Extremality in higher spin gravity and $\mathcal{W}(2, \frac{5}{2}, 4)$ unitarity bounds.

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Abstract

In [10] a new proposal for black hole extremality was put forward and it was shown that supersymmetry does not require extremality in higher spin gravity, finding agreement with the "large-c" limit of the $W_{3(2)}$ CFT. We review this proposal and discuss how these solutions can be considered as exact solutions at finite-$c$. We then apply this proposal to spin-4 charged black holes in AdS$_3$ hypergravity and study their extremality properties. In [11] it was shown that this theory allows for an upper bound on the spin-4 charge derived from the entropy solely. It precise nature thus remained unclear. We show that when extremality is defined in terms of the Jordan classes of the holonomy instead [10], the same bound is recovered as an extremality bound. We then turn to the dual $W(2,\frac{5}{2},4)$ CFT and study its unitary representations. We derive the appearance of a semiclassical unitarity bound in the NS sector, that agrees with the extremality bound in the limit $\mathcal{L} \to \infty$.  

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1 Introduction

Higher spin theories are an extension of Einstein gravity, and are most prominently known from Vasiliev gravity\[1\]. These theories are characterized by the highly non-linear interactions of the metric field $g_{\mu\nu}$ to a tower of massless higher spin fields on AdS and are notoriously difficult to study explicitly in general dimensions. In some sense, the theories lie in the middle between ordinary gravity and string theory and provide a framework to study the non-linear dynamics that are expected to arise in theories of quantum gravity, i.e. string theory. In string theory there is also an infinite tower of massive higher spin fields. In this sense, string theory is believed to be a spontaneously broken phase of the massless higher spin theory. Through a Higgs mechanism the spontaneously broken symmetries will then give mass to the higher spin fields that are present in the string theory.

It goes without justification why one is thus interested in studying black holes in such theories. However, under the increased amount of higher spin gauge symmetries concepts such as "geometry" and "causality" are not invariant, and notions such as "black holes" become elusive. Nonetheless, AdS/CFT holography has provided an interesting framework to study these theories. In this context the higher spin gauge fields, translate into conserved currents in the dual CFT. Because of this increased number of increased currents the CFT duals of higher spin gravity are expected to be tractable and makes higher spin holography an interesting framework to understand the related problems. An important class of such holographic dualities for AdS$_3$/CFT$_2$, are the proposals of Gaberdiel and Gopakumar\[2\][3] that relate the interacting Prokushkin-Vasiliev higher spin theory\[4\][5] to the "large-c" limit of the $W_N$ minimal model coset CFT\[6\].

The gravity side of the correspondence, contains additional matter fields that couple to the higher spin gauge fields, but there exist consistent truncations in which case these matter fields decouple. In that case, the higher spin theory can be described by a three-dimensional Chern-Simons theory, similar to ordinary gravity\[6\] which becomes topological in three dimensions. The fully fledged higher spin gravity theory on AdS$_3$ can then be described by two copies of the infinite dimensional $hs(\lambda)$ gauge algebra, in which case the conserved currents in the dual CFT form an $W_{\infty}[\lambda]$ chiral algebra, extending the Virasoro algebra of normal gravity. This class of $W$ algebras, have a complex and rich structure and have been shown to be related to the tensionless limit of string theory on AdS$_3 \times S^3 \times \mathbb{T}^4$. In this thesis we shall be interested in a particular, and easier, class of such dualities which is when $\lambda = \pm N$ is an integer. In that case, the higher spin Chern Simons gauge algebra can be truncated to $\mathfrak{s}(N)$. The infinite tower of massless higher spin fields, truncates in this case to $N - 1$ gauge fields of spins $s \leq N$. The boundary dynamics are then described by a $W_N$ CFT.

That gravity can be described by a Chern Simons theory in three dimensions, has further advantages and it allows one to define notions such as "black holes" in a gauge invariant way. The first proposal was made by Gutperle and Kraus in\[8\] who proposed that a higher spin black hole should be defined as a flat connection with a trivial holonomy along the thermal Euclidean cycle on the torus. The latter are a generalization of the smoothness condition of the Euclidean horizon in metric formulation and lead to a consistent thermodynamics. Furthermore, its associated partition function was shown to match a perturbative CFT calculation in\[9\]. The holonomy along the non-contractable spatial cycle on the other hand, was proposed to define different black hole solutions.

In this thesis, we shall be interested in a particular aspect of black holes, which is extremality. In metric formulation, these form a specific class of black holes that are characterized by a saturation of the bound $M \leq |Q|$, that ensures an absence of a naked singularity. In particular, for such solutions, the two event horizons coincide and they are at zero temperature. Given the above considerations, it is then natural to ask for a definition of extremality that exploits the topological nature of the (higher spin) Chern Simons gauge algebra.
theory. Such a definition was given in the recent work [10]. There it was proposed that an extremal higher spin black hole should be defined by a non-diagonalizable spatial component of the connection. This allows for a natural classification of extremality in terms of the Jordan classes of the holonomy.

The above described holographic dualities and notions extend naturally to the supersymmetric case, in which the higher spin theory is described by a suitable super-gauge algebra and the boundary dynamics by an extension of the super-Virasoro algebra. In this context, extremality is of interest because of its relation to BPS bounds in the dual CFT. In standard gravitational theories, it is a known fact [10], that solutions that preserve supersymmetry, must necessarily be at zero temperature and therefore extremal. It came then as quite a surprise, that when a large enough gauge algebra is taken, $\mathfrak{sl}(3|2)$ to be precise, the theory allows for non-extremal BPS solutions. These solutions where found to saturate the semiclassical limit of the $W_{(3|2)}$ BPS bound. In this work we will discuss in more detail the properties of these solutions at finite-c. In particular, we will discuss the implications once finite-c corrections, dictated by the CFT BPS bound are taken into account.

In the second portion of this thesis we will turn our attention to hypergravity on AdS$_3$. This theory is a higher spin extension of supergravity and contains a spin-$5/2$ field. In addition, the theory contains a spin-4 field, $U$, that is required for consistency. It was shown in [11], that this theory allows for two extremality bounds on the spin-4 charge. The lower of these bounds, arises naturally as a semiclassical BPS bound from the asymptotic $W(2,\frac{5}{2},4)$ algebra. No fundamental explanation however, has so far been found for the appearance of the upper bound on the spin-4 charge, which was derived solely from the entropy. In this thesis, we will apply the extremality proposal of [10] to this hypergravity theory. We show that saturation of this bound is achieved by a set of Jordan classes allowing for a more fundamental interpretation as an extremality bound.

In the final and last portion of this thesis, we will turn to the dual $W(2,\frac{5}{2},4)$ CFT of hypergravity, where we shall be interested in its unitary representations. We will show how, in the Neveu-Schwarz sector, a semiclassical upper bound on the spin-4 charge appears naturally as a unitarity bound, very similar the the $W_3$ theory. The unitarity bound in the Neveu-Schwarz sector, shows a striking resemblance to the extremality bound. The two are found to coincide at the quadratic level in $L \to \infty$.

The structure of this work will be as follows. In section 2 we introduce AdS-space, the BTZ black hole and the AdS/CFT correspondence. Section 3 will reformulate this framework to Chern-Simons formulation, which will allow for a natural generalization to higher spin gravity in section 4. We discuss the importance of the choice of embedding of the gravitational $\mathfrak{sl}(2)$ subalgebra and discuss the emergence of a centrally extended $W_3 \otimes W_3$ algebra describing the boundary dynamics. Subsection 4.5 will generalise the BTZ black hole to include higher spin charges. Section 5 will include an introduction to supersymmetry and supergravity after which we discuss the $W_{(3|2)}$ theory of [10] in section 6. Hypergravity will be discussed in section 7 and the $W(2,\frac{5}{2},4)$ algebra is discussed in section 8.
2 Anti-de-Sitter Gravity

Anti-de Sitter space is the vacuum solution to the Einstein-Hilbert action with a negative cosmological constant $\Lambda$:

$$S_{EH} = \frac{1}{16\pi G_N} \int_M d^d x \sqrt{-g} (R - 2\Lambda). \quad (2.1)$$

This vacuum solution has acquired much interest due to its role in the AdS/CFT duality. The purpose of this section will be as follows. In subsection [2.1] we will discuss aspects of AdS as an embedding space. We will further pay attention to the isometries of AdS and discuss their role in the AdS/CFT correspondence.

Then, in subsection [2.2] we will discuss the BTZ black hole. We will discuss that this is not a black hole in the conventional sense and discuss asymptotically AdS spacetimes. In subsection [2.2.1] we will discuss the thermodynamics of the BTZ black hole. We will close this section with a brief discussion of the 2d CFT and how it is related to solutions of AdS$_3$. The material in this chapter will follow references such as [12][13][14].

2.1 AdS$_d$ as an embedding space

AdS$_d$ is most easily thought of as the Lorentzian generalization of the hyperboloid. For this we consider $\mathbb{R}^{d-1,2}$ with metric $\eta_{MN} = (-1,1,\cdots,1,-1)$ and denote the coordinates of the embedding space by $X^M$.

The Lorentzian metric of constant negative curvature can then be obtained from the metric:

$$ds^2 = \eta_{MN} dX^M dX^N \quad \Rightarrow \quad ds^2 = -dX_0^2 - dX_1^2 + \cdots + dX_{d-1}^2, \quad (2.2)$$

via the embedding:

$$-X_0^2 - X_1^2 + \cdots + X_{d-1}^2 = -l^2. \quad (2.3)$$

The parameter $l$ is called the 'AdS-radius' and is related to the cosmological constant as $\Lambda = -\frac{1}{l^2}$.

This representation of AdS makes it manifest that its isometry group is $SO(d-1,2)$. The $SO(d-1)$ factor rotates the spatial coordinates $X^i$, $i = 1 \cdots d-1$ whereas the $SO(2)$ factor rotates the time-like coordinates. The $(d+1)d\over 2$ killing vectors of AdS$_d$ that generate the isometries of the hyperboloid can be most easily found in terms of the coordinates $X^M$ of the embedding space. Specifying to $d = 3$, we denote the embedding coordinates by $(U, X, Y, V)$ with metric $(-1,1,1,-1)$. The following 6 killing vectors then generate an $so(2,2)$ lie algebra:

$$J_{01} = V \partial_U - U \partial_V, \quad J_{23} = X \partial_V - Y \partial_X, \quad J_{02} = X \partial_V + V \partial_X, \quad (2.4)$$

$$J_{12} = X \partial_U + U \partial_X, \quad J_{03} = Y \partial_V + V \partial_Y, \quad J_{13} = Y \partial_V - U \partial_Y.$$

Of these $J_{01}$ generates time translations and $J_{23}$ generates rotations in the $x - y$ plane. The other four generate spatial rotations and boosts. Explicitly, the $so(2,2)$ lie algebra reads:

$$[J_{AB}, J_{CD}] = \eta_{BC}J_{AD} + \text{permutations}. \quad (2.5)$$

**Inönü Wigner contraction:** Recovering Poincare symmetry.

In the limit of vanishing cosmological constant, or infinite AdS radius, the AdS algebra should reduce to the to the Poincare algebra of flat Minkowski space. This reduction of the isometry algebra can be implemented by performing what is called a **Inönü Wigner contraction**. We divide the generators $J_{AB}$, $A, B = 0, 1, 2, 3$ into $J_{ab}$ and $J_{a3}$ for $a, b = 0, 1, 2$ followed by a rescaling $J_{a3} \rightarrow l P_a$. One then obtains the following rescaled $so(2,2)$ algebra:

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \text{permutations}, \quad [J_{ab}, P_c] = P_a \eta_{bc} - P_b \eta_{ac}, \quad [P_a, P_b] = \frac{1}{l^2} J_{ab}. \quad (2.6)$$
This algebra, is characterized by the fact that the translations commute to the Lorentz transformations and is often referred to as the AdS algebra. Upon using the three dimensional dualization:

$$J_a = \frac{1}{2} \epsilon_{abc} J^b c \leftrightarrow J^{ab} = -\epsilon^{abc} J_c,$$

we may cast the algebra as:

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \frac{1}{l^2} \epsilon_{abc} J^c.$$  

(2.7)

Thus indeed, in the limit \( l \to \infty \) the AdS algebra reduces to the Poincare algebra of flat Minkowski.

**Global coordinates**

A set of coordinates solving the embedding constraint (2.3) is given by:

$$X_0 = l \cosh(\rho) \cos(t), \quad X_1 = l \sinh(\rho) \sin(\phi), \quad X_2 = l \sinh(\rho) \cos(\phi), \quad X_3 = l \cosh(\rho) \sin(t),$$

(2.9)

where \( t \in [0, 2\pi], \phi \in [0, 2\pi], \) and \( \rho \in [0, \infty) \). The metric in these coordinates becomes:

$$ds^2 = l^2 \left( d\rho^2 - \cosh^2(\rho) dt^2 + \sinh^2(\rho) d\phi^2 \right).$$

(2.10)

There is one problem though. In this form, the time coordinate \( t \) is periodic, allowing for closed time like curves. This can be solved by considering what is called the universal cover of AdS, obtained by unwinding the time direction and thus extending its range to \( \mathbb{R} \). Making the additional coordinate transformation

$$\sinh(\rho) = \tan(\theta), \quad \theta \in [0, \pi/2)$$

(2.13)

where for simplicity we set the AdS radius \( l = 1 \) the metric can be put into Fefferman-Graham form [16]:

$$ds^2 = \frac{l^2}{\cos^2(\theta)} \left( -dt^2 + d\theta^2 + \sin^2(\theta) d\phi^2 \right).$$

(2.14)

This representation of global AdS will be most convenient for our purposes.

**Poincaré coordinates and isometries at the conformal boundary**

Poincaré coordinates can be defined through:

$$X_0 = \frac{lt}{2}, \quad X_d = \frac{l^2}{2z} \left( z^2 + \sum_i x_i^2 - t^2 \right),$$

$$X_i = \frac{lx_i}{z}, \quad X_{d-1} = \frac{l^2}{2z} \left( z^2 - \sum_i x_i^2 + t^2 \right).$$

(2.15)
\( \forall i = 1 \ldots d - 2 \) and where \( \tilde{x}, t \) denote the coordinates on the hyperboloid. This brings the metric to the following form
\[
ds^2 = i^2 \left( \frac{dz^2 - dt^2 + dx^2}{z^2} \right) = i^2 (d\rho^2 + e^{2\rho} (-dt^2 + dx^2)),
\]
where in the second equality we have set \( z = e^{-\rho} \). The hyperbolid coordinates \( x^0 = t, x^i \) range form \((-\infty , \infty )) \) and \( -\infty < z < \infty \) or \(-\infty < z < 0 \). This restricted range of \( z \) is to maintain a single valued map and we thus need several patches to describe the whole spacetime. These coordinates are called Poincaré coordinates because the surfaces of constant \( z \) are conformal to two dimensional slices of respectively the positive or negative region of Minkowski spacetime depending on the range of \( z \). In these coordinates however, only half of the spacetime is covered, because \( z \) divides the hyperboloid in two pieces: \( z < 0, z > 0 \). The conformal boundary of Poincaré AdS lies at \( z = 0 \). This boundary, has the topology of a plane since both coordinates \( (x,t) \) range from \((-\infty, +\infty)) \).

Using this metric it is now fairly easy to identify all the continuous symmetries of AdS\(_d\). We denote \( x^\nu = (t, \tilde{x}) \). Then the symmetries are:

- Translations of the \( d - 1 \) coordinates: \( \delta x^\nu = a^\nu \).
- The \((d-1)(d-2)/2\) different Lorentz transformations: \( \delta x^\nu = \omega^\nu_\mu x^\mu \).
- A scale transformation of all coordinates. \( \delta x^\nu = \rho x^\nu, \delta z = \rho z \).
- \( d - 1 \) distinct transformations \( \delta x^\nu = e^\nu (x \cdot x + z^2) - 2\epsilon \cdot xx^\nu, \delta z = -2\epsilon \cdot xz \).

The extra \( z \) coordinate, that is characteristic for AdS, is what is responsible for the enhancement of invariance with respect to the flat Minkoskian line element. In the limit \( z \to 0 \), the above symmetries reduce to the well known conformal symmetries of flat Minkowski. The isometry group of the Poincaré patch of AdS\(_d\) thus acts as the full conformal group on its \( d - 1 \) Minkowski boundary.

Let us do a count of the symmetries. AdS has \( d(d+1)/2 = 10 \) killing vectors, Minkowski space on the other hand has at most \( d(d-1)/2 = 6 \) killing vectors. If we add the translations and Lorentz transformations we find \((d-1)(d-2)/2 + (d-1) = d(d-1)/2 \) isometries, which already accounts for all the Minkowski killing vectors. The latter two arise at the boundary as additional conformal symmetries. The conformal boundary thus inherits from AdS not only Poincaré invariance but in fact a full invariance under conformal transformations. These identifications of the \( SO(2,2) \) AdS\(_3\) isometries as the global \( SO(2,2) \) conformal group on the boundary are roughly speaking what makes AdS/CFT work. However, for the special case of 3 dimensions which we consider here, this is not yet the end of the story, and we will discuss in section 2.4 that the actual symmetry at the boundary of AdS\(_3\) is governed by a centrally extended Virasoro algebra with central charge \( c = \frac{3}{2G} \) as was first shown by Brown and Henneaux\(^{[17]}\).

### 2.2 BTZ black hole

Having discussed two coordinate representations of empty AdS we will now discuss aspects of a more interesting solution which is the BTZ black hole. As we mentioned in the introduction, 3d gravity has no propagating degrees of freedom. It therefore came as quite a surprise when a black hole solution was found\(^{[13]}\), which, after the authors, has been dubbed the BTZ black hole. However, this black hole differs from its 4d Schwarzschild counterpart on several aspects. Firstly, the curvature of AdS is everywhere constant and finite making it impossible to have a singularity in 3d. Instead, the BTZ is constructed from global identifications of AdS and its singularity is one in the causal structure of the spacetime. Secondly, whereas the 4d Schwarzschild black hole is asymptotically flat, the BTZ is asymptotically AdS. Despite these differences, there are also some striking similarities, which are in favour of calling the BTZ a \(^2\)“proper” black hole. For example, it has an event horizon and one can assign thermodynamical quantities to it, such as

\(^2\)Although we will not prove this fact until we come to discuss higher spin gravity in the next section we will use this fact already several times.
an entropy and temperature. In this subsection we will first introduce the metric of the BTZ. Then we will pay some attention to asymptotically AdS spacetimes after which we will discuss the thermodynamics of the BTZ black hole.

**Metric of the BTZ black hole**

The metric of the BTZ black hole is given by:[13]

\[
ds^2 = -\left(\frac{r^2 - r_+^2}{l^2} + \frac{l^2}{r^2} - r_+^2\right)dt^2 + \frac{l^2}{r_+^2}dr^2 + r^2\left(\frac{d\phi}{r_+} - \frac{r_+}{l^2}dt\right)^2,
\]

with \(0 < r < \infty\). Its ADM mass and angular momentum are

\[
M = \frac{r_+^2 + r_0^2}{8G_3l^2}, \quad J = \frac{r_+r_0}{4G_3l},
\]

Introducing the lapse and shift functions:

\[
N^2(r) = -8MG_3 + \frac{r^2}{l^2} + \frac{16G_5J^2}{r^2}, \quad N^0(r) = -\frac{4GJ}{r^2},
\]

we can bring the metric into a more generic form

\[
ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(N^0dt + d\phi)^2 \quad \text{(2.19)}
\]

\[
= -\left(-8G_3M + \frac{r^2}{l^2} + \frac{16G_5J^2}{r^2}\right)dt^2 + \left(-8G_3M + \frac{r^2}{l^2} + \frac{16G_5J^2}{r^2}\right)^{-1}dr^2 + r^2\left(-\frac{4GJ}{r^2}dt + d\phi\right)^2
\]

\[
= -\left(-8G_3M + \frac{r^2}{l^2}\right)dt^2 + \left(-8G_3M + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\phi^2.
\]

This metric becomes singular at two values of \(r\) for which \(N^2(r) = 0\):

\[
r_{\pm} = 4G_3Ml^2 \left[1 \pm \sqrt{1 - \left(\frac{J}{4M^2}\right)^2}\right].
\]

Here \(r_+\) is the event horizon of the black hole and \(r_-\) is the inner horizon. In order for these horizons to exist we must further require \(M > 0\), \(|J| \leq Ml\). As for the Schwarzschild black hole, the horizons represent merely coordinate singularities. The only true singularity appears at \(r = 0\), which we recall is a singularity in the causal structure. When the bound

\[
|J| < Ml,
\]

is saturated, we speak of an extremal black hole, in which case the horizons coincide.

Let us now analyze three interesting limit of the non-rotating \((J = 0)\) BTZ black hole.

- **Massless BTZ** This is obtained by letting \(M \to 0\). In this limit the black hole disappears. There is no event horizon but a conical singularity remains. This limit describes the ground state of the BTZ or just empty space. The black hole spectrum is found for \(M \geq 0\).

- **Global AdS** This spacetime is recovered by letting \(M = -1/8G_3\). This is the vacuum solution.

- **Conical defects** These solutions correspond to \(-1/8G_3 < M < 0\). In this limit the outer horizon \(r_+\) disappears and a naked conical singularity arises.

Usually, solutions containing a naked singularity must be excluded from the physical spectrum. In this sense, global AdS \(_3\) emerges as a "bound state" from the black hole spectrum by a mass gap of \(\Delta M = 1/8G_3\). This configuration cannot be continuously obtained from the black hole spectrum since it would involve going through a series of conical singularities that must be excluded from the configuration.
space. Nonetheless, conical defects are often studied because they allow for an interpretation of a solution containing a point mass\cite{18}\cite{19}.

### A general solution

Before continuing with further aspects of the BTZ and discussing asymptotically AdS spacetimes, it will prove to be convenient to repackage the above discussed solutions into a single metric. In \cite{20} it was shown that the following metric provides the most general solution for Einstein gravity with a negative cosmological constant:

$$ds^2 = \ell^2 \left[ d\rho^2 + \frac{2\pi L(x^+)}{k} (dx^+)^2 + \frac{2\pi L(x^-)}{k} (dx^-)^2 + \left( e^{2\rho} - e^{-2\rho} \frac{(2\pi)^2 L(x^+) L(x^-)}{k^2} dx^+ dx^- \right) \right]. \quad (2.22)$$

with $x^\pm = t/\ell \pm \phi$ and $\rho$ representing the radial coordinate. The constant $k$, is called the Chern-Simons level. It is related to the AdS radius and Newtons constant as $k = \frac{l^4}{3\pi G}$. We will discuss it in further detail in the next chapter. For now it acts merely as a normalization and is not important. This solution parametrizes the whole space of solutions which are asymptotically AdS. Different choices for the functions $L(x^+)$ and $\bar{L}(x^-)$ correspond to physically different solutions. Not only is this a very efficient representation of the solutions of AdS but it will also allow for a straightforward generalization when we come to discuss higher spin gravity. For example, empty AdS or the massless BTZ if we identify $3\phi \sim \phi + 2\pi$, correspond to $L = \bar{L} = 0, (M = J = 0)$ whereas global AdS is recovered for $2\pi L = 2\pi \bar{L} = -\frac{k}{4}, (J = 0)$.

Meanwhile, the BTZ black hole \cite{17} may be recovered via the following change of variables\cite{20}:

$$r^2 = r_+^2 \cosh^2(\rho - \rho_0) - r_+^2 \sinh^2(\rho - \rho_0), \quad e^{2\rho_0} = \frac{r_+^2 - r_-^2}{4\ell^2}, \quad (2.23)$$

with the additional identification

$$2\pi L = \frac{1}{2} (Ml - J), \quad 2\pi \bar{L} = \frac{1}{2} (Ml + J). \quad (2.24)$$

with $L, \bar{L}$ constant. $M$ and $J$ are again the ADM mass and angular momentum in \cite{21}.

### A stress tensor for Asymptotically AdS spacetimes

The BTZ is an example of an asymptotically AdS spacetime (A-AdS) and in the presence of a boundary we need to ensure that we have a well-defined variational principle. In \cite{14}\cite{21} the authors describe what boundary terms need to be added to the Einstein-Hilbert action in order to achieve this. Along the way we will determine the stress tensor for AdS$_3$ and see that the above function $L, \bar{L}$ are related to the zero modes of the boundary CFT stress tensor.

We start by writing the metric of an asymptotically AdS$_3$ spacetimes as:

$$ds^2 = d\rho^2 + g_{ij} dx^i dx^j. \quad (2.25)$$

Note that the metric \ref{eq:metric_adS} already has this form. $\rho$ represents as before the radial coordinate and $g_{ij}$ is an arbitrary function of the $x^i$ for $i = 1, 2$. It is the induced metric at the boundary. Upon varying the Einstein-Hilbert action and using this expression for the metric one finds a boundary term that is inconsistent with a variational principle where the induced metric on the boundary is held fixed, but not its normal derivative. This can be solved by adding the Gibbons-Hawking term to the action. It is given by:

$$I_{GH} = \frac{1}{8\pi G_3} \int_{\partial M} d^2x \sqrt{\mathcal{g}} \text{Tr} K \quad K_{ij} = \frac{1}{2} \partial_\rho g_{ij}. \quad (2.26)$$

\footnote{This identification we will discuss in section \ref{sec:stress_tensor}.}
Here $K$ is the extrinsic curvature. Varying the total action $I_{EH} + I_{GH}$ will then give two contributions. A bulk piece that vanishes when the equations of motion are satisfied, and a boundary piece, which will define a stress tensor:

$$\delta(I_{EH} + I_{GH}) = \frac{1}{2} \int_{\partial M} d^2x \sqrt{g} \delta g^{ij} T_{ij} = -\frac{1}{8\pi G_3} \left( K^{ij} - \text{Tr} K g^{ij} \right). \tag{2.27}$$

So far, we have not used any particular facts about asymptotically AdS$_3$ spacetimes and the above expression is valid in general. When we specialise to asymptotically AdS$_3$ spacetimes, we will require the addition of a second boundary term. We write our ansatz metric such that it grows as $r^2$ at infinity in agreement with $\text{(2.16)}$, which in our present coordinates translates to a growth of $e^{2\rho/r}$:

$$g_{ij} = e^{2\rho/r} \left( g^{(0)}_{ij} + g^{(2)}_{ij} + \mathcal{O}(e^{-\rho/r}) \right), \tag{2.28}$$

which is known as a Fefferman-Graham expansion. It is $g^{(0)}_{ij}$ that we identify with the metric of the conformal boundary where the dual CFT lives. It is defined up to Weyl transformations via a redefinition of the radial coordinate $\rho$. We should thus consider a variational principle in which this boundary metric is held fixed whereas the subleading terms in $\text{(2.28)}$ are allowed to vary. However, when using this ansatz in $\text{(2.27)}$ a new counter term is needed to cancel a divergence in the large $\rho$ limit:

$$I_{ct} = -\frac{1}{8\pi G_3} \int_{\partial M} d^2x \sqrt{\bar{g}}. \tag{2.29}$$

Finally, the variation of the action now becomes:

$$\delta(I_{EH} + I_{GH} + I_{ct}) = \frac{1}{2} \int d^2x \sqrt{g} \delta g^{(0)} T^{(0)}_{ij} \delta g_{ij}, \quad T_{ij} = \frac{1}{8\pi G_3} \left( g^{(2)}_{ij} - \text{Tr}(g^{(2)}) g^{(0)}_{ij} \right), \tag{2.30}$$

with $T_{ij}$ the final AdS$_3$ stress tensor that we identify with the CFT stress tensor. Next it is convenient to take $g^{(0)}_{ij}$ as the flat metric on the cylinder and work in complex coordinates: $\bar{g}^{(0)}_{ij} dx^i dx^j = d\bar{z}d\bar{\bar{z}}$. The stress tensor then has the following non-vanishing components:

$$T_{zz} = \frac{1}{8\pi G_3} \bar{g}^{(2)}_{zz} = \bar{\mathcal{L}}, \quad T_{\bar{z}\bar{z}} = \frac{1}{8\pi G_3} g^{(2)}_{\bar{z}\bar{z}} = \bar{\bar{\mathcal{L}}}, \tag{2.31}$$

where we used $\kappa = l/(4G_3)$ and we used the metric $\text{(2.22)}$. From this we define the Virasoro generators to find:

$$L_n - \frac{c}{24} \delta_{n,0} = \oint d\bar{z} e^{-nz} T_{zz} = 2\pi \mathcal{L}, \quad \bar{L}_n - \frac{c}{24} \delta_{n,0} = \oint d\bar{\bar{z}} e^{-nz} T_{\bar{z}\bar{z}} = 2\pi \bar{\bar{\mathcal{L}}}. \tag{2.32}$$

where the last equalities hold for $\mathcal{L}, \bar{\bar{\mathcal{L}}}$ constant.

Indeed, the conserved charges $M, J$ are then given in terms of the Virasoro zero modes by virtue of $(\text{2.24})$ are then given by the eigenvalues of zero modes of the Virasoro generators:

$$h - \frac{c}{24} = \frac{1}{2} (ML - J), \quad \bar{h} - \frac{c}{24} = \frac{1}{2} (ML + J), \tag{2.33}$$

where $h$ denotes the $L_0$ eigenvalue on the plane. Note that on global AdS$_3$ with mass $ML = -1/8G_3 = -c/12$, and $J = 0$ we have $L_0 = \bar{L}_0 = 0$ in agreement with its invariance under the global part of conformal algebra which is spanned by $\{L_0, L_{\pm}\}$.

---

1 See e.g. [22] for a derivation.
2 One can indeed show that it has a non-zero trace:

$$\text{Tr}(T) = -\frac{1}{8\pi G_3} \text{Tr}(g^{(2)}) = \frac{1}{2} \mathcal{R}^{(0)}$$

which reproduces the Weyl anomaly. Comparing to the CFT Weyl anomaly $\text{Tr}(T_{CFT}) = -\frac{1}{32\pi^2} R$ one recovers the Brown-Henneaux central charge $c$.
3 The relation $\kappa = l/(4G_3)$ will be proven later, when we discuss the Chern-Simons formulation of the theory.
4 These are Virasoro is shown by performing the contour integrations. Then one finds that the $L_n$ generate a Virasoro algebra.
5 Note the conventional shift by $c/24$ which comes from working on the cylinder.
6 Which are the mass and angular momentum

10
2.2.1 Thermodynamics of the BTZ black hole

To discuss the thermodynamics of the BTZ we compactify the time coordinate by Wick rotating to Euclidean time via $t \rightarrow i t_E$, $J \rightarrow i J_E$ and $r_+ \rightarrow r_-$. The inner horizon $r_-$ thus becomes imaginary in Euclidean signature. That we must Wick rotate also $r_+$ can be seen from the third term in the metric \((2.34)\). If we where to Wick rotate only $t \rightarrow i t_E$ then the cross term would be imaginary. Regularity at the Euclidean horizon then imposes that the angular and Euclidean time component are both periodic $r \sim \Omega$, $t \sim \Omega$.

This can be shown\(^{[23]}\) by considering the Euclidean metric and taking the near horizon limit by changing variables to \(r \rightarrow r_i = \frac{x^2}{4r_+} \) and Taylor expanding around $x \approx 0$. Doing so, the Euclidean metric takes the following form:

\[
\begin{align*}
    ds_E^2 &= \frac{1}{2} \left( \frac{\beta^2}{r_+^2 - r_-^2} \right) dx^2 + \frac{(r_+^2 - r_-^2) x^2}{2 l r_+^2} dt_E^2 + r_+^2 \left( d\phi + \frac{|r_-|}{l r_+} dt_E \right)^2. \\
    \text{If we now redefine} \quad \sqrt{\frac{\beta^2}{r_+^2 - r_-^2}} x &= \bar{x} \equiv x \text{ the new Euclidean horizon metric (2.34) reduces to:} \\
    ds_E^2 &= dx^2 + \frac{r_+^2 - r_-^2}{l^2 r_+^2} x^2 dt_E^2 + r_+^2 \left( d\phi + \frac{|r_-|}{r_+ l} dt_E \right)^2, \\
    &\equiv dx^2 + x^2 (\kappa dt_E)^2 + r_+^2 \left( d\phi + \frac{l|r_-|}{r_+^2 - r_-^2} \kappa dt_E \right)^2. 
\end{align*}
\]

To get from the first to the second line we absorbed all remaining variables into a new variable $\kappa$.\(^{[20]}\) Now, the first part of the metric we recognise as flat space $\mathbb{R}^2$ in polar coordinates, provided we assume that the Euclidean time coordinate is periodic with a periodicity $t_E \sim t_E + \frac{2\pi}{\beta}$. If this is not the case, the metric describes a cone rather than flat space. Since the last term in the Euclidean metric describes simply a sphere we must furthermore require $\phi$ to be periodic: \(\phi \sim \phi + \frac{2\pi l^2 |r_-|}{r_+^2 - r_-^2}\). Regularity thus demands that:

\[
(\phi, t_E) \sim (\phi + \Omega, t_E + \beta), \quad \beta = \frac{2\pi l^2 r_+}{r_+^2 - r_-^2}, \quad \Omega = \frac{2\pi l^2 |r_-|}{r_+^2 - r_-^2}. 
\]

Two important quantities we can assign to a black hole at the Hawking temperature and Bekenstein-Hawking entropy:

\[
T_H = \frac{1}{\beta} = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}, \quad S_{BH} = A = \frac{2\pi r_+}{4G},
\]

where $A$ is the area of the black hole horizon. Note that in the extremal limit, $r_+ \rightarrow r_-$, the Hawking temperature $T_H \rightarrow 0$. This observation will be important in this thesis: **Extremal black holes are at zero temperature.** All these thermodynamical quantities of a black hole we have just discussed, are related to each other by the **first law of black hole thermodynamics**:

\[
dM = T dS + \Omega dJ.
\]

Euclidean continuation and modular parameter

For later purposes we discuss here the thermodynamics in terms of the Euclidean light-cone coordinates $z, \bar{z}$ via $x^+ \rightarrow z = \phi + i t_E$, and $x^- \rightarrow \bar{z} = -\phi + i t_E$. The manifold now has the topology of a solid torus where $z, \bar{z}$ describe the boundary coordinates. The regularity conditions \((2.36)\) then translate into

\[
z \sim z + 2\pi \z \sim z + 2\pi \tau,
\]

where $\tau$ is the modular parameter of the torus. In terms of the inverse temperature $\beta$ and the angular velocity of the horizon $\Omega$ one has:

\[
\tau = \frac{i}{2\pi} (\beta + \Omega),
\]

\(^{[20]}\kappa\) represents the surface gravity.
and Vol($T^2$) = $4\pi^2\text{Im}(\tau)$ is the volume of the torus. The relation between $\tau$ and $L$ is given by:

$$\tau = \frac{i}{2} \sqrt{\frac{k}{2\pi L}}.$$ (2.41)

We will revisit this relation when we discuss thermodynamics in Chern Simons formulation in the next sections. Let us lastly remark that out of the two inequivalent cycles defining the torus, i.e. the angular and thermal cycle, the $\phi$ cycle is non-contractable, whereas the thermal cycle becomes contractable. This can be seen straight from the metric (2.35) by noting that near the horizon, $x \sim 0$, in which case the time cycle shrinks to zero size.

**Entropy from the partition function**

The formula for the entropy as given by the Bekenstein-Hawking area law, is an inherently geometrical definition, and importantly it will not hold when we come to discuss black holes in higher spin gravity in the next section. For this purpose we now discuss a more useful way to calculate the black hole entropy, which will carry over to the higher spin case. If we assume the first law of thermodynamics to hold, then the entropy can be computed from the partition function which reads:

$$Z(\tau, \bar{\tau}) = \text{Tr} \exp^{\frac{4\pi^2i}{2\pi L}(\tau \mathcal{L} - \bar{\tau} \bar{\mathcal{L}})},$$ (2.42)

where we used the relation between the partition functions of AdS and the CFT. $H$ denotes the CFT Hilbert space and $q = \exp(2\pi i \tau)$. The charge $L$ can then be extracted from the partition function as:

$$L = \frac{1}{4\pi^2i} \frac{\partial \ln Z}{\partial \tau}, \quad \bar{L} = -\frac{1}{4\pi^2i} \frac{\partial \ln Z}{\partial \bar{\tau}}.$$ (2.43)

These relations are to be understood as expectation values, from the point of view of the CFT. In the thermodynamical limit the entropy is found to be:

$$S = \ln Z - 4\pi^2i(\tau \mathcal{L} - \bar{\tau} \bar{\mathcal{L}}).$$ (2.44)

Using then relation between $L$ and $\tau$ given by (2.41), we can find $\ln Z$ by integrating (2.43). Then we obtain:

$$\ln Z = 4\pi^2i \left( \int \mathcal{L} d\tau - \int \bar{\mathcal{L}} d\bar{\tau} \right).$$ (2.45)

$$= 4\pi^2i \frac{k}{8\pi} \left( \frac{1}{\tau} - \frac{1}{\bar{\tau}} \right).$$

$$= \pi \sqrt{2\pi k \mathcal{L}} + \pi \sqrt{2\pi k \bar{\mathcal{L}}},$$

from which we find the entropy of the BTZ:

$$S_{\text{th}} = 2\pi \sqrt{2\pi k \mathcal{L}} + 2\pi \sqrt{2\pi k \bar{\mathcal{L}}}.$$ (2.46)

In terms of the CFT charges the entropy reads:

$$S = 2\pi \sqrt{\frac{c}{6} \left( h - \frac{c}{24} \right)} + 2\pi \sqrt{\frac{c}{6} \left( \bar{h} - \frac{c}{24} \right)},$$ (2.47)

which is the famous Cardy formula\textsuperscript{11} for the entropy in a 2d CFT in the high temperature limit\textsuperscript{26}.

---

\textsuperscript{11}The derivation of the Cardy formula can be found in the discussion.
2.3 The dual 2D CFT

In the last portion of this section we will pay attention to the dual 2D conformal field theory of AdS$_3$. In the previous sections we have seen that global AdS$_3$ has an $so(2,2) \simeq sl(2) \times sl(2)$ isometry algebra, which is the same as the global conformal algebra in two dimensions. However the full isometry algebra of a 2D CFT is a centrally extended Virasoro algebra of local conformal transformations. This apparent mismatch in AdS$_3$/CFT$_2$ was resolved by Brown and Henneaux\cite{17} who showed that the asymptotic symmetry algebra of AdS$_3$ is in fact an infinite dimensional extension of the $sl(2) \times sl(2)$ isometry algebra. Although this derivation was originally done in the metric formulation, we will discuss it in section 4.4.1 in the context of Chern Simons gauge theory\cite{12}. For now we will use the fact that the asymptotic symmetries of AdS$_3$ are described by a centrally extended Virasoro algebra. The Virasoro generators $L_{-1}, L_0, L_1$ span the finite $sl(2)$ sub algebra which together form the isometry algebra of AdS$_3$. Brown and Henneaux calculated the central charge of the centrally extended asymptotic symmetry algebra to be:

$$c = \frac{3l}{2G_3}.$$  \hspace{1cm} (2.48)

With these facts we can now relate some AdS quantities to their dual states in the 2D CFT. For this we need to recall the relation between the ADM mass and angular momentum in terms of the Virasoro generators, which is found from (2.33):

$$Ml = L_0 + \bar{L_0} - \frac{c}{12}, \hspace{1cm} J = L_0 - \bar{L_0}.$$  \hspace{1cm} (2.49)

From the CFT perspective these relations are very natural. Considering radial quantization, we know that $L_0 + \bar{L_0}$ corresponds to time translations and it is thus natural to associate this to energy/mass in AdS. Similarly, $L_0 - \bar{L_0}$ generates rotations in a CFT, and thus should be identified with angular momentum in AdS. The remaining ambiguity is now the factor of $\frac{c}{12}$. However, when we discussed the BTZ black hole, we saw that there is a mass gap between the massless BTZ black hole and global AdS. In \cite{25} this mass gap is explained in the context of supersymmetric CFT’s as a Casimir energy. They showed that the zero mass BTZ $M = 0$ arises as the ground state of the Ramond sector, whereas global AdS arises as the ground state of the Neveu-Schwarz sector. The $M > 0$ BTZ black hole corresponds to a thermal state in the dual CFT. However, as is shown in \cite{14} the $M > 0$ BTZ black hole with modular parameter $\tau$ is related to thermal AdS with modular parameter $-1/\tau$ via a coordinate transformation, that effectively interchanges the $\phi$ and $t$ cycles. As a result, they showed that in the low temperature regime the partition function will be dominated by thermal AdS whereas the BTZ black hole will dominate the partition function in the high temperature limit. Therefore both configurations correspond to a thermal state in the CFT, but one will be preferred over the other depending on the temperature. Such a change in the most probable classical solution is known as a Hawking-Page phase transition.

\footnote{Although we will do it for the $sl(3)$ higher spin theory, we will see that Virasoro algebra appear as a subalgebra of the centrally extended $W_3$ algebra.}
3 2+1 dimensional gravity as a Chern Simons theory

The purpose of this section will be to reformulate the results of the previous section in the context of Chern Simons gauge theory. This will allow for a rather straightforward generalization to higher spin gravity in section 4. The setup of this section will be as follows. In section 3.1 we will recall the main concepts of gauge theories, then in section 3.2 we will discuss how 3d gravity can be formulated as a Chern Simons theory, the actual proof of which is given in appendix C. With this result we then discuss how the solutions and concepts of the previous section can be reformulated in Chern Simons theory. This section will furthermore, use results from the vielbein formalism and readers not familiar with this are referred to appendix C.

3.1 General aspects of gauge theory

We consider a theory that is invariant under global infinitesimal transformations $g(\alpha) \in G$: \[ g(\alpha) = e^{i\alpha^a T_a} \simeq 1 + i\alpha^a T_a. \] (3.1)

Here $\alpha^a$ is an infinitesimal parameter and $T_a$ are the generators of the Lie-algebra $g$ associated to the symmetry group $G$. Noether’s theorem tells us that for each continuous symmetry we have a conserved charge. These conserved charges, here denoted by $T_a$, generate the symmetries and span a Lie-algebra $g$: \[ [T_a, T_b] = f^c_{ab} T_c. \] (3.2)

We next promote this global symmetry to a local gauge symmetry by letting the infinitesimal parameters be spacetime dependent, i.e. $\alpha \to \alpha(x)$. To maintain invariance under transformations $g(\alpha(x))$ we introduce the covariant derivative $D_\mu$

\[ D_\mu \equiv \partial_\mu - B^a_\mu T_a = \partial_\mu - A_\mu, \] (3.3)

with $B^a_\mu$ a gauge field. There will be one gauge field for every gauged symmetry of the theory. Imposing that the covariant derivative satisfies \[ (D_\mu g) \Phi = g (D_\mu \Phi), \] (3.4)

we find the transformation law for $A_\mu$

\[ A_\mu \to g A_\mu g^{-1} + g (\partial_\mu g^{-1}). \] (3.5)

Note that (3.5) shows that the gauge fields transform in the adjoint representation. Infinitesimally this becomes

\[ A_\mu \to A_\mu - D_\mu \alpha \equiv A_\mu - (\partial_\mu \alpha + [A_\mu, \alpha]). \] (3.7)

3.2 2+1 dimensional AdS$_3$ gravity as a Chern Simons theory

We will now apply the above discussed aspects of gauge theory to (2+1)-dimensional AdS$_3$ gravity. We take $A_\mu \in \mathfrak{so}(2,2)$ and we may then separate it in terms of the generators as

\[ A_\mu = e^a_\mu P_a + \omega^a_\mu M_a \in \mathfrak{so}(2,2). \] (3.8)

Here $e^a_\mu$ is the gauge field for local translations and $\omega^a_\mu$ is the gauge field associated to local Lorentz transformations. As we shall discuss momentarily, they are identified with the vielbein and spin-connection familiar from the first order formulation of Einstein gravity. $P_a$ and $M_a$ are as before the generators of

\[ \delta_\alpha B^a_\mu = - (\partial_\mu \alpha^a + \alpha^b B^a_\mu f^{bc}_a), \] (3.6)

which is particular convenient to derive the transformation laws of the gauge fields if the explicit algebra is at hand.
translations and Lorentz transformations with their algebra given by \( (2.8) \) and invariant bilinear form given by:
\[
(M_a, M_b) = \delta_{ab}, \quad (M_a, P_b) = 0, \quad (P_a, P_b) = \frac{1}{l^2} \delta_{ab}.
\] (3.9)
The novel observation in \([6]\) was that the Einstein Hilbert action with negative cosmological constant in the first order formalism can be written as a Chern Simons action:\(^{14}\)
\[
S_{EH} = S_{CS}[A],
\] (3.10)
where
\[
S_{CS}[A] = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
\] (3.11)
Here \( k \) denotes the Chern-Simons level, the integration is over a 2+1 dimensional manifold \( \mathcal{M} \) and \( \text{Tr} \) represents the contraction of the \( \mathfrak{so}(2,2) \) generators with the Lie algebra metric, c.f. (3.9).

From now on it will be convenient to use that \( \mathfrak{so}(2,2) \) becomes reducible in 2+1 dimensions and splits into two mutually commuting \( \mathfrak{sl}(2) \) copies. This splitting is made explicit by writing
\[
L^+_a = \frac{1}{2} (M_a \pm iP_a).
\] (3.12)
The connection \( A \) then decomposes into two pairs of \( \mathfrak{sl}(2) \) connections
\[
A = A^a L^+_a + \tilde{A}^a L^-_a,
\] (3.13)
with
\[
A = (\omega^a + \frac{1}{l} e^a) L^+_a, \quad \tilde{A} = (\omega^a - \frac{1}{l} e^a) L^-_a.
\] (3.14)
Using the \( \mathfrak{so}(2,2) \) commutation relations, \( L^+_a, L^-_a \) are easily seen to obey the \( \mathfrak{sl}(2) \) algebra:\(^{15}\)
\[
[L^+_a, L^+_b] = \epsilon_{abc} L^+_c, \quad [L^-_a, L^-_b] = \epsilon_{abc} L^-_c, \quad [L^-_a, L^+_b] = 0.
\] (3.15)
The relation (3.10) then becomes:
\[
S_{EH} = S_{CS}[A] - S_{CS}[\tilde{A}],
\] (3.16)
where
\[
S_{CS}[A] = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
\] (3.17)
\( \text{Tr} \) now denotes the contraction with the Lie algebra metric of the \( \mathfrak{sl}(2) \) generators which is easily found from (3.9) and (3.12). The proof of this relation can be found in appendix C.1. For this one substitutes (3.14) in the RHS of (3.16) and identifies the Chern-Simons level with the AdS radius \( l \) as:
\[
k = \frac{l}{4G_3} = \frac{c}{6},
\] (3.18)
and identifies the gauge fields \( e^a \) and \( \omega^a \) as the vielbein and spin-connection from the first order formulation of GR. Note though, that the from the gravity point of view, for the relation (3.8) to hold, it is crucial that it three dimension it is possible to dualize the spin connection in its Lorentz indices so that it acquires the same index structure as the vielbein.

Before we can claim complete victory though, there are a few aspects that are worth verifying. One of these is whether the Chern-Simons action is actually gauge invariant. A second fact we might worry about is that the gauge transformations on each side of (3.16) are actually the same. These two aspects

\(^{14}\)Here we will be primarily interested in AdS. A similar proof holds for Minkowski space by taking \( l \to \infty \) and de Sitter space by taking \( l \to il \), i.e. positive cosmological constant. In these cases gravity is a gauge theory for respectively the groups ISO(2,2) and SO(3,1).

\(^{15}\)For the explicit matrix representation of the generators see appendix A.1.
will be addressed in the upcoming paragraphs. To start with, we will check that the equations of motion are the same.

**Equations of motion**

The equations of motion are found by varying the action with respect to $A, \bar{A}$:

$$dA + A \wedge A = F = 0 \quad d\bar{A} + \bar{A} \wedge \bar{A} = F = 0.$$  \hfill (3.19)

Thus solutions to the Chern-Simons equations of motion are flat connections. When rewritten in terms of the vielbein and spin connection these equations reproduce the equations of motion of the EH action. For this we use that $A = \omega + e/l, \bar{A} = \omega - e/l, e = e^a L_a$ and $\omega = \omega^a L_a$ to find:

$$0 = \frac{1}{2}(F + \bar{F}) = d\omega + \omega \wedge \omega + \frac{1}{2} e \wedge e = R + \frac{1}{2} e \wedge e.$$  \hfill (3.20)

$$0 = \frac{1}{2}(F - \bar{F}) = de + \omega \wedge e.$$  \hfill (3.21)

(3.20) is the vacuum Einstein equation in the presence of a cosmological constant. (3.21) is the zero torsion constraint (C.12). This identifies $\omega$ as the torsionless Levi-Civita connection.

