Pricing of Contingent Convertible Bonds

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Abstract

Contingent Convertible bonds (CoCos) are bonds designed to convert to equity when a bank is close to becoming insolvent. This conversion is typically triggered by an accounting ratio falling below some threshold. However, in the existing literature on the pricing of CoCos no difference is made between accounting values and market values. In this thesis a model is proposed which attempts to fill this gap. In the proposed model, debt is valued under the assumption that the only information available is noisy accounting information, which is only received at discrete moments in time. In this way, it is possible to distinguish between market values and book values in the valuation of CoCos. Another important contribution of the model is the inclusion of the MDA regulations concerning the payment of coupons.

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Introduction

A Contingent Convertible bond (CoCo) is a special type of bond, which is designed to absorb losses when the capital of the issuing bank becomes too low. When this happens, the bond converts into equity or is (partially) written down. In both cases, the debt of the issuing bank is reduced and its equity is raised. The conversion of the bond is triggered by a specified trigger event, for example the capital ratio of the bank falling below a certain threshold. CoCos were first issued in 2009, when Lloyds Banking group offered some of its debt holders the possibility to exchange their bonds for bonds which possibly would convert into shares. After the financial crisis in 2007 it was realized that stronger regulation of the banking sector was necessary, this lead to the Basel III framework. In this new framework, contingent convertible instruments were included as a part of Capital, Additional Tier 1 Capital to be specific, which lead to an increasing popularity of CoCo bonds.

Contingent convertible bonds

A Contingent Convertible bond is a bond which converts into equity or is (partially) written down at the conversion date. This means that the design of a CoCo contract is specified by two main characteristics:

- The trigger event: when does conversion happen?
- The conversion mechanism: what does happen at conversion?

Trigger event

The trigger event specifies at which moment the conversion takes place. As in [9], we can distinguish three types of trigger events: an accounting trigger, a market trigger and a regulatory trigger.

In case of an accounting trigger, the conversion is triggered by an accounting ratio, e.g. the Common Equity Tier 1 Ratio (defined as the fraction of common equity and risk weighted assets) falling below a certain barrier. This type of trigger is typical in practice, it is however criticised by the academic world. By Flannery [11], it is argued that a book value will only be triggered long after the damage has already be done, because book values are not up-to-date at any moment. Therefore, the market trigger is proposed in the literature. In case of a market trigger, the conversion would happen if a market value, e.g. the share price of the issuing bank, falls below a certain threshold. A market price would better reflect the current situation of the issuing bank, because a market price is a forward looking parameter; it reflects the market’s opinion on the future of the
bank. However, the market trigger also has its shortcomings. Sundaresan and Wang [18] and Glasserman and Nouri [13] point out that a market trigger could lead to a multiple equilibria problem for the pricing of a CoCo if the terms of conversion are beneficial to CoCo holders. In this case, a market trigger could also encourage CoCo holders to short-sell shares of the issuing bank, to profit from a conversion, which could lead to a “death spiral”. Although the market trigger is preferred by a substantial part of the academic world, there exist no CoCos with a market trigger in practice.

A third type of trigger is the regulatory trigger, which allows the regulator to call for a conversion. In reality, CoCos often have a trigger which is a combination of an accounting trigger and a regulatory trigger.

Conversion mechanism

The conversion mechanism specifies what happens at the moment of conversion, there are two possibilities: a (partial) principal write-down or a conversion into shares. In case of a (partial) principal write-down mechanism, the principal of the CoCo bond is (partially) written down at the moment of conversion, to strengthen the capital position of the issuing bank. In case of a conversion into shares, the principal of the CoCo bond is converted into an amount of shares. Of course, it needs to be specified how many shares a CoCo holder receives at conversion. This conversion rule can be designed in two different ways. One possibility is that the CoCo holder receives a fixed number of shares for every dollar of principal. Another option is a variable number of shares, in this case the CoCo holder would “buy” a number of shares against a conversion price. This conversion price can be set as the market price of shares, which possibly leads to an infinitely large dilution of the existing shareholders. A way to avoid this, is to place a cap on the conversion price.

Pricing of CoCos

Due to the hybrid nature of Contingent Convertible bonds, a lot of different pricing models have been proposed in the literature. Following [19], these can be roughly grouped into three categories: structural models (see e.g. [1], [6],[12], [17]), equity derivative models (see e.g. [8], [9]) and credit derivative models or reduced form models (see e.g. [9]). For a comprehensive overview of the existing literature on the pricing of CoCos, see Chapter 1. As pointed out in this literature overview, the above mentioned models differ in applications and complexity (jumps/no jumps, constant/stochastic risk free rate, finite maturities/perpetuities, fixed/variable shares at conversion, etc.). However, they all have in common that a conversion trigger based on market values is used, while in reality all CoCos have a trigger based on the regulatory capital ratio, a book value.

Maximum Distributable Amount

As Contingent Convertible bonds qualify as a form of capital in the Basel III regulations, they are also affected by the concept of the Maximum Distributable Amount (MDA),
which requires regulators to stop earnings distributions when the bank’s capital becomes too low. An example of these earnings distributions are dividends. However, as CoCos qualify as a form of capital, the coupons on CoCos are also affected by the MDA. This means that when the bank’s capital falls below some threshold, higher than the CoCo’s conversion trigger, the payment of coupons is stopped until the bank’s capital is again above the MDA trigger.

Contents of this thesis

This thesis is organized as follows. In the first chapter an overview of the existing literature on the pricing of Contingent Convertibles is provided. In the second chapter a structural model, allowing for jumps in the asset process, is described. This is basically the model proposed by Chen, Glasserman, Nouri and Pelger [7], for which all the mathematical details left out in the paper are filled in. This pricing model is a very rich model, which is still tractable due to the use of exponential distributions and leads to closed form solutions. However, as all the models in the existing literature, it makes no distinction between market values and book values in the valuation of CoCos.

In Chapter 3 of this thesis a model is proposed which does make a distinction between market values and accounting values and in which also early cancelling of coupons, as caused by the above described MDA regulations, is considered. The model developed in this chapter thus contributes in two different ways to the existing literature; it distinguishes between market and book values of assets in the valuation of CoCos and it allows the coupons of CoCos to be already cancelled at a moment before the conversion date. The model is based on the model by Duffie and Lando [10], in which debt is valued under the assumption that the only information available is noisy accounting information which is received at selected moments in time. This setting is particularly relevant for the pricing of CoCos since, as pointed out above, conversion triggers are always based on imperfect accounting ratios observed at discrete moments in time, rather than on continuously observable market prices. The first part of Chapter 3 is devoted to a comprehensive description of the model proposed by Duffie and Lando, including the derivation of all the relevant formulas and proofs left out in the original paper. After this, in the sections 3.5 and 3.6, this thesis goes beyond the paper by Duffie and Lando, as explicit formulas and algorithms for the pricing of CoCos are provided. The setting is applied to the valuation of different kinds of CoCo bonds, namely CoCos with a (partial) principal write down and CoCos with a conversion into shares. Also a distinction is made between CoCos with a regulatory trigger, for which conversion could happen at any moment in time, and CoCos that can only be triggered at one of the accounting dates. The model does not lead to closed form solutions, but the expressions for CoCo prices involve integrals that are computed using MCMC-methods.

The last Chapter of this thesis is devoted to some applications of the model described in Chapter 3. The impact of several model parameters on the price of a CoCo is examined and the impact of the issuance of CoCos on the capital structure and on incentives for shareholders is investigated. Finally, the model is applied in an attempt to explain
the big downward price jump that CoCos of Deutsche Bank suffered at the beginning of 2016 after the release of a profit warning. In this particular case the added value of the proposed model becomes most clear as it allows for the announcement of a bad accounting report. Also, the sudden price drop seemed to be more out of fear for the MDA trigger than for the conversion trigger, as the conversion trigger was still far out of reach. So the two most important contributions of the model, the notion of accounting reports and the inclusion of the MDA trigger, seem to be very relevant to this case.
1 Literature overview

Due to the hybrid nature of Contingent Convertible bonds, a lot of different pricing models have been proposed in the literature. Following [19], these can be roughly grouped into three categories: structural models (see e.g. [1], [6], [12], [17]), equity derivative models (see e.g. [8], [9]) and credit derivative models or reduced form models (see e.g. [9]).

In a structural model one starts to describe the value of the assets of a firm by introducing a stochastic process. Also the liabilities are described and the capital is given by the difference between the assets and the liabilities. Conversion of CoCos occurs if a specified trigger event happens, for example if the market value of the firm’s assets [1,6] or the firm’s capital ratio [17] falls below a predetermined threshold.

Albul, Jaffee and Tchistyi [1] consider a model in which the firm’s value of assets $A_t$ follows a geometric Brownian motion process under the risk-neutral measure, given by

$$dA_t = \mu A_t dt + \sigma A_t dW_t.$$  

Here $\mu$ and $\sigma$ are constants and $W$ is a standard Brownian motion. Furthermore, the risk free rate is assumed to be constant. The firm issues two types of debt; a straight bond and a CoCo bond, both with perpetual maturities. Both pay coupons continuously at a constant rate. The CoCo is assumed to convert at the first time the value of assets drops below some specified threshold $a_c$, i.e. the conversion time is given by

$$\tau(a_c) = \inf\{t \geq 0 : A_t \leq a_c\}.$$  

It is assumed that the CoCo fully converts into equity against market value of shares at a specified conversion ratio $\lambda$ ($\lambda$ equal to 1 means the CoCo holders receive equity, valued at its market price, equal to the face value of the bond). Liquidation of the firm is also incorporated in the model, by assuming that the equity holders liquidate the firm when the value of assets falls below some optimal threshold, chosen by the shareholders to maximize equity value. Furthermore, it is assumed that default cannot occur before conversion. Now the value of the various claims (including the CoCos) is given by the risk-neutral expectation of the discounted future cashflows regarding the claim. In case of the CoCos this leads to a value at time $t < \tau(a_c)$, given by

$$V_t = \mathbb{E}_Q \left( \int_t^{\tau(a_c)} e^{-r(s-t)} c_c ds + e^{-r(\tau(a_c)-t)} \frac{c_c}{r} \right).$$  

Here the first term represents the discounted coupon payments, paid at rate $c_c$, until conversion, while the second term accounts for the equity received by the CoCo holders.
at conversion. Note that the face value of the CoCo is given by $c_c/r$, such that the CoCo holders receive $\lambda c_c/r$ at conversion. Due to the tractable setting (geometric Brownian motion, constant risk free rate), this leads to easy closed form solutions. For instance the value of a CoCo at time $t$ reads

$$V_t = \frac{c_c}{r} \left( 1 - \left( \frac{A_t}{a_c} \right)^{-\gamma} \right) + \left( \frac{A_t}{a_c} \right)^{-\gamma} \lambda \frac{c_c}{r},$$

where

$$\gamma = \frac{1}{\sigma^2} \left( (\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2r\sigma^2} \right).$$

Chen, Glasserman, Nouri and Pelger [6] propose a more involved model in which the market value of assets follows not only a geometric Brownian motion, but in which the asset value process also involves jumps, with a distinction between market-wide jumps and firm-specific jumps. For tractability it is assumed that the log-values of the jump sizes have exponential distributions. Again, the risk free rate is taken as a constant. The bank issues four kinds of debt: insured deposits, senior and subordinated debt and contingent convertible bonds. All pay coupons (interest in case of the deposits) continuously at a constant rate. It is assumed that all debt has an exponential distributed maturity, which could easily be replaced by perpetuities, tractability would be retained in that case, by taking the rate parameter of the exponential distribution equal to zero. This leads to a setting in which the par value of outstanding debt remains constant for all types of debt. Again, conversion of CoCos into equity is triggered the first time the value of assets falls below some specified threshold. In contrast to the variable shares feature in [1], at conversion the CoCo holders receive a fixed number of shares for every dollar of principal. The model also involves a notion of default, endogenously. Similar to the situation in [1], it is assumed that the shareholders declare the firm bankrupt when the value of assets falls below some optimal threshold, chosen by the equity holders to maximize equity value. Again the firm’s liabilities are valued by discounting their future cashflows and taking expectations under the risk neutral measure. In case of a CoCo, these future cashflows are the coupons paid until either maturity or conversion, the principal paid at maturity if the bond matures before conversion and the value of equity received at conversion if conversion occurs before maturity. Due to the use of exponential distributions (for both jumpsizes and maturities) this valuation leads to closed form solutions. For a detailed description of a simplified version of this model, see chapter 2.

Pennacchi [17] also considers a jump-diffusion process for the dynamics of the market value of assets, however only one type of jumps is considered, no distinction between marketwide or firm-specific jumps is made. To be specific, the instantaneous rate of return $A_t^*$ earned on the firm’s assets, which have time $t$ value $A_t$, is modeled by the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$\frac{dA_t^*}{A_t^*} = (r_t - \lambda_t k_t)dt + \sigma dW_t + (Y_{q_t} - 1)dq_t.$$
Here $W$ is a standard Brownian motion under $Q$ and $q$ is a Poisson process with intensity $\lambda_t$ and $k_t := \mathbb{E}_Q(Y_{q_t} - 1)$ is the expected proportional jump in case of a Poisson event. Furthermore, the risk free rate is not taken as a constant, but assumed to evolve stochastically in time, the dynamics given by a Cox-Ingersoll-Ross model. The firm again issues deposits, straight bonds and CoCos. The deposits of the bank have instantaneous maturities, meaning that they are very short-term sources for funding of the bank, while the straight debt and the CoCos are perpetual bonds. The market value of deposits outstanding, denoted by $D_t$, follows a process, which relates the growth of deposits positively to the asset-to-deposit ratio $x_t = A_t/D_t$, given by

$$dD_t = g(x_t - \hat{x})dt.$$ 

Here, $\hat{x} > 1$ is the target asset-to-deposit ratio and $g > 0$ is a constant. This can be interpreted as follows; when the asset-to-deposit ratio lies above the target, the bank will issue more deposits, while if the asset-to-deposit ratio is below the target, the value of outstanding deposits will decrease. In this way, a mean-reversion is created in which the asset-to-deposit ratio will converge to the target. This setting is supported by empirical evidence that banks have target capital ratios and that deposits increase when there is a capital surplus, while they decrease in case of a capital shortage. Furthermore, the deposits pay out interest plus deposit premiums continuously at some stochastic time dependent rate, given by an insurance premium on top of the risk free rate. Failure of the bank is also incorporated in the model, occurring the first time the value of assets drops below the deposit value. The bonds (either straight debt or CoCos) continuously pay fixed or floating coupons and have maturity $T$. The CoCos are triggered if the market value of the bank’s asset-to-deposit ratio $x_t = A_t/D_t$ falls below some specified threshold $\bar{x}$. It is assumed that the face value of the CoCo is fully converted into equity, following a conversion mechanism specified by two parameters, $p$ and $\alpha$. The first parameter $p$ accounts for the maximum proportion of the par value that CoCo holders can receive in the form of new shares and $\alpha$ for the maximum proportion of all shares that CoCo holders can receive. In this way, the model is suitable for both a variable number of shares (as in [1]) and a fixed number of shares (as in [6]). Denote by $\tau_c$ the time of conversion, i.e. $\tau_c = \inf\{t \geq 0 : x_t \leq \bar{x}\}$, and by $B$ the face value of the CoCo, then the value of the contingent capital at conversion is given by

$$V_{t_c} = \begin{cases} 
pB & \text{if } pB < \alpha(A_{t_c} - D_{t_c}) \\
\alpha \tau_c(A_{t_c} - D_{t_c}) & \text{if } 0 < \alpha(A_{t_c} - D_{t_c}) \leq pB \\
0 & \text{if } A_{t_c} - D_{t_c} \leq 0 \end{cases} \quad (1.1)$$

It is also assumed that the CoCo pays coupons continuously at rate $c_t$. Now the value of the CoCo at time 0 is given by

$$V_0 = \mathbb{E}_Q \left( \int_0^T e^{-\int_0^t r_s ds} v(t) dt \right),$$

where $v(t)$ denotes the cashflow per unit time paid at date $t$. This cashflow is $c_t B$ as long no conversion has happened. If at time $T$ no conversion has happened and also the
bank did not fail, there is a final cashflow, the payment of the principal $B$. If $t_c \leq T$ there is a cashflow at $t_c$, with a value given by equation (1.1). Due to the richness of the model, no closed form solution is obtained, but a Monte Carlo simulation method is proposed to evaluate the above expectation.

All of the above models make use of a conversion trigger based on the market price of a firm’s equity. Because the market price of equity depends on the firm’s capital structure itself, which changes in case of conversion, questions rise about the internal consistency of this type of trigger. This problem is analysed by Glasserman and Nouri [13]. They consider a post-conversion firm, for which the contingent capital is already converted prior to time zero and a no-conversion firm, which is not subject to a conversion at all. The question then is whether there exists a stock price process for the original firm that is equal to the no-conversion stock price before the trigger event and equal to the post-conversion value afterwards. Such a stock price is then called an equilibrium stock price. A second question is whether such an equilibrium is unique or if there are multiple equilibria. Reasonable conditions are stated under which existence and uniqueness hold. For existence it is required that the no-conversion price is higher than the post-conversion price, when both are above the trigger. This condition can be interpreted to mean that shareholders suffer from a conversion. For uniqueness it is required that the likelihood that the no-conversion price reaches the trigger is sufficiently large. The results hold in a setting in which the asset value process is a geometric Brownian motion. By a small modification of the precise conditions, it is shown that a unique equilibrium also exists if jumps are incorporated in the asset value process.

All the above described models have a similar structure, but they differ in applications and complexity (jumps/no jumps, constant/stochastic risk free rate, finite maturities/perpetuities, fixed/variable shares at conversion, etc.). However, they all have in common that a conversion trigger based on market values is used, while in reality almost all CoCos have a trigger based on the regulatory capital ratio, a book value. An attempt to distinguish between market values and book values of assets is made by Glasserman and Nouri [12]. They consider a model in which the conversion of contingent capital is partial and ongoing. That means, every time the capital ratio falls below a threshold, just enough conversion takes place to retain the capital ratio at the minimum level required. This is in contrast with the situation in reality and the above models in which conversion takes place all at once, the first time the trigger event occurs. The dynamics of the book value of assets are modeled by a geometric Brownian motion

$$\frac{dV_t}{V_t} = (r - \delta)dt + \sigma dW_t,$$

where $r$ is the risk-free rate and $\delta$ is a constant pay out rate. The key assumption made to relate book values to market values, denoted by $A_t$, is that they largely agree on whether a bank is solvent. So it is assumed that the market value of assets is greater than the total debt outstanding whenever the book value of assets is greater than the total debt outstanding, that is

$$A_t > B_t + D \text{ whenever } V_t > B_t + D,$$
where $D$ denotes the book value of the straight debt (assumed to be constant) and $B_t$ denotes the book value of contingent capital outstanding. Then a second geometric Brownian motion $U$ is introduced

\[ U_t = U_0 \exp(\theta_u t + \sigma_u W'_t), \]

where $W'$ is a second Brownian motion and the instantaneous correlation between $W$ and $W'$ is given by $\rho$. $U$ can be roughly interpreted as a market-to-book ratio. That is, the difference of the market value of assets and total debt is equal to this geometric Brownian motion times the difference of the book value of assets and the total debt outstanding:

\[ A_t - B_t - D = U_t(V_t - B_t - D). \]

This method to relate book values and market values introduces two extra parameters; the volatility $\sigma_u$ of the second geometric Brownian motion (the “book-to-market volatility”) and the instantaneous correlation $\rho$ between the two Brownian motions. These parameters should be calibrated using market values of the bank's debt and equity and book values from financial statements. This leads to a problem regarding the book values; using the above approach, it is assumed that book values can be observed continuously. In practice however, regulatory capital ratios are only calculated quarterly.

Brigo, Garcia and Pede [4] also consider a trigger event which is not based solely on market values, but is, as in reality, related to regulatory capital. They propose a model in which the value of the firm is modeled by a geometric Brownian motion, where the volatility is allowed to be time-dependent. After this, a process for the regulatory capital of the firm, denoted by $c_t$ is needed, because the CoCos are triggered when this regulatory capital falls below some threshold. Instead of modeling the regulatory capital directly, it is seen as an exogenous variable. A proxy for its value is then estimated by a linear regression, where it is assumed that the asset-to-equity ratio is the driver for the regulatory capital, that is

\[ c_t = \alpha + \beta X_t + \epsilon_t, \]

where $\epsilon$ is the residual term and $X_t$ is the asset-to-equity ratio, defined by $X_t = A_t/(A_t - L_t)$, where $A_t$ denotes the value of assets at time $t$ and $L_t$ the value of liabilities.

The above described models can all be categorized as structural models. In [9] two whole different approaches to the pricing of CoCos are described. The first one is a credit derivatives model, which is set up from the point of view of a fixed income investor. A fixed income investor would compute how much yield is needed on top of the risk free rate to compensate for the possible loss in case the CoCo is triggered. The second proposed approach is an equity derivatives model. In this model the problem of pricing a CoCo is approached from the point of view of an equity derivatives specialist, who will see a CoCo as a long position in shares that are knocked in when the CoCo is triggered. For the first case, a reduced form approach is used. One describes the likeliness of a trigger event by a trigger intensity $\lambda$. A Black-Scholes setting is used to determine the value for the intensity. That is, the stock price $S_t$ is assumed to follow a geometric Brownian motion

\[ dS_t = \mu S_t dt + \sigma S_t dW_t. \]
It is assumed that the CoCo converts when the stock price falls below a certain threshold $S^*$. Due to the Black-Scholes setting there is a closed form solution for the probability $p^*$ that the threshold is reached during the lifetime of the contingent convertible. The value for the trigger intensity can then be derived from this probability, by

$$\lambda = -\frac{\log(1 - p^*)}{T}.$$  

After this, it is possible to compute the credit spread needed on top of the risk free rate to compensate for the possible loss in case of a trigger event. This credit spread is given by the trigger intensity times the fraction of the face value that is lost at conversion, that is

$$cs = \lambda \left(1 - \frac{S^*}{C_p}\right),$$

where $C_p$ is the conversion price.

In the equity derivatives approach, one tries to replicate the payoff of the CoCo by using equity derivatives. The CoCo is seen as the following combination of equity derivatives

$$\text{CoCo} = \text{Straight Bond} + \text{Knock-In Forwards} - \text{Binary Down-In Options}.$$  

Here the long position in knock-in forwards corresponds to the possible purchase of shares (against the conversion price) in case the trigger event occurs (that is, the forwards are knocked-in). The short position in the binary down-in options reflects the reduction of coupons after conversion. Now the values of the knock-in forwards and BDI options can be computed in a Black-Scholes setting, which leads to a closed form solution for the price of a CoCo. Using this approach, it is assumed that the CoCo investor receives forwards at conversion. However, in reality the investor receives shares at conversion. This makes a major difference if the trigger event occurs a long time before the expiration date of the CoCo. For example, shares would entitle the investor immediately to dividends and voting rights, while this is not the case for a forward on those shares. Under the reasonable assumption that dividends will be low after the trigger event occurs, this equity derivatives approach will be an acceptable model. Another drawback of the models proposed in [9] is the fact that both models use a trigger driven by the stock price of the company, which should in some way be linked to the actual accounting ratio trigger. However, it is unclear how these two quantities should be linked. Furthermore, both of the approaches make use of a Black-Scholes setting, in which the stock price process follows a geometric Brownian motion. However, CoCos come with a lot of fat tail risk, which can not easily be handled in the Black-Scholes model, so other, better fitting, processes should be considered to improve the models. This will come at the cost of replacing the closed form solutions for simulation based solutions. Corcuera, De Spiegeleer, Ferreiro-Castilla, Kyprianou, Madan and Schoutens [8] work this out, now the equity derivative approach is not applied in a Black-Scholes setting, but the stock price dynamics follow an exponential Lévy process incorporating jumps and fat tails.
2 A structural model involving jumps

In this chapter the structural model proposed by Chen et al. [7], which is an updated version of [6] and contains a simplified version of the original model, is considered.

