lambda)\psi'(\lambda)\right)(\xi) = \tilde{\alpha}(\lambda)\left(\xi(\dot{x}(\lambda), \dot{t}(\lambda))\right) = \begin{pmatrix} \alpha(\lambda) & D_t H(t(\lambda), x(\lambda)) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi(\dot{x}(\lambda)) \\ \xi(\dot{t}(\lambda)) \end{pmatrix} = \begin{pmatrix} \xi\left(\alpha(\lambda)\dot{x}(\lambda) + \dot{t}(\lambda)D_t H(t(\lambda), x(\lambda))\right) \\ \xi(\dot{t}(\lambda)) \end{pmatrix} = \xi\begin{pmatrix} 0 \\ \dot{t}(\lambda) \end{pmatrix},
\]
where the latter line is due to the fact that \( \alpha(\lambda)\dot{x}(\lambda) + \dot{t}(\lambda)D_t H(t(\lambda), x(\lambda)) = 0 \). Therefore, \( \tilde{\alpha}(\lambda)\psi'(\lambda) = (0, \dot{t}(\lambda)) \) for \( \lambda \in [a, b] \), then (2.2.2) becomes
\[
\sigma(\tilde{\alpha}, [a, b]) = \text{sgn}(\alpha(\lambda))\text{sgn}(\dot{t}(\lambda))
\]
(Note that by Theorem 1.2.7 \( \tilde{\alpha}(a)\psi'(a) = (0, \dot{t}(a)), \tilde{\alpha}(b)\psi'(b) = (0, \dot{t}(b)) \in \text{GL}(\mathbb{R}, \{0\} \times \mathbb{R}) \), and hence \( \dot{t}(a) \neq 0, \dot{t}(b) \neq 0 \).

First assume \( \dot{t}(a) = 0 \), then for every \( \lambda > a, \dot{t}(\lambda) > \dot{t}(a) \), because \( \dot{t}(\cdot) \) takes values in \([0, 1]\). Thus \( \dot{t}(a) > 0 \). If \( \dot{t}(b) = \dot{t}(a) = 0 \), then by the same
reason for every \( \lambda < b \), \( t(\lambda) > t(b) \), and \( \dot{t}(b) < 0 \). Hence \( \sigma(\bar{\alpha}, [a, b]) = -1 \).

And if \( t(b) = 1 \), then for every \( \lambda < b \), \( t(\lambda) < t(b) \), and \( \dot{t}(b) > 0 \). Hence for \( t(b) \neq t(a) \), \( \sigma(\bar{\alpha}, [a, b]) = 1 \). The cases \( t(a) = t(b) = 1 \) and \( t(a) = 1, t(b) = 0 \) are similar. \( \square \)

**Proof of Theorem 2.2.1.** If \( \overline{\Omega} \cap H(0,.)^{-1}(p) = \emptyset \) and \( \overline{\Omega} \cap H(1,.)^{-1}(p) = \emptyset \), then \( \text{deg}(H(0,\cdot),\Omega,p) = \text{deg}(H(1,\cdot),\Omega,p) = 0 \) by definition. Thus from now on, we assume that \( \overline{\Omega} \cap H(0,.)^{-1}(p) \neq \emptyset \). If we take \( \overline{\Omega} \cap H(1,.)^{-1}(p) \neq \emptyset \) as the initial hypothesis, then you can repeat the proof below verbatim.

Since \( \Omega \subset X \) is bounded, then \([0, 1] \times \Omega \) is bounded. Thus by Remark 2.1.1 \( H, H(0,\cdot) \) and \( H(1,\cdot) \) are proper. Moreover, the set \( ([0, 1] \times \Omega) \cap H^{-1}(p) \) is nonempty and compact from properness of \( H \), and since \( p \notin H([0, 1] \times \partial \Omega) \), \( ([0, 1] \times \overline{\Omega}) \cap H^{-1}(p) = ([0, 1] \times \Omega) \cap H^{-1}(p) \). On the other hand, by \( p \) being a regular value of \( H \), the Implicit Function Theorem implies that if \( (t_0, x_0) \in H^{-1}(p) \), then there exist neighborhoods \( U \) of \( t_0 \) and \( V \) of \( x_0 \) and a \( C^1 \) function \( g : U \rightarrow V \) such that \( H(t, g(t)) = 0 \) if and only if \( x = g(t) \) for every \( (x, t) \in U \times V \). This means that \( ([0, 1] \times \Omega) \cap H^{-1}(p) \) is a one-dimensional submanifold of \([0, 1] \times \Omega \), because \( x = g(t) \) is one-dimensional manifold. Likewise the case when \( X \) and \( Y \) are finite dimensional, we derive that \( ([0, 1] \times \Omega) \cap H^{-1}(p) \) is the disjoint union of finitely many (because of compactness) connected components, which each of them has a \( C^1 \) parametrization \( (t(\lambda), x(\lambda)), \lambda \in [a, b] \) such that \( (t(\lambda), \dot{x}(\lambda)) \neq (0, 0) \) for \( \lambda \in [a, b] \).

Set
\[ \Omega \cap H(0,.)^{-1}(p) = \{ x_{01}, \ldots, x_{0k_0} \} \]
with \( 1 \leq k_0 < \infty \), because \( H(0,\cdot) \) is proper and \( p \) is a regular value of \( H(0,\cdot) \).

And set
\[ \Omega \cap H(1,.)^{-1}(p) = \{ x_{11}, \ldots, x_{1k_1} \} \]
with \( 0 \leq k_1 < \infty \) with the same reason.

Fix \( 1 \leq i \leq k_0 \) and call \( K_{0i} \) the connected component of \( ([0, 1] \times \Omega) \cap H^{-1}(p) \) which contains \( (0, x_{0i}) \). Then \( K_{0i} \) connects \( (0, x_{0i}) \) either to \( (0, x_{0j}) \) with \( j \neq i \), or connects \( (0, x_{0i}) \) to \( (1, x_{1i}) \) for some index \( 1 \leq i \leq k_1 \), i.e. \( K_{0i} \) is either equal to the set
\[
\{(t(\lambda), x(\lambda)) \in ([0, 1] \times \Omega) \cap H^{-1}(p); \lambda \in [a, b], (t(\lambda), \dot{x}(\lambda)) \neq (0, 0), (t(a), x(a)) = (0, x_{0i}) \& (t(b), x(b)) = (0, x_{0j}), j \neq i \}
\]
or to the set
\[
\{(t(\lambda), x(\lambda)) \in ([0, 1] \times \Omega) \cap H^{-1}(p); \lambda \in [a, b], (t(\lambda), \dot{x}(\lambda)) \neq (0, 0),
(t(a), x(a)) = (0, x_0) \& (t(b), x(b)) = (1, x_U), 1 \leq l \leq k_1\}.
\]

Later on when we say \(K_{0i}\) connects a point to another we mean connection in above sense. Furthermore, no point \((1, x_U)\) (respectively \((0, x_{0j})\)) is contained in \(K_{0j}\) for more than one index (respectively two indices) \(j\) (cf. Lloyd [12]).