**Gauge invariance of the action**

The Chern-Simons action is invariant under gauge transformations of the form (3.5). Noting that we have used that $SO(2,2) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ the first factor $S_{CS}[A]$ is invariant under "left" gauge transformations of the form

$$A_\mu \rightarrow L(x)(A_\mu + \partial_\mu)L^{-1}(x),$$  \hfill (3.22)

whereas the second factor $S_{CS}[\bar{A}]$ is invariant under "right" gauge transformations

$$\bar{A}_\mu \rightarrow R^{-1}(x)(\bar{A}_\mu + \partial_\mu)R(x).$$  \hfill (3.23)

Here $L, R \in SL(2, \mathbb{R})$ are the analogs of (3.1) and the $T_a$ become the generators of the two commuting $\mathfrak{sl}(2)$ copies that have been denoted by $L_a^\pm$ in the previous subsection. To prove invariance, we focus on a single copy, say the left, and for notational simplicity denote the gauge transformation as $A \rightarrow g(A + d)g^{-1}$.

Then the action transforms as, using trace cyclicity\footnote{Restoring the explicit anti-symmetrisation over the indices we have}

$$S_{CS}[A] \rightarrow S_{CS}[A] - \frac{k}{4\pi} \int_M \text{Tr} \left( g^{-1} A dg + \frac{1}{3} g^{-1}(dg) g^{-1}(dg) g^{-1}(dg) \right).$$  \hfill (3.24)

The first term is a mere total derivative, and hence vanishes under suitable boundary conditions on $A$. The second term however, in the non-abelian case, defines the so-called *winding number density* of $g$:

$$w(g) = \frac{1}{24\pi^2} e^{\mu\nu\rho} \text{Tr} \left( g^{-1}(dg) g^{-1}(dg) g^{-1}(dg) \right).$$  \hfill (3.25)

The integral of $w(g)$ vanishes for gauge transformations $g$ that leave the boundary invariant, hence implying a gauge invariant action. For gauge transformations that act non-trivially at the boundary\footnote{In section 4.4.1 we will refer to these two types of gauge transformations as proper and improper gauge transformations.} it can be proven\footnote{Under such gauge transformations, the action thus transforms as $S_{CS} \rightarrow S_{CS} - 2\pi k N$. Since one is always interested in the} that the integral of the winding number is an integer say $N \in \mathbb{Z}$. Under such gauge transformations, the action thus transforms as $S_{CS} \rightarrow S_{CS} - 2\pi k N$. Since one is always interested in the
path integral where the action appears as \( \exp(iS) \) path integral will be invariant under the assumption that the Chern-Simons level \( k \) is quantized.

Matching gauge transformations

Lastly, we must verify that the gauge transformations in both formulations of the theory are the same\[2\]. In the Chern-Simons formulation gauge transformations are generated by a zero form

\[
u = \rho^a P_a + \tau^a M_a \in \mathfrak{so}(2, 2), \tag{3.26}\]

where \( \rho_a \) and \( \tau_a \) are respectively the parameters of infinitesimal local translations and infinitesimal local Lorentz transformations. From the transformation law of \( A \), its decomposition in terms of \( e_a \) and \( \omega_a \) and the \( \mathfrak{so}(2, 2) \) algebra we can then determine the transformation laws of the latter under local translations:

\[
\delta \rho^a_{\mu} = - \partial_{\mu} \rho - [e_{\mu}, \rho] = - \partial_{\mu} \rho - \epsilon^{abc} \omega_{\mu b} \rho_c, \tag{3.27}
\]

\[
\delta \omega^a_{\mu} = - \partial_{\mu} \rho - [\omega_{\mu}, \rho] = - \epsilon^{abc} \omega_{\mu b} \rho_c, \tag{3.28}
\]

and local Lorentz transformations

\[
\delta \rho_{\mu} = - \partial_{\mu} \rho - [e_{\mu}, \rho] = - \epsilon^{abc} \omega_{\mu b} \rho_c, \tag{3.29}
\]

\[
\delta \omega_{\mu} = - \partial_{\mu} \rho - [\omega_{\mu}, \rho] = - \partial_{\mu} \tau - \epsilon^{abc} \omega_{\mu b} \tau_c. \tag{3.30}
\]

Now, clearly there is no problem with the local Lorentz transformations. In the frame formulation Local Lorentz transformations with infinitesimal parameter \( \alpha_a^b \) act as:

\[
\delta e_{\mu}^a = \alpha_a^b e_{\mu}^b, \tag{3.31}
\]

\[
\delta \omega_{\mu} = \alpha_a^b \omega_{\mu}^b - \frac{1}{2} \epsilon^{abc} \partial_{\mu} \alpha_{bc}, \tag{3.32}
\]

and thus infinitesimal Local Lorentz transformations with parameter \( \alpha_a^b \) in the frame formulation are thus seen to correspond to Local Lorentz transformations generated by an infinitesimal parameter \( \tau_a \) in the Chern Simons formulation if we identify:

\[
\alpha_a^b = - \epsilon^{abc} \tau_c, \quad \leftrightarrow \quad \tau_a = \frac{1}{2} \epsilon^{abc} \alpha_{bc}. \tag{3.33}
\]

The problem is with the infinitesimal local translations generated by \( \rho_a \) in the Chern Simons formulation which have no obvious counterpart in the frame formulation. As we will show next, on shell they can be matched to some combination of local Lorentz transformations and diffeomorphisms. Now, diffeomorphisms generated by an infinitesimal vector field \( -v^a \) act as:

\[
\delta A_{\mu} = - v^a \partial_{a} A_{\mu} - A_{\nu} \partial_{\mu} v_{\nu}. \tag{3.34}
\]

If we then consider the difference of (3.34) with the transformation law under local translations and let \( \rho^a = v^a e_{\mu}^a \) we find, focusing only on the vielbein, the spin connection is done similarly:

\[
(\delta - \delta) e_{\mu}^a = - v^a (\partial_{a} e_{\mu}^a - \partial_{\mu} e_{\mu}^a) + \epsilon^{abc} v^b e_{\mu}^b \omega_{\mu c} \tag{3.35}
\]

\[
= - v^a (D_{a} e^a_{\mu} - D_{\mu} e^a_{\mu}) + \epsilon^{abc} v^b e_{\mu}^b \omega_{\mu c}, \tag{3.36}
\]

where we used (C.6) and \( \omega_{ab} = - \epsilon^{abc} \omega_c \). The first term vanishes if we assume a torsionless connection, (C.12) whereas the second term is again an infinitesimal local Lorentz transformation with parameter

\[
\tau_{\mu} = v^a \omega_{\mu}^a, \quad \leftrightarrow \quad \alpha_{ab} = \epsilon^{abc} v^c \omega_{\mu c}, \tag{3.37}
\]

as we see from (3.29) and (3.31).
3.3 Classical solutions to AdS3 Chern Simons theory

As discussed we can also describe the AdS solutions discussed in the previous section in Chern-Simons formulation. For this one needs only observe that:

\[ g_{\mu\nu} = \frac{1}{2} \text{Tr} \left[ (A - \bar{A})_\mu (A - \bar{A})_\nu \right], \quad (3.38) \]

which follows from \( A = \omega + e/l \), \( \bar{A} = \omega - e/l \) and \( g_{\mu\nu} = \frac{1}{2} \text{Tr}(\epsilon_{\mu\nu} \epsilon_{\rho\sigma}) \). Using this fact, we may re-express the general solution (2.22) as:

\[ A = b^{-1} \left( L_1 - \frac{2\pi \mathcal{L}}{k} \right) dx^+, \quad \bar{A} = b(\rho) \left( L_1 - \frac{2\pi \bar{\mathcal{L}}}{k} \right) dx^- \quad (3.39) \]

where \( b(\rho) = \exp(\rho L_0) \). Here we have employed the gauge freedom of the theory to gauge away the radial part of the connection\(^{18}\). The function \( b \) is thus regarded as a gauge transformation and we will therefore refer to this form of the connection as the radial gauge. \( a, \bar{a} \) are lastly given by:

\[ a = \left( L_1 - \frac{2\pi \mathcal{L}}{k} \right) dx^+, \quad \bar{a} = \left( L_1 - \frac{2\pi \bar{\mathcal{L}}}{k} \right) dx^- \quad (3.41) \]

As before, all solutions may be recovered from this connection by choosing the functions \( L, \bar{L} \).

3.3.1 Holonomies

As we have seen, classical solutions of the Chern-Simons theory are given by flat connections \( A \). This means that locally they may be written as pure gauge:

\[ A = g^{-1} dg. \quad (3.42) \]

Globally, however this statement is not true as the spacetime may have some non-trivial topology. This obstruction to writing the solutions globally as pure gauge is captured by the holonomy around a non-contractable cycle \( C \) of the spacetime defined as\(^{20}\):

\[ \text{Hol}_C(A) = \mathcal{P} \exp \left( \oint_C A \right) \in G, \quad (3.43) \]

where \( G \) represents the gauge group and \( \mathcal{P} \) represents path ordering. The holonomy transforms by conjugation under gauge transformations. When the holonomy it is non-trivial, it is impossible to find a globally defined gauge transformation \( g \), such that \( A = g^{-1} dg \). If one would try to do so, then \( g \) would not be single valued around the cycle \( C \), but instead pick up a factor of the holonomy. This means that classical solutions in Chern-Simons theory are uniquely specified by the holonomies around the cycles of the manifold, up to an overall gauge transformation\(^{19}\). The solutions of the Chern Simons theory are, in Euclidean signature, specified by two cycles: The spatial cycle and the thermal cycle. The \( \phi \) coordinate will represent the non-contractable cycle. Flat connections \( a \) are thus be uniquely specified by the non-trivial holonomy:

\[ \text{Hol}_\phi(A) = b^{-1} \exp \left( \oint a \right) b. \quad (3.44) \]

Now, what about the thermal cycle? As we have seen in section 2.2.1 smoothness of the Euclidean solution demands that the thermal cycle is contractable, which the present language implies that the holonomy around the thermal cycle is trivial, i.e. lies in the center of the gauge group \( G \):

\[ \text{Hol}_\tau(a) = \mathcal{P} \left( \exp \left( \oint a_\tau dz + \oint a_\bar{\tau} \bar{d}z \right) \right) = \exp(2\pi i L_0). \quad (3.45) \]

\(^{18}\)See [24] for a simple proof of this fact.

\(^{19}\)In more mathematical language: The solutions are uniquely specified by maps from the fundamental group \( \pi_1(M) \), into the gauge group, modulo overall conjugation by \( G \).
Here \( L_0 \) denotes the Cartan element of \( \mathfrak{sl}(2) \). Under the natural assumption that it is in its diagonal form we may write:

\[
\tau a_z + \bar{\tau} a_{\bar{z}} = iL_0 \implies \tau \lambda_z + \bar{\tau} \lambda_{\bar{z}} = iL_0, \tag{3.46}
\]

where \( \lambda_z \) and \( \lambda_{\bar{z}} \) denote diagonal matrices that have the eigenvalues of \( a_z \) and \( a_{\bar{z}} \) on the diagonal. This condition results in equations that define the modular parameter \( \tau \) and constrain the connection. For example, evaluated on the connection (3.41) we find, for the non-rotating solution \( \tau = \bar{\tau} \)

\[
\tau \sqrt{\frac{2\pi L}{k}} = \frac{i}{2}, \tag{3.47}
\]

reproducing (2.41). For future reference we define \( \omega = 2\pi (\tau a_z + \bar{\tau} a_{\bar{z}}) \), which is called the holonomy matrix, and we demand its eigenvalues to be equal to \((\pm 2\pi i)\), i.e. the eigenvalues of \( L_0 \).
4 (2+1)-d Higher Spin gravity: Theoretical Background

Interacting higher spin gravity is known for its notorious difficulties and there appear to be several no-go theorems that forbid their existence\cite{30}. For example, Weinberg’s theorem forbids the long range interactions between massless higher spin particles. Nonetheless, Vasiliev constructed a certain type of higher spin theories in general dimension and containing an infinite tower of higher spin fields that were not forbidden by these no-go theorems. These "Vasiliev higher spin theories" are considered as some type of tensionless limit of string theory, although in the full string theory the higher spin symmetry is broken. These no-go theorems however, only apply for dimensions \(d > 3\), and therefore in three dimensions they can be surpassed and therefore it should be possible to formulate a consistent higher spin theory without the need for an infinite tower of higher spin fields.

The purpose of the next subsection will be to give an overview of the problems that arises with general interacting higher spin theories, and to motivate the advantages of three dimensions. In subsection 4.2 we will discuss how the spectrum of the theory depends on how one chooses to embed the spin-2 sector in the full theory. Then in section 4.4, for the principal embedding, we discuss the asymptotic symmetries of the theory, which as shown in \cite{24}, becomes a classical \(W_3 \otimes W_3\) algebra extending the Virasoro algebra. Lastly, the BTZ black hole will be generalized in section 4.5 to a proper black hole in the context of higher spin theory. We will review aspects of two types of black holes that have been constructed in the literature and comment on their differences. We will as well discuss their thermodynamics. Lastly, in section 4.6 we will discuss a new proposal given in \cite{10} suitable to define extremality in the higher spin context.

4.1 The free higher spin theory and coupling to gravity

Fronsdal \cite{31} was the first to construct equations of motion on a Minkowski background for massless bosonic spin-\(s\) fields, that where later extended to fermions\cite{32}. This field is described by a fully symmetric rank-\(s\) tensor \(\phi_{\mu_1...\mu_s}\) and satisfies the following second order field equation\cite{24}:

\[
F_{\mu_1...\mu_s} \equiv \Box \phi_{\mu_1...\mu_s} - \partial_{(\mu_1} \partial_{\lambda} \phi_{\mu_2...\mu_s)\lambda} + \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3...\mu_s)\lambda} = 0,
\]

which is invariant under the gauge transformations:

\[
\delta \phi_{\mu_1...\mu_s} = \partial_{(\mu_1} \xi_{\mu_2...\mu_s)} \xi_{\mu_1...\mu_s+3\lambda} = 0.
\]

Note that for \(s = 2\), \(F_{\mu\nu}\) is the linearized Ricci tensor and that the gauge transformations are linearized diffeomorphisms. These gauge transformations, ensure that in \(d = 3\) the theory contains no dynamical degrees of freedom while in \(d > 3\) that the theory has the correct amount of degrees of freedom. They are referred to as higher spin diffeomorphisms. Imposing a double trace constraint on the spin-\(s\) field, Fronsdal wrote down an action that leads to the above field equation.

To couple the theory to gravity we now make the minimal substitution \(\eta_{\mu\nu} \rightarrow g_{\mu\nu}\) and \(\partial \rightarrow \nabla\). It is required that the resulting theory admits the same gauge symmetries as on an Minkowski background to ensure consistency. It is at this point where the problems and inconsistencies arise, as was shown by Aragone and Deser\cite{33}. At the heart of the obstruction lies the following decomposition of the Riemann tensor in a generic background and general dimension:

\[
R_{\mu\nu\rho\gamma} = S_{\mu\nu\rho\gamma} + E_{\mu\nu\rho\gamma} + C_{\mu\nu\rho\gamma},
\]

with

\[
S_{\mu\nu\rho\gamma} = \frac{2R}{d(d-1)} \left( g_{[\mu[\rho} g_{\gamma]\nu]} \right),
\]

\[
E_{\mu\nu\rho\gamma} = \frac{2}{d+2} \left( g_{[\mu[\rho} S_{\gamma]\nu]} - g_{[\nu[\rho} S_{\gamma]\mu]} \right).
\]
The first two terms are fixed entirely by the Ricci tensor and Ricci scalar and are thus fixed by the
equations of motion. The third term is the Weyl tensor and is left undetermined by the field equations.
It is thus this term that captures the dynamical information of the theory. In general dimensions this
tensor will not be zero and so the Riemann tensor is not proportional to the Ricci tensor. Now, one
can compute the gauge variation of the Einstein-Hilbert action to find that it is proportional to the Ricci
tensor. On the other hand, the gauge variation of the action of the minimally coupled higher spin action is
proportional to the full Riemann tensor instead and as a consequence the contribution of the two actions
can, on generic backgrounds, not be cancelled out, not even when the Einstein equations are impose.

There is a way out however, as was shown by Fronsdal and Vasiliev in [34]. They showed that with a non-
vanishing cosmological constant $\Lambda$, it is actually possible to modify the field equations. This introduces
extra higher derivative terms that depend on negative $\Lambda$ and cancel the offending Riemann curvature
terms. This observation eventually led Vasiliev [35] to his "Vasiliev equations" that describe the full non-
linear interaction of an infinite tower of higher spin fields on constant curvature backgrounds in general
dimensions. Consistency of the field equations required the presence of this infinite higher spin tower and
make the theory technically notoriously complicated in general dimensions.

Something special happens in three dimension however. In this case the Weyl tensor vanishes and thus the
Riemann tensor is proportional to the Ricci tensor. As a consequence it is possible to circumvent the
above mentioned difficulties and to formulate consistent interactions of massless higher spin fields with
gravity. In particular, there is no need to resort to an infinite tower of higher spin fields to obtain consist-
ent interactions. From this point on we will focus on three dimensions. In the next section we will review
the construction of the theory describing interacting fields of spins $s \geq 2$. For this it will be convenient to
formulate the linearized theory in the first order formalism. Then a spin-$s$ field freely propagating on a
in a constant curvature background is described by a pair of generalized one-forms:

$$\epsilon^{a_1 \ldots a_{s-1}}_{\mu}, \omega^{a_1 \ldots a_{s-1}}_{\mu}.$$ \hspace{1cm} (4.6)

The generalized spin-connections are auxiliary fields acting as a generalization of local Lorentz invariance.
One can then combine these into a gauge connection, very similar to what we discussed before, by con-
tracting the higher spin vielbein and spin connection with a set of higher spin generators that extend the
gravitational $\mathfrak{s}(2)$ gauge algebra to a larger algebra $\mathfrak{g}$. This set of generators is not entirely arbitrary.
They must transform as irreducible $\mathfrak{s}(2)$ tensors. It is very important that the $\mathfrak{s}(2)$ forms a subalgebra
of the extended gauge algebra, because this is what defines the gravitational sector. We will see that
different theories arise depending on how one chooses to embed $\mathfrak{s}(2) \hookrightarrow \mathfrak{g}$. If the total set of generators
then admits an invariant bilinear form one can write down a Chern Simons action for the theory. In the
following section we will pay attention to extended $\mathfrak{s}(N)$ gauge algebra describing the interactions of
higher spin fields with spin $s \leq N$. We will pay in particular emphasize the importance of the embedding
$\mathfrak{s}(2) \hookrightarrow \mathfrak{g}$.

---

21It is only for $s = 3/2$ that this problem does not occur, which is vital for the construction of supergravity [35].
This is because the variation of the Rarita-Schwinger action for the spin-3/2 field is independent of the Weyl tensor,
i.e proportional to the Ricci tensor. However, as soon as one considers $s > 3/2$ the problem of non-vanishing Weyl
tensor in $d > 3$ reintroduces itself. We will come back to this in section [7] when we discuss hypergravity.

22There has been some progress in the second order metric formulations. See [36].
A slight subtlety to note here is that although $\mathfrak{s}(N)$ becomes the infinite dimensional higher spin algebra $\mathfrak{hs}(\lambda)$
in the limit $N \to \infty$, $\mathfrak{s}(N)$ is not itself a consistent truncation of $\mathfrak{hs}(\lambda)$ if $N > 2$ because the generators do not
form a subalgebra. However, if one enforces a truncation of $\mathfrak{hs}(\lambda)$ by sending the higher spin fields to zero then
the resulting truncated algebra is isomorphic to $\mathfrak{s}(N)$. In addition, there does exist a well defined continuation
procedure to send $\mathfrak{s}(N)$ to $\mathfrak{hs}(\lambda)$ and therefore $\mathfrak{s}(N)$ may be considered as an appropriate higher spin extension of
$\mathfrak{s}(2)$.
4.2 Embeddings of the gravitational sector

After choosing an appropriate gauge algebra, one is then left with the question how to embed the gravitational $\mathfrak{sl}(2)$ into the extended $\mathfrak{g} = \mathfrak{sl}(N)$ higher spin gauge algebra. Different embeddings correspond to physically different theories, and the number of inequivalent embeddings equals the number of partitions of $N$. We will exemplify this fact here for $N = 3$, in which case there are two different embeddings which are called the principal embedding and the diagonal embedding. In general, the different embeddings are classified by how the fundamental representation of $\mathfrak{sl}(N)$ branches into irreducible $\mathfrak{sl}(2)$ representations. Characterizing for the principal embedding is that the fundamental representation of $\mathfrak{sl}(N)$ becomes an irreducible representation of the embedded $\mathfrak{sl}(2)$ algebra. The diagonal embedding is characterized by the fact that the embedded $\mathfrak{sl}(2)$ algebra takes a block diagonal form inside $\mathfrak{sl}(N)$. However, of special interest to determine the spectrum of the theory is the branching of the adjoint representation which can be deduced from the branching rules of the fundamental representation. There are two reasons for this. Firstly, it is the adjoint representation under which the gauge fields transform, and secondly the dimension of the adjoint representation is equal to the number of generators of the gauge algebra, and hence this representation naturally provides information about the structure of the gauge algebra. One of the purposes of this section will be to illustrate how, for the principal embedding, one can find the spectrum of the theory.

4.3 Spectrum of the principal embedding

We will now discuss the spectrum of the $\mathfrak{sl}(3)$ higher spin theory with $\mathfrak{sl}(2)$ principally embedded. It is explained in[40] how the branching rules for the adjoint representation are acquired. Here we will only quote the results, as we are interested in the spectra.

Principal embedding

In the principal embedding the adjoint representation branches as[40]:

$$\text{Adj}_N \simeq 3_2 \oplus 5_2 \oplus ... \oplus (2N - 1)_2 = \bigoplus_{j=1}^{N-1} (2j+1)_2. \quad (4.7)$$

The representations are labeled by their $\mathfrak{sl}(2)$ spin-$S$ and the indices within each multiplet run from $-S$ to $S$. We thus have 3 generators in the spin-1 representation, i.e. the AdS gravitational sector, 5 generators in the spin-2 representation, ... and $2N - 1$ generators in the spin-$(N - 1)$ representation of $\mathfrak{sl}(2)$.

This branching induces then a decomposition of the gauge field:

$$A_\mu = \sum_{s=2}^{N} t^{a_1,a_2,...,a_{s-1}}_{j} T_{a_1,a_2,...,a_{s-1}}$$
$$= f_\mu^a L_a + \sum_{s=3}^{N} t^{a_1,a_2,...,a_{s-1}}_{j} T_{a_1,a_2,...,a_{s-1}}, \quad (4.8)$$

and similar for the other sector and where we have explicitly taken out the $\mathfrak{sl}(2)$ factor in the second step. The $T_{a_1,a_2,...,a_{s-1}}$ are the generators of the spin-$S = (s - 1)$ representation of $\mathfrak{sl}(2)$. The $t^{a_1,a_2,...,a_{s-1}}_{j}$ represent combinations of the vielbein and spin connection and their higher spin generalizations, c.f. (4.6) and transform under the corresponding spin-$(s - 1)$ representation. Explicitly they are given by:

$$t^{a_1,...,a_{s-1}}_{j} = \omega^{a_1,...,a_{s-1}}_{\mu} + \frac{1}{t} t^{a_1,...,a_{s-1}}, \quad n^{a_1,...,a_{s-1}} = \omega^{a_1,...,a_{s-1}}_{\mu} - \frac{1}{t} t^{a_1,...,a_{s-1}}. \quad (4.9)$$

To determine the spectrum one then linearizes the equations of motion $dA + A \wedge A = 0$ around empty AdS. For this, consider the following fluctuations around the backgrounds $e^{a_1,a_2,...,a_{s-1}}_{\mu}$ and $\omega^{a_1,a_2,...,a_{s-1}}_{\mu}$:

$$e^{a_1,a_2,...,a_{s-1}}_{\mu} \rightarrow \tilde{e}^{a_1,a_2,...,a_{s-1}}_{\mu} + \delta e^{a_1,a_2,...,a_{s-1}}_{\mu}, \quad \omega^{a_1,a_2,...,a_{s-1}}_{\mu} \rightarrow \tilde{\omega}^{a_1,a_2,...,a_{s-1}}_{\mu} + \delta \omega^{a_1,a_2,...,a_{s-1}}_{\mu}. \quad (4.10)$$
Once the expressions for the connection are substituted into the CS field equation and only terms that are linear in fluctuations are kept, we acquire a set of linearized equations of motion. These linearized equations can then be identified with the Fronsdal equations of motion describing the free-propagation of a spin-s field $\phi_{\mu_1 \ldots \mu_s}$ on an AdS$_3$ background:\[24].

\[
\phi_{\mu_1 \ldots \mu_s} = \frac{1}{8} \epsilon^{[a_1 \ldots [e_{\mu_1} e_{\mu_{s-2}] e_{\mu_2} e_{\mu_s}] a_1 \ldots a_{s-1}}. \tag{4.11}
\]

The metric spin 2-field and the spin-3 field can be recovered from the vielbein as:

\[
g_{\mu_1 \mu_2} \equiv \phi_{\mu_1 \mu_2} = \frac{1}{2} \text{Tr}(e_{(\mu_1} e_{\mu_2)}), \quad \phi_{\mu_1 \mu_2 \mu_3} = \frac{1}{3} \text{Tr}(e_{(\mu_1} e_{\mu_2} e_{\mu_3)}). \tag{4.12}
\]

where as usual the round brackets $(\ldots)$ denote full symmetrisation of the indices.

What we have seen here is very general. The branching rule for the adjoint representation of $\mathfrak{sl}(N)$ under the principal embedding of $\mathfrak{sl}(2)$ will give irreducible $\mathfrak{sl}(2)$ representations of spin-$(s - 1)$ that give rise to spin-s gauge fields in the spectrum of the bulk theory.

### 4.3.1 Relating the Higher spin Chern-Simons level to the AdS radius.

Besides the spectrum of the theory, the choice of embedding also affects how the constants in front of the actions are related. Denoting the Chern Simons level in front of the higher spin Chern-Simons action with gauge group $\text{SL}(N) \times \text{SL}(N)$ by $k_{cs}$, it can be related to the lower spin Chern-Simons level $k$ via the identification:

\[
k_{cs} = \frac{k}{2\text{Tr}(L_0 L_0)}, \quad k = \frac{L_0}{4G_3}. \tag{4.13}
\]

This normalisation factor arises because of the fact that in the derivation in \[C.1\] we used the $\mathfrak{sl}(2, \mathbb{R})$ algebra and metric. When $\mathfrak{sl}(2, \mathbb{R})$ is considered as a subalgebra of $\mathfrak{sl}(N)$ however, the $\mathfrak{sl}(2)$ generators obey a different metric depending on the chosen embedding. To compensate for this fact we must therefore add a normalization factor as in \[\text{(4.13)}\] denoted by $\epsilon = 2\text{Tr}(L_0 L_0)$. Here $L_0$ is the Cartan generator of the $\mathfrak{sl}(2)$ subalgebra embedded in $\mathfrak{sl}(N)$.

### 4.4 $\mathcal{W}$-algebras as asymptotic symmetries

In the previous section we have discussed two possibilities for formulating a higher spin theory on AdS$_3$. In the Chern Simons formalism the required input is an appropriate gauge algebra $\mathfrak{g}$ accompanied with a specified $\mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ embedding that defines the gravitational sector. The field content is then found from the branching rule of $\mathfrak{g}$ under the adjoint action of this $\mathfrak{sl}(2)$ embedding.

In this section we will discuss the asymptotic symmetries of AdS$_3$ by imposing suitable boundary conditions. It is a priori clear that these boundary conditions mean that the allowed number of gauge transformations must be restricted such that the boundary conditions are left invariant. This in turn means that configurations that where gauge equivalent before, now become physically distinct at the boundary. Thus, although in the bulk there are no propagating degrees of freedom, at the boundary we will have dynamical degrees of freedom. We will discuss that the boundary dynamics of a Chern-Simons theory, is described by a CFT with an affine Lie-symmetry algebra, $\mathfrak{g}_\infty$, also known as the Wess-Zumino-Witten CFT's. Not all solutions of the Chern Simons theory will be admissible as classical configurations though. We must instead restrict to asymptotically AdS$_3$ configurations which will impose further constraints. By gauge fixing these constraints the asymptotic affine Lie algebra can be turned into a classical $\mathcal{W}$ algebra. This procedure of deriving $\mathcal{W}$ algebras from affine Lie algebra goes under the name of classical Drinfeld Sokolov reduction. We will only discuss this superficially.

Not only the spectrum of the theory, but also the asymptotic symmetry algebra relies heavily on the chosen $\mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ embedding. In this section we will focus on the principal embedding, $\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(3)$ and show how a centrally extended $\mathcal{W}_3$ emerges. This section will mostly follow the discussion of \[20, 24\] and \[11\].
4.4.1 Boundary terms and global charges

To define what is meant by asymptotic/global charges we consider the Chern-Simons action on a manifold with topology \( \mathcal{M} = \mathbb{R} \times \Sigma \), where \( \Sigma \) is a two-manifold with boundary \( \partial \Sigma \). The time coordinate \( t \) parametrizes \( \mathbb{R} \), while the disk \( \Sigma \) is parametrized by \( \rho, \phi \). The gauge field \( A = A_\mu dx^\mu \) can then be split as:

\[
A^a = A^a_t dt + A^a_i dx^i.
\]

The action then becomes:

\[
S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} d^4 x \epsilon^{i j k} \left( A_i \frac{\partial}{\partial x_j} F_{k j} - A_j \frac{\partial}{\partial x_i} F_{k j} \right) + B.
\]

Here \( B \) represents a boundary term and \( (a = 1,...,N, i = 1,2) \). The \( x^i \) are local coordinates on the spatial surface \( \Sigma \). There are \( 2N \) dynamical fields \( A^a_i \) that satisfy the equal time Poisson bracket:

\[
[A^a_i(x), A^b_j(y)]_{PB} = \frac{2\pi}{k} \epsilon_{ij} \delta^a_b (x - y) \delta^{ab},
\]

where \( \delta^{ab} \) is the Cartan-Killing metric on the gauge algebra \( \mathfrak{g} \). The Poisson bracket of two differentiable functionals \( F[A_i] \) and \( H[A_i] \) is computed with:

\[
[F, H]_{PB} = \frac{2\pi}{k} \int_{\Sigma} d^2 x \epsilon^{ij} \frac{\delta F}{\delta A^a_i(x)} \frac{\delta H}{\delta A^b_j(x)} \delta^{ab}.
\]

Since there are no time derivatives terms in the action, \( A_t \) acts as a Lagrange multiplier whose equation of motion leads to the constraint:

\[
G^{(0)}_a = \frac{k}{4\pi} \epsilon_{ij} F_{ij} \delta^{ab} = 0.
\]

We can then define a smeared integral of the constraint with a parameter \( \eta \):

\[
G^{(0)}[\eta] = \int_{\Sigma} d^2 x \eta^a G^{(0)}_a.
\]

If \( \Sigma \) is closed, i.e. there is no boundary, then these constraints generate gauge transformations:

\[
\left[ G^{(0)}[\eta], A^a_j \right]_{PB} = \epsilon^{ij} \delta^{ab} \left[ G^{(0)}[\eta], A^b_j \right] = D_\eta \eta^a = \delta^a_b A^b_j,
\]

where \( D_\eta \) is the gauge covariant derivative and we used (4.22) Among themselves, the smeared generators satisfy the algebra:

\[
[G^{(0)}[\eta], G^{(0)}[\chi]]_{PB} = G^{(0)}[\zeta(\eta, \chi)], \quad \zeta = [\eta^a, \chi^b] = f^{ab}_{cd} \eta^c \chi^d,
\]

with the \( f^{ab}_{cd} \) the structure constants. However, in the presence of a boundary (4.20) will no longer hold. The functional derivative of the smeared generator is in that case ill defined and its variation contains a boundary term that does not vanish if the gauge parameter is non-zero at the boundary.

\[
\delta G^{(0)}[\eta] = \frac{k}{4\pi} \int_{\Sigma} d^2 x \epsilon^{ij} \eta^a \delta F^a_{ij} = \frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} \eta^a \partial_i (\delta A_j^a) = \frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} D_i (\eta_a \delta A_j^a) - \frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} D_i \eta_a \delta A_j^a = \frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} D_i \eta_a \delta A_j^a = \frac{k}{2\pi} \int_\Sigma d^2 \eta \epsilon^{ij} D_i \eta_a \delta A_j^a
\]

From the first to the second line we used that \( F_{ij} = [D_i, D_j] \). From the third to the fourth line we used that \( \eta_a \delta A_j^a \) is a scalar under the action of \( D_i \) and thus it acts as a normal derivative. In the last line
Stokes was applied. The first term is the boundary term $B$ that arises in the presence of a boundary and if the gauge parameter $\eta \neq 0$ it spoils the poisson algebra. In particular we see that in the presence of a boundary and for $\eta \neq 0$, $G^0$ cannot be a generator of gauge transformations. We can cure this problem by redefining: $G[\eta] = G^0[\eta] + Q[\eta]$ where $Q[\eta]$ is given by:

$$Q[\eta] = -\frac{k}{2\pi} \int_{\partial \Sigma} dx^i \eta_a A^a_i.$$  

(4.23)

This new functional $G[\eta]$ now has a well-defined variation with respect to $A$. Its Poisson bracket with itself is found to be:

$$\{G[\eta], G[\xi]\}_{PB} = \{G[\zeta(\eta, \chi)]\} + \frac{k}{2\pi} \int_{\partial} dx^i \eta_a \partial_j \xi^a.$$  

(4.24)

This extra term with respect to the Poisson brackets of the original $G^0$ is called the central extension. This centrally extended algebra can be reduced to a smaller algebra by imposing additional suitable boundary conditions. We discuss this in further detail in the next subsection.

Let us now make a few important observations and in particular define the distinction between proper and improper gauge transformations important in the presence of a boundary. In the language of gauge theories, gauge transformations are generated by so-called first class constraints. Roughly speaking, a constraint is called first class if its poisson bracket reads

$$\{\phi_a, \phi_b\} = C_{ab} \phi_c,$$

(4.25)

with $C_{ab} = [\phi_a, \phi_b]$ and if $\phi_a = 0$ then the matrix $C_{ab}$ is weakly zero (or zero entirely). Here an example of such a generator of gauge symmetries is $G_0$, generated by the constraint $F = 0$. On shell these generators vanish. On the contrary, global symmetries are generated by quantities that do not vanish on shell, i.e. $Q[\eta]$. By definition, the space of physical solutions is given by the space of all classical solutions, modulo the gauge transformations. Without a boundary, the physical phase space is trivial. The general solution would be of the form $A = g^{-1}dg$ which is gauge equivalent to the empty solution $A = 0$, yielding no local degrees of freedom. All symmetries are generated by the constraints, $F = 0$, and hence represent gauge symmetries generated by $G_0$. One can then use this gauge freedom to set $g = 1$ and hence the only solution is $A = 0$. If a boundary is present though, then a part of the symmetries is not generated by constraints, but instead by the on-shell non-zero $Q[\eta]$ that do not vanish when the constraint $F = 0$ is imposed. While in the bulk it is still true that we can relate any flat $A$ to the empty solution, they do not represent the same physical configuration. At the boundary, both $A \neq 0$ and $A = 0$, are solutions to the equations of motion but they are physically distinguishable and related by a global symmetry of the action generated by $Q[\eta]$. This quantity $Q[\eta]$ is called the boundary charge and they generate these global symmetries through:

$$\{Q[\eta], A^a_k\}_{PB} = \delta_k A^a_k.$$  

(4.26)

They satisfy the same Poisson algebra as the $G[\eta]$, i.e.:

$$\{Q[\eta], Q[\chi]\}_{PB} = Q[\zeta(\eta, \chi)] + \frac{k}{2\pi} \int_{\partial \Sigma} \eta_a d\chi^a,$$  

(4.27)

and one can use this algebra to read of the asymptotic symmetry algebra. Note though that this bracket is now defined on the physical phase space, i.e. after gauge fixing. The presence of a boundary thus makes a clear separation of actual gauge transformations and global symmetries. Those transformations that represent true gauge symmetries and are generated by constraints are called "proper". Those that are not, are called improper and they generate the global symmetries at the boundary.

\footnote{Of course there are the global ones that are measured by the holonomies, but these are not important for the present discussion.}
We will begin by reviewing what boundary conditions have to be imposed on the connections $A, \bar{A}$ to have a well-defined action upon adding a boundary. As usual we demand that the on-shell variation of the Chern Simons action vanishes at the boundary. Working in light-cone coordinates this is easily found to be:

$$\delta S_{CS}[A] = -\frac{k}{4\pi} \int_{\partial \mathcal{M}} d^2x \text{ Tr}(A_+ \delta A_- - A_- \delta A_+),$$

where $\partial \mathcal{M} = \mathbb{R} \times S^1$. This boundary contribution can be set to zero if we impose

$$A_- \bigg|_{\partial \mathcal{M}} = 0.$$  

(4.29)

Next we need to determine the physical phase space of solutions, which given our previous discussion, means that we must consider the full space of classical solutions and mod out the gauge transformations. We thus need to pick a gauge. A particular useful one is the radial gauge we have discussed before [24].

$$A_\rho = b^{-1}(\rho) \partial_\rho b(\rho) \quad b(\rho) = e^{\rho L_0},$$

(4.30)

with $L_0 \in \mathfrak{sl}(2)$. It can be shown that this gauge choice completely removes all of the gauge degrees of freedom of $A_\rho$ [24]. Next we impose the equations of motion. From varying the action to $A_t$ we find:

$$F_{\rho \phi} = \partial_\rho A_\phi + [A_\rho, A_\phi] = 0.$$  

(4.31)

This has a solution for

$$A_\rho(t, \rho, \phi) = b^{-1}(\rho) \tilde{a}(t, \phi) b(\rho),$$

(4.32)

with $a(t, \phi)$ an arbitrary $\mathfrak{g}$ valued function of $t$ and $\phi$. Next the radial dependence of $A_t$ is determined by the $F_{\rho \rho} = 0$ which also has a solution of the above form. The boundary condition (4.29) next implies that $A_t = A_\phi$ on $\partial \mathcal{M}$, but since we completely fixed the $\rho$ dependence by $b(\rho)$, this must hold not only on $\partial \mathcal{M}$ but on all of $\mathcal{M}$:

$$A_- = \frac{1}{2}(A_t - A_\phi) = 0 \quad \text{Everywhere.}$$

(4.33)

Lastly we use the final equation of motion

$$F_{t \phi} = \partial_\phi A_\rho - \partial_\rho A_\phi + [A_t, A_\phi] = (\partial_\rho - \partial_\phi) A_+ = 0,$$

(4.34)

from which we see that $\partial_- a(t, \phi) = 0$ and therefore $a$ is a function of $x^+$ only. The other sector is treated similarly, with the difference being that one must impose the boundary condition $A_+ = 0$ to ensure that the vielbein is invertible [24]. In summary:

$$A = b^{-1}a(x^+)b + b^{-1}db$$

(4.35)

$$\bar{A} = b\bar{a}(x^-)b^{-1} + bdb^{-1},$$

(4.36)

with the functions $a(x^+), \bar{a}(x^-)$ parametrising the reduced (gauge fixed) phase space of the theory. Different choices represent gauge inequivalent solutions. We emphasize once again, that without a boundary, the solution would be $A = b^{-1}db$ which is gauge equivalent to $A = 0$, i.e. generated by a first class constraint with $G = G^0 = 0$. With a boundary, we have the solutions parametrized by $a$, which can also be related to the trivial solution, but by a gauge transformation not generated by a first class constraint, i.e. $G = Q \neq 0$, thus acting as a global symmetry on the space of solutions.

Asymptotic symmetry Poisson algebra

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24As discussed before, this is always possible. One can start from an arbitrary $A'_\rho$ and solve the equation $g^{-1}A'_\rho + g^{-1}\partial_\rho g = b^{-1}\partial_db$. The solution for this can be shown to be given by a path ordered exponential for any $b(\rho)$ valued in the gauge group.
As we mentioned, going to the radial gauge completely fixed the gauge degrees of freedom, there are however those left that represent global symmetries on the space of solutions, i.e. after gauge fixing. The gauge transformations that preserve the gauge choice \((4.30)\), are parametrized by functions \(\Lambda\), that satisfy
\[
\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda] = 0
\]
and are thus again of the form:
\[
\Lambda(t, \rho, \phi) = b^{-1}(\rho)\lambda(t, \phi)b(\rho).
\]

Requiring that they preserve the boundary condition \(\delta A_- = 0\) fixes \(\lambda = \lambda(x^+)\). Now we are in a position to apply what we have discussed in the previous section. Making the replacement \(\eta \to \Lambda\), we have an explicit expression for the global charge \(Q(\Lambda)\) given by \((4.23)\) and with its algebra given by \((4.27)\) where we recall that this is now a bracket on the reduced phase space of physical solutions since we have fixed a gauge. As we have discussed, the physical phase space, the basic variables are not \(A\) anymore but rather \(a(x^+)\). Using the expression for \(A_\rho\) in \((4.32)\) the global charge thus reads:
\[
Q(\Lambda) = -\frac{k}{2\pi} \int d\phi \text{Tr}(\Lambda(\phi)A_\phi(\phi)) = -\frac{k}{2\pi} \int d\phi \text{Tr}(\Lambda(\phi)a_\phi(\phi)).
\]

Note that we only wrote down the \(\phi\)-dependence since we are integrating over \(\partial\Sigma\) and the time dependence is entirely set by \(\phi\). Now, transformations on \(a\) are of the form:
\[
\delta_{\lambda}a = \partial_{\lambda} + [a, \lambda]
\]

On the other hand we should find the same expression from the transformation law:
\[
\delta_{\lambda}a = \{Q(\Lambda), a(\phi)\} = -\frac{k}{2\pi} \int \lambda(x')\delta_{ab}\{a'^\lambda(x'), a(\phi)\}.
\]

If we expand in a generic basis \(T_a\) of \(\mathfrak{g}\), \(a = a^a T_a\), then the two transformations can be shown to be compatible:
\[
\{a^a(\phi), a^b(\phi')\} = \frac{2\pi}{k} \eta^{ab}\delta(\phi - \phi') - f_{\phi'}^{\phi a} \partial^a(\phi)\delta(\phi - \phi'),
\]
where the \(f_{\phi'}^{\phi a}\) are the structure constants of \(\mathfrak{g}\) in a chosen basis of the \(T_a\) and \(\eta^{ab}\) represents the inverse killing metric. If we then expand in terms of the modes \(a^a(\phi) = \frac{1}{k} \sum_{p \in Z} a^{\phi}_{a,p}e^{ip\phi}\) the algebra becomes:
\[
\{a^a_{\mu}, a^b_{\nu}\} = -f^{\phi}_{\mu \nu} a^{\phi}_{\mu + q} + ipk \eta^{ab} \delta_{\mu, -q}.
\]

This algebra is called an affine Lie algebra, which is denoted by \(\hat{\mathfrak{g}}_k\). They form the symmetry algebra of the Wess-Zumino-Witten CFT’s. The parameter \(k\) is called the level. Thus the boundary dynamics of a Chern Simons theory with gauge group \(G\) is described by a CFT with an affine lie algebra \(\hat{\mathfrak{g}}_k\), denoted \(\hat{\mathfrak{g}}\) as its symmetry algebra. These algebras are also known as current algebras or kac-moody algebras. We will discuss somewhat more about them in appendix \([E.4.1]\) where we discuss the Sugawara construction of the stress energy tensor.

**Asymptotically AdS boundary conditions**

There is still a problem with the discussion of the previous section, since we know that the asymptotic symmetries of 3d gravity are not described by an affine lie algebra but instead by a Virasoro algebra. It was proposed in \([24]\) that the boundary conditions:
\[
A_- = 0 \quad A_\rho = b^{-1}(\rho)\partial_\rho b(\rho) \quad A_+ = b^{-1}(\rho)a(x^+)b(\rho),
\]
should be supplemented by an additional asymptotic fall-of condition at \(\rho \to \infty\):
\[
(A - A_{\text{AdS}})_{\text{boundary}} = \mathcal{O}(1).
\]
The combined boundary conditions, (4.43) and (4.44), are referred to as Asymptotically AdS boundary conditions. The inclusion of the latter allows one to reduce the affine lie algebra to a smaller algebra whilst guaranteeing that the Virasoro algebra is included. Depending on the gauge algebra under consideration, this reduced algebra will be Virasoro itself for $\mathfrak{g} = \mathfrak{sl}(2)$ or an extension in the case of an extended gauge algebra containing $\mathfrak{sl}(2)$. Note though that this combined set of boundary conditions, i.e (4.43) and (4.44), is not unique. One can show that there exists other choices of boundary conditions that lead to the same asymptotic structure. However, there is a set of consistency conditions on the allowed boundary conditions that must be satisfied restricting the possibilities. Firstly, they must be invariant under the isometry group and secondly, they should include asymptotically AdS solutions that are of physical interest (for example the BTZ black hole) since otherwise they would be too restrictive. Third and finally, they must lead to well-defined asymptotic charges\textsuperscript{33}. The boundary conditions (4.43) and (4.44) simply provide an efficient set that does the trick.

4.4.3 The emergence of $\mathcal{W}_3$ algebras

We now specify to $\mathfrak{sl}(3)$ with the gravitational $\mathfrak{sl}(2)$ principally embedded. We discussed that a generic solution is parametrised by $a(x^+)$ solely. Using then (4.8) we can decompose the connection in terms of the $\mathfrak{sl}(3,R)$ generators as:

$$a(x^+) = \sum_{i=-1}^{1} l^i(x^+) L_i + \sum_{m=-2}^{2} w^m(x^+) W_m.$$  \hfill (4.45)

Imposing (4.44) we must set $l^1 = 1$, $w^1 = w^2 = 0$. These conditions are found by restoring the radial dependence of the connection and observing that these terms interfere with the asymptotic behaviour in the $\rho \to \infty$ limit. By this we mean that they have higher order fall-off conditions with the radial coordinate. Computing the Poisson brackets of $l^i, w^m$, one finds that they define first class constraints\textsuperscript{24,25}. They therefore generate gauge transformations and we can fix this freedom by setting $l^0 = w^0 = w^{-1} = 0$. The remaining components $l^{-1}$ and $w^{-2}$ can lastly be conveniently normalised such that we can express a generic asymptotically AdS solution as:

$$a(x^+) = L_1 + \frac{2\pi L(x^+)}{k} L_{-1} + \pi W(x^+) W_{-2}.$$  \hfill (4.46)

Just as before, different choices for the functions $\mathcal{L}$ and $\mathcal{W}$ correspond to physically different solutions. As this point there are two different ways to proceed, which we will discuss next\textsuperscript{24}.

**Approach 1**

We expand a general gauge transformation as:

$$\lambda(x^+) = \sum_{i=-1}^{1} \epsilon^i(x^+) L_i + \sum_{m=-2}^{2} \chi^m(x^+) W_m,$$  \hfill (4.47)

where we have already taken out the radial dependence. The task is now to identify all those gauge transformations that leave the structure of (4.46) invariant. This results in 6 equations between the parameters $\epsilon^i$ and $\chi^m$\textsuperscript{24}, which can be expressed in terms of two independent parameters $\epsilon = \epsilon^1, \chi = \chi^2$. Then, under these transformations one can read of the corresponding transformation laws for $\mathcal{L}$ and $\mathcal{W}$.

\textsuperscript{25}I.e. there Poisson brackets are of the form of (4.25).
One finds:

$$\delta_\epsilon \mathcal{L} = \epsilon \mathcal{L}' + 2\epsilon' \mathcal{L} + \frac{k}{4\pi} \epsilon''$$,
$$\delta_\omega \mathcal{W} = \omega \mathcal{W}' + 3\omega' \mathcal{W},$$
$$\delta_\chi \mathcal{L} = 2\chi \mathcal{V}' + 3\chi' \mathcal{W},$$
$$\delta_\chi \mathcal{W} = \frac{1}{3} \left( 2\chi \mathcal{L}'' + 9\chi' \mathcal{L}' + 15\chi'' \mathcal{L} + \frac{k}{4\pi} \chi^{(5)} + \frac{64\pi}{k} (\chi \mathcal{L}' + \chi' \mathcal{L}^2) \right).$$

The transformation rules under a variation with the parameter $\epsilon$ and $\chi$ thus identify $\mathcal{L}$ and $\mathcal{W}$ as the CFT stress tensor of conformal weight $(2,0)$ and a primary field of conformal weight $(3,0)$. With these explicit expressions for the variations and we can now determine the asymptotic surface charges, using (4.23):

$$Q[\epsilon] = - \int_{\partial\Omega} \epsilon \mathcal{L}, \quad Q[\chi] = - \int_{\partial\Omega} \chi \mathcal{W},$$

where $Q[\epsilon]$ and $Q[\chi]$ are respectively the spin-2 and spin-3 charges. They generate the above transformations via:

$$\delta_\epsilon \mathcal{L} = [Q[\epsilon], \mathcal{L}]_{PB}, \quad \delta_\chi \mathcal{W} = [Q[\chi], \mathcal{W}]_{PB},$$

$$\delta_\epsilon \mathcal{L} = [Q[\epsilon], \mathcal{L}]_{PB}, \quad \delta_\chi \mathcal{W} = [Q[\chi], \mathcal{W}]_{PB}. \quad (4.49)$$

Using the relation

$$\delta_\epsilon \mathcal{L} = [Q[\epsilon], \mathcal{L}(x^+)]_{PB} = - \int dx^+ \epsilon [\mathcal{L}(x^+), \mathcal{L}(x^+)]_{PB}$$

and similar for $\xi$ and $\mathcal{W}$ one can read of the Poisson brackets between the functions $\mathcal{L}$ and $\mathcal{W}$. It can be made most explicit by writing it in terms of the Fourier modes of $\mathcal{L}$ and $\mathcal{W}$.

$$\mathcal{L} = - \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathcal{L}_n e^{-inx^+}, \quad \mathcal{W} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathcal{W}_n e^{-inx^+}. \quad (4.50)$$

Shifting the $\mathcal{L}$ zero mode according to:

$$\mathcal{L}_0 \rightarrow \mathcal{L}_0 - \frac{k}{4}$$

(4.51)

where $k = c/6$, to obtain the standard form of the Virasoro algebra and furthermore performing the substitution $[\ldots]_{PB} \rightarrow -i[\ldots]_{com}$ we obtain the classical $\mathcal{W}_3$ algebra:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{m+n} + \frac{c}{12} (n^3 - n) \delta_{m+n,0},$$
$$[\mathcal{L}_n, \mathcal{W}_m] = (2n-m)\mathcal{W}_{m+n},$$
$$[\mathcal{W}_n, \mathcal{W}_m] = \frac{1}{3} \left[ (n-m)(2n^2 + 2m^2 - nm - 8)\mathcal{L}_{m+n} + \frac{96}{c} (n-m) \Lambda_{n+m} \right] + \frac{c}{12} n(n^2 - 1)(n^2 - 4) \delta_{n+m,0},$$

where

$$\Lambda_n = \sum_{k \in \mathbb{Z}} \mathcal{L}_{n+k}. \quad (4.52)$$

Thus the asymptotic symmetry algebra of a spin-3 field coupled to gravity in an asymptotically AdS background is a classical $\mathcal{W}_3 \otimes \mathcal{W}_3$ algebra.