2.1 The firm’s asset value process

Consider a firm generating cash continuously at a rate \( \{ \delta_t, t \geq 0 \} \). The dynamics of the income flow are taken as a jump-diffusion process, given by

\[
\frac{d \delta_t}{\delta_t^-} = \mu dt + \sigma d\tilde{W}_t + d \left( \sum_{i=1}^{N_t} (\tilde{Y}_i - 1) \right). \tag{2.1}
\]

Here \( \delta_t^- \) is the value of the income flow just before a possible jump at time \( t \), \( \tilde{\mu}, \tilde{\sigma} \) are constants and \( \tilde{W} = \{ \tilde{W}_t : t \geq 0 \} \) is a standard Brownian motion. Furthermore, \( \tilde{N} = \{ \tilde{N}_t : t \geq 0 \} \) is a Poisson process with intensity \( \tilde{\lambda} \), which drives the jumps with sizes given by \( \{ \tilde{Y}_i : i = 1, 2, \ldots \} \). Because only downward jumps are relevant concerning a trigger event, it is assumed that \( \tilde{Y}_i < 1 \) for every \( i \). For the sake of tractability, the jump sizes are taken to be log-exponentially distributed, that is

\[ \tilde{Z}_i := -\log(\tilde{Y}_i) \sim \exp(\tilde{\eta}) \]

for some \( \tilde{\eta} > 0 \).

Furthermore, it is assumed that the jump sizes \( \{ Y_i : i = 1, 2, \ldots \} \), \( \tilde{W} \) and \( \tilde{N} \) are all independent of each other and that the risk free rate \( r \) is constant. Denote by \( \mathcal{F}_t \) the sigma-algebra generated by \( \delta_t \), that is \( \mathcal{F}_t = \sigma(\delta_s, s \leq t) \).

Now the dynamics of the asset value process can be stated explicitly, following Kou [15].

**Theorem 2.1.** In a rational expectations framework, the equilibrium price of a claim on future income of the firm is given by the expected value of the discounted payoff of the claim under a risk-neutral measure \( \mathbb{Q} \). It follows that the value of the firm’s assets \( V_t \) at time \( t \) is given by

\[
V_t = \mathbb{E}_\mathbb{Q} \left( \int_t^\infty e^{-r(s-t)} \delta_s ds | \mathcal{F}_t \right).
\]

Furthermore, \( V_t/\delta_t \) is a constant, denote it by \( \delta \), and the \( \mathbb{Q} \)-dynamics of \( V_t \) are given by

\[
\frac{dV_t}{V_t^-} = \left( r - \delta + \frac{\lambda}{1 + \eta} \right) dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} (Y_i - 1) \right), \tag{2.2}
\]
where, under \( \mathbb{Q} \), \( W \) is a Brownian motion and \( N \) is a Poisson process with intensity \( \lambda \). The distribution of the jump sizes \( Y_i \) is identical to that of \( \tilde{Y}_i \), but now with a different parameter \( \eta \).

The proof of the theorem and explicit expressions for the new parameters can be found in [15].

2.2 The firm’s capital structure

The assets of the firm are financed through two types of debt, straight debt and contingent convertible debt (CoCos), and also through equity. In this setting deposits are a special case of straight debt, in which the maturity of straight debt corresponds to the time until depositors withdraw their money.

2.2.1 Straight debt

The straight debt issued by the firm is the most senior claim at default, i.e. at default the holders of straight debt have the first claim at the firm’s assets. We assume that the firm continuously issues straight debt at rate \( p_1 \). That is, the par value of the debt issued in the interval \((t, t + dt)\) is given by \( p_1 dt \). Furthermore it is assumed that the maturity of the debt is an exponentially distributed variable with rate parameter \( m \).

The assumptions that the maturity is exponential and that the issuance rate is constant lead to a setting in which the total par value of debt outstanding is constant, given by

\[
P_1 := \int_{-\infty}^{\infty} \int_{-\infty}^{t} p_1 m e^{-m(s-u)} du ds = \frac{p_1}{m}
\]

This follows from the fact that in the interval \((t, t + dt)\) debt of value \( p_1 dt \) is issued, but for all \( s \geq 0 \) a portion of \( m \exp(-ms)ds \) of the total value \( p_1 dt \) matures in the interval \((t + s, t + s + ds)\). The debt also pays a coupon continuously at rate \( c_1 \) per unit of the par value of debt. The coupon payments are tax-deductible, where the marginal tax rate is given by \( \kappa_1 \), \( 0 \leq \kappa_1 < 1 \). It follows that the net value of coupon payments is given by \((1 - \kappa_1)c_1 P_1\).

2.2.2 Contingent convertible bonds

For the issuance and maturity of CoCos the same approach as for the straight debt is used. That is, CoCos are issued continuously at rate \( p_2 \) and their maturity is again exponentially distributed with rate parameter \( m \). This leads, similar as above, to a constant total par value of CoCos outstanding, denoted by \( P_2 \). Furthermore, the CoCos also pay a coupon continuously at rate \( c_2 \). Note that, to capture the case that straight debt and CoCos have perpetual maturities, it suffices to set \( m = 0 \), which implies \( T = \infty \).

Conversion of the CoCos is triggered when the asset value of the firm falls below a specific threshold \( v_c \). That is, conversion occurs at

\[
\tau_c = \inf\{t \geq 0 : V_t \leq v_c\}.
\]

(2.3)
In reality a CoCo converts if the capital ratio of the firm falls below some given threshold. This could be implemented in this setting by saying that the CoCos convert if

\[
\frac{V_t - P_1 - P_2}{V_t} \leq \rho,
\]

where \( \rho \in (0, 1) \). This is compatible with the setting in equation (2.3), by taking

\[
v_c = \frac{P_1 + P_2}{1 - \rho}.
\]

It should be noted that in this setting a market value of the capital ratio is used regarding the trigger event, while in reality this is a book value.

At conversion, the CoCo holder receives a fixed number of shares, denoted by \( \Delta \), for every dollar of principal value, so a total of \( \Delta P_2 \) shares is provided to the CoCo holders. By normalizing the number of shares before conversion to 1, it follows that the CoCo holders own a fraction \( \frac{\Delta P_2}{1 + \Delta P_2} \) of the firm after conversion.

### 2.2.3 Default

The firm is declared bankrupt the first time the asset value falls at or below some threshold \( v_b \). Thus, bankruptcy occurs at

\[
\tau_b = \{ t \geq 0 : V_t \leq v_b \}.
\]

At bankruptcy a fraction \( (1 - \alpha), 0 \leq \alpha \leq 1 \) of the asset value of the firm is lost to bankruptcy costs. This means that at the moment of bankruptcy the asset value is given by \( \alpha V_{\tau_b} \). We assume that conversion takes place before the firm defaults, i.e. \( v_b \leq v_c \).

This is a natural assumption in the sense that CoCos are actually designed to prevent the firm from defaulting.

### 2.3 Valuation of the firm’s liabilities

The model described above leads to closed-form solutions for the value of both the straight debt and the CoCos.

#### 2.3.1 Valuation of the straight debt

The value at time \( t \) of straight debt with unit face value and time to maturity \( T \) is given by

\[
b(V_t, T) = \mathbb{E}_Q \left( e^{-rT} 1_{\{\tau_b > T+t\}} \big| \mathcal{F}_t \right) + \mathbb{E}_Q \left( e^{-r(\tau_b-t)} 1_{\{\tau_b \leq T+t\}} \frac{\alpha V_{\tau_b}}{P_1} \big| \mathcal{F}_t \right)
\]

\[
+ \mathbb{E}_Q \left( \int_t^{\tau_b \wedge (T+t)} c_1 e^{-r(s-t)} ds \big| \mathcal{F}_t \right)
\]

(2.4)
In this valuation, different cases are considered. The first term on the right describes the principal payment if no default occurs before the debt matures. The second term denotes the payment at default when default occurs before maturity; at default the asset value is given by $\alpha V_{\tau b}$, which has to be divided among all the straight debt holders, so a holder of straight debt with unit face value will receive $\frac{\alpha V_{\tau b}}{P_1}$. The last term represents the discounted value of the coupon payments. From now on, we will take $t = 0$ to simplify notation (the conditional expectations become ordinary expectations in this case). It should be kept in mind that the value of both straight debt and CoCos depends on the current value of assets, denoted by $V$. Now, because the total par value of straight debt is $P_1$ and the maturity satisfies $T \sim \exp(m)$, the total market value of straight debt is given by

$$B(V) = P_1 \int_0^\infty b(V,T) e^{-mT}dT.$$ 

By inserting equation (2.4) in the above it follows that

$$B(V) = P_1 \int_0^\infty \mathbb{E}_Q(e^{-rT}1_{\{\tau > T\}}) me^{-mT}dT + P_1 \int_0^\infty \mathbb{E}_Q\left(e^{-r\tau b}\frac{\alpha V_{\tau b}}{P_1}\right) me^{-mT}dT$$

$$+ P_1 \int_0^\infty \mathbb{E}_Q\left(\int_0^T c_1 e^{-rs}ds\right) me^{-mT}dT. \quad (2.5)$$

Here, the first integral on the right hand side of equation (2.5) is given by

$$P_1 \int_0^\infty \mathbb{E}_Q(e^{-rT}1_{\{\tau > T\}}) me^{-mT}dT = mP_1 \mathbb{E}_Q\left(e^{-(r+m)T}\right)$$

$$= \frac{mP_1}{m+r} \mathbb{E}_Q\left(1 - e^{-(m+r)\tau b}\right).$$

Furthermore, the second integral is given by

$$P_1 \int_0^\infty \mathbb{E}_Q\left(e^{-r\tau b}\frac{\alpha V_{\tau b}}{P_1}\right) me^{-mT}dT = m\mathbb{E}_Q\left(e^{-r\tau b}\alpha V_{\tau b} \int_0^\infty e^{-mT}dT\right)$$

$$= \mathbb{E}_Q\left(\alpha V_{\tau b} e^{-(m+r)\tau b}\right).$$

And the last integral is given by

$$P_1 \int_0^\infty \mathbb{E}_Q\left(\int_0^{\tau b} c_1 e^{-rs}ds\right) me^{-mT}dT = P_1 mc_1 \mathbb{E}_Q\left(\int_0^{\tau b} \int_0^T e^{-rs} e^{-mT}dsdT\right)$$

$$+ P_1 mc_1 \mathbb{E}_Q\left(\int_0^\infty \int_0^{\tau b} e^{-rs} e^{-mT}dsdT\right)$$

$$= P_1 mc_1 \mathbb{E}_Q\left(\int_0^{\tau b} \frac{1}{r}(1 - e^{-rT}) e^{-mT}dT\right)$$

$$+ P_1 mc_1 \mathbb{E}_Q\left(\int_0^\infty \frac{1}{r}(1 - e^{-r\tau b}) e^{-mT}dT\right).$$
\[
= \frac{P_1 c_1}{r} E_Q \left( 1 - e^{-m r} \right) \\
- \frac{P_1 m c_1}{r (m + r)} E_Q \left( 1 - e^{-(m + r) r} \right) \\
+ \frac{P_1 c_1}{r} E_Q \left( e^{-m r} - e^{-(m + r) r} \right) \\
= \frac{c_1 P_1}{m + r} E_Q \left( 1 - e^{-(m + r) r} \right). 
\]

Inserting these three expressions into equation (2.5) implies that the market value of total straight debt outstanding is given by

\[
B(V) = \frac{P_1 (m + c_1)}{m + r} \left( 1 - e^{-(m + r) r} \right) + E_Q \left( \alpha V_{\tau} e^{-(m + r) r} \right). 
\]  

(2.6)

From the expression in equation (2.6) it follows that the key to valuation of the straight debt is the joint Laplace transform of \( \tau_b \) and the log-asset value \( \log V_t \). In Section 2.4, it will be shown that this transform has a closed form solution, which leads to a closed form solution for the value of straight debt.

2.3.2 Valuation of the contingent convertibles

Again we start by computing the market value of a CoCo with a unit face value and maturity \( T \), which is given by

\[
d(V, T) = E_Q \left( e^{-r T} 1_{\{\tau_c > T\}} \right) + E_Q \left( \int_0^{T \wedge \tau_c} c_2 e^{-r s} ds \right) + \frac{\Delta}{\Delta P_2 + 1} E_Q \left( e^{-r \tau_c} E^{PC}(V_{\tau_c}) 1_{\{\tau_c < T\}} \right). 
\]  

(2.7)

Here \( E^{PC}(v) \) is the value of the firm’s equity after conversion, at asset value \( v \). So this value corresponds to a firm with a total par value of straight debt given by \( P_1 \) and with no CoCos. At conversion, all the CoCo investors together own a fraction \( \frac{\Delta P_2}{\Delta P_2 + 1} \) of the firm, so the holder of a CoCo with unit face value will obtain a fraction \( \frac{\Delta P_2}{\Delta P_2 + 1} \) of the firm, this explains the last term in equation (2.7). The first term represents the payment of the principal if the CoCo matures before conversion is triggered, while the second term accounts for the coupon payments until either maturity or conversion.

In the same way as before, the total market value of CoCos is now given by

\[
D(V) = P_2 \int_0^\infty d(V, T) me^{-m T} dT.
\]

Following exactly the same calculations as in the case for straight debt, it follows that

\[
D(V) = \frac{P_2 (m + c_2)}{m + r} \left( 1 - e^{-(m + r) \tau_c} \right) + E_Q \left( \frac{\Delta P_2}{\Delta P_2 + 1} E^{PC}(V_{\tau_c}) e^{-(m + r) \tau_c} \right). 
\]  

(2.8)

Now the post-conversion value of equity at conversion \( E^{PC}(V_{\tau_c}) \) still needs to be computed. To this end, first the post-conversion firm value is computed and thereafter the
value of debt is substracted to obtain the equity value. Note that after conversion, straight debt is the only debt remaining, so it follows that

\[ E^{PC}(V_{\tau_e}) = F^{PC}(V_{\tau_e}) - B(V_{\tau_e}). \]

Now note that the firm value upon conversion is given by

\[
F^{PC}(V_{\tau_e}) = V_{\tau_e} + \mathbb{E}_Q \left( \int_{\tau_e}^{\tau_b} \kappa_1 c_1 p_1 e^{-r(s-\tau_e)} ds \bigg| \mathcal{F}_{\tau_e} \right) - \mathbb{E}_Q \left( e^{-r(\tau_b-\tau_e)} (1-\alpha) V_{\tau_b} \bigg| \mathcal{F}_{\tau_e} \right) = V_{\tau_e} + \frac{\kappa_1 c_1 p_1}{r} \mathbb{E}_Q \left( 1 - e^{-r(\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right) - \mathbb{E}_Q \left( e^{-r(\tau_b-\tau_e)} (1-\alpha) V_{\tau_b} \bigg| \mathcal{F}_{\tau_e} \right).
\]

Here the first term is just the unleveraged firm value, i.e. the firm’s value if it would carry no debt and would be entirely financed through equity. The second term accounts for the tax benefits, while the third term represents bankruptcy costs. Furthermore, note that the conversion does not affect the value of straight debt, such that equation (2.6) applies. By modifying this equation to the setting in which the present time is \( \tau_e \), it follows that

\[
B(V_{\tau_e}) = \frac{p_1(m + c_1)}{m + r} \mathbb{E}_Q \left( 1 - e^{-(m+r) (\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right) + \mathbb{E}_Q \left( \alpha V_{\tau_b} e^{-(m+r) (\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right).
\]

This leads to the following expression for the equity value upon conversion

\[
E^{PC}(V_{\tau_e}) = V_{\tau_e} + \frac{\kappa_1 c_1 p_1}{r} \mathbb{E}_Q \left( 1 - e^{-r(\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right) - \mathbb{E}_Q \left( e^{-r(\tau_b-\tau_e)} (1-\alpha) V_{\tau_b} \bigg| \mathcal{F}_{\tau_e} \right) - \frac{p_1(m + c_1)}{m + r} \mathbb{E}_Q \left( 1 - e^{-(m+r) (\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right) - \mathbb{E}_Q \left( \alpha V_{\tau_b} e^{-(m+r) (\tau_b-\tau_e)} \bigg| \mathcal{F}_{\tau_e} \right).
\]

So, as before, the valuation boils down to finding a formula for the joint Laplace transform of \( \tau_e, \tau_b \) and \( \log V_t \). This is considered in the next section.

### 2.4 Computing the transforms

In this section closed form solutions for the transforms needed in the valuation of the firm’s liabilities are derived following the method proposed by Cai et al [5]. In this section all dynamics, expressions and expectations considered are with respect to the risk-neutral measure \( \mathbb{Q} \). Recall from equation (2.2) that the dynamics of the asset value process are given by

\[
\frac{dV_t}{V_{t^-}} = \left( r - \delta + \frac{\lambda}{1 + \eta} \right) dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} (Y_i - 1) \right).
\]

This stochastic differential equation is solved by

\[
V_t = V_0 \exp \left( \left( r - \delta + \frac{\lambda}{1 + \eta} - \sigma^2/2 \right) t + \sigma W_t \right) \prod_{i=1}^{N_t} Y_i.
\]
Now denote $X_t = \log(V_t)$, $\mu = \left( r - \delta + \frac{\lambda}{\eta} - \sigma^2/2 \right)$ and recall that $-\log(Y_t) = Z_t \sim \exp(\eta)$, it then follows that

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} (-Z_i).$$

Also, we again denote $F_t = \sigma(V_s : s \leq t) = \sigma(X_s : s \leq t)$. Now note that $X$ is a Lévy process with Lévy exponent

$$\xi(s) := \frac{1}{t} \log \mathbb{E} \left( \exp(sX_t) \right) = \mu s + \frac{1}{2} \sigma^2 s^2 - \frac{\lambda}{\eta} \frac{s}{\eta + s},$$

which satisfies the following lemma, due to Kou and Wang [16].

**Lemma 2.1.** The equation $\xi(s) = a$ has three distinct real roots $\beta, -\gamma_1, -\gamma_2$, where $\beta, \gamma_1, \gamma_2 > 0$ and all these roots are different from $\eta$.

**Proof.** Note that $\xi(s)$ is convex on the interval $(-\eta, \infty)$. Also $\xi(0) = 0$, $\lim_{s \downarrow -\eta} \xi(s) = \infty$ and $\lim_{s \to \infty} \xi(s) = \infty$. So it follows there exists a unique $-\gamma_1 \in (-\eta, 0)$ such that $\xi(-\gamma_1) = a$ and there exists a unique $\beta \in (0, \infty)$ such that $\xi(\beta) = a$. Furthermore it holds that $\lim_{s \uparrow -\eta} \xi(s) = -\infty$ and $\lim_{t \to -\infty} \xi(s) = \infty$, so there must be at least 1 root of $\xi(s) = a$ on $(-\infty, -\eta)$. But $(\eta + s)\xi(s)$ is a polynomial of order three, so there are at most three real roots of the equation. Hence there exists a unique $-\gamma_2 \in (-\infty, -\eta)$ such that $\xi(-\gamma_2) = \eta$.

Now denote $\tau_x = \inf \{ t \geq 0 : X_t \leq x \}$ for a constant $x$. Note that $X$ can reach or cross the barrier $x$ in two ways; with or without a jump at $\tau_x$. Let $J_0$ denote the event that the barrier is reached without a jump at $\tau_x$ and $J_1$ the event that the barrier $x$ is crossed with a jump at $\tau_x$. We want to say something about the overshoot $x - X_{\tau_x}$ in the second case, so define the events $F_0 := \{ X_{\tau_x} = x \} \cap J_0$, $F_1 := \{ X_{\tau_x} < x + y \} \cap J_1$ for some negative $y$. As mentioned above, to find solutions for the pricing of the liabilities, the only quantities that still need to be evaluated are of the form

$$u_i(x_0) = \mathbb{E} \left( e^{-\alpha_{\tau_x} + \theta X_{\tau_x}} 1_{F_i} \bigg| X_0 = x_0 \right), \ i = 0, 1,$$

(2.10)

for constants $a \geq 0$ and $\theta$.

To this end, we first need the following two lemmas, of which the first one is a modified version of a result by Kou and Wang [16].

**Lemma 2.2.** The joint distribution of $\tau_x$ and the overshoot $x - X_{\tau_x}$ satisfies, for any $y > 0$:

(i) $\mathbb{P} (\tau_x \leq t, x - X_{\tau_x} \geq y) = e^{-\eta y} \mathbb{P} (\tau_x \leq t, x - X_{\tau_x} > 0)$.

(ii) $\mathbb{P} (x - X_{\tau_x} \geq y | J_1) = e^{-\eta y}$.

(iii) $\mathbb{P} (\tau_x \leq t, x - X_{\tau_x} \geq y | J_1) = \mathbb{P} (\tau_x \leq t | J_1) \mathbb{P} (x - X_{\tau_x} \geq y | J_1)$.
Proof.

(i) Denote by $T_1, T_2, \ldots$ the arrival times of the poisson process $N$. Note that for $y > 0$, $x - X_{\tau_x} \geq y$ can only happen with a jump at $\tau_x$, hence $\tau_x$ is one of this arrival times. So we can write

$$P(\tau_x \leq t, x - X_{\tau_x} \geq y) = \sum_{n=1}^{\infty} P(T_n = \tau_x \leq t, x - X_{T_n} \geq y).$$

Now write

$$P_n := P(T_n = \tau_x \leq t, x - X_{T_n} \geq y)$$

and observe that

$$P_n = \mathbb{P}\left(\min_{0 \leq s < T_n} X_s > x, (-X_{T_n}) \geq y - x, T_n \leq t\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{(-X_{T_n} \geq y-x)} \mathbf{1}_{\left\{\min_{0 \leq s < T_n} X_s > x, T_n \leq t\right\}}|T_n, \mathcal{F}_{T_n}^{-}\right)\right) = \mathbb{E}\left(\mathbb{P}(\mathbb{1}_{(-X_{T_n} \geq y-x})|T_n, \mathcal{F}_{T_n}^{-}) \mathbf{1}_{\left\{\min_{0 \leq s < T_n} X_s > x, T_n \leq t\right\}}\right),$$

where $\mathcal{F}_{T_n}$ is defined as $\mathcal{F}_{T_n} = \sigma(\mathcal{F}_0 \cup \{A_s \cap \{s < T\} : A_s \in \mathcal{F}_s, s \geq 0\})$ for a stopping time $T$.

Furthermore note that

$$-X_{T_n} + X_0 + \mu T_n + \sigma W_{T_n} - \sum_{i=1}^{n-1} Z_i = Z_n \sim \exp(\eta),$$

from which it follows that

$$\mathbb{P}(-X_{T_n} \geq y-x|T_n, \mathcal{F}_{T_n}^{-}) = \exp\left(-\eta\left(y-x+X_0+\mu T_n+\sigma W_{T_n} - \sum_{i=1}^{n-1} Z_i\right)\right) = e^{-\eta y} \mathbb{P}(-X_{T_n} > y-x|T_n, \mathcal{F}_{T_n}^{-}).$$

Hence

$$P_n = e^{-\eta y} \mathbb{E}\left(\mathbb{P}(\mathbb{1}_{(-X_{T_n} \geq y-x})|T_n, \mathcal{F}_{T_n}^{-}) \mathbf{1}_{\left\{\min_{0 \leq s < T_n} X_s > x, T_n \leq t\right\}}\right)$$

$$= e^{-\eta y} \mathbb{P}\left(\min_{0 \leq s < T_n} X_s > x, x - X_{T_n} > 0, T_n \leq t\right)$$

$$= e^{-\eta y} \mathbb{P}(x - X_{T_n} > 0, T_n = \tau_x \leq t).$$

So we conclude that

$$\mathbb{P}(\tau_x \leq t, x - X_{\tau_x} \geq y) = \sum_{n=1}^{\infty} P_n$$

$$= \sum_{n=1}^{\infty} e^{-\eta y} \mathbb{P}(x - X_{T_n} > 0, T_n = \tau_x \leq t)$$

$$= e^{-\eta y} \mathbb{P}(\tau_x \leq t, x - X_{\tau_x} > 0),$$

which proves the first part.
(ii) First note that \( J_1 = \{ x - X_{\tau x} > 0 \} \). Furthermore, by letting \( t \to \infty \) in (i) and noting that \( \tau_x < \infty \) on the set \( J_1 \) by definition, we have

\[
P(x - X_{\tau x} \geq y) = e^{-\eta y}P(x - X_{\tau x} > 0),
\]

which implies

\[
P(x - X_{\tau x} \geq y | J_1) = \frac{P(x - X_{\tau x} \geq y)}{P(x - X_{\tau x} > 0)} = e^{-\eta y}.
\]

(iii) From (i) and (ii) it follows that

\[
P(\tau_x \leq t, x - X_{\tau x} \geq y | J_1) = \frac{P(\tau_x \leq t, x - X_{\tau x} \geq y)}{P(x - X_{\tau x} > 0)} = e^{-\eta y}P(\tau_x \leq t | J_1) = P(\tau_x \leq t | J_1)P(x - X_{\tau x} \geq y | J_1).
\]

Lemma 2.3. For any \( a > 0 \) and \( l \in i\mathbb{R} \), it holds that

\[
M_t := \exp(-at + lX_t) - \exp(lX_0) - (\xi(l) - a)\int_0^t \exp(-as + lX_s)ds
\]

is a zero-mean martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\).