Let \(1 \leq k \leq k_0\) denote the number of distinct points \((0, x_{0i})\) which are connected by \(K_{0i}\) to some point \((1, x_U)\). Obviously, if \(k_1 = 0\), then \(k = 0\). And if \(k_1 \geq 1\), we may exchange the roles of the points \((0, x_{0i})\) and \((1, x_U)\), and denote \(K_U\) the connected component of \(([0, 1] \times \Omega) \cap H^{-1}(p)\) which contains \((1, x_U)\). Whenever \(K_{0i}\) connects \((0, x_{0i})\) to \((1, x_U)\) we have \(K_{0i} = K_U\). If we assume that \(K_U\) connects \((1, x_U)\) to \((0, x_{0i})\) this relation also holds. In other words, the number of indices \(l\) such that \(K_U\) connects \((1, x_U)\) to some point \((0, x_{0i})\) coincides with the index \(k\) above. Otherwise, \(K_U\) connects \((1, x_U)\) to another point \((1, x_{1m})\) with \(m \neq l\).

Therefore, if we rearrange indices if it is needed, we can assume that, for \(1 \leq i \leq k\), \(K_{0i} = K_{1i}\) connects \((0, x_{0i})\) to \((1, x_{1i})\). For \(k + 1 \leq j \leq k_0\), \(K_{0i}\) connects \((0, x_{0i})\) to \((0, x_{0j})\) with \(j \neq i\) and \(k + 1 \leq j \leq k_0\), and for \(k + 1 \leq l \leq k_1\), \(K_U\) connects \((1, x_{1i})\) to \((1, x_{1m})\) with \(m \neq l\) and \(k + 1 \leq m \leq k_1\).

Let \(\alpha_{0i} : ([a_{0i}, b_{0i}], \{a_{0i}, b_{0i}\}) \rightarrow (\Phi_0(X, Y), \text{GL}(X, Y))\) be any curve with \(\alpha_{0i}(a_{0i}) = L\) and \(\alpha_{0i}(b_{0i}) = D_xH(0, x_{0i})\), and \(\alpha_{1i} : ([a_{1i}, b_{1i}], \{a_{1i}, b_{1i}\}) \rightarrow (\Phi_0(X, Y), \text{GL}(X, Y))\) be any curve with \(\alpha_{1i}(a_{1i}) = L\) and \(\alpha_{1i}(b_{1i}) = D_xH(1, x_{1i})\). Then by definition of the degree at regular values, we get
\[
\deg(H(0, .), \Omega, p) = \sum_{i=1}^{k_0} \sigma(\alpha_{0i}, [a_{0i}, b_{0i}]),
\] (2.2.3)
and
\[
\deg(H(1, .), \Omega, p) = \sum_{i=1}^{k_1} \sigma(\alpha_{1i}, [a_{1i}, b_{1i}]),
\] (2.2.4)
where as usual, if \(k_1 = 0\), the sum is 0 and \(\deg(H(1, .), \Omega, p) = 0\).

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Lemma 2.2.2 asserts that a parametrization of $K_0$ such that $t_i(b_{0i}) = 0$, $x_i(b_{0i}) = x_0$, $t_i(b_{1i}) = 1$, and $x_i(b_{1i}) = x_{1i}$. Now if we let

$$\alpha_{ai} = \begin{cases} \alpha_{0i}(\lambda) & \lambda \in [a_{0i}, b_{0i}] \\ \gamma_i(\lambda) & \lambda \in [b_{0i}, b_{1i}] \end{cases},$$

then we have $\alpha_{ai}(a_{0i}) = \alpha_{0i}(a_{0i}) = L$, and $\alpha_{ai}(b_{1i}) = \gamma_i(b_{1i}) = D_x H(1, x_{1i})$. Hence by putting $\alpha_{ai} = a_{0i}$, Theorem 1.2.3 implies that $\sigma(\alpha_{ai}, [a_{0i}, b_{0i}]) = \sigma(\alpha_{0i}, [a_{0i}, b_{0i}]) \sigma(\gamma_i, [b_{0i}, b_{1i}])$. But Lemma 2.2.2 asserts that $\sigma(\gamma_i, [b_{0i}, b_{1i}]) = 1$, since $0 = t_i(b_{0i}) \neq t_i(b_{1i}) = 1$. Thus $\sigma(\alpha_{0i}, [a_{0i}, b_{0i}]) = \sigma(\alpha_{ai}, [a_{1i}, b_{1i}])$.

On the other hand,

$$\sigma(\alpha_{0i}, [a_{0i}, b_{0i}]) = -\sigma(\alpha_{0j}, [a_{0j}, b_{0j}]),$$

whenever $k + 1 \leq i \leq k_0$ and $K_0$ connects $(0, x_{0i})$ to $(0, x_{0j})$ with $j \neq i$ and $k + 1 \leq j \leq k_0$. Because let $\gamma_i(\lambda) = D_x H(t_i(\lambda), x_i(\lambda))$, $\lambda \in [b_{0i}, b_{0j}]$, where $(t_i(\lambda), x_i(\lambda))$ be a $C^1$ parametrization of $K_0$ such that $t_i(b_{0i}) = 0$, $x_i(b_{0i}) = x_{0i}$, $t_i(b_{0j}) = 0$, and $x_i(b_{0j}) = x_{0j}$. Now if we let

$$\alpha_{0j} = \begin{cases} \alpha_{0i}(\lambda) & \lambda \in [a_{0i}, b_{0i}] \\ \gamma_i(\lambda) & \lambda \in [b_{0i}, b_{0j}] \end{cases},$$

then we have $\alpha_{0j}(a_{0i}) = \alpha_{0i}(a_{0i}) = L$, and $\alpha_{0j}(b_{0j}) = \gamma_i(b_{0j}) = D_x H(0, x_{0j})$. Hence by putting $\alpha_{0j} = a_{0i}$, Theorem 1.2.3 implies that $\sigma(\alpha_{0j}, [a_{0j}, b_{0j}]) = \sigma(\alpha_{0i}, [a_{0i}, b_{0i}]) \sigma(\gamma_i, [b_{0i}, b_{1i}])$. But Lemma 2.2.2 asserts that $\sigma(\gamma_i, [b_{0i}, b_{1i}]) = -1$, since $t_i(b_{0i}) = t_i(b_{0j}) = 0$. Thus $\sigma(\alpha_{0i}, [a_{0i}, b_{0i}]) = -\sigma(\alpha_{0j}, [a_{0j}, b_{0j}]).$