**Approach II**

Alternatively, one can take a second approach. Recall that the canonical phase space was given by (4.42). We then imposed a set of additional first class constraints by (4.44) and turned them into so called “second class” constraints by gauge fixing to the highest weight gauge. These constraints are used to turn the affine Lie algebra into a $\mathcal{W}$ algebra. We denote the constraints by $\chi_\alpha$, which we recall demand that all
the modes of $a$, except for the lowest ones, have to vanish. From this we define the matrix $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$, with the bracket defined by \[4.42\]. We then define a new bracket on the constraint surface to be:

$$\{f, g\} = \{f, g\} - \{f, \chi_\alpha\}(C^{-1})^{\alpha\beta}\{\chi_\beta, g\}.$$  

(4.57)

Expanding $l^m(\phi)$ and $w^n(\phi)$ in Fourier modes, one can then determine the induced Poisson structure between $l_{p-1}^m$ and $w_{p-2}^n$, which precisely reproduces a $W_3$ algebra. This method goes under the name of (classical) Drinfeld-Sokolov reduction[74].

4.4.4 Quantum asymptotic algebra

Before we close this section, a few remarks about the composite operator \[4.56\] are in order. The algebra quoted above is classical. When we turn to the quantum theory, we need to resort to an appropriate normal ordering prescription:

$$: \Lambda :_n = \sum_{p \geq -1} L_{n-p}L_p + \sum_{p < -1} L_pL_{n-p}. \quad (4.58)$$

However, if we assume this quantum normal ordering prescription and the asymptotic algebra \[4.55\] one finds that the algebra is no longer associative, i.e. the Jacobi identities are no longer satisfied, which is clearly a problem since the Jacobi identities are a defining property of any algebra. The most straightforward approach to solve this problem is to assume that the quantum algebra takes the same structure as the classical algebra with the above quantum normal ordering prescription but with new structure constants[3]. These are then determined by demanding that the Jacobi identities are again satisfied. In particular, this results in $O(1/c)$ corrections for the “old” structure constants and it is in this sense that the classical algebra corresponds to the large $c$ limit of the quantum algebra. In the specific case at hand, one will find:

$$[W_n, W_m] = -\frac{1}{3} \left[ (n-m)f(n,m)L_{m+n} + \frac{96}{c+\frac{22}{5}} (n-m) : \Lambda :_m + \frac{c}{12} n(n^2 - 1)(n^2 - 4)\delta_{n+m,0} \right]. \quad (4.59)$$

with

$$f(n,m) = \left( 2n^2 + 2m^2 - nm - 8 - \frac{144}{5c+22} (n+m+3)(n+m+2) \right). \quad (4.60)$$

We thus see that the quantum algebra takes a different form than the classical algebra. However, the additional factors that appear in the structure constants $f(n,m)$ may be absorbed via an additional redefinition of the normal ordered composite to let the quantum algebra take the same form. For the $W$ algebra \[4.55\] this implies a redefinition:

$$: \hat{\Lambda} :_n =: \Lambda :_n - \frac{3}{10} (n+3)(n+2)L_n. \quad (4.61)$$

From the point of view of the CFT, this redefinition is very natural because unlike $: \Lambda :_n$, which is not a quasi-primary, the redefined composite operator $: \hat{\Lambda} :_n$ is. In terms of $: \hat{\Lambda} :_n$ the quantum algebra takes then the following form:

$$[W_n, W_m] = -\frac{1}{3} \left[ (n-m)(2n^2 + 2m^2 - nm - 8)L_{m+n} + \frac{96}{c+\frac{22}{5}} (n-m) : \hat{\Lambda} :_m + \frac{c}{12} n(n^2 - 1)(n^2 - 4)\delta_{n+m,0} \right]. \quad (4.62)$$

which is known from the literature, e.g. [74].
4.5 Black holes in (2+1)-dimensional Higher Spin gravity

The purpose of this section will be to generalise the BTZ black hole in the context of higher spin gravity by endowing it with higher spin charges. Geometrical definitions such as "causality", "horizons" and "curvature" loose gauge invariance in higher spin gravity and we must thus resort to other ways to define higher spin black holes in a gauge invariant manner\(^{26}\). Firstly, higher spin black holes are defined to be **stationary** solutions\(^{[8]}\), restricting the functions \(L(x^+), W(x^+)\) in (\(\ref{2.42}\)) to be constant. Smoothness of the horizon, as we have seen for the BTZ, can be formulated in terms of the holonomy smoothness condition. This definition therefore can naturally be extended to the higher spin case. However, imposing this condition on the above connection one would find that these constrain \(W \to 0\). Therefore, as soon as higher spin charges are assigned to a black hole, it becomes necessary to incorporate a source in the connection for this higher spin charge. As we will discuss in this section, for the spin-2 charge, this source can be incorporated in the modular parameter \(\tau\), allowing one to have a connection involving only the charges. For higher spin charges however, this will no longer remain possible and the question then arises how the sources (and charges) are incorporated in the connection. In particular it becomes necessary to turn on the \(a_\ell, \bar{a}_\ell\) component of the connection. One then has the choice whether the charges remain incorporated in the \(a_\ell\) component, of wether they are incorporated in the \(a_0 = a_z + \bar{a}_\ell\) component instead. Furthermore, in the context of AdS/CFT, this requires deforming the CFT by a term \(\mu W\), where \(\mu\) sources the spin-3 charge. In \([46]\) the authors showed, that these two choices amount to different boundary conditions which map to corresponding partition functions in the CFT. By carefully comparing the bulk flatness conditions to the Ward identities in the CFT, they have shown that the choice to incorporate the charges in the angular component \(a_\ell\) corresponds to deformations of the CFT Hamiltonian:

\[
H = H_{CFT} + \int d\phi \sum_s \mu_s J_s = \int d\phi \sum_s \bar{\mu}_s \bar{J}_s \tag{4.63}
\]

while the choice to incorporate them in the \(a_z\) component corresponds to deformations of the CFT action:

\[
I = I_{CFT} + \int d^2 w \sum_s \mu_s J_s = \int d^2 w \sum_s \bar{\mu}_s \bar{J}_s \tag{4.64}
\]

where \(J_s = \mu_s w^i 2 \pi^i (\tau L - \bar{\tau} \bar{L} + \alpha N - \bar{\alpha} \bar{N})\) and \(\alpha, \bar{\alpha}\) their corresponding sources. The sum runs over the particular spectrum of the theory under consideration. These two distinct choices of incorporating the charges in the connections then give rise to respectively the canonical black hole and the holomorphic black hole. In the upcoming section we will derive a formula for the entropy of a higher spin black hole very similar to what we did for the BTZ. In the higher spin case however, we will see that an additional condition must be imposed which goes under the name of the **integrability condition**. Then we will write down the criteria that should define a higher spin black hole as recorded in \([8]\).

**Entropy and integrability condition**

As for the BTZ, black holes that arise as solutions from higher spin gravity should remain to obey the first law of thermodynamics \((2.38)\). On the other hand the entropy as computed from the Bekenstein-Hawking area law loses validity in the higher spin case. Fortunately, it remains possible to compute the entropy from the partition function \((2.42)\) but now generalised to include higher spin charges\(^{[8]}\).

\[
Z(\tau, \bar{\tau}, \alpha, \bar{\alpha}) = \text{Tr}_H e^{2 \pi i (\tau L - \bar{\tau} \bar{L} + \alpha \lambda - \bar{\alpha} \bar{\lambda})} = \text{Tr}_H q^{J_0 - \frac{\alpha}{\lambda}} q^{\bar{J}_0 - \frac{\bar{\alpha}}{\bar{\lambda}}} u^W \bar{u} \bar{W}, \tag{4.65}
\]

\(^{26}\)This was shown in e.g. \([29]\) where the authors showed that in the \(SL(2)\) case, performing a gauge transformation on the connections \((A, \bar{A})\) leads to mere diffeomorphisms of the metric. In the \(SL(3)\) case however, gauge transformations will in general mix the metric and higher spin field. They demonstrate that gauge transformations can change the causal structure of the spacetime and that one can create smooth geometries from a singularity and vice versa.
where again $H$ denotes the CFT Hilbert space, $q = \exp(2\pi i \tau)$ and $u = \exp(2\pi i \alpha)$. The $SL(3)$ black hole solution will be characterised by 4 independent global charges $L, \bar{L}, W, \bar{W}$, which are again expectation values from the point of view of the CFT. The parameters $\alpha$ and $\tau$ are the potentials conjugate to these charges. In the next section we will discuss how this potential is related to the source $\mu$ for the higher spin charge. The charges are again recovered from the partition function as:

$$L = \frac{1}{2\pi i} \frac{\partial \ln Z}{\partial \tau}, \quad W = \frac{1}{2\pi i} \frac{\partial \ln Z}{\partial \alpha},$$

and similar expressions for the barred section. A novel feature however, is that these expressions are seen to satisfy:

$$\left. \frac{\partial L}{\partial \alpha} \right|_{\tau} = \left. \frac{\partial W}{\partial \tau} \right|_{\alpha}.$$  (4.67)

(4.67) is called the integrability condition which the black hole solution has to satisfy in order to associate to the black hole a partition function as in (4.65). In other words, it guarantees that they satisfy the first law of thermodynamics. It can be used as a consistency check on the solutions. In order to perform this check, one needs to find the global charges and potentials explicitly in terms of the parameters appearing in the connection. The global charges, as we have seen in the previous section can easily read of from the connection when this is written in the highest weight gauge. $\tau$ and $\alpha$ can be completely fixed in terms of the charges by the smoothness condition.

Now we have all the necessary information to define what we mean by a higher spin black hole[^8]:

**Definition 4.1.** A higher spin black hole is a classical solution of Chern Simons theory carrying higher spin charges and must satisfy the following properties:

1. The connections obey asymptotically AdS boundary conditions.
2. The components are constant and the connection thus describes stationary solutions.
3. The higher spin solution should be smoothly connected to the BTZ black hole when the higher spin charges are send to zero.
4. They should have a smooth Lorentzian horizon and regular Euclidean continuation.
5. They carry charges and chemical potentials which are real in Lorentzian signature.

The first two conditions we have already discussed. The third is also very natural. Since the BTZ black hole is a, albeit rather trivial, solution of the theory, and the higher spin charges are independent we should be able to continuously send them to zero while keeping the mass and angular momentum, i.e. $\mathcal{L}$ finite. The fourth condition is a generalisation of the holonomy condition[^28]. Denoting again $\omega = 2\pi(\tau a_z + \bar{\tau} \bar{a}_z)$ we thus demand that the eigenvalues of $\omega$ are $(2\pi i, 0, -2\pi i)$ where these are now the eigenvalues of the $L_0$ Cartan element embedded in $\mathfrak{s}(3)$. An equivalent condition is to impose:

$$\det(\omega) = 0, \quad \text{Tr}(\omega^2) + 8\pi^2 = 0.$$  (4.68)

The advantage of demanding this instead of the former is that we do not need to know the explicit eigenvalues. However, it will in general result in more solutions, one for each possible ordering of the eigenvalues. Each of these solutions corresponds to a different thermodynamical branch, and one must

[^27]: Again, depending on the chosen embedding, the physical spectrum traced over, as well as the definitions of the operators $L_0$ and $W_0$ differ. For the principal embedding the states organise into representations of the $W_3$ algebra and $L_0$ and $W_0$ are respectively the zero mode of the boundary CFT stress tensor and the zero mode of a $(3,0)$ primary. For the diagonal embedding the states are organised into $W_\Delta^0$ representations and now $W_0$ is the zero mode of a $(1,0)$ primary[^29].

[^28]: Horizon smoothness statements involving the metric are not gauge invariant at this stage so this definition is crucial.
choose the one that reduces to the BTZ black hole in a suitable limit. This is referred to as the BTZ branch. In the former, i.e. demanding the eigenvalues to equal \((2\pi i, 0, -2\pi i)\) there will be one solution as a choice for the ordering has been made. One might question, why we did not include the integrability condition into this definition. The reason it is that, once the holonomy condition is solved, this automatically ensures that the integrability condition \((4.67)\) is fulfilled.

### 4.5.1 Holomorphic Vs Canonical black hole

In this section we will review aspects of two types of black holes that have been constructed in the literature and go under the names of the holomorphic and the canonical black hole. Since in the remainder of this thesis, the focus will be on primarily the canonical black hole, for reasons that will be explained later, we will mostly focus on the canonical black hole.

**The holomorphic black hole**

The holomorphic black hole was in fact the first that was constructed in the literature\[8\]. For this black hole the charges are incorporated in the highest weight components of the \(a_x^+\) component, reflecting the holomorphic nature of the Lagrangian theory. The sources are then incorporated in the \(x^-\) component so that the flatness conditions reproduce the correct Ward identities in the CFT. This constrains the connection to schematically read

\[
a^+_x = L_1 + Q, \quad a^- = M.
\]

(4.69)

\(Q\) contains the charges as highest weight components and \(M\) is linear in the source \(\mu\) as lowest weight component and contains higher weight terms that are fixed by the equations of motion. A similar expression holds for the barred sector. Such a combined pair is referred to as a "Drinfeld-Sokolov pair" consisting of one component of the connection carrying the charges/currents as highest weights and a conjugate component that carries the corresponding sources as lowest weights. The explicit connection for the first holomorphic black hole that was constructed in the literature reads

\[
a = \left( L_1 - L_{-1} - WW_{-2} \right) dx^+ + \mu \left( W_2 - 2\omega W_0 + \omega^2 W_{-2} + 8\omega W_{-1} \right) dx^-,
\]

(4.70)

with again a similar expression for the barred sector. Note however, that this connection violates the boundary condition \(A^-\) of the previous section. This connection generalises the BTZ and we identify \(W\) as the spin-3 charge and \(\mu\) is its associated chemical potential. Furthermore, as we will see later the chemical potential \(\mu\) is related to the source \(\alpha\) in \((4.65)\) and it sources the spin-3 current in the CFT. It is furthermore clear from this connection that we recover the BTZ connection as we take \(W, \mu \to 0\) as well as their barred counterparts. Lastly, by explicit computation one easily verifies that it satisfies the integrability condition. One simply evaluates the holonomy matrix \(\omega\) and substitutes it into \((4.68)\). This gives two equations in terms of \(\mathcal{L}, W, \mu, \tau\). Using then that \(\alpha = \tau\mu\), it is a trivial task of taking derivatives to verify that in the integrability condition is satisfied.

**The canonical black hole**

A black hole with canonical boundary conditions was first constructed in \[44\]. They motivate the inclusion of the higher spin sources from a Hamiltonian point of view where one instead works on fixed time slices and thus with the coordinates \(t\) and \(\phi\) separately instead of with the previously used the lightcone coordinates \(x^\pm\). The charges generating the asymptotic symmetries, are defined on these constant \(t\) slices and most importantly, should not depend on how time slices evolve into each other. The evolution from
a slice at $t$ to a slice $t + \delta t$ is given by the infinitesimal gauge transformation with parameter $a_t$:

$$\partial_\phi a_t + [a_\phi, a_t] = \delta_t a_\phi = 0. \quad (4.71)$$

The definition of the charges can therefore only rely on the component $A_r, A_\phi$ and not on the Lagrange multiplier $A_t$. This reduces the connection to:

$$a_\phi = L_1 - \mathcal{L}L_{-1} - WW_{-2}, \quad (4.72)$$

and furthermore means that the boundary conditions will be preserved at each constant time slice. The sources are then incorporated in the time component. This way the flatness conditions reproduce the correct Ward identities in the CFT. One obvious solution to (4.71) is taking $a_\phi = a_t$, but under the inclusion of chemical potentials in the time components, more solutions are possible. In the same notation as above, the connection takes the following form:

$$a_\phi = L_1 - W, \quad a_t = a_\phi + M. \quad (4.73)$$

Explicitly, $M$ can be calculated from trace invariants of $a_\phi$:

$$M = \mu \left[ (a_\phi)^2 - \frac{1}{3} \text{tr}(a_\phi)^2 \right]. \quad (4.74)$$

### Canonical black hole: Revisiting the boundary conditions

In a new set of generalised boundary conditions is discussed which are more appropriate on the $t = \text{const.}$ slices and that allow for the inclusion of chemical potentials:

$$A_\phi(t, \phi) \xrightarrow{\rho \to \infty} L_1 - \frac{1}{k} \mathcal{L}(t, \phi)L_{-1} - \mathcal{W}(t, \phi)W_{-2}$$

$$\mathcal{L}(t, \phi) \xrightarrow{\rho \to \infty} \mathcal{L}(t, \phi) + O(1/\rho)$$

$$\mathcal{W}(t, \phi) \xrightarrow{\rho \to \infty} \mathcal{W}(t, \phi) + O(1/\rho)$$

$$A_\rho \xrightarrow{\rho \to \infty} O(1/\rho) \quad (4.75)$$

In fact, if we focus on just the $sl(2)$ part, $(W \to 0)$, these boundary conditions can be obtained from those originally identified by Brown and Henneaux in metric formulation. Now, these fall-off conditions are preserved under a restricted set of gauge transformations of the form $\delta a_\phi = d\lambda + [a_\phi, \lambda]$, with $\lambda = \lambda(\epsilon, \chi)$ lie algebra valued gauge parameter and $\epsilon, \chi$ are arbitrary functions of $t, \phi$ defined on each time slice as long as they satisfy fall-off conditions similar to those of $L, W$. One may then determine, the explicit form of $\lambda$ and the corresponding transformation law for the fields $L, W$ in the same way as we did before.

Then at spatial infinity, the time evolution of $a_\phi$ is a gauge transformation with parameter equal to the Lagrange multiplier $a_t$. The most general Lagrange multiplier preserving the asymptotic condition is of the form

$$a_t = \lambda[\xi, \mu]. \quad (4.76)$$

This way the asymptotic symmetries remain conserved under time evolution. Here the "chemical potentials" $\xi, \mu$ are also arbitrary functions of $t, \phi$ and are assumed to be fixed at the boundary. Consistency of the Lagrange multipliers under gauge transformations means that the functions on the asymptotic region satisfy the same variation but now with the gauge parameters replaced by the chemical potentials. Furthermore, the gauge parameters describing the asymptotic symmetries are required to satisfy so called
"deformed chirality conditions", which in general become differential equations of first order in time and impose a condition on $\epsilon, \chi$ and $\xi, \mu$.

In the next paragraph we will review a method described in [47] and [46] to determine the entropy of the canonical black hole, though the Euclidean action. In particular, we will discuss why after Euclidean continuation $t \rightarrow i\ell_E$, there is no need to incorporate the chemical potential for the spin-2 charge explicitly in the connection, but that it can instead be incorporated as the modular parameter $\tau$ of the boundary torus.

**Entropy from the Euclidean action**

In [46] the authors discuss a how the entropy and free energy can be obtained from the Euclidean action by using a variational principle. The CFT partition function can, in the limit of large central charge and temperature, be obtained from the on shell value of the Euclidean action.

$$\ln Z = -f^{(E)}_{\alpha\sigma} = -\left( I^{(E)}_{\text{CS}} + I^{(E)}_{B} \right)_{\alpha\sigma}. \quad (4.77)$$

$I_{\text{CS}}$ denotes the Chern Simons action and $I_{B}^{(E)}$ is the Euclidean continuation of a boundary term that needs to be added for a well defined action upon adding the sources. Recall that we had to impose $A_\tau = 0$ to have a well defined variational principle. When we include chemical potentials into the connection however, the variation of the Chern Simons action will no longer be zero at the boundary and we thus have to add a boundary term. For the canonical black hole this boundary term is given by [46]:

$$I_{B}^{(E)} = -\frac{k_{cs}}{2\pi} \int_{M} d^2z \text{Tr} \left[ (a_z + \bar{a}_\bar{z} - 2L_1) a_{\bar{z}} - \frac{k_{cs}}{2\pi} \int_{\partial M} d^2z \text{Tr} \left[ (\bar{a}_\bar{z} + a_z - 2L_{-1}) \bar{a}_z \right], \quad (4.78)$$

so that (4.77) has a well-defined variation when the sources are held fixed. Next we are interested in the variation of the on-shell Euclidean action. However, we have to be careful here because the modular parameter $\tau$ is incorporated through the identification of coordinates and is itself varying. This problem can be circumvented by working instead with coordinates with a fixed periodicity:

$$z = \frac{1 - i\tau}{2} w + \frac{1 + i\tau}{2} \bar{w}, \quad \bar{z} = \frac{1 - i\bar{\tau}}{2} w + \frac{1 + i\bar{\tau}}{2} \bar{w}, \quad (4.79)$$

where now the identification reads $w \simeq w + 2\pi \simeq w + 2\pi i$. Performing the variation and afterwards changing back to $(z, \bar{z})$ coordinates results in the following variation of the on-shell Euclidean action:

$$\delta f^{(E)}_{\alpha\sigma} = -2\pi k_{cs} \int_{\partial M} d^2z \text{Tr} \left[ \frac{1}{2} (a_z + \bar{a}_{\bar{z}})^2 \delta \tau + (a_z + \bar{a}_{\bar{z}} - L_1) \delta ((\bar{\tau} - \tau) a_z) + \text{other sector} \right]$$

$$= -2\pi k_{cs} \int \frac{1}{2} (a_z + \bar{a}_{\bar{z}})^2 \delta \tau + (a_z + \bar{a}_{\bar{z}} - L_1) \delta ((\bar{\tau} - \tau) a_z) + \text{other sector} \quad (4.80)$$

where we used that $\text{Vol}(\partial M) = 4\pi^2 \text{Im}(\tau)$ and that for constant connections $a, \bar{a}$ the integral can be explicitly evaluated. Now we also see that the variation of the action takes the schematic form

$$\int_{\partial M} d^2z \text{EV's} \delta \text{(sources)}, \quad (4.81)$$

such that (4.80) vanishes when the sources are held fixed at the boundary. We can now read of the expectation value of the stress tensor coupling to the modular parameter $\tau$.

$$2\pi \mathcal{L} = \frac{1}{2\pi i} \frac{\partial \ln Z}{\partial \tau} = \frac{k_{cs}}{2} \text{Tr} \left[ (a_z + \bar{a}_{\bar{z}})^2 \right]. \quad (4.82)$$

With this we can then write down the correct Euclidean version of (4.73):

$$a_z + a_{\bar{z}} = L_1 + Q \quad (\bar{\tau} - \tau) a_{\bar{z}} = \bar{M} + ..., \quad (4.83)$$

35
with the difference being that upon the transition to Euclidean formalism the sources have been rescaled by \( \tau - \bar{\tau} \). The matrix \( \tilde{M} \) now contains the actual sources \( \alpha \sim (\bar{\tau} - \tau)\mu \). The matrix \( \tilde{M} \) further does not contain the spin 2 source, because that has been incorporated in the modular parameter.

**Free energy and entropy of the canonical black hole**

Now that we have the variation of the Euclidean action at our disposal, we can determine the entropy and free energy. The free energy is obtained from (4.77) and \( \beta F = -\ln Z = I_{E_{0}}^{\phi} \). Then:

\[
\beta F = 2\pi i k_{cs} \text{Tr} \left[ \frac{1}{2}(a_{z} + a_{\bar{z}})^{2}\tau + (\bar{\tau} - \tau)a_{z} + \text{other sector} \right].
\]

(4.84)

The thermal entropy can then be obtained from the free energy by means of a Legendre transform. This gives:

\[
S = \ln Z - 2\pi i k_{cs} \text{Tr} \left[ \frac{1}{2}(a_{z} + a_{\bar{z}})^{2}\tau + (\bar{\tau} - \tau)a_{z} + \text{other sector} \right]
\]

(4.85)

\[
= -2\pi i k_{cs} \text{Tr} [(a_{z} + a_{\bar{z}})(\tau a_{z} + \bar{\tau}a_{\bar{z}}) + \text{other sector}],
\]

(4.86)

for the thermal entropy. We can write the free energy and entropy in a more insightful way however. If we use (4.82) and the fact that the normalisation of the charges and sources can be chosen such that

\[
k_{cs}(\bar{\tau} - \tau)\text{Tr}[Qa_{\bar{z}}] = \sum_{s=3}^{N} 2\pi \alpha_{s} W_{s} = \sum_{s=3}^{N} \frac{s^{2}(s-1)(\bar{\tau} - \tau)2\pi \mu_{s} W_{s}},
\]

(4.87)

\[
k_{cs}(\bar{\tau} - \tau)\text{Tr}[La_{\bar{z}}] = \sum_{s=3}^{N} 2\pi \alpha_{s} W_{s} = \sum_{s=3}^{N} (\bar{\tau} - \tau)2\pi \mu_{s} W_{s},
\]

(4.88)

then the free energy (4.84) and entropy (4.85) can be written in terms of the charges as:

\[
\beta F = 4\pi^{2} i \left[ \tau L + \sum_{s=3}^{N} (s - 1)\alpha_{s} W_{s} \right]
\]

(4.89)

\[
S = -4\pi^{2} i \left[ 2\tau L + \sum_{s=3}^{N} s\alpha_{s} W_{s} \right].
\]

(4.90)

Furthermore, if we note that \( a_{z} + a_{\bar{z}} = a_{\phi} \) and use the holonomy conditions, i.e. \( (\tau a_{z} + \bar{\tau}a_{\bar{z}}) = iL_{0} \) the entropy can be compactly written as:

\[
S = 2\pi k_{cs} \text{Tr}[(\lambda_{\phi} - \bar{\lambda}_{\phi})L_{0}],
\]

(4.91)

where \( \lambda_{\phi}, \bar{\lambda}_{\phi} \) are diagonal matrices that contain the eigenvalues of \( a_{\phi}, \bar{a}_{\phi} \). This formula now allows us to compute the entropy using solely the eigenvalues of the angular component of the connection, and relates nicely to the fact that all information characterising a smooth black hole should be contained in the holonomy. This formula we refer to as the *Canonical entropy formula*, for the simple reasons that the canonical components (i.e. \( a_{\phi} \)), are used to evaluate the entropy.

### 4.6 Black hole extremality

Having defined what we mean by a higher spin black hole and having discussed their thermodynamics, in the remaining portion of this thesis we shall be primarily interested in extremality. As we have seen in the lower spin setup, extremality is characterized by the saturation of a bound (2.21). Equivalently, this implied a confluence of the inner and outer horizons and in particular had zero temperature as a consequence. Naturally, one might then define extremality in the higher spin context by simply remaining
to demand that they are zero temperature solutions. However, this definition would not utilise the
topological nature of the Chern Simons theory and furthermore, does not relate directly to the degeneracy
of the parameters of the solution at extremality. For this reason, another definition for extremality was
proposed in [10], which in particular has zero temperature as a consequence. They proposed that an
extremal higher spin black hole be defined as follows:

Definition 4.2. Extremality
An extremal black hole in higher spin gravity should satisfy in addition to the conditions given in definition
4.1:
• The angular component of at least one of \( a_\phi, \bar{a}_\phi \), say \( a_\phi \) is non-diagonalizable.

This definition is in fact rather natural. Namely, suppose that both \( a_\phi \) and \( \bar{a}_\phi \) were diagonalizable. Then
flatness of the solution would tell us that \([a_\phi, a_{tE}] = 0\) and so the time component can be diagonalized
simultaneously. It is then possible to solve the smoothness condition
\[
e^{(\tau a_z + \bar{a}_z)} = e^{iL_0}
\]
and at the same time find a non-zero and well defined temperature and chemical potentials as a function
of the charges. If, however, one of \( a_\phi, \bar{a}_\phi \) is non-diagonalizable, then \( a_{tE} \) will be non-diagonalizable as
well. As we demand the smoothness condition to be still satisfied, both features can only be compatible
if we take \( \tau \to \infty \), i.e. the zero temperature limit. Although not true in general, it can be shown that the
special form of the connection dictated by the boundary conditions, guarantees that non-diagonalizability
of the connection, is equivalent to equating eigenvalues[10]. In this sense, equating eigenvalues can be un-
derstood as the confluence of horizons. This is very convenient, because it allows us to encode extremality
in terms of the Jordan classes of the holonomy that encode which eigenvalues are degenerate.

Lastly, a convenient object in the analysis will be the discriminant of the characteristic polynomial, since
its roots correspond to the degeneracy of eigenvalues:
\[
\Delta(p_\lambda) = \prod_{i<j}^n (\lambda_i - \lambda_j).
\]
(4.93)
It then follows that \( a_\phi \) can only be diagonalised when \( \Delta \neq 0 \), in which case there exists a similarity matrix
\( V \) such that
\[
V^{-1}a_\phi V = a_\phi^D.
\]
(4.94)
If on the other hand \( a_\phi \) is non-diagonalizable then \( \Delta = 0 \) and the Jordan decomposition becomes instead:
\[
V^{-1}a_\phi V = a_\phi^D + a_\phi^N,
\]
(4.95)
where \( a_\phi^N \) is a nilpotent matrix commuting with \( a_\phi^D \), whose precise form will depend on the number
of zero’s of \( \Delta \), i.e. the number of repeated eigenvalues. It is unique up to similarity transformations, that
obviously leave \( a_\phi^D \) invariant. We will use this definition in chapters 6 and 7.

Before we close this section, let us comment on a few aspects of this definition that are worth highlighting.
• Firstly, we note that it is quite convenient that extremality can be defined terms of the angular
\((\phi)\) component of the connection. In the remaining portion of this thesis we will adopt canonical
boundary conditions, in which case the charges are carried by the \( \phi \) component solely. Consequently,
the extremality bounds can be formulated in terms of the charges only. As we will discuss in the
next section, in supersymmetric theories extremality is of interest because its relation to so called
BPS conditions in the dual CFT. The latter can be derived straight from the operator/mode algebra
and thus are relations between only the charges also. This allows for a clean comparison between
the two conditions.
• Secondly, formulating extremality in terms of the angular component of the connection, allows for an easy generalization to other solutions which are not black holes and do not include sources, such as conical defects.
5 Introduction to AdS supersymmetry and Supergravity

Supersymmetric theories exhibit a symmetry between bosons and fermions. Bosons and fermions always come in pairs in these types of theories and can be accommodated within an irreducible representation of the super-Poincaré algebra, if one considers flat Minkowski, or the super-AdS algebra. A well-studied example of a consistent supersymmetric theory is supergravity. This theory contains the graviton together with its associated superpartner which is a massless spin-3/2 field known as the gravitino/Rarita-Schwinger field. Considering our discussion of section 4.1, the case of spin-3/2 is rather special, because it is a consistent theory in any dimension \( d \). The action describing the spin-3/2 gravitino is known as the Rarita-Schwinger action. Surprisingly, the gauge variation of the Rarita-Schwinger action is not proportional to the full Riemann tensor, but to the Einstein tensor. This variation is then precisely compensated by the variation of the Einstein-Hilbert action.

This section is intended for readers not familiar with the basic notions from supergravity. Readers familiar with the concepts of supergravity, can freely continue to section 6. The setup of this section will be as follows. First we will discuss the conceptual ideas behind supersymmetry and, from a physics point of view, discuss how the AdS algebra can be extended to incorporate supersymmetry generators. Such algebras are known as Lie-superalgebras. For some more background on these algebras, we refer the reader to appendix D. Then we will turn attention to supergravity, in which supersymmetry appears as a gauged symmetry. We will pay attention to the Chern Simons formulation of the theory. Some extra background concerning the vielbein formulation of the theory can be found in appendix C.2. Having discussed this, we turn to two interesting aspects of supergravity which are killing spinors, and BPS bounds. The material in this section will be broadly based on [15][27][48].

Global supersymmetry and the Super-AdS algebra

In global (rigid) supersymmetry the AdS algebra \( \mathfrak{so}(2,2) \), generated by \( M_{ab} \) and \( P_a \), is extended to include additional spinor supercharges \( Q_\alpha^i \) which are the generators of supersymmetry. Their symmetry parameters are denoted by \( \epsilon_\alpha \), which are constant spinor parameters and independent of the coordinate \( x \). The index \( \alpha \) is a spacetime spinor index and \( i = 1, 2, \ldots N \) labels the distinct supercharges. When \( N > 1 \), we speak of \( N \) extended supersymmetry. These generators together join in a new algebraic structure called a Lie-Superalgebra. These are \( \mathbb{Z}_2 \) graded Lie algebras and they are characterised by two types of generators: even graded (bosonic) generators \( B \) and odd graded (fermionic) generators \( F \) which follow the general pattern:

\[
[B_i, B_j] = f^{ij}_{kl} B_k, \quad [B_i, F_\alpha] = f^i_{\alpha\beta} F_\beta, \quad \{F_\alpha, F_\beta\} = f^i_{\alpha\beta} B_i.
\]

The bosonic generators thus form a conventional Lie-algebra. Now in general, finding such an supersymmetric extension of any given Lie algebra, in our case the AdS algebra, is a hard mathematical problem whose solution lies in the classification of superalgebra’s. Instead, as pointed out in [48], one a can take a more practical approach by starting from the representations one is interested in from the start, based on physical reasons, and trying to determine the explicit form of the fermionic generators such that one obtains the smallest supersymmetric extension, by demanding closure of the algebra. A useful fact for this, is that the gamma matrices provide a natural representation of the AdS algebra in any dimension.

---

31There is in fact a second example of a supersymmetric theory containing the graviton, which is known as hypergravity. In this theory the superpartner of the graviton is a massless spin-5/2 field, but the miracle of supergravity does not repeat itself. One again finds a variation proportional to the full Riemann tensor and it is necessary to work in three dimensions. We will discuss this theory in section 7.

32For further definitions and properties of Lie Super-algebras see appendix D.

33This representation is known as the spinorial representation.
Explicitly, in terms of the \( \mathfrak{sl}(2) \) generators \( L_a^\pm = \frac{1}{2}(M_a \pm iP_a) \), \( a = 0, 1, 2 \), one can choose:

\[
L_a = \frac{1}{2} \gamma_a. \tag{5.2}
\]

In \((2+1)\)-dimensions the \( \gamma \) matrices may be expressed in terms of the Pauli matrices. Explicitly, they can be taken to be:

\[
\gamma_0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.3}
\]

The indices \( a, b \) label as before the non-coordinate tangent frame indices. The \( \gamma \) matrices obey the Clifford algebra, and satisfy

\[
\gamma_{ab} = \frac{1}{2} \{ \gamma_a, \gamma_b \} = 2\epsilon_{abc} \gamma_c, \quad \epsilon^{012} = 1.
\]

Now, the most trivial extension is of course to extend the generators (5.2) by one row and one column of zeros. The new fermionic generators, denoted as \( Q_\gamma \), take their values in these new entries, and they carry a single spinor index so that they transform in the spin-1/2 representation of the Lorentz group. These new fermionic generators are defined such that they have a single non-vanishing entry in the \( \gamma \)-th row of the last (additional) column. Schematically, it is of the form:

\[
Q_\gamma = \begin{pmatrix} 0_{2 \times 2} & \delta^\gamma_0 \\ -C_\gamma & 0 \end{pmatrix}, \tag{5.4}
\]

\( C \) is known as the charge conjugation matrix. Its form is fixed by demanding that one obtains the smallest supersymmetric extension of the AdS algebra. In general dimensions, there are two ways to restrict the dimension of a representation that is consistent with Lorentz invariance. These are chirality and reality. In odd dimensions though, chirality is not of much use since the \( \gamma_5 \) matrix appearing in the projection operators is proportional to the identity. The reality condition defines the relation between a spinor and its conjugate:

\[
\bar{\psi}_\alpha = C_{\alpha\beta} \psi_\beta, \tag{5.5}
\]

which is the well known Majorana condition. \( C \) is invertible and can be viewed as defining a metric in the space of Majorana spinors. It is also used to define the conjugate of the fermionic generator \( Q_\gamma \):

\[
\bar{Q}_\gamma = C^{\alpha\beta} Q_\beta. \tag{5.6}
\]

By demanding closure of the algebra, in particular the \( \{ Q_\gamma, Q_\alpha \} \) anti-commutator, one can determine then the explicit form of the charge conjugation matrix. This procedure can be extended to include fermionic generators, by including more extra new columns in (5.2). In this case however, closure of the algebra requires the addition of additional bosonic generators, known as the \( R \)-symmetry generators. These \( R \)-generators rotate the supercharges amongst each other, and define an internal automorphism of the algebra. All in all, the algebra for \( \mathcal{N} \)-extended supersymmetry, for one chiral copy reads:

\[
\begin{align*}
[L_a, L_b] &= \epsilon_{abc} L^c \\
[L_a, t^{ij}] &= 0 \\
[t^{ij}, t^{kl}] &= \delta^{ik} t^{jl} - \delta^{il} t^{jk} - \delta^{jk} t^{il} + \delta^{jl} t^{ik} \\
[L_a, Q_{\alpha}^i] &= \frac{1}{2} (\gamma^a)^i_{\beta} Q_{\beta}^i \\
\{Q_{\alpha}^i, Q_{\beta}^j\} &= 2\delta^{ij} (\gamma^a)_{\alpha\beta} L_a + \eta_{\alpha\beta} t^{ij} \\
[t^{ij}, Q_{\alpha}^i] &= \delta^{ij} Q_{\alpha}^i - \delta^{\alpha\beta} Q_{\beta}^i.
\end{align*}
\tag{5.7}
\]

This set of generators and corresponding commutation relations define the \( \mathfrak{osp}(\mathcal{N}|2) \) algebra which is the \( \mathcal{N} \) supersymmetrization of the \( \mathfrak{sp}(2) \cong \mathfrak{sl}(2) \) algebra. As explained above, there are extra bosonic generators.

---

34See appendix C.2

35For now we will consider the \( \mathcal{N} = 1 \) unextended case. We will comment later on the extended case.

36One property it satisfies is that it is antisymmetric \( C^T = -C \)
present. They are the spin-0 \(O(N)\) generators that will yield spin-1 gauge fields in the Chern Simons theory. Of course, the \(L_a\) span as before an \(\mathfrak{sl}(2)\) algebra. We will not work with the Super-AdS algebra in this explicit form though. Instead, similar to the \(\mathfrak{sl}(2)\) case, we will work with a convenient representation of this algebra, (or rather its higher spin extension), that allows for a Chern-Simons formulation of the theory. Thus, whilst for this thesis the above discussion is not of practical use, it nicely describes from a physics point of view, why the superalgebra one deals with in supergravity has the form it does. As a last remark we note that, the above Super-AdS algebra reduces to the super-Poincare algebra in the limit of infinite AdS radius \(l \to \infty\), by an Inon"u-Wigner contraction, very similar to what we saw for the AdS algebra in (2.8). For this one needs to make an additional rescaling \(Q \to \sqrt{\ell}Q\) and \(t \to t/\sqrt{\ell}\).

5.1 Pure Supergravity in \(\text{AdS}_3\): Chern-Simons formulation

Building upon the ideas of section 3.1, gauged supersymmetry is expected to yield an extension of GR, in which the graviton acquires the spin-3/2 gravitino as its superpartner. That gauged SUSY includes gravity is easily seen from the algebra. The SUSY generators are joined into the superalgebra together with \(P_a\) and \(M_{ab}\), which must therefore necessarily also be gauged, thus implying gravity. Thus to write down a theory of supergravity we simply let also the SUSY parameter \(\epsilon^\alpha\) become local: \(\epsilon^\alpha \to \epsilon^\alpha(x)\). The gravitino \(\psi_\alpha^\mu\) is the gauge field for this local symmetry. It is a vector-spinor field. As before the gauge field describing gravity is the vielbein \(e^a_\mu\).

Having identified the gauge fields of the \(\mathcal{N} = 1\) theory, we now combine them into a single gauge "superconnection" and choose a matrix representation for the generators:

\[
\Gamma = L_1 - L_{-1} + \psi S_{-1/2},
\]

which is valued in the \(\mathcal{N} = 1\) super AdS algebra \(\mathfrak{osp}(2|1)\). Since the \(\mathfrak{osp}\) algebra furthermore acquires a non-degenerate invariant bilinear form\(^\text{[6]}\) we may construct a Chern-Simons action from this superconnection

\[
L_{\text{CS}} = s\text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) + \text{other sector}.
\]

In this notation, \(s\text{Tr}\) denotes this invariant bilinear form and refers to the super trace in the fundamental matrix representation defined in appendix D. One can verify explicitly that this action reduces to the SUGRA action discussed in appendix C.2. Unlike in the non-SUSY case, the field equation \(F = d\Gamma + \Gamma \wedge \Gamma\) does no longer impose a vanishing torsion, but instead one schematically has \(\tilde{T}^a \equiv T^a - \frac{1}{4} \psi \gamma^a \bar{\psi}\) where \(T^a = de^a + \epsilon^{abc} \omega_b c_c\) is the usual torsion, that vanishes for the the Levi-Civita spin-connection. The transformation rules of the gauge fields may again be determined by considering an infinitesimal gauge parameter \(u = \rho^\alpha P_\alpha + \tau^a M_a + \epsilon^\alpha Q_\alpha\).

For \(\mathcal{N}\) extended supergravity, the procedure is the same. The superconnection for the \(\mathfrak{osp}(2|\mathcal{N})\) algebra becomes:

\[
\Gamma = L_1 - \tilde{L}_{-1} + \psi I S^I_{-1/2} + J^{IJ} t_{IJ},
\]

where the index \(I\) runs up to \(\mathcal{N}\) and \(J^{IJ}\) is the gauge field corresponding to the internal R-symmetry\(^\text{[38]}\). A convenient representation for the \(t^{IJ}\) generators can be formulated in terms of appropriate contractions of kronecker delta functions. Note that this time we wrote a \(\sim\) on the spin-2 charge. It is related to the previously discussed (Virasoro) \(\mathcal{L}\) as:

\[
\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{5} J_{IJ} J^{IJ}.
\]

\(^\text{37}\)Alternatively, we may also write the connection as \(le^a P_\alpha + \omega^a M_a + \psi^a Q_\alpha\). The non-vanishing brackets can be found in \[\text{[35]}\].

\(^\text{38}\)Note that we have denoted this gauge field by \(A_\mu\) in appendix C.2.
The expressions differ by what is known as the Sugawara stress energy tensor. It arises because the CFT currents generate their own symmetry algebra known as a Kac Moody algebra, and thus define their own stress energy tensor. This construction is discussed in appendix E.4.1.

**Asymptotic symmetries of** $\mathcal{N} = 1$ osp(1|2) **Supergravity.**

In section 4.4.2 we reviewed the procedure to find the asymptotic symmetry algebra, which in the case of pure gravity led to the Virasoro algebra and for spin-3 gravity a centrally extended $\mathcal{W}_3$ algebra. This procedure can be extended to determine also the asymptotic symmetries of a $\mathcal{N}$ extended supergravity theory. In [50] this was done for $\mathcal{N} = 1$, where the asymptotic symmetry algebra for osp(1|2) supergravity was shown to be the $\mathcal{N} = 1$ superconformal algebra discussed in appendix E.39. The main novelty, compared to the purely bosonic case, is that one must specify appropriate boundary conditions on the additional fermionic fields. This involves, besides specifying appropriate fall-off conditions with the radial coordinate, a chiral projection which makes the induced boundary spinors chiral in 2 dimensions. Explicitly:

$$
\Psi_t \sim r^{-1/2}[1 + \gamma_1]\chi_{t,\phi}, \\
\Psi_\phi \sim r^{-1/2}[1 + \gamma_1]\chi_{t,\phi}, \\
\Psi_r \sim r^{-5/2}[1 + \gamma_1]\chi_{t,\phi}.
$$

The explicit asymptotic boundary conditions one must then impose are:

$$
A_+ = 0 \quad A_r = b^{-1}\partial_r b \quad \Psi_+ = \Psi_r = 0.
$$

Note that $\Psi$ used here is not the Rarita-Schwinger field. Instead, we adopted the notation $\Gamma = A + \Psi$. Following the same steps as before, one will find a centrally extended classical $\mathcal{N} = 1$ superconformal algebra as the asymptotic symmetry algebra, c.f (E.39).

**Preserved supersymmetries: Killing spinors**

In gravitational theories isometries are generated by killing vectors. General solutions to the equations of motion will not preserve any symmetry of the AdS vacuum and hence admit no killing vectors, whereas AdS as maximally symmetric space time will preserve the most symmetries and therefore supports the maximal amount of killing vectors. This reasoning continues to hold in theories of supergravity and the fermionic analogue of killing vectors are called killing spinors. Global AdS is still the vacuum configuration, admitting besides the maximal amount of killing vectors also the maximal amount of killing spinors. Again general solutions will not preserve all the supersymmetries of the vacuum and admit less, or no, killing spinors.

In the following we will discuss how killing spinors can in general be found. In the section 6.2 we will specialize this for Chern Simons supergravity. In general, killing spinors are found directly from the transformation rules under arbitrary spinor functions $\epsilon(x)$ evaluated on the background field configurations. In the most general case these supersymmetry transformation rules are of the form:

$$
\delta_\epsilon B = i\epsilon f(B)F + \mathcal{O}(F^3), \quad \delta_\epsilon F = g(B)\epsilon + \mathcal{O}(F^2),
$$

\[\text{In the next section we will discuss higher spin supergravity in which case the asymptotic symmetry algebra becomes an super-}W\text{ algebra.}\]

\[\text{For a comparison with the metric formulation see appendix E.2}\]
where \( f \) and \( g \) are functions of the bosons and their derivatives. \( B \) denotes the bosonic fields in the theory, i.e. \( \phi^a, \omega^a_\mu \) and \( F \) denotes the fermionic rarita-schwinger fields. Typically though, one is interested in classical solutions to the supergravity equations of motion for which the fermionic background fields vanish, \( F_0 \equiv 0 \). Then trivially, \( \delta B = 0 \). However, from demanding \( \delta F = 0 \) one finds a non-trivial condition that needs to be satisfied:

\[
\delta F = g(B)\epsilon = 0.
\]  

(5.15)

for the background fields to remain unchanged. This equation is what is referred to as the killing spinor equation and its nonzero linearly independent solutions \( \epsilon(x) \) are the killing spinors. In general, the killing spinor equations yield a set of coupled differential equations involving the bosonic fields and killing spinors. More specifically, the killing spinor equation reads:

\[
\delta \psi = D_\mu \epsilon = 0.
\]

(5.16)

with \( D_\mu \) the spinor covariant derivative introduced in appendix C.2. Using this approach, it was shown in [25] that for \((1,1)\) supergravity the only configurations that preserve some supersymmetry of the vacuum AdS are the massless BTZ, preserving 2 supersymmetries and extremal black holes with \( lM = |J| \) preserving a single supersymmetry. Then, for the extended \((2,0)\) theory, the authors in [51] found a class of charged extreme black holes that where supersymmetric. The charge is half integral. In addition they identified a family of charged and non-charged conical defects. As a final remark let me mention that solutions that admit killing spinors are often referred to as BPS solutions. This will be the topic of the next section.

**BPS bounds and extremality**

"BPS" is an abbreviation for Bogomol’nyi, Prasad and Sommerfield and historically has its origin outside the context of supergravity. In a supersymmetry or supergravity theory, the term BPS designates a solution of the theory that is annihilated by some non-zero odd element of the supersymmetry algebra. In this work, we shall be interested in such BPS bounds of super-conformal algebras in the context of AdS/SCFT. To explain this, it is most instructive to take the \( \mathcal{N} = 2 \) superconformal algebra (E.40) as an explicit example.