Proof. First note that for \( s < t \)

\[
\mathbb{E}(M_t | \mathcal{F}_s) = M_s + \mathbb{E}\left( \exp(-at + lX_t) - \exp(-as + lX_s) - (\xi(l) - a)\int_s^t \exp(-au + lX_u) | \mathcal{F}_s \right).
\]

By definition of \( \xi \) it holds that \( \mathbb{E}e^{lX_t} = e^{\xi(l)t} \) and we know that \( X \) has independent and stationary increments, so it follows that

\[
\mathbb{E}\left( (\xi(l) - a)\int_s^t e^{-au + lX_u}du \middle| \mathcal{F}_s \right) = (\xi(l) - a)e^{lX_s} \mathbb{E}\left( \int_s^t e^{-a(u-s) + l(X_u - X_s)}du \middle| \mathcal{F}_s \right)
\]

\[
= (\xi(l) - a)e^{lX_s} \mathbb{E}\left( \int_s^t e^{-a(u-s) + l(X_u - X_s)}du \right)
\]

\[
= (\xi(l) - a)e^{lX_s} \mathbb{E}\left( \int_s^t e^{\xi(l)(u-s) - a(u-s)}du \right)
\]

\[
= e^{lX_s} \mathbb{E}e^{\xi(l)(t-s) - a(t-s) - 1}
\]

\[
= e^{lX_s} \mathbb{E}e^{lX_{t-s} - a(t-s) - 1}.
\]
Also, it holds that
\[
\mathbb{E}(\exp(-at + lX_t) - \exp(-as + lX_s)|\mathcal{F}_s) = e^{lX_s - as}\mathbb{E}\left(e^{-a(t-s) + l(X_t - X_s)} - 1|\mathcal{F}_s\right) \\
= e^{lX_s - as}\left(\mathbb{E}e^{lX_{t-s} - a(t-s)} - 1\right).
\]

Hence
\[
\mathbb{E}\left(\exp(-at + lX_t) - \exp(-as + lX_s) - (\xi(l) - a)\int_s^t \exp(-au + lX_u)|\mathcal{F}_s\right) = 0,
\]
so we conclude that \(\mathbb{E}(M_t|\mathcal{F}_s) = M_s\) and that \(M\) has zero mean. \(\square\)

Now, define a matrix \(M\) by
\[
M := \begin{pmatrix}
e^{-\gamma_1 x} & e^{-\gamma_2 x} \\
e^{-\gamma_2 x} & e^{-\gamma_2 x}
\end{pmatrix}
\]
and note that \(M\) is invertible, because the roots \(-\gamma_1, -\gamma_2\) are not equal. Recall that the goal of this section is to find explicit expressions for \(u_i(x_0), i = 0, 1\), defined by equation (2.10). The matrix \(M\) is used to compute these expressions, as stated in the following theorem.

**Theorem 2.2.** Let \(a > 0\) and consider the negative roots \(-\gamma_1, -\gamma_2\) of the equation \(\xi(s) = a\). Let \(w(x_0) := (\exp(-\gamma_1 x_0), \exp(-\gamma_2 x_0))^\top\) and define
\[
D := \begin{pmatrix}
e^{\theta x} & 0 \\
e^{\theta x} & e^{(\theta + \eta) y}
\end{pmatrix}.
\]

Then it holds that
\[
\begin{pmatrix}
u_0(x_0) \\
u_1(x_0)
\end{pmatrix} = DM^{-1}w(x_0).
\]

**Proof.** First note that
\[
\mathbb{E}(\exp(-a\tau_x + \theta X_{\tau_x})1_{F_0}|X_0 = x_0) = e^{\theta x}\mathbb{E}(\exp(-a\tau_x)1_{J_0}|X_0 = x_0). \quad (2.11)
\]

Also, by lemma 2.2, \((ii)\) and \((iii)\), we see that conditional on \(J_1\), \(x - X_{\tau_x}\) is exponentially distributed with rate parameter \(\eta\) and is independent of \(\tau_x\). This leads to
\[
\mathbb{E}(\exp(-a\tau_x + \theta X_{\tau_x})1_{F_1}|X_0 = x_0)
\]
\[
= e^{\theta x}\mathbb{E}(\exp(-a\tau_x + \theta(x - X_{\tau_x}))1_{J_1}1_{\{X_{\tau_x} < x + y\}}|X_0 = x_0)
\]
\[
= e^{\theta x}\mathbb{E}(\exp(-a\tau_x + \theta(x - X_{\tau_x}))1_{\{X_{\tau_x} < x + y\}|J_1}|X_0 = x_0)
\]
\[
= e^{\theta x}\mathbb{E}(\exp(-a\tau_x)1_{J_1}|X_0 = x_0)\mathbb{E}(\exp(-\theta(x - X_{\tau_x}))1_{\{x - X_{\tau_x} > y\}|J_1})
\]
\[
= e^{\theta x}(\exp(-a\tau_x)1_{J_1}|X_0 = x_0)\frac{\eta}{\theta + \eta}e^{(\theta + \eta) y}. \quad (2.12)
\]

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So it is sufficient to find expressions for \(v_i\), defined by

\[ v_i(x_0) := \mathbb{E}(\exp(-a\tau_x)1_{J_i} | X_0 = x_0), \quad i = 0, 1. \]

Now consider the martingale \(M\) from lemma 2.3. By the optional sampling theorem it follows that \(\mathbb{E}(M_{\tau_x} | X_0 = x_0) = 0\), that is

\[
\mathbb{E}(\exp(-a\tau_x + lX_{\tau_x}) | X_0 = x_0) = e^{lx_0} - (\xi(l) - a)\mathbb{E} \left( \int_0^{\tau_x} \exp(-as + lX_s) ds | X_0 = x_0 \right) = 0.
\tag{2.13}
\]

From setting \(y = 0\) in equation (2.12) it follows that the first term on the left hand side can be written as

\[
\mathbb{E}(\exp(-a\tau_x + lX_{\tau_x}) | X_0 = x_0) = \mathbb{E}(\exp(-a\tau_x + lX_{\tau_x})1_{J_0} | X_0 = x_0) + \mathbb{E}(\exp(-a\tau_x + lX_{\tau_x})1_{J_1} | X_0 = x_0)
\]

\[
= e^{lx_0} \mathbb{E}(\exp(-a\tau_x)1_{J_0} | X_0 = x_0) + e^{lx_0} \mathbb{E}(\exp(-a\tau_x)1_{J_1} | X_0 = x_0) \frac{\eta}{l + \eta} + (\xi(l) - a)\mathbb{E} \left( \int_0^{\tau_x} \exp(-as + lX_s) ds | X_0 = x_0 \right).
\tag{2.14}
\]

Inserting this into equation (2.13) leads to

\[
0 = e^{lx_0} \mathbb{E}(\exp(-a\tau_x)1_{J_0} | X_0 = x_0) + e^{lx_0} \mathbb{E}(\exp(-a\tau_x)1_{J_1} | X_0 = x_0) \frac{\eta}{l + \eta} - e^{lx_0} - (\xi(l) - a)\mathbb{E} \left( \int_0^{\tau_x} \exp(-as + lX_s) ds | X_0 = x_0 \right).
\tag{2.14}
\]

Now let \(h(l)\) denote the right hand side of equation (2.14), then \(h(l) = 0\), for all \(l \in i\mathbb{R}\). Define \(H(l) = (l + \eta)h(l)\), then \(H\) is well-defined and analytic in \(\mathbb{C}\) and \(H(l) = 0\) for all \(l \in i\mathbb{R}\). Then, by the identity theorem of holomorphic functions, we have that \(H(l) = 0\) for all \(l \in \mathbb{C}\), which implies that \(h(l) = 0\) for all \(l \in \mathbb{C} \setminus \{-\eta\}\). Now we can choose \(l = -\gamma_j\), \(j = 1, 2\), which gives \(\xi(l) - a = 0\). From \(h(l) = 0\) then follows

\[
e^{-\gamma_j x_0} = e^{-\gamma_j x} \mathbb{E}(\exp(-a\tau_x)1_{J_0} | X_0 = x_0) + e^{-\gamma_j x} \mathbb{E}(\exp(-a\tau_x)1_{J_1} | X_0 = x_0) \frac{\eta}{\eta - \gamma_j},
\]

\[
e^{-\gamma_j x_0} = e^{-\gamma_j x} v_0(x_0) + e^{-\gamma_j x} \frac{\eta}{\eta - \gamma_j} v_1(x_0), \quad j = 1, 2.
\]

Which is equivalent to

\[
w(x_0) = M \begin{pmatrix} v_0(x_0) \\ v_1(x_0) \end{pmatrix}.
\]

Now note that, by equations (2.11) and (2.12), it follows that

\[
\begin{pmatrix} u_0(x_0) \\ u_1(x_0) \end{pmatrix} = D \begin{pmatrix} v_0(x_0) \\ v_1(x_0) \end{pmatrix} = DM^{-1}w(x_0),
\]

which concludes the proof. \(\square\)
Note that in order to compute equation (2.8), one has to compute iterated expressions of the form
\[ E \left( e^{-a_1 \tau_{x_1} + \theta_1 X_{\tau_{x_1}}} \right) = E \left( e^{-a_2 (\tau_{x_2} - \tau_{x_1}) + \theta_2 X_{\tau_{x_2}}} \frac{\lambda}{F_{\tau_{x_1}}} \right), \]
where \( \tau_{x_1} \leq \tau_{x_2} \).

The remainder of this section shows how to modify the computations in the proof of theorem 2.2 to evaluate the \( F_{\tau_{x_1}} \)-conditional expectation. First we want to compute the conditional expectations
\[ u_i(X_{\tau_{x_1}}) := E \left( e^{-a_2 \tau_{x_2} + \theta_2 X_{\tau_{x_2}}} \frac{\lambda}{F_{\tau_{x_1}}} \right), \]

Now denote, similarly as before:
\[ D_2 := \begin{pmatrix} e^{\theta x_2} & 0 \\ 0 & e^{\theta x_2} \frac{\eta}{\eta + \theta} e^{(\theta + \eta) y} \end{pmatrix}, \]
and
\[ M_2 := \begin{pmatrix} e^{-\gamma_1^{(2)} x_2} & e^{-\gamma_1^{(2)} x_2} \frac{\eta}{\eta - \gamma_1^{(2)}} \\ e^{-\gamma_2^{(2)} x_2} & e^{-\gamma_2^{(2)} x_2} \frac{\eta}{\eta - \gamma_2^{(2)}} \end{pmatrix}, \]
where \( -\gamma_j^{(2)}, j = 1, 2 \) are the roots of the equation \( \xi(s) = a_2 \). Then in the same way as in the proof of theorem 2.2 the computation of the \( u_i \) boils down to finding expressions for
\[ u_i(X_{\tau_{x_1}}) := E \left( \exp(-a_2 \tau_{x_2}) \frac{\lambda}{F_{\tau_{x_1}}} \right), \]
where
\[ \begin{pmatrix} u_0(X_{\tau_{x_1}}) \\ u_1(X_{\tau_{x_1}}) \end{pmatrix} = D_2 \begin{pmatrix} v_0(X_{\tau_{x_1}}) \\ v_1(X_{\tau_{x_1}}) \end{pmatrix}. \]

Now adapting equation (2.13) to the \( F_{\tau_{x_1}} \)-conditional setting leads to
\[ 0 = E \left( \exp(-a_2 \tau_{x_2} + l X_{\tau_{x_2}}) \frac{\lambda}{F_{\tau_{x_1}}} \right) - \exp(-a_2 \tau_{x_1} + l X_{\tau_{x_1}}) \]
\[ - (\xi(l) - a_2) E \left( \int_{\tau_{x_1}}^{\tau_{x_2}} \exp(-as + l X_s) ds \frac{\lambda}{F_{\tau_{x_1}}} \right), \]
which implies that equation (2.14) modifies into
\[ 0 = e^{l x_2} \mathbb{E} \left( e^{-a_2 \tau_{x_2}} \frac{\lambda}{F_{\tau_{x_1}}} \right) + e^{l x_2} \mathbb{E} \left( e^{-a_2 \tau_{x_2}} \frac{\lambda}{F_{\tau_{x_1}}} \frac{\eta}{l + \eta} \right) - \exp(-a_2 \tau_{x_1} + l X_{\tau_{x_1}}) \]
\[ - (\xi(l) - a_2) E \left( \int_{\tau_{x_1}}^{\tau_{x_2}} \exp(-as + l X_s) ds \frac{\lambda}{F_{\tau_{x_1}}} \right). \]
Following the same arguments as in the proof of theorem 2.2, we have
\[ w(X_{\tau_{x_1}}) = e^{a_2 \tau_{x_1}} M_2 \begin{pmatrix} v_0(X_{\tau_{x_1}}) \\ v_1(X_{\tau_{x_1}}) \end{pmatrix}. \]
so we conclude that
\[
\begin{pmatrix}
u_0(X_{\tau x_1}) \\
u_1(X_{\tau x_1})
\end{pmatrix} = e^{-a_2\tau x_1} D_2 M_2^{-1} w(X_{\tau x_1}).
\]

Hence
\[
\mathbb{E} \left( e^{-a_2(\tau x_2 - \tau x_1) + \theta_2 X_{\tau x_2} 1_{F_0}} \bigg| F_{\tau x_1} \right) = (D_2 M_2^{-1} w(X_{\tau x_1}))_1,
\]
\[
\mathbb{E} \left( e^{-a_2(\tau x_2 - \tau x_1) + \theta_2 X_{\tau x_2} 1_{F_1}} \bigg| F_{\tau x_1} \right) = (D_2 M_2^{-1} w(X_{\tau x_1}))_2
\]
and
\[
\mathbb{E} \left( e^{-a_1\tau x_1 + \theta_1 X_{\tau x_1}} \mathbb{E} \left( e^{-a_2(\tau x_2 - \tau x_1) + \theta_2 X_{\tau x_2}} 1_{F_0} \bigg| F_{\tau x_1} \right) \right) = \mathbb{E} \left( e^{-a_1\tau x_1 + \theta_1 X_{\tau x_1}} (D_2 M_2^{-1} w(X_{\tau x_1}))_1 \right),
\]
(2.15)
\[
\mathbb{E} \left( e^{-a_1\tau x_1 + \theta_1 X_{\tau x_1}} \mathbb{E} \left( e^{-a_2(\tau x_2 - \tau x_1) + \theta_2 X_{\tau x_2}} 1_{F_1} \bigg| F_{\tau x_1} \right) \right) = \mathbb{E} \left( e^{-a_1\tau x_1 + \theta_1 X_{\tau x_1}} (D_2 M_2^{-1} w(X_{\tau x_1}))_2 \right).
\]
(2.16)

Now note that \((D_2 M_2^{-1} w(X_{\tau x_1}))_i, i = 1, 2,\) are linear combinations of the terms \(\exp(-\gamma_1^{(2)} X_{\tau x_1}),\) \(\exp(-\gamma_2^{(2)} X_{\tau x_1})\) such that the expectations in equations (2.15) and (2.16) are solved by applying theorem 2.2.
3 A model with imperfect accounting information

In this chapter a structural model is considered in which debt is valued under the assumption that the only information available is noisy accounting information which is received at selected times. The setting is that of Duffie and Lando [10]. The first four sections are used to describe the model proposed by Duffie and Lando and to provide all the formulas and proofs that are left out in the original paper. In the last two sections the setting is applied to the valuation of different forms of Contingent Convertible bonds.

3.1 Description of the model

The value of assets of the firm, denoted by \( V_t \), is modeled by a geometric Brownian motion, that is
\[
\frac{dV_t}{V_t} = \mu dt + \sigma dW_t.
\]
Define \( Z_t = \log V_t \) and \( m = \mu - \sigma^2/2 \), then we can write
\[
Z_t = Z_0 + mt + \sigma W_t.
\]

The firm issues straight debt with a total value \( P_1 \). The straight debt has a perpetual maturity and pays coupons continuously at rate \( c_1 \). Furthermore it is assumed that the risk free interest rate \( r \) is constant. As before, default occurs the first time the value of assets falls below some trigger \( v_b \), which means that the firm defaults at \( \tau_b \), defined by
\[
\tau_b = \inf \{ s \geq 0 : V_s \leq v_b \}.
\]

As mentioned before, the bond investors do not have all the information about the asset value, instead they receive imperfect accounting information at times \( t_1 < t_2 < \ldots \) (typically every three months). At every observation date there is an imperfect accounting report of the asset value available, denoted by \( \hat{V}_t \), where \( \log \hat{V}_t \) and \( \log V_t \) are assumed to be joint normal. This means that we can write
\[
Y_t := \log \hat{V}_t = Z_t + U_t,
\]
where \( U_t \) is normally distributed and independent of \( Z_t \). The information available to bond investors is now described by the filtration \( \mathcal{H}_t \), where
\[
\mathcal{H}_t = \sigma ( \{ Y_{t_1}, \ldots, Y_{t_n}, 1_{\tau_b \leq s} : s \leq t \} ),
\]
Lemma 3.1. The probability \( Z \) find an expression for the conditional distribution of \( V \) observed whether the firm is liquidated at \( t \), where \( \phi \) for the probability \( \psi \) conditional density of \( Z \). Also we state

In this section \( t \) is a fixed time at which the only noisy accounting value \( Y_t \) is observed, also we stave \( Z_0 = z_0 \), for some \( z_0 \in \mathbb{R} \). The goal is to compute \( g_t(\cdot|Y_t, \tau_b > t) \), the conditional density of \( Z_t \) given \( Y_t \) and \( \tau_b > t \). To this end, we first need an expression for the probability \( \psi(z_0, x, \sigma \sqrt{t}) \) that \( \min \{ Z_s : s \leq t \} > 0 \), conditional on \( Z_0 = z_0 > 0 \) and \( Z_t = x > 0 \). This expression is stated in the following lemma.

**Lemma 3.1.** The probability \( \psi(z_0, x, \sigma \sqrt{t}) \) that \( \min \{ Z_s : s \leq t \} > 0 \), conditional on \( Z_0 = z_0 > 0 \) and \( Z_t = x > 0 \), is given by

\[
\psi(z_0, x, \sigma \sqrt{t}) = 1 - \exp\left( -\frac{2z_0x}{\sigma^2t} \right).
\]

**Proof.** To prove this Lemma, we will rely on the following result by Harisson [14, Chapter 1.8]. Denote by \( X_t \) a Brownian motion with drift \( \mu \), variance \( \sigma^2 \) and \( X_0 = 0 \). Furthermore define \( M_t := \max \{ X_s : 0 \leq s \leq t \} \). Then the joint distribution of \( X_t \) and \( M_t \) satisfies

\[
P(X_t \in dx, M_t \leq y) = \frac{1}{\sigma \sqrt{t}} \exp\left( \frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2} \right) \left( \phi\left( \frac{x}{\sigma \sqrt{t}} \right) - \phi\left( \frac{x - 2y}{\sigma \sqrt{t}} \right) \right) dx,
\]

where \( \phi \) denotes the standard normal density function. Now define \( X_t = -Z_t + z_0 \), then \( X_t = -mt - \sigma W_t \), which is a Brownian motion with drift \( -m \), variance \( \sigma^2 \) and \( X_0 = 0 \). Denote by \( f_{X_t} \) its density, which is normal with mean \( -mt \) and variance \( \sigma^2 t \). Also it holds that

\[
\min \{ Z_s : 0 \leq s \leq t \} > 0 \iff M_t = \max \{ X_s : 0 \leq s \leq t \} < z_0.
\]

Then by Bayes’ rule and equation (3.1) it follows that

\[
\psi(z_0, x, \sigma \sqrt{t}) = P(\min \{ Z_s : 0 \leq s \leq t \} > 0 | Z_t = x)
\]

\[
= P(M_t < z_0 | X_t = z_0 - x)
\]

\[
= \frac{\int_{(z_0 - x)} f_{X_t}(z_0 - x) dx}{\int_{(z_0 - x)} f_{X_t}(z_0 - x) dx}
\]

\[
= \exp\left( -\frac{m(z_0 - x)}{\sigma^2} - \frac{m^2 t}{2\sigma^2} \right) \left( \exp\left( -\frac{(z_0 - x)^2}{2\sigma^2 t} \right) - \exp\left( -\frac{(z_0 + x)^2}{2\sigma^2 t} \right) \right)
\]

\[
= 1 - \exp\left( -\frac{(z_0 + x)^2}{2\sigma^2 t} + \frac{(z_0 - x)^2}{2\sigma^2 t} \right)
\]

3.2 Conditioning on one noisy accounting report

In this section \( t \) is a fixed time at which the only noisy accounting value \( Y_t \) is observed, also we stave \( Z_0 = z_0 \), for some \( z_0 \in \mathbb{R} \). The goal is to compute \( g_t(\cdot|Y_t, \tau_b > t) \), the conditional density of \( Z_t \) given \( Y_t \) and \( \tau_b > t \). To this end, we first need an expression for the probability \( \psi(z_0, x, \sigma \sqrt{t}) \) that \( \min \{ Z_s : s \leq t \} > 0 \), conditional on \( Z_0 = z_0 > 0 \) and \( Z_t = x > 0 \). This expression is stated in the following lemma.
= 1 - \exp \left( -\frac{2\alpha x}{\sigma^2 t} \right) .

Of course we can write the stopping time \( \tau_b \) as

\[ \tau_b = \inf \{ s \geq 0 : Z_s \leq z_b \}, \]

for \( z_b = \log v_b \). Next we can compute the density \( b(\cdot|Y_t) \) of \( Z_t \), for \( \tau_b > t \), conditional on \( Y_t \). That is, \( b(\cdot|Y_t) \) satisfies

\[ b(x|Y_t) \, dx = P(\tau_b > t, Z_t \in dx|Y_t), \text{ for } x \geq z_b. \]

Recall that \( Y_t = U_t + Z_t \) and that \( Z_t \) and \( U_t \) are independent. Furthermore, it holds that

\[ \tau_b > t \iff \min \{ Z_s : 0 \leq s \leq t \} > z_b. \]

So by Bayes’ rule it follows that

\[ P(\tau_b > t, Z_t \in dx|Y_t) = P(\tau_b > t|Y_t) P(Z_t \in dx|Y_t) \]

\[ = \psi(z_0 - z_b, x - z_b, \sigma \sqrt{t}) f_{Z_t}(x|Y_t) \, dx \]

\[ = \psi(z_0 - z_b, x - z_b, \sigma \sqrt{t}) \frac{f_{Y_t}(Y_t|Z_t = x) f_{Z_t}(x)}{f_{Y_t}(Y_t)} \, dx \]

\[ = \psi(z_0 - z_b, x - z_b, \sigma \sqrt{t}) \frac{f_{U_t}(Y_t - x) f_{Z_t}(x)}{f_{Y_t}(Y_t)} \, dx, \]

which is equivalent to writing

\[ b(x|Y_t) = \frac{\psi(z_0 - z_b, x - z_b, \sigma \sqrt{t}) f_{U_t}(Y_t - x) f_{Z_t}(x)}{f_{Y_t}(Y_t)}, \tag{3.2} \]

where \( f_{U_t}, f_{Z_t} \) and \( f_{Y_t} \) denote the densities of \( U_t, Z_t \) and \( Y_t \), respectively. These are all normal, with respective means \( u_t = \mathbb{E} U_t, m_t + z_0 \) and \( m_t + z_0 + u_t \), and respective variances \( \sigma^2 = \text{Var}(U_t), \sigma^2 t \) and \( \alpha^2 + \sigma^2 t \). Note that the standard deviation \( \alpha \) of \( U_t \) determines how noisy the accounting reports are.