By establishing the above argument in the exact same way, we get

$$\sigma(\alpha_{ai}, [a_{al}, b_{al}]) = -\sigma(\alpha_{ai}, [a_{ai}, b_{ai}]),$$

whenever $k + 1 \leq i \leq k_1$ and $K_1$ connects $(1, x_{1i})$ to $(1, x_{1m})$ with $m \neq l$ and $k + 1 \leq l \leq m \leq k_1$. Keep in mind that indices $(i, j)$ and $(l, m)$ as in (2.2.6) and (2.2.7) occur in distinct pairs, thus by (2.2.3) and (2.2.4) we have

$$\deg(H(0, \Omega, p)) = \sum_{i=1}^{k} \sigma(\alpha_{0i}, [a_{0i}, b_{0i}]),$$

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as well as
\[ \deg(H(1,.), \Omega, p) = \sum_{l=1}^{k} \sigma(\alpha_{1l}, [a_{1l}, b_{1l}]). \]

Therefore (2.2.5) implies that \( \deg(H(0,.), \Omega, p) = \deg(H(1,.), \Omega, p) \). \( \square \)

Now, if we assume that the \( L \)-homotopy \( H \) is \( C^2 \), we may drop the hypothesis that \( p \) is a regular value of \( H \) in Theorem 2.2.1. To ensure that the regular values are dense, the \( C^2 \) restriction is necessary. Because if \( H \) is \( C^r \), proper and Fredholm of index \( k \), and \( r > \max(k,0) \), then the set of regular values of \( H \) is open and dense in \( Y \) (Smale’s density Theorem [14]. Note that we adapt the version quoted by Elworthy and Tromba (1970) to suit our needs). Since \( D_x H(t,x) \) is Fredholm of index 0, \( DH(t,x) \) is either Fredholm of index 0 or 1, which implies that \( H \) could possibly be Fredholm of index 1. Therefore we need \( H \) to be \( C^2 \) to ensure denseness of regular values.

**Corollary 2.2.3.** Let \( \Omega \subset X \) be open and bounded, and \( H : [0,1] \times \overline{\Omega} \rightarrow Y \) be \( C^2 \), \( L \)-homotopy. Moreover, let \( p \in Y, \ p \notin H([0,1] \times \partial \Omega) \), be a regular value of \( H(0,. \) and \( H(1,. \). Then
\[ \deg(H(0,.), \Omega, p) = \deg(H(1,.), \Omega, p). \]

**Proof.** Since \( p \) is a regular value of \( H(0,. \) and \( H(1,. \), by Theorem 2.1.4 we can replace \( p \) by any point \( q \) of a ball \( B_r(p) \) with center \( p \) and small enough radius \( r > 0 \). Then \( q \) is still a regular value of \( H(0,. \) and \( H(1,. \),
\[ \deg(H(0,.), \Omega, p) = \deg(H(0,.), \Omega, q), \]
and \( \deg(H(1,.), \Omega, p) = \deg(H(1,.), \Omega, q). \)

On the other hand, since \( H \) is \( C^2 \), we can invoke Smale’s density Theorem (stated above) to ascertain that we can choose \( q \in B_r(p) \) to be a regular value of \( H \). Therefore, By Theorem 2.2.1 \( \deg(H(0,.), \Omega, q) = \deg(H(1,.), \Omega, q), \) and thus \( \deg(H(0,.), \Omega, p) = \deg(H(1,.), \Omega, p). \) \( \square \)

### 2.3 General definition of degree and homotopy invariance

With open and bounded subset \( \Omega \subset X \) and mapping \( F \) as in previous sections, we consider \( F : \overline{\Omega} \rightarrow Y \) to be \( C^2 \). Now, we shall extend the definition of \( \deg(F, \Omega, p) \) when \( p \in Y, \ p \notin F(\partial \Omega) \), is a regular value of \( F \) (see (2.1.1)) to the case when \( p \) is a singular value, i.e. a point in \( Y \) which is not a regular value. This can be done in the obvious way, setting

\[ \deg(F, \Omega, p) = \deg(F, \Omega, q), \quad (2.3.1) \]
for every regular value \( q \) of \( F \) close enough to \( p \). This definition is justified by existence of such regular value which is due to the fact that \( F \) is \( C^2 \) together with Smale’s density Theorem [14], and independence of the right hand side of (2.3.1) upon the specific choice of \( q \). To show that the right hand side of (2.3.1) is independent of the choice \( q \), recall that \( F(\partial \Omega) \) is closed, hence for small radius \( r > 0 \), if \( q \in B_r(p) \), \( q \not\in F(\partial \Omega) \), because \( Y \setminus F(\partial \Omega) \) is open. Now, if \( q_1, q_2 \in B_r(p) \) be regular values of \( F \), and if we set \( H(t, x) = F(x) - (1 - t)q_1 - t q_2 \) for \( (t, x) \in [0, 1] \times \Omega \), then 0 is a regular value of \( H(0, \cdot) \) and \( H(1, \cdot) \) (because if \( x \in (H(0, \cdot))^{-1}(0) \), then \( H(0, x) = 0 \), which means that \( F(x) - q_1 = 0 \), and hence \( F(x) = q_1 \). Therefore, \( x \in F^{-1}(q_1) \) and hence since \( q_1 \) is a regular value of \( F \), \( F'(x) \) is invertible. Thus, \( D_x H(0, \cdot) = D_x (F(x) - q_1) = F'(x) \) is invertible which implies that 0 is a regular value of \( H(0, \cdot) \). 0 being a regular value of \( H(1, \cdot) \) is similar. Moreover, since \( q_1 \not\in F(\partial \Omega) \), and \( q_2 \not\in F(\partial \Omega) \), and since \( Y \setminus F(\partial \Omega) \) is path connected, the line segment \((1 - t)q_1 + t q_2 \not\in F(\partial \Omega) \), for every \( t \in [0, 1] \). Hence \( 0 \not\in F(\partial \Omega) - (1 - t)q_1 - t q_2 \), for every \( t \in [0, 1] \), which means that \( 0 \not\in H([0, 1] \times \partial \Omega) \). Also, \( H \) is a \( C^2 \) homotopy because \( F \) is \( C^2 \). Now we are allowed to use Corollary 2.2.3 which implies that

\[
\deg(H(0, \cdot), \Omega, 0) = \deg(H(1, \cdot), \Omega, 0).
\]