Consider the anti-commutator between two fermionic generators in the NS sector:

\[
\{G^+_{-r}, G^-_r\} |h, q\rangle = \left(2h - 2rq + \frac{c}{3} \left(r^2 - \frac{1}{4}\right)\right) |h, q\rangle, \quad r \in \mathbb{Z} + \frac{1}{2}.
\]  

(5.17)

Demanding unitarity, and setting \( r = 1/2 \) it follows then that

\[
\langle h, q | \{G^+_{-\frac{1}{2}}, G^-_{\frac{1}{2}}\} |h, q\rangle = 2h - q \geq 0,
\]  

(5.18)

---

41This notation means that each factor in the gauge group has \( \mathcal{N} = 1 \) supersymmetry. This is possible because the AdS group is reducible in three dimensions and one can supersymmetrize each \( sl(2) \) factor independently. These theories are known as \((p,q)\) supergravities, and \( p + q = \mathcal{N} \). It was shown that the \((1,1)\) and \((2,0)\) theories, result in 2 distinct theories.

42In its full generality it refers to an inequality for solutions of partial differential equations, such as the Einstein-equations. Among these, a special type of solutions are those for which equality holds, and these are called the BPS solutions. They where first encountered in the study of magnetic monopoles. These solutions satisfy a simpler set of differential equations and one can deduce already much information from these solutions. Now, as we have argued above, supersymmetric solutions admit killing spinors for which the killing spinor equation (5.16) is non-trivially solved. Often it is easier to find classical solutions to the equations of motion by studying the killing spinor equations, rather then the full equations of motion themselves. Both types of approaches, involve differential equations between the bosonic components of the background solutions and it is because of this comparison between the two approaches that supersymmetric solutions are often referred to as BPS solutions. Another way to see the comparison is by embedding magnetic monopoles in a supergravity theory to find that they preserve half of the supersymmetry.
and thus

\[ h \geq q/2. \]  \tag{5.19}

A bound on the CFT charges, of this sort is what we will henceforth refer to as a BPS(-like) bound and states saturating it, are the NS BPS states. In the NS sector, the BPS states are more commonly known as chiral primaries \( |h = q/2\rangle \). Note that, as mentioned above, we need only consider the lowest fermionic generator present in the \textit{global part} of the super-conformal algebra. A similar bound may be derived in the Ramond sector:

\[ \{G^+_\sigma, G^-_\tau\} |h, q\rangle = \left( 2h - 2rq + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \right) |h, q\rangle, \quad r \in \mathbb{Z}. \]  \tag{5.20}

Demanding again unitarity and setting \( r = 0 \) one finds:

\[ \langle h, q | \{G^+_\sigma, G^-_\tau\} |h, q\rangle = 2h - \frac{c}{12} \geq 0, \]  \tag{5.21}

implying

\[ h \geq c/24. \]  \tag{5.22}

The state saturating this bound, is what we will refer to as the Ramond ground state \( |h = c/24 ; q\rangle \). This phenomenon, that BPS states are annihilated by some of the supersymmetry generators, causes them to assemble into so called short-representations of the SUSY-algebra, whose dimension is smaller then the average length of a representation.

In the context of holography, it should now not come as a surprise that the previously discussed BPS states of the CFT have a dual AdS configuration that admits globally defined killing spinors. One can explicitly show, by identifying the charges in the bulk and CFT, that global AdS is dual to the NS ground state \( |h = 0 ; q=0\rangle \), the neutral extremal BTZ is dual to the R ground state \( |h = c/24 ; q=0\rangle \) and that the charged extremal BTZ is dual to the charged Ramond ground state \( |h = c/24 ; q\rangle \).

The last topic of this section, is how all of the above relates to extremality. That there is a relation is not surprising, since extremality may also be formulated as a saturation between the charges of the background, similar to the BPS conditions. In fact, supersymmetry/BPS \textit{implies} extremality. The argument goes as follows[10]. Suppose we are at finite temperature. Then we have discussed that the Euclidean time cycle becomes contractable in the bulk. However, on such a contractable cycle, bosons are periodic and fermions anti-periodic, which is in contradiction with the supersymmetry relating the two kinds of fields. Thus, we are forced to conclude that supersymmetry and finite temperature are incompatible. In other words, if we consider solutions that are supersymmetric, they must necessarily be at zero temperature and hence extremal. Note though, that whereas supersymmetry \textit{does} imply extremality, the converse \textit{does not} hold in general. Saturation of the extremality bound between the black hole charges, can in general be satisfied for more values of the CFT charges then just those for which also the BPS bound is saturated.
6 Higher spin supergravity

In this section we will turn our attention to the higher spin extension of supergravity. We have seen the \(\mathfrak{osp}(N, 2)\) algebra appearing as the \(\mathcal{N}\) extended isometry algebra of supergravity on AdS\(3\). One can show that there are in fact 7 different Lie superalgebras\(^{[52]} [53] [54] [55]\) that can be used to formulate supergravity on AdS\(3\). These superalgebras must satisfy two basic requirements. Denoting \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) equipped with its \(Z_2\) grading they must satisfy the following conditions:

- The even part of \(\mathfrak{g}\) must contain the gravitational \(\mathfrak{sl}(2, \mathbb{R})\)
- The fermionic generators of \(\mathfrak{g}_1\) must transform in the 2, i.e. spin-1/2 spinor representation of \(\mathfrak{sl}(2, \mathbb{R})\)

In this thesis we shall be only interested in the \(\mathfrak{osp}(N, 2)\) to describe \(\mathcal{N} = 2\) supergravity. To avoid confusion in the notation, let me reserve the \(\mathcal{N}\) symbol, to denote the amount of supersymmetry, and use \(N\) to specify the maximum spin of the higher spin fields.

To acquire a higher spin extension of the supergravity theory we must allow for higher spin generators in the odd sector \(\mathfrak{g}_1\). In this section, we will focus on the superalgebra \(\mathfrak{sl}(N|N - 1)\), which can be seen as a supersymmetrization of \(\mathfrak{sl}(N)\). "Normal" supergravity is recovered for \(N = 2\), in which case we find \(\mathfrak{sl}(2|1) \simeq \mathfrak{osp}(2, 2)\). As for "ordinary" higher spin gravity, the spectrum of the theory depends again on the embedding of the \(\mathfrak{sl}(2)\) subalgebra. In subsection 6.1 we will discuss how this can be done for the \(\mathfrak{sl}(N|N - 1)\) higher spin theory similar to section 4.2. Then in subsection 6.2 we will discuss a prescription to determine killing spinors that is more appropriate to the Chern Simons theories. Subsection 6.3 we will turn to the specific case of \(N = 3\). We will review the findings of [10] where the authors showed that in higher spin supergravity, supersymmetry does not require zero temperature. Put differently, they constructed an \(\mathfrak{sl}(3|2)\) black hole solution, that was BPS and nonetheless non-extremal. Subsection 6.5 will finally, discuss these solutions from the CFT perspective by taking corrections dictated by the CFT BPS bound into account.

6.1 \(\mathcal{N} = 2\) Extended Higher spin SUGRA: \(\mathfrak{sl}(3|2)\) algebra

We consider the principal embedding of the gravitational \(\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(N|N - 1)\). Now, similar to the non-supersymmetric case, we want to know the spectrum of the theory under the principal embedding. In this case, the branching into \(\mathfrak{sl}(2)\) irreducible representations reads\(^{[56]}\):

\[
\mathfrak{sl}(N|N - 1) = \mathfrak{sl}(2, \mathbb{R}) \oplus \left( \bigoplus_{s=3}^{N} \mathfrak{g}_s^s \right) \oplus \left( \bigoplus_{s=1}^{N-1} \mathfrak{g}_s^s \right) \oplus 2 \times \left( \bigoplus_{s=1}^{N-1} \mathfrak{g}_s^{s+\frac{3}{2}} \right).
\]  

(6.1)

As before, the representations are labeled by their \(\mathfrak{sl}(2)\) spin-\(S = (s - 1)\). The generators \(g^{(s)}\) transform in these representations under the adjoint action of \(\mathfrak{sl}(2, \mathbb{R})\). The indices in each multiplet as before run from \(-S\) to \(S\). In the notation of section 4.2 this becomes, setting \(N = 3\):

\[
\text{adj} = 5_2 \oplus 2 \cdot 3_2 \oplus 1_2 \oplus 2 \cdot 2_2 \oplus 2 \cdot 4_2.
\]  

(6.2)

Note from (6.1) that theories with \(N > 2\) will necessarily contain higher spin fermionic fields. It is only for \(N = 2\) that we are dealing with only the spin-1/2 fermionic generators. From the branching rule (6.2) we can now find the spectrum of the \(\mathfrak{sl}(3|2)\) theory\(^{[43]}\). We find a spin-2 multiplet \(W_m\), 2 spin-1 multiplets \(L_i\) and \(A_i\) and a spin-0 singlet \(J\). Together these span the bosonic subsector \(\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{u}(1)\), with the \(\mathfrak{sl}(3)\) spanned by \(W_m\) and \(\frac{1}{2}(L_i + A_i)\) and \(\mathfrak{sl}(2)\) spanned by \(\frac{1}{2}(L_i - A_i)\). Note that this \(\mathfrak{sl}(2)\) is not the gravitational \(\mathfrak{sl}(2)\), which is spanned by the \(L_i\) alone. The abelian \(\mathfrak{u}(1)\) is of course generated by \(J\). The fermionic sector on the other hand contains 2 spin-1/2 multiplets \(H_r\), \(G_r\) and 2 spin-3/2 multiplets \(T_s, S_s\). The \(\mathcal{N} = 2\) supergravity truncation to \(\mathfrak{sl}(2|1)\) is furthermore spanned by \(L_i, J, H_r, G_r\).

\(^{43}\)Recall from section 4.2 that the spin of the corresponding bulk field is \(s = S + 1\).
and the \( \mathcal{N} = 1 \) supergravity truncation to \( \mathfrak{osp}(2|1) \) is spanned by \( L_i, \frac{1}{\sqrt{3}}(H_r + G_r) \). Conventions on the matrix representations of the generators can be found in appendix A of [10]. Having defined the principal embedding, to fully specify the Lorentzian theory one must specify in addition the real form of \( \mathfrak{sl}(3|2) \), which is taken as \( \mathfrak{su}(2, 1|1, 1) \). We refer to the appendix in [10] for the explicit matrix representations of the \( \mathfrak{sl}(3|2) \) and \( \mathfrak{su}(2, 1|1, 1) \) superalgebras.

### 6.2 Killing spinors in Chern Simons formulation

In the spirit of Chern Simons theory, fermionic symmetries are naturally defined by demanding that the bosonic background connection is left invariant under gauge transformations supported by the odd elements of the real form of the gauge super algebra. Writing \( \epsilon \) for such an odd transformation parameter, we should thus demand, c.f. (3.1), (3.5):

\[
\delta \psi \equiv \partial_\mu \epsilon + [a_\mu, \epsilon] = D\epsilon = 0,
\]

where \( D \) denotes the covariant derivative in the presence of the background \( a_\mu \). Since we are only interested in stationary backgrounds, the flatness condition \([a_\phi, a_\phi] = 0\) allows us to immediately write down a general solution to (6.4):

\[
\epsilon(t, \phi) = e^{-a_{\phi} t} e^{a_\phi} \epsilon_0 e^{a_{\phi} t + a_\phi} \phi,
\]

with \( \epsilon_0 \) a constant odd element of the real form of the gauge super algebra. Locally, killing spinors exist for any connection and fermionic generators. Globally, only those that have the correct \( \phi \) periodicity are admissible. They can be periodic in the R sector of anti-periodic in the NS sector imposing constraints on both \( \epsilon_0 \) and \( a_\phi \). Most of the focus will be on the \( \phi \) dependence of the killing spinor. We will use the shorthand \( \epsilon(\phi) \equiv \epsilon(0, \phi) \). To find the globally defined killing spinors, it is now convenient to write \( a_\phi \) in terms of its Jordan decomposition:

\[
V^{-1} a_\phi V = a_\phi^D + a_\phi^N,
\]

where \( a_\phi^D + a_\phi^N \) denotes the Jordan normal form of \( a_\phi \). We then have

\[
\epsilon(\phi) = V^{-1} \epsilon(0) V, \quad \epsilon_0 = V^{-1} \epsilon_0 V.
\]

and accordingly

\[
\epsilon(\phi) = e^{-a_\phi^D \phi} e^{a_\phi^N \phi} \epsilon_0 e^{a_\phi^D \phi} e^{a_\phi^N \phi},
\]

where we used that \([a_\phi^D, a_\phi^N] = 0\) together with the BCH formula. A subtlety to stress though, is that after performing the similarity transformation, both \( a_\phi^D \) and \( a_\phi^N \) as well as the spinors \( \epsilon, \epsilon_0 \) need not belong to the real form of the superalgebra under consideration. Furthermore, since \( a_\phi^N \) is nilpotent, the series expansion \( e^{a_\phi^N \phi} \) is truncated at some finite order. This however, introduces a polynomial \( \phi \) dependence which is neither periodic, nor anti-periodic. We must therefore require also:

\[
[a_\phi^N, \epsilon_0] = 0.
\]

The next step is to go to the Cartan-Weyl basis introduced in appendix [D.3]. Then \( a_\phi^D \) may be expressed in terms of the Cartan elements \( H_{\alpha} \). The advantage of this basis is that it diagonalizes the adjoint action of
of the Cartan elements, as can be seen from \((D.17)\) and \((D.18)\) and that in particular
\[
\text{Ad}_{\alpha}^\dagger(E_{IJ}) = [a^D_j, E_{IJ}] = \omega_{ij} E_{IJ}.
\]
which may be solved for \(\omega_{ij}\) using the commutation relations in the Cartan Weyl basis. Lastly, we may express the constant spinor \(\varepsilon_0\) in terms of the \(E_{IJ}\) basis
\[
\varepsilon_0 = \sum_{ij} \varepsilon_{ij} E_{ij}.
\]
The indices \((i,j)\) run over the dimensions of the two bosonic sub-algebras of the superalgebra. If we denote this as \(\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2\) then the indices take values \(i = 1 \ldots \text{dim}(\mathfrak{g}_1), j = 1 \ldots \text{dim}(\mathfrak{g}_2)\) where \(\text{dim}\) denotes the dimension of the bosonic generators in the fundamental representation. When the bosonic subalgebra contains in addition an \(\mathfrak{u}(1)\) we can make an additional decomposition in \(U(1)\) eigenstates: \(\varepsilon_0 = \varepsilon_0^+ + \varepsilon_0^-\). Making such an eigenstate decomposition however, one has to ensure that when undoing the similarity transformation the killing spinor \(\epsilon\) lies in the real part of the gauge algebra, which will tie the two \(U(1)\) sectors by complex conjugation and thereby thus the components of the killing spinor. This happens for example in the case of the \(\mathfrak{sl}(3|2)\) theory, which we shall discuss below.

At this stage there are thus \(\text{dim}(\mathfrak{g}_1) \cdot \text{dim}(\mathfrak{g}_2)\) complex parameters \(\varepsilon_{ij}\) and a fully symmetric background preserves twice this number of real parameters. Finally, we find thus for \(\varepsilon(\phi)\) in \((6.9)\) the following expression:
\[
\varepsilon(\phi) = \sum_{ij} \varepsilon_{ij} E_{ij} e^{-\omega_{ij} \phi}.
\]
where we used \((6.11)\) and \((6.10)\). From this we see that, in order to agree with the required periodicities, the frequencies \(\omega_{ij}\) must be quantized into integer or half integer imaginary values:
\[
\omega_{ij} \in \begin{cases} i\mathbb{Z} & \text{R sector.} \\ i(\mathbb{Z} + 1/2) & \text{NS sector.} \end{cases}
\]
This condition will in general not be automatically fulfilled. When the frequencies are explicitly determined in terms of the charges carried by the background, the quantization condition will translate into constraints over these charges, or equivalently, the eigenvalues. When these constraints cannot be fulfilled this will further restrict the number of supersymmetries.

### 6.2.1 Counting supersymmetries

With the killing spinor expressed as
\[
\varepsilon(\phi) = \sum_{ij} \varepsilon_{ij} E_{ij} e^{-\omega_{ij} \phi}
\]
there are thus two conditions that restrict the number of supersymmetries. These are \((6.10)\) and \((6.14)\). After these conditions have been imposed, we count the number of supersymmetries by determining the number of independent complex parameters \(\varepsilon_{ij}\). As an example, we consider a fully symmetric background

\(\text{In a more general sense, the frequencies are determined by the roots of the superalgebra. Namely, let } \alpha_j \text{ denote such a root and let } a_j^D \text{ as above. Since } a_j^D \in \mathfrak{h}, \text{ we may can associate a root vector } \Lambda_\alpha \in \mathfrak{h}^* \text{ in the dual root space with it. Similarly, using the isomorphism between } \mathfrak{h} \text{ and the root space } \mathfrak{h}^* \text{ we may also associate an element } H_j \in \mathfrak{h} \text{ with the root } \alpha_j. \text{ Then using the bilinear form on } \mathfrak{h}^* \text{, in } (D.23), i.e. } <\alpha, \beta> = (H_\alpha, H_\beta) = \text{Tr}[\text{adj} H_\alpha \text{adj} H_\beta] \text{ the frequencies can be written as:}
\[
\omega_j = <\Lambda_\alpha, \alpha_j>
\]

\(\text{For the superalgebra } \mathfrak{osp}(1|4) \text{ this bosonic subalgebra is } \mathfrak{so}(1) \oplus \mathfrak{sp}(4) \text{ and thus } i \in \{1\} \text{ and } j \in \{2,3,4,5\} \text{ whereas for the superalgebra } \mathfrak{sl}(3|2) \text{ it is } \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{u}(1) \text{ and so we have } i \in \{1,2,3\} \text{ and } j \in \{4,5\}\)
(e.g. AdS) that preserves 12 real supercharges. The killing spinor will have 6 independent complex components and this background is fully BPS. Now we put a black hole inside AdS. The killing spinor of this background will, after imposing (6.10) and (6.14), have generically less independent components. If e.g. it has 2 independent complex components, then this solutions is 1/3 BPS.

6.2.2 Displaying the killing spinors

The explicit form of the killing spinor is found by undoing the similarity transformation once (6.13) is found. Then one can use the BPS condition (6.3) to determine the most general form of a fermionic gauge parameter $\epsilon = \epsilon_r G^i$, which imposes relations between the $\epsilon_r$. Here we used a short hand $G_i$ to collectively denote all the fermionic generators. The advantage of displaying the killing spinor in this way is that becomes explicit which of the supercharges, the background preserves. To motivate this way of presenting the killing spinor, we discuss an interesting example from the $\mathfrak{sl}$ next subsection we will fully specify to the $\mathfrak{sl}$ supercharges in the higher spin multiplet.

Consider now the case $\epsilon = \epsilon_r G^i$, which imposes relations between the $\epsilon_r$. Here we used a short hand $G_i$ to collectively denote all the fermionic generators. The advantage of displaying the killing spinor in this way is that becomes explicit which of the supercharges, the background preserves. To motivate this way of presenting the killing spinor, we discuss an interesting example from the $\mathfrak{sl}(3|2)$ theory of [10]. Recalling that the spin-1/2 generators are denoted by $H, G_r, r = \pm \frac{1}{2}$ and the higher spin-3/2 generators by $S_r, T_r, s = \pm \frac{1}{2}, \pm \frac{3}{2}$, the killing spinor takes the form

$$
\epsilon(\phi) = \epsilon^{+}_{-\frac{1}{2}}(\phi)H_{\frac{1}{2}} + \epsilon^{-}_{-\frac{1}{2}}(\phi)G_{\frac{1}{2}} + \epsilon^{+}_{-\frac{3}{2}}(\phi)T_{\frac{3}{2}} + \epsilon^{-}_{-\frac{3}{2}}(\phi)S_{\frac{3}{2}} (6.16)
$$

with the highest weight terms all fixed in terms of the lower weight components. The reality condition $\epsilon \in \mathfrak{su}(2,1|1,1)$ lastly ties the parameters by complex conjugation in particular relating $\epsilon^{-}_{-\frac{1}{2}} = i\epsilon^{+}_{-\frac{1}{2}}$ and $\epsilon^{-}_{-\frac{3}{2}} = -i\epsilon^{+}_{-\frac{3}{2}}$. Consequently, one can fully define the killing spinor by defining $\epsilon^{+}_{-\frac{1}{2}}$ and $\epsilon^{+}_{-\frac{3}{2}}$.

Consider now the case $\epsilon^{+}_{-\frac{3}{2}} = 0$. This would make one naïvely suspect that the killing spinor supports supercharges in the $\mathfrak{su}(2|1)$ truncation only. However, when the higher weight components are written explicitly in terms of $\epsilon^{+}_{-\frac{1}{2}}$ and $\epsilon^{+}_{-\frac{3}{2}}$, one observes that the lower spin $\epsilon^{+}_{-\frac{3}{2}}$ can by itself induce components in the killing spinors that concatenate with the higher spin generators. This can be seen explicitly from the expressions for $t_{-\frac{1}{2}}$ and $s_{-\frac{1}{2}}$. The explicit expression for $t_{-\frac{1}{2}}$ is

$$
t_{-\frac{1}{2}} \equiv -\frac{1}{6} \partial^2 \epsilon^{+}_{-\frac{1}{2}} - \frac{1}{2} i Q_{1} \partial \epsilon^{+}_{-\frac{3}{2}} + \frac{1}{3} \partial^2 \epsilon^{+}_{-\frac{3}{2}} + \frac{1}{2} (\frac{7}{3} \mathcal{L} - Q_2 + Q_1^2) \partial \epsilon^{+}_{-\frac{3}{2}} + \frac{2}{3} i Q_{1} \partial \epsilon^{+}_{-\frac{3}{2}} (6.19)
$$

In the second equality we have set $\epsilon^{+}_{-\frac{3}{2}} = 0$. $\mathcal{L}, Q_1, Q_2$ are the global charges, defined in (6.22). This shows explicitly that $\epsilon^{+}_{-\frac{3}{2}}$ can turn on components in the killing spinor $\epsilon(\phi)$ that are supported by generators in the higher spin multiplet.

Having formulated a prescription to determine the global killing spinors of general connections, in the next subsection we will fully specify to the $\mathfrak{sl}(3|2)$ theory. We will start by discussing the findings of [10] where this prescription was applied to study the supersymmetries of $\mathfrak{sl}(3|2)$ black holes. We will contrast

---

47 The asymptotic generators are charged under the $U(1)$ generator $J$ as:

$$
[J, G_r] = G_r, \quad [J, H_r] = -H_r, \quad [J, S_r] = S_r, \quad [J, T_r] = -T_r
$$

Note that their $U(1)$ charge assignments, are opposite to the used index notation for the $U(1)$ eigenstate decomposition $\epsilon = \epsilon^+ + \epsilon^-$. Nonetheless, we have adopted the conventions of [10].

48 The explicit expression can be found in [10] eqn 3.44

49 Recall that the bosonic subalgebra of $\mathfrak{sl}(3|2)$ contains a diagonal $u(1)$.

50 The components $s_r = -i \epsilon_r$ are related by complex conjugation.
these to their extremality properties and discuss how supersymmetry does not require zero temperature in higher spin gravity.

6.3 The $\mathfrak{s}(3|2)$ theory: Non-extremal BPS solutions

Given that where are interested in comparing extremality to supersymmetry, and the extremality condition is expressed as non-diagonalizability of the $\phi$ component of the connection, it is convenient to adopt canonical boundary conditions, c.f. (4.73), so that the charges carried by the background are incorporated in the $\phi$ component. This way, both the BPS condition as well as the extremality condition are relations between the charges only, allowing for a transparent comparison of the two conditions.

After having gauge fixed the radial component, the angular component reads

$$a_\phi = L_1 - LL_{-1} - iQ_1 J - Q_2 A_{-1} - iQ_3 W_{-2}.$$  (6.22)

The sources are thus incorporated as lowest weight components in $i a_{\alpha_k} + a_\phi$ such that the flatness condition $[a_\phi, a_{\alpha_k}] = 0$ is satisfied. Its explicit expression is not important for the present discussion, as we shall not be discussing the thermodynamics of this solution in detail. Similar to the previously discussed solutions, the charges carried by the background are related to the (eigenvalues of the) zero modes of the currents in the CFT generating the $W_{(3|2)}$ algebra on the plane:

$$L = \frac{6}{c} \left( h - \frac{c}{24} - \frac{3}{2c} q^2 + \frac{1}{2} \kappa q^2 \right),$$

$$Q_1 = -\frac{3}{c} q,$$  (6.23)

$$Q_2 = -\frac{9}{5c} \kappa q_2,$$

$$Q_3 = \frac{3}{5c} \kappa \left( q_1 - \frac{6}{c} q_2 \right).$$

Here $h$ is the zero mode of the stress tensor $T$, $q$ of a $U(1)$ current $J$, $q_2$ of a primary $V$ of dimension 2 and $q_3$ of a current $W$ of dimension 3. As discussed, we see in this again the shift by the Sugawara stress tensor. The constant $\kappa$, appears in the OPE relations on the plane as a self-coupling constant of the higher spin multiplet with itself. It is defined in (6.41). It is fixed in terms of the central charge and the large $c$-limit is understood in the above expressions.

Applying the prescription of section 4.6 to $a_\phi + iQ_1 J$, where the $U(1)$ piece is already subtracted because it is already diagonal and commutes with the other generators, one finds that there are 10 Jordan classes in which a general $\mathfrak{s}(3|2)$ connection can live. Since the bosonic subalgebra is $\mathfrak{s}(3) \oplus \mathfrak{s}(2) \oplus \mathfrak{u}(1)$, the connection takes a block diagonal form and there are thus two discriminants $\Delta_{2/3}$, whose distinct roots label the different classes. Explicitly they read:

$$\Delta_3 = 64 \left( 4(L + Q_3)^3 + 27Q_3^2 \right),$$  (6.24)

$$\Delta_2 = 4(L - Q_2) = \lambda_3^2.$$  (6.25)

The insertion of the factors of $i$ in the connection is not entirely trivial. They are inserted such that, given the conventions for the generators in [10], $a_\phi$ lies on the real form $\mathfrak{su}(2,1|1,1)$. It secondly ensures that in the $(2,0)$ supergravity truncation the kinetic Rarita-Schwinger term in the Lagrangian, has the right sign.
with the chosen ordering of the eigenvalues as

$$\text{eigen}(a_\phi + iQ_1, J) = \left[ \lambda_1, -\lambda_1 + \lambda_2, -\lambda_2, \frac{1}{2}\lambda_3, -\frac{1}{2}\lambda_3 \right].$$  \hspace{1cm} (6.26)

Here we used the following relation between the global charges and the eigenvalues:

$$\mathcal{L} = \frac{1}{8}(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2 + \lambda_3^2),$$
$$Q_2 = \frac{1}{8}(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2 - \lambda_3^2),$$
$$Q_3 = -\frac{i}{8}(-\lambda_1 + \lambda_2)\lambda_1\lambda_2.$$  \hspace{1cm} (6.27)

Requiring next that the global charges are real in Lorentzian signature, puts the following reality condition on the eigenvalues:

$$\lambda_1 = \lambda_2^*, \quad \lambda_3 = \lambda_3^*.$$  \hspace{1cm} (6.28)

Consequently, there are thus only 6 non-empty classes, since the choices $2\lambda_1 = \lambda_2$ and $2\lambda_2 = \lambda_1$ conflict with the reality condition. The physical classes are displayed in the first three columns of table 1. To study the supersymmetries of these solutions the prescription discussed in subsection 6.2 is applied. For this one needs the explicit Cartan-Weyl basis for the $\mathfrak{sl}_3(\mathbb{C})$ algebra, which is given in [10]. Using (6.11) the frequencies for this background are then found to be:

$$\omega_{14} = \lambda_1 - \frac{\lambda_1}{2} + iQ_1,$$
$$\omega_{15} = \lambda_1 + \frac{\lambda_1}{2} + iQ_1,$$
$$\omega_{24} = -\lambda_1 + \lambda_2 - \frac{\lambda_3}{2} + iQ_1,$$
$$\omega_{25} = -\lambda_1 + \lambda_2 + \frac{\lambda_3}{2} + iQ_1,$$
$$\omega_{34} = -\lambda_2 - \frac{\lambda_1}{2} + iQ_1,$$
$$\omega_{35} = -\lambda_2 + \frac{\lambda_1}{2} + iQ_1.$$  \hspace{1cm} (6.29)

These frequencies should now be integer or half integer quantised c.f. (6.14), which given the reality condition (6.28) may or may not be possible to fulfil. The number of supercharges preserved by the solutions in the different classes, are listed in table 1, where we recall that a full-BPS solution can preserve a total of 12 real supercharges.

<table>
<thead>
<tr>
<th>Class</th>
<th>Eigenvalue condition</th>
<th>Extremal?</th>
<th>Quantization condition</th>
<th>BPS?</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\lambda_1 = \lambda_2 = 0$</td>
<td>Yes</td>
<td>$-Q_1 = \eta + \frac{1}{2}$</td>
<td>1/3</td>
</tr>
<tr>
<td>II</td>
<td>$\lambda_1 = -\lambda_2 \neq 0$</td>
<td>Yes</td>
<td>$i\lambda_1 - Q_1 = \eta + \frac{1}{2}$</td>
<td>1/3</td>
</tr>
<tr>
<td>III</td>
<td>$\lambda_1 \neq -\lambda_2$</td>
<td>Yes</td>
<td>$-2i\lambda_1 - Q_1 = \eta + \frac{1}{2}$</td>
<td>1/6</td>
</tr>
<tr>
<td>IV</td>
<td>$\lambda_1 = \lambda_2 = 0$</td>
<td>Yes</td>
<td>None</td>
<td>Not BPS</td>
</tr>
<tr>
<td>V</td>
<td>$\lambda_1 = -\lambda_2 \neq 0$</td>
<td>Yes</td>
<td>None</td>
<td>Not BPS</td>
</tr>
<tr>
<td>VI</td>
<td>$\lambda_1 \neq -\lambda_2$</td>
<td>No</td>
<td>$\frac{1}{2}(\lambda_1 - \lambda_2) - Q_1 = \eta + \frac{1}{2}$ and $\lambda_3 = \pm(\lambda_1 + \lambda_2)$</td>
<td>1/3 BPS</td>
</tr>
</tbody>
</table>

Table 1: The BPS properties of $\mathfrak{sl}(3|2)$ black holes. The eigenvalues are parametrized as $\text{eigen}(a_\phi + iQ_1, J) = [\lambda_1, -\lambda_1 + \lambda_2, -\lambda_2, \frac{1}{2}\lambda_3, -\frac{1}{2}\lambda_3]$, with the reality condition $\lambda_1^* = \lambda_2$ and $\lambda_3 \in \mathbb{R}$. Given are also the supersymmetry properties of these solutions. $\eta + \frac{1}{2}$ represents the quantization parameter. It is integer in the Ramond sector and half integer in the NS sector.
There are two notable features of the content of this table that we will discuss next.

- The first observation concerns non-extremal class VI. For this class, the eigenvalues satisfy:

  \[ \lambda_1 \neq -\lambda_2 \quad \lambda_3 \neq 0. \]

Therefore, for general eigenvalues, it will not be possible to fulfill any of the quantization conditions and a general solution in this class will not preserve any supersymmetries. However, when we set

  \[ \lambda_3 = \pm (\lambda_1 + \lambda_2). \]

it becomes possible to quantize the frequencies \( \omega_{14}, \omega_{15}, \omega_{35}, \omega_{34} \). Thus, whilst solutions in this class, according to the general definition \( \text{(4.2)} \), are non-extremal, setting \( \lambda_3 = \pm (\lambda_1 + \lambda_2) \) allows for BPS solutions in this class. Given our discussion of the previous section, where we argued that supersymmetry and finite temperature are incompatible, this is at the very least a peculiar and unnatural feature of the \( \mathfrak{sl}(3|2) \) higher spin theory. In the following subsection we will expand more on these solutions.

- The \( \mathcal{N} = 2, \mathfrak{sl}(1|2) \) supergravity truncation is recovered for \( \lambda_1 = \lambda_2 = \lambda_3 \). The \( \mathfrak{sl}(1|2) \) theory, thus resides in class I and the non-BPS subsector of class VI. The charged BTZ, resides only in class I though, and it is \( 1/3 \) BPS. Now note, that if we set

  \[ \lambda_1 = \frac{i}{3}(\eta_2 - \eta_1) \quad Q_1 = -\frac{1}{3}(2\eta_1 + \eta_2) - \frac{1}{2}, \quad (6.30) \]

we can construct solutions in class II, that allow for a simultaneous fulfillment of the two quantization conditions listed. Consequently, this implies an enhancement to 6 supersymmetries in class II. In class I, both conditions are also simultaneously fulfilled, but there, no supersymmetry enhancement occurs because the two conditions coincide. This means that the charged BTZ, is not the most supersymmetric solution in higher spin gravity. We will revisit these solutions in the last portion of this chapter, when we consider finite-\( c \) corrections dictated by the CFT BPS bound.

### Class VI: Non-extremal BPS solutions

As reviewed above, setting \( \lambda_3 = \pm (\lambda_1 + \lambda_2) \) within class VI, allows for BPS solutions in the non-extremal sector. A slightly reassuring feature, is that the gravitational \( \mathfrak{sl}(2|1) \) subsector of class VI, does not allow for these solutions. Thus the usual notions from the gravitational theory, relating supersymmetry to zero temperature remain unchanged. Nonetheless, the fact that in the higher spin setup, supersymmetry does no longer require extremality/zero temperature is a peculiar feature. In [10] the authors point out two reasons why these solutions should not be a priori discarded.

- In the gravitational setup we argued that, finite temperature and supersymmetry are incompatible by noting that on the contractable Euclidean thermal cycle, bosons are periodic and fermions anti-periodic, contradicting the supersymmetry relating them [10]. In the higher spin setup however, the authors suspect that the non-linearities of the \( \mathfrak{sl}(3|2) \) gauge algebra, are allowing for the solution to balance the periodic bosons and anti-periodic fermions along the contractable thermal cycle, consequently allowing the solution to remain at finite temperature. These non-linearities can be explicitly derived when writing the killing spinor in the form \( \text{(6.18)} - \text{(6.21)} \). From this one can then determine the corresponding (non-linear) transformation rules of the charges, using the same techniques as outlined in section [1,3]. These non-linearities should be contrasted to standard supergravity, where the BPS conditions are always linear in the fields, c.f. [5,14].

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\( ^{52} \)This is found from requiring that the higher spin charges and chemical potentials vanish for this choice and that \( \tau = \frac{i}{2\sqrt{2}} \).
In the next subsection, we will discuss these BPS solutions, from the point of view of the $W_{(3|2)}$ CFT. For now, we mention that by computing the semiclassical limit of the CFT BPS bound, the same quantization condition on the frequencies was found. Somewhat more precisely, the semiclassical BPS bound, written in terms of the bulk data, and at level $\bar{\eta} + \frac{1}{2}$ reported in [10] reads:

$$\det M^{(\eta + \frac{1}{2})} = \frac{\epsilon^2}{2304} \left[ \lambda_3^2 + \left( 1 + 2\eta - 2i(\lambda_1 + iQ_1) \right)^2 \right] \left[ \lambda_3^2 + \left( 1 + 2\eta - 2i(\lambda_1 - iQ_1) \right)^2 \right]$$

$$\cdot \left[ \lambda_3^2 + \left( 1 + 2\eta - 2i(\lambda_2 - \lambda_3 + iQ_1) \right)^2 \right]$$

$$= \frac{\epsilon^2}{36} \prod_{i,j} \left( i\omega_{ij} - \left( \eta + \frac{1}{2} \right) \right) \geq 0. \quad (6.31)$$

Here the frequencies are given in terms of the eigenvalues in (6.29) which are in turn related to the CFT charges via (6.27) and (6.23). $\epsilon$ is a parameter that appears in the definition of hermitian conjugation of the higher spin currents and $\eta$ is the spectral flow parameter. The quantization parameter $\eta + \frac{1}{2}$ is integer in the R sector and half integer in the NS sector. The important information contained in (6.31), is that it reproduces the BPS bounds as found on the gravity side, and in particular that of class IV:

$$\lambda_3 = \pm (\lambda_1 + \lambda_2), \quad \eta + \frac{1}{2} = \frac{i}{2} (\lambda_1 - \lambda_2) - Q_1 \quad (6.32)$$

Before we discuss the CFT BPS bound (6.31) and its quantum analogue in more detail, we note that we can group the BPS conditions listed in table 1 into two families.

**BPS family 1:**

We define BPS family 1 by the following conditions:

$$\lambda_3 = \pm (\lambda_1 + \lambda_2), \quad \eta + \frac{1}{2} = \frac{i}{2} (\lambda_1 - \lambda_2) - Q_1. \quad (6.33)$$

Alternatively, we may express this in terms of the global charges as:

$$4 \left( \mathcal{L} + \frac{5}{2} Q_2 \right) Q_2^2 + 9Q_3^2 = 0, \quad -\frac{3}{2} \frac{Q_3}{Q_2} + Q_1 = \eta + \frac{1}{2}. \quad (6.34)$$

This BPS-family is thus accessible to the $\frac{1}{2}$-BPS solutions in classes I, II and VI. Depending on the choice of sign, this is equivalent to demanding quantization of $\omega_{14}, \omega_{15}, \omega_{35}, \omega_{34}$. Note however, that although (6.33) implies (6.34) the converse does not hold in the most general sense. Besides $\lambda_3 = \pm (\lambda_1 + \lambda_2)$, the implication $\text{(6.34)} \rightarrow \text{(6.33)}$ allows for also the possibilities $\lambda_3 = \pm (\lambda_1 - 2\lambda_2)$, $\lambda_3 = \pm (2\lambda_1 - \lambda_2)$. However, as we have discussed, these choices define the empty and unphysical classes as they conflict with (6.28). Therefore, if we restrict ourselves to the physical solutions of the theory, then the above two statements can in fact be treated as equivalences. We can take this comparison one step further, by relating these conditions to the CFT charges, $h, q, q_2, q_3$, using (6.23) and (6.27). For this we make the observation that the second condition in (6.34) translates in terms of the CFT charges identically to

$$q_2 = q_3, \quad (6.35)$$

and solutions in this family are therefore defined by $q_2 = q_3$ at the semiclassical level. The full BPS condition (6.34), restricting to the physical classes, becomes in terms of the CFT charges:

$$h = q/2, \quad q_2 = q_3. \quad (6.36)$$

---

52 The spectral flow automorphism for the $\mathcal{N} = 2$ super-Virasoro algebra is discussed in appendix E.3.3.
In subsection 6.4 we will discuss the implications of this observation when we discuss the $W_{(3|2)}$ CFT and its BPS bound.

BPS family 2

The second BPS family we define by

$$\lambda_3 = 0, \quad \text{and} \quad -i(\lambda_1 - \lambda_2) - Q_1 = \eta + \frac{1}{2}, \quad (6.37)$$

and it is accessible to class I and the $\frac{1}{6}$-BPS solutions of classes II and III. It corresponds to quantization of $\omega_{24}$ and $\omega_{25}$. In terms of the global charges, this BPS condition translates into:

$$\lambda_3 = 0 \quad \mathcal{L} - Q_2 = 0. \quad (6.38)$$

However, when we turn to the dual CFT, it will tell us that solutions of this family are non physical as they conflict with reality. Explicitly:

$$\mathcal{L} = Q_2 \quad \leftrightarrow \quad q_2 = -\frac{i(c^2 - 24ch + 36q^2)}{48c} \quad (6.39)$$

Thus, solutions in class III and the $1/6$ BPS sector in class II are in fact non-physical. In particular, this means that the naive supersymmetry enhancement in class II is a non-physical effect. The charged BTZ in the $\mathcal{N} = 2$ supergravity truncation remains the most supersymmetric solution. The only physical solutions that belong to this family reside in class I, for which $\mathcal{L} = Q_2 = 0$.

6.4 The $W_{(3|2)}$ CFT

The purpose of this section will be to contrast the BPS conditions found in the $sl(3|2)$ higher spin theory, to the BPS bound that can be derived in a CFT with $W_{(3|2)}$ symmetry. The structure will be as follows. We will first define the operator spectrum of the CFT. Then, we will discuss the derivation of its finite-\(c\) BPS bound, whose semiclassical limit, in terms of the eigenvalues, is given by (6.31). We will then discuss how this BPS bound, relates to our observation in (6.36). We will see that we can draw some interesting conclusions from this. In subsection 6.5 we will then discuss the finite-\(c\) corrections to the BPS bounds, that are dictated by the CFT.

The $W_{(3|2)}$ CFT and its operator spectrum

The $W_{(3|2)}$ is a higher spin extension of the $\mathcal{N} = 2$ Super-Virasoro algebra discussed in appendix E.3.3. It is also known as the $\mathcal{N} = 2$-super $W_3$ algebra. In addition to the super-Virasoro generators, \{\(T, J, G^+, G^-\)}, this algebra contains an $\mathcal{N} = 2$ multiplet, that is generated by a superconformal primary \(V\) of dimension 2. The currents in this multiplet are denoted by \{\(V, U^+, U^-, W\}\}. The current \(W\) has conformal dimension 3 and the fermionic super currents have conformal dimension $\frac{5}{2}$. We refer the reader to appendix B.1 of [10] for the explicit commutation relations of this algebra. The hermiticity properties of the currents are found by requiring the OPE relations on the plane to be invariant under hermitian conjugation. For the $\mathcal{N} = 2$ superconformal generators on finds:

$$L_n^1 = L_{-n}, \quad J_n^1 = J_{-n} \quad (G^\pm)_n^1 = (G^\mp)_{-n}, \quad (6.40)$$

The hermiticity properties of the higher spin generators are more subtle and depend on the value of the coupling constant $\kappa$ of the higher spin multiplet with itself:

$$\kappa = \pm \frac{(c + 3)(5c - 12)}{\sqrt{2}(c + 6)(c - 1)(2c - 3)(15 - c)} \quad \text{as} \quad c \to \infty \quad \rightarrow \pm \frac{5i}{2}. \quad (6.41)$$

53
Depending on its value the hermiticity properties are found to be:

\[(W_n)^\dagger = \epsilon W_{-n}, \quad (V_n)^\dagger = \epsilon V_{-n}, \quad (U^\pm_r)^\dagger = \epsilon U^\mp_r, \quad (6.42)\]

where

\[\epsilon = \begin{cases} +1, & \text{if} \ \kappa \in \mathbb{R}(-6 < c < 1 \cup \frac{3}{2} < c < 15), \\ -1, & \text{if} \ \kappa \text{ is imaginary} (c > 15). \end{cases} \quad (6.43)\]

A small remark is in order here. In the semiclassical limit, the parameter \(\epsilon = -1\) and consequently, the semiclassical BPS bound \((6.31)\), picks up an overall minus sign, indicating that the \(W_{(3|2)}\) algebra is in fact non-unitary in the "large-c" limit. Nonetheless, all solutions of interest here, saturate, the semiclassical BPS bound, for which this feature is not relevant.

In the next paragraph we will discuss how a highest weight representation for this algebra is constructed. Then, we will review the derivation of the finite-c BPS bound whose semiclassical limit is \((6.31)\). Before we do this a note is in order. Due to the presence of an increased number of fermionic zero modes in the Ramond sector compared to the \(\mathcal{N} = 2\) super-Virasoro algebra, its highest weight representation is quite involved. However, since the algebra admits a spectral flow automorphism, all results can be obtained from the NS sector by half integer units of spectral flow.

**Highest weight representation of the NS sector.**

The zero modes in the NS sector are \(L_0, J_0, V_0\) and \(W_0\). Both the sets \(\{L_0, J_0, V_0\}\) and \(\{L_0, J_0, W_0\}\) consist of mutually commuting operators. However, the zero mode operators \(V_0\) and \(W_0\) do not commute identically:

\[\left[ V_0, W_0 \right] = \mathcal{C}_0^{[4]} \quad (6.44)\]

Here \(\mathcal{C}_0^{[4]}\) consists of normal ordered composite operators. In all of the following I will denote similar normal ordered composites symbolically by \(\mathcal{B}_n\) and \(\mathcal{C}_n\), whose expressions can be found in appendix B of [10]. From \((6.44)\), one might then suspect that a highest weight representation could not be labeled by the simultaneous eigenvalues of the set of operators \(\{L_0, J_0, V_0, W_0\}\). This turns out not to be case case though. Starting with a highest weight representation of the set \(\{L_0, J_0, V_0\}\) only, one will find that this is automatically a highest weight state of the full set \(\{L_0, J_0, V_0, W_0\}\) as well.

We thus assume a highest weight state \(|h.w.\rangle = |h, q, q_2\rangle\) obeying:

\[
L_n |h.w.\rangle = J_n |h.w.\rangle = V_n |h.w.\rangle = 0, \quad n > 0, \quad (6.45)
\]

\[
G^+_r |h.w.\rangle = 0, \quad r > 0. \quad (6.46)
\]

From the mode algebra one then finds:

\[
U^\pm_r |h.w.\rangle = 0, \quad r > 0. \quad (6.47)
\]

\[
W_n |h.w.\rangle = 0, \quad n > 0. \quad (6.48)
\]

This can be shown as follows:

The first follows from the commutation relation \([G^+_{-t}, V_0] = \mp U^+_r\), combined with the highest weight conditions \(V_0 |h.w.\rangle = q_2 |h.w.\rangle\) and \(G^+_{r} |h.w.\rangle = 0\) for \(r > 0\). Using then this condition combined with the relation \(\{G^+_{n-\frac{1}{2}}, U^+_r \} |h.w.\rangle = (\mp (3n - 4t)|V_n + 2W_n |h.w.\rangle\) for \(t = \frac{1}{2}\) and \(n > 0\), one finds \(U^+_r G^+_{n-\frac{1}{2}} |h.w.\rangle = 2W_n |h.w.\rangle\). If we then combine this with \(G^+_{n-\frac{1}{2}} |h.w.\rangle = 0\) for \(n > 0\), we find that \(W_n |h.w.\rangle = 0\) for \(n > 0\) as well.
The last step is to check the action of \( V \). Since \( \{L_0, J_0, W_0\} \) forms a commuting set, we automatically have:
\[
L_n |\phi_W\rangle = 0, \quad J_n |\phi_W\rangle = 0, \quad n > 0.
\]
(6.50)

Using (6.47) we also see that:
\[
G_r^+ |\phi_W\rangle = G_r^+ (W_0 |h.w.) = [G_r^+, W_0] |h.w.) = 2rU_{r+n}^+ |h.w) = 0, \quad r > 0,
\]
(6.51)

where in the third equality we used the commutator:
\[
[G_r^+, W_n] = (2r - \frac{n}{2}) U_{r+n}^+.
\]
(6.52)

The action of the modes \( V_n, W_n \) on \( |\phi_W\rangle \) is more complex. Their commutators with \( W_0 \) read:
\[
[V_n, W_0] = \mathcal{C}_n^{(4)} + 2n\mathcal{C}_n^{(3)} + n(n^2 - 1)\mathcal{C}_n^{(1)},
\]
(6.53)
\[
[W_n, W_0] = n\mathcal{B}_n^{(4)} + 2n(n^2 - 4)\mathcal{B}_n^{(2)}.
\]
(6.54)

Since \( W_n |h.w) = V_n |h.w) = 0 \) for \( n > 0 \), these conditions become, for \( n > 0 \):
\[
V_n |\phi_W\rangle = V_n (W_0 |h.w) = \left( \mathcal{C}_n^{(4)} + 2n\mathcal{C}_n^{(3)} + n(n^2 - 1)\mathcal{C}_n^{(1)} \right) |h.w.) ,
\]
(6.55)
\[
W_n |\phi_W\rangle = W_n (W_0 |h.w) = \left( n\mathcal{B}_n^{(4)} + 2n(n^2 - 4)\mathcal{B}_n^{(2)} \right) |h.w.) .
\]
(6.56)

Since all positive modes of all currents annihilate the highest weight state, we find:
\[
V_n |\phi_W\rangle = W_n |\phi_W\rangle = 0, \quad n > 0.
\]
(6.57)

The last step is to check the action of \( V_0 \) on \( |\phi_W\rangle \). From (6.53) we find:
\[
V_0 |\phi_W\rangle = \left( V_0 + W_0 V_0 \right) |h.w) = \mathcal{C}_0^{(4)} |h.w) + q_2 |\phi_W\rangle .
\]
(6.58)

From the explicit mode expansion of \( \mathcal{C}_0^{(4)} \), one will find that \( \mathcal{C}_0^{(4)} |h.w) = 0 \) and thus
\[
V_0 |\phi_W\rangle = q_2 |\phi_W\rangle .
\]
(6.59)

Then we finally have that also
\[
U_{r}^+ |\phi_W\rangle = 0, \quad r > 0,
\]
(6.60)
as follows from \( [G_r^+, V_0] = \mp U_{r}^+ \) and (6.59).