Now we can move forward to the main result of this subsection, an expression for the conditional density \( g(\cdot|Y_t, \tau_b > t) \) of \( Z_t \), given \( Y_t \) and \( \tau_b > t \).

**Theorem 3.1.** The conditional density \( g_t(\cdot|Y_t, \tau_b > t) \) of \( Z_t \), given \( Y_t \) and \( \tau_b > t \), is given by

\[ g_t(x|y, \tau_b > t) = \frac{\sqrt{\frac{\beta_2}{\beta_3}} e^{-J(y, \hat{y}, \hat{z}_0)} (1 - \exp \left( -\frac{2\hat{z}_0 \hat{x}}{\sigma^2 t} \right))}{\exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( \frac{\beta_1}{\sqrt{2\beta_0}} \right) - \exp \left( \frac{\beta_2^2}{4\beta_0} - \beta_3 \right) \Phi \left( -\frac{\beta_2}{\sqrt{2\beta_0}} \right)}, \tag{3.3} \]
where \( \tilde{y} = y - z_b - u_t, \tilde{x} = x - z_b, \tilde{z}_0 = z_0 - z_b, \) \( \Phi \) denotes the standard normal distribution function,

\[
J(\tilde{y}, \tilde{x}, \tilde{z}_0) = \frac{(\tilde{y} - \tilde{x})^2}{2a^2} + \frac{(\tilde{z}_0 + mt - \tilde{x})^2}{2\sigma^2 t},
\]

and

\[
\begin{align*}
\beta_0 &= \frac{a^2 + \sigma^2 t}{2a^2 \sigma^2 t}, \\
\beta_1 &= \frac{\tilde{y}}{a^2} + \frac{\tilde{z}_0 + mt}{\sigma^2 t}, \\
\beta_2 &= -\beta_1 + 2\frac{\tilde{z}_0}{\sigma^2 t}, \\
\beta_3 &= \frac{1}{2} \left( \frac{\tilde{y}}{a^2} + \frac{(\tilde{z}_0 + mt)^2}{\sigma^2 t} \right).
\end{align*}
\]

**Proof.** First note that

\[
\mathbb{P}(\tau_b > t|Y_t) = \int_{z_b}^{\infty} b(z|Y_t) \, dz,
\]

and recall that

\[
b(x|Y_t)dx = \mathbb{P}(\tau_b > t, Z_t \in dx|Y_t).
\]

Using Bayes’ rule and equation (3.2), we can compute

\[
g_t(x|y, \tau_b > t) = \frac{b(x|Y_t = y)}{\int_{z_b}^{\infty} b(z|Y_t = y) \, dz} = \frac{\psi(\tilde{z}_0, \tilde{x}, \sigma \sqrt{t}) f_{U_t}(y-x) f_{Z_t}(x)}{\int_{z_b}^{\infty} \psi(\tilde{z}_0, z - z_b, \sigma \sqrt{t}) f_{U_t}(y-z) f_{Z_t}(z) \, dz}, \tag{3.4}
\]

where the numerator is given by

\[
\psi(\tilde{z}_0, \tilde{x}, \sigma \sqrt{t}) f_{U_t}(y-x) f_{Z_t}(x) = \frac{1 - \exp \left( -\frac{2\tilde{z}_0 \tilde{x}}{\sigma^2 t} \right)}{\sqrt{2\pi a^2 2\pi \sigma^2 t}} \exp \left( -\frac{(y - x - u_t)^2}{2a^2} - \frac{2(z - m t - z_0)^2}{2\sigma^2 t} \right) \tag{3.5}
\]

Furthermore, the denominator of equation (3.4) can be written as

\[
\int_{z_b}^{\infty} \frac{1 - \exp \left( -\frac{2\tilde{z}_0 (z - z_b)}{\sigma^2 t} \right)}{\sqrt{2\pi a^2 2\pi \sigma^2 t}} \exp \left( -\frac{(y - z - u_t)^2}{2a^2} - \frac{2(z - m t - z_0)^2}{2\sigma^2 t} \right) \, dz = (I) - (II),
\]

where

\[
(I) = \frac{1}{\sqrt{2\pi a^2 2\pi \sigma^2 t}} \int_{z_b}^{\infty} \exp \left( -\frac{(y - z - u_t)^2}{2a^2} - \frac{2(z - m t - z_0)^2}{2\sigma^2 t} \right) \, dz
\]
\[
\frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \int_0^\infty \exp \left( -\frac{(\bar{y} - z)^2}{2a^2} - \frac{(\bar{z}_0 + mt - z)^2}{2\sigma^2 t} \right) dz \\
= \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \int_0^\infty \exp \left( -\frac{(\beta_0 z^2 - \beta_1 z + \beta_3)}{\beta_0} \right) dz \\
= \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\
= \frac{1}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( \frac{\beta_1}{\sqrt{2\beta_0}} \right),
\]

and where
\[
(II) = \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \int_0^\infty \exp \left( -\frac{2\bar{z}_0 (z - \bar{z}_0)}{\sigma^2 t} - \frac{(y - z - u_t)^2}{2a^2} - \frac{(z - mt - \bar{z}_0)^2}{2\sigma^2 t} \right) dz \\
= \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \int_0^\infty \exp \left( -\frac{2\bar{z}_0 z}{\sigma^2 t} - \frac{(\bar{y} - z)^2}{2a^2} - \frac{(\bar{z}_0 + mt - \bar{z}_0)^2}{2\sigma^2 t} \right) dz \\
= \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \exp \left( -\frac{2\bar{z}_0 z}{\sigma^2 t} + \frac{(\bar{y} - z)^2}{2a^2} - \frac{(\bar{z}_0 + mt - \bar{z}_0)^2}{2\sigma^2 t} \right) dz \\
= \frac{1}{\sqrt{2\pi a^2 \sigma^2 t}} \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\
= \frac{1}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( -\frac{\beta_2}{\sqrt{2\beta_0}} \right).
\]

Hence the denominator of equation (3.4) is equal to
\[
\frac{1}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \left( \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( \frac{\beta_1}{\sqrt{2\beta_0}} \right) - \exp \left( \frac{\beta_2^2}{4\beta_0} - \beta_3 \right) \Phi \left( -\frac{\beta_2}{\sqrt{2\beta_0}} \right) \right). 
\]

(3.6)

Now it follows from equations (3.4), (3.5) and (3.6) that
\[
g_t(x | y, \tau_b > t) = \frac{1}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \exp \left( \frac{x^2}{2\sigma^2 t} \right) \left( 1 - \exp \left( -\frac{2\bar{z}_0 \beta_1}{\sigma^2 \beta_0} \right) \right) \\
= \frac{1}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \left( \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( \frac{\beta_1}{\sqrt{2\beta_0}} \right) - \exp \left( \frac{\beta_2^2}{4\beta_0} - \beta_3 \right) \Phi \left( -\frac{\beta_2}{\sqrt{2\beta_0}} \right) \right) \\
= \frac{\sqrt{\frac{2\sigma_0}{\pi}} \exp \left( \frac{x^2}{\beta_0} \right)}{\sqrt{2\pi (a^2 + \sigma^2 t)}} \left( 1 - \exp \left( -\frac{2\bar{z}_0 \beta_1}{\sigma^2 \beta_0} \right) \right)
\]

\[
= \exp \left( \frac{\beta_1^2}{4\beta_0} - \beta_3 \right) \Phi \left( \frac{\beta_1}{\sqrt{2\beta_0}} \right) - \exp \left( \frac{\beta_2^2}{4\beta_0} - \beta_3 \right) \Phi \left( -\frac{\beta_2}{\sqrt{2\beta_0}} \right),
\]

which concludes the proof.
3.3 Survival probability and default intensity

3.3.1 Survival probability

Denote by \( p(t, s) \) the \( \mathcal{H}_t \)-conditional probability of survival until time \( s > t \), that is
\[
p(t, s) = \mathbb{P}(\tau_b > s | \mathcal{H}_t), \text{ for } s > t.
\]

To obtain an expression for the survival probability, first consider the probability \( \pi(t, x) \) that \( Z \) hits 0 before time \( t \), starting from \( x > 0 \). This probability is given by the following lemma.

**Lemma 3.2.** The probability \( \pi(t, x) \) that \( Z \) hits 0 before time \( t \), starting from \( x > 0 \), is given by
\[
\pi(t, x) = 1 - \Phi \left( \frac{x + mt}{\sigma \sqrt{t}} \right) + e^{-2mx/\sigma^2} \Phi \left( \frac{-x + mt}{\sigma \sqrt{t}} \right).
\]

**Proof.** To prove this, we will rely on the following result by Harrison [14, Chapter 1.8]. For a Brownian motion \( X \) with drift \( \mu \) and variance \( \sigma^2 \), denote \( T_x(t) = \inf \{ t \geq 0 : X_t = x \} \). Then the probability that \( X \) did not hit \( x > 0 \), starting from 0, before time \( t \) is given by
\[
\mathbb{P}(T_x(t) > t) = \Phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) - e^{2\mu x/\sigma^2} \Phi \left( \frac{-x - \mu t}{\sigma \sqrt{t}} \right).
\]

Now the expression for \( \pi(t, x) \) follows directly from this result by noting that the probability of hitting 0 before time \( t \), starting from \( x > 0 \), with drift \( m \), is equal to the probability of hitting \( x > 0 \) before time \( t \), starting from 0, with drift \( -m \).

Stationarity of \( Z \) now implies that the \( \mathcal{H}_t \)-conditional survival probability \( p(t, s) \) for time \( t < \tau_b \), can be written as
\[
p(t, s) = \int_{z_b}^{\infty} (1 - \pi(s - t, x - z_b)) g_t(x | Y_t, \tau_b > t) dx.
\]

3.3.2 Default intensity

One of the advantages of the current setting, is the fact that it is compatible with a reduced form approach. That is, it is possible to define a stochastic intensity for default. This is possible because \( \tau_b \) is a totally inaccessible \( \mathcal{H}_t \)-stopping time, which means that for any sequence of \( \mathcal{H}_t \)-stopping times dominated by \( \tau_b \), the probability that the sequence approaches \( \tau_b \), is zero. First consider the following definition.

**Definition 3.1.** A progressively measurable process \( \lambda = (\lambda_t)_{t \geq 0} \), is called an intensity process for a stopping time \( \tau \), with respect to a filtration \( (\mathcal{G}_t)_{t \geq 0} \), if it satisfies \( \int_0^t \lambda_s ds < \infty \) a.s. for all \( t \geq 0 \) and \( \{ 1_{\{ \tau \leq t \}} - \int_0^t \lambda_s ds \} \) is a \( \mathcal{G}_t \)-martingale.

The intuitive meaning of such an intensity process is what one would expect from an intensity, that is
\[
\mathbb{P}(\tau \in (t, t + dt) | \mathcal{G}_t) = \lambda_t dt.
\]
This intuitive meaning is made rigorous in the following result, due to Aven [2].
Lemma 3.3. Let $\tau$ be a stopping time with respect to a filtration $(\mathcal{G}_t)_{t \geq 0}$. Define

$$Y_n(s) = \frac{1}{h_n} \mathbb{P}(\tau \leq s + h_n | \mathcal{G}_s) 1_{\tau > s},$$

where $(h_n)_{n \in \mathbb{N}}$ is a sequence decreasing to 0. For $(\lambda_t)_{t \geq 0}$ and $(\gamma_t)_{t \geq 0}$ nonnegative measurable processes, assume that

(i) For all $t \geq 0$, $\lim_{n \to \infty} Y_n(t) = \lambda_t$ a.s.

(ii) For all $t \geq 0$, for almost all $\omega$, there exists an $n_0 = n_0(t, \omega)$ such that for all $s \leq t$, $n \geq n_0$ it holds that

$$|Y_n(s, \omega) - \lambda_s(\omega)| \leq \gamma_s(\omega).$$

(iii) $\int_0^t \gamma_s ds < \infty$ a.s., for all $t \geq 0$.

Then it follows that $\{1_{\{\tau \leq t\}} - \int_0^t \lambda_s ds\}$ is a $\mathcal{G}_t$-martingale, i.e. the intensity process of $\tau$ with respect to $(\mathcal{G}_t)_{t \geq 0}$ is given by

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(\tau \leq t + h | \mathcal{G}_t) 1_{\{\tau > t\}}.$$

From the results in the previous section it follows that, for every pair $(\omega, t)$ such that $\tau_b(\omega) > t$, the $\mathcal{H}_t$-conditional distribution of $Z_t$ admits a continuously differentiable conditional density $f(t, \cdot, \omega)$, which is zero in $z_b$ and therefore has a derivative $f_x(t, x, \omega) := \frac{\partial}{\partial x} f(t, x, \omega)$ which is positive at $z_b$. This can be seen as follows. For a time $t$ before the first accounting report at time $t_1$, assuming $t < \tau_b$, the density of $Z_t$ can be written down explicitly and does not depend on $\omega$. As a completion to [10], we will now derive an explicit expression for this density. Denote this density by $\tilde{f}(t, \cdot, z_0)$, then it needs to satisfy

$$\mathbb{P}(Z_t \in dx | \tau_b > t) = \tilde{f}(t, x, z_0) dx.$$

By Bayes’ rule we can write

$$\mathbb{P}(Z_t \in dx | \tau_b > t) = \frac{\mathbb{P}(Z_t \in dx, \tau_b > t)}{\mathbb{P}(\tau_b > t)}.$$

The denominator of this expression is given by

$$\mathbb{P}(\tau_b > t) = 1 - \pi(t, z_0 - z_b) = \Phi \left( \frac{z_0 - z_b + mt}{\sigma \sqrt{t}} \right) - e^{-2m(z_0 - z_b)/\sigma^2} \Phi \left( \frac{z_b - z_0 + mt}{\sigma \sqrt{t}} \right)$$

and the numerator can be computed using the same method as in the proof of Lemma 3.1. That is, denote $X_t = -Z_t + z_0$, which is a Brownian motion with drift $-m$, variance $\sigma^2$ and $X_0 = 0$. Furthermore, denote $M_t = \max\{X_s : 0 \leq s \leq t\}$. Then equation (3.1) implies that

$$\mathbb{P}(Z_t \in dx, \tau_b > t) = \mathbb{P}\left( Z_t \in dx, \inf_{0 \leq s \leq t} Z_s > z_b \right).$$
Theorem 3.2. Define a process $H$ the noisy accounting reports filtration (of this, we can define an intensity process for the default stopping time, with respect to $\tau$ be defined for $t > 0$ and that the integrability conditions are met. Note that the intensity process can only have to show that on the event $\{\tau = t\}$ the density of $x$ $t$ive $\tilde{\sigma}$ can then be assumed that $Z$ then the density of $x$ $t$ density at the time $\tilde{\sigma}$ satisfies $\tilde{\sigma}$ which implies $\tilde{\sigma}$ So it follows that the $f$-conditional density $\tilde{f}$ with respect to $x$ and has a derivative with respect to $x$, uniformly bounded on $[t, T]$ for all $t > 0$. Because of this, we can define an intensity process for the default stopping time, with respect to the noisy accounting reports filtration $(H_t)_{t \geq 0}$.

**Theorem 3.2.** Define a process $\lambda$ by

$$
\lambda_t(\omega) = \frac{1}{2} \sigma^2 f_x(t, x, \omega) 1_{\{\tau_0 > t\}}(\omega), \text{ for } t > 0.
$$

Then $\lambda$ is an intensity process of $\tau_0$ with respect to $(H_t)_{t \geq 0}$.

**Proof.** Without loss of generality we can assume that $z_b = 0$, for which we will denote $\tau_0 = \inf\{t \geq 0 : Z_t = 0\}$. To prove the result, we will rely on lemma 3.3. That is, we have to show that on the event $\{\tau_0 > t\}$ it holds that

$$
\lim_{h \to 0} \frac{1}{h} \mathbb{P}(\tau_0(\omega) \leq t + h | H_t) = \frac{1}{2} \sigma^2 f_x(t, 0, \omega)
$$

and that the integrability conditions are met. Note that the intensity process can only be defined for $t > 0$, because at $t = 0$ we have perfect accounting information, which implies that $\tau_0$ is not totally inaccessible. So we can only prove the existence of an

$$
\mathbb{P}(X_t \in d(z_0 - x), M_t \leq z_0 - z_b) = \frac{1}{\sigma \sqrt{t}} \exp\left(\frac{-m(z_0 - x)}{\sigma^2} - \frac{m^2 t}{2 \sigma^2}\right) \left(\phi\left(\frac{z_0 - x}{\sigma \sqrt{t}}\right) - \phi\left(\frac{-z_0 - x + 2z_b}{\sigma \sqrt{t}}\right)\right) dx.
$$

(3.8)

So we conclude that

$$
\tilde{f}(t, x, z_b) = \frac{1}{\sigma \sqrt{t}} \exp\left(\frac{-m(z_0 - x)}{\sigma^2} - \frac{m^2 t}{2 \sigma^2}\right) \left(\phi\left(\frac{z_0 - x}{\sigma \sqrt{t}}\right) - \phi\left(\frac{-z_0 - x + 2z_b}{\sigma \sqrt{t}}\right)\right) \cdot \Phi\left(\frac{z_0 - x + mt}{\sigma \sqrt{t}}\right) - e^{-2m(z_0 - z_b)/\sigma^2} \Phi\left(\frac{z_0 - x + mt}{\sigma \sqrt{t}}\right).
$$

(3.9)

This density satisfies $\tilde{f}(t, z_b, 0) = 0$ and is differentiable with respect to $x$, with derivative $\tilde{f}_x(t, \cdot, z_b)$ that is bounded uniformly on $[t, t_1]$ for all $t > 0$. The $H_t$-conditional density at the time $t_1$ of the first accounting report is given by theorem 3.1, denoted by $g_1(x|Y_{t_1}, \tau_0 > t_1)$. This density $g_1(x|Y_{t_1}(\omega), \tau_0 > t_1)$ equals zero at $x = z_b$ and has a bounded derivative with respect to $x$, for all $\omega$. Now let $s > 0$ such that $t_1 < t_1 + s < T$, then the density of $Z_{t_1+}$ is given by

$$
f(t_1 + s, x, \omega) = \int_{z_b}^{\infty} \tilde{f}(s, x, u) g_1(u|Y_{t_1}(\omega), \tau_0(\omega) > t_1) du,
$$

which implies

$$
f_x(t_1 + s, x, \omega) = \int_{z_b}^{\infty} \tilde{f}_x(s, x, u) g_1(u|Y_{t_1}(\omega), \tau_0(\omega) > t_1) du.
$$

So it follows that the $H_t$-conditional density $f(t, x, \omega)$ of $Z_t$ also equals zero at $x = z_b$ and has a derivative with respect to $x$, uniformly bounded on $[t, T]$, for all $t > 0$. Because of this, we can define an intensity process for the default stopping time, with respect to the noisy accounting reports filtration $(H_t)_{t \geq 0}$.
intensity process for \( t > 0 \), this will be done by proving existence on compact intervals \([t, T]\), for all \( t > 0\).

Define \( Y_n(t) = \frac{1}{h_n} \mathbb{P}(\tau_0(t) \leq t + h_n | H_t) \mathbbm{1}_{\{\tau_0 > t\}}(\omega) \), for a sequence \( (h_n)_{n \in \mathbb{N}} \) such that \( h_n \downarrow 0 \) as \( n \to \infty \). Recall that by \( \pi(t, x) \) we denote the probability that \( Z \) hits 0 before time \( t \), starting from \( x > 0 \). So we can write

\[
\lim_{n \to \infty} Y_n(t, \omega) = \lim_{n \to \infty} \frac{1}{h_n} \mathbb{P}(\tau_0(\omega) \leq t + h_n | H_t) \mathbbm{1}_{\{\tau_0 > t\}}(\omega)
= \lim_{n \to \infty} \frac{1}{h_n} \int_{(0,\infty)} \pi(h_n, x) f(t, x, \omega) dx \mathbbm{1}_{\{\tau_0 > t\}}(\omega)
= \lim_{n \to \infty} \int_{(0,\infty)} \pi(h_n, \sqrt{h_n} z, \omega) f(t, \sqrt{h_n} z, \omega) \sigma^2 z dz \mathbbm{1}_{\{\tau_0 > t\}}(\omega),
\]

where the last equation follows by the substitution \( z = \frac{x}{\sigma \sqrt{h_n}} \).

Now denote

\[
G_1(z, h) = \pi(h, \sqrt{h} z), \quad G_2(z, h, \omega) = \frac{f(t, \sqrt{h} z, \omega)}{\sigma \sqrt{h} z} \sigma^2.
\]

and note that by lemma 3.2 we have

\[
G_1(z, h) = 1 - \Phi \left( z + \frac{m \sqrt{h}}{\sigma} \right) + e^{-2m \sqrt{h} z / \sigma} \Phi \left( -z + \frac{m \sqrt{h}}{\sigma} \right).
\]

Furthermore, because \( f(t, 0, \omega) = 0 \), it holds that

\[
\lim_{h \downarrow 0} G_2(z, h, \omega) = f_x(t, 0, \omega) \sigma^2.
\]

Then, assuming that the dominated convergence theorem can be applied, it follows that

\[
\lim_{n \to \infty} Y_n(t, \omega) = \lim_{n \to \infty} \int_{(0,\infty)} G_1(z, h_n) G_2(z, h_n, \omega) z dz \mathbbm{1}_{\{\tau_0 > t\}}(\omega)
= \int_{(0,\infty)} (1 - \Phi(z) + \Phi(-z)) f_x(t, 0, \omega) \sigma^2 z dz \mathbbm{1}_{\{\tau_0 > t\}}(\omega)
= \sigma^2 f_x(t, 0, \omega) \int_{(0,\infty)} (1 - \Phi(z) + \Phi(-z)) z dz \mathbbm{1}_{\{\tau_0 > t\}}(\omega).
\]

Since

\[
\int_{(0,\infty)} (1 - \Phi(z) + \Phi(-z)) z dz = \int_{(0,\infty)} 2\Phi(-z) z dz
= [\Phi(-z)^2]_0^\infty + \int_{(0,\infty)} \phi(-z) z^2 dz
= 0 + \frac{1}{2} \int_{(-\infty,\infty)} \phi(z) z^2 dz
= \frac{1}{2}.
\]

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we conclude that
\[
\lim_{n \to \infty} Y_n(t, \omega) = \frac{1}{2} \sigma^2 f_x(t, 0, \omega) 1_{\{\tau_0 > t\}}(\omega) = \lambda_t(\omega).
\]

The justification of the use of dominated convergence is as follows. Because \( f_x(t, 0, \omega) \) is bounded, there exists a constant \( C > 0 \) such that
\[
\left| \frac{f(t, \sigma \sqrt{h}, \omega) \sigma \sqrt{h}}{\sigma \sqrt{h}} \right| \leq C.
\]
Also, for \( h < 1 \) it holds that
\[
|G_1(z, h)| \leq G(z) := 1 - \Phi\left( z - \frac{|m|}{\sigma} \right) + e^{2|m|z/\sigma} \Phi(-z).
\]
So it follows that for all \( h < 1 \)
\[
|G_1(z, h)G_2(z, h)| < C\sigma^2 zG(z).
\]
Now note that \( 1 - \Phi(x) \) behaves like \( \phi(x)/x \) as \( x \to \infty \), which implies that \( G(z) \to 0 \) exponentially fast as \( z \to \infty \), such that \( C\sigma^2 zG(z) \) provides an integrable upper bound, which justifies the application of the dominated convergence theorem.