But this means

\[
\deg(F - q_1, \Omega, 0) = \deg(F - q_2, \Omega, 0).
\]

Then by Remark 2.1.3 we get \( \deg(F, \Omega, q_1) = \deg(F, \Omega, q_2) \).

Let us consider some elementary properties of the degree before proving the homotopy invariance. The elementary properties of the degree that we shall consider below are additivity on domain, normalization, and constancy of the degree on connected components of \( Y \setminus F(\partial \Omega) \).

Additivity on domain. Let \( \Omega_1 \) and \( \Omega_2 \) be two disjoint open subsets of \( \Omega \). Then both are bounded and hence \( F_{|\Omega_1} \) and \( F_{|\Omega_2} \) are proper. Furthermore, \( F : \Omega_1 \to Y \) and \( F : \Omega_2 \to Y \) are \( C^2 \). If \( p \in Y \), \( p \not\in F(\Omega_1 \cup \Omega_2) \), then \( p \not\in F(\partial \Omega_1) \) and \( p \not\in F(\partial \Omega_2) \). Assume \( x \in F^{-1}(p) \cap \Omega \), \( p \not\in F(\Omega_1 \cup \Omega_2) \) implies that \( x \in F^{-1}(p) \cap \Omega_1 \) and since \( \Omega_1 \) and \( \Omega_2 \) are disjoint, either \( x \in F^{-1}(p) \cap \Omega_1 \) or \( x \in F^{-1}(p) \cap \Omega_2 \). Therefore, if \( p \) is a regular value of \( F \) and hence of \( F_{|\Omega_1} \) and \( F_{|\Omega_2} \) then
$$\deg(F, \Omega, p) = \sum_{x \in \Omega \cap F^{-1}(p)} \sigma(\alpha, [a, b])$$

$$= \sum_{x \in \Omega_1 \cap F^{-1}(p)} \sigma(\alpha, [a, b]) + \sum_{x \in \Omega_2 \cap F^{-1}(p)} \sigma(\alpha, [a, b])$$

$$= \deg(F, \Omega_1, p) + \deg(F, \Omega_2, p).$$

Moreover, properness of $F$ guarantees that the condition $p \notin F(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$ is stable under small perturbations of $p$. Hence the above relation remains valid for definition of degree at singular values given in (2.3.1).

Normalization. Let $p \in Y$, $p \notin F(\partial \Omega)$. Then if $\deg(F, \Omega, p) \neq 0$, there is $x \in \Omega$ such that $F(x) = p$. On the other hand, let $F = L \in \text{GL}(X, Y)$, hence proper and $C^2$. Then every value $p \notin L(\partial \Omega)$ is regular. In addition, if $p \in L(\Omega)$, there is only one $x \in \Omega \cap L^{-1}(p)$ and we have $\deg(L, \Omega, p) = \sigma(\alpha, [a, b])$, where $\alpha(a) = L$ and $\alpha(b) = DL(x) = L$. Thus $\sigma(\alpha, [a, b]) = 1$, and

$$\deg(L, \Omega, p) = 1. \ (2.3.2)$$

Invariance of the degree on connected components. The set $Y \setminus F(\partial \Omega)$ is open, because $F$ is a closed mapping. Its connected components are also arcwise connected, i.e. any two points $p_1$ and $p_2$ in a given component $\Xi$ are joined by a curve $p(.) \in C^2([0, 1]; \Xi)$ lying entirely on $\Xi$. Let $H(t, x) = F(x) - p(t)$, then 0 is a regular value of $H(0, .)$ and $H(1, .)$ (because if $x \in (H(0, .))^{-1}(0)$, then $H(0, x) = 0$, which means that $F(x) - p(0) = F(x) - p_1 = 0$, and hence $F(x) = p_1$. Therefore, $x \in F^{-1}(p_1)$ and hence since $p_1$ is a regular value of $F$, $F'(x)$ is invertible. Thus, $D_x H(0, .) = D_x (F(x) - p_1) = F'(x)$ is invertible which implies that 0 is a regular value of $H(0, .)$. 0 being a regular value of $H(1, .)$ is similar.). Furthermore, $p(t) \notin F(\partial \Omega)$, for every $t \in [0, 1]$. Hence $0 \notin F(\partial \Omega) - p(t)$, for every $t \in [0, 1]$, which means that $0 \notin H([0, 1] \times \partial \Omega)$. Also, $H$ is a $C^2$ homotopy because $F$ and $p$ are $C^2$. Thus Corollary 2.2.3 yields that $\deg(H(0, .), \Omega, 0) = \deg(H(1, .), \Omega, 0)$. or equivalently $\deg(F, \Omega, p_1) = \deg(F, \Omega, p_2)$. If $p_1$ or $p_2$ or both be a singular value of $F$ we can replace them by a nearby regular value and then this relation remains unchanged. In particular, the degree is locally constant with respect to values $p \in Y$, $p \notin F(\partial \Omega)$.