To sum up, in the NS sector, the state \( |\phi_W\rangle \) carries the same quantum numbers \( h, q, q_2 \) and satisfies the same highest weight conditions as the state \( |h,w) \) itself and must therefore be proportional to it in order for the representation to be irreducible. Put differently, a highest weight representation build from \( \{L_0, J_0, V_0, W_0\} \) in the NS sector, will automatically be a highest weight representation of the full set \( \{L_0, J_0, V_0, W_0\} \) as well.

The highest weight state can thus be defined as the state \( |h.w) = |h, q, q_2, q_3 \rangle \) obeying:
\[
L_0 |h.w) = h |h.w) \quad J_0 |h.w) = q |h.w) \quad V_0 |h.w) = q_2 |h.w) \quad W_0 |h.w) = q_3 |h.w)
\]
(6.61)

which is annihilated by all positive modes.

55
The $\mathcal{W}_{(3|2)}$ BPS bound

To derive the BPS bound of the $\mathcal{W}_{(3|2)}$ algebra we must consider the matrix of inner products of the level-1/2 fermionic descendents:

$$|\alpha^\pm\rangle = G_{-1/2}^\pm |h.w.\rangle \quad |\beta^\pm\rangle = U_{-1/2}^\pm |h.w.\rangle$$

(6.63)

The relevant anti-commutators will be the following:

$$\{G_r^+, G_s^+\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r,-s},$$

(6.64)

$$\{U_r^+, U_s^+\} = D_{r+s}^0 + (r-s)D_{r+s}^{|3|} + \left(3r^2 - 4rs + 3s^2 - \frac{9}{2}\right)D_{r+s}^{|2|}$$

(6.65)

$$+ (r-s)(r^2 + s^2 - \frac{5}{2})D_{r+s}^{|1|} + \frac{c}{12}(r^2 - \frac{1}{4})(r^2 - \frac{9}{4})\delta_{r,-s},$$

$$\{U_r^+, U_s^\pm\} = (\mathcal{E}^\pm)_{r+s},$$

(6.66)

$$\{G_r^+, U_s^\pm\} = \pm (3r-s)V_{r+s} + 2W_{r+s}.$$  

(6.67)

The anti-commutator $\{U_r^+, U_s^\pm\}$ between two higher spin fermionic generators of the same $U(1)$ charge, is in general non-vanishing, since there exist suitable spin-4, charge 2 operators that can appear on the right hand side:

$$\mathcal{E}^\pm = -\frac{6}{c-1}\langle\partial G^+G^\pm\rangle - \frac{12\kappa}{c+3}(G^+U^\pm),$$

(6.68)

One will find though, that the action of this operator on the highest weight state is zero by virtue of $\{G^+, G^\pm\} = \{G^\pm, U^\pm\} = 0$. Therefore, states of different $U(1)$ charge are orthogonal, and we can focus on a single charged sector. The matrix of inner products then becomes:

$$K^{(1/2)} = \begin{pmatrix}
\langle\alpha^+|\alpha^+\rangle & \langle\alpha^+|\beta^+\rangle \\
\langle\beta^+|\alpha^+\rangle & \langle\beta^+|\beta^+\rangle
\end{pmatrix} = \begin{pmatrix}
2h - q & 2(q_3 - q_2) \\
2c(q_3 - q_2) & -\frac{1}{2}(D_0^{|4|} - D_0^{|3|} - 2D_{r+s}^{|2|} + 2D_0^{|1|})
\end{pmatrix}.$$  

(6.69)

The brackets denote the expectation value in the highest weight state $|h.w.\rangle$ defined above. The explicit expression for the action of the highest weight state of the normal ordered composites $D_0^{|4|}, D_0^{|3|}, D_{r+s}^{|2|}, D_0^{|1|}$ can be found in appendix B.3 of [19]. We have verified the correctness of these expressions as well as their actions on the highest weight state. The quantum BPS bound thereby becomes:

$$\text{det}K^{(1/2)} = \epsilon \left(2h - q\right)\left(\frac{3h^2}{3c} - \frac{18}{2c} + q\right) + \frac{2c}{3}(c(q_3 - q_2) + 4q_2) > 0.$$  

(6.70)

In particular, we see from this BPS bound, that all the effects of the semiclassical limit are incorporated in $D_0 \equiv (D_0^{|4|} - D_0^{|3|} - 2D_{r+s}^{|2|} + 2D_0^{|1|})$. In the semiclassical limit, it reads

$$D_{(0)}^\text{semi} = \frac{27}{2c}(2h - q)\left(\frac{h}{3} - \frac{q^2}{3c} - \frac{1}{18}\left(2c - q\right)^2\right) + \frac{2c}{5c}(c(q_3 - q_2) + 4q_2 - 3q_2).$$

(6.71)

with $\kappa = \frac{5\pi}{4}$, c.f. [6.41]. Furthermore, it is now not hard to see that an exact solution at finite-$c$, to the BPS bound (6.70) is given by:

$$h = q/2 \quad q_2 = q_3.$$  

(6.72)

Recall now though, that this is the defining condition for BPS family 1, i.e. (6.36). In other words, the BPS solutions of family 1, and in particular class VI, constitute a set of exact solutions to the finite-$c$ CFT BPS bound, and in particular, the non-extremal BPS solutions of class VI can be seen as exact solutions to the BPS bound.

Let us now take a closer look at $(D_{(0)}^\text{semi})$. In particular, under $q_2 = q_3$, it becomes:

$$\frac{-1}{4c^2}\left(2h - q\right)(2c^2 + 72q^2 + 3c(-18h + q - 36q_0)).$$

(6.73)

The corresponding expressions at finite-$c$ can be found in [10].
This means that, semi-classically, BPS family 1 does not distinguish between solutions for which \( h = q/2 \), or those for which \( D_0 = 0 \) since they amount to the same condition. The other possibility

\[
2c^2 + 72q^2 + 3c(-18h + q - 36iq) = 0,
\]

(6.74)
is not accessible to the general physical solutions of BPS family I. In fact, one can explicitly verify, that this condition is fulfilled only by solutions that live in the empty Jordan classes for which \( \lambda_3 = \pm(\lambda_1 - 2\lambda_2) \), \( \lambda_3 = \pm(\lambda_2 - 2\lambda_1) \). The one exception for which this condition is fulfilled by a physical class occurs for class I. This is most easily to see when translating (6.74) to the global charges:

\[
\mathcal{L} = -\frac{1}{36} (1 + 2Q_1)^2 + Q_2,
\]

(6.75)
which can only be fulfilled by class I for which \( \mathcal{L} = Q_2 = 0, Q_1 = -\frac{1}{6} \). Nonetheless, in terms of the CFT charges this amounts to \( h = c_1/12, q = c_6 \) and there is thus not much interesting information captured here, since it once again tells us that \( h = q/2 \).

In summary, the above discussion shows that at the semiclassical level, BPS solutions in family I, are **exact** solutions defined by \( q_2 = q_3, h = q/2 \). It furthermore raises the following questions, that we will address in the next subsections.

1. Exact solution in the bulk belonging to BPS family 1, exists for \( q_2 = q_3 \) and \( h = q/2 \) exactly. Semi-classically, this condition does not distinguish between solutions for which \( q_2 = q_3 \) and \( (D_0)_{semi} = 0 \). For that reason, we might wonder, whether solutions with \( q_2 = q_3 \) and \( D_0 = 0 \), exist also at finite-\( c \).

2. Semi-classically, a necessary condition for family 1, is that \( q_2 = q_3 \). This raises the question whether in the CFT \( q_2 = q_3 \) remains a necessary condition as well, or if it is only a sufficient condition. In other words, do solutions at finite-\( c \) exist for which \( q_2 \neq q_3 \).

In the next section we will address these two questions by taking into account the finite-\( c \) contributions to the CFT BPS bound.

### 6.5 Finite-\( c \) corrections from the BPS bound

As we have discussed, the BPS solutions in the bulk, saturate the semiclassical CFT BPS bound (6.31). In this section we will take into account the finite-\( c \) contributions to (6.31). The motivation for this is that it will tell us whether the BPS solutions identified in the bulk, remain well defined solutions at finite-\( c \), or whether they are an effect of the "large-\( c \)" limit\(^{55}\). Along the way we will pay attention to the two questions raised above.

#### Strategy

The strategy will be to solve the quantum CFT BPS bound perturbatively. Now, from the point of view of the CFT one would most naturally use \( \hbar \) as the perturbative expansion parameter, since this is the fundamental variable that characterizes quantum effects. However, when written in terms of the CFT data, the BPS bound is much less illuminating then when it is written in terms of the eigenvalues of the bulk holonomy. For family 1, with the above observations it is easy to solve the BPS bound at zeroth order. However, the zeroth order relations between the CFT charges for family 2 become more cumbersome and in particular, it is less transparent how we can distinguish between the different classes belonging to a single BPS family. In terms of the eigenvalues this becomes far more transparent, as is seen from (6.31). In the following we will show why an expansion in \( \hbar \to 0 \) in terms of the CFT variables,

\(^{55}\)Initially, this was in particular motivated by class VI. It was only at a later stage during this project, that I found out about the identifications made in the previous section, that allowed for a quick interpretation of class VI as an exact solution at finite-\( c \). We will see this solution reappearing from perturbation theory in the next section.
is equivalent to an expansion in $1/c$, $c \to \infty$, in terms of the bulk variables.

1. Expansion in terms of the bulk variables.

For this, let us first recall how the semiclassical limit is defined\[^{[83]}\]. More details can be found in appendix G. We must first restore the appropriate factors of $\hbar$ by rescaling all the currents and fundamental parameters of the $\mathcal{W}$ algebra via:

$$J = \hbar^{-1} \tilde{J}, \quad c = \hbar^{-1} \tilde{c}. \quad (6.76)$$

whilst the rescaled generators $\tilde{J}$ and $\tilde{c}$ are to be held fixed. Here $J$ denotes a collective set of charges $\{h, q, q_2, q_3\}$. In the following we will take the conformal weight $h$ as an explicit representative. Similar reasonings hold for the other three charges. The next step is to assume an expansion of the CFT charges given as:

$$\tilde{h} = h^{(0)} + \hbar \tilde{h}^{(1)} + \mathcal{O}(\hbar^2), \quad \implies \quad h = h^{-1} \tilde{h}^{(0)} + \tilde{h}^{(1)} + \mathcal{O}(\hbar). \quad (6.77)$$

Then, in the semiclassical limit ($\hbar \to 0$), the BPS bound will thus take the schematic form:

$$\det K_{\hbar \to 0} = P(\tilde{h}^{(0)}, \tilde{q}_1^{(0)}, \tilde{q}_2^{(0)}, \tilde{q}_3^{(0)}) + \mathcal{O}(\hbar). \quad (6.78)$$

where $P$ denotes a polynomial.

The last step is now to define how this limiting procedure translates to the eigenvalues, i.e. in terms of $h(\lambda)$. In other words, given (6.76) and the relation between the CFT charges and the eigenvalues, we would like to define a classical definition of the eigenvalues, which we will denote as $\lambda$, and a quantum definition, which we will denote as $\tilde{\lambda}$. Using now that we know that $h(\lambda)$ is given by:

$$h(\lambda) = \frac{c}{6} \left( \frac{5}{24} (\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2) + \frac{1}{8} (\lambda_1^3 - \lambda_1 \lambda_2 + \lambda_2^3) + Q_1^2 + \frac{1}{4} \right), \quad (6.79)$$

we assume the following perturbative expansion for the eigenvalues and $U(1)$ charge\[^{[56]}\]

$$\lambda_i = \tilde{\lambda}_i^{(0)} + \frac{1}{c} \tilde{\lambda}_i^{(1)} + \mathcal{O}(\frac{1}{c^2}) \quad \text{and} \quad Q_1 = \tilde{Q}_1^{(0)} + \frac{1}{c} \tilde{Q}_1^{(1)} + \mathcal{O}(\frac{1}{c^2}). \quad (6.80)$$

Then one observes that the correct powers of $\hbar$ in (6.76) appear through the rescaling of the overall factor of the central charge $c = \hbar^{-1} \tilde{c}$.

Therefore we can conclude that:

- an expansion at the level of the CFT variables in $\hbar$ can be considered equivalent as an expansion in large $c$ at the level of the eigenvalues.

2. The spectral flow parameter at finite $c$

The BPS bound in (6.31) was obtained by spectral flowing\[^{[57]}\] the semiclassical level-1/2 bound. In the semiclassical regime, this spectral flowed bound can be expected to contain all the relevant information in analogy to the Super-Virasoro case where the BPS bound dominates the semiclassical unitarity bounds. However, since we are now interested in the quantum BPS bound, it will generically not be the case, that the spectral flowed quantum BPS bound will be the most stringent bound, since in the quantum regime it is necessary to consider all descendant states at any given level to obtain the full set of unitarity bounds. Therefore, we cannot assume that a spectral flowed quantum BPS bound will contain all relevant information, i.e we might be ignoring various unitarity issues. At the quantum level, we can thus consider only the level-1/2 bound, which amounts to setting $\eta = 0$ as we go beyond the semiclassical level.

\[^{[56]}\]Note that the $\tilde{\lambda}_i^{(0)}, \tilde{Q}_1^{(0)}$ are known. They saturate (6.31)

\[^{[57]}\]See appendix E.3.3 for the definition of the spectral flow automorphism for the $\mathcal{N} = 2$ Super conformal algebra. The analogous automorphism for the $\mathcal{W}(3|2)$ algebra can be found in \[10\].
6.5.1 Finite-\(c\) corrections

With these considerations in mind we now turn to the task of finding the finite-\(c\) corrections to the BPS bounds. As discussed, we will do this by solving the quantum BPS bound perturbatively, with \(1/c\) acting as the expansion parameter. In the specific case at hand, the first step is thus to extract from the quantum BPS bound, all terms that are of the same order in \(1/c\). Doing so, we find that there are contributions up to \(\mathcal{O}(1/c^2)\) and we schematically write the quantum BPS bound as:

\[
K = e^2 K^{(0)} + cK^{(1)} + K^{(2)} + \frac{1}{c} K^{(3)} + \frac{1}{c^2} K^{(4)}. \tag{6.81}
\]

The contributions at the different orders, denoted \(K^{(i)}\) are all functions of the eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) and \(U(1)\) charge \(Q_1\). As a consistency check, we note that the \(\mathcal{O}(c^2)\) contribution \(K^{(0)}\), reproduces the result \(6.31\). The next step is to assume the perturbative expansion for the eigenvalues and \(U(1)\) charge, as given in \(6.30\):

\[
\lambda_i = \tilde{\lambda}_i^{(0)} + \frac{1}{c} \tilde{\lambda}_i^{(1)} + \mathcal{O}\left(\frac{1}{c^2}\right) \quad \text{and} \quad Q_1 = \tilde{Q}_1^{(0)} + \frac{1}{c} \tilde{Q}_1^{(1)} + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{6.82}
\]

with \(i = 1, 2, 3\). Thus combining these two expansions, we must solve for the BPS condition at each order in \(1/c\). For example, at \(\mathcal{O}(c^2)\), we solve

\[
c^2 K^{(0)} \left(\tilde{\lambda}_i^{(0)}, \tilde{Q}_1^{(0)}\right) = 0, \tag{6.83}
\]

which is automatically fulfilled, by virtue of the semiclassical BPS bounds. Then at \(\mathcal{O}(c)\), we find the first non-trivial correction. Schematically, the \(\mathcal{O}(c)\) contribution of the combined expansion, is:

\[
c^2 K^{(0)} \left(\frac{1}{c} \tilde{\lambda}_i^{(1)}, \frac{1}{c} \tilde{Q}_1^{(1)}, \tilde{\lambda}_i^{(0)}, \tilde{Q}_1^{(0)}\right) + eK^{(1)} \left(\tilde{\lambda}_i^{(0)}, \tilde{Q}_1^{(0)}\right) = 0. \tag{6.84}
\]

This condition will constrain the \(\tilde{\lambda}_i^{(1)}, \tilde{Q}_1^{(1)}\) in terms of the \(\tilde{\lambda}_i^{(0)}, \tilde{Q}_1^{(0)}\), and thus define us first order corrections to the holonomy eigenvalues and \(U(1)\) charge. In turn, these corrections, will define corrections to the global charges via \(6.27\). The charges however, must at all orders in perturbation theory, remain real valued. This may not be possible to fulfil given the zeroth order definitions of the different classes. This means that such a BPS solution, must be regarded as unphysical when finite-\(c\) effects are taken into account. In a similar fashion, we find the next order corrections, by extracting the \(\mathcal{O}(1)\) contributions from the expanded BPS bound and equating this to zero.

We will structure this section as follows. Each of the BPS classes will be discussed separately. We can specify to a BPS class by choosing the \(\tilde{\lambda}_i^{(0)}\), which we recall where subject to \(\tilde{\lambda}_1^{(0)} = (\tilde{\lambda}_2^{(0)})^*, \) and \(\tilde{\lambda}_3^{(0)} \in \mathbb{R}\). We next use the zeroth order BPS bound \(6.31\) to specify to a specific BPS branch. Here by ‘branch’ we refer to the 6 different factors \(\left(\omega_{ij} - \frac{1}{2}\right)\) in \(6.31\) labeled by the frequencies. As explained, the spectral flow parameter has been set to zero. For example, we may quantize the frequency \(\omega_{14}\) by using that

\[
\tilde{Q}_1^{(0)} = \frac{i}{2} (2\tilde{\lambda}_1^{(0)} - \tilde{\lambda}_3^{(0)} - 1). \tag{6.85}
\]

Under this condition, \(6.83\) is automatically solved for. In the next paragraph we will apply this to class VI belonging to BPS family I. Since classes I and II may recovered from class VI, I will not discuss these cases separately. As noted, we need not consider BPS family II, as this family is already unphysical semi-classically.

**BPS family I**

BPS family I is contains classes I, II and IV. The results for classes I and II may be recovered from
the results in this class by setting $a = b = 0$ and $a = 0$ respectively, with $a, b$ defined in (6.87).

Class VI

We recall that class VI is defined by

$$\tilde{\lambda}^{(0)}_3 \neq 0, \quad \tilde{\lambda}^{(0)}_1 = \left(\tilde{\lambda}^{(0)}_2\right)^*,$$

which, for convenience, we parametrize as:

$$\tilde{\lambda}^{(0)}_1 = a + bi, \quad \tilde{\lambda}^{(0)}_2 = a - bi, \quad a, b \in \mathbb{R}/\{0\}.$$ (6.87)

The BPS sector we recover by taking in addition

$$\tilde{\lambda}^{(0)}_3 = \tilde{\lambda}^{(0)}_1 + \tilde{\lambda}^{(0)}_2.$$ (6.88)

As we remarked before, these BPS solutions belong to family I corresponding to quantization of the frequencies $\omega_{14}, \omega_{15}, \omega_{34}, \omega_{35}$. We will take $\omega_{14}$ as a representative, which is related to the other three frequencies through the following interchanges

$$\tilde{\lambda}^{(0)}_1 \leftrightarrow \tilde{\lambda}^{(0)}_2 \quad \tilde{\lambda}^{(0)}_3 \leftrightarrow \pm \tilde{\lambda}^{(0)}_3,$$ (6.89)

which lead to equivalent conclusions.

At $O(c)$ we find a contribution that schematically reads:

$$\left(\tilde{\lambda}^{(0)}_3 - \tilde{\lambda}^{(0)}_1 - \tilde{\lambda}^{(0)}_2\right)(\cdots),$$ (6.90)

and which already vanishes by assumption.

At $O(1)$, we will find more structure. Demanding the $O(1)$ BPS condition to vanish, can be achieved in three distinct ways. The first possibility reads simply:

$$\tilde{\lambda}^{(0)}_1 = \tilde{\lambda}^{(0)}_2 = 0$$ (6.91)

which we should disregard as it contradicts with (6.87). Next there are two other conjugate possibilities to solve the BPS bound condition, which can be expressed in terms of a correction to the $U(1)$ charge $\tilde{Q}^{(1)}_1$. Explicitly:

$$\tilde{Q}^{(1)}_1 = \frac{1}{2a^2(a^2 + 9b^2)} \left(48b^3 + 3a^2b(16 + 3b(\tilde{\lambda}^{(1)}_1 - \tilde{\lambda}^{(1)}_2)) + ia^4(\tilde{\lambda}^{(1)}_1 - \tilde{\lambda}^{(1)}_2)\right)$$

$$\pm \frac{1}{2a^2(a^2 + 9b^2)} \left[2304a^4b^2 + 4608a^2b^4 + 2304b^6 - a^3(2a^2 + 9b^2)(\tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 - \tilde{\lambda}^{(1)}_3)\right]^{1/2},$$ (6.92)

where we used the above parametrization for the zeroth order eigenvalues. Note that we need not worry about divergencies here, since $a^2(a^2 + 9b^2) = 0$ is not admissible to solutions of class 6. We will refer to these solutions as respectively the "plus" and "minus" solution.

There are a few things to note here. The first is that the second term under the square root is troublesome\footnote{We note that this term is absent for the truncations to classes I and II.}. If it grows sufficiently large, the square root expression becomes imaginary, and it is therefore necessary to demand that class VI solutions obey:

$$2304a^4b^2 + 4608a^2b^4 + 2304b^6 \geq -a^3(2a^2 + 9b^2)(\tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 - \tilde{\lambda}^{(1)}_3).$$

$$\left(a(a^2 + 9b^2)(\tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 - \tilde{\lambda}^{(1)}_3) + 32(a^2 + b^2)\right).$$ (6.93)
Before we say more about this, note that a sufficient condition is to set $\tilde{\lambda}_3^{(1)} = \tilde{\lambda}_1^{(1)} + \tilde{\lambda}_2^{(1)}$ which corresponds to the limiting case. In the following we consider this possibility after which we will come back to the inequality. Note that, we may also consider the other limiting possibility:

$$\tilde{\lambda}_3^{(1)} = \tilde{\lambda}_1^{(1)} + \tilde{\lambda}_2^{(1)} + \frac{32(a^2 + b^2)}{a^2 + 9ab^2}. \tag{6.94}$$

This choice however, gives rise to conditions that conflict with reality at $O(1/c)$, and therefore does not correspond to physical solutions.

The "plus" solution: setting $\tilde{\lambda}_3^{(1)} = \tilde{\lambda}_1^{(1)} + \tilde{\lambda}_2^{(1)}$.

Under this condition, the two contributions in (6.92) nicely combine so as to leave us with:

$$\tilde{Q}_1^{(1)} = \frac{i}{2} (\tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)}) \implies \tilde{\lambda}_1^{(1)} = \left(\tilde{\lambda}_2^{(1)}\right)^*, \quad \tilde{\lambda}_3^{(1)} \in \mathbb{R}/\{0\}. \tag{6.95}$$

Given our discussion in the previous section, this solution is nothing but the exact solution, that in terms of the CFT variables reads: $q_2 = q_1$ and $h = q/2$ at first order in $h$. At higher orders, we find that the condition instead reads

$$\tilde{Q}_1^{(2)} = \frac{1}{6b} \left(3b(\tilde{\lambda}_1^{(2)} - \tilde{\lambda}_2^{(2)}) + a(\tilde{\lambda}_1^{(2)} + \tilde{\lambda}_2^{(2)} - \tilde{\lambda}_3^{(2)})\right). \tag{6.96}$$

leaving the relation between $\tilde{\lambda}_1^{(2)}, \tilde{\lambda}_2^{(2)}, \tilde{\lambda}_3^{(2)}$ undetermined.

The "minus" solution: setting $\tilde{\lambda}_3^{(1)} = \tilde{\lambda}_1^{(1)} + \tilde{\lambda}_2^{(1)}$.

For this solution, the two contributions to (6.92) instead combine to yield:

$$\tilde{Q}_1^{(1)} = \frac{1}{2a^2(a^2 + 9b^2)} \left(ia^4 (\tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)}) + 3a^2 b (32 + 3b(\tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)})) + 96b^3\right). \tag{6.97}$$

Reality thus enforces:

$$\tilde{\lambda}_1^{(1)} = \left(\tilde{\lambda}_2^{(1)}\right)^* \implies \tilde{\lambda}_3^{(1)} \in \mathbb{R}/\{0\}, \tag{6.98}$$

allowing us to parametrize:

$$\tilde{\lambda}_1^{(1)} = e + fi, \quad \tilde{\lambda}_2^{(1)} = e - fi, \quad e, f \in \mathbb{R}. \tag{6.99}$$

(6.97) then becomes:

$$\tilde{Q}_1^{(1)} = \frac{48b}{a^2 + 9b^2} \left(a^2 + b^2\right) - f \in \mathbb{R}. \tag{6.100}$$

This solution, as we have explicitly verified, corresponds in terms of the CFT variables to $q_2 = q_1, D_0 = 0$ in (6.70). Recall now from (6.73) that, semi classically, BPS family I can not distinguish between solutions with $h = q/2$ and $D_0 = 0$, as we noted that they amount to the same condition. The above derivation shows that this is no longer the case at finite-$c$ and they can be regarded as different solutions. Before we move on to the strict inequality let me mention that when the next order $O(1/c)$, corrections to this solution are considered there are no troublesome features appearing that make this solution unphysical. One finds only a constraint on the parameter $f$ in (6.99). When this is implemented into an expansion of the charges $L, Q_2, Q_3$, one finds that the only condition that has to be imposed is $\tilde{\lambda}_3^{(2)} \in \mathbb{R}$. We expect this solution to remain well defined at higher orders as well.

All in all, we conclude that, although in the bulk the only solution in class VI, is defined by $q_2 = q_3, h = q/2$,

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59 I will not discuss this possibility here explicitly. The expressions become in this case far too long to be displayed properly.
in the CFT this solution allows for more structure. In particular, the CFT allows for solutions for which \( q_2 = q_3, D_0 = 0 \) at finite-\( c \) which at the semiclassical level, where not visible. As a final consistency check, we note that the same corrections on the \( U(1) \) charge are recovered, if the \( \mathcal{O}(1) \) correction resulting from the BPS bound, is expressed in terms of the eigenvalues instead. If one then expands the charges \( \mathcal{L}, Q_2, Q_3 \), and one restricts to the sufficient limiting cases described above, one finds that their reality dictates the same correction on the \( U(1) \) charge and condition on the eigenvalues.

To answer the question whether solutions with \( q_2 \neq q_3 \) exists at finite-\( c \) we must look at the strict inequality in (6.93). This inequality is satisfied for several cases, in all of which we have:

\[
\tilde{\lambda}^{(1)}_1, \tilde{\lambda}^{(1)}_2, \tilde{\lambda}^{(1)}_3 \in \mathbb{R}.
\] (6.101)

Then there are the following possibilities:

\[
\begin{align*}
 b &= \pm a/3, \quad a \neq 0, \quad \tilde{\lambda}^{(1)}_2 \neq \tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 + \frac{80}{9d}, \\
 \left( b < -\frac{a}{3} \lor b > \frac{a}{3} \right), \quad a > 0, \\
 -a/3 < b < a/3, \quad a > 0, \quad \tilde{\lambda}^{(1)}_3 < \tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 - \frac{16 (a^2 + b^2) (\sqrt{a^2 - 9b^2} - a)}{a^4 + 9a^2b^2}, \\
 -a/3 < b < a/3, \quad a > 0, \quad \tilde{\lambda}^{(1)}_3 > \tilde{\lambda}^{(1)}_1 + \tilde{\lambda}^{(1)}_2 - \frac{16 (a^2 + b^2) (\sqrt{a^2 - 9b^2} - a)}{a^4 + 9a^2b^2}.
\end{align*}
\] (6.102)

These conditions correspond, at finite-\( c \), to solutions for which \( q_2 \neq q_1 \), in the BPS condition (6.70). Unfortunately, explicit analysis for these conditions becomes very cumbersome due to the inequalities. Nonetheless, given the structure of the global charges, the reality condition on the eigenvalues, and the ranges of the parameters, it is not hard to see that these conditions will not result in any unphysical behaviour of the first order corrections to the charges \( \mathcal{L}, Q_2, Q_3 \), where by unphysical we mean divergencies or discontinuities.

In particular, we conclude from this that in the CFT allows for solutions with \( q_2 \neq q_3 \), provided that they fulfil the above inequalities.
7 Black hole extremality in AdS$_3$ hypergravity

In the previous section we have paid attention to supergravity, containing the spin-3/2 graviton, and a higher spin extension. Supergravity however, is not the unique theory that contains a superpartner for the graviton. In the childhood of supergravity another theory was considered as a possible supersymmetric theory of gravity. This theory contains a massless spin-5/2 field as the superpartner of the graviton. It has been dubbed the hypertini and the resulting theory hypergravity. However, in general dimensions the miracle of supergravity does not repeat itself and the theory was soon found to suffer from various inconsistencies\cite{60}. In 1984, Aragone and Deser then showed that in 3 dimensions, in which case the Weyl tensor vanishes, a consistent theory on a Minkowski background can indeed be written down, describing consistent interactions between gravity and the spin-5/2 field\cite{61}. However, when the theory was put on an AdS$_3$ background, it was soon found to suffer again from inconsistencies. In \cite{61}\cite{62} it was then discussed that consistency could be recovered if additional massless fields of spin-4 are included. In this section we shall be interested in the Chern Simons formulation of the theory. The gauge group for this theory is $\text{OSP}(1|4) \times \text{OSP}(1|4)$. The $\text{osp}(1|4)$ algebra is the hypersymmetric extension of the $\text{sp}(4)$ algebra\cite{27} and with $N = 1$ hypersymmetry in each factor. Under the principal embedding $\text{sl}(2) \rightarrow \text{sp}(4)$ we have a spin-1 multiplet $L_i$ that span the gravitational $\text{sl}(2)$, a spin-3 multiplet $U_m, m = 0, \pm 1, \pm 2, \pm 3$ and a spin-3/2 multiplet $S_p, p = \pm \frac{1}{2}, \pm \frac{3}{2}$. The spins of the corresponding bulk fields are thus 2 and 4, and 5/2.

Our conventions for the $\text{osp}(1|4)$ algebra are given in appendix D.2. Note that the matrix representations differ from those given in \cite{11}. This choice of basis will relate better to the Cartan Weyl basis of the $\text{osp}(1|4)$ algebra. The setup of this section will be as follows. In subsection 7.1 we will define black hole solutions in AdS$_3$ hypergravity and apply the extremality definition 4.2 to determine the different extremality classes. Then, in subsection 7.1.1 we will study their thermodynamics. Subsection 7.1.2 will finally study their hyper symmetries by applying the formalism discussed in 6.2 to these solutions. For completeness, we also study conical defects in subsection 7.2.

7.1 $\text{osp}(1|4)$ solutions

For the same reasons as in the previous section, we will adopt canonical boundary conditions, c.f. (4.73). After gauge fixing the radial component, the global charges are thus incorporated in the angular component of the connection which in highest weight gauge reads:

$$a_\phi = a_z + a_{\bar{z}} = L_1 - \mathcal{L} L_{-1} + U U_{-3}.$$  \hspace{1cm} (7.1)

A similar expression holds for the other sector $\bar{a}_\phi$, which we will omit in the following discussion. The global charges in the connection are related to the (eigenvalues of the ) zero modes of the currents in the CFT generating the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra on the plane as:

$$\mathcal{L} = \frac{h}{c} \left( h - \frac{c}{24} \right), \quad U = \frac{2}{c} \sqrt{\frac{21}{5}} q.$$  \hspace{1cm} (7.2)

Here $h$ is the zero mode of the stress energy tensor $T$ and $q$ is that of the spin-4 primary/current $U$. This map has been derived by explicit comparison of the asymptotic symmetry algebra of hypergravity to the semiclassical limit of the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra. We will discuss this in more detail at the end of this section when we make the transition to the CFT.

Extremality

\footnote{It was discussed in \cite{63} how the truncated tower of spins described by $\text{sl}(N)$ may be truncated further to include only even spins $s = 2, 4 \ldots 2n = N$. The resulting gauge algebra is in that case $\text{sp}(2n)$.}
As explained in section 4.6, the extremality classes for this black hole are determined by the roots of the determinant of its characteristic polynomial, c.f. (4.93). We thus compute the eigenvalues of \(\phi\) which are found to be:

\[
\lambda_{\pm} = \sqrt{5L \pm 2\sqrt{4L^2 - 15U}},
\]

with the ordering chosen as

\[
\lambda_\phi = [0, \lambda_+, \lambda_-, -\lambda_+, -\lambda_-].
\]

As before, \(\lambda_\phi\) denotes the matrix that contains the eigenvalues on its diagonal. The charges are then given in terms of the eigenvalues as

\[
L = \frac{\lambda_+^2 + \lambda_-^2}{10}, \quad U = -\frac{(\lambda_- - 3\lambda_+)(3\lambda_- - \lambda_+)(3\lambda_+ + \lambda_-)(\lambda_- + 3\lambda_+)}{6000},
\]

which are expressions symmetric in the eigenvalues. Note that there are four choices for the eigenvalues that accomplish a vanishing spin-4 charge. However, given our choice for the ordering of eigenvalues, (7.4), only one of these will accomplish a vanishing spin-4 chemical potential, and corresponds to the BTZ branch.

Evaluating the characteristic polynomial \(p(\lambda)\) we find it to be

\[
p(\lambda) = -\lambda(9L^2 + 60U - 10L\lambda^2 + \lambda^4)
\]

\[
= -\lambda(\lambda - \lambda_+)(\lambda + \lambda_+)(\lambda - \lambda_-)(\lambda + \lambda_-).
\]

The discriminant then becomes:

\[
\Delta \sim 110592 (4L^2 - 15U)^2 (3L^2 + 20U)^3
\]

\[
\sim 16(\lambda_+)^6(\lambda_-)^6(\lambda_-^2 - \lambda_+^2)^4.
\]

We can thus put a generic \(OSp(1|4)\) black hole, in one of the following different classes:

<table>
<thead>
<tr>
<th>Class</th>
<th>Eigenvalue condition</th>
<th>Extremal?</th>
<th>Global charges</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(\lambda_- = 0), (\lambda_+ \neq 0)</td>
<td>Yes</td>
<td>(U = -\frac{1}{15}L^2)</td>
</tr>
<tr>
<td>II</td>
<td>(\lambda_- = 0), (\lambda_+ = 0)</td>
<td>Yes</td>
<td>(U = L = 0)</td>
</tr>
<tr>
<td>III</td>
<td>(\lambda_- \neq 0), (\lambda_+ = 0)</td>
<td>Yes</td>
<td>(U = -\frac{1}{15}L^2)</td>
</tr>
<tr>
<td>IV</td>
<td>(\lambda_- = \lambda_+ \neq 0)</td>
<td>Yes</td>
<td>(U = \frac{4}{15}L^2)</td>
</tr>
<tr>
<td>V</td>
<td>(\lambda_- = -\lambda_+ \neq 0)</td>
<td>Yes</td>
<td>(U = \frac{4}{15}L^2)</td>
</tr>
<tr>
<td>VI</td>
<td>(\lambda_- \neq \lambda_+ \neq 0) or (\lambda_- \neq -\lambda_+ \neq 0)</td>
<td>No</td>
<td>Arbitrary</td>
</tr>
</tbody>
</table>

Table 2: The 6 classes of the \(OSp(1|4)\) theory.

Note that, when written in terms of the global charges, there is a priori no distinction between classes I and III and between classes IV and V. As we will see though, the entropy is sensitive to the corresponding choices for the eigenvalues.

7.1.1 Thermodynamics

Next we will discuss the thermodynamics of the spin-4 black hole by resorting to the Euclidean description, and letting the connections become complex valued with the constraint \(\bar{A} = -A^*\). Since we have included the charges as highest weight components of \(\phi\), the Euclidean solution will contain the sources in the lowest weight components of \(ia_{\mu} + a_\phi\). For this we write down the most general expression

\[
ia_{\mu} + a_\phi = \mu_2 L_1 + gL_0 + \nu U_3 + aU_2 + bU_1 + cU_0 + dU_{-1} + eU_{-2} + fU_{-3} + mL_{-1}.
\]

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The functions \{L, U, \mu_2, \nu, a, b, c, d, e, f, g, m\} are arbitrary and are allowed to depend on \(t, \phi\). They are fixed by the flatness condition \(F_{t\phi} = 0\). The resulting expressions are given in appendix B.1. Taking into account that at the end we must consider a stationary solution, and that for the Euclidean solution the spin-2 source can be incorporated in the modular parameter, we find:

\[
ia_{t\phi} + a_\phi = \nu \left[ U_3 - 3LU_1 - \left( \frac{11}{3} U + L^2 \right) LU_{-3} + (5U + 3L^2)U_{-1} + 3UL_{-1} \right]. \tag{7.9}
\]

Next we solve the smoothness condition for the Euclidean black hole. Similarly to the \(sl(3)\) example, demanding that the holonomy of the connection along the thermal cycle is trivial, fixes the chemical potentials in terms of the charges. However, some caution is needed here since the center of the bosonic subalgebra, \(so(1) \oplus sp(4)\), is

\[
\Gamma^\pm = \begin{pmatrix} 1 & 0 \\ 0 & \pm 4\times 4 \end{pmatrix}. \tag{7.10}
\]

The choice of sign in the \(sp(4)\) block determines whether the fermions satisfy periodic or anti-periodic boundary conditions. Since a contractable cycle allows only for the possibility of anti-periodic fermions we shall adopt:

\[
\mathcal{H}_r = \Gamma^- . \tag{7.11}
\]

Note that in particular \(exp(2\pi i L_0) = \Gamma^-\) which agrees with (3.45). If we would have instead used the second Cartan element \(U_0\), given our choice for the ordering of the eigenvalues, we would not be able to accommodate for the BTZ branch. Solving then the smoothness condition we find:

\[
\tau = i \left( \frac{\lambda_- + 3\lambda_+}{100\lambda_-\lambda_+ (\lambda_-^2 - \lambda_+^2)} \right) , \tag{7.12}
\]

\[
\gamma = i \left( \frac{3\lambda_- - 9\lambda_+}{10\lambda_-\lambda_+ (\lambda_-^2 - \lambda_+^2)} \right) , \tag{7.13}
\]

where we introduced the thermal source for the chemical potential, i.e. \(\nu = \frac{2\pi i}{\lambda_-}\) and we considered the non-rotating case, \(\bar{\tau} = -\tau\). As a non-trivial check, we see from (7.12) that all the classes identified as being extremal are indeed at zero temperature (\(\tau \to \infty\)). The BTZ branch is now recovered from \(\lambda_+ = 3\lambda_-\), yielding

\[
\mathcal{L} = \lambda_-^2, \quad U = 0, \tag{7.14}
\]

and accordingly

\[
\tau \to \frac{i}{2\lambda_-} = \frac{i}{2\sqrt{\mathcal{L}}}, \quad \gamma \to 0. \tag{7.15}
\]

The extremal BTZ belongs to class II for which \(\lambda_- = \lambda_+ = 0\).

Entropy and extremal limit

Next we turn to the entropy. Using (4.91) we find it to be:

\[
S = \frac{1}{5} \pi k (\lambda_- + 3\lambda_+) + \text{barred sector} \tag{7.16}
\]

\[
= \frac{\pi k}{\sqrt{5}} \sqrt{\mathcal{L}} \left[ \sqrt{1 - \frac{4}{5}} \sqrt{1 - \frac{15k}{4\mathcal{L}^2}} + 3 \sqrt{1 + \frac{4}{5}} \sqrt{1 - \frac{15k}{4\mathcal{L}^2}} \right] + \text{barred sector} \tag{7.17}
\]

where we used \(k_{es} = k/(2\pi Tr(L_0L_0)) = k/10\). On the BTZ branch, \(\lambda_+ = 3\lambda_-\), \(U \to 0\), the entropy reduces to

\[
S_{BTZ} = 2\pi k \sqrt{\mathcal{L}} + 2\pi k \sqrt{\mathcal{L}}. \tag{7.18}
\]

Note that this entropy vanishes in the extremal BTZ limit \(\mathcal{L} \to 0\). 

\[^6\text{The first corrections in } U \sim 0 \text{ are: } \frac{25\pi k L^2}{16\mathcal{L}^2} + O(\mathcal{L}^3).\]
Let us see next how the entropy behaves in the different extremality classes. As we noted above, the global charges are symmetric under interchange of the eigenvalues. Consequently, there is a priori no distinction between the extremality classes I \((\lambda_- = 0, \lambda_+ \neq 0)\) and III, \((\lambda_- \neq 0, \lambda_+ = 0)\). Note though, that the entropy as written in (7.16), is sensitive to this fact. The class with \(\lambda_+ = 0\) is slightly preferred over that with \(\lambda_- = 0\) as it has a lower entropy. When the entropy is written in terms of the global charges, this feature would not be visible. A similar thing occurs between extremality classes IV and V. Note also, that except for extremality class II, which contains the BTZ, the entropy from the extremal sector remains finite. We lastly note, that the maximum entropy is recovered in class IV. It would be interesting to explore whether the hypergravity theory allows for Hawking-page phase transitions along the lines of [64].

**Lorentzian signature**

All of these considerations took place in Euclidean signature. When we translate back to Lorentzian signature, we must demand that the global charges and chemical potentials are real so that the connection lies on the real form \(osp(1|4\mathbb{R})\). Reality of the potentials (7.12) requires the eigenvalues to be real.

**Extremality from the entropy**

The extremality classes that we have discussed, can be formulated as bounds on the spin-4 charge. For extremality classes I, II and III we have:

\[ U = -\frac{3}{20}L^2, \]

whereas for extremality classes IV and V we have:

\[ U = \frac{4}{15}L^2. \]

This second bound, was actually already derived in [11]. There it was interpreted as an extremality bound, the motivation being that it gives the maximal values of the global charges, beyond which the entropy (7.17) ceases to be real. The entropy however, is a derived quantity and therefore not be most natural object to use as a definition for extremality. Defining extremality in terms of the different Jordan classes of the holonomy on the other hand, leads to the same bound on the charges, i.e. extremality classes IV and V, and in particular has as a consequence that the entropy remains real valued. Put differently, we have shown that the above bound can indeed be interpreted as an extremality bound finding, on this point, agreement between both approaches to define extremality. Nonetheless, we have also seen, that when extremality is formulated in terms of the global charges only, one would not be able to distinguish between the classes I and III, and between IV and V. In that sense, we consider the extremality proposal of [10] as a more precise definition of extremality than that in [11].

### 7.1.2 Unbroken hyper symmetries: Killing spinors

In this section we will investigate the BPS properties of the extremal solutions identified in the previous section. We will use the techniques developed in subsection 6.2 and make use of the Cartan basis discussed in appendix D.2.

We may then express \(a_\phi^D\) as:

\[ a_\phi^D = \frac{3}{5} \lambda_+ \left( L_0 + \frac{U_0}{3} \right) + \frac{3}{5} \lambda_- \left( \frac{L_0}{3} - U_0 \right) \]

\[ = -\lambda_+ E_{42} - \lambda_- E_{33} = \lambda_+ H_1 + \lambda_- H_2. \]
Then using \([a^D_\phi, E_{IJ}] = \omega_{IJ} E_{IJ}\) we can extract the odd frequencies as:

\[
\begin{align*}
\omega_{12} &= \lambda_+ , \\
\omega_{14} &= -\lambda_+ , \\
\omega_{13} &= \lambda_- , \\
\omega_{15} &= -\lambda_- .
\end{align*}
\] (7.22) (7.23)

We write the killing spinor as in (6.13)

\[
\varepsilon (\phi) = \sum_{ij} \varepsilon_{ij} E_{ij} e^{\omega_{ij} \phi} = \sum_j \varepsilon_j E_{1j} e^{\omega_j \phi} ,
\] (7.24)

upon which we have to impose \([a^N_\phi, \varepsilon_0] = 0\), with \(\varepsilon_0 = \varepsilon_{ij} E_{ij}\). As discussed in subsection 6.2.2, when the killing spinor is found, we will present these in the language of the asymptotic symmetries:

\[
\varepsilon (\phi) = \varepsilon_3 S_3 + \varepsilon_+ S_+ + \varepsilon_- S_- + \varepsilon_5 S_5 ,
\] (7.25)

Demanding that under such a fermionic gauge transformation \(\delta a_\phi = \partial \phi \varepsilon + [a_\phi, \varepsilon] = 0\), (7.26)

we find that this is the most general killing spinor provided the parameters are fixed algebraically in terms of \(\varepsilon_{\frac{3}{2}}\) as:

\[
\begin{align*}
\varepsilon_{\frac{1}{2}} &= -\partial \phi \varepsilon_{\frac{3}{2}} , \\
\varepsilon_{-\frac{1}{2}} &= -\frac{1}{2} (\partial^2 \varepsilon_{\frac{3}{2}} + 3 \mathcal{L} \varepsilon_{\frac{3}{2}}) , \\
\varepsilon_{-\frac{3}{2}} &= -\frac{1}{6} \left( \partial^3 \varepsilon_{\frac{3}{2}} - 7 \mathcal{L} \partial \phi \varepsilon_{\frac{3}{2}} \right) .
\end{align*}
\] (7.27)

and with \(\varepsilon_{\frac{3}{2}}\) satisfying the following differential equation:

\[
\partial^4 \varepsilon_{\frac{3}{2}} = -3 (3 \mathcal{L}^2 + 20 \mathcal{U}) \varepsilon_{\frac{3}{2}} + 10 \mathcal{L} \partial^2 \varepsilon_{\frac{3}{2}} .
\] (7.28)

Next we will go through the classes of black hole solutions and classify their BPS properties.

**Class I: \(\lambda_- = 0, \lambda_+ \neq 0\)**

In this class we have

\[
a^D_\phi = -\lambda_+ E_{42} , \quad a^N_\phi = e_{35} = -\frac{1}{2} E_{55} .
\] (7.29)

The charges in this class are

\[
\mathcal{L} = \frac{\lambda^2_+}{10} , \quad \mathcal{U} = -\frac{3 \lambda^4_+}{2000} .
\] (7.30)

The condition \([a^N_\phi, \varepsilon_0] = 0\) then becomes using the commutation relation

\[
[E_{jk}, E_i] = G_{ij} E_k + G_{ik} E_j ,
\] (7.31)

which translates into:

\[
[E_{55}, \sum_i \varepsilon_i E_i] = 2 \varepsilon_3 E_5 = 0 \implies \varepsilon_3 = 0 .
\] (7.32)

We thus have

\[
\varepsilon (\phi) = \varepsilon_2 E_2 e^{\omega_2 \phi} + \varepsilon_4 E_4 e^{\omega_4 \phi} + \varepsilon_5 E_5 ,
\] (7.33)

However, due to the reality constraint on the eigenvalues it is impossible to satisfy the quantization condition on the frequencies. Therefore we need to set \(\varepsilon_2 = \varepsilon_4 = 0\) as well and thus:

\[
\varepsilon (\phi) = \varepsilon_5 E_5 .
\] (7.34)
Since the killing spinor possesses a single independent parameter, out of a total of four, this solution is 1/4 BPS. After undoing the similarity transformation, exchanging the coefficient $\varepsilon_5$ for $\varepsilon_2$ and casting the generators as in (7.25) we find that the killing spinor reads:

$$\epsilon(\phi) = \varepsilon_2 \frac{s}{2} + \varepsilon_\frac{1}{2} \frac{s}{2}.$$  \hspace{1cm} (7.35)

**Class II:** $\lambda_- = \lambda_+ = 0$

In this class we have:

$$a_\phi^D = 0 \quad a_\phi^N = 2\varepsilon_3 + \sqrt{3}(\varepsilon_3 - \varepsilon_4) = E_3 - \sqrt{3}E_5.$$  \hspace{1cm} (7.36)

The charges in this class are

$$L = 0, \quad U = 0.$$  \hspace{1cm} (7.37)

From $[a_\phi^N, \varepsilon_0] = 0$ we find that:

$$-2\varepsilon_2 E_3 - \sqrt{3}(\varepsilon_3 E_2 - \varepsilon_4 E_5) = 0,$$

implying $\varepsilon_5 = \varepsilon_3 = \varepsilon_4 = 0$. The killing spinor is thus:

$$\epsilon(\phi) = \varepsilon_2 E_2.$$  \hspace{1cm} (7.39)

Solutions in this class preserve again 1/4 hypersymmetry. In terms of the asymptotic generators, the (7.25) becomes

$$\epsilon(\phi) = \varepsilon_2 \frac{s}{2}.$$  \hspace{1cm} (7.40)

**Class III:** $\lambda_- \neq 0, \lambda_+ = 0$

This class is very similar to class I.

$$a_\phi^D = \lambda_- E_{53} \quad a_\phi^N = -\frac{1}{2} E_{44}.$$  \hspace{1cm} (7.41)

The charges in this class are

$$L = \frac{\lambda_-^2}{10}, \quad U = -\frac{3\lambda_+^4}{2000}.$$  \hspace{1cm} (7.42)

The condition $[a_\phi^N, \varepsilon_0] = 0$ then translates into:

$$[E_{44}, \sum \varepsilon_i E_i] = 2\varepsilon_2 E_4 = 0 \implies \varepsilon_2 = 0.$$  \hspace{1cm} (7.43)

Leaving us with:

$$\epsilon(\phi) = \varepsilon_3 E_3 e^{\omega_{13} \phi} + \varepsilon_4 E_4 + \varepsilon_5 E_5 e^{\omega_{15} \phi}.$$  \hspace{1cm} (7.44)

Reality of the eigenvalues again forces us to set $\varepsilon_3 = \varepsilon_5 = 0$ as well and the killing spinor becomes:

$$\epsilon(\phi) = \varepsilon_4 E_4.$$  \hspace{1cm} (7.45)

Once again the background preserves 1/4 hypersymmetry. Undoing the similarity transformation, exchanging the coefficient $\varepsilon_4$ for $\varepsilon_2$ and casting the generators as in (7.25) we find that the killing spinor reads:

$$\epsilon(\phi) = \varepsilon_2 \frac{s}{2} + \varepsilon_\frac{1}{2} \frac{s}{2}.$$  \hspace{1cm} (7.46)

**Class IV, V, VI:**

In these classes all the eigenvalues are non zero and the killing spinors therefore become identically zero.