It now suffices to check the conditions of lemma 3.3, to conclude the proof. Because \( f_x(t, 0, \omega) \) is uniformly bounded on \([t, T]\) it follows that
\[
\int_t^T |\lambda_s(\omega)| ds < \infty
\]
and the above discussion implies that
\[
\int_t^T |Y_n(s, \omega)| ds < \infty,
\]
for almost all \( \omega \) and all \( n \) such that \( h_n < 1 \). So, if we choose \( n_0 \) such that \( h_n < 1 \) whenever \( n > n_0 \), it follows that for all \( n > n_0 \)
\[
|Y_n(s, \omega) - \lambda_s(\omega)| \leq |Y_n(s, \omega)| + |\lambda_s(\omega)|, \text{ where } \int_t^T |Y_n(s, \omega)| + |\lambda_s(\omega)| ds < \infty,
\]
which shows that the conditions of lemma 3.3 are met. Hence \( \lambda_t \) defines an intensity process for \( \tau_0 \).

\[\square\]

3.4 Conditioning on several noisy accounting reports

The results in the previous sections hold in a setting in which only one single noisy accounting value \( Y_t \) is observed at time \( t \). In this section, this is extended to a setting
in which noisy accounting reports \( Y_i := Y_{t_i} \) arrive at successive dates \( t_1 < t_2 < \ldots \), typically every three months in practice. Of course, it is reasonable that there exists some correlation between the accounting noise \( U_1, U_2, \ldots \). To be more specific, following Duffie and Lando [10], it is assumed that \( Y_i = Z_i + U_i \), where
\[
U_i = \kappa U_{i-1} + \epsilon_i,
\]
for some fixed \( \kappa \in \mathbb{R} \) and independent and identically distributed \( \epsilon_1, \epsilon_2, \ldots \), which have a normal distribution with mean \( \mu_\epsilon \in \mathbb{R} \) and variance \( \sigma_\epsilon^2 > 0 \), and are independent of \( Z \). Consider the following notation for the relevant random vectors and their realisations:
\[
Z^{(n)} = (Z_1, Z_2, \ldots, Z_n) \text{ and its realisation } z^{(n)} = (z_1, z_2, \ldots, z_n),
\]
\[
Y^{(n)} = (Y_1, Y_2, \ldots, Y_n) \text{ and its realisation } y^{(n)} = (y_1, y_2, \ldots, y_n),
\]
\[
U^{(n)} = Y^{(n)} - Z^{(n)} \text{ and its realisation } u^{(n)} = y^{(n)} - z^{(n)}.
\]
Consistent with the previous notation, denote by \( b_n(\cdot|Y^{(n)}) \) the conditional density of \( Z^{(n)} \) for \( \tau_b > t_n \), conditional on \( Y^{(n)} \), that is
\[
b_n(z^{(n)}|Y^{(n)})dz^{(n)} = P(Z^{(n)} \in dz^{(n)}, \tau_b > t_n|Y^{(n)}).
\]
Note that \( (Z_n)_{n \in \mathbb{N}} \) and \( (U_n)_{n \in \mathbb{N}} \) are Markov processes and denote by \( p_Z(z_n|z_{n-1}) \) and \( p_U(u_n|u_{n-1}) \) their respective transition densities for realisations \( z^{(n)}, u^{(n)} \). Furthermore, denote by \( p_Y(y_n|y^{(n-1)}) \) the conditional density of \( Y_n \) given \( Y^{(n-1)} = y^{(n-1)} \).
By Bayes’ rule it follows that
\[
b_n(z^{(n)}|y^{(n)})dz^{(n)} = \frac{P(\tau_b > t_n, Z^{(n)} \in dz^{(n)}, Y^{(n)} \in dy^{(n)})}{P(Y^{(n)} \in dy^{(n)})}.
\]
Now note that
\[
\{\tau_b > t_n, Z^{(n)} \in dz^{(n)}, Y^{(n)} \in dy^{(n)}\} = A \cap B,
\]
where
\[
A = \{\tau_b > t_n, Z_n \in dz_n, Y_n \in dy_n\},
\]
\[
B = \{\tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)}\}.
\]
This implies that
\[
b_n(z^{(n)}|y^{(n)})dz^{(n)} = \frac{P(A|B)P(B)}{p_Y(y_n|y^{(n-1)})dy_nP(Y^{(n-1)} \in dy^{(n-1)})} = \frac{P(A|B)P(Z^{(n-1)}|y^{(n-1)})}{p_Y(y_n|y^{(n-1)})dy_n}dz^{(n-1)},
\]
where
\[
P(A|B) = P(\tau_b > t_n, Z_n \in dz_n, Y_n \in dy_n|\tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
\[
= P(\tau_b > t_n|Z_n \in dz_n, Y_n \in dy_n, \tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
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\[
\mathbb{P}(Y_n \in dy_n | Z_n \in dz_n, \tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
\[
\mathbb{P}(Z_n \in dz_n | \tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
\[
= \psi(z_{n-1} - z_b, z_n - z_b, \sigma\sqrt{t_n - t_{n-1}})
\]
\[
\times \mathbb{P}(U_n | Z_n \in dz_n, \tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
\[
\times \mathbb{P}(Z_n \in dz_n | \tau_b > t_{n-1}, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)})
\]
\[
= \psi(z_{n-1} - z_b, z_n - z_b, \sigma\sqrt{t_n - t_{n-1}})p_U(y_n - z_n | y_{n-1} - z_{n-1})p_Z(z_n | z_{n-1})dy_n dz_n.
\]

So we obtain the following expression for \( b_n(z^{(n)}|y^{(n)}) \):
\[
\frac{\psi(z_{n-1} - z_b, z_n - z_b, \sigma\sqrt{t_n - t_{n-1}})p_Z(z_n | z_{n-1})p_U(y_n - z_n | y_{n-1} - z_{n-1})b_{n-1}(z^{(n-1)}|y^{(n-1)})}{p_Y(y_n | y^{(n-1)})}.
\] (3.10)

Similarly as in the case with only one noisy accounting observation, treated in section 3.2, it now follows that the conditional density \( g_n(\cdot|Y^{(n)}, \tau_b > t_n) \) of \( Z^{(n)} \) is given by
\[
g_n(z^{(n)}|y^{(n)}, \tau_b > t_n) = \frac{b_n(z^{(n)}|y^{(n)})}{\int_{A_n} b_n(z^{(n)}|y^{(n)})dz^{(n)}},
\] (3.11)

where
\[
A_n = \{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n : z_1 \geq z_b, \ldots, z_n \geq z_b \}.
\]

It should be noted that there is no explicit expression for the integral in the denominator of equation (3.11), such that it needs to be evaluated using numerical integration. Now the marginal conditional density of \( Z_n \) at time \( t_n \) is given by
\[
g_n(z_n|y^{(n)}, \tau_b > t_n) = \int_{A_{n-1}} g_n(z^{(n)}|y^{(n)}, \tau_b > t_n)dz^{(n-1)}.
\] (3.12)

For \( \tau_b > t_n \), the \( \mathcal{H}_{t_n} \)-conditional survival probability is then given by
\[
p(t_n, s) = \int_{z_b}^{\infty} (1 - \pi(s - t_n, z_n - z_b))g_n(z_n|Y^{(n)}, \tau_b > t_n)dz_n 
\] (3.13)

where \( \pi(t, x) \) denotes again the probability that \( Z \) hits \( 0 \) before time \( t \), starting from \( x > 0 \), as computed in Lemma 3.2. Also the notion of default intensity can be extended to the case in which we have several accounting reports, as follows. At time \( t_n < t < t_{n+1} \) the \( \mathcal{H}_t \)-conditional density of \( Z_t \) is given by
\[
f(t, x, \omega) = \int_{z_b}^{\infty} \tilde{f}(t - t_n, x, z_n)g_n(z_n|Y^{(n)}(\omega), \tau_b(\omega) > t_n)dz_n,
\]

where as before \( \tilde{f}(t, \cdot, z_0) \) is the \( \mathcal{H}_t \)-conditional density before the first accounting report, given by equation (3.9). The density \( f(t, \cdot, \omega) \) again has a derivative with respect to \( x \) given by
\[
f_x(t, x, \omega) = \int_{z_b}^{\infty} \tilde{f}_x(t - t_n, x, z_n)g_n(z_n|Y^{(n)}(\omega), \tau_b(\omega) > t_n)dz_n.
\]
Then also the default intensity is extended to the case in which we have several accounting reports, i.e. the default intensity $\lambda$ with respect to $H_t$ is given by

$$\lambda_t(\omega) = \frac{1}{2} \sigma f_x(t, z_b, \omega) 1_{\{t_b > t\}}(\omega), \quad \text{for } t > 0.$$  

(3.14)

The proof of this result immediately follows from the proof of theorem 3.2, because we only have a finite number of accounting reports.

### 3.5 Valuation of CoCos with a principal write-down

In this section, the model with several noisy accounting reports of the previous section will be applied to the valuation of contingent convertible bonds with a (partial) principal write-down at conversion. It is assumed that, additionally to the straight bonds, the firm issues contingent convertible debt with a total par value outstanding of $P$ and maturity time $T$.

#### 3.5.1 CoCos with a regulatory trigger

Banks have the obligation to report it to their supervisor at the moment they are approaching a trigger. Then the regulator will call for conversion, this is called a point of non-viability. Of course, this type of conversion can also happen in between accounting report dates. This type of conversion thus occurs when the value of assets falls for the first time below some threshold $v_c$, i.e. the conversion time is given by

$$\tau_c = \inf\{t \geq 0 : V_t \leq v_c\}.$$  

Also, the firm pays coupons continuously at rate $c$ until either maturity or conversion. To start with, we assume that the CoCo bond suffers a principal write-down upon conversion. Later on, in Section 3.6, the case of a conversion into a fixed number of shares will be considered. For now, a fraction $1 - R$ of the principal value is written down at conversion, while a fraction $R$ is recovered to the bond holder, for $R \in [0, 1)$. Furthermore it is assumed that the risk free rate is constant, denoted by $r$.

Of course, the bond investors can observe whether the CoCos have converted. This means that $H_t$ is now defined as

$$H_t = \sigma(\{Y_{t_1}, \ldots, Y_{t_n}, 1_{\{\tau_c \leq s\}} : s \leq t\}).$$

Denote by $p_c(t, s)$ the $H_t$-conditional probability that the CoCos did not convert until time $s > t$, that is

$$p_c(t, s) = \mathbb{P}(\tau_c > s | H_t).$$

As before, for $z_c = \log v_c$, we can also write

$$\tau_c = \inf\{t \geq 0 : Z_t \leq z_c\}.$$
In analogy to equation (3.13), we have for \( t_n < \tau_c \) and \( s > t_n \)

\[
p_c(t_n, s) = \mathbb{P}(\tau_c > s | \tau_c > t_n, Y^{(n)}) = \int_{z_c}^{\infty} (1 - \pi(s - t_n, z_n - z_c)) g_n(z_n | Y^{(n)}, \tau_c > t_n) dz_n.
\]

For a more general time \( t \) such that \( t_n \leq t < t_{n+1} \) and \( \tau_c > t \), this conditional probability reads

\[
p_c(t, s) = \int_{z_c}^{\infty} (1 - \pi(s - t, x - z_c)) f(t, x, \omega) dx, \tag{3.15}
\]

for

\[
f(t, x, \omega) = \int_{z_c}^{\infty} \tilde{f}(t - t_n, x, z_n) g_n(z_n | Y^{(n)}(\omega), \tau_c(\omega) > t_n) dz_n, \tag{3.16}
\]

where \( \tilde{f}(t, x, z_0) \) is, in analogy to equation (3.9), given by

\[
\tilde{f}(t, x, z_0) = \frac{1}{\sigma \sqrt{t}} \exp \left( -\frac{m(z_0 - x)}{\sigma^2} - \frac{m^2 t}{2 \sigma^2} \right) \left( \phi \left( \frac{z_0 - z_0 + mt}{\sigma \sqrt{t}} \right) - e^{-2m(z_0 - z_0)/\sigma^2} \Phi \left( \frac{z_0 - z_0 + mt}{\sigma \sqrt{t}} \right) \right).
\]

Now the value at time \( t < \tau_c \) of the CoCos, given imperfect accounting information \( \mathcal{H}_t \), is given by

\[
C(t) = \mathbb{E} \left( P e^{-r(T-t)} 1_{\{\tau_c > T \}} | \mathcal{H}_t \right) + \mathbb{E} \left( \int_t^T c P e^{-r(u-t)} 1_{\{\tau_c > u \}} du | \mathcal{H}_t \right) \\
+ \mathbb{E} \left( R P e^{-r(\tau_c-t)} 1_{\{\tau_c \leq T \}} | \mathcal{H}_t \right) \\
= P e^{-r(T-t)} \mathbb{P}(\tau_c > T | \mathcal{H}_t) + c P \int_t^T e^{-r(u-t)} \mathbb{P}(\tau_c > u | \mathcal{H}_t) du \\
+ R P \int_t^T e^{-r(u-t)} \mathbb{P}(\tau_c \leq u | \mathcal{H}_t) \\
= P e^{-r(T-t)} p_c(t, T) + c P \int_t^T e^{-r(u-t)} p_c(t, u) du - R P \int_t^T e^{-r(u-t)} p_c(t, du). \tag{3.17}
\]

Here the first term represents the payment of the principal, in case conversion does not happen before maturity, while the second term accounts for the payment of coupons until either conversion or maturity. The last term values the recovery of the principal at conversion. The integral in this last term can be written as

\[
\int_t^T e^{-r(u-t)} p_c(t, du) = \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} p_c(t, u) du \\
= \int_t^T e^{-r(u-t)} \int_{z_c}^{\infty} \frac{\partial}{\partial u} (1 - \pi(u - t, x - z_c)) f(t, x, \omega) dx du
\]
\[ \int_{z_c}^{\infty} f(t, x, \omega) \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} (1 - \pi(u-t, x-z_c)) \, du \, dx = \int_{z_c}^{\infty} f(t, x, \omega) I(x) \, dx, \]  

(3.18)

where

\[ I(x) = \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} (1 - \pi(u-t, x-z_c)) \, du. \]  

(3.19)

Furthermore, the integral in the second term of equation (3.17) can be written as

\[ \int_t^T e^{-r(u-t)} p_c(t, u) \, du = \int_t^T e^{-r(u-t)} \int_{z_c}^{\infty} (1 - \pi(u-t, x-z_c)) f(t, x, \omega) \, dx \, du \]

\[ = \int_{z_c}^{\infty} f(t, x, \omega) \int_t^T e^{-r(u-t)} (1 - \pi(u-t, x-z_c)) \, du \, dx \]

\[ = \int_{z_c}^{\infty} f(t, x, \omega) \tilde{I}(x) \, dx, \]  

(3.20)

where

\[ \tilde{I}(x) = \int_t^T e^{-r(u-t)} (1 - \pi(u-t, x-z_c)) \, du \]

\[ = \left[ -\frac{1}{r} e^{-r(T-t)} (1 - \pi(T-t, x-z_c)) \right]_{u=t}^{T} + \frac{1}{r} I(x) \]

\[ = -\frac{1}{r} e^{-r(T-t)} (1 - \pi(T-t, x-z_c)) + \frac{1}{r} I(x). \]  

(3.21)

Putting equations (3.17), (3.19) and (3.20) together allows us to write the CoCo price \( C(t) \) as a single integral, weighted by the density \( f(t, x, \omega) \), as follows

\[ C(t) = \int_{z_c}^{\infty} \left( P e^{-r(T-t)} (1 - \pi(T-t, x-z_c)) + c P \tilde{I}(x) - R P I(x) \right) f(t, x, \omega) \, dx \]  

(3.22)

\[ = \int_{z_c}^{\infty} \left( \frac{r-c}{r} P e^{-r(T-t)} (1 - \pi(T-t, x-z_c)) + \frac{c}{r} P + \left( \frac{cP}{r} - R P \right) I(x) \right) f(t, x, \omega) \, dx. \]  

(3.23)

It now remains to find an analytical expression for \( I(x) \). First consider

\[ \frac{\partial}{\partial u} (1 - \pi(u-t, x-z_c)) \]

\[ = \frac{\partial}{\partial u} \left( \frac{\phi \left( \frac{x-z_c + m(u-t)}{\sigma \sqrt{u-t}} \right) - e^{-2m(x-z_c)/\sigma^2} \phi \left( \frac{-(x-z_c) + m(u-t)}{\sigma \sqrt{u-t}} \right)}{2\sigma \sqrt{u-t}} - \frac{x-z_c}{2\sigma(u-t)^{3/2}} \right) \]  

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\[
-e^{-2m(x-z_c)/\sigma^2} \phi \left( \frac{-x - z_c + m(u - t)}{\sqrt{u-t}} \right) \left( \frac{m}{2\sigma\sqrt{u-t}} + \frac{x - z_c}{2\sigma(u-t)^{3/2}} \right) \\
= \frac{z_c - x}{\sigma(u-t)^{3/2}} \phi \left( \frac{x - z_c + m(u - t)}{\sqrt{u-t}} \right),
\]

which implies

\[
I(x) = \int_t^T e^{-r(u-t)} \frac{z_c - x}{\sigma(u-t)^{3/2}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x - z_c + m(u - t))^2}{2\sigma^2(u-t)} \right) \, du \\
= \frac{z_c - x}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{m(x - z_c)}{\sigma^2} \right) \int_0^{T-t} e^{-ru} \frac{1}{u^{3/2}} \exp \left( -\frac{(x - z_c)^2}{2\sigma^2 u} - \frac{m^2 u}{2\sigma^2} \right) \, du \\
= \frac{z_c - x}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{m(x - z_c)}{\sigma^2} \right) \int_0^{T-t} 1 \frac{1}{u^{3/2}} \exp \left( -\frac{(x - z_c)^2}{2\sigma^2 u} - \frac{m^2 u}{2\sigma^2} \right) \, du \\
= 2 \frac{z_c - x}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{m(x - z_c)}{\sigma^2} \right) \int_{(T-t)^{-1/2}}^{\infty} \exp \left( -A v^2 - B \frac{1}{v^2} \right) \, dv,
\]

(3.24)

where the last line follows by substitution of \( v = u^{-1/2} \) and by setting

\[
A = \frac{(x - z_c)^2}{2\sigma^2}, \quad B = \frac{m^2}{2\sigma^2} + r.
\]

Now, by noting that \((A v^2 + B/v^2) = (\sqrt{A}v - \sqrt{B}/v)^2 + 2\sqrt{AB},\) as well as \((A v^2 + B/v^2) = (\sqrt{A}v + \sqrt{B}/v)^2 - 2\sqrt{AB},\) the remaining integral can be evaluated, by doing the substitutions \( u = \sqrt{A} v - \sqrt{B}/v \) and \( u = \sqrt{A} v + \sqrt{B}/v,\) as follows

\[
\int_{(T-t)^{-1/2}}^{\infty} \exp \left( -A v^2 - B \frac{1}{v^2} \right) \, dv \\
= \frac{1}{2\sqrt{A}} \int_{(T-t)^{-1/2}}^{\infty} \exp \left( -\left( \sqrt{A} v - \sqrt{B}/v \right)^2 - 2\sqrt{AB} \right) \left( \sqrt{A} + \sqrt{B} \frac{1}{v^2} \right) \, dv \\
+ \frac{1}{2\sqrt{A}} \int_{(T-t)^{-1/2}}^{\infty} \exp \left( -\left( \sqrt{A} v + \sqrt{B}/v \right)^2 + 2\sqrt{AB} \right) \left( \sqrt{A} - \sqrt{B} \frac{1}{v^2} \right) \, dv \\
= \frac{1}{2\sqrt{A}} e^{-2\sqrt{AB}} \int_{\sqrt{A/(T-t)} - \sqrt{B/(T-t)}}^{\infty} e^{-u^2} \, du \\
+ \frac{1}{2\sqrt{A}} e^{2\sqrt{AB}} \int_{\sqrt{A/(T-t)} + \sqrt{B/(T-t)}}^{\infty} e^{-u^2} \, du \\
= \sqrt{\frac{\pi}{4\sqrt{A}}} \left( e^{-2\sqrt{AB}} \text{erfc} \left( \sqrt{A/(T-t)} - \sqrt{B/(T-t)} \right) \\
+ e^{2\sqrt{AB}} \text{erfc} \left( \sqrt{A/(T-t)} + \sqrt{B/(T-t)} \right) \right),
\]

(3.25)

where \(\text{erfc}(x)\) is the complementary error function, which is defined by

\[
\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} \, du
\]
and satisfies
\[ \frac{1}{2} \text{erfc}(x/\sqrt{2}) = 1 - \Phi(x). \]

Combining equations (3.24) and (3.25) and substituting back the expressions for \( A \) and \( B \), finally leads to the following expression for \( I(x) \):

\[
I(x) = 2 \frac{z_c - x}{\sqrt{2\pi} \sigma^2} \exp \left( -\frac{m(x - z_c)}{\sigma^2} \right) \int_{(T-t)^{-1/2}}^{\infty} \exp \left( -\frac{Av^2 - B}{v^2} \right) dv \\
= \exp \left( -\frac{m(x - z_c)}{\sigma^2} \right) \left( -e^{-2\sqrt{AB} \frac{1}{2} \text{erfc} (\sqrt{A/(T-t)} - \sqrt{B/(T-t)})} \right) \\
- e^{2\sqrt{AB} \frac{1}{2} \text{erfc} (\sqrt{A/(T-t)} + \sqrt{B/(T-t)})} \right) \\
= \exp \left( -\frac{m(x - z_c) + (x - z_c)\sqrt{m^2 + 2r\sigma^2}}{\sigma^2} \right) \left( \Phi \left( \frac{x - z_c - \sqrt{m^2 + 2r\sigma^2}(T-t)}{\sigma\sqrt{T-t}} \right) - 1 - 1 \right) \\
+ \exp \left( -\frac{m(x - z_c) - (x - z_c)\sqrt{m^2 + 2r\sigma^2}}{\sigma^2} \right) \left( \Phi \left( \frac{x - z_c + \sqrt{m^2 + 2r\sigma^2}(T-t)}{\sigma\sqrt{T-t}} \right) - 1 \right) .
\]

(3.26)

To summarize, in the above we have proven the following theorem.

**Theorem 3.3** (Price of a CoCo with a regulatory trigger and a principal write-down).
The secondary market price of the CoCo at time \( t < \tau_c \) is given by

\[
C(t) = \int_{z_c}^{\infty} h(x) f(t, x) dx ,
\]

(3.27)

where \( f(t, x) \) is given by equation (3.16) for \( t_n \leq t < t_{n+1} \) and \( h(x) \) is defined as

\[
h(x) := \frac{r - c}{r} P e^{-r(T-t)} (1 - \pi(T-t, x - z_c)) + \frac{c}{r} P + \left( \frac{cP}{r} - RP \right) I(x),
\]

(3.28)

in which \( I(x) \) is given by equation (3.26).

Now we can move on with the computation of this integral. Let us first consider

the computation of \( C(t_n) \), i.e. the price of the CoCo at the time of the \( n \)th accounting report. In this case, the conditional density \( f(t_n, \cdot) \) is equal to the marginal density \( g_{t_n}(z|\tau_c > t_n, Y^{(n)}) \), as can be seen from equation (3.16). Recall from equation (3.12) that the marginal density \( g_{t_n}(z_n|\tau_c > t_n, Y^{(n)}) \) is given by

\[
g_{t_n}(z_n|y^{(n)}, \tau_c > t_n) = \int_{A_{n-1}} g_{t_n}(z^{(n)}|y^{(n)}, \tau_c > t_n) dz^{(n-1)},
\]

where \( A_n \) is defined as

\[
A_n = \{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n : z_1 \geq z_c, \ldots, z_n \geq z_c \}. \]

(3.29)
So it follows that $C(t_n)$ satisfies

$$
C(t_n) = \int_{z_c}^\infty h(z_n)f(t_n, z_n)dz_n \\
= \int_{A_n} h(z_n)g_{tn}(z^{(n)}|y^{(n)}, \tau_c > t_n)dz^{(n)}.
$$

(3.30)

This integral cannot be evaluated analytically, so this will be done using Monte Carlo simulation. If we can simulate a sample $((z^{(n)})^1, \ldots, (z^{(n)})^G)$ from $g_{tn}(z^{(n)}|y^{(n)}, \tau_c > t_n)$, it is possible to approximate $C(t_n)$ as

$$
C(t_n) \approx \frac{1}{G} \sum_{g=1}^G h(z^g_n),
$$

(3.31)

where $z^g_n$ denotes the $n$th coordinate of $(z^{(n)})^g$, for $g = 1, \ldots, G$.