**Remark 2.3.1.** Note that the relation $\deg(F - p, \Omega, 0) = \deg(F, \Omega, p)$, remains valid for singular values because of the definition given in (2.3.1).
Theorem 2.3.2. Let $\Omega \subset X$ be open and bounded, and $H : [0,1] \times \overline{\Omega} \rightarrow Y$ be $C^2$, L-homotopy. Moreover, let $p \in Y$, $p \notin H([0,1] \times \partial \Omega)$. Then

$$\deg(H(0,.),\Omega,p) = \deg(H(1,.),\Omega,p).$$

Proof. If $p$ is a regular value of $H(0,.)$ and $H(1,.)$, then Corollary 2.2.3 implies $\deg(H(0,.),\Omega,p) = \deg(H(1,.),\Omega,p)$. If $p$ be a singular value of $H(0,.)$ or $H(1,.)$, or both, then since $H$ is $C^2$ we can replace $p$ by a point $q$ close enough to $p$ such that $q$ be a regular value of $H(0,.)$ and $H(1,.)$. Thus Corollary 2.2.3 yields that $\deg(H(0,.),\Omega,q) = \deg(H(1,.),\Omega,q)$. On the other hand, we can use definition given in (2.3.1) or Theorem 2.1.4 if they are needed in each case and get $\deg(H(0,.),\Omega,p) = \deg(H(0,.),\Omega,q)$ and $\deg(H(1,.),\Omega,p) = \deg(H(1,.),\Omega,q)$, which means $\deg(H(0,.),\Omega,p) = \deg(H(1,.),\Omega,p)$. \hfill \square

Remark 2.3.3. By assumption that $H(t,.)$ be $C^2$ for $t \in [0,1]$, We can drop the condition of $H$ being $C^2$. (cf. \cite{8})

2.4 Fitzpatrick-Pejsachowicz Degree Theory for general Fredholm mappings

Although we adopt the Fitzpatrick-Pejsachowicz Degree Theory for explaining our degree theory for the class of mappings that we chose in previous sections, it is slightly different since it is for general Fredholm mappings. In this section we will explain their work briefly.

The class of our mappings were closed mappings which together with boundedness of domain yields properness (cf. \cite{15}, Lemma 2.1, p.449). But, in 1992, P. M. Fitzpatrick, J. Pejsachowicz, and P. J. Rabier, gave a degree theory for general Fredholm mappings. They could assume closeness of mappings and boundedness of domain to ensure properness, but they assumed the properness of mappings in their work. Hence they say $F$ is $\Omega$-admissible, if $F|_{\overline{\Omega}}$ is proper.

Let $y \in Y$, $y \notin F(\partial \Omega)$ and suppose first that $DF(x) \in GL(X,Y)$ for no $x \in X$, so that $F^{-1}(y) = \emptyset$ if $y$ is a regular value of $F$. In this case we introduce the absolute degree $|\deg |(F,\Omega,y)$, by setting

$$|\deg |(F,\Omega,y) = 0.$$

If there exists $p \in X$ such that $Df(p) \in GL(X,Y)$, we call such a point a base point of $F$. With the same argument as in section 2.1 the set $\Omega \cap F^{-1}(y)$
is finite. We set
\[ \Omega \cap F^{-1}(y) = \{x_1, \ldots, x_k\} \]

where \(0 \leq k < \infty\).

Let \(p \in X\) be a base point. We define a degree \(\deg_p(F, \Omega, y)\) depending on \(p\) by setting
\[ \deg_p(F, \Omega, y) = \sum_{i=1}^{k} \sigma_i, \]
where \(\sigma_i = \sigma(DF \circ \gamma_i, [a_i, b_i]) = \pm 1\), \(1 \leq i \leq k\), and \(\gamma_i : [a_i, b_i] \to X\) is any continuous curve such that \(\gamma_i(a_i) = p\) and \(\gamma_i(b_i) = x_i\) \(1 \leq i \leq k\) (less formally, \(\gamma_i\) is any continuous curve in \(X\) joining \(p\) to \(x_i\)).

Suppose that \(q \in X\) is another base point, so that
\[ \deg_q(F, \Omega, y) = \sum_{i=1}^{k} \tilde{\sigma}_i, \]
and \(\tilde{\sigma}_i = \sigma(DF \circ \tilde{\gamma}_i, [\tilde{a}_i, \tilde{b}_i])\) where \(\tilde{\gamma}_i\) is a continuous curve in \(X\) joining \(q\) to \(x_i\). But such a curve can be obtained by concatenation of two paths \(\gamma_i\) joining \(q\) to \(p\) and \(\gamma\) joining \(p\) to \(x_i\). Then by Theorem 1.2.3 (The parity of concatenation of paths), with \(\gamma\) being defined over \([a, b]\) and letting \(\tilde{a}_i = a\) and \(\tilde{b}_i = b_i\), \(\tilde{\sigma}_i = \sigma(DF \circ \gamma_i, [a, b])\sigma_i\), and thus
\[ \deg_q(F, \Omega, y) = \epsilon \deg_p(F, \Omega, y), \]
where \(\epsilon = \sigma(DF \circ \gamma_i, [a, b]) = \pm 1\). This shows that it is legitimate to set
\[ |\deg|(F, \Omega, y) = |\deg_p(F, \Omega, y)|, \]
irrespective of the base point \(p \in X\).

All the results for this degree are the same as ours but homotopy invariance which reads as follows: Let \(H : [0, 1] \times X \to Y\) be \(C^2\) such that \(D_xH(t, x) \in \Phi_0(X, Y)\) for every \((t, x) \in [0, 1] \times X\). We call \(H\) an \(\Omega\)-admissible homotopy, if \(H|_{[0,1] \times \partial\Omega}\) is proper.

**Theorem 2.4.1.** Let \(H\) be \(C^2\) and \(\Omega\)-admissible homotopy. Let \(y \in Y\), \(y \notin H([0,1] \times \partial\Omega)\), be a regular value of \(H(0,.)\) and \(H(1,.)\). On the other hand, let \(p \in X\) be such that \(D_xH(0, p), D_xH(1, p) \in GL(X, Y)\), so that \(p\) is a base point for both \(H(0,.)\) and \(H(1,.)\). Then
\[ \deg_p(H(0,.), \Omega, y) = \epsilon \deg_p(H(1,.), \Omega, y), \]
where \(\epsilon = \sigma(D_xH(., [0,1]), y) = \pm 1\).
Chapter 3

Application to nonlinear elliptic equations

Degree theory has some different applications in the Analysis. We will discuss application of the degree theory explained in Chapter 2 in the context of nonlinear elliptic equations. In section 1, we consider the Laplace equation with Dirichlet boundary condition and start by proving the criteria which is needed to apply the degree. In following section we apply the degree to prove the existence of the solution to $p$-Laplace equation with Dirichlet boundary condition.