---

\textsuperscript{62}Via a similarity transformation $a_\phi^N$ may be brought to the form $a_\phi^N = \varepsilon_2$ + $\varepsilon_3$ + $\varepsilon_4$ + $\varepsilon_5$. However, with the purpose of expressing $a_\phi^N$ in terms of the Cartan basis, it is more convenient to express it in the above form.
since for none of the four exponentials the quantization condition can be satisfied. The charges in classes IV and V read:

\[ \mathcal{L} = \frac{\lambda_+^2}{5}, \quad \mathcal{U} = \frac{4\lambda_+}{375}, \]

whilst in class VI we have:

\[ \mathcal{L} = \frac{\lambda_+^2 + \lambda_-^2}{10}, \quad \mathcal{U} = -\frac{1}{15} \left( \frac{(\lambda_+^2 - \lambda_-^2)^2}{16} - \frac{(\lambda_+^2 + \lambda_-^2)^2}{25} \right). \] (7.48)

### 7.1.3 Supersymmetry Vs Extremality.

The hyper-symmetries of the classes discussed in the previous subsection are summarised in the table below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Eigenvalue condition</th>
<th>Extremal?</th>
<th>Extremal charges</th>
<th>BPS?</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( \lambda_- = 0 ), ( \lambda_+ \neq 0 )</td>
<td>Yes</td>
<td>( \mathcal{U} = -3\mathcal{L}^2/20 )</td>
<td>( 1/4 )</td>
</tr>
<tr>
<td>II</td>
<td>( \lambda_- = 0 ), ( \lambda_+ = 0 )</td>
<td>Yes</td>
<td>( \mathcal{L} = \mathcal{U} = 0 )</td>
<td>( 1/4 )</td>
</tr>
<tr>
<td>III</td>
<td>( \lambda_- \neq 0 ), ( \lambda_+ = 0 )</td>
<td>Yes</td>
<td>( \mathcal{U} = -3\mathcal{L}^2/20 )</td>
<td>( 1/4 )</td>
</tr>
<tr>
<td>IV</td>
<td>( \lambda_- = \lambda_+ \neq 0 )</td>
<td>Yes</td>
<td>( \mathcal{U} = 4\mathcal{L}^2/15 )</td>
<td>No</td>
</tr>
<tr>
<td>V</td>
<td>( \lambda_- = -\lambda_+ \neq 0 )</td>
<td>Yes</td>
<td>( \mathcal{U} = 4\mathcal{L}^2/15 )</td>
<td>No</td>
</tr>
<tr>
<td>VI</td>
<td>( \lambda_- \neq \lambda_+ \neq 0 ) or ( \lambda_- \neq -\lambda_+ \neq 0 )</td>
<td>No</td>
<td>None</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 3: The 6 classes of the \( OSP(1|4) \) theory and their hypersymmetries.

In terms of the eigenvalues of the connection \( a_\phi \) we can group the extremal black holes into two families. The first one is BPS and it is accessible to classes I, II and III for which

\[ \lambda_+ = 0 \quad \text{and / or} \quad \lambda_- = 0. \] (7.49)

The solutions in this family are \( 1/4 \) BPS and are always extremal:

\[ \mathcal{U} = -\frac{3}{20} \mathcal{L}^2 \implies \Delta = 0. \] (7.50)

BPS in spin-4 hypergravity thus always implies extremality. The converse is not true however, since black holes in classes IV and V are extremal but not BPS. Their eigenvalues and charges satisfy

\[ \lambda = \pm \lambda_- \neq 0, \quad \mathcal{U} = \frac{4}{15} \mathcal{L}^2 \] (7.51)

leading to no hypersymmetries.

### 7.2 Conical defects

Having discussed black holes and their hyper symmetries, we will now pay attention to conical defects. In this case we must demand a trivial holonomy along the spatial cycle:

\[ \mathcal{H}_\phi = 1^\pm. \] (7.52)

Since the topology of conical defect backgrounds is the same as that for AdS\(_3\), for which the \( \phi \) cycle is contractable, smoothness of the fermionic fields at the origin of AdS\(_3\) would require \( \mathcal{H}_\phi = 1^- \). For the purpose of comparing to the dual CFT in the next section though, we will allow for also the possibility of periodic fermions. The above holonomy condition immediately implies that that the eigenvalues are purely imaginary.

\[ \lambda_+ = \pm in_1, \quad \lambda_- = \pm in_2. \] (7.53)
The parameters $n_1, n_2$ label whether the solution supports periodic or anti-periodic fermions. If $n_1, n_2 \in \mathbb{Z}$ we have $\mathcal{H}_\phi = \Gamma^+$, whereas for $n_1, n_2 \in \mathbb{Z} + \frac{1}{2}$ we have that $\mathcal{H}_\phi = \Gamma^-$. The global charges in terms of these parameters read:

$$\mathcal{L} = -\frac{n_1^2 + n_2^2}{10}, \quad \mathcal{U} = -\frac{9n_1^4 + 82n_1^2n_2^2 - 9n_2^4}{6000}. \quad (7.54)$$

To determine which values $n_1$ and $n_2$ can take, we use that conical defects as solutions for which

$$-\frac{1}{4} < \mathcal{L} < 0. \quad (7.55)$$

This constrains $n_1, n_2$, to satisfy:

$$n_1^2 + n_2^2 < \frac{5}{2}. \quad (7.56)$$

This allows for three different configurations listed in the table below:

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$\mathcal{L}$</th>
<th>$\mathcal{U}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$-1/5$</td>
<td>$4/375$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-1/10$</td>
<td>$-3/2000$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$-1/20$</td>
<td>$1/1500$</td>
</tr>
</tbody>
</table>

Table 4: The three different allows configurations for conical defects in AdS$_3$ hypergravity.

In the limiting case of $n_1 = 3/2, n_2 = 1/2$ we recover global AdS for which $\mathcal{L} = -1/4, \mathcal{U} = 0$. Let us now analyze the supersymmetries of these four configurations. Since the smoothness condition implies that the frequencies are automatically quantized, these solutions are always maximally hyper-symmetric preserving all four hyper symmetries. The killing spinor is simply given by (7.25) and (7.27). Note though, that the solution with $n_1 = n_2 = 1$ does not saturate the lower BPS bound. This is similar to what happens in $\mathcal{N} = 1, (2 + 1)$-supersymmetry\[25\], where they identified conical defect solutions which are supersymmetric, but nonetheless do not saturate the BPS bounds. The anti-periodic configuration, as we will discuss, saturates a BPS bound in the NS sector.

### 7.3 The asymptotic symmetries of hypergravity

In anticipation of the next section where we will discuss unitarity of the dual CFT of the hypergravity theory, we will now discuss the asymptotic symmetries of the hypergravity theory. The asymptotic symmetry algebra was computed in [11] where they showed that it corresponds to the semiclassical limit of the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra\[63\]. An important fact here, is that this $\mathcal{W}$ algebra is of the generic type\[64\] i.e. defined for all values of the central charge, which allows one to take a semiclassical limit. As already briefly mentioned in the introduction of this section, by comparing this asymptotic symmetry algebra to the semiclassical limit of the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra we have found the following map between the global charges and the CFT charges on the plane:

$$\mathcal{L} = \frac{6}{c} \left( h - \frac{c}{24} \right), \quad \mathcal{U} = \frac{2}{c} \sqrt{\frac{21}{5}} q. \quad (7.57)$$

This map we will use in the next section to relate the semiclassical unitarity bounds of the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra to the results discussed in this section. Before we conclude this chapter, there is one last interesting feature we would like to point out. Namely, recall from the introduction that the necessity for the spin-4 field arose when putting the theory on an AdS$_3$ background. Thus, one would expect that, in the limit of infinite AdS radius $l \to \infty$ one can consistently get rid of the spin-4 field in the asymptotic algebra. Indeed

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63 We checked explicitly that the given expressions are correct.

64 See also appendix E.4
this turns out to be the case. By making an appropriate rescaling of the asymptotic generators\textsuperscript{65}, similar to what we discussed before, and sending $l \to \infty$, one can consistently decouple the spin-4 field from the asymptotic algebra, to recover the theory of \textsuperscript{61}. In the CFT a similar thing can be accomplished. The spin-4 self coupling constant, vanishes for $c = -13/14$, in which case one obtains the $W(2, \frac{5}{2})$ algebra\textsuperscript{65}. This algebra however, is of the exotic type, i.e. admits no large-$c$ limit.

\textsuperscript{65}See also appendix (E.3.4)
8 $\mathcal{W}(2, \frac{5}{2}, 4)$ unitarity bounds

In this section we will discuss semiclassical unitarity for the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra. As we have discussed, the hypergravity theory allows for two extremality bounds on the spin-4 charges. In this section we will discuss how, a similar bounds is dictated by the CFT as a semiclassical unitarity bound.

The reader is referred to appendix E for general background on 2d conformal field theory, $\mathcal{W}$ algebras as well as properties of the Kac-determinant. In particular we discuss there how $\mathcal{W}$ algebras can be constructed. The OPE’s and current algebra of $\mathcal{W}(2, \frac{5}{2}, 4)$ can be found in appendix F. The Neveu-Schwarz sector will be discussed in subsection 8.1 and the Ramond sector in subsection 8.2.

Hermiticity conditions

Before we discuss the highest weight representation of the Neveu Schwarz and Ramond sector, we need a prescription for Hermitian conjugation that will define us an inner product on the Verma module. For this we must demand that the OPE relations on the plane given in appendix F, are invariant under Hermitian conjugation. This implies the following hermiticity conditions:

\[(Q_r)^\dagger = Q_{-r}, \quad (L_n)^\dagger = L_{-n}, \quad (U_n)^\dagger = \epsilon U_{-n}.\] (8.1)

Here $Q_r$ is the mode of the spin-5/2 current, $L_n$ of the spin-2 stress energy tensor, $U_n$ that of a spin 4 current and $\epsilon$ is defined as:

\[\epsilon = \begin{cases} -1 & \text{if } \kappa \text{ is imaginary } \left(-\frac{22}{5} < c < -\frac{13}{14}\right), \\ +1 & \text{if } \kappa \in \mathbb{R}.\end{cases}\] (8.2)

Here $\kappa$ defines the coupling constant of the higher spin multiplet to itself, i.e. for $QQ \sim U$, $UQ \sim U$ and $UU \sim U$. It is defined as

\[\kappa = \frac{1}{\sqrt{(14c + 13)(5c + 22)}}.\] (8.3)

Note in particular that $\kappa \in \mathbb{R}$ in the semiclassical limit $c \to \infty$, in which case all operators satisfy the standard hermiticity properties on the plane.

Approach: Roots of the Kac determinant

When the matrix of inner products of states, known as the Gram matrix, is evaluated, the problem of determining unitarity bounds comes down to requiring this matrix to be positive definite, which ensures that all of its eigenvectors (eigenstates) have positive eigenvalues (norm). The common approach is then to require its determinant, i.e. the Kac determinant, to be positive.

For this, it is convenient to write the Kac determinant in terms of its roots, which we will denote as

\[K(h, q, c) = \prod_i \left(q - f_i(h, c)\right).\] (8.4)

Written like this, one can then determine a region where the Kac determinant is manifestly positive. Saturation of a condition $q = f(c, h)$ corresponds to a zero of the Kac determinant, i.e. a null state in the Verma module, and the number of roots is in one-to-one correspondence to the number of null states in the Verma module. Consequently, this tells us that, as such a "plane $q = f(c, h)$" of vanishing determinant is crossed, the Kac determinant picks up a minus sign, and it will allow us to determine the regions where

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66 We will formulate the unitarity bounds as bounds on the spin-4 charge $q$, for simply the reason that this parameter appears with the lowest order in the determinant.
the Kac determinant is positive.

**Quantum Vs Semiclassical**

In this work we shall restrict ourselves to a semiclassical analysis, which we recall entails:

\[ h \rightarrow h/\hbar, \quad q \rightarrow q/\hbar, \quad c \rightarrow c/\hbar, \quad \hbar \rightarrow 0 \]  \hspace{1cm} (8.5)

The motivation for this is two fold:

- From the point of view of the CFT one is of course most interested in unitarity bounds at finite-\(c\). However, as can already be seen from the matrix elements that we will give and the structure constants of the \(W\) algebra at finite-\(c\), given the approach pursued here, this is an impossible task. Deriving unitarity bounds at finite-\(c\), using numerical methods is in principle not too hard. However, already above level 1 one will encounter the first problems. The complexity of the matrix elements combined with the dimension of the Verma module make it impossible to be even able to compute the determinant. Furthermore, the unitarity structure at finite-\(c\) differs from the semiclassical structure in the sense that at finite-\(c\) one usually has to go to higher levels to acquire the full set of unitarity constraints. For example, in the case of the \(\mathcal{N} = 2\) super-Virasoro algebra, all information is in the semiclassical limit captured by the lowest level BPS bound, which at finite-\(c\) is certainly not the case\cite{10}.

- Since we shall be interested in a comparison of the unitarity bounds to the bulk extremality bounds, semiclassical unitarity bounds will suffice for our purposes.

### 8.1 Neveu-Schwarz sector

**Highest weight representation**

In the Neveu-Schwarz the zero mode algebra is spanned by \(L_0\) and \(U_0\). They mutually commute and so we can label a highest weight state by their simultaneous eigenvalues. We will denote the highest weight state by \(|h, q\rangle\). As usual it satisfies:

\[ L_0 |h, q\rangle = h |h, q\rangle, \quad U_0 |h, q\rangle = q |h, q\rangle, \]  \hspace{1cm} (8.6)

and with the additional condition:

\[ L_n |h, q\rangle = U_n |h, q\rangle = 0, \quad n > 0, \]  \hspace{1cm} (8.7)

\[ Q_r |h, q\rangle = 0, \quad r > 0. \]
At level-1/2 there is one fermionic descendant.

\(|\alpha\rangle \equiv Q_{-1/2} |h,q\rangle. \tag{8.8}\)

Unitarity then results in a level-1/2 BPS bound:

\[-\frac{1}{5}h + \frac{27}{5c + 22} : \Lambda : \geq 0 + \sqrt{\frac{6(14c + 13)}{5c + 22}} q \geq 0. \tag{8.9}\]

where we used (F.15). In the semiclassical limit, (8.5), this becomes:

\[-\frac{1}{5}h + \frac{27}{5c} h^2 + 2 \sqrt{\frac{21}{5}} q \geq 0. \tag{8.10}\]

The NS vacuum is \(|0\rangle\), for which \(Q_{-1/2} |0\rangle_{NS} = 0\). Its dual is global AdS3. The NS vacuum is invariant under the global part of the \(\mathcal{W}\) algebra, i.e. under:

\[\{L_0, L_{\pm 1}, U_0, U_{\pm 1}, U_{\pm 2}, U_{\pm 3}, Q_{\pm 1/2}, Q_{\pm 3/2}\}\]

which spans the \(\mathfrak{osp}(1|4)\) algebra. In subsection 8.2 we will derive a similar BPS bound in the Ramond sector. As we shall see, in the Ramond sector, the BPS bound does not allow for positively charged BPS state, in contrast to the bound (8.9). The maximum value of the spin-4 charge is attained for

\[h = \frac{1}{54}(c - 1), \tag{8.11}\]

in which case the spin-4 charge becomes:

\[q = \frac{(c - 1)^2}{108\sqrt{6}\sqrt{5c + 22}\sqrt{14c + 13}}. \tag{8.12}\]

Note though, that there is a priori no assumption or motivation why the spin-4 charge should be positive. Before moving on to level-1 we note that the BPS state with

\[\left(\frac{24h}{c}, \frac{\sqrt{105}q}{c}\right) = \left(\frac{4}{5}, \frac{1}{600}\right)\]

is dual to the anti-periodic conical defect with \(n_1 = n_2 = 1/2\) we identified in the bulk as being hypersymmetric. See table 4.

A basis for the level-1 Verma module is spanned by the states:

\[|\delta\rangle \equiv Q_{-1/2}Q_{-1/2} |h,q\rangle, \quad |\beta\rangle \equiv L_{-1} |h,q\rangle, \quad |\gamma\rangle \equiv U_{-1} |h,q\rangle. \tag{8.14}\]
Upon explicit computation, using the commutator algebra (F.12), the matrix elements are found to be:

\[
\langle \beta | \beta \rangle = 2h, \\
\langle \gamma | \beta \rangle = 4q, \\
\langle \delta | \beta \rangle = 2 \langle \alpha | \alpha \rangle, \\
\langle \delta | \delta \rangle = \left(- \langle \alpha | \alpha \rangle - \frac{1}{5} (h + \frac{1}{2}) + \frac{27}{5c + 22} \left( \frac{1}{5} (h + \frac{1}{2}) + (h + \frac{1}{2})^2 \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} \right) \langle \alpha | \alpha \rangle, \\
\langle \gamma | \delta \rangle = \left( 4a - 18b + 48d + 60f + g(-2 - 2h) + (2 + h)(4c - 5h_{\text{algebra}}) \right) \langle \alpha | \alpha \rangle, \\
\langle \gamma | \gamma \rangle = \left( \frac{2 \cdot 24}{1680} \left( 36 \cdot \frac{D}{12} - 20 \cdot E \right) : \Lambda \cdot \omega_3 - 2 \left( 36 \cdot \frac{H}{12} - 20f \right) U_0 + 2 \left( F : \Xi_0 + G : \Delta \cdot \omega_3 + J : \Omega \cdot \omega_3 + K : \Gamma \omega_3 \right) \right),
\]

with \(\langle \alpha | \alpha \rangle\) given by the NS level-1/2 matrix element (8.9). Here \(\langle \ldots \rangle\) denotes the expectation value in the highest weight representation. Normal ordering of composite operators is denoted by \(\ldots\). Our normal ordering prescription is defined in (E.22). The action of the zero modes of the normal ordered composite operators are given in (F.25) and (F.28) and (F.29). All coefficients are given in appendix F. Matrix elements related to the above by Hermitian conjugation, we have not explicitly written. Note that we have written a subscript with the OPE coefficient \(h_{\text{algebra}}\), to distinguish it from the conformal weight \(h\).

The semiclassical Kac determinant at level-1

As explained above, we will now focus on the semiclassical Kac determinant, in which case all expressions become more tractable. Furthermore, the fact that the matrix elements that involve \(\langle \delta | \delta \rangle\) are all proportional to the level-1/2 bound \(\langle \alpha | \alpha \rangle\), means that we can already factor out a first root. The remaining portion of the Kac determinant is now second order in \(q\) and takes on a particularly nice form. Explicitly one finds:

\[
K^{NS}_{1}(q, h, c) = \frac{4(30h - c)(3c^2h^2 - 2\sqrt{105}c^2hq - 840c^2q^2 - 177ch^3 + 84\sqrt{105}ch^2q + 2592h^4)}{105c^3}
\]

\[
= \frac{32}{c}(c - 30h)(q - f_1(h, c))(q - f_1(h, c))
\]

where

\[
f_1(h, c) = \sqrt{105}ch - 27\sqrt{105}h^2, \quad f_1(h, c) = \frac{32\sqrt{105}h^2 - \sqrt{105}ch}{140c}
\]

(8.16)

(8.17)

The subscript \(i\) denotes here the level where the null state corresponding to \(q = f_i(h, c)\) first appear. The first of these we recognize once again as the level-1/2 BPS bound (8.10). Coupled to the BPS bound \(q \geq f_\frac{1}{2}(h, c)\), one then observes that positivity of the Kac determinant in the region \(h \geq \frac{c}{30}\) requires:

\[
q \leq f_1(h, c) = \frac{32\sqrt{105}h^2 - \sqrt{105}ch}{140c}
\]

which defines an upper bound on the spin-4 charge. The points of intersection of these two bounds are located at:

\[
\left( \frac{24h}{c}, \frac{\sqrt{105}q}{c} \right) = (0, 0), \quad \text{and} \quad \left( \frac{24h}{c}, \frac{\sqrt{105}q}{c} \right) = \left( 4, \frac{1}{600} \right).
\]

The first of these is nothing but the ground state of the NS sector, dual to global AdS_5. The second state is once again dual to the NS conical defect. Note that the multiplicative factor of \((c - 30h)\), also intersects this point. The regions that define a positive Kac determinant are shown in fig. [I]. The blue line defines the level-1/2 BPS bound, whereas the red line shows the upper bound in (8.18). The regions that define a
positive Kac determinant are shaded in light-blue. The rightmost region defines our region of asymptotic unitarity. The left unitarity region, arises due to a double crossing of zero’s of the Kac determinant.

![Figure 1](image-url)

Figure 1: The region that defines a positive Kac determinant, is shaded in light blue. The blue line defines the level-1/2 BPS bound, whereas the red line shows the upper bound in (8.18). The regions that define a positive Kac determinant are shaded in light-blue. The rightmost region defines our region of asymptotic unitarity. The left unitarity region, is due to the crossing of the zero $h = \frac{c}{30}$ of the Kac determinant. The loci where the bounds cross correspond to the NS vacuum, dual to global AdS$_3$ and the state $|\frac{24h}{c} = \frac{1}{4}, \frac{\sqrt{105}q}{c} = \frac{1}{600}\rangle$ dual to the NS conical defect.

**Level 3/2**

At level-3/2, the Verma module is spanned by the following set of states:

$$\begin{align*}
|\beta\rangle &= Q_{-1/2} L_{-1} |h, q\rangle , \\
|\delta\rangle &= Q_{-1/2} Q_{-1/2} Q_{-1/2} |h, q\rangle , \\
|\gamma\rangle &= U_{-1} Q_{-1/2} |h, q\rangle , \\
|\chi\rangle &= Q_{-3/2} |h, q\rangle .
\end{align*}$$

(8.21)

Note how we have ordered our states. We have assumed the conventions of [74], for the lexicographic ordering of the higher spin operators:

$$W^{(s_{i_1})}_{-m_1-x_{i_2}} \ldots W^{(s_{i_n})}_{-m_n-x_{i_n}} |h,q\rangle ,$$

(8.22)

with $s_{i_j} \geq s_{i_{j+1}}$, $i_j = i_{j+1} \implies m_j \geq m_{j+1}$ and $m_j \geq 0$.

Due to the increased length of the explicit expressions for the matrix elements, they have been included in appendix H.1. The Kac determinant is in this case rather more lengthy and the explicit expression hardly illuminating. One finds:

$$K^{NS}_{3/2} (q, h, c) = c^7 (c - 27h)^2 (q - f_2 (h, c))^3 (f_1 (h, c) - q)(q - f_2 (h, c))^2$$

(8.23)

with $f_2 (h, c)$ and $f_1 (h, c)$ given in (8.18). The newly appearing root at this level is given by:

$$f_2 (h, c) = - \frac{3(\sqrt{105} ch + 3\sqrt{105} h^2)}{70c} .$$

(8.24)

---

$^{68}$We have omitted numerical pre factors.
Note that (8.24) is fully subdominant to the BPS bound for $\forall h > 0$ indicating that in the semiclassical limit the BPS bound is the dominant lower bound on the spin-4 charge. Nonetheless, level-3/2 provides us with no information concerning the unitarity regions identified at level-1, since the remaining pre-factor $(c - 27h)$ appears squared. This newly appearing bound is plotted in fig 2. Before we move on to level-2 it will prove useful to take a careful look at the mode algebra. As we will discuss, there are a few very useful recursions taking place, that give some insight in the structure that is at play.

**Intermezzo: Recursions from the mode algebra.**

At this point it will prove useful to take a closer look at the commutator algebra because it will tell us which operators where responsible for the derived unitarity bounds. We emphasize though, that the recursions discussed here hold only semi-classically. We start by noting that from the Kac determinant constructed from solely the operators $L_{-n}$ and $U_{-n}$ one can derive two null conditions.

\[
q = -\frac{(-1 + h + n^2)(-9 + 9h + n^2)}{384} \\
q = \frac{(-1 + h + n^2)(-4 + 4h + n^2)}{96}
\]  

(8.25)

For convenience of displaying the expressions I have written these expressions in terms of $h' = \frac{24h}{c}$, $q' = \sqrt{\frac{105}{c}}$. I will refer to these null conditions as the $LU$ ‘lower’ and ‘upper’ null conditions respectively. The next step now consists of noticing that for the $LU$ lower null conditions we have the following recursion within the mode algebra:

\[
L_{-n} + aU_{-n} \leftrightarrow Q_{-\frac{n}{2}}
\]

(8.26)

Even more so, one can show that also from the commutator $[U_{n}, Q_{-n/2}]$ accessible in the NS sector for odd $n$, that this is related to the upper condition from the $L_{-n}, U_{-n}$ Gram matrix:

\[
\frac{n (4h + n^2 - 4)}{c}
\]

(8.27)
This explains why the Kac determinant at level-1 took such a nice form. Because all the entries where related to each other via the above recursion relations, all terms in the Kac determinant combined in a rather nice way, so as to leave us at level-1 with only the $L_{n-1}, U_{n-1}$ upper condition as a new bound. On the contrary, at level-3/2, one can check explicitly that this root arises from the fermionic operator $Q_{-3/2}$.

Given then these considerations, we can form somewhat of a suspicion for the bounds that will appear at higher levels. At the half integer levels $r$ we expect that the bounds that arise are those from states $Q_{-r}$ which rapidly become subdominant to the BPS bound. Similarly, at the odd integer levels we might expect that the new unitarity bound to be a corresponding upper bound from the operators $L_{-n}$ and $U_{-n}$, c.f. (8.25). We emphasize here that we expect this for the odd levels, since for even $n$, the recursion relations hold for fermionic operators that belong to the Ramond sector, and there is thus a priori no reason why even levels should follow this pattern. In particular, if this feature of the mode algebra will persist at odd integer levels, we expect that, as we go beyond level-2, no more stringent bounds will arise, as the upper bounds at odd integer levels will follow this pattern and the most dominant contribution arises for $n = 1$. One should note here though, that this argument/suspicion is far from rigorous. Although it seems plausible that the structure of the bounds will be the same at similar levels, one should keep in mind that we are drawing conclusions focusing only on a subset of operators. Unfortunately, as explained in the next paragraph where we discuss level-2, given the approach taken in this thesis, explicit analysis from the Kac determinant becomes hardly possible at increasing levels, primarily due to computational reasons.

It would be very interesting to determine whether this pattern for the odd integer levels will indeed be persistent, or whether it is only a nice coincidental feature of level-1. For this, a different (representation theoretical) technique will have to be used, which was not addressed in this thesis. We will come back in somewhat more detail to the recursion relations just described, in the discussion. In particular we will discuss there that the bulk extremality bound is related to the semiclassical mode algebra in a rather surprising way.

### Level-2

A basis for the Verma module at level-2 is spanned by the states:

\[
|\beta\rangle = L_{-1}L_{-1}|h, q\rangle, \quad |\delta\rangle = L_{-2}|h, q\rangle, \quad |\kappa\rangle = Q_{-1/2}Q_{-1/2}L_{-1}|h, q\rangle, \\
|\gamma\rangle = U_{-1}U_{-1}|h, q\rangle, \quad |\chi\rangle = U_{-2}|h, q\rangle, \quad |\xi\rangle = U_{-1}Q_{-1/2}Q_{-1/2}|h, q\rangle, \\
|\rho\rangle = U_{-1}L_{-1}|h, q\rangle, \quad |\sigma\rangle = Q_{-3/2}Q_{-3/2}|h, q\rangle, \quad |\phi\rangle = Q_{-1/2}Q_{-1/2}Q_{-1/2}|h, q\rangle.
\]

(8.28)

At this level however, all expressions become too long to be displayed properly, even in an appendix, and I will therefore refrain from citing them explicitly. Furthermore, compared to the lower levels, the dimension of the problem has clearly increased significantly and we hit somewhat of a problem at this point. The Kac determinant, becomes in this case far more complex then at the lower levels and naive computational manipulations, e.g. determining the roots, are not of use. For that reason we shall have to be more pragmatic. Instead of searching for the explicit bounds dictated by the Kac determinant, we will probe whether this level will contain more stringent information or not. The most brute force way to do this, is to explicitly evaluate the Kac determinant at points inside the regions of interest, i.e. the blue shaded regions in fig. [1]. The rationale behind this is as follows. If more stringent bounds will appear at level-2, then that means that the level-2 Kac determinant will be negative in regions inside the shaded areas in fig. [1]. If though we can be certain, that the Kac determinant is positive everywhere inside such a region, then this tells us that no more stringent bounds will arise at this level. One remark with this approach should be made though. As explained above, at level-2 the Kac determinant became too large for explicit analysis. One should keep in mind though, that this method used here has an actual meaning, when one knows that the Kac determinant one is working with is the correct Kac determinant. In this case however, it was found to be hardly possible to check with absolute certainty that this was the case. Its not too hard to see that the BPS bound forms again a zero of the Kac determinant, by observing that...
e.g. all matrix elements including $|\phi\rangle$ are proportional to the BPS bound. A similar quick argument could not be used for the upper unitarity bound however, and even more so, Mathematica has not been able to do a full simplification upon its substitution. Nonetheless, we are confident that the computation of the matrix elements has been done with enough care. Since this analysis has been done entirely with the use of Mathematica, I will quote only the result and drawn conclusions.

- **Right region:** In the right shaded unitarity region in fig[1] the Kac determinant has been evaluated at various places below the upper bound. No indications for the appearance of a more stringent bound have been found.

- **Left region:** In the left shaded region, on the other hand, we expect that level-2 will contain more stringent bounds, as the Kac determinant was not found to be manifestly positive in this region. Given that, this region arose due to a bound of the form $h \geq \frac{c}{30}$, this region will most likely be ruled out due to a similar bound.

In the next paragraph, we will compare unitarity to extremality. For the reasons described above we will work with the following set of bounds.

$$q = \sqrt{105}c - 27\sqrt{105}h^2 \quad \frac{210c}{2}$$

$$q = \frac{32\sqrt{105}h^2 - \sqrt{105}ch}{140c} \quad h \geq \frac{c}{30}$$

(8.29)

where we recall that the first corresponds to the BPS bound, whereas the latter two are level-1 unitarity bounds.

**Unitarity Vs Extremality**

The unitarity region, together with the extremality bound are plotted in fig[3] The extremality bound is represented in green and lies always inside the unitarity window. Note furthermore, that when written in
terms of the global charges, the upper unitarity bound \(8.19\) reads:

\[
\mathcal{U} = \frac{4L^2}{15} + \frac{L}{12} + \frac{1}{240}.
\]

(8.30)

On the contrary, the extremality bound identified in the bulk reads:

\[
\mathcal{U} < \frac{4}{15} L^2.
\]

(8.31)

As \(L \to \infty\), the extremality bound, asymptotes to the unitarity bound but it will never cross it. We can thus conclude that NS (extremal) black holes will always be unitary.

We also note that the NS conical defect identified in the bulk, lies at the frontier of the unitarity window and is represented by the blue dot at \(\frac{24h}{\mathcal{L}} = \frac{4}{5}\). The blue dot at \(h = q = 0\) represents global AdS3.

8.2 Ramond sector

As for the NS sector, we will start by constructing a highest weight representation, which in this case becomes slightly more subtle due to the presence of a fermionic zero mode \(Q_0\).

Highest weight representation

As always, a highest weight state \(\vert h.w. \rangle\) will be annihilated by all the positive modes of the algebra.

\[
L_n \vert h.w. \rangle = U_n \vert h.w. \rangle = Q_r \vert h.w. \rangle = 0 \quad n, r > 0
\]

(8.32)

In total there are three zero modes, \(L_0, U_0, Q_0\). \(L_0\) commutes with both so we may represent it by a number \(h\), i.e. the conformal weight. Under the action of the fermionic \(Q_0\), the highest weight representation become 2 dimensional. Namely, since \(Q_0\) commutes with the \(L_0\) operator, the states

\[
\vert h \rangle \quad \text{and} \quad Q_0 \vert h \rangle,
\]

(8.33)

have equal eigenvalue under \(L_0\):

\[
L_0 (Q_0 \vert h \rangle) = Q_0 L_0 \vert h \rangle = Q_0 h \vert h \rangle = h (Q_0 \vert h \rangle).
\]

(8.34)

and the Verma module is spanned by the two orthogonal highest weight states \(8.33\). The spin-4 zero mode, \(U_0\) is a little bit more subtle, and on first sight one does not expect it to commute with \(Q_0\). Its zero mode commutator reads:

\[
\left[ U_0, Q_0 \right] \sim Q_0 \vert h \rangle.
\]

(8.38)

Note further that the zero modes of the composite operators contain an infinite sum of operators. However, keeping in mind that we always apply these to a highest weight state, only a few of them remain. Acting on a highest weight state \(\vert h \rangle\) or \(Q_0 \vert h \rangle\), by virtue of \(8.34\), we have:

\[
\left( \partial TQ \right)_0 \vert h \rangle = \left( 2h + \frac{5}{2} \right) Q_0 \vert h \rangle,
\]

(8.36)

\[
\left( TQ \right)_0 \vert h \rangle = \left( h + \frac{5}{2} \right) Q_0 \vert h \rangle,
\]

(8.37)

and thus

\[
\left[ U_0, Q_0 \right] \vert h \rangle \sim Q_0 \vert h \rangle.
\]

(8.38)
In the case at hand, we are in fact quite fortunate. One can explicitly verify that the commutator (8.35) vanishes identically. Therefore, in the highest weight representation, the zero mode algebra is commutative and we can label both highest weight states by \( h \) and \( q \) which are the eigenvalues under \( L_0 \) and \( U_0 \) respectively:

\[
L_0 \vert h, q \rangle = h \vert h, q \rangle, \quad L_0(Q_0 \vert h, q \rangle) = h(Q_0 \vert h, q \rangle), \tag{8.39}
\]
\[
U_0 \vert h, q \rangle = q \vert h, q \rangle, \quad U_0(Q_0 \vert h, q \rangle) = q(Q_0 \vert h, q \rangle). \tag{8.40}
\]

**More generally**

Since the vanishing of a commutator like (8.35) will not be the case in general, let me comment on what the approach would have been when (8.35) would have been non zero. In this case, the action of \( U_0 \) on the highest weight state \( Q_0 \vert h, q \rangle \) must be slightly adapted. \( L_0 \) and \( U_0 \) are still commutative, so we may define as above the highest weight state as \( \vert h, q \rangle \) which now carries eigenvalue \( q \) under \( U_0 \). The difference lies in the action of \( U_0 \) on the highest weight state \( Q_0 \vert h, q \rangle \). A quick computation shows:

\[
U_0(Q_0 \vert h, q \rangle) = Q_0 U_0 \vert h, q \rangle + [U_0, Q_0] \vert h, q \rangle \equiv (q + \Delta q)Q_0 \vert h, q \rangle \tag{8.41}
\]

where we have defined \( \Delta q \) to be the coefficient in the commutator (8.35). That this commutator is proportional to \( Q_0 \) again is important, and it ensures that the state \( Q_0 \vert h, q \rangle \) is again an eigenstate of \( U_0 \), but now with eigenvalue \( q + \Delta q \). Setting \( \Delta q = 0 \) one recovers the case at hand.

**Normalization prescription**

The last ingredient we need, is a normalization prescription. I will denote the two highest weight states by:

\[
\vert h, q \rangle \equiv \vert + \rangle \quad \text{and} \quad Q_0 \vert h, q \rangle \equiv \vert - \rangle. \tag{8.42}
\]

Adopting the normalization for \( \vert + \rangle \) as unity, this implies that the norm of the \( \vert - \rangle \) state, is given by the eigenvalue of the \( Q_0^2 \) operator in the h.w. representation. Consistency with the mode algebra requires it to be:

\[
Q_0^2 \vert h, q \rangle = \frac{1}{2} \{Q_0, Q_0\} \vert h, q \rangle
\]
\[
= \frac{1}{2} \left( \frac{9c}{960} - \frac{9}{20} h + \frac{27}{5c + 22} \left( h^2 + \frac{1}{5} h \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} q \right) \vert h, q \rangle
\]
\[
= g(c, h, q) \vert h, q \rangle. \tag{8.43}
\]

In the following we will use the shorthand \( g(c, h, q) \equiv g \) and thus assume the following normalisation for the highest weight states:

\[
\langle + | + \rangle = 1, \quad \langle - | - \rangle = g. \tag{8.44}
\]

**Level-0**

At level-0 we can derive a BPS bound from (F.15):

\[
\frac{3c}{320} - \frac{9}{20} h + \frac{27}{5c + 22} \left( h^2 + \frac{1}{5} h \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} q \geq 0. \tag{8.45}
\]

In the semiclassical limit,

\[
h \to h/\hbar \quad q \to q/\hbar \quad c \to c/\hbar, \quad \hbar \to 0.
\]

(8.45) becomes:

\[
\frac{27}{5c} \left( h - \frac{c}{24} \right)^2 + 2\sqrt{\frac{27}{5}} q \geq 0. \tag{8.47}
\]
We define the Ramond ground state in a similar way as we did for the $N = 2$ Super-Virasoro algebra i.e. (5.22). At finite-$c$, it is given by:

$$h = \frac{1}{120} \left( 10 \pm \sqrt{1010 - c + 5c} \right), \quad q = 0.$$  \hspace{1cm} (8.48)

Note though, that at finite-$c$ the ground state is thus degenerate. Taking the semiclassical limit, we find:

$$h = \frac{c}{24}, \quad q = 0.$$  \hspace{1cm} (8.49)

saturating (8.47) and in which case the degeneracy disappears. This state, i.e. $|h = \frac{c}{24}, q = 0\rangle$, is invariant under the action of $\{L_0, Q_0\}$, including a corresponding shift of $L_0 \rightarrow L_0 - \frac{c}{24}$. Its bulk dual is the hypersymmetric massless BTZ for which $\mathcal{L} = \mathcal{U} = 0$. Before we move on to level-1, we will make to final remarks. Firstly, all states saturating (8.45) or its semiclassical version, necessarily carry negative spin-4 charges, the maximal value attained for the ground state. This is in contrast to the NS sector, where there was a small range of parameters where the spin-4 charge was allowed to be positive. Secondly, for those states that saturate BPS condition, the highest weight representation discussed above becomes one-dimensional.

**Level-1**

A basis for the level-1 Verma module is given by:

$$|\delta_+\rangle = Q_{-1}|+\rangle, \quad |\beta_+\rangle \equiv L_{-1}|+\rangle, \quad |\gamma_+\rangle = U_{-1}|+\rangle,$$

$$|\delta_-\rangle = Q_{-1}|-\rangle, \quad |\beta_-\rangle \equiv L_{-1}|-\rangle, \quad |\gamma_-\rangle = U_{-1}|-\rangle.$$  \hspace{1cm} (8.50)

At first sight, the dimension of the problem appears to be growing be rapidly and we have to take 6 states into account. However, we can divide the dimension of the problem in half by assuming the following ordering of the basis states:

$$K^R_1(h, q, c) = \det \begin{pmatrix}
    \langle \delta_-|\delta_- \rangle & \langle \delta_-|\beta_+ \rangle & \langle \delta_-|\gamma_+ \rangle & \langle \delta_-|\delta_+ \rangle & \langle \delta_-|\beta_- \rangle & \langle \delta_-|\gamma_- \rangle \\
    \langle \beta_-|\delta_- \rangle & \langle \beta_-|\beta_+ \rangle & \langle \beta_-|\gamma_+ \rangle & \langle \beta_-|\delta_+ \rangle & \langle \beta_-|\beta_- \rangle & \langle \beta_-|\gamma_- \rangle \\
    \langle \gamma_-|\delta_- \rangle & \langle \gamma_-|\beta_+ \rangle & \langle \gamma_-|\gamma_+ \rangle & \langle \gamma_-|\delta_+ \rangle & \langle \gamma_-|\beta_- \rangle & \langle \gamma_-|\gamma_- \rangle \\
    \langle \delta_+|\delta_- \rangle & \langle \delta_+|\beta_+ \rangle & \langle \delta_+|\gamma_+ \rangle & \langle \delta_+|\delta_+ \rangle & \langle \delta_+|\beta_- \rangle & \langle \delta_+|\gamma_- \rangle \\
    \langle \beta_+|\delta_- \rangle & \langle \beta_+|\beta_+ \rangle & \langle \beta_+|\gamma_+ \rangle & \langle \beta_+|\delta_+ \rangle & \langle \beta_+|\beta_- \rangle & \langle \beta_+|\gamma_- \rangle \\
    \langle \gamma_+|\delta_- \rangle & \langle \gamma_+|\beta_+ \rangle & \langle \gamma_+|\gamma_+ \rangle & \langle \gamma_+|\delta_+ \rangle & \langle \gamma_+|\beta_- \rangle & \langle \gamma_+|\gamma_- \rangle \\
\end{pmatrix}. \hspace{1cm} (8.51)
$$

With this ordering, the non-zero matrix elements, will make (8.51) block diagonal. Most of the matrix elements have already been computed in the NS sector. For the remaining ones we find:

$$\langle \delta_+|\beta_- \rangle = 5 \cdot g,$$  \hspace{1cm} (8.52)

$$\langle \delta_-|\delta_- \rangle = \langle \beta_-|\beta_- \rangle = \langle \gamma_-|\gamma_- \rangle = 1,$$

$$\langle \beta_-|\gamma_+ \rangle = g \cdot \left( 4a - 15b + 35d + \left( h + \frac{5}{2} \right) \left( 4e - \frac{9h_{\text{algebra}}}{2} \right) - \frac{315f}{8} - g_{\text{algebra}} \left( 2h + \frac{5}{2} \right) \right) \equiv g \cdot \kappa,$$

$$\langle \gamma_+|\gamma_+ \rangle = \left( U_{-1} - 2 \right) \left( 36 \cdot \frac{D}{12} - 30 \cdot E \right) : \Lambda : 35 \cdot \frac{H}{12} - 20I ) + 2 \left( F : \Xi : + G : \Delta : 35 \cdot \frac{H}{12} - 20I + K : \Gamma : 35 \cdot \frac{H}{12} - 20I \right) \equiv \gamma.$$  

Again $\langle \ldots \rangle$ denotes the expectation value between the highest weight states (8.42). $g$ denotes the normalization as given in (8.44). Normal ordering of composite operators is denoted by $: \cdot :$. Our normal ordering prescription is defined in (E.22). The action of the zero modes of the normal ordered composite operators
are given in (F.25)–(F.30) and (8.37). All OPE coefficients are given in appendix F and matrix elements related to the above by normalization or Hermitian conjugation, we have not explicitly written. We have furthermore denoted the structure constants \( h, g \) by a subscript \( g \) algebra, to distinguish them from the conformal weight \( h \) and normalization \( g \). Thus, \( h, g \) represent the conformal weight and normalization whereas \( g \) algebra, \( h \) algebra represent the OPE coefficients. All in all the matrix of inner products becomes:

\[
K_1^R(h, q, c) = \begin{vmatrix}
\delta \cdot g & \frac{1}{2} g & \kappa g & 0 & 0 & 0 \\
\frac{1}{2} g & 2h & 4q & 0 & 0 & 0 \\
\kappa g & 4q & \gamma & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & \frac{1}{2} g & \kappa g \\
0 & 0 & 0 & \frac{1}{2} g & 2h & 4qg \\
0 & 0 & 0 & \kappa g & 4qg & \gamma g
\end{vmatrix}.
\]

(8.53)

Since both blocks differ only by an overall multiplicative factor of the normalization \( g \), it will be sufficient to focus on the positivity of one block solely.

In this sector though, the level-1 Kac determinant is not nearly as compact as in the NS sector because the recursion relations are not of much use. Naively, attempting to compute the roots of the Kac determinant, one finds that they schematically yield us the level-0 BPS condition, arising from the normalization of the states, together with a set of 3 highly intractable roots, that are useless for all practical purposes. Furthermore, a multiplicative factor that puts a bound on the conformal weight is absent in this sector. Thus, as in the NS sector at level-2, we shall also here have to be somewhat less rigorous. We shall be interested in determining whether Kac determinant will dictate an upper bound like (8.19), on the spin-4 charge and whether or not the BPS bound remains the most stringent bound.

Now, schematically, the Kac determinant takes the following form:

\[
\left(3c^2 - 144ch + 128\sqrt{105}cq + 1728h^2\right)P(q, h, c),
\]

(8.54)

where \( P(q, h, c) \) is the remaining contribution to the Kac determinant, which is 3rd order in \( q \). A new set of bounds can be found by substituting back into the Kac determinant:

\[
q = -\sqrt{\frac{1}{128c}(c - 24h)^2} + \epsilon, \quad \epsilon > 0.
\]

(8.55)

Clearly, \( \epsilon > 0 \) strictly, as the Kac determinant vanishes identically under this substitution if \( \epsilon = 0 \). Now, since the first term will be manifestly positive under this substitution this allows us to determine what the effect of \( P(h, c, q) \) is to leading order in \( \epsilon \to 0 \). Thus requiring that also \( P(h, c, q) \geq 0 \), one finds:

\[
\frac{\sqrt{\frac{\epsilon}{32}\epsilon(3c - 80h)(c - 40h)^3}}{204800c} \geq 0.
\]

(8.56)

Consequently, this constrains \( h \) to be bounded by

\[
h \leq \frac{c}{40}, \quad h \geq \frac{3c}{80},
\]

(8.57)

and thus also \( c \geq 0 \). The next step now consists of determining an approximate (upper) bound on the spin-4 charge. For this one can take several approaches.

- Analogously to NS level-2, one might again substitute points inside the Kac determinant. This is not the best approach however in this case, because we are not yet in the possession of an upper bound, and it will only provide us with a very rough estimate of the form of the upper bound.
• The second approach one might take is to determine an approximation of the upper bound numerically. The idea behind the approach that one can take is the following. As before one can pick a region where the Kac determinant is positive. As we then start varying parameters, then Kac determinant will at some point become zero and flip sign. If one implements this numerically, one can determine an approximation for an upper bound by means of an interpolation. There is a word of caution though, concerning what we mean by "varying the central charge". As we are in the semiclassical limit, the operator algebra contains corrections of $O(1/c)$ that we are neglecting. Clearly, this means that we can only consider large values of the central charge as otherwise these corrections would become large. However, varying the central charge is a continuous way, does not have much meaning if we are restricting to large values. Nonetheless, we can fix the central charge to some large value, and determine the form of the potential upper bound on $q$, as we vary $h$, (compared to the central charge).

Unfortunately, near the end of this project, there was not enough time left to do this with full numerical rigorousness\footnote{I will comment more on this in the discussion.} and we leave this as a future project. We have however, found compelling evidence that there will most definitely be an upper bound on the spin-4 charge.

This has been done in Mathematica and I will again describe only the observations.

• $h \in \left[0, \frac{c}{40}\right]$  
  - In this region, as the central charge is taken to be large, the Kac determinant is not found to become negative if $q$ is roughly restricted to negative values. There are some small ambiguities around $q = 0$ though but as soon as $q$ grows sufficiently positive, the Kac determinant will always be negative.

• $h \in \left[\frac{3c}{80}, \infty\right]$  
  - In this region we again keep the central charge large. Similar to the first region, one easily verifies that the Kac determinant remains always positive if we $q$ is taken to be negative. For positive values of $q$, we have more freedom in this case. It does not take too much work to convince oneself, that as we increase $q$, we must also take larger values of the conformal weight $h$ for the Kac determinant to remain positive. In other words, we must approximately have:

\[
q < f(h)
\]

where $f$ is some function of $h$. For relatively small values, this bound is approximately linear in $h$, but once the values of both $h$ and $q$ increase it becomes more cumbersome to judge what the leading behaviour of the bound is.

Closing words on the Ramond sector: Unitarity and extremality

With the rough form of the bounds that have been found in this sector, we can again compare the unitarity constraints to extremality. With absolute certainty, one can conclude that out of the two conical defect solutions in this sector:

\[
\left(\frac{24h}{c}, \frac{\sqrt{105}q}{c}\right) = \left(\frac{1}{5}, \frac{2}{75}\right) \vee \left(\frac{3}{5}, -\frac{3}{800}\right)
\]

the first of these is non-unitary. Substituting these values into the level-1 Kac determinant one finds it to be negative. The second conical defect on the other hand saturates the BPS bound, and as we have seen, defines a zero of the Kac determinant. Furthermore, in the region $h \geq \frac{3c}{80}$, found by re-substitution of the level-0 bound, we have not found evidence that more stringent bounds then the BPS bound will appear. Consequently, we expect that BPS solutions will always be unitary, in particular the BPS conical defects and BPS black holes.
9 Summary and discussion

In this thesis we have discussed several topics. We reviewed the Chern-Simons formulation of AdS$_3$ gravity and after that turned to the higher spin extension of gravity. We saw how classical $\mathcal{W}$ algebras arise as the asymptotic symmetry algebras of the higher spin theory. We discussed also a new definition for black hole extremality first proposed in [10]. This defined extremality in terms of the Jordan classes of the holonomy, and is more in line with the degeneracy of the parameters at extremality.