The sample $((z^{(n)})^1, \ldots, (z^{(n)})^G)$ is obtained by execution of the following MCMC-algorithm.

**Algorithm 3.1** (Random walk Metropolis-Hastings).

1. In each iteration $g, g = 1, \ldots, n_0 + G$, given the current value $(z^{(n)})^g$, the proposal $(z^{(n)})'$ is drawn according to

$$
(z^{(n)})' = (z^{(n)})^g + X, \text{ for } X \sim N_n(0, \Sigma),
$$

where the $n \times n$-covariance matrix $\Sigma$ is chosen to reach some desired acceptance rate.

2. Set

$$(z^{(n)})^{(g+1)} = \begin{cases} (z^{(n)})' & \text{with prob. } \alpha((z^{(n)})^g, (z^{(n)})') \\ z^{(n)} & \text{with prob. } 1 - \alpha((z^{(n)})^g, (z^{(n)})') \end{cases},$$

where the acceptance-probability $\alpha(z^{(n)}, (z^{(n)})')$ is given by

$$
\alpha(z^{(n)}, (z^{(n)})') = \min \left\{ 1, \frac{g_{tn}(z^{(n)}|y^{(n)}, \tau_c > t_n)}{g_{tn}(z^{(n)}|y^{(n)}, \tau_c > t_n)} \right\} = \min \left\{ 1, \frac{b_{tn}(z^{(n)}|y^{(n)})}{b_{tn}(z^{(n)}|y^{(n)})} \right\}.
$$

3. Discard the draws from the first $n_0$ iterations (because the Markov chain needs a burn-in period to converge to the target distribution) and save the sample $(z^{(n)})^{n_0+1}, \ldots, (z^{(n)})^{n_0+G}$.

In order to compute $C(t)$ for $t_n \leq t < t_{n+1}$ a similar procedure can be followed. First note that we can write

$$
C(t) = \int_{z_c}^\infty h(x)f(t, x)dx
$$
\[
\int_{z_c}^\infty h(x) \int_{z_c}^\infty \tilde{f}(t-t_n, x, z_n) g_{t_n}(z_n) Y(n), \tau_c > t_n \, dz_n \, dx \\
= \int_{z_c}^\infty \int_{A_n} h(x) \tilde{f}(t-t_n, x, z_n) g_{t_n}(z_n) Y(n), \tau_c > t_n \, dz_n \, dx \\
= \int_{A_{n+1}} h(z_{n+1}) \tilde{f}(t-t_n, z_{n+1}, z_n) g_{t_n}(z_n) Y(n), \tau_c > t_n \, dz^{(n+1)}_n.
\]

(3.32)

So now we need a sample \((z^{(n+1)})^1, \ldots, (z^{(n+1)})^G\) from the \((n+1)\)-dimensional distribution on \(A_{n+1}\) with density \(f(t-t_n, z_{n+1}, z_n) g_{t_n}(z_n) Y(n), \tau_c > t_n\), in order to approximate \(C(t)\) as

\[
C(t) \approx \frac{1}{G} \sum_{g=1}^{G} h(z^{(n+1)}_n).
\]

(3.33)

The algorithm used to obtain the sample, now modifies into

**Algorithm 3.2 (Random walk Metropolis-Hastings).**

1. In each iteration \(g, g = 1, \ldots, n_0+G\), given the current value \((z^{(n+1)})^g\), the proposal \((z^{(n+1)})'\) is drawn according to

\[
(z^{(n+1)})' = (z^{(n+1)})^g + X, \text{ for } X \sim N_{n+1}(0, \Sigma),
\]

where the \((n+1) \times (n+1)\)-covariance matrix \(\Sigma\) is chosen to reach some desired acceptance rate.

2. Set

\[
(z^{(n+1)})^{(g+1)} = \begin{cases} (z^{(n+1)})' & \text{with prob. } \alpha((z^{(n+1)})^g, (z^{(n+1)})') \\ (z^{(n+1)})^g & \text{with prob. } 1 - \alpha((z^{(n+1)})^g, (z^{(n+1)})') \end{cases},
\]

where the acceptance-probability \(\alpha(z^{(n+1)}, (z^{(n+1)})')\) is given by

\[
\alpha(z^{(n+1)}, (z^{(n+1)})') = \min \left\{ 1, \frac{\tilde{f}(t-t_n, z_{n+1}', z_n') g_{t_n}(z_n) Y(n), \tau_c > t_n}{\tilde{f}(t-t_n, z_{n+1}, z_n) g_{t_n}(z_n) Y(n), \tau_c > t_n} \right\}
\]

\[
= \min \left\{ 1, \frac{\tilde{f}(t-t_n, z_{n+1}', z_n') b_n((z^{(n)})'|y^{(n)})}{\tilde{f}(t-t_n, z_{n+1}, z_n) b_n((z^{(n)})|y^{(n)})} \right\}.
\]

3. Discard the draws from the first \(n_0\) iterations and save the sample \((z^{(n+1)})^{n_0+1}, \ldots, (z^{(n+1)})^{n_0+G}\).

In both algorithms the acceptance probability involves the term \(\frac{b_n((z^{(n)})'|y^{(n)})}{b_n((z^{(n)})|y^{(n)})}\). It follows from equations (3.2) and (3.10) that this fraction is explicitly given by

\[
\frac{b_n((z^{(n)})'|y^{(n)})}{b_n((z^{(n)})|y^{(n)})} = \prod_{i=1}^{n} \psi(z_{i-1}', z_{i}' - z_i, z_i, \sigma \sqrt{t_i - t_{i-1}}) p_Z(z_i | z_{i-1}') p_U(y_i - z_i') y_{i-1}' - z_{i-1}'),
\]

\[
\prod_{i=1}^{n} \psi(z_{i-1}' - z_i, z_i - z_{i-1}, \sigma \sqrt{t_i - t_{i-1}}) p_Z(z_i | z_{i-1}) p_U(y_i - z_i) y_{i-1} - z_{i-1}'),
\]

"\[46\]
under the convention that \( t_0 = 0 \) and \( p_U(\cdot|u_0) = p_U(\cdot) \) is a Gaussian density with mean \( u \) and variance \( \sigma^2 \) (as in section 3.2). Note that \( p_Z|Z_{i-1} \) is a Gaussian density with mean \( z_{i-1} + m(t_i - t_{i-1}) \) and variance \( \sigma^2(t_i - t_{i-1}) \), that \( p_U|u_{i-1} \) is a Gaussian density with mean \( \kappa u_{i-1} + \mu \epsilon \) and variance \( \sigma^2 \), and that \( \psi \) is given in Lemma 3.1.

### Early cancelling of coupons

In the valuation of the firm’s convertible debt in equation (3.17), it is assumed that coupons are paid until conversion. However, as pointed out in the Introduction, CoCos are affected by the Maximum Distributable Amount (MDA), which requires regulators to stop earnings distributions when the firm’s total capital falls below some trigger, higher than the conversion trigger. This we could model by introducing a trigger \( z_{cc} > z_c \). If \( Z \) is below \( z_{cc} \) the firm will not pay coupons, while if \( Z \) is above \( z_{cc} \) the firm still pays coupons. To value the CoCo in this case, only the second term in equation (3.17) needs to be adjusted. In this case, coupons are only paid at time \( u \) if \( Z_u > z_{cc} \), so the term

\[
E \left( \int_t^T e^{-r(u-t)} 1_{\{\tau_c > u\}} du | \mathcal{H}_t \right)
\]

needs to be replaced with

\[
E \left( \int_t^T e^{-r(u-t)} 1_{\{\tau_c > u, Z_u > z_{cc}\}} du | \mathcal{H}_t \right).
\]  

(3.34)

For \( \tau_c > t \) and \( t_n \leq t < t_{n+1} \) this term equals

\[
cP \int_t^T e^{-r(u-t)} \mathbb{P} \left( \tau_c > u, Z_u > z_{cc} | Y^{(n)}, \tau_c > t \right) du.
\]

So now we need to compute \( \mathbb{P} \left( \tau_c > u, Z_u > z_{cc} | Y^{(n)}, \tau_c > t \right) \). In order to compute this conditional probability, we first need the following result.

**Lemma 3.4.** The joint probability \( \tilde{\pi}(t, x, y) \) that \( Z \), starting from \( x > 0 \), does not hit 0 before time \( t \) and that \( Z_t > y \) is given by

\[
\tilde{\pi}(t, x, y) := \mathbb{P} \left( \inf_{0 \leq s \leq t} Z_s > 0, Z_t > y \right) = \Phi \left( \frac{x - y + mt}{\sigma \sqrt{t}} \right) - e^{-2mx/\sigma^2} \Phi \left( \frac{-x - y + mt}{\sigma \sqrt{t}} \right).
\]  

(3.35)

**Proof.** To prove this lemma, we will rely on the following result by Harrison [14, Chapter 1.8]. For \( X \) a Brownian motion with drift \( \mu \), variance \( \sigma^2 \) and \( X_0 = 0 \), define \( M_t = \sup_{0 \leq s \leq t} X_s \). Then the joint distribution of \( M_t \) and \( X_t \) is given by

\[
\mathbb{P}(X_t \leq x, M_t \leq y) = \Phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) - e^{2\mu y/\sigma^2} \Phi \left( \frac{x - 2y - \mu t}{\sigma \sqrt{t}} \right).
\]  

(3.36)
Now define $X_t = x - Z_t$, then $X_t$ is a Brownian motion with drift $-m$, variance $\sigma^2$ and $X_0 = 0$. And $M_t = \inf_{0 \leq s \leq t} Z_s$. Hence
\[
\tilde{\pi}(t, x, y) := \mathbb{P}(\inf_{0 \leq s \leq t} Z_s > 0, Z_t > y) = \mathbb{P}(M_t \leq x, X_t \leq x - y),
\]
which proves the result, by applying equation (3.36).

Now, similarly to equation (3.15), we can write
\[
\mathbb{P}\left(\tau_c > u, Z_u > z_{cc}\right) = \int_{\infty}^{u} \tilde{\pi}(u - t, x - z_c, z_{cc} - z_c) \tilde{f}(t, x) dx. \quad (3.37)
\]
As in equation (3.32), this can be written as
\[
\int_{A_{n+1}} \tilde{\pi}(u - t, z_{n+1} - z_c, z_{cc} - z_c) \tilde{f}(t - t_n, z_{n+1}, z_{n}) \hat{g}_n(z^{(n)}|Y^{(n)}, \tau_c > t_n) dz^{(n+1)}.
\]
So, for a sample $((z^{(n+1)}), \ldots, (z^{(n+1)})^G)$ from the $(n+1)$-dimensional distribution with density \(\tilde{f}(t - t_n, z_{n+1}, z_{n}) \hat{g}_n(z^{(n)}|Y^{(n)}, \tau_c > t_n)\), we can approximate the desired conditional probability by
\[
\mathbb{P}\left(\tau_c > u, Z_u > z_{cc}\right) \approx \frac{1}{G} \sum_{g=1}^{G} \tilde{\pi}(u - t, z_{n+1} - z_c, z_{cc} - z_c).
\]
The necessary sample is again obtained using Algorithm 3.2.

Recall that we wanted to compute
\[
cP \int_t^{T} e^{-r(u-t)} \mathbb{P}\left(\tau_c > u, Z_u > z_{cc}\right) dy_n(t, x) \mathbb{P}(\tau_c > t) du,
\]
which now can be done by performing numerical integration over $u$.

The other two terms in equation (3.17) do not change, so the CoCo price at time $t < \tau_c$ is given by the sum of the new term in (3.34) and the unchanged part
\[
P e^{-r(T-t)} p_c(t, T) - R \int_t^{T} e^{-r(u-t)} p_c(t, du).
\]
By an adaption of equation (3.27) it is seen that this unchanged part can be written as
\[
\int_{\infty}^{\infty} \tilde{h}(x) \tilde{f}(t, x) dx,
\]
where
\[
\tilde{h}(x) = P e^{-r(T-t)}(1 - \pi(T - t, x - z_c)) - R I(x),
\]
in which $I(x)$ is given by equation (3.26). So this part can be computed performing the same computations as in the old case, by replacing $h(x)$ with $\tilde{h}(x)$. 

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3.5.2 CoCos with only a book value trigger

There exists also CoCos which conversion trigger solely depends on accounting reports, for example the CoCos issued by Barclays. This means that conversion happens when the reported value of the capital ratio falls below some threshold and hence conversion can only happen at one of the accounting report dates $t_1, t_2, \ldots$. This corresponds to a setting in which the conversion time is defined as

$$\tau_c = \inf\{t_i \geq 0 : Y_{t_i} \leq y_c\},$$

for some threshold $y_c \geq 0$. In this case the available information at time $t$ would reduce to

$$\mathcal{H}_t = \sigma(Y_{t_1}, \ldots, Y_{t_n}),$$

for the largest $n$ such that $t_n \leq t$. This means that we are interested in the probability that, given $n$ accounting reports and $\tau_c > t_n$, the $(n + i)$th accounting report will cause a trigger event, for $i = 1, 2, \ldots$. So we are primarily interested in the conditional density $p_Y(y_{n+i}, \ldots, y_{n+1} | y^{(n)})$ of $Y_{n+i}, \ldots, Y_{n+1}$, given $Y^{(n)} = y^{(n)}$, where $y_i > y_c$, for $1 \leq i \leq n$.

Denote by $\Delta t$ the time between to successive accounting reports and recall from section 3.4 the notation $Y_i = Y_{t_i}, Z_i = Z_{t_i}, U_i = U_{t_i}$ and that $Y_i = Z_i + U_i$, where

$$U_i = \kappa U_{i-1} + \epsilon_i,$$

for some fixed $\kappa \in \mathbb{R}$ and independent and identically distributed $\epsilon_1, \epsilon_2, \ldots$, which have a normal distribution, say with mean $\mu_\epsilon$ and variance $\sigma_\epsilon^2$, and are independent of $Z$.

This allows us to write

$$Y_{n+1} = Z_{n+1} + U_{n+1}$$

$$= Z_{n+1} + \kappa U_n + \epsilon_{n+1}$$

$$= Z_{n+1} - \kappa Z_n + \epsilon_{n+1} + \kappa Y_n,$$

which leads to the following expression

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \kappa^2 Z_n + \epsilon_{n+2} + \kappa \epsilon_{n+1} + \kappa^2 Y_n \\ Z_{n+1} - \kappa Z_n + \epsilon_{n+1} + \kappa Y_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & \kappa \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_{n+2} - Z_{n+1} \\ Z_{n+1} - Z_n \\ \epsilon_{n+2} \\ \epsilon_{n+1} \end{pmatrix} + \begin{pmatrix} \kappa^2 \\ \kappa \end{pmatrix} Y_n + \begin{pmatrix} 1 - \kappa^2 \\ 1 - \kappa \end{pmatrix} Z_n.$$

Now note that $Z_{n+2} - Z_{n+1}, Z_{n+1} - Z_n, \epsilon_{n+2}$ and $\epsilon_{n+1}$ are all Gaussian and independent of $Y^{(n)}, Z^{(n)}$ and each other. Hence, conditional on $Y^{(n)} = y^{(n)}, Z^{(n)} = z^{(n)}$, the vector $(Z_{n+2} - Z_{n+1}, Z_{n+1} - Z_n, \epsilon_{n+2}, \epsilon_{n+1})$ follows a multivariate normal distribution. So the above implies that the conditional density $p_Y(y_{n+2}, y_{n+1} | y^{(n)}, z^{(n)})$ of $y_{n+2}, y_{n+1}$ given $Y^{(n)} = y^{(n)}$ and $Z^{(n)} = z^{(n)}$, is the density of a bivariate normal distribution with certain
mean vector and covariance matrix, which is presented below for the general case. The above can be extended, for \( i = 1, 2, \ldots \), as follows

\[
\begin{pmatrix}
Y_{n+i} \\
\vdots \\
Y_{n+1}
\end{pmatrix}
= M \begin{pmatrix}
Z_{n+i} - Z_{n+i-1} \\
\vdots \\
Z_{n+1} - Z_{n} \\
\epsilon_{n+i} \\
\vdots \\
\epsilon_{n+1}
\end{pmatrix}
+ \begin{pmatrix}
\kappa^i \\
\vdots \\
\kappa \\
1 - \kappa^i
\end{pmatrix} Y_n
+ \begin{pmatrix}
1 - \kappa \\
\vdots \\
1 - \kappa
\end{pmatrix} Z_n,
\]

where \( M \) denotes the \((i \times 2i)\)-matrix defined by

\[
M = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & \kappa & \kappa^2 & \cdots & \kappa^{i-1} \\
0 & 1 & \cdots & 1 & 1 & 0 & \kappa & \cdots & \kappa^{i-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \ \\
0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 1 & \kappa \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\]

and where the vector \((Z_{n+i} - Z_{n+i-1}, \ldots, Z_{n+1} - Z_n, \epsilon_{n+i}, \ldots, \epsilon_{n+1})\) follows a multivariate normal distribution with 2\(i\)-dimensional mean vector \( \mu'_i \) and \((2i \times 2i)\)-dimensional covariance matrix \( \Sigma'_i \), defined by

\[
\mu'_i = \begin{pmatrix}
m\Delta t \\
\vdots \\
m\Delta t \\
\mu_\epsilon
\end{pmatrix}, \quad \Sigma'_i = \text{Diag}(\sigma^2 \Delta t, \ldots, \sigma^2 \Delta t, \sigma^2_\epsilon, \ldots, \sigma^2_\epsilon).
\]

Hence it follows that the conditional density \( p_Y(y_{n+i}, \ldots, y_{n+1}|y^{(n)}, z^{(n)}) \) of \( y_{n+i}, \ldots, y_{n+1} \) given \( Y^{(n)} = y^{(n)} \) and \( Z^{(n)} = z^{(n)} \), is the density of a multivariate normal distribution with mean vector

\[
\hat{\mu}_i = M\mu'_i + \begin{pmatrix}
\kappa^i \\
\vdots \\
\kappa \\
1 - \kappa^i
\end{pmatrix} y_n
+ \begin{pmatrix}
1 - \kappa \\
\vdots \\
1 - \kappa
\end{pmatrix} z_n,
\]

and covariance matrix

\[
\Sigma_i = M\Sigma'_i M^\top.
\]

Recall that the goal was to find an expression for \( p_Y(y_{n+i}, \ldots, y_{n+1}|y^{(n)}) \), which can now be done as follows

\[
p_Y(y_{n+i}, \ldots, y_{n+1}|y^{(n)}) = \int_{\mathbb{R}^n} p_Y(y_{n+i}, \ldots, y_{n+1}|y^{(n)}, z^{(n)}) p_Z(z^{(n)}|y^{(n)}) dz^{(n)},
\]
where the conditional density $p_Z(z^{(n)}|y^{(n)})$ of $Z^{(n)}$, given $Y^{(n)} = y^{(n)}$, can be computed in the same way as $b_n(z^{(n)}|y^{(n)})$ in section 3.4, which leads to

$$p_Z(z^{(n)}|y^{(n)}) = \frac{p_Z(z_n|z_{n-1})p_U(y_n - z_n|y_{n-1} - z_{n-1})p_Z(z^{(n-1)}|y^{(n-1)})}{p_Y(y_n|y^{(n-1)})}$$

$$= \prod_{i=1}^{n} \frac{p_Z(z_i|z_{i-1})p_U(y_i - z_i|y_{i-1} - z_{i-1})}{p_Y(y_n|y^{(n-1)})}, \quad (3.38)$$

under the convention that $t_0 = 0$ and $p_U(\cdot|u_0) = p_U(\cdot)$ is a Gaussian density with mean $u$ and variance $\sigma^2$ (as in section 3.2). Also, note that $p_Z(z_i|z_{i-1})$ is a Gaussian density with mean $z_{i-1} + m(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$ and that $p_U(u_i|u_{i-1})$ is a Gaussian density with mean $\kappa u_{i-1} + \mu$ and variance $\sigma^2$.

This leads to an expression for the survival probability until time $t_{n+i}$, given survival until time $t_n \leq t < t_{n+1}$, that is

$$\mathbb{P}(\tau_c > t_{n+i}|Y^{(n)} = y^{(n)}) = \int_{(y_c, \infty)^i} p_Y(y_{n+i}, \ldots, y_{n+1}|y^{(n)})dy_{n+1}, \ldots, dy_{n+i}$$

$$= \int_{\mathbb{R}^n} \mathbb{P}(\xi \in (y_c, \infty)^i)p_Z(z^{(n)}|y^{(n)})dz^{(n)}, \quad (3.39)$$

where $\xi$ denotes a multivariate normal distributed random variable with mean vector $\hat{\mu}_i$ and covariance matrix $\Sigma_i$. To indicate the dependence of $\xi$ on $z_n$, we denote it as $\xi(z_n)$.

For a sample $((z^{(n)})^1, \ldots, (z^{(n)})^G)$ from $p_Z(z^{(n)}|y^{(n)})$, this survival probability can be computed as

$$\mathbb{P}(\tau_c > t_{n+i}|Y^{(n)} = y^{(n)}) \approx \frac{1}{G} \sum_{g=1}^{G} \mathbb{P}(\xi(z_g^n) \in (y_c, \infty)^i). \quad (3.40)$$

The necessary sample is again obtained using a MCMC-algorithm, as follows.

**Algorithm 3.3 (Random walk Metropolis-Hastings).**

1. In each iteration $g$, $g = 1, \ldots, n_0 + G$, given the current value $(z^{(n)})^g$, the proposal $(z^{(n)})'$ is drawn according to

   $$(z^{(n)})' = (z^{(n)})^g + X, \quad \text{for } X \sim N_n(0, \Sigma),$$

   where the $n \times n$-covariance matrix $\Sigma$ is chosen to reach some desired acceptance rate.