3.1 Laplace equation with Dirichlet boundary condition

Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial D$. Consider the problem

$$
\begin{cases}
-\Delta u = g(x, u, \nabla u) & x \in D, \\
u = 0 & x \in \partial D.
\end{cases}
$$

(3.1.1)

For the nonlinearity $g$ we assume that $g$ is $C^\infty$ of arguments and
\[ |g(x, u, p)| \leq C + C|p|^{\gamma}, \quad \gamma < 1, \]  
\[ |g_u(x, u, p)| \leq C + C|p|^{\gamma}, \quad \gamma < 1, \]  
\[ |g_p(x, u, p)| \leq C, \]  
\[ |g_{uu}(x, u, p)| \leq C + C|p|^{\gamma}, \quad \gamma < 1, \]  
\[ |g_{up}(x, u, p)| \leq C, \]  
\[ |g_{pp}(x, u, p)| \leq C, \]  
uniformly in \( x \in \mathcal{D}, u \in \mathbb{R}, \) and \( \nabla u \in \mathbb{R}^n \). Let \( X = H^1_0(D) \) with norm \( \|u\|_X = \left( \int_D |\nabla u|^2 \, dx \right)^{1/2} \), and \( Y = H^{-1}(D) \) with norm \( \|u\|_Y = \sup_{\|v\|_X = 1} \int_D u \cdot v \, dx \).

If we let \( F(u) = -\Delta u - g(x, u, \nabla u) \), then \( F \) is a mapping from \( X \) to \( Y \), i.e. \( F : X \rightarrow Y \).

Lemma 3.1.1. The Laplace operator, i.e. \( -\Delta \) is an isomorphism and \( C^2 \), hence \( C^2 \) linear Fredholm operator of index 0.

Proof. To prove \( -\Delta : H^1_0(D) \rightarrow H^{-1}(D) \), is an isomorphism, note that if \( -\Delta u = 0 \), then theory of linear Partial Differential Equations implies that \( u = 0 \). Thus \( -\Delta \) is one to one. Now assume \( f \in H^{-1}(D) \), then the equation \( -\Delta u = f \) reads the weak form \( B[u, v] = f(v) \), for every \( v \in H^1_0(D) \), where \( B[u, v] = \int_D \nabla u \cdot \nabla v \, dx \). Furthermore, \( B[u, v] = \langle u, v \rangle_{H^1_0(D)} \) and hence Riesz representation theorem implies that there exists a unique \( u \in H^1_0(D) \) such that \( \langle u, v \rangle_{H^1_0(D)} = f(v) \), for every \( v \in H^1_0(D) \). Therefore \( -\Delta \) is onto. Moreover, \( -\Delta \) is a linear operator and hence \( C^2 \). Thus \( -\Delta \) is a \( C^2 \) isomorphism. \( \square \)

Also, under conditions on \( g \) we have following result:

Lemma 3.1.2. Under the assumptions (3.1.2)-(3.1.7) the Nemytskii map \( u \mapsto g(x, u, \nabla u) \) is a \( C^2 \) compact operator.

Proof. First note that for given \( u \) the mapping \( g(x, u(x), \nabla u(x)) : \mathcal{D} \rightarrow \mathbb{R} \) is in \( L^p(D) \) for \( p \geq 1 \). Now consider the mapping

\[ \Upsilon : H^1_0(D) \rightarrow L^2 \times (L^2)^n \xrightarrow{N_g} L^2 \rightarrow H^{-1}(D) \]

\[ u \mapsto (u, \nabla u) \mapsto g(x, u(x), \nabla u(x)), \]

Where the first mapping is a linear mapping which is bounded and hence is continuous and \( C^2 \). On the other hand Theorem 10.58 of [13] together with (3.1.2) implies that \( N_g \) is a continuous map from \( (L^2)^{n+1} \) to \( L^2 \). And finally
the embedding from $L^2$ to $H^{-1}(D)$ is also continuous and $C^2$. Therefore $\Upsilon$ is a composition of continuous maps and thus is continuous. Observe that since the last map is a compact embedding $\Upsilon$ is a compact mapping from $H_0^1(D)$ to $H^{-1}(D)$. On the other hand, estimates (3.1.3)-(3.1.7) together with Theorem 10.58 of [13] and also Theorem 2.6 of [5] implies that $N_g$ is $C^2$, and thus $\Upsilon$ is a composition of $C^2$ maps and hence is $C^2$.

Therefore, the mapping $F$ is in the form $L + k$ where $L = -\Delta u$ is linear Fredholm operator of index 0, and $k = g$ is a compact operator. Moreover, note that if $u \in H_0^1(D)$,

$$\| - \Delta u \|_Y = \sup_{\|v\|_X = 1} \int_D (-\Delta u) v dx,$$

which by integration by part and the fact that $u, v \in H_0^1(D)$ we get

$$\| - \Delta u \|_Y = \sup_{\|v\|_X = 1} \int_D \nabla u \nabla v dx = \sup_{\|v\|_X = 1} \langle u, v \rangle_{H_0^1(D)}.$$

Now if we let $v = \frac{u}{\|u\|_X}$, then

$$\langle u, v \rangle_{H_0^1(D)} = \left\langle u, \frac{u}{\|u\|_X} \right\rangle_{H_0^1(D)} = \frac{\|u\|_X^2}{\|u\|_X} = \|u\|_X,$$

and hence, $\sup_{\|v\|_X = 1} \langle u, v \rangle_{H_0^1(D)} \geq \|u\|_X$. On the other hand, for every $v \in H_0^1(D)$ with $\|v\|_X = 1$, The CauchySchwarz inequality implies that

$$\langle u, v \rangle_{H_0^1(D)} \leq \|u\|_X \cdot \|v\|_X = \|u\|_X.$$

Hence $\sup_{\|v\|_X = 1} \langle u, v \rangle_{H_0^1(D)} \leq \|u\|_X$, and thus

$$\| - \Delta u \|_Y = \sup_{\|v\|_X = 1} \langle u, v \rangle_{H_0^1(D)} = \|u\|_X.$$