We then turned our attention to (higher spin) supergravity, and reviewed the main results of [10]. We discussed how the $\mathfrak{sl}(3|2)$ higher spin theory allows for non-extremal (finite-temperature) BPS solutions within family I, that saturate a semiclassical BPS bound in the dual CFT with $\mathcal{W}(3|2)$ symmetry, indicating that in higher spin gravity, supersymmetry does not require extremality. One of the goals of this thesis was to understand whether these solutions might have been an artefact of the large-$c$ limit. To answer this question we turned to perturbation theory. Soon however, we found that perturbation theory told us that the non-extremal BPS solutions could in fact be seen as a class of exact solutions at finite-$c$, in particular meaning that they were not an effect of the large-$c$ limit. This brought an insightful relation between the bulk global charges and CFT charges to our attention. These finite temperature BPS solutions were seen to be defined by the conditions $q_2 = q_3$ and $h = q/2$ in terms of the CFT charges. Looking back at the finite-$c$ BPS bound:

$$(h - q/2)D_0 - (q_2 - q_3)^2,$$

(9.1)

this allowed for a quick interpretation as a class of exact solutions. Since all effects of the semiclassical limit were included in $D_0$, whose schematic form reads:

$$(D_0)_{\text{semi}} = (h - q/2)(\ldots),$$

(9.2)

this meant in addition that in the semiclassical limit, one could not make a distinction between solutions for which $h = q/2$ exactly, or those for which $(D_0)_{\text{semi}} = 0$ as they amount to the same condition. This raised the following questions that were addressed with perturbation theory:

- Exact solution in the bulk belonging to BPS family 1, exists for $q_2 = q_3$ and $h = q/2$ exactly. Semi classically, this condition does not distinguish between solutions for which $q_2 = q_3$ and $(D_0)_{\text{semi}} = 0$. For that reason, we addressed the question whether solutions with $q_2 = q_3$ and $D_0 = 0$, exist also at finite-$c$.

- Semi-classically, a necessary condition for family 1, is that $q_2 = q_3$. This raised the question whether in the CFT $q_2 = q_3$ remains a necessary condition as well, or if it is only a sufficient condition. In other words, do solutions at finite-$c$ exist for which $q_2 \neq q_3$.

The answer to both questions was found to be positive. We furthermore discussed, that BPS family II was in fact non-physical once the CFT is taken into account. Consequently, the charged BTZ contained in class I remains the most supersymmetric solution in the $\mathfrak{sl}(3|2)$ higher spin theory.

Nonetheless, although the discussion in this work has shown that these finite-temperature BPS solutions remained well-defined at finite-$c$ these solutions may open doors to address a different question. To discuss this we will first need to learn about the Cardy formula that gives us the entropy in a CFT[59]. Cardy’s insight was that

$$Z_0(\tau) = \text{Tr} e^{2\pi i (L_0 - \frac{c}{12}) \tau}$$

(9.3)

is modular invariant under $\tau \rightarrow -1/\tau$. Now we look at the partition function on the torus with modular parameter $\tau$:

$$Z(\tau) = \text{Tr} e^{2\pi i L_0} = \sum_h \rho(h) e^{2\pi i h \tau},$$

(9.4)
where $\rho$ is the density of states with $L_0$ eigenvalue $h$. The entropy can as usual be extracted from the density of states as:

$$S(h) = \ln(\rho(h)).$$

(9.5)

Thus we need to extract the $\rho(h)$ from the above formula. This is most easily done with a contour integration, and letting $\tau$ become complex. Then one finds:

$$\rho(h) = \frac{1}{2\pi i} \int dq \frac{q^h}{q^{h+1}} Z(q) = \int d\tau e^{-2\pi i h \tau} Z(\tau),$$

(9.6)

where we used the standard notation $q = e^{2\pi i \tau}$ and thus $dq = 2\pi i e^{2\pi i \tau} d\tau$. Now we will make use of modular invariance. Note firstly that:

$$Z(\tau) = e^{\frac{2\pi i c}{24}\tau} Z_0(\tau).$$

(9.7)

Applying modular invariance to $Z_0$ we find that:

$$Z(\tau) = e^{\frac{2\pi i c}{24}\tau} Z_0(-1/\tau) = e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z(-1/\tau).$$

(9.8)

Plugging this into the expression for the density of states one finds that:

$$\rho(h) = \int d\tau e^{-2\pi i h \tau} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z(-1/\tau).$$

(9.9)

The next step is now to use a saddle point approximation in the high temperature limit $\tau \to 0$ and in the limit of large $h$. In this approximation, the integrand is dominated by the extremum of the exponent, which found to lie at:

$$\tau = i \sqrt{\frac{c}{24h}} = \frac{i}{2} \sqrt{\frac{c}{6h}} \quad \text{or} \quad h = \frac{c}{24} \frac{1}{\tau^2} \sim T^2.$$

(9.10)

Substituting this back into the integrand, one finds the density of states in the high temperature limit to be:

$$\rho(h) \simeq \exp \left(2\pi \sqrt{\frac{ch}{6}}\right),$$

(9.11)

thus yielding for the entropy

$$S(h) = 2\pi \sqrt{\frac{ch}{6}}.$$

(9.12)

This is the Cardy formula for the entropy in a 2d CFT.

With these results we can then derive the first law of thermodynamics. For this we need to remember the following relations:

$$h = \frac{c}{24} \frac{1}{\tau^2}, \quad \tau = \frac{\beta}{2\pi}, \quad \beta = 1/T \quad \implies \quad h = \frac{c}{6} \pi^2 T^2.$$  

(9.13)

Now we compute:

$$\frac{dS(h)}{dh} = 2\pi \frac{d}{dh} \sqrt{\frac{ch}{c}} = 2\pi \sqrt{\frac{c}{6}} \frac{1}{2\sqrt{h}} = \frac{1}{T},$$

(9.14)

where we used the above relation between the energy and the temperature. Rewriting we find the first law of thermodynamics:

$$dh = T dS.$$

(9.15)

Let us now turn to the gravity side. As we discussed in subsection 2.3 the bulk thermodynamics are dominated by the BTZ in the high temperature limit, whereas thermal AdS dominates the thermodynamics in the low temperature limit. Thus, for holographic CFT’s, this explains why the entropy of the BTZ takes on the form of the Cardy formula. The temperature where the black hole no longer dominates the bulk thermodynamics defines the Hawking Page Phase transition and it is in fact known the Cardy formula applies up to that temperature, (for holographic CFT’s). Furthermore, the relation (9.10) found from
the saddle point approximation in the high temperature limit, is in the bulk captured by the holonomy condition that told us that:
\[ \tau = \frac{i}{2} \frac{1}{\sqrt{L}}. \] (9.16)

To summarise, we have derived the holonomy condition from the CFT partition function in the high temperature saddle point approximation.

What we would now like to do, is to do the same thing for the higher spin case. I.e. we would like to derive the higher spin analog of the holonomy condition in the bulk higher spin theory, from the CFT partition function,
\[ d(h, q_3, \ldots) = \int d\tau d\alpha \ldots e^{-2\pi i h \tau} e^{-2\pi i q_3 \alpha} \ldots Z(\tau, \alpha, \ldots), \] (9.17)

using a similar saddle point approximation. In (9.17), \( q_3 \) denotes a higher spin charge and \( \alpha \) is its associated potential. Unlike in the spin-2 case though, the higher spin partition function is not known to have any nice modular invariant properties, which makes it very hard to generalise the above procedure and derive the higher spin holonomy condition from the CFT. It has only been shown that the gravity and CFT results agree by treating \( \alpha \) perturbatively[57][58]. Nonetheless, one approach to determine whether the higher spin holonomy condition arises from the CFT, would be to require that the CFT partition function in the saddle point approximation reproduces the holonomy condition in the bulk, and to consequently “guess” what the modular transformation law of the partition function should be if it where to reproduce this result. However, this is easier said then done. In the \( \mathfrak{sl}(2) \) case, we had only one global charge to deal with and the holonomy condition could be solved in terms of the charges only. In the higher spin case, with more charges present, the holonomy condition is in general solved implicitly in terms of the eigenvalues, which poses an additional obstacle. Now, let us specialize to the \( \mathfrak{sl}(3|2) \) theory. Analogously to the \( \mathfrak{sl}(2) \) case, one can write down a higher spin first law of thermodynamics:
\[ dh = TdS + \mu_1 dq + \mu_2 dq_2 + \mu_3 dq_3. \] (9.18)

As before, \( h, q, q_2, q_3 \) are the CFT charges and \( \mu_i \) the associated sources. The holonomy condition ensures that this equation is fulfilled. Now, in the lower spin case, the bulk extremality condition will tell us that the solution necessarily has to be at zero temperature, for the first law to hold. In the higher spin case, we have seen finite-\( T \) solutions with a relative simple relation between their charges: \( q_2 = q_3, h = q/2 \).

Looking at the first law of thermodynamics though, one might then question whether these solutions are truly at finite-\( T \) also at finite-\( c \), or whether perhaps this might have been a large-\( c \) effect. If these solutions are truly at finite-\( T \), then the relatively simple relation between the charges, might allow one to make an educated guess for the modular properties of the higher spin partition function, such that it reproduces the holonomy conditions in the saddle point approximation. We leave the question whether this will be possible for future directions.

The upper bound of hypergravity

The second topic addressed in this thesis was the \( \mathfrak{osp}(1|4) \) hypergravity theory. This theory contained besides the spin-2 field a spin-5/2 field, and when put on an AdS\(_3\) background, consistency of the field equations required the inclusion of a spin-4 field \( U \). This theory was of interest because of an earlier publication [11]. In this paper, the authors identified two extremality bounds on the spin-4 charge. The lower bound arose as a proper BPS bound, in the semiclassical limit of the dual CFT. The nature of the second upper bound however:
\[ U < \frac{4}{15} L^2 \]
was less clear. In [11], it was derived by requiring reality of the entropy. However, the entropy is a derived quantity and therefore does not provide a satisfactory explanation for the appearance of this upper
extremality bound. One of the purposes of this work was to understand the nature of this upper bound at a more fundamental level. We firstly showed that when the extremality proposal of [10] is applied to the hypergravity theory, the Jordan classes define the same extremal values for the global charges, c.f. table 2. We found furthermore, that whilst BPS $\Rightarrow$ extremality, the converse was not the case in agreement with the conventional gravitational notions. The class of BPS black holes was found to be 1/4 BPS. We also considered conical defects. In the NS sector we identified a maximally symmetric conical defect saturating the BPS bound. In the Ramond sector, we identified two maximally symmetric conical defects. Only one of them saturated the BPS bound.

The dual $\mathcal{W}(2, \frac{5}{2}, 4)$ CFT.

We then turned to the dual $\mathcal{W}(2, \frac{5}{2}, 4)$ CFT of the hypergravity theory and studied its unitary representations. The reason for this was two fold. Firstly, we where interested in determining which black holes and conical defects of the hypergravity theory where also allowed by unitarity. Secondly, motivated by the $\mathcal{W}_3$ higher spin theory, whose extremality bound is closely related to the unitarity bound [10], we suspected that the CFT would dictate an upper unitarity bound upon us, explaining the appearance of the upper bound of [11]. At the beginning of this project we set out to determine a set of unitarity bounds at finite-$c$. Soon however, given the approach taken in this thesis, this was found to be a rather impossible task, and we restricted ourselves to a semiclassical analysis. In that case, the algebra became tractable enough for explicit results at the lower levels. In the NS sector we derived a BPS bound at level-1/2 and an upper unitarity bound at level-1 that showed a striking resemblance to the upper extremality bound. The two where found to coincide in the limit $\mathcal{L} \to \infty$. The BPS bound was found to be always the most dominant lower bound at the semiclassical level, similar to what happens for the $\mathcal{N} = 2$ Super-Virasoro algebra. As emphasised in the main text, rigorous results have only been possible up to level-3/2 in this sector. At level-2, the dimension of the Kac determinant and its complexity made explicit analysis highly cumbersome. We therefore took a different approach and verified whether or not bounds more stringent then the level-1 upper bound would appear or not. The answer was found to be negative.

The Ramond sector was a different story on the other hand, and already at level-1 we hit the problem that an analysis of the Kac determinant in terms of its roots would not be possible. Unfortunately the course of events in this sector turned out to be rather unfortunate, and explicit bounds from the Kac determinant required more time then was possible. We therefore tackled the Kac determinant "by hand" to determine a rough estimate of the form of the bounds. We found evidence that the Kac determinant remains manifestly positive if $q$ is restricted to negative values. We furthermore concluded that there will most definitely be an upper bound in the Ramond sector.

The explanation for the difference at level-1 between the Ramond and Neveu-Schwarz sector, could be found in the semiclassical mode algebra. In the NS sector, the mode algebra exhibited some useful recursion relations that allowed for the matrix elements in the NS sector level-1 to combine in a rather nice way. To some extend, this explained why the Ramond sector at level-1 differed so substantially from the NS sector. The same recursion relations also made us form a suspicion for the types of bounds that will appear at half integer and odd integer levels in the NS sector.

The extremality bound from the mode algebra

We recall here that we found from the Gram matrix constructed from solely the operators $L_{-n}$ and $U_{-n}$ two null conditions. Expressed in terms of $h' = \frac{2h}{c}$, $q' = \sqrt{\frac{9}{c}}$:

$$q' = \frac{\left(-1 + h' + n^2\right)\left(-9 + 9h' + n^2\right)}{384} \quad \text{and} \quad q' = \frac{\left(-1 + h' + n^2\right)\left(-4 + 4h' + n^2\right)}{96} \quad (9.19)$$
One then notices that the lower condition satisfies:

\[ L_{-n} + aU_{-n} \leftrightarrow Q_{-\frac{a}{2}}. \]  

(9.20)

We can make quite a remarkable observation from (9.19). We first note that they intersect in the following set of points:

\[ \left( \frac{24h}{c}, \sqrt{105 \theta} \right) = \left( 1 - n^2, 0 \right), \left( 1 - \frac{n^2}{5}, \frac{n^4}{600} \right). \]  

(9.21)

The first series of points is not so much interesting. We can only note from it that the lines intersecting it follow the pattern:

\[ LU_n^{\text{lower}} = LU_n^{\text{upper}} = LU_{2n}^{\text{upper}} = LU_{3n}^{\text{lower}} \]  

(9.22)

The second series of points on the other hand, are located all on the extremality bound identified in the bulk. For \( n = 1, 2 \), these intersections correspond to the extremal NS and R conical defects. The upper null conditions intersect the Ramond BPS condition tangentially in the series of points:

\[ \left( \frac{24h}{c}, \sqrt{105 \theta} \right) = \left( 1 - \frac{2n^2}{5}, -\frac{3n^4}{800} \right) \]  

(9.23)

which for \( n = 1 \) yields the BPS R conical defect. The lower conditions also intersect the Ramond BPS bound but here we should make a distinction between \( n \in \mathbb{Z}/2\mathbb{Z} \), with an associated NS \( Q_{n/2} \) and \( n \in 2\mathbb{Z} \) with a Ramond \( Q_n \):

\[
\begin{align*}
    n &\in \mathbb{Z}/2\mathbb{Z}, \quad \text{NS fermionic } Q \\
    &\left( \frac{24h}{c}, \sqrt{105 \theta} \right) = \left( 1 - \frac{n^2}{10}, -\frac{3n^4}{1200} \right) \\
    2\mathbb{Z}, \quad \text{R fermionic } Q &\left( \frac{24h}{c}, \sqrt{105 \theta} \right) = \left( 1 - \frac{2n^2}{5}, -\frac{3n^4}{800} \right)
\end{align*}
\]  

(9.24)

Note that in the Ramond case we intersect also (9.23). The structure thus discussed is illustrated in fig 4a. The region on the left is bounded by the Ramond BPS bound and the black dots represent the intersections described in the text. It is quite surprising that the exact expression for the bulk extremality bound is related to the semiclassical mode algebra in such a way. What is certainly interesting, is that for \( n = 1, 2 \) the bulk conical defects are amongst the intersections. Nonetheless, we emphasize again that one should carefully question the meaning of results that are based on only subsets of operators. It would be interesting to understand if there is indeed a deeper meaning to this pattern or whether it is merely a coincidental feature of the mode algebra. We leave this for future studies.
(a) The null conditions derived from the Gram matrix constructed from the operators $L_{-n}, U_{-n}$ solely, which follow a recursion pattern $L_{-n}U_{-n} \leftrightarrow Q_{-2}$. Here plotted up to $n = 6$ and $r = 4$. To the right the level decreases. The lines that bound this region are the extremality bound and the level-0 Ramond BPS bound.

(b) The same recursion pattern restricted to $n = 1, 2$. The dots represent the three conical defects.

Figure 4
10 Acknowledgements

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A Matrix Conventions

A.1 $SL(2, \mathbb{R})$

We denote the 3 $s(2)$ generators as $\{L_0, L_{\mp 1}, L_{-1}\}$. They satisfy the algebra:

\[
[L_a, L_b] = \epsilon_{abc} L^c, \quad L^c = \delta^{cd} L_d,
\]  

(A.1)

where $\epsilon_{abc}$ is a completely antisymmetric tensor and $\epsilon_{0+} = 1$. The metric on the Lie algebra is given by

\[
\delta_{00} = \frac{1}{2}, \quad \delta_{+} = \delta_{-} = -1.
\]  

(A.2)

In the fundamental representation one can take the matrices to read:

\[
L_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]  

(A.3)
B  Building the $\mathfrak{osp}(1|4)$ black hole connection.

In this appendix we will construct the connection used for the spin-4 black hole in section 7. Consider the most general connection:

$$a_\phi = a_z + a_\bar{z} = L_1 - LL_1 + UU_{-3},$$

$$ia t_\psi + a_\phi = 2a_t = \mu_2 L_1 + gU_0 + \nu U_3 + aU_2 + bU_1 + cU_0 + dU_{-1} + eU_{-2} + fU_{-3} + mL_{-1},$$

where for now the functions $\{L, U, \mu_2, \nu, a, b, c, d, e, f, g, m\}$ are arbitrary and are allowed to depend on $t, \phi$. The explicit matrix representations of the $\mathfrak{osp}(1|4)$ generators are given in D.10. Upon imposing flatness $da + a \wedge a = 0$ we find the following conditions:

$$m = 30\nu U - L\mu_2 + \frac{1}{2} \partial_\phi^2 \mu_2,$$

$$g = - \partial_\phi \mu_2,$$

$$a = - \partial_\phi \nu,$$

$$b = \frac{1}{2} \left[ \partial_{\bar{\phi}}^2 \nu - 6\mathcal{L}\nu \right],$$

$$c = - \frac{1}{6} \partial_{\bar{\phi}}^2 \nu + \partial_\phi (\mathcal{L}\nu) + \frac{5}{3} \mathcal{L}\partial_\phi \nu,$$

$$d = \left[ -\frac{1}{12} \partial_{\bar{\phi}}^2 \nu + \partial_\phi (\mathcal{L}\nu) + \frac{5}{3} \partial_{\bar{\phi}} (\mathcal{L}\partial_\phi \nu) + 2\mathcal{L} \partial_\phi^2 \nu - 12\mathcal{L}^2 \nu - 2\mathcal{M}\nu \right],$$

$$e = \left[ -\frac{1}{24} \partial_{\bar{\phi}}^2 \nu + \frac{1}{4} \partial_{\bar{\phi}} (\mathcal{L}\nu) + \frac{5}{12} \partial_{\bar{\phi}} (\mathcal{L}\partial_\phi \nu) - \frac{1}{12} \mathcal{L}\partial_\phi^2 \nu + 3\mathcal{L}^2 \nu + 5\mathcal{M}\nu - \mathcal{L}\partial_{\bar{\phi}} \frac{1}{2} \nu - 3\mathcal{L} \partial_\phi (\mathcal{L}\nu) + 5\mathcal{L}^2 \partial_\phi \nu + 10\mathcal{M} \partial_\phi \nu \right],$$

$$f = \frac{1}{6} \left[ \partial_\phi \nu + 2\mathcal{L} \partial_\nu - 4\mathcal{M} \partial_\nu - 6\mathcal{L} \mu_2 \right].$$

Setting the charges and sources to constant values, then yields the connection used in section (7.1.1). The charges are further required to satisfy:

$$-\mathcal{D} \mathcal{L} = -2L \partial_\mu - \mu \partial L + 30\nu \partial U + 40\partial \nu U + \frac{1}{2} \partial^4 \mu,$$

$$-\mathcal{D} \mathcal{U} = \partial_\phi U_{\mu_2} + 40 \partial_{\bar{\phi}} U_{\mu_2} + \frac{8}{5} \left[ 2L^3 \partial_\nu + 3L^2 \partial \nu \right]$$

$$- \frac{2}{3} \left[ \frac{59}{120} \nu \partial \mathcal{L} + \frac{13}{60} \partial^4 \mathcal{L} + \frac{59}{72} (\partial \mathcal{L})^2 \partial \nu + \frac{44}{45} \mathcal{L} \partial \mathcal{L} \partial \nu + \frac{49}{30} \mathcal{L} \partial \mathcal{L} \partial^2 \nu + \frac{49}{90} \mathcal{L}^2 \partial^3 \nu \right]$$

$$+ \frac{14}{3} \left[ \partial \mathcal{L} \partial \nu + \mathcal{L} \partial \nu + 2\mathcal{L} \partial \nu \right] - \frac{1}{6} \left[ \partial^4 \nu + 5\partial \nu \partial^2 \nu + 9\partial^2 \nu \partial \nu + 6 \partial^3 \nu \right]$$

$$+ \frac{1}{120} \left[ \partial^4 \mathcal{L} + \partial \mathcal{L} \partial \mathcal{L} + \frac{56}{3} \partial^3 \mathcal{L} \partial^2 \nu + 28 \partial^2 \mathcal{L} \partial^3 \nu + \frac{70}{3} \partial \mathcal{L} \partial^4 \nu + \frac{28}{3} \mathcal{L} \partial^5 \nu - \frac{\partial^5 \nu}{720} \right].$$

These relations, are to be compared to the Ward identities in the $\mathcal{W}(2, \frac{5}{2}, 4)$ CFT. Note that in the limit of vanishing sources, the global charges obey the correct chirality conditions $\partial \mathcal{L} = 0$ and $\partial \mathcal{U} = 0$, which are the Ward identities, in the undeformed theory, i.e. vanishing higher spin sources.

Using these transformation laws one can derive the same map, i.e. (7.2), between the bulk and CFT operators using the method outlined in [18]. For this one needs to know the centrally extended Poisson algebra of the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra. This can be computed using the method outlined in [87] i.e. with a free field realisation. This however, is a very long and time-consuming computation and for that reason we took the approach described in the main text i.e. by comparing the asymptotic symmetry algebra to the CFT mode algebra. It was only much later during this project, when we found out that the computation had already been done in [31]. See also section 7.3.
C First order formulation of Einstein Gravity

In this appendix we will discuss the necessary concepts from the first order, or vielbein, formalism needed to formulate 3d gravity as a gauge theory\(^{66}\). For this one introduces a new non-coordinate basis with orthonormal basis vectors \(\hat{e}_a = \frac{\partial}{\partial y^a}\) spanning a tangent space \(T_x\), defined at each point \(x \in M\), with \(M\) some (curved) manifold of dimension \(d\). This tangent space \(T_x\) is then a \(d\) dimensional flat tangent space of Lorentzian signature \((-+++)\) and \(a = 1, \ldots, d\). This tangent space \(T_x\) represents a good approximation of the manifold in a small enough region of \(x \in M\). The new basis vectors of \(T_x\) are related to the coordinate basis vectors \(\hat{e}_\mu = \partial_x\) of \(M\) through:

\[
\hat{e}_\mu = e_\mu^a \hat{e}_a \quad \leftrightarrow \quad \frac{\partial y^a}{\partial x^\mu} = e_\mu^a(x). \quad \text{(C.1)}
\]

The greek indices \((\mu)\) represent the curved coordinate indices and the latin indices \((a)\) represent the non-coordinate indices. The Jacobian matrix \(e_\mu^a\) between the tangent space \(T_x\) and \(M\) defines an isomorphism between them allowing one to express tensors on \(M\) in terms of tensors on \(T_x\) and vice versa. The inverse vielbein is defined via the relations

\[
e_a^\mu e_\mu^b = \delta^b_a \quad \text{and} \quad \epsilon_a^\mu \epsilon^a_\nu = \delta^\nu_\mu. \quad \text{(C.2)}
\]

Given the flat metric \(\eta_{ab}\) of \(T_x\), the vielbein induces the metric on \(M\)

\[
g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad \text{(C.3)}
\]

in accordance with the signature of the manifold. Because of this relation, the vielbein is often called the "square root" of the metric. Given the vielbein, the metric \(g_{\mu\nu}\) is uniquely determined. However, starting from the metric \(g_{\mu\nu}\) there are infinitely many choices of the vielbein reproducing the same metric. This is related to the freedom one has in choosing a basis \(e^a\) for the tangent spaces \(T_x\). Internal rotations of \(T_x\) should be undetectable from the point of view of the manifold\(^{70}\). Indeed, note that this relation is invariant under

\[
e_\mu^a \rightarrow \Lambda^a_b(x)e_\mu^b, \quad \text{(C.4)}
\]

where \(\Lambda\) represent a Lorentz transformation with infinitesimal parameter \(\alpha^b_a\). Thus, working in a non-coordinate basis we now have the freedom to perform a local Lorentz transformation. These local Lorentz transformations however, are quite different from the conventional Lorentz transformations. In fact, the transformation \((C.4)\) indicates that there is a redundancy in the choice of variables in \((C.3)\). Local Lorentz transformations should thus be seen as gauge symmetries in the vielbein formulation.

Using the vielbein we can transform tensors in a curved coordinate basis to the flat non-coordinate basis.

\[
V^\nu_c = e_a^c e_\nu^b V_b^a. \quad \text{(C.5)}
\]

To differentiate tensors in the non-coordinate basis, we introduce in analogy to the Christoffel connection \(\Gamma^\lambda_\mu\nu\) the spin connection denoted by \(\omega^a_\mu\) such that

\[
D_\mu X^a_b = \partial_\mu X^a_b + \omega^a_{\lambda\nu}X^\nu_b - \omega^b_{\lambda\nu}X^a_\nu. \quad \text{(C.6)}
\]

The spin connection encodes parallel transport of tensors between tangent spaces of two nearby points, \(T_x \rightarrow T_{x+dx}\). It is related to the Christoffel connection via

\[
\Gamma^\nu_\mu\lambda = e_a^\nu \partial_\mu e^a_\lambda + e^a_\mu e_\nu^b \omega^a_{\lambda\nu}. \quad \text{(C.7)}
\]

\(^{70}\)Off course we are here interested in the particular case of 3 dimensions but for generality let us write \(d\), the construction can be done in any number of dimension. Nonetheless, it is only for \(d = 3\) that the identification with the Einstein Hilbert action holds.

\(^{71}\)The same may be done for Euclidean signature, but for the sake of the argument we consider Lorentzian signature.

\(^{72}\)Note that the vielbein has \(d^2\) components, whereas the metric has only \(d(d + 1)/2\) independent components. This apparent mismatch of \(d(d-1)/2\) components is precisely the number of independent rotations in \(d\) dimensions.
Under local Lorentz transformations it transforms as

\[ \omega^{a}_{\mu b} \rightarrow (\Lambda^{-1})^{a}_{c} \omega^{c}_{\mu d} A^{d}_{b} - (\Lambda^{-1})^{a}_{c} \partial_{\mu} \Lambda^{c}_{b} \]  

or more schematically

\[ \Lambda^{-1} d\Lambda + \Lambda^{-1} \omega \]  

The condition of metric compatibility in the coordinate basis \( \nabla_{\mu} g_{\nu\rho} \) translates to the non-coordinate basis as \( D_{\mu} \eta_{ab} = 0 \) which means:

\[ D_{\mu} \eta_{ab} = \partial_{\mu} \eta_{ab} - \omega^{c}_{a \mu} \eta_{cb} - \omega^{c}_{b \mu} \eta_{ac} = 0 \implies \omega_{\mu ab} = -\omega^{ba}_{\mu} \]

and so the spin connection is anti-symmetric in its frame indices. The Riemann tensor and the torsion are finally rewritten as:

\[ R^{a}_{\mu \nu b} = \partial_{\mu} \omega^{a}_{\nu b} - \partial_{\nu} \omega^{a}_{\mu b} + [\omega^{c}_{\mu c}, \omega^{a}_{\nu b}] \]  

\[ T^{a}_{\mu \nu} = \partial_{\mu} e^{a}_{\nu} - \partial_{\nu} e^{a}_{\mu} + \omega^{a}_{\mu b} e^{b}_{\nu} - \omega^{a}_{\nu b} e^{b}_{\mu} = D_{[\mu} e^{a}_{\nu]} \]

These equations are known as the Cartan-Structure equations. From these one recovers the equations of motion \( R = 0 \). In general relativity, based on observational evidence, zero torsion is assumed, leading to the Levi-civita/Christoffel connection.

**First Vs the second order formulation**

The most conventional formulation of GR is the second order formulation in which the metric tensor is treated as the fundamental variable describing gravity\(^{73}\). The name second order refers to the fact that the gravitational field equations are second order in derivatives of \( g_{\mu \nu} \) or \( e^{a}_{\mu} \). In the first order formulation on the other hand, one treats both \( e^{a}_{\mu} \) and \( \omega_{\mu ab} \) as independent variables and the field equation are first order in derivatives. The field equations then set \( \omega_{\mu ab} = \omega^{ab}_{\mu}(e) \), where \( \omega^{ab}_{\mu}(e) \) is a solution to \( T^{a}_{\mu \nu} = 0 \). Substituting this back into the action, one recovers the action from the second order formulation. In both formulations the curvature tensor and covariant derivatives are constructed from the torsion free connections \( \Gamma^{\rho}_{\mu \nu}(g) \) and \( \omega_{\mu ab}(e) \).

**C.1 First order formulation of Einstein Gravity**

We here prove that gravity with a negative cosmological constant can be formulated as a Chern-Simons theory up to a boundary term\(^{74}\). Thus, we want to show the following equivalence:

\[ S_{EH} = \frac{1}{16\pi G_{3}} \int d^{3}x \sqrt{g} \left( R + \frac{2}{l^{2}} \right) \]

\[ = S_{CS}[A] - S_{CS}[\tilde{A}] \]

\[ = \frac{k}{4\pi} \left[ \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \int \text{Tr} \left( \tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right) \right] \]

with as usual \( \Lambda = -\frac{1}{l^{2}} \). Substituting

\[ A = (\omega^{a} + \frac{1}{l} e^{a}) L^{a}_{\mu} \]

\[ \tilde{A} = (\omega^{a} - \frac{1}{l} e^{a}) L^{a}_{\mu} \]

\( ^{73} \)As we will discuss next, in the presence of fermions one must resort to the vielbein description.

\( ^{74} \)As pointed out in the main text, the proof for zero cosmological constant and positive cosmological constant proceeds in a similar manner.
into the RHS we obtain the following:

\[
\text{Tr} \left[ A \wedge dA - \bar{A} \wedge d\bar{A} \right] = \text{Tr}[L_a L_b] \left[ A^a \wedge dA^b - \bar{A}^a \wedge d\bar{A}^b \right]
\]

\[= \epsilon_R \left[ A^a \wedge dA_a - \bar{A}^a \wedge dA_a \right]
\]

\[= \epsilon_R \frac{2}{7} \left[ \omega^a \wedge d\omega_a + e^a \wedge d\omega_a \right]
\]

\[= \epsilon_R \left[ \frac{4}{7} (e^a \wedge d\omega_a) - \frac{2}{7} d(e^a \wedge d\omega_a) \right] \quad (C.15)
\]

where we used \(\text{Tr}[L_a L_b] = \epsilon_R \delta_{ab}\) and \(\epsilon_R\) is a normalization factor that depends on the chosen representation for the \(\mathfrak{sl}(2)\) generators. For the remaining terms we find:

\[
\frac{2}{3} \text{Tr} \left[ A \wedge A \wedge A - \bar{A} \wedge \bar{A} \wedge \bar{A} \right] = \frac{2}{3} \text{Tr}[L_a L_b L_c] \left[ A^a \wedge A^b \wedge A^c - \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c \right]
\]

\[= \frac{2}{3} \epsilon_{abc} \frac{2}{7} \left( \frac{1}{l} e^a \wedge e^b \wedge e^c + \frac{3}{l} e^a \wedge \omega^b \wedge \omega^c \right) \quad (C.16)
\]

where we used \(\text{Tr}[L_a L_b L_c] = \epsilon_R \epsilon_{abc}/2\). Combining then \(C.15\) and \(C.16\) and omitting the explicit boundary term in \((C.15)\) we have:

\[
S_{CS}[A] - S_{CS}[\bar{A}] = \epsilon_R k \frac{1}{4\pi} \int \left( \frac{4}{7} (e^a \wedge d\omega_a) + \frac{\epsilon_{abc}}{2} \left[ \frac{4}{7} e^a \wedge \omega^b \wedge \omega^c - \frac{4}{10} e^a \wedge e^b \wedge e^c \right] \right) \quad (C.17)
\]

Noticing now the following relation:

\[
\frac{4}{7} \left[ e^a \wedge (d\omega_a + \frac{\epsilon_{abc}}{2} \omega^b \wedge \omega^c) \right] = \frac{4}{7} (e^a \wedge R_a) = \frac{2}{7} \epsilon_{abc} e^a \wedge R^{bc} \quad (C.18)
\]

where

\[
R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega^b \wedge \omega^c, \quad R_a = \frac{2}{l} \epsilon_{abc} R^{bc} \quad (C.19)
\]

we can write \((C.17)\) as:

\[
\epsilon_R k \frac{1}{4\pi} \int \left[ \frac{2}{7} \epsilon_{abc} e^a \wedge R^{bc} + \frac{2 \epsilon_{abc}}{3l} e^a \wedge e^b \wedge e^c \right] \quad (C.20)
\]

If we now use the following identities \[69\]

\[\epsilon_{abc} e^a \wedge R^{bc} = \sqrt{|g|} R d^3 x \quad \text{and} \quad \epsilon_{abc} e^a \wedge e^b \wedge e^c = 3! \sqrt{|g|} d^3 x \quad (C.21)\]

we find:

\[
\epsilon_R \int \frac{k}{4\pi} \left( \frac{2}{7} \sqrt{|g|} R d^3 x + \frac{4}{l} \sqrt{|g|} d^3 x \right) = \epsilon_R \int \frac{k}{4\pi} \left( \frac{2}{7} \left( \sqrt{|g|} R + \frac{2}{l} \right) \sqrt{|g|} d^3 x \right) \quad (C.22)
\]

Thus identifying

\[
k = \frac{l}{\epsilon_R G_{3R}} \quad (C.23)
\]

we see that we recover the Einstein Hilbert action as claimed, upon using the conventional normalization for the fundamental representation of the \(\mathfrak{sl}(2)\) generators \(\text{Tr}(L_a L_b) = \frac{1}{2} \delta_{ab}\) \(\epsilon_R = \frac{1}{2}\).

All that remains now is to prove the identities given in \((C.21)\). The second one is rather straightforward so we will prove the first.

\[
\epsilon_{abc} e^a \wedge R^{bc} = \epsilon_{abc} \epsilon^{\lambda \mu \nu} e_\lambda^a \frac{1}{2} R^{bc} d^3 x
\]

\[= \epsilon_{abc} \epsilon^{\lambda \mu \nu} e_\lambda^a e_\mu^b e_\nu^c \frac{1}{2} R^{bc} d^3 x \]

now use \(|e| = \frac{1}{3!} \epsilon_{abc} \epsilon^{\lambda \mu \nu} e_\lambda^a e_\mu^b e_\nu^c \rightarrow \epsilon_{abc} \epsilon^{\lambda \mu \nu} e_\lambda^a e_\mu^b e_\nu^c = |e| \epsilon_{\lambda \rho \sigma}\)

\[= |e| \frac{1}{3} \epsilon_{abc} \epsilon^{\lambda \mu \nu} \epsilon_{\lambda \rho \sigma} R_{\mu \sigma} d^3 x
\]

\[= |e| \epsilon^{\lambda \mu \nu} \epsilon_{\lambda \rho \sigma} R_{\mu \sigma} d^3 x
\]

\[= \sqrt{|g|} R d^3 x
\]

where we used:

\[\epsilon_{\lambda \rho \sigma} = 3! \quad \epsilon^{\lambda \mu \nu} \epsilon_{\lambda \rho \sigma} = 2 \delta_{\rho \lambda \lambda \rho}, \quad (C.24)\]
C.2 Vielbein formalism and fermions

In the previous section we have introduced the vielbein formalism and used it to write the Einstein Hilbert action as Chern Simons gauge theory. Whilst this has proven useful for discussing higher spin gravity the primary, “historical advantage” of the vielbein formalism has been to couple gravity to fermions. Their transformation rules are hard to generalise to curved backgrounds and the above discussed formalism allows one to define them in the tangent space, work with their usual Lorentz properties in flat space, and then translate back to the coordinate (curved) background using the vielbein. Although in this thesis we have worked solely with the Chern-Simons formulation of the theory, similar to what we did for ordinary gravity, this appendix will outline the main concepts from the equivalent metric formulation of the theory. The material will broadly be based on [27].

Transformation rule and covariant derivative

As has been discussed in the main text, in SUGRA the global supersymmetry is promoted to a local symmetry, and the gauge field of this gauge symmetry is denoted by $\psi_\alpha^\mu$. In three dimensions it is a Majorana vector-spinor. They transform as scalars under general coordinate transformations and in a spinor representation $R$ of the local Lorentz group.

$$\psi'_\alpha(x) = R(\Lambda(x))_\beta^\alpha \psi_\beta(x).$$  \hspace{1cm} (C.25)

The Lorentz generators $M_{ab}$ in the spinorial representation are then given by $\frac{1}{2} \gamma^{ab}$, with $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$, and thus the transformation rule (C.25) becomes:

$$\psi'(x) = \exp\left(-\frac{1}{2} \lambda_{ab}(x) M_{ab}\right) \psi(x) = \exp\left(-\frac{1}{4} \lambda_{ab}(x) \gamma_{ab}\right) \psi(x).$$  \hspace{1cm} (C.27)

Acting on spinors, the covariant derivative reads:

$$D_\mu \psi \equiv \left(\partial_\mu + \frac{1}{2} \omega_{ab}^{\mu} M_{ab}\right) \psi = \left(\partial_\mu + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab}\right) \psi.$$  \hspace{1cm} (C.28)

Kinetic term for the fermions

A kinetic term for $\psi_\mu^\alpha$ was first written down by Rarita and Schwinger who proposed adding the following action to the Einstein-Hilbert action.

$$S_{3/2} = \frac{1}{16\pi G_3} \int d^3 x \bar{\psi}_\mu \epsilon^{\mu\nu\rho} D_\nu \psi_\rho,$$  \hspace{1cm} (C.29)

with for now $D_\mu$ the Lorentz covariant derivative defined with the torsion less spin connection. The symmetry transformation rules for the total action $S_{EH} + S_{3/2}$, as they have been cited in the main text, are given by \[\delta \epsilon_\mu = \frac{1}{2} \epsilon^\alpha \psi_\mu, \quad \delta \psi_\mu = D_\mu \epsilon(x),\]  \hspace{1cm} (C.30)

\[\text{footnote}{\text{I will not however include the proof of invariance under local supersymmetry transformations, which we will discuss momentarily, as this goes beyond the purpose of this thesis. The explicit proof is given in [27] for the case of } d = 4 \text{ in which they make exhaustive use of explicit transposition properties of the }\gamma\text{ matrices and reality properties of the Majorana spinors. Off course we are here interested in } d = 3, \text{ which somewhat changes these subtleties. However, as pointed out there, the method easily modifies to } d = 3 \text{ by using instead appropriate chirality properties of the Majorana spinors and using specific properties of the gamma matrices in three dimensions.}}\]
where $\epsilon(x)$ is the infinitesimal parameter for local supersymmetry transformations, and as discussed in the main text, the killing spinor equation is found from $\delta_\epsilon \psi = 0$. However, these equations do not yet yield a fully supersymmetric theory and one can show that an explicit four fermion contact term must be added to the action. In fact, this can be most easily found by resorting to the first order formalism in which $e^a_\mu$ and $\omega^{ab}_\mu$ appear again as a priori independent fields in $S_{EH} + S_{3/2}$. This simplifies the computations of proving invariance under the supersymmetry transformations but it requires one to determine an additional field equation for the spin connection. Similarly to what we discussed for pure gravity, one can show that an explicit four fermion contact term must be added to the action. In fact, this can be most easily found by resorting to the first order formalism in which $e^a_\mu$ and $\omega^{ab}_\mu$ appear again as a priori independent fields in $S_{EH} + S_{3/2}$. This simplifies the computations of proving invariance under the supersymmetry transformations but it requires one to determine an additional field equation for the spin connection. Similarly to what we discussed for pure gravity, one can solve the field equation for $\omega^{ab}_\mu$, this time however, resulting in a non-zero torsion $(C.12)$, that defines the spin connection as:

$$\omega^{ab}_\mu = \omega^{ab}_\mu(e) + K^{ab}_\mu(\psi).$$  

(C.31)

Here $\omega^{ab}_\mu(e)$ is the torsion free spin connection present in normal GR and $K^{ab}_\mu(\psi)$ is a bilinear in the fermion fields arising from the RS action. It is called the contorsion. If one then substitutes this solution back into the action one can rewrite the result in terms of a torsion-free spin connection but now with an explicit four fermion interaction term present. The advantage thus, of working with a spin connection with torsion is that it captures the same effects of the quartic fermion term in the second order formulation but that one can now work with only $(C.29)$ and $(C.30)$ where the covariant derivative is now defined with the torsion-full spin connection.

If one were now interested in proving invariance of the action under the supersymmetry transformations the brute straightforward approach would be to start from the action including the quartic fermion term, or as we argued above the torsion-full spin connection, and to proof invariance under the transformation rules $(C.30)$. However, varying this action leads to terms that are first, third and fifth order in the fermions. The latter appears due to the explicit dependence $\omega(e)$. In the first order formalism this fifth order fermion term is avoided, but one needs to evaluate $\delta \omega^{ab}_\mu$ since it is now an independent variable. For "simple" supergravity theories, this is not a hard task, but the computation can become complicated when extended theories are considered. The most convenient way to prove invariance though is to work in the so called "1.5 order formalism". Morally, one stays in the second order formalism, since only $e^a_\mu$ and $\psi_\mu$ are independent variables. However, since the second order formulation is obtained from the first order formulation by substituting $\omega$ as in $(C.31)$, the proof of invariance can be simplified by preserving the original grouping of terms when varying the action. To understand this we write down the explicit variations of the action in first and second order formulation:

$$\delta S_{\text{first}} = \frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \omega} \delta \omega + \frac{\delta S}{\delta \psi} \delta \psi$$  

(C.32)

$$\delta S_{\text{second}} = \frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \omega} \left( \frac{\delta \omega}{\delta e} \delta e + \frac{\delta \omega}{\delta \psi} \delta \psi \right) + \frac{\delta S}{\delta \psi} \delta \psi$$  

(C.33)

However, we can think of the spin connection $\omega = \omega(e, \psi)$ as being defined by its field equation $\delta S/\delta \omega = 0$, for which the second terms in the second order formalism vanishes and one can do the computation of verifying invariance whilst neglecting all contributions coming from variation w.r.t. the spin connection. This goes under the name of the "1.5 order formalism".

Supergravity on AdS

Specifying to an AdS background, the covariant derivative is modified to

$$D_\mu \psi = \left( \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} - \frac{1}{32} e^a_\mu \gamma_a \right) \psi.$$  

(C.34)

The last term is needed to ensure that AdS remains a solution to $R_{\mu\nu} = 0$ upon adding to the Einstein Hilbert action the cosmological constant $\Lambda = -\frac{1}{l^2}$. Let us next discuss the action for an $\mathcal{N} = 2$ supergravity theory $(C.20)$. In an $\mathcal{N} = 2$ supergravity theory we have two, two-component real spinor fields which may be
combined into a single complex spinor. Note that this implies that the supersymmetry transformations are also parametrized by a single complex two component spinor parameter $\epsilon$. Furthermore, we discussed in the main text that in the case of extended supergravity the algebra contains additional internal $R$-symmetry generators. Thus we have an additional $O(2) \simeq U(1)$ gauge field $A_\mu$. The isomorphism follows from the fact that the 2 real spinors are combined into a single complex spinor. The action thus becomes:

$$S = \frac{1}{16\pi G_3} \int d^3x \left[ e \left( R + \frac{2}{l^2} \right) + ie^{\mu\nu\rho} \bar{\psi}_\mu D_\nu \psi_\rho - l e^{\mu\nu} A_\mu \partial_\nu A_\rho \right].$$

(C.35)

Consequently, the covariant derivative is modified to include an additional $iA_\mu$:

$$D_\mu = \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} + i A_\mu - \frac{1}{2l} e^a_\mu \gamma_a.$$  

(C.36)

Using these killing spinor equations, $\delta \psi = D\epsilon$, one can then determine which spacetimes are supersymmetric. In the case of $(1,1)$ supergravity, it was shown in [25] that the only supersymmetric spacetimes are zero mass, non-rotating black hole, and extremal rotating black holes. The $(2,0)$ supergravity theory was discussed in [51]. Under the supersymmetric spacetimes they identify are global AdS with $M = -1$ and the zero mass BTZ with $M = 0$ and a class of half integer charged black holes. In addition, they find a class of supersymmetric conical defects.

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[^77]: Note the difference in notation with respect to [51]. We have that $m = \frac{1}{2l}$ and their $\gamma$ matrices are imaginary. This is due to the opposite sign of the metric they employ.
D Lie Super Algebras

In this appendix we gather some useful facts regarding Lie-superalgebras after which we will discuss the particular case of the superalgebra $\mathfrak{osp}(1|4)$ used in the main text. We will mostly follow [71]. Conventions regarding the $\mathfrak{sl}(3|2)$ superalgebra and its real form $\mathfrak{su}(2,1|1,1)$ can be found in [10]. To start, let us define what we mean by a superalgebra.

**Definition D.1.** A Lie algebra $\mathfrak{g}$, is called a Lie-superalgebra, or $\mathbb{Z}_2$ graded algebra, if it can be written as a direct sum of the form:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$  
(D.1)

with

$$\mathfrak{g}_0 \cdot \mathfrak{g}_0 \subset \mathfrak{g}_0, \quad \mathfrak{g}_0 \cdot \mathfrak{g}_1 \subset \mathfrak{g}_1, \quad \mathfrak{g}_1 \cdot \mathfrak{g}_1 \subset \mathfrak{g}_0.$$  
(D.2)

We define $|x|$ to be the degree of $x$. Elements $x \in \mathfrak{g}_0$ are called even graded, $|x| = 0$ and elements $x \in \mathfrak{g}_1$ are called odd graded, $|x| = 1$. Lie superalgebras inherit the usual multiplication and addition laws.

The Lie-superbracket or supercommutator is defined as

$$[x,y] = -(-1)^{|x||y|}[y,x],$$  
(D.3)

and is required to satisfy the super-Jacobi identity.

$$(-1)^{|x||y|}[x,[y,z]] + (-1)^{|y||z|}[y,[z,x]] + (-1)^{|z||x|}[z,[x,y]] = 0.$$  
(D.4)

A sub-superalgebra of a superalgebra is defined similarly as for non-graded algebra’s:

**Definition D.2.** A graded subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a non-empty set $\mathfrak{h} \subset \mathfrak{g}$ which is a superalgebra which the two composition laws $(+, \cdot)$ induced by $\mathfrak{g}$ such that $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and $\mathfrak{h}_1 \subset \mathfrak{g}_1$.

D.1 Orthosymplectic superalgebras

In this work we are primarily interested in the orthosymplectic matrix superalgebras $\mathfrak{osp}(m,2n)$. Their elements are of the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  
(D.5)

with $A \in \mathfrak{gl}(m)$, $D \in \mathfrak{gl}(n)$, and $B, C$ respectively $m \times n$ and $n \times m$ rectangular matrices. We define the supertrace, denoted $s\text{Tr}$, for these algebras as:

$$s\text{Tr} = \text{Tr}(A) - \text{Tr}(D).$$  
(D.6)

Even graded elements have $B = C = 0$ whereas odd graded elements have $A = D = 0$. They satisfy the following conditions:

$$A^t = -A, \quad D^t G = -GD, \quad B = C^t G,$$  
(D.7)

where $X^t$ denotes transposition of $X$ and the matrix $G$ is defined by:

$$G = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$  
(D.8)

These orthosymplectic superalgebras can be divided into three different classes. Here we will be interested in only one of them which is the superalgebra $B(m,n)$ or $\mathfrak{osp}(2m+1|2n)$. The other two classes are the superalgebra $C(n+1)$ or $\mathfrak{osp}(2|2n)$ and the superalgebra $D(m,n)$ or $\mathfrak{osp}(2m,2n)$.