2. Set

   $$(z^{(n)})^{(g+1)} = \begin{cases} (z^{(n)})' & \text{with prob. } \alpha((z^{(n)})^g, (z^{(n)})') \\ z^{(n)} & \text{with prob. } 1 - \alpha((z^{(n)})^g, (z^{(n)})') \end{cases},$$

   where the acceptance-probability $\alpha(z^{(n)}, (z^{(n)})')$ is given by

   $$\alpha(z^{(n)}, (z^{(n)})') = \min \left\{ 1, \frac{p_Z((z^{(n)})'|y^{(n)})}{p_Z(z^{(n)}|y^{(n)})} \right\}$$

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can be written as
\[ P_{\text{principal recovery rate}} \]

For \( R \) As in equation (3.17), this CoCo has secondary market price
\[
C'(t) = E \left( P e^{-r(T-t)} 1_{\{\tau_c > T\}} | \mathcal{H}_t \right) + E \left( \int_t^T cPe^{-r(u-t)} 1_{\{\tau_c > u\}} du | \mathcal{H}_t \right) + E \left( RP e^{-r(\tau_c - t)} 1_{\{\tau_c \leq T\}} | \mathcal{H}_t \right).
\]

Finally, it is now possible to value the contingent convertible bond with, as before, principal \( P \), continuous coupon rate \( c \), maturity \( T \) and a principal write-down with recovery rate \( R \). As in equation (3.17), this CoCo has secondary market price

\[ C'(t) = P e^{-r(T-t)} \mathbb{P}(\tau_c > T | Y^{(n)} = y^{(n)}) + \int_t^T cPe^{-r(u-t)} \mathbb{P}(\tau_c > u | Y^{(n)} = y^{(n)}) du \\
+ RP \sum_{i=1}^m e^{-r(t_{n+i-1} - t)} \mathbb{P}(\tau_c = t_{n+i} | Y^{(n)} = y^{(n)}) \\
= P e^{-r(T-t)} \mathbb{P}(\tau_c > t_{n+m} | Y^{(n)} = y^{(n)}) \\
+ cP \left( \sum_{i=1}^{m-1} \int_{t_{n+i-1}}^{t_{n+i}} e^{-r(u-t)} du \mathbb{P}(\tau_c > t_{n+i} | Y^{(n)} = y^{(n)}) + \int_{t_{n+i}}^{t_{n+m}} e^{-r(u-t)} du \right) \\
+ RP \sum_{i=1}^m e^{-r(t_{n+i})} \left( \mathbb{P}(\tau_c > t_{n+i-1} | Y^{(n)} = y^{(n)}) - \mathbb{P}(\tau_c > t_{n+i} | Y^{(n)} = y^{(n)}) \right) \\
= P e^{-r(T-t)} \mathbb{P}(\tau_c > t_{n+m} | Y^{(n)} = y^{(n)}) \\
+ \sum_{i=1}^{m-1} \frac{cP}{r} (e^{-r(t_{n+i-1} - t)} - e^{-r(t_{n+i} - r(t_{n+i+1} - t))}) \mathbb{P}(\tau_c > t_{n+i} | Y^{(n)} = y^{(n)}) \\
+ \frac{cP}{r} (1 - e^{-r(t_{n+1} - t)}) \\
+ RP \sum_{i=1}^m e^{-r(t_{n+i})} \left( \mathbb{P}(\tau_c > t_{n+i-1} | Y^{(n)} = y^{(n)}) - \mathbb{P}(\tau_c > t_{n+i} | Y^{(n)} = y^{(n)}) \right) \\
= (1 - R)P e^{-r(T-t)} \mathbb{P}(\tau_c > t_{n+m} | Y^{(n)} = y^{(n)}) \\
+ \sum_{i=1}^{m-1} \left( \frac{cP}{r} - RP \right) (e^{-r(t_{n+i-1} - t)} - e^{-r(t_{n+i} - r(t_{n+i+1} - t))}) \mathbb{P}(\tau_c > t_{n+i} | Y^{(n)} = y^{(n)}) \\
+ \frac{cP}{r} (1 - e^{-r(t_{n+1} - t)}) + RP e^{-r(t_{n+1} - t)},
\]

(3.42)
where the only things left to compute are of the form $P(\tau_c > t_{n+i}|Y^{(n)} = y^{(n)})$ for $1 \leq i \leq m$, which can be computed using equation (3.40).

### Early cancelling of coupons

As in the previous subsection, we can also consider the case in which coupons are already cancelled at a moment before the conversion date. It is now assumed that coupons over the time interval $[t_i, t_{i+1})$ are only paid if $Y_i > y_{cc}$, for some trigger level $y_{cc} > y_c$. To value the CoCo in this case, the second term in equation (3.41) needs to be changed to

$$e \left( \sum_{i=1}^{m-1} \int_{t_{n+i}}^{t_{n+i+1}} cPe^{-r(u-t)}1_{\{\tau_c > u, Y_{n+i} > y_{cc}\}}du + 1_{\{Y_{n}>y_{cc}\}} \int_{t}^{t_{n+1}} cPe^{-r(u-t)}du \right),$$

where $t_{n} \leq t < t_{n+1}$, $T = t_{n+m}$ for some $m \in \mathbb{N}$.

For $Y^{(n)} = y^{(n)}$, where $y_i > y_c, 1 \leq i \leq n$, this can be written as

$$\int_{t_{n+i}}^{t_{n+i+1}} e^{-r(u-t)}\mathbb{P}(\tau_c > u, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)})du + 1_{\{Y_{n}>y_{cc}\}} \int_{t}^{t_{n+1}} cPe^{-r(u-t)}du$$

$$= \sum_{i=1}^{m-1} \frac{cP}{r} (e^{-r(t_{n+i}-t)} - e^{-r(t_{n+i+1}-t)})\mathbb{P}(\tau_c > t_{n+i}, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)})$$

$$+ 1_{\{Y_{n}>y_{cc}\}} \frac{cP}{r} (1 - e^{-r(t_{n+1}-t)}),$$

(3.43)

where, similar to equation (3.39),

$$\mathbb{P}(\tau_c > t_{n+i}, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)}) = \mathbb{P}(Y_{n+1} > y_c, \ldots, Y_{n+i-1} > y_c, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)})$$

$$= \int_{\mathbb{R}^p} \mathbb{P}(\xi \in (y_c, \infty)^{i-1} \times (y_{cc}, \infty)) p_Z(z^{(n)}|y^{(n)})dz^{(n)}.$$

This integral can as before be computed by

$$\mathbb{P}(\tau_c > t_{n+i}, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)}) \approx \frac{1}{G} \sum_{g=1}^{G} \mathbb{P}(\xi(z^{(n)}_g) \in (y_c, \infty)^{i-1} \times (y_{cc}, \infty)),$$

for a sample $((z^{(n)})^1, \ldots, (z^{(n)})^G)$ from $p_Z(z^{(n)}|y^{(n)})$.

The CoCo price of equation (3.42) then modifies for the current setting into

$$C'(t) = Pe^{-r(T-t)}\mathbb{P}(\tau_c > t_{n+m}|Y^{(n)} = y^{(n)}) + 1_{\{Y_{n}>y_{cc}\}} \frac{cP}{r} (1 - e^{-r(t_{n+1}-t)})$$

$$+ \sum_{i=1}^{m-1} \frac{cP}{r} (e^{-r(t_{n+i}-t)} - e^{-r(t_{n+i+1}-t)})\mathbb{P}(\tau_c > t_{n+i}, Y_{n+i} > y_{cc}|Y^{(n)} = y^{(n)})$$

$$+ RP \sum_{i=1}^{m} e^{-r(t_{n+i}-t)} \left( \mathbb{P}(\tau_c > t_{n+i-1}|Y^{(n)} = y^{(n)}) - \mathbb{P}(\tau_c > t_{n+i}|Y^{(n)} = y^{(n)}) \right).$$

(3.44)
3.6 Valuation of CoCos with a conversion into shares

In this section we consider the valuation of contingent convertible bonds which convert into equity at the conversion date. As before the firm issues two types of debt; straight debt and contingent convertible debt. The total par value of straight debt outstanding, is denoted by $P_1$, over which coupons are paid continuously at rate $c_1$. Furthermore, the straight bonds have a perpetual maturity and, as before, default occurs at $\tau_b = \inf\{t \geq 0 : Z_t \leq z_b\}$.

At the moment of default a fraction $(1 - \alpha)$, for $\alpha \in (0, 1)$, of the firm’s asset value is lost to bankruptcy costs, so a fraction $\alpha$ of the asset value is recovered and distributed among the senior debt holders.

The total par value of CoCos outstanding is denoted by $P_2$, over which coupons are paid continuously at rate $c_2$. Furthermore, the maturity of the contingent convertible bonds is denoted by $T$. As in Subsection 3.5.1 the conversion date is denoted by $\tau_c = \inf\{t \geq 0 : Z_t \leq z_c\}$, where $z_c > z_b$, to ensure that conversion happens before default. Note that we consider here a regulatory trigger, which means that conversion can happen in between accounting dates, as in subsection 3.5.1. As in Chapter 2, following [6], the CoCo holders receive $\Delta$ shares for every dollar of principal at the moment of conversion. This means that, if we normalize the number of shares before conversion to 1, the CoCo holders own a fraction $\Delta P_2 / (\Delta P_2 + 1)$ of the firm’s equity after conversion.

As before, the information in the market at time $t$ is described by

$$H_t = \sigma(\{Y_t, \ldots, Y_{t_n}, 1_{\{\tau_c \leq s\}}, 1_{\{\tau_b \leq s\}} : s \leq t\})$$

for $t_n \leq t < t_{n+1}$.

In analogy to equation (3.17), the market price of the CoCos is given by

$$C(t) = \mathbb{E}\left(P_2 e^{-r(T-t)} 1_{\{\tau_c > T\}} | H_t\right) + \mathbb{E}\left(\int_t^T c_2 P_2 e^{-r(u-t)} 1_{\{\tau_c > u\}} du | H_t\right)$$

$$+ \mathbb{E}\left(\frac{\Delta P_2}{\Delta P_2 + 1} e^{\Delta P_2 (T - \tau_c)} 1_{\{\tau_c \leq T\}} | H_t\right).$$

(3.45)

Of course only the third term has changed compared to equation (3.17), because this term describes what happens at the moment of conversion (note that the second term needs to be replaced by the corresponding term in equation (3.34), if we want to include early cancelling of coupons). The third term now describes that the CoCo holders obtain a fraction $\Delta P_2 / (\Delta P_2 + 1)$ of the firm’s post-conversion equity, denoted by $E^{PC}(\tau_c)$. This post conversion equity satisfies

$$E^{PC}(\tau_c) = V_{\tau_c} - D(\tau_c) - \mathbb{E}\left(e^{-r(\tau_b - \tau_c)} (1 - \alpha)V_{\tau_b} | H_{\tau_c}\right).$$
That is, the firm’s value of assets minus the value of straight debt, denoted by \( D(\tau_c) \), and bankruptcy costs, described by the last term. Note that the value of straight debt at conversion is given by

\[
D(\tau_c) = E \left( \int_{\tau_c}^{\infty} c_1 P_1 e^{-r(u-\tau_c)} 1_{\{\tau_b > u\}} du | \mathcal{H}_{\tau_c} \right) + E \left( \alpha V_{\tau_b} e^{-r(\tau_b-\tau_c)} | \mathcal{H}_{\tau_c} \right),
\]

where the first term accounts for the continuous payment of coupons and the second term describes the payment at default. It follows that the post-conversion equity value is given by

\[
E^{PC}(\tau_c) = V_{\tau_c} - \frac{\Delta P_2}{\Delta P_2 + 1} E^{PC}(\tau_{c-t}) e^{-r(\tau_{c-t})} 1_{\{\tau_{c-t} \leq T\}} | \mathcal{H}_t
\]

So for \( \tau_c > t \), the third term in equation (3.45) can be written as

\[
E \left( \frac{\Delta P_2}{\Delta P_2 + 1} E^{PC}(\tau_{c-t}) e^{-r(\tau_{c-t})} 1_{\{\tau_{c-t} \leq T\}} | \mathcal{H}_t \right)
\]

Note that the first integral in the last line of this equation is already computed in subsection 3.5.1 and given by equation (3.18) as

\[
e^{z_c} \int_t^{T} e^{-r(u-t)} \mathbb{P}(\tau_c \in du | \tau_c > t, Y(n))
\]

where \( I(x) \) is given by equation (3.26).

To compute the other integrals in equation (3.46), it is sufficient to find expressions for

\[
\mathbb{P}(\tau_c \leq T, \tau_b > u | \tau_c > t, Y^{(n)} = y^{(n)}) \quad \text{and} \quad \mathbb{P}(\tau_c \leq u, \tau_b > u | \tau_c > t, Y^{(n)} = y^{(n)})
\]

In order to find expressions for this joint probabilities, we first need the following lemma.
Lemma 3.5. The joint probability \( \gamma(x, y, z, t_1, t_2) \) that \( Z \), starting from \( x \), does not hit \( z \) before time \( t_1 \) but does hit \( y \) before time \( t_2 \), is for \( x > y > z \) given by

\[
\gamma(x, y, z, t_1, t_2) = \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s < y )
\]

\[
= \begin{cases} 
\pi(t_2, x - y) - \pi(t_1, x - z) & \text{for } t_1 \leq t_2, \\
1 - \pi(t_1, x - z) - \int_{y}^{\infty} (1 - \pi(t_1 - t_2, \tilde{z} - z)) \tilde{f}(x, y, \tilde{z}, t_2) d\tilde{z} & \text{for } t_1 > t_2,
\end{cases}
\]

where

\[
\tilde{f}(x, y, \tilde{z}, t_2) = \frac{1}{\sigma \sqrt{t_2}} \exp\left( \frac{-m(x - \tilde{z})}{\sigma^2} - \frac{m^2 t_2}{2 \sigma^2} \right) \left( \phi\left( \frac{x - \tilde{z}}{\sigma \sqrt{t_2}} \right) - \phi\left( \frac{-x - \tilde{z} + 2y}{\sigma \sqrt{t_2}} \right) \right)
\]

(3.48)

Proof.

- For \( t_1 \leq t_2 \), we can write

\[
\mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s < y ) = \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s < y ) - \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s < y, \inf_{0 \leq s \leq t_1} Z_s > z )
\]

\[
= \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s < y ) - \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s < z )
\]

\[
= \pi(t_2, x - y) - \pi(t_1, x - z).
\]

- For \( t_1 > t_2 \), note that

\[
\mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s < y ) = \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z ) - \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s > z, \inf_{0 \leq s \leq t_1} Z_s > y )
\]

\[
= 1 - \pi(t_1, x - z) - \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s > y ),
\]

where

\[
\mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s > y )
\]

\[
= \int_{y}^{\infty} \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z, \inf_{0 \leq s \leq t_2} Z_s > y | Z_{t_2} = \tilde{z} ) \mathbb{P}( Z_{t_2} \in d\tilde{z} )
\]

\[
= \int_{y}^{\infty} \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s - Z_{t_2} > z - \tilde{z} ) \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s > y, Z_{t_2} \in d\tilde{z} )
\]

\[
= \int_{y}^{\infty} \mathbb{P}( \inf_{0 \leq s \leq t_1} Z_s > z - \tilde{z} + x ) \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s > y, Z_{t_2} \in d\tilde{z} )
\]

\[
= \int_{y}^{\infty} (1 - \pi(t_1 - t_2, \tilde{z} - z)) \mathbb{P}( \inf_{0 \leq s \leq t_2} Z_s > y, Z_{t_2} \in d\tilde{z} ),
\]

where is used that \( Z \) has independent and stationary increments.

Now the result follows by noting that by a modification of equation (3.8) to the current setting, it holds that

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\[ \mathbb{P} \left( \inf_{0 \leq s \leq t_2} Z_s > y, Z_{t_2} \in d\tilde{z} \right) = \hat{f}(x, y, \tilde{z}, t_2)d\tilde{z}. \]

Now the desired probabilities are, in analogy to equation (3.15), given by

\[ \mathbb{P}(\tau_e \leq T, \tau_b > u | \tau_e > t, Y^{(n)}) = \int_{z_e}^{\infty} \gamma(x, z_c, z_b, u - t, T - t) f(t, x) dx \]

and

\[ \mathbb{P}(\tau_e \leq u, \tau_b > u | \tau_e > t, Y^{(n)}) = \int_{z_e}^{\infty} \gamma(x, z_c, z_b, u - t, u - t) f(t, x) dx. \]

Recall that the objective was to compute the last two integrals in equation (3.46). Let us first consider the second one, that is

\[ - \int_t^{\infty} e^{-r(u-t)} \mathbb{P}(\tau_e \leq T, \tau_b \in du | \tau_e > t, Y^{(n)}) \mathbb{P}(\tau_e \leq u | \tau_e > t, Y^{(n)}) du = (I) + (II), \]

where

\[ (I) = \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} \mathbb{P}(\tau_e \leq T, \tau_b > u | \tau_e > t, Y^{(n)}) du \]

\[ = \int_{z_e}^{\infty} f(t, x) \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} \gamma(x, z_c, z_b, u - t, T - t) du dx \]

\[ = \int_{z_e}^{\infty} f(t, x) \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} (-\pi(u - t, x - z_b)) du dx \]

\[ = \int_{z_e}^{\infty} f(t, x) I_b(x) dx, \]

in which

\[ I_b(x) = \int_t^T e^{-r(u-t)} \frac{\partial}{\partial u} (-\pi(u - t, x - z_b)) du \]

\[ = \exp \left( -\frac{m(x - z_b) + (x - z_b)\sqrt{m^2 + 2r\sigma^2}}{\sigma^2} \right) \left( \Phi \left( \frac{x - z_b - \sqrt{m^2 + 2r\sigma^2}(T - t)}{\sigma\sqrt{T - t}} \right) - 1 \right) \]

\[ + \exp \left( -\frac{m(x - z_b) - (x - z_b)\sqrt{m^2 + 2r\sigma^2}}{\sigma^2} \right) \left( \Phi \left( \frac{x - z_b + \sqrt{m^2 + 2r\sigma^2}(T - t)}{\sigma\sqrt{T - t}} \right) - 1 \right), \tag{3.49} \]

which follows from equation (3.26), by replacing \( z_c \) by \( z_b \). Furthermore, we have

\[ (II) = \int_t^{\infty} e^{-r(u-t)} \frac{\partial}{\partial u} \mathbb{P}(\tau_e \leq T, \tau_b > u | \tau_e > t, Y^{(n)}) du \]

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\[
\int_{z_c}^{\infty} f(t, x) \int_{T}^{\infty} e^{-r(u-t)} \frac{\partial}{\partial u} \gamma(x, z_c, z_b, u-t, T-t) du dx
\]
\[
= \int_{z_c}^{\infty} f(t, x) \int_{T}^{\infty} e^{-r(u-t)} \frac{\partial}{\partial u} (-\pi(u-t, x-z_b)) du dx
\]
\[
- \int_{z_c}^{\infty} \int_{z_c}^{\infty} f(t, x) \hat{f}(x, z_c, \tilde{z}, T-t) \int_{T}^{\infty} e^{-r(u-t)} \frac{\partial}{\partial u} (1-\pi(u-T, \tilde{z}-z_b)) du \tilde{z} dx
\]
\[
= \int_{z_c}^{\infty} f(t, x) (J_b(x) - I_b(x)) dx
\]
\[
- \int_{z_c}^{\infty} \int_{z_c}^{\infty} f(t, x) \hat{f}(x, z_c, \tilde{z}, T-t) e^{-r(T-t)} J_b(\tilde{z}, T) d\tilde{z} dx,
\]
where
\[
J_b(x) = \int_{t}^{\infty} e^{-r(u-t)} \frac{\partial}{\partial u} (1-\pi(u-t, x-z_b)) du
\]
\[
= -\exp \left( -\frac{m(x-z_b) + (x-z_b)\sqrt{m^2+2r\sigma^2}}{\sigma^2} \right), \tag{3.50}
\]
where the last line follows by taking \( T \to \infty \) in equation (3.49). This leaves us with an expression for the last integral in equation (3.46).

Similarly, the other integral simplifies to
\[
\int_{t}^{\infty} e^{-r(u-t)} \mathbb{P}(\tau_c \leq T \land u, \tau_b > u | \tau_c > t, Y^{(n)}) du = (III) + (IV),
\]
where
\[
(III) = \int_{t}^{\infty} e^{-r(u-t)} \mathbb{P}(\tau_c \leq u, \tau_b > u | \tau_c > t, Y^{(n)}) du
\]
\[
= \int_{z_c}^{\infty} f(t, x) \int_{t}^{\infty} e^{-r(u-t)} \gamma(x, z_c, z_b, u-t, t) du dx
\]
\[
= \int_{z_c}^{\infty} f(t, x) \int_{t}^{\infty} e^{-r(u-t)} (\pi(u-t, x-z_c) - \pi(u-t, x-z_b)) du dx
\]
\[
= \int_{z_c}^{\infty} f(t, x) (\tilde{I}_b(x) - \tilde{I}(x)) dx
\]
in which \( \tilde{I}(x) \) is defined in equation (3.21) and \( \tilde{I}_b(x) \) is equivalently defined as
\[
\tilde{I}_b(x) = \int_{t}^{\infty} e^{-r(u-t)} (1-\pi(u-t, x-z_b)) du
\]
\[
= \left[ -\frac{1}{r} e^{-r(u-t)} (1-\pi(u-t, x-z_b)) \right]_{u=t}^{T} + \frac{1}{r} I_b(x)
\]
\[
= -\frac{1}{r} e^{-r(T-t)} (1-\pi(T-t, x-z_b)) + \frac{1}{r} + \frac{1}{r} I_b(x). \tag{3.51}
\]
Furthermore, we have

\begin{align*}
(IV) &= \int_T^\infty e^{-r(u-t)}\mathbb{P}(\tau_e \leq T, \tau_b > u|\tau_e > t, Y^{(n)})du \\
&= \int_T^\infty \int_{z_e}^\infty f(t, x) \int_T^\infty e^{-r(u-t)}(1 - \pi(u-t, x - z_b))dudx \\
&= \int_T^\infty f(t, x) \int_T^\infty \int_{z_e}^\infty e^{-r(u-t)}(1 - \pi(u-t, x - z_b))dudx \\
&= \int_T^\infty f(t, x)(\tilde{J}_b(x) - \tilde{I}_b(x))dx \\
&= \int_T^\infty f(t, x)\tilde{J}_b(x)d\tilde{x}dx,
\end{align*}

in which

\[
\tilde{J}_b(x) = \int_t^\infty e^{-r(u-t)}(1 - \pi(u-t, x - z_b))du \\
= \left[-\frac{1}{r}(1 - \pi(u-t, x - z_b)\right]_u=t + \frac{1}{r}J_b(x) \\
= \frac{1}{r} + \frac{1}{r}J_b(x). \tag{3.52}
\]

Putting all the above together leads to an expression for the last two integrals in equation (3.46), given by

\[-c_1P_1\int_t^\infty e^{-r(u-t)}\mathbb{P}(\tau_e \leq T \land u, \tau_b > u|\tau_e > t, Y^{(n)})du \\
- e^{z_b}\int_t^\infty e^{-r(u-t)}\mathbb{P}(\tau_e \leq T, \tau_b \in du|\tau_e > t, Y^{(n)}) \\
= e^{z_b}(I + (II)) - c_1P_1((III) + (IV)) \\
= \int_T^\infty f(t, x)\left(e^{z_b}J_b(x) + c_1P_1\tilde{I}(x) - c_1P_1\tilde{J}_b(x)\right)dx \\
+ \int_T^\infty \int_{z_e}^\infty f(t, x)\tilde{f}(x, z_e, \tilde{z}, T-t)e^{-r(T-t)}(c_1P_1\tilde{J}_b(\tilde{z}) - e^{z_b}J_b(\tilde{z}))d\tilde{x}dx. \tag{3.53}
\]

Finally, by combining equations (3.46), (3.47) and (3.53), it follows that the third term in equation (3.45), i.e.

\[
\mathbb{E}\left(\frac{\Delta P_2}{\Delta P_2 + 1}E^{\Pi C}(\tau_e)e^{-r(\tau_e - t)}1_{\{\tau_e \leq T\}}|H_t\right),
\]

is given by

\[
\int_T^\infty f(t, x)h_1(x)dx + \int_T^\infty \int_{z_e}^\infty f(t, x)\tilde{f}(x, z_e, \tilde{z}, T-t)h_2(\tilde{z})d\tilde{x}dx, \tag{3.54}
\]

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where
\[ h_1(x) = \frac{\Delta P_2}{\Delta P_2 + 1} \left( e^{z_b} J_b(x) + c_1 P_1 \tilde{I}(x) - c_1 P_1 \tilde{J}_b(x) - e^{z_c} I(x) \right), \]
\[ h_2(\tilde{z}) = \frac{\Delta P_2}{\Delta P_2 + 1} e^{-r(T-t)} (c_1 P_1 \tilde{J}_b(\tilde{z}) - e^{z_b} J_b(\tilde{z})), \]

in which \( \tilde{I}(x), I(x), J_b(x) \) and \( \tilde{J}_b(x) \) are respectively given by equations (3.21), (3.26), (3.50) and (3.52). Also, note that the first two terms of equation (3.45) are equal to the integral in equation (3.27), by taking \( R = 0, c = c_2 \) and \( P = P_2 \) in equation (3.28) (we denote the function corresponding to this choice by \( h_0 \) in the following theorem).

To summarize, in the above we have proven the following theorem.