Furthermore, $\|u\|_X \leq C \|u\|_{H^1_0(D) \cap H^2(D)}$ for a $C > 0$ (because there is a compact embedding from $H_0^1(D) \cap H^2(D)$ to $H_0^1(D)$), if $u$ is a solution of (3.1.1) then $-\Delta u = g$, hence

$$\|g\|_Y = \| - \Delta u \|_Y = \|u\|_X \leq C \|u\|_{H^1_0(D) \cap H^2(D)},$$

where $\|u\|_{H^1_0(D) \cap H^2(D)} = (\int_D |\Delta u|^2 dx)^{1/2} = (\int_D g(x, u, \nabla u)|^2 dx)^{1/2}$, and hence

$$\|g\|_Y \leq C \|g\|_{L^2(D)}.$$
Moreover, by the estimate (3.1.2) on \( g \) we have that

\[
\|g\|_{L^2(D)}^2 = \int_D |g(x,u,\nabla u)|^2 \, dx \leq \int_D |C + C|\nabla u|^2 \|u\|^\gamma \, dx \\
\leq C \int_D [1 + |\nabla u|^{2\gamma}] \, dx \\
\leq C \int_D [1 + |\nabla u|^2]^{\gamma} \, dx \\
\leq C \left( \int_D 1 + |\nabla u|^2 \, dx \right)^\gamma \\
\leq C \left( 1 + \|u\|_X^2 \right)^\gamma,
\]

which proves that for \( u \in X, F(u) \in Y \), and hence \( F : X \rightarrow Y \) is well defined. Furthermore, using the above estimate on \( \|g\|_{L^2(D)} \) we obtain an a priori estimate on the solutions:

\[
\|u\|_X = \| - \Delta u \|_Y = \|g\|_Y \leq C \|g\|_{L^2(D)} \leq C \left( 1 + \|u\|_X^2 \right)^{\gamma/2},
\]

which, since \( \gamma < 1 \), implies that there exists \( R > 0 \) such that \( \|u\|_X \leq R \) (because \( \|u\|_X \) is bounded by a sub linear function of \( \|u\|_X \), and it does happen only for amounts of \( \|u\|_X \) less than a constant which is where the equality in above estimate ocurrs, since after this constant \( \|u\|_X \) grows faster than the sub linear function and thus contradicts the above estimate.). Define the domain \( \Omega = B_{2R}(0) \subset X \). By Lemma 3.1.1 and 3.1.2 \( F \) is a \( C^2 \) map from \( \Omega \) to \( Y \). By the above a priori estimate \( F^{-1}(0) \subseteq B_R(0) \), and hence \( 0 \notin F(\partial \Omega) \), and therefore the degree theory of Chapter 2

\[
\deg(F, \Omega, 0)
\]

is well-defined. Now to compute this degree, we consider the following homotopy:

\[
F_t(u) = -\Delta u - tg(x,u,\nabla u), \quad t \in [0,1].
\]

Observe that above homotopy is obviously a \( C^2 - \Delta u \)-homotopy and by the same a priori estimates, \( F_t^{-1}(0) \subseteq B_R(0) \), for every \( t \in [0,1] \), and therefore \( 0 \notin F_t(\partial \Omega) \), for every \( t \in [0,1] \). Then, homotopy invariance of degree implies that

\[
\deg(F, \Omega, 0) = \deg(-\Delta u, \Omega, 0),
\]

in which by Lemma 3.1.1 \(-\Delta u \) is an isomorphism and (2.3.2) yields that

\[
\deg(F, \Omega, 0) = \deg(-\Delta u, \Omega, 0) = 1.
\]

Thus \( \deg(F, \Omega, 0) \neq 0 \) and together with Normalization property of degree, we get \( F^{-1}(0) \neq \emptyset \). Therefore equation (3.1.1) has a solution \( u \in X \).
3.2 \( p \)-Laplace equation with Dirichlet boundary condition

Let \( D \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial D \). Consider the problem

\[
\begin{cases}
-\Delta_p u = g(x, u, \nabla u) & x \in D, \\
u = 0 & x \in \partial D,
\end{cases}
\tag{3.2.1}
\]

where \(-\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\). For the nonlinearity \( g \) we assume that \( g \) is \( C^\infty \) of arguments,

\[
|g(x, u, p)| \leq C + C|p|^\gamma, \quad \gamma < p - 1, \tag{3.2.2}
\]

and conditions (3.1.3)-(3.1.7) hold. Let \( X = H_0^{1,p}(D) \) with norm \( \|u\|_X = \left( \int_D |\nabla u|^p dx \right)^{1/p} \), and \( Y = H^{-1,q}(D) \) with norm \( \|u\|_Y = \sup_{\|v\|_X = 1} \int_D u.vdx \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). If we let \( F(u) = -\Delta_p u - g(x, u, \nabla u) \), then \( F \) is a mapping from \( X \) to \( Y \), i.e. \( F : X \longrightarrow Y \). Moreover, note that if \( u \in H_0^{1,p}(D) \),

\[
\| -\Delta_p u\|_Y = \sup_{\|v\|_X = 1} \int_D (-\Delta_p u).vdx,
\]

which by integration by part and the fact that \( u, v \in H_0^{1,p}(D) \) we get

\[
\| -\Delta_p u\|_Y = \sup_{\|v\|_X = 1} \int_D \text{div}(|\nabla u|^{p-2}\nabla u).vdx = \sup_{\|v\|_X = 1} \int_D |\nabla u|^{p-2}\nabla u.\nabla vdx.
\]

Now if we let \( v = \frac{u}{\|u\|_X} \), then

\[
\int_D |\nabla u|^{p-2}\nabla u.\nabla \frac{u}{\|u\|_X} dx = \frac{1}{\|u\|_X} \int_D |\nabla u|^{p-2}\nabla u.\nabla udx = \frac{1}{\|u\|_X} \int_D |\nabla u|^p dx = \frac{\|u\|_X^p}{\|u\|_X} = \|u\|_X^{p-1},
\]

and hence, \( \sup_{\|v\|_X = 1} \int_D |\nabla u|^{p-2}\nabla u.\nabla vdx \geq \|u\|_X^{p-1} \). On the other hand, for every \( v \in H_0^{1,p}(D) \) with \( \|v\|_X = 1 \), The Hölder inequality implies that
\[
\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \leq \left( \int_D |\nabla v|^p dx \right)^{1/p} \left( \int_D |\nabla u|^{q(p-1)} dx \right)^{1/q} 
\]
\[
= \|v\|_X \left( \int_D |\nabla u|^p dx \right)^{1/q} \left( \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q(p-1) = p \right) 
\]
\[
= \|u\|_{X}^{p/q} \left( \frac{1}{p} + \frac{1}{q} = 1 \right) 
\]

Hence \( \sup_{\|v\|_X=1} \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \leq \|u\|_{X}^{p-1} \), and thus

\[
\| - \Delta_p u \|_Y = \sup_{\|v\|_X=1} \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \|u\|_{X}^{p-1}. 
\]

Furthermore, \( \|u\|_X \leq C \|u\|_{H^{1,p}_0(D) \cap H^{2,p}(D)} \) for a \( C > 0 \) (because there is a compact embedding from \( H^{1,p}_0(D) \cap H^{2,p}(D) \) to \( H^{1,p}_0(D) \)), if \( u \) is a solution of (3.2.1) then \(-\Delta_p u = g\), hence

\[
\|g\|_Y = \| - \Delta_p u \|_Y = \|u\|_{X}^{p-1} \leq C \|u\|_{H^{1,p}_0(D) \cap H^{2,p}(D)},
\]

where \( \|u\|_{H^{1,p}_0(D) \cap H^{2,p}(D)} = (\int_D |\Delta_p u|^p dx)^{1/p} = (\int_D |g(x,u,\nabla u)|^p dx)^{1/p} \), and hence

\[
\|g\|_Y \leq C \|g\|_{L^p(D)}. 
\]

Moreover, by the estimate (3.2.2) on \( g \) we have that

\[
\|g\|_{L^p(D)} = \int_D |g(x,u,\nabla u)|^p dx \leq \int_D [C + C|\nabla u|^\gamma]^p dx 
\]
\[
\leq C \int_D [1 + |\nabla u|^\gamma] dx 
\]
\[
\leq C \int_D [1 + |\nabla u|^\gamma] dx 
\]
\[
\leq C \left( \int_D 1 + |\nabla u|^p dx \right)^\gamma 
\]
\[
\leq C \left( 1 + \|u\|_{X}^{\gamma} \right)^\gamma, 
\]

which proves that for \( u \in X, F(u) \in Y \), and hence \( F : X \rightarrow Y \) is well defined. Furthermore, using the above estimate \( \|g\|_Y \) we obtain an a priori estimate on the solutions:
\[ \|u\|_X = \| - \Delta_p u \|_{Y}^{1/p-1} = \|g\|_{Y}^{1/p-1} \leq C\|g\|_{L^p(D)}^{1/p-1} \leq C(1 + \|u\|_X^{\gamma/p(p-1)}) , \]

which, since \( \gamma < p - 1 \), implies that there exists \( R > 0 \) such that \( \|u\|_X \leq R \)(because \( \|u\|_X \) is bounded by a sub linear function of \( \|u\|_X \), and it does happen only for amounts of \( \|u\|_X \) less than a constant which is the equality in above estimate occurs, since after this constant \( \|u\|_X \) grows faster than the sub linear function and thus contradicts the above estimate.). Define the domain \( \Omega = B_{2R}(0) \subset X \). Moreover we have following result

**Lemma 3.2.1.** The \( p \)-Laplace operator is a \( C^2 \) operator, and also it has a base point \( b \) in \( X \).

**Proof.** The derivative of the \( p \)-Laplace operator is given by

\[ (-\Delta_p u)'|_{u=u_0} = (\text{div}(\|u\|^{p-2}\nabla u))|_{u=u_0} = (p-1)\text{div}(\|u_0\|^{p-2}\nabla.) = \text{div}(a(x)\nabla.), \]

and thus is in the form of linear elliptic partial differential equations. Moreover, for \( u_0 \approx 0 \), \( DF(u_0)\varphi = \text{div}(a(x)\nabla \varphi) - Dg(x, u_0(x), \nabla u_0(x))\varphi, \varphi \in H^1_0(D) \). Since \( u_0 \approx 0 \), the part \( a(x) \) in the derivative of \(-\Delta_p u\) has required conditions for existence of the solution for \( \text{div}(a(x)\nabla \varphi) = f \) where \( f \in H^{-1,q}(D) \). On the other hand, \( u_0 \approx 0 \) implies that \( Dg(x, u_0(x), \nabla u_0(x)) \approx Dg(x, 0, 0) \) and hence \( Dg(x, u_0(x), \nabla u_0(x)) \) is a linear functional in \( H^{-1,q}(\Omega) \).

Thus \( DF(u_0)\varphi = f(\varphi) \) with \( f \in H^{-1,q}(D) \) and \( \varphi \in H^1_0(D) \) has a unique solution \( \varphi \in H^1_0(D) \) and therefore if we let \( b = u_0 \), then \( DF(b) \in GL(X,Y) \) and hence \( b \) is a base point of the mapping \( F \). Furthermore, it is easy to check that \( -\Delta_p \) is a \( C^2 \) operator. \( \square \)

By Lemma \[3.2.1\] and \[3.1.2\] \( F \) is a \( C^2 \) map from \( \overline{\Omega} \) to \( Y \). By the above a priori estimate \( F^{-1}(0) \subset B_R(0) \), and hence \( 0 \notin F(\partial \Omega) \), and therefore the degree theory of Chapter 2

\[ \text{deg}_b(F, \Omega, 0) \]

is well-defined. Similar to what we did in previous section \( \text{deg}_b(F, \Omega, 0) \neq 0 \) and hence the equation \[3.2.1\] has a solution \( u \in X \). Now to compute this degree, we consider the following homotopy:

\[ F_t(u) = -\Delta_p u - tg(x, u, \nabla u), \quad t \in [0,1], \]

Observe that above homotopy is obviously a \( C^2 \), \( \Omega \)-admissible homotopy and by the same a priori estimates, \( F_t^{-1}(0) \subset B_R(0) \), for every \( t \in [0,1] \),

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and therefore $0 \not\in F_t(\partial \Omega)$, for every $t \in [0,1]$. Then, homotopy invariance of degree implies that

$$\text{deg}(F, \Omega, 0) = \text{deg}(-\triangle_p u, \Omega, 0),$$

in which $-\triangle_p u$ is an isomorphism (the same argument as in Lemma 3.1.1 to prove $-\triangle$ being an isomorphism) and thus $(-\triangle_p)^{-1}(0) = \{0\}$ and hence

$$\text{deg}(F, \Omega, 0) = \text{deg}(-\triangle u, \Omega, 0) = \sigma(D(-\triangle_p) \circ \gamma, [a,b]) = \pm 1,$$

where $\gamma$ is a path in $X$ connecting $b$ to 0. Thus $\text{deg}(F, \Omega, 0) \neq 0$ and together with Normalization property of degree, we get $F^{-1}(0) \neq \emptyset$. Therefore equation (3.1.1) has a solution $u \in X$. 

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Bibliography