**Theorem 3.4** (Price of a CoCo with a regulatory trigger and a conversion into shares).

The secondary market price at time \( t < \tau_c \) of the CoCo with a regulatory trigger and a conversion into shares is given by

\[ C(t) = \int_{z_c}^{\infty} \left( h_0(x) + h_1(x) \right) f(t, x) \, dx + \int_{z_c}^{\infty} \int_{z_c}^{\infty} f(t, x) \tilde{f}(x, z_c, \tilde{z}, T-t) h_2(\tilde{z}) \, d\tilde{z} \, dx, \]

(3.55)

where \( \tilde{f}(x, y, \tilde{z}, t_2) \) is given by equation (3.48) and

\[ h_0(x) = \frac{r - c_2}{r} P_2 e^{-r(T-t)} (1 - \pi(T-t, x, z_c)) + \frac{c_2 P_2}{r} - c_1 P_1 I(x), \]
\[ h_1(x) = \frac{\Delta P_2}{\Delta P_2 + 1} \left( e^{z_b} J_b(x) + c_1 P_1 \tilde{I}(x) - c_1 P_1 \tilde{J}_b(x) - e^{z_c} I(x) \right), \]
\[ h_2(\tilde{z}) = \frac{\Delta P_2}{\Delta P_2 + 1} e^{-r(T-t)} (c_1 P_1 \tilde{J}_b(\tilde{z}) - e^{z_b} J_b(\tilde{z})), \]

in which \( \tilde{I}(x), I(x), J_b(x) \) and \( \tilde{J}_b(x) \) are respectively given by equations (3.21), (3.26), (3.50) and (3.52).

Now note that the first integral in equation (3.55) can be approximated in the same way as in equations (3.32) and (3.33), by replacing \( h(x) \) with \( h_0(x) + h_1(x) \). Furthermore, the second integral in equation (3.55) can, in analogy to equation (3.32), be written as

\[ \int_{z_c}^{\infty} \int_{A_{n+1}} h_2(\tilde{z}) \tilde{f}(z_{n+1}, z_c, \tilde{z}, T-t) \tilde{f}(t-t_n, z_{n+1}, z_n) g_n(z^{(n)}|Y^{(n)}, \tau_c > t_n) \, dz^{(n+1)} \, d\tilde{z} = \int_{A_{n+2}} h_2(z_{n+2}) \tilde{f}(z_{n+1}, z_c, z_{n+2}, T-t) \tilde{f}(t-t_n, z_{n+1}, z_n) g_n(z^{(n)}|Y^{(n)}, \tau_e > t_n) \, dz^{(n+2)}, \]

(3.56)

for \( A_n \) as defined in equation (3.29).

Now note that, by definition of \( \tilde{f} \), it holds that

\[ \int_{z_c}^{\infty} \tilde{f}(z_{n+1}, z_c, z_{n+2}, T-t) \, dz_{n+2} = \int_{z_c}^{\infty} \mathbb{P} \left( \inf_{0 \leq s \leq T-t} Z_s > z_c, Z_{T-t} \in d z_{n+2} \mid Z_0 = z_{n+1} \right) \]

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by the above we know that 

\[ f(z_{n+1}, z_c, z_{n+2}, T-t) \max{(t-t_n, z_{n+1}, z_n)} \max{(z^{(n)})|Y^{(n)}}, \tau_c > t_n \] 

is not a density function on \( A_{n+2} \), so it is not possible to proceed in the same way as before. However, by the above we know that

\[
\frac{\hat{f}(z_{n+1}, z_c, z_{n+2}, T-t) \hat{f}(t-t_n, z_{n+1}, z_n) \max{(z^{(n)})|Y^{(n)}}, \tau_c > t_n)}{1 - \pi(T-t, z_{n+1} - z_c)}
\]

is a density function on \( A_{n+2} \).

So if we have a sample \( ((z^{(n+2)})t, \ldots, (z^{n+2})G) \) from the \( (n + 2) \)-dimensional distribution with this density, we can approximate the integral in equation (3.56) by

\[
\frac{1}{G} \sum_{g=1}^{G} h_2(z_{n+2}^g)(1 - \pi(T-t, z_{n+1}^g - z_c)).
\]

This sample is, in analogy to Algorithm 3.2, obtained by the following MCMC-algorithm.

**Algorithm 3.4** (Random walk Metropolis-Hastings).

1. In each iteration \( g, g = 1, \ldots, n_0 + G \), given the current value \( (z^{(n+2)})g \), the proposal \( (z^{n+2})g' \) is drawn according to

\[
(z^{(n+2)})g' = (z^{(n+2)})g + X, \quad \text{for } X \sim N_{n+2}(0, \Sigma),
\]

where the \( (n + 2) \times (n + 2) \)-covariance matrix \( \Sigma \) is chosen to reach some desired acceptance rate.

2. Set

\[
(z^{(n+2)})g+1 = \begin{cases} 
(z^{(n+2)})g' \quad \text{with prob. } \alpha((z^{(n+2)})g, (z^{(n+2)})g') \\
(z^{(n+2)})g \quad \text{with prob. } 1 - \alpha((z^{(n+2)})g, (z^{(n+2)})g')
\end{cases}
\]

where the acceptance-probability \( \alpha((z^{(n+2)}), (z^{(n+2)})g') \) is given by

\[
\min \left\{ 1, \frac{\hat{f}(z_{n+1}', z_c', z_{n+2}', T-t) \hat{f}(t-t_n, z_{n+1}', z_n') b_n((z^{(n)})g'| y^{(n)})(1 - \pi(T-t, z_{n+1} - z_c))}{\hat{f}(z_{n+1}, z_c, z_{n+2}, T-t) \hat{f}(t-t_n, z_{n+1}, z_n) b_n(z^{(n)}| y^{(n)})(1 - \pi(T-t, z_{n+1} - z_c))} \right\}.
\]

3. Discard the draws from the first \( n_0 \) iterations and save the sample \( (z^{(n+2)})n_0+1, \ldots, (z^{(n+2)})n_0+G \).
4 Applications

4.1 Impact of model parameters

In this section the impact of several model parameters on the CoCo price is investigated. Unless stated otherwise the parameters are as in Table 4.1 and a CoCo has a regulatory trigger as in Section 3.6. For the choice of the base case parameters, some restraints should be taken into account. For example, the conversion trigger should be higher than the default trigger. Also, a CoCo should pay a higher coupon than straight debt, to compensate for the higher risk. We have no empirical evidence for a reasonable level of accounting noise, so the volatility of accounting noise is set equal to the base case parameter chosen by Duffie and Lando [10].

Furthermore, in Section 3.6 it is assumed that the CoCo holders own a fraction \( \frac{\Delta P_2}{\Delta P_2 + 1} \) of the firm after conversion, in this section we will denote this fraction by \( \rho \) and call it the dilution ratio, as in [6]. A dilution ratio of \( \rho = 0 \) means that the CoCo suffers a principal write-down (PWD) at conversion, while \( \rho = 1 \) corresponds to the extreme case that the original shareholders are completely wiped out at conversion.

To compute prices for PWD CoCos, we make use of Theorem 3.3. The involving integral is approximated as in equation (3.33), for which the necessary sample is obtained by using Algorithm 3.2. To compute prices for CoCos with a conversion into shares, we make use of Theorem 3.4, where the first term in the pricing formula follows again by using Algorithm 3.2 and the second term is approximated as in equation (3.57), for which the necessary sample is obtained by execution of Algorithm 3.4. Then the figures are produced by repeatedly following this procedures for different values of the parameters.
<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
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<tr>
<td>Initial asset value, $V_0$</td>
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</tr>
<tr>
<td>Number of acc. reports until time $t$, $n$</td>
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<tr>
<td>Conversion trigger, $v_c$</td>
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<tr>
<td>Default trigger, $v_b$</td>
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<tr>
<td>Recovery rate at default, $\alpha$</td>
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<td>Total principal straight debt, $P_1$</td>
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<tr>
<td>Coupon straight debt, $c_1$</td>
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<tr>
<td>Total principal CoCos, $P_2$</td>
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<tr>
<td>Coupon CoCos, $c_2$</td>
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</tr>
<tr>
<td>Maturity CoCos, $T$</td>
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<tr>
<td>Volatility asset process, $\sigma$</td>
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<tr>
<td>Mean accounting noise, $\mu_\epsilon$</td>
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</tr>
<tr>
<td>Volatility accounting noise, $\sigma_\epsilon$</td>
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</tr>
<tr>
<td>Risk free interest rate, $r$</td>
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</tr>
</tbody>
</table>

Table 4.1: Base case parameters.

4.1.1 Volatility

In Figure 4.1 several CoCo prices are plotted against the volatility of assets, $\sigma$. The solid line corresponds to a PWD CoCo. Clearly, the price of a PWD CoCo decreases when assets become more volatile. This is of course as one would expect, as an increasing $\sigma$ increases the probability of the principal write-down happening, causing the CoCo price to decrease. The dashed line, corresponding to $\rho = 0.5$, shows already that this negative effect from volatility on the CoCo price is weaker when terms of conversion are more favorable to the CoCo investor. In the extreme case that shareholders are completely wiped out at conversion, corresponding to the dashed-dotted line, this negative effect is even partially reversed. In this case, the price first increases with volatility, as the price increases when the favorable conversion becomes more likely. However, for higher volatility levels the increasing probability of default pushes the price down again.

4.1.2 Accounting noise

Recall that the size of the noise of the noisy accounting reports is determined by the volatility $\sigma_\epsilon$. Now let us consider the relationship between the amount of accounting noise and the price of a CoCo. In the upper panel of Figure 4.2 the price of a PWD CoCo is plotted against $\sigma_\epsilon$. The figure shows that the CoCo price decreases when the amount of accounting noise increases. This is in line with the results of Duffie and Lando [10], as they find that the default probability decreases when the reports become more noisy. In our CoCo setting, this would mean that the probability of conversion increases when $\sigma_\epsilon$ increases, causing the CoCo price to decrease. In the upper panel of Figure 4.2 a CoCo with a regulatory trigger is considered, but it is also interesting to look at a CoCo
which conversion trigger only depends on the noisy accounting reports, as in subsection 3.5.2. This type of CoCo is priced by the formula given in equation (3.42), which is computed using the approximation in equation (3.40), for which the necessary samples are obtained by using Algorithm 3.3. In the lower panel of Figure 4.2, the dashed line shows that for a trigger which only depends on accounting reports, the CoCo price is much more sensitive to accounting noise than in case of a regulatory trigger.

Figure 4.1: CoCo price versus $\sigma$, for different values of $\rho$. 

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Figure 4.2: PWD CoCo price versus $\sigma_\epsilon$, the volatility of accounting noise. The upper panel considers only a regulatory trigger, the lower panel considers both a regulatory trigger and a book value trigger.
4.1.3 The conversion trigger

In Figure 4.3, the CoCo price is plotted against the conversion trigger. The solid line corresponds to a PWD CoCo. As one would expect, the price of a PWD CoCo is lower for a higher conversion trigger, as a higher conversion trigger increases the probability of a principal write-down. However, the other lines show that if conversion terms are favorable enough to the coco investor, the price will increase with the conversion trigger. In the extreme case that the dilution ratio $\rho$ equals one, represented by the dashed line, the CoCo price increases strongly with the conversion trigger, while for other cases this positive effect is weaker ($\rho = 0.5$, dashed-dotted line) or terms of conversion are not favorable enough for CoCo holders to create this effect ($\rho = 0.1$, dotted line).

![Figure 4.3: CoCo price versus $v_c$, for different values of $\rho$.](image)

4.1.4 The number of shares received at conversion

In Figure 4.4, the price of a CoCo is plotted against $\Delta$, the number of shares received at conversion per unit of principal, for different values of straight debt in the firm’s capital structure. $\Delta = 0$ corresponds to a principal write-down CoCo, while $\Delta = \infty$ corresponds to the case in which all of the original shareholders are wiped out at conversion and the CoCo investors are the only shareholders left. As becomes clear from Figure 4.4, the CoCo price increases with $\Delta$. This is of course as expected, as a higher $\Delta$ means a higher payout at conversion. Furthermore, the figure shows that a CoCo with a conversion into shares has a higher price when there is a lower amount of straight debt issued. Hence
the CoCo is more valuable when the firm has a lower leverage. This can also be easily explained, as the CoCo investors receive a fraction of the firm’s equity value at conversion and of course the equity value is higher in case there are less liabilities.

![Graph showing CoCo price versus Δ for different values of $P_1$.](image)

**Figure 4.4: CoCo price versus Δ, for different values of $P_1$.**

### 4.2 Impact CoCos on capital structure

In this section we examine the impact of the issuance of CoCos on the capital structure of the bank and on incentives for shareholders. The computation of the prices and the production of the figures is performed following the same procedures as in Section 4.1.

#### 4.2.1 Replacing debt with CoCos

First consider the case in which straight debt is replaced with CoCos. In Figure 4.5 the change in equity value occurring when 5 units of straight debt are replaced with 5 units of CoCos, is plotted against the conversion trigger. It is seen from the solid line and the dotted line that shareholders only benefit from replacing debt with CoCos, when the terms of conversion are favorable enough to the shareholders and the trigger is high enough. The dashed line and the dashed-dotted line show that shareholders have no incentive to issue highly dilutive CoCos.
4.2.2 Replacing equity with CoCos

Similarly, we could consider the case in which equity is replaced with CoCos and the proceeds of the CoCo issue are used to buy back equity. This case gives similar results as when we replace debt with CoCos. Again, the shareholders only have incentive to issue CoCos, when the conversion terms are not too favorable to the CoCo investors, as we see by Figure 4.6. By comparing the Figures 4.5 and 4.6, it is seen that shareholders have a higher incentive to replace debt with PWD CoCos than to replace equity with PWD CoCos. However, if they have to issue more dilutive CoCos, they should choose to replace equity with CoCos, instead of replacing debt with CoCos.

4.2.3 Risk taking incentives

We can examine the impact of the introduction of Contingent Convertible bonds on the risk taking incentives of the shareholders. This can be done by looking at the relation between equity value and $\sigma$, the volatility of the asset process. If equity value increases with volatility, this would mean that the equity holders have an incentive to take on more risk. In the upper panel of Figure 4.7, the value of equity is plotted against $\sigma$ for a total straight debt of $P_1 = 50$. The solid line corresponds to a firm with no CoCos, which shows that in this situation equity value decreases when volatility increases. That is, in this case the shareholders have no risk taking incentive. However, if 5 units of
Figure 4.6: Change in equity value when 5 units of equity are replaced with 5 units of CoCos (in market value).

equity are replaced with 5 units of principal write-down CoCos, then the firm’s value of equity increases with volatility, as shown by the dashed line. This suggests that the issuance of principal write-down CoCos could give shareholders an incentive to take on more risk. The dashed-dotted line corresponds to an issuance of CoCos with a conversion into shares for a dilution ratio $\rho = 1$, which means that the original shareholders are completely wiped out. The dotted line corresponds to a situation including CoCos with a conversion into shares for a dilution ratio $\rho = 0.5$. These two lines show that this risk taking incentive does not appear in case of an issuance of CoCos with a conversion into shares. This difference can of course be explained by the fact that the write-down of a CoCo is beneficial to the shareholders, as equity will raise in this case, while a conversion into shares may be disadvantageous to the original shareholders.

In the lower panel of Figure 4.7 the same situation is considered, now for a firm with a higher leverage; where the total amount of straight debt is 60. Again, the solid line corresponds to the case where no CoCos are issued. In contrast to the case with lower leverage, the shareholders already have an incentive to take on more risk without CoCos in the capital structure. It is seen by the dashed line that PWD CoCos certainly are not able to reverse this incentive, while the CoCos with a conversion into shares are only able to neutralize this incentive for higher volatility levels, as can be seen from the dashed line and the dashed-dotted line. This is not in line with Chen et al [6], who find that CoCos are able to reverse this risk taking incentive. This difference could be explained by the fact that in their model, the maturity of debt is rolling over, while we use a fixed
perpetual maturity. Also, they make a distinction between jump risk and diffusion risk, which is not possible in our model. To summarize, we see that at the lower leverage level there is no risk taking incentive for the shareholders in a firm without CoCos, but the issuance of PWD CoCos could create such an incentive. The incentive does not appear when conversion terms of CoCos are beneficial enough to the CoCo investors. In case of a higher leverage, the risk taking incentive is already there in a firm without CoCos and it can only be partially neutralized by issuing CoCos with a conversion into shares and a sufficiently high dilution ratio.

4.2.4 Investment incentives

In this subsection it is investigated if shareholders have an incentive to invest in the firm. In a structural model without CoCos, the shareholders do not have an incentive to invest exactly at the moment the firm most needs an investment: when the firm is near bankruptcy. In this case, the firm would need an investment to avoid bankruptcy, however almost all of the value of the investment will be captured by the debt holders, as the value of debt increases when the probability of a bankruptcy is reduced. This phenomenon is called debt overhang. This problem of debt overhang could possibly be solved by the issuance of CoCos as, in the presence of CoCos, the shareholders may have an incentive to make an investment, to avoid conversion.

The investment incentives of shareholders can be investigated by looking at what happens when assets are increased by one unit, financed through one unit of equity (in market value). If equity raises by more that one unit, the shareholders would make a profit when they invest, giving them an incentive to do so. However, when equity increases by less than one unit, investment is not beneficial to shareholders. Consider the case in which a new accounting report has just be released, with an asset value of $Y_t = 100$, we can then examine what happens when this asset value increases by one unit. The profit of this investment of one unit is plotted against the conversion trigger in Figure 4.8. The solid black line shows the concept of debt overhang: in the case of no CoCos the shareholders do not make a profit when they invest, they actually suffer a small loss. Furthermore, the dashed-dotted lines show that when the terms of conversion are favorable to shareholders, the shareholders do not have an incentive to invest in the firm. The black dashed-dotted line corresponds to the existence of a PWD CoCo in the capital structure of the firm and shows that the PWD CoCo even makes the investment incentive for shareholders more negative, especially close to the conversion trigger. This shows that CoCos with a principal write-down or less dilutive CoCos are not capable of solving the problem of debt overhang. However, highly dilutive CoCos do give the shareholders an incentive to invest to avoid conversion. This is shown by the dashed lines in Figure 4.8, which correspond to highly dilutive CoCos and show that the shareholders make a profit when they invest. Especially close to the conversion trigger, the shareholders have a big incentive to invest in a last attempt to avoid conversion. To summarize, when terms of conversion are beneficial enough to CoCo investors, CoCos are capable of creating an investment incentive for the shareholders. However, PWD CoCos and less dilutive CoCos are certainly not capable of creating such an incentive.
Figure 4.7: Equity value versus volatility, where equity is replaced by different types of CoCos. The upper panel corresponds to a lower level of leverage than the lower panel.
Figure 4.8: Profit (a negative profit means a loss) made by the shareholders, following from the investment of one unit, plotted against the conversion trigger.

4.3 The role of the MDA trigger

In this section we will investigate the role of the MDA trigger in the valuation of the CoCos. This means that in the valuation of a CoCo, we apply Theorem 3.4 and Algorithm 3.2, but with a different coupon term, as in equation (3.34). This computation is performed as described under the header Early cancelling of coupons at the end of Subsection 3.5.1, by using Algorithm 3.4. To demonstrate the relevance of the inclusion of this trigger in the valuation of CoCos, we will look at the big price drop that the CoCos of Deutsche Bank suffered at the beginning of 2016. On January 28 Deutsche Bank reported a net loss of 2.1 Billion EUR over the last quarter of 2015. The relevant report furthermore reported for its Risk-Weighted Assets a value of 397 Billion EUR, down from 408 Billion EUR in the previous accounting report. Also, the Common Equity Tier 1 (CET1) ratio (defined as the fraction of the common equity and the RWA) fell from 11.5% to 11.1%, primarily reflecting the net loss over the quarter. The particular report caused a big downward move in the price of the CoCos of Deutsche Bank. At this time, Deutsche Bank had four different CoCos issued (two in USD, one in EUR, one in GBP, all PWD CoCos).

To simplify the case, we will only consider the EUR CoCo. This CoCo’s write-down is

\[ \text{Information comes from the Financial Data Supplement 4Q2015, which can be found at https://www.db.com/ir/en/download/FDS_4Q2015_11_03_2016.pdf} \]
triggered when the CET1-ratio hits the level of 5.125% and it pays a coupon of 6%. As is clear from the above, the CET1-ratio did not even come close to the low trigger level. Still, the CoCo price tumbled 19.5% percent within a week after the announcement of the report. Probably, this happened out of fear for reaching the MDA trigger and the subsequent cancelling of coupon payments. The model proposed in Chapter 3 seems particularly relevant to analyze this case, as we can include the announcement of a bad accounting report in the valuation, as well as the early cancelling of coupons when the MDA trigger is hit. The precise value of the MDA trigger is not publicly known, so it is not possible to use the real value of the MDA trigger. However, it is still interesting to examine how much of a price drop the model can explain by taking the MDA trigger close to the reported values.

Unless stated otherwise, we use the same parameters as in Table 4.1. Before the bad accounting report arrives, we are in the case that there is one accounting report, with a value $Y_{t_1} = 408\text{ bn}$, after the new accounting report arrives, we have two accounting reports with values $Y_{t_1} = 408\text{ bn}$ and $Y_{t_2} = 397\text{ bn}$. The triggers are chosen such that they correspond with CET1 ratios at the moment of the accounting report. That is, we choose $v_c$ such that it corresponds to a CET1 ratio of 5.125%. As we know the CET1 ratio is 11.1% where RWA is 397, this means that the total amount of debt (only CoCos and straight debt in the model) is $397 \times 0.889 = 352.93$. So a CET1 ratio of 5.125% would then correspond to a RWA value of $352.93 / (1 - 0.05125) = 372$, which is thus the value of the conversion trigger $v_c$. The value of the MDA trigger $v_{cc}$ can be chosen in the same way, a MDA trigger at a CET1 ratio of 10% would correspond to a RWA value of $352.93 / (1 - 0.1) = 392$. The coupon of the CoCo is of course chosen as $c_2 = 0.06$. As the relevant CoCo has a perpetual maturity, we choose the first call date, 10/10/18, as the maturity. Because we assumed that the second accounting report arrives at 01/28/16, $t = 0$ corresponds to 07/28/15. Hence $T = 3 + 2/12 + 13/365$. In Figure 4.9 the price change after the announcement of a bad accounting report is illustrated for different choices of the MDA trigger. The solid line corresponds to the case that the MDA-trigger is not included in the model, in this case only a drop of 14.3% in the CoCo price occurs, when looking at the price just before the release of the accounting report and afterwards. However, if we add the MDA trigger to the model, a more negative price change can be found. The dashed line corresponds to the case that we take the MDA trigger at 11%, i.e. just beneath the reported CET1 value. This gives a price drop of 18.7%. If we take the MDA trigger to be 10%, the price drops by 17.3%, which is illustrated by the dotted line. However, we have to take the MDA trigger above the reported CET1 ratio of 11.1% to create a price drop of 19.5%, this is showed by the dashed-dotted line. That is, a price drop of 19.5% corresponds in the model to the situation that the MDA trigger is already breached, which was not the case. However, it is clear that a significant part of the price change is driven by the MDA trigger, not by the conversion trigger. The above illustrates the added value of taking the MDA trigger into account in the valuation of a CoCo, especially when the MDA trigger is coming close, but the conversion trigger is still far away.
Figure 4.9: CoCo price around the release of the bad accounting report, for different choices of the MDA trigger. The downfall in the middle of the figure reflects the influence of the accounting report. The solid line suffers a downfall of 14.3%, the dotted line drops by 17.3%, the dashed line by 18.7% and the dashed-dotted line drops by 19.5%.
Popular summary

In this thesis different pricing models are studied for the pricing of Contingent Convertible bonds (CoCos). These are special type of bonds, which convert into equity or are written down, when the capital of the issuing bank becomes too low. In this way, outstanding debt is reduced and capital is raised, to strengthen the capital position of the bank. In practice, this conversion is typically triggered by the capital ratio of the issuing bank falling below some threshold or by a regulator calling for conversion. However, in all of the existing pricing models the conversion of CoCos is triggered by a market value, like a stock price, falling below some threshold. In the first two chapters of this thesis the existing literature is examined and one particular model is studied in detail. In the third chapter of this thesis a new CoCo pricing model is proposed which takes into account the difference between market values and book values. In this model, the market cannot observe the true asset value process, but it only has access to noisy accounting reports, which are only published at discrete moments in time, typically every three months in practice. In this way, the price of CoCos can only be based on the information from the accounting reports, not on the true asset process. The model does not lead to closed form solutions for CoCo prices, but Markov Chain Monte Carlo methods are used to compute prices.
Bibliography


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