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78In the context super supersymmetry these are also called bosonic and fermionic.
D.2 The \( \mathfrak{osp}(2m + 1|2n) \) superalgebra

The superalgebra \( \mathfrak{osp}(2m + 1|2n) \) is defined for \( m \geq 0, n \geq 1 \). Its even part \( \mathfrak{g}_0 \) is the Lie algebra \( \mathfrak{so}(2m + 1) \oplus \mathfrak{sp}(2n) \). It has rank \( m + n \) and dimension \( 2(m + n)^2 + m + 3n \). In what follows we will mostly specify to \( m = 0, n = 2 \), i.e. \( \mathfrak{osp}(1|4) \), but keep the discussion general when convenient.

Commutation relations of \( \mathfrak{osp}(1|4) \)

In the principal embedding of \( \mathfrak{sl}(2) \) in \( \mathfrak{osp}(1|4) \) the superalgebra is decomposed into the spin-1 \( \mathfrak{sl}(2) \) generators \( L_i \), a spin-3 multiplet \( U_m \) and a spin-3/2 multiplet \( S_r \). As in the main text, spin denotes the \( \mathfrak{sl}(2) \) spin \( S \). The indices within each multiplet range from \( -S \) to \( S \). This structure is encoded in the commutation relations\([11]\):

\[
\begin{align*}
[L_i, L_j] &= (i - j)L_{i-j}, \\
[L_i, U_m] &= (3i - m)U_{i+m}, \\
[L_i, S_r] &= \left(\frac{3i}{2} - r\right)S_{i+r}, \\
[U_m, U_n] &= \frac{1}{12}(m - n)\left((m^2 + n^2 - 4)\left(m^2 + n^2 - \frac{2}{3}mn - 9\right) - \frac{2}{3}(mn - 6)mn\right)L_{m+n} \\
&\quad + \frac{1}{6}(m - n)(m^2 - mn + n^2 - 7)U_{m+n}, \\
[U_n, S_r] &= \frac{1}{24}(2n^3 - 8n^2r + 20nr^2 + 82r - 23n - 40r^3)S_{n+r}, \\
\{S_r, S_s\} &= U_{r+s} + \frac{1}{12}(6r^2 - 8rs + 6s^2 - 9).
\end{align*}
\]

Matrix representation

We have adopted the following matrix representation for the \( \mathfrak{osp}(1|4) \) generators in the fundamental
representation

\[
L_0 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix},
L_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{3}
\end{pmatrix},
\]

\[
U_0 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 3
\end{pmatrix},
U_1 = \frac{2}{\sqrt{3}} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
U_2 = \frac{5}{\sqrt{3}} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
S_{-\frac{1}{2}} = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 \\
5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
S_{\frac{1}{2}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
S_{\frac{3}{2}} = \frac{1}{\sqrt{3}} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0
\end{pmatrix},
S_{\frac{5}{2}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[S_{\frac{7}{2}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad S_{\frac{9}{2}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (D.10)
\]

Note that there matrices obey:

\[L_i = (-1)^i L_i, \quad U_m^\dagger = (-1)^m U_{-m}. \quad (D.11)\]

### D.3 Cartan-Weyl basis

In this section we will discuss the Cartan-Weyl basis that has been used in the main text.\(^79\)

Any simple Lie-algebra\(^80\) \(\mathfrak{g}_0\) can be put in the standard Cartan-Weyl basis. To construct this basis let \(\{\mathcal{H}_i | i = h, 2, \ldots, r\}\) be a maximal set of linearly independent elements of \(\mathfrak{g}_0\) that can be diagonalized simultaneously, such that

\[\mathcal{H}_i, \mathcal{H}_j \in \mathfrak{h}\]  \(i, j = 1, \ldots, h,\) \(D.12\)

The generator \(\mathcal{H}_i\) span the Cartan subalgebra \(\mathfrak{h}\), with \(\dim(\mathfrak{h}) = h = \text{rank}(\mathfrak{g}_0)\). The remaining generators are chosen such that they satisfy:

\[\mathcal{H}_i, \mathcal{E}_\alpha = \alpha(i) \mathcal{E}_\alpha. \quad (D.13)\]

\(^79\)Note that this representation differs from the one given in [11]. This representation is more suitable for the purpose of relating them to the Cartan-Weyl basis we will discuss in the next section.

\(^80\)Recall that "simple" means that there are no proper ideals, meaning that there is no subalgebra \(\mathfrak{h} \subset \mathfrak{g}\) such that \([\mathfrak{h}, \mathfrak{g}] \in \mathfrak{h}\) other then the trivial ideals 0 and \(\mathfrak{g}\). A lie algebra is semisimple if it is a direct sum of simple Lie algebras.
where $\alpha_{(i)}$ is a number. Each $E_\alpha$ can be associated with a vector $\vec{\alpha} = (\alpha_{(1)}, ..., \alpha_{(h)})$, which is called a root. It is an element of the dual space of the Cartan subalgebra denoted by $\mathfrak{h}^*$ since it maps an element $H_i \in \mathfrak{h}$ to the number $\alpha_i$ via:

$$\alpha(H_i) = \alpha_i. \quad (D.14)$$

Under the adjoint representation we can define an inner product on the Lie algebra: $(\cdot, \cdot): \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$

$$K(X,Y) = \text{Tr}(\text{Ad}_X \circ \text{Ad}_Y), \quad (D.15)$$

where the adjoint action of $X$ on $\mathfrak{g}_0$ is defined as the map:

$$\text{ad}_X: \mathfrak{g}_0 \to \mathfrak{g}_0. \quad (D.16)$$

$$\text{ad}_X(Y) = [X,Y]. \quad (D.17)$$

Using this and $(D.13)$ we then have:

$$\text{ad}_{H_i}(E_\alpha) = \alpha_i E_\alpha. \quad (D.18)$$

The inner product $(D.15)$ on the Lie algebra is known as the Cartan-Killing form. For semi-simple Lie algebras the Killing form is non-degenerate. This can be seen as a different definition of semi-simplicity.\footnote{As a remark we note that it is often normalised by $1/2g$ where $g$ is the dual Coxeter number. This can be defined by the number of distinct roots of the algebra, divided by the rank.}

The standard basis is chosen to be orthonormal with respect to the Killing form and so are the generators of the Cartan subalgebra. In the Cartan-Weyl basis the Killing forms will obey:

$$K(H_i, H_j) = \delta_{ij}, \quad K(H_i, E_\beta) = 0, \quad K(E_\alpha, E_\beta) = \delta_{\alpha,-\beta}. \quad (D.20)$$

For completeness let us mention that this Killing form is used to establish an isomorphism between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. The Killing form $(H_i, ...)$, $i$ fixed, maps every element of the Cartan subalgebra onto a number. To every root $\vec{\alpha} \in \mathfrak{h}^*$, we may then associate an element $H_\alpha = \vec{\alpha} \cdot H = \sum_i \alpha_i H_i \in \mathfrak{h}$ through:

$$(H_i, H_\alpha) = \alpha_i. \quad (D.21)$$

For a general element $H = \sum_i h_i H_i \in \mathfrak{h}$ we have then:

$$(H, H_\alpha) = \alpha(H) = \sum_i h_i \alpha_{(i)} \in \mathfrak{h}^*. \quad (D.22)$$

With this isomorphism the killing form naturally induces a positive definite scalar product in the Cartan dual space $\mathfrak{h}^*$ denoted $<\cdot, \cdot>$ as:

$$<\vec{\alpha}, \vec{\beta}> = (H_\alpha, H_\beta) = \sum_i \alpha_{(i)} \beta_{(i)}. \quad (D.23)$$

For further background concerning aspect of the Cartan-Weyl basis we refer to the literature. We will next introduce the quadratic Casimir invariant that can be constructed for any semisimple Lie algebra. This element is uniquely characterized by its commutativity with all generators of the algebra. In a generic basis of generators $T_a$ it can be checked to be given by:

$$C_2 = \sum_{a,b=1}^{\dim \mathfrak{g}} [K(T_a, T_b)]^{-1} T^a T^b, \quad (D.24)$$

with $K$ the killing form $(D.15)$ that is non-degenerate for semi-simple Lie algebras. In the orthonormal basis one simply has:

$$C_2 = \sum_a T^a T^a. \quad (D.25)$$
We note though that $C_2$ is not an element of $g$ itself. Instead, it lies in the center of the universal enveloping algebra $U(g)$ of $g$ with the set of all formal power series in elements of $g$. Of course, we are very familiar with quadratic Casimir invariants. A well-known example from quantum mechanics is the $su(2)$ angular momentum operator $L^2 = L_x^2 + L_y^2 + L_z^2$ that commutes with all generators. This is, for $su(2)$ the only Casimir operator. However, for higher rank algebras there are Casimir algebras of higher degrees. We discuss this when we review the construction of $W$ algebras as Casimir algebras where the higher spin fields are realised as higher order Casimir operators.

**D.3.1 Cartan-Weyl basis for the $osp(1|4)$ superalgebra.**

A Cartan-Weyl basis for $osp(M|N)$, $(M = 2m + 1, N = 2n)$ is constructed by taking a basis of $(M + N)^2$ matrices $(e_{IJ})_{KL} = \delta_{IL} \delta_{KL}$, $IJKL = 1, \ldots, M + N$ and defining the following graded matrix [71]:

$$G_{IJ} = \begin{pmatrix}
0 & 1_m & 0 \\
1_m & 0 & 0 \\
0 & 0 & 1 \\
0 & 1_n \\
-1_n & 0
\end{pmatrix}. \quad \text{(D.26)}$$

We next divide the indices $I, J$ into small unbarred indices $i, j = 1, \ldots, M$ and small barred indices $\bar{i}, \bar{j}$ from $M + 1, \ldots, M + N$. The generators of $osp(1|4)$ are then given by:

$$E_{ij} = G_{ik} e_{kj} - G_{jk} e_{ki},$$
$$E_{i\bar{j}} = G_{ik} e_{\bar{k}j} + G_{\bar{j}k} e_{ki},$$
$$E_{i\bar{j}} = E_{\bar{j}i} = G_{ik} e_{\bar{k}j}. \quad \text{(D.27)}$$

The $E_{ij}$ generate the $so(M)$ subalgebra, and are antisymmetric in $i, j$. The $E_{i\bar{j}}$ generate the $sp(N)$ subalgebra, and are symmetric in $\bar{i}, \bar{j}$. The even graded elements are generated by the $E_{ij}$ and $E_{i\bar{j}}$; and the odd graded elements are generated by the $E_{i\bar{j}}, E_{\bar{j}i}$. The Cartan subalgebra is lastly represented as

$$H_i = e_{i+1,i+1} - e_{i+3,i+3}. \quad \text{(D.28)}$$

The commutation relations for general $M, N$ are listed in [71]. Here we are interested in $M = 1$, in which case they reduce to:

$$[E_{i\bar{j}}, E_{i\bar{k}}] = -G_{i\bar{j}} e_{i\bar{k}} - G_{i\bar{k}} e_{i\bar{j}} - G_{i\bar{k}} e_{i\bar{j}} - G_{i\bar{j}} e_{i\bar{k}}.$$
$$[E_{i}, E_{\bar{j}}] = -G_{i\bar{j}} E_{\bar{k}} - G_{i\bar{k}} E_{\bar{j}}.$$
$$\{E_{i}, E_{\bar{j}}\} = E_{i\bar{j}}. \quad \text{(D.29)}$$

where the $E_i \equiv E_{1i}$.
E  Extended symmetry algebras

In this appendix we will gather some needed results regarding 2d CFT’s, Super Virasoro algebras and (super)-W algebras. We will start by reviewing aspects of the Virasoro algebra and afterwards turn to its extensions. Since the focus will be on 2 dimensions only, we will use the terms conformal weight \( h, \bar{h} \), conformal dimension \( \Delta \) and spin-\( s \) interchangeably. In the present case these three notions are equivalent, as we will focus on a single chiral copy. In general dimension the conformal dimension is \( \Delta = h + \bar{h} \), whereas the spin is \( s = h - \bar{h} \). This section will follow references such as [72][73], and [74].

E.1 Basic Virasoro results

In an 2d CFT, the main object is the conserved stress energy tensor \( T_{\mu\nu}(\vec{x}) \). It defines a conserved current \( j_\mu = T_{\mu\nu}\epsilon^\nu \) associated with a conformal symmetry \( x^\mu \to x^\mu + \epsilon^\mu \). This can be used to show that \( T_{\mu\nu} \) is conserved, symmetric and traceless. Converting to complex coordinates \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \), the conformal transformations become analytic transformations \( z \to f(z), \bar{z} \to \bar{f}(\bar{z}) \). The zero trace condition implies that stress tensor splits into components \( T \) and \( \bar{T} \). Such primaries satisfy the following OPE with the stress energy tensor:

\[
\phi(z) \phi(\bar{z}) = \frac{h \phi(z)}{(z-\bar{z})^2} + \frac{\partial \phi(z)}{z-\bar{z}} + \ldots
\]  

(E.3)

We will in this thesis only focus on the holomorphic sector. There is an infinite number of conserved currents:

\[
\partial(\epsilon(z)T(z)) = 0 \quad \partial(\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})) = 0.
\]  

(E.2)

Later we shall discuss how conserved currents like these with higher/half integer spin will yield extensions of the Virasoro algebra. First we will discuss the Virasoro algebra in more detail.

The stress tensor satisfies the following short distance operator product expansion with itself:

\[
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots
\]  

(E.3)

where the \( \ldots \) represent regular terms in the limit \( z \to w \) in the OPE. The parameter \( c \) is as usual the central charge. Let us next define the primary fields. A field \( \phi \) is called primary of conformal dimension \( (h, \bar{h}) \) if it transforms under conformal transformations \( z \to f(z), \bar{z} \to \bar{f}(\bar{z}) \) as

\[
\phi'(z, \bar{z}) = \phi(f(z), \bar{f}(\bar{z}))(\frac{df}{dz})^h (\frac{d\bar{f}}{d\bar{z}})^{\bar{h}}.
\]  

(E.4)

Such primaries satisfy the following OPE with the stress energy tensor \( T(z) \), which generates local scale transformations:

\[
T(z)\phi_h(w) = \frac{h \phi_h(w)}{(z-w)^2} + \frac{\partial \phi_h(w)}{z-w} + \ldots
\]  

(E.5)

Note that the stress tensor is not a primary itself as is seen from [E.3]. It is a quasi primary, which we shall define momentarily. We define the Laurent modes of the stress energy tensor, denoted \( L_n \), as

\[
T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad L_n = \oint_{\gamma_0} \frac{dz}{2\pi i} z^{n+1} T(z)
\]  

(E.6)

with the contour of integration \( \gamma_0 \) surrounding the origin counterclockwise. Using this, the OPE [E.5] translates into the famous Virasoro algebra:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}.
\]  

(E.7)

It is the central extension of the Witt-algebra generated by the local analytic transformations

\[
l_n = -z^{n+1} \frac{d}{dz}
\]  

(E.8)
of $z$. The globally well defined $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of translations, global scale transformations, rotations and special conformal transformations is spanned by $L_{\pm 1}, L_0$, in which case the central charge vanishes. The OPE for a primary $\phi_h(z)$ translates into:

$$[L_n, \phi_h(z)] = (n + 1) z^n h \partial \phi_h(z) + z^{n+1} \partial \phi_h(z).$$  \hspace{1cm} (E.9)

Using this we call $\phi_h(z)$ a quasi-primary field if the above relation holds for $n = 0, \pm 1$ only. This indeed identifies the stress energy tensor as a quasi primary of conformal dimension $h = 2$. As a final ingredient, we mention that those fields that are not (quasi)-primary are known as descendants and they are constructed from derivatives of (quasi)-primaries, i.e. by acting with the raising operators. In the next section we will discuss some further aspects of the Virasoro algebra.

**E.1.1 Verma Modules and the Kac determinant**

Next we are interested in the Hilbert space of physical states of a CFT. Conformal invariance tells us that all states assemble into representations of the Virasoro-algebra. The relevant ones are those for which the Hamiltonian $L_0 + \bar{L}_0$ is bounded from below, which means that the highest weight representations are what we should study.

The highest weight vector is an eigenstate of $L_0$, with the lowest eigenvalue. We denote it by $\left| h, \bar{h} \right>$ and it is defined by the following conditions:

$$L_0 \left| h, \bar{h} \right> = h \left| h, \bar{h} \right>, \quad L_{-n} \left| h, \bar{h} \right> = 0 \quad n > 0.$$ \hspace{1cm} (E.10)

$$\bar{L}_0 \left| h, \bar{h} \right> = \bar{h} \left| h, \bar{h} \right>, \quad \bar{L}_{-n} \left| h, \bar{h} \right> = 0 \quad n > 0.$$ \hspace{1cm} (E.11)

In the following we will again focus on just the holomorphic part.

Using the highest weight state we can define the Verma Module, denoted $V(h,c)$. It consists of all finite linear combinations of the descendant states:

$$V(h,c) = \text{span}_C \left\{ L_{-n_1} L_{-n_2} \ldots L_{-n_k} |h> \mid n_i > 0 \right\}.$$ \hspace{1cm} (E.12)

The Verma modules admit an $L_0$ eigenspace decomposition

$$V(h,c) = \bigoplus_{N \geq 0} V(h,c)_N$$ \hspace{1cm} (E.13)

with

$$V(h,c)_N = \{ |v> \in M(h,c) : L_0 |v> = (h + N) |v> \}$$ \hspace{1cm} (E.14)

which follows from $[L_0, L_{-m}] = mL_{-m}$ and (E.11). A basis for such an eigenspace $V(h,c)_N$ is given by the states

$$L_{-n_1} \ldots L_{-n_k} |h, c>,$$ \hspace{1cm} \sum_{i=1}^{k} n_i = N, \quad n_1 \geq n_2 \geq \ldots \geq n_k > 0.$$ \hspace{1cm} (E.15)

The number $N$ is known as the level. The dimension of the eigenspace $V(h,c)_N$ at level $N$ is given by the Euler partition function $p(N)$.

The hermiticity conditions $L_{-n} = L_n^\dagger$ together with a normalization $\langle h, c|h, c \rangle = 1$ defines a symmetric bilinear form $\langle \cdot | \cdot \rangle$ on the Verma Module $V(h,c)$. The above eigenspace decomposition, (E.13), is orthogonal with respect to this bilinear form, meaning that states at different levels are orthogonal.

**Irreducibility and unitarity of the Verma module**

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82One can explicitly show, that these globally defined generators, agree with the killing vectors of AdS in Poincare coordinates at the conformal (planar) boundary $z = 0$, with $z$ now playing the role of the radial coordinate.
In general the Verma module will not be irreducible, i.e. it will contain invariant subspaces. These invariant subspaces are generated by null-states, i.e. zero norm states. The null states, as well as their descendants, are orthogonal to all other states in the Verma Module and they can thus be safely modded out. One can prove that once the null submodules are modded out from the Verma Module, one is left with an irreducible highest weight module

\[ L(h, c) = \frac{V(h, c)}{\{\forall |v\rangle | \langle v|v\rangle = 0\}}. \]  

(E.16)

The second issue of the Verma module that we discuss is unitarity. Unitarity of the Verma module means that all of its states must have a positive norm. This condition puts bounds on the parameters \( h, c \). For example from \( \langle h, c | L_1 L_{-1} | h, c \rangle \) we find that \( h > 0 \). A second condition can be found from the two point function of the stress energy tensor \( T = L_{-1} \). It tells us that: \( \langle h, c | L_2 L_{-2} | h, c \rangle \geq 0 \implies c \geq 0 \). However, we must also guarantee the absence of linear combinations with negative norm. To have a more systematic approach, we turn to the matrix \( M(h, c) \), which is the matrix of inner products of a set of basis vectors of \( V(h, c) \). Since the decomposition (E.13) is orthogonal, this matrix takes a block diagonal form with blocks \( M_N(h, c) \) for each level \( N \). The dimension of \( M_N(h, c) \) is \( p(N) \). The condition that a state \( |v\rangle = \sum_i \Lambda_i |i\rangle \) has a vanishing norm means that:

\[ ||v||^2 = \sum_{i,j} \Lambda_i \langle i|j\rangle \Lambda_j = \Lambda^T M \Lambda = 0 \]  

(E.17)

implying that \( \Lambda \) is an eigenvector of \( M \) with 0 eigenvalue. Requiring that all such states have a positive norm means that we must require this matrix to be positive definite and by virtue of (E.13), we may do so at each level \( N \). This means that we must require that:

\[ \det M_N(h, c) > 0. \]  

(E.18)

(E.18) is called the Kac determinant. In fact, there is a general formula for the Kac determinant. With a careful analysis it can be shown that unitarity restricts the values of \( h, c \) of irreducible representations to:

- \( c \geq 1, \quad h \geq 0 \)
- \( c = \left(1 - \frac{6}{m(m+1)}\right) \quad m = 2, 3, \ldots \) in which case there are only a finite number of allowed values for \( h \)

E.2 Normal ordering prescription

Before we discuss extensions of the Virasoro algebra in the next section, we will define a normal ordering prescription, that is appropriate for local quantum operators. With such a prescription we can define a regular product of local operators at the same point in spacetime.

As a first step we decompose a product of operators in its regular and singular parts:

\[ A(z)B(w) = \text{regular part} + \sum_{r=0}^{\infty} \frac{(z-w)^r}{r!} : A\partial^r B : (w) \quad r \in \mathbb{Z}. \]  

(E.19)

The singular part is also called the (Wick) contraction, obtained for the negative values for \( r \). It defines the OPE of the local operators \( A(z) \) and \( B(w) \). For the purpose of defining normal ordering it is not important. Here : : denotes normal ordering\(^{83}\).

It is the constant, \( z \) independent, term in (E.19) that we need to define our regular product of local

\(^{83}\)Other common notations used in the literature are \( N(AB), (AB) \) or \( [AB]_0 \)
operators. We can find a formula for \( AB : (w) \) by applying \( \frac{1}{2\pi i} \oint dz (z - w)^{-1} \) to (E.19). This way we pick up the \( r = 0 \) term and find:

\[
AB : (w) = \oint_{C(w)} \frac{dz}{2\pi i} A(z)B(w).
\]

(E.20)

As is well known, an alternative definition of normal ordering is to require that the annihilation operators are always placed to the right of the creation operators. Let us see next how these two definitions relate. For this we consider the Laurent mode expansions of the normal ordered product given by:

\[
AB : (z) = \sum_{n \in \mathbb{Z}} z^{-n-h_A-h_B} AB : \gamma_n
\]

(E.21)

Then an explicit expression for the Laurent modes of the normal ordered product : \( AB : (z) \) may be found by using (E.20) inside (E.21) and performing the contour integrals. This results in the following formula for the Laurent modes of a normal ordered product:

\[
AB : \gamma_n = \sum_{p \leq -h^A} A_p B_{n-p} + \sum_{p > -h^A} B_{n-p} A_p.
\]

(E.22)

Note though that normal ordering is not commutative : \( AB : (z) \not\equiv BA : (z) \). Normal ordered products of more than two currents are defined recursively.

\[
AB : \gamma_n = : A : B \gamma_n.
\]

(E.23)

In this thesis we only need a formula for three operators in which case we have:

\[
ABC : \gamma_n = : A : BC : \gamma_n.
\]

(E.24)

Furthermore, from (E.26) we easily find a formula for the modes of derivatives of operators:

\[
(\partial A)_n = -(n + h_A) A_n.
\]

(E.25)

Similar to (E.6) a current of conformal dimension \( h \) may be expanded in its Laurent modes as:

\[
A(z) = \sum_{m \in \mathbb{Z}} A_m z^{-m-h_A}, \quad A_m = \oint_{C_0} \frac{dz}{2\pi i} z^{m+h_A-1} A(z).
\]

(E.26)

E.2.1 The Operator Product Expansion

In general the OPE for two primary fields can be expressed in terms of the other primaries of the theory and their descendants. The structure of these OPE expressions can become quite complicated and we refer to [74] for the general expressions. In the case of two quasi-primaries the structure simplifies, because one needs only consider the descendants (derivatives) of quasi-primaries\(^{84}\). For \( \phi^i, \phi^j \) of conformal dimensions \( h_i, h_j \) it takes the general form:

\[
\phi_i(z)\phi_j(w) = \sum_{k,n \geq 0} C^{ij}_{nk} a_n^{(ijk)} \frac{\partial^n \phi^k(w)}{(z-w)^{h_i+h_j+h_k-n}}.
\]

(E.27)

The coefficients \( a_n^{(ijk)} \) are given in terms of the conformal weights as:

\[
a_n^{(ijk)} = \frac{(h_i-h_j+h_k)_n}{(2h_k)_n}.
\]

(E.28)

with the notation \( (x)_n = \Gamma(x+n)/\Gamma(x) \) and the coefficients \( C^{ij}_{nk} \) are known as the structure constants. The latter are algebraic expressions of the central charge \( c \). They can be determined by requiring the operator algebra to be associative. Alternatively, one may impose that the corresponding commutator algebra of the modes satisfies the Jacobi identity:

\[
[A_m, [B_n, C_r]] + [C_r, [A_m, B_n]] + [B_n, [C_r, A_m]] = 0.
\]

(E.29)

E.3 Extensions of the Virasoro algebra

As we have seen in the previous sections the Virasoro algebra has conserved currents $J^{(2)} = \epsilon(z)T(z)$ satisfying $\partial J^{(2)}(z) = 0$. There is a rather simple way to extend the success of the Virasoro algebra, which is by adding primary fields $Q^{(s)}(z)$ with integer or half integer spin-$s$. Such CFT’s will have extra conserved currents $J^{(s)}(z) = \xi(z)Q^{(s)}(z)$ that satisfy $\partial J^{(s)}(z) = 0$. These additional conserved currents will then extend the Virasoro algebra with an additional infinite set of generators. In this section we will review the $N = 1$ and $N = 2$ superconformal algebras where additional currents of weight 1/2 are added and after that discuss $W$ algebras. These extend the Virasoro algebra by additional higher spin currents of integer conformal weight.

E.3.1 Higher Spin currents: Zamolodchikovs construction

The first systematic approach for higher spin symmetries was formulated by Zamolodchikov[75]. In this section we will review this construction and discuss several examples.

We assume that, besides the stress energy tensor, there are additional holomorphic spin-$s$ fields $Q^{(s)}$. When no other fields are present their operator product expansion must schematically read:

$$Q^{(s)}Q^{(s)} \sim a[1] + bQ^{(s)}.$$  

Here the notation $[\phi]$ denotes the contribution of the entire conformal family associated to a primary $\phi$, i.e. all its descendants. For those above we have:

$$[1] = x^{-2s}(1 + x^2\beta_2^{(0)}L_{-2} + x^4\beta_4^{(0)}L_{-4} + \ldots)\,1,$$  

$$[Q^{(s)}] = x^{-s}(1 + x\beta_1^{(s)}L_{-1} + x^2\beta_2^{(s)}L_{-2} + \ldots)Q^{(s)},$$

where we introduced the shorthand $x = z - w$. This is easily seen to generalise to

$$[\phi] = x^{-2s+b_c}(1 + x\beta_1^{(s)}L_{-1} + x^2\beta_2^{(s)}L_{-2} + \ldots)\phi.$$  

Note that the stress-energy tensor in contained in $[E.30]$, since it is a descendant of the identity, i.e. $T = L_{-2}1$. The coefficients $\beta$ in front of the descendants, are completely fixed by conformal invariance in terms of the central charge $c$. The value of $a$ is merely a normalization of the field $Q^{(s)}$ and is conventionally chosen to be $a = c/s$. The only free parameters are therefore the central charge $c$ and the coefficient $b$.

They are constrained by associativity of the operator algebra. By calculating the coefficients $\beta_{\{i,j\}}^{(s)}$ and the first number of descendants one can find the following general OPE:

$$Q^{(s)}(z)Q^{(s)}(w) = \frac{a}{(z-w)^{2s}} + \frac{2T(w)}{(z-w)^{2s-2}} + \frac{\partial T(w)}{(z-w)^{2s-3}} + \frac{1}{(z-w)^{2s-4}} \left[ \frac{3}{10} \partial^2 T(w) + 2\gamma \Lambda(w) \right] + \ldots$$

$$+ \frac{1}{(z-w)^{2s-n}} \left[ \frac{1}{15} \partial^3 T(w) + \gamma \partial \Lambda(w) \right] + \ldots$$

$$+ \frac{bQ^{(s)}(w)}{(z-w)^s} + \frac{b\partial Q^{(s)}(w)}{(z-w)^{s-1}} + \ldots$$

where

$$\gamma = \frac{5s + 1}{22 + 5c},$$

and

$$\Lambda(z) = : TT : (z) - \frac{3}{10} \partial^2 T.$$  

Note that if $Q^{(s)}$ has half integer spin then $b = 0$. Also when $s$ is an odd integer we must take $b = 0$ since otherwise there would be a contradiction with the $z \leftrightarrow w$ symmetry. The complete operator algebra is now given by $TT, TQ^{(s)}$ and $Q^{(s)}Q^{(s)}$. However, for this full algebra to be associative, the free parameters need to be chosen appropriately. Restrictions one their values come from crossing symmetry of the four point function, of equivalently, imposing the Jacobi identity on the mode algebra. It turns out that in general it will not always be possible to ensure associativity for all values of $b, c$. In the following we will look at a few examples of extended symmetry algebras following this approach.
### E.3.2 Spin-3/2 current and the $N=1$ superconformal algebra

In the first example we look at we take $s = 3/2$ and define $Q^{(3/2)} \equiv G$. In this case becomes:

$$G(z)G(w) = \frac{2c}{3(z - w)^2} + \frac{2T(w)}{(z - w)} + \ldots, \quad (E.38)$$

with the \ldots representing the regular terms. One can show that by imposing crossing symmetry on the four point function, or the Jacobi identity on the mode algebra, there are no restrictions on the central charge $c$. We can translate the $T(z)G(w)$ OPE and the above into mode (anti)-commutators to find:

$$[L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r}, \quad (E.39)$$

$$[G_r, G_s] = 2L_{r+s} + \frac{1}{3}c\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.$$  

Combined with the $[L_m, L_n]$ commutator these commutation relations define the $N=1$ superconformal algebra. The Neveu-Schwarz sector is defined by $r, s \in \mathbb{Z} + \frac{1}{2}$ and in the Ramond sector we has $r, s \in \mathbb{Z}$. The globally defined $\mathfrak{osp}(1|2)$ subalgebra is spanned by $\{L_0, L_{\pm}, G_{\pm}\}$ and belongs to the Neveu-Schwarz sector. The vacuum $|0\rangle$ is invariant under this subalgebra. The ground state of the Ramond sector is the state with $h = \frac{c}{24}$. Thus, by adding a spin-3/2 field we have obtained a supersymmetric extension of the Virasoro algebra. We call $G$ the supercurrent.

### E.3.3 $N=2$ superconformal algebra

To construct conformal field theories with $N$-extended supersymmetry, we must add several spin-3/2 fields $G^i$, $i = 1, 2, \ldots, N$. Besides that, it turns out that closure of the operator algebra requires the inclusion of additional spin-1 current $J$, that transform the supercharges into one another. These spin-1 currents generate by themselves an internal Kac-moody current algebra and are known as the $R$-symmetry generators. If we restrict to $N=2$, we thus need two supercharges $G^{\pm}$ both of spin-3/2, and a $SO(2) \sim U(1)$ spin-1 current $J$. Constructing the OPE relations with a generalization of $[E.30]$ and $[E.35]$ and translating them into commutation relations, we find:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$

$$[L_m, J_n] = -nJ_{m+n},$$

$$[L_m, G^{\pm}_r] = \left( \frac{m}{2} - r \right) G^{\pm}_{m+r}, \quad (E.40)$$

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0},$$

$$[J_m, G^{\pm}_r] = \pm G^{\pm}_r,$$

$$\{G^{+}_r, G^{-}_s\} = 2L_{r+s} + (r - s)J_{r+s} + \frac{1}{3}c\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},$$

$$\{G^{\pm}_r, G^{\pm}_s\} = 0.$$  

$N=2$ algebras have a very interesting, and very useful, property, known as spectral flow. Spectral flow refers to an automorphism of the algebra, which for the specific $N=2$ case reads:

$$L_n' = L_n + \eta J_n + \frac{\eta}{6}\delta_{n,0},$$

$$J_n' = J_n + \frac{c}{3}\eta\delta_{n,0},$$

$$G^{\pm}_r' = G^{\pm}_r.$$  

---

\[\text{Footnote: We will define this in the next section. See } (E.45).\]
for any \( \eta \). The spectral flow interpolates between the Ramond sector and Neveu-Schwarz sector by choosing \( \eta \in \mathbb{Z} \) or \( \eta \in \mathbb{Z} + \frac{1}{2} \). The spectral flow automorphism is useful, because it allows one to perform computations in the NS sector, and obtain the equivalent result in the Ramond sector via spectral flow, without the need of dealing with fermionic zero modes.

E.3.4 Spin-5/2: \( \mathcal{W}(\frac{5}{2}) \) algebra

Adding a field of spin-5/2 does not yield any interesting extensions. One can show, that associativity of the operator algebra uniquely fixes the central charge to \( c = -\frac{13}{14} \). In the next section we will discuss an extension of this algebra by adding a spin-4 field. This algebra is \( \mathcal{W}(2, \frac{5}{2}, 4) \) and exists for any value of the central charge.

E.3.5 Spin-3 current and the \( \mathcal{W}_3 \) algebra

The last example we will discuss is when we add a field of spin \( s = 3 \). As explained above we must take \( b = 0 \) because a term \( \mathcal{W}^{(3)}(z)\mathcal{W}^{(3)}(w) \) OPE would be in contradiction with \( z \leftrightarrow w \) symmetry. We can then write down the following OPE:

\[
\mathcal{W}^{(3)}(z)\mathcal{W}^{(3)}(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^3} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \left[ \frac{3}{10} \partial^2 T(w) + 2\gamma_3 \Lambda(w) \right] + \frac{1}{z-w} \left[ \frac{1}{15} \partial^3 T(w) + \gamma_3 \partial \Lambda(w) \right].
\]

with now \( \gamma_3 = \frac{16}{22+5c} \). Imposing crossing symmetry on the four point function one finds that the operator algebra is associative for all values of the central charge. Converting the OPE to commutation relations one finds:

\[
[W_n, W_m] = (m-n) \left[ \frac{1}{15} (n+m+2)(n+m+3) \frac{1}{6} (m+2)(n+2) \right] L_{n+m} + \frac{16}{22+5c} (n-m) \delta_{n+m} + \frac{c}{360} (m^4 - 4)(m^2 - 1) m \delta_{n,-m},
\]

with \( \Lambda_m \) the modes of \( \mathcal{W}_3 \). This algebra, together with the commutation relations for \( L_n \) define the \( \mathcal{W}_3 \) algebra. Important to note though, is that the mode algebra does not close on the simple fields only. It is unavoidable to introduce the composite quasi primary operators to close the algebra. Formally, we say that the \( \mathcal{W} \) algebra closes in the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) which consists of all formal power series in elements of \( \mathfrak{g} \). This feature is responsible for the non-linear nature of \( \mathcal{W} \) algebras. In fact, quantum \( \mathcal{W} \) algebras are not Lie-algebras at all, due to these non-linearities. A lie-algebra is recovered in the limit \( c \to \infty \) though, and as we have seen, they play an important role in the context of higher spin gravity.

The unitary representations of this algebra have also been classified. The minimal unitary models are found for the following discrete series of the central charge:

\[
c = 2 \left( 1 - \frac{12}{m(m+1)} \right) \quad m = 4, 5, \ldots
\]

E.4 More background on \( \mathcal{W} \) algebras

In this section we will give an overview of some more results on \( \mathcal{W} \) algebras. Firstly, we can distinguish two types of \( \mathcal{W} \) algebras.

- "Generic" \( \mathcal{W} \) algebras, that exist for any value of the central charge \( c \).
- "Exotic" \( \mathcal{W} \), that exist only for certain values of \( c \).
An example of the first type is the $W_3$ algebra \[^{86}\] and an example of the second type is the $W(2, \frac{3}{2})$ algebra that only closes for $c = -\frac{12}{7}$. 

Secondly, there are several ways to construct $W$ algebras. In the previous section, we discussed the "brute force" approach. We added higher spin currents, wrote down the general OPE using conformal symmetry and imposed closure of the algebra/associativity of the four-point function. Clearly, it is the last step, associativity, that makes this approach very complicated as the spins increase. However, algorithms for this construction have been made which has opened the doors for the construction of many new $W$ algebras. In the following sections we will "superficially" discuss two other methods to construct $W$ algebras. The first of these, which we will not discuss, is quantum Drinfeld-Sokolov reduction. This can be seen by realizing the currents in terms of free bosons as \[^{87}\] and using that the stress energy tensor has the form

\[^{73}\] where we also used $d_{ab}j^b = \dim g$. This construction of the stress energy tensor, starting from an affine lie algebra, is known as the Sugawara construction. The crucial point to note, is that it \[^{46}\] defines in fact the second order Casimir element \[^{D.24}\]. The construction we discussed here, applies to simple lie algebras $g$ of rank $l = 1$. For those algebras, there is a single Casimir element which is precisely given by
With this observation though, it is in fact rather straightforward to extend this construction to higher rank simple algebras. This goes under the name of the Casimir construction and naturally gives rise to a class of \( W \) algebras known as Casimir algebras. Before discussing this generalization we will first discuss the coset construction, or Goddard-Kent-Olive coset construction. This construction also naturally generalizes to higher rank algebras which we will also discuss in the next section.

**E.4.2 The Coset construction, a.k.a. GKO construction**

We consider a Lie algebra \( \mathfrak{g} \) with a proper subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). We write \( \hat{\mathfrak{g}}_{\mathfrak{h}} \) and \( \hat{\mathfrak{h}}_{\mathfrak{g}} \) for their associated Kac Moody algebras which are generated by currents \( j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \) and \( j^b_{\hat{\mathfrak{h}}_{\mathfrak{g}}} \) respectively. The Sugawara stress energy tensors then read:

\[
T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} = \frac{1}{2(k_\mathfrak{g} + C_\mathfrak{g})} \sum_{a=1}^{\text{dim} \mathfrak{g}} : (j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}} j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}}) : (z), \tag{E.48}
\]

\[
T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} = \frac{1}{2(k_\mathfrak{h} + C_\mathfrak{h})} \sum_{b=1}^{\text{dim} \mathfrak{h}} : (j^b_{\hat{\mathfrak{h}}_{\mathfrak{g}}} j^b_{\hat{\mathfrak{h}}_{\mathfrak{g}}}) : (z), \tag{E.49}
\]

where we have chosen an orthonormal basis \( d_{ab} = \delta_{ab} \). Note that the current \( j^b_h \) corresponding to \( \hat{\mathfrak{h}}_{\mathfrak{g}} \) is a primary of dimension \( h = 1 \) with respect to both energy momentum tensors:

\[
T_g(z) j^b_h (w) \equiv T_h(z) j^b_h (w) \approx \frac{j^b_h (w)}{(z-w)^2} + \frac{\partial_w j^b_h (w)}{z-w} + \ldots, \tag{E.50}
\]

Taking the difference of these OPE’s we end up with:

\[
(T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} - T_{\hat{\mathfrak{h}}_{\mathfrak{g}}}) (z) j^b_h (w) = \text{regular},
\]

\[
(T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} - T_{\hat{\mathfrak{h}}_{\mathfrak{g}}}) (z) T_h (w) = \text{regular}. \tag{E.51}
\]

This motivates to make a decomposition:

\[
T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} / \hat{\mathfrak{h}}_{\mathfrak{g}} + T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} \quad \Rightarrow \quad T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} / \hat{\mathfrak{h}}_{\mathfrak{g}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} - T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} \tag{E.52}
\]

which decomposes the Virasoro algebra generated by \( T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \) into two mutually commuting Virasoro subalgebras, generated by \( T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} / \hat{\mathfrak{h}}_{\mathfrak{g}} \) and \( T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} \), since the OPE’s \( (E.51) \) are regular. In terms of the modes this simply means that:

\[
\{ L_{m,\mathfrak{g}}^{\mathfrak{h}} / \mathfrak{h}, \mathfrak{g} \} = 0. \tag{E.53}
\]

The central charge of the coset model generated by \( T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \), is found from either employing the relation:

\[
T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} + T_{\hat{\mathfrak{h}}_{\mathfrak{g}}} T_{\hat{\mathfrak{g}}_{\mathfrak{h}}}, \tag{E.54}
\]

or computing \( [L_{m,\mathfrak{h}}^{\mathfrak{g}}, L_{n,\mathfrak{h}}^{\mathfrak{g}}] \). Either way, one finds a central charge:

\[
c_{\mathfrak{g} / \mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}} = \frac{k_{\mathfrak{g}} \text{ dim } \mathfrak{g}}{k_{\mathfrak{g}} + C_{\mathfrak{g}}} - \frac{k_{\mathfrak{h}} \text{ dim } \mathfrak{h}}{k_{\mathfrak{h}} + C_{\mathfrak{h}}}. \tag{E.55}
\]

An important type of coset theories are those based on the diagonal cosets \( \hat{\mathfrak{g}}_{\mathfrak{h}} \ subgroup \). In this case the algebras \( \hat{\mathfrak{g}}_{\mathfrak{h}} \) are generated by currents \( j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \) and \( \hat{\mathfrak{g}}_{\mathfrak{h}} \) is generated by \( j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \) and \( j^b_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \). Then by definition we have \( [j^a_{\hat{\mathfrak{g}}_{\mathfrak{h}}}, j^b_{\hat{\mathfrak{g}}_{\mathfrak{h}}}] = 0 \) and so the structure constants and level \( k \) of the Kac moody algebra \( \hat{\mathfrak{g}}_{\mathfrak{h}} \) of the combined currents are simply \( f^{abc} = f_1^{abc} + f_2^{abc} \) and \( k = k_1 + k_2 \). Similar to the previous discussion, the energy momentum tensor for this coset model is:

\[
T_{\hat{\mathfrak{g}}_{\mathfrak{h}} / \hat{\mathfrak{g}}_{\mathfrak{h}}} = T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} + T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} - T_{\hat{\mathfrak{g}}_{\mathfrak{h}}} \tag{E.56}
\]

and the central charge is this given by:

\[
c = \text{ dim } \mathfrak{g} \left( \frac{k_1}{k_1 + C_{\mathfrak{g}}} + \frac{k_2}{k_2 + C_{\mathfrak{g}}} - \frac{k_1 + k_2}{k_1 + k_2 + C_{\mathfrak{g}}} \right). \tag{E.57}
\]
An interesting, and one of the simplest, examples is when $\hat{g} = \mathfrak{su}(2)$. It turns out that the coset
\[
\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}},
\]
with $k \geq 1$, gives rise to all the Virasoro minimal models with coset central charge:
\[
c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)} \quad k \geq 1,
\]
where we used that for $\text{dim}(\mathfrak{su}(2)) = 3$, and $C_{\mathfrak{su}(2)} = 3$. Comparing this to (E.11) with $m = k + 2$ these are indeed precisely the values for the central charge of the Virasoro minimal models! Note that, the coset theory does not contain any currents, just as the Virasoro algebra. There is no subalgebra generated by $j_1^+ + j_2^+$ that commutes with all $j^+ = j_1^+ + j_2^+$ and so the coset theory does not contain any currents. Note further that $\mathfrak{su}(2)$ just has a single Casimir operator of degree 2, which is precisely the Sugawara energy momentum tensor. Following the same steps, one can next show that all the $\mathcal{N} = 1$ super-Virasoro minimal models can be found from the coset:
\[
\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_2}{\mathfrak{su}(2)_{k+1}},
\]
from which we find:
\[
c = \frac{3}{2} \left( 1 - \frac{8}{(k+2)(k+4)} \right) \quad k \geq 1.
\]

### E.4.3 Generalizing the Sugawara and coset construction: Casimir algebras

The starting point for generalizing the Sugawara construction is to consider the following generalized Casimir operators:
\[
W^s(z) = \frac{1}{n!} \eta^s(g, k) \sum_{a,b,c...} d_{abc...}(\eta^a(\eta^b(\eta^c(\ldots))))(z)
\]
We have denoted $(\ldots)$ to indicate normal ordering. $\eta^s(g, k)$ is some normalization constant and $d_{abc...}$ is a completely symmetry traceless $g$ invariant tensor of rank $s_i$. If rank $g = l$ then $i = 1, 2, \ldots, l$. The index $s_i = i + 1$. The tracelessness condition on the tensor $d$ ensures that the Casimir operators $W^s(z)$ $i = 2, \ldots, l$ are primary fields with respect to the Sugawara stress energy tensor that is recovered for $i = 1$:
\[
T(z) = W^s(z).
\]
Let us now illustrate this procedure with a specific example of the rank-2 algebra $\tilde{A}_2 = \mathfrak{su}(3)_k$. In this case there are two independent Casimir invariants of order 2 and
\[
\tilde{A}_2 \quad \text{A second order Casimir is simply} \quad T(z) = W^s(z).
\]

The OPE of the third order Casimir with itself can be shown to be:
\[
W^3(z)W^3(w) = \frac{c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^4}
\]
\[
+ \frac{1}{(z-w)^2} \left[ 2\beta \Lambda(w) + \frac{3}{10} \partial^2 T(w) + R^{(4)}(w) \right]
\]
\[
+ \frac{1}{(z-w)} \left[ \beta \Lambda(w) + \frac{1}{15} \partial^3 T(w) + \frac{1}{2} \partial R^{(4)}(w) \right]
\]
Since for $g = \mathfrak{su}(3)$ we have $\text{dim} \; g = 3$ and $C_g = 3$ the central charge is, using (E.17), $c(\mathfrak{su}(3), k) = \frac{8k}{k+3}$.

Note though, the appearance of a new spin-4 primary field $R^{(4)}(z)$. It cannot be written in terms of the Casimirs $T, W^{(3)}$ and, more importantly, it does not vanish for generic level $k$. However, one can show that for $k = 1$, this spin-4 field becomes a null field and thus decouples from the algebra from which we

\[\text{sl}(N) \cong \mathfrak{su}(N) \cong \mathfrak{so}(N), \quad \text{rk} \mathfrak{g} = \frac{N}{2}.
\]

\[\mathfrak{sl}(N) \cong \mathfrak{su}(N) \cong \mathfrak{so}(N), \quad \text{rk} \mathfrak{g} = \frac{N}{2}.
\]
conclude that the Casimir algebra closes only for $c = 2$. One concludes from this that the $A_2 = \hat{\mathfrak{su}}(3)$ Casimir algebra at level $k = 1$ is the $W_3$ algebra of Zamolodchikov [E.42] (with $c = 2$).

The coset construction discussed for the Virasoro minimal models can also be generalized. The central charge of the $W_3$ minimal models [E.44] are the same as those for the diagonal coset model [77]

$$\hat{su}(3)_k \oplus \hat{su}(3)_1 \oplus \hat{su}(2)_{k+1}$$ (E.64)

Following the same approach as for the Virasoro case, the authors constructed a spin-3 primary in $\hat{su}(3)_k \oplus \hat{su}(3)_1$ that has regular OPE with the diagonal $\hat{su}(3)_{k+1}$. As such it is thus a primary of the coset model.

Computing the central charge for this coset model, using $\text{dim} \, \mathfrak{su}(3) = 8$ and $C_{\mathfrak{su}(3)} = 3$, one finds the same discrete series as for the Zamolodchikov’s $W_3$ algebra (E.44) with $m = k + 3$. In fact, we can generalise this even further to the $W_N$. From the coset

$$\hat{su}(N)_k \oplus \hat{su}(N)_1 \oplus \hat{su}(N)_{k+1},$$ (E.65)

one finds the $W_N$ minimal models with central charge:

$$c = (N - 1) \left( 1 - \frac{N(N + 1)}{p(p + 1)} \right),$$ (E.66)

with $p = k + N \geq N + 1$ As before, the Casimir invariants that generate the $W_N$ algebra are those of $\hat{su}(N)_k \oplus \hat{su}(N)_1$ and they commute with the diagonal $\hat{su}(N)_{k+1}$.

Remark

In the literature the notation that is often used to denote the Casimir algebra is to write e.g. $WA_2$ for the Casimir algebra of $A_2$. We note that only a few of such Casimir algebras have been explicitly constructed in the literature. These are the Virasoro and Super-Virasoro algebra that correspond to $WA_1$ and $WB_1$, Zamolodchikov’s $W_3$ algebra corresponding to $WA_2$, and very recently $WA_3$[78]. In the next section we discuss the $WB_2$ algebra which can be realised as the Casimir algebra of $B_2$.

$W_N$ minimal models and higher spin holography

(E.66) is the final equation of this section. The above discussed coset $W$ minimal models based with central charge given by [E.66] turn out to play an important role in higher spin holography. This relation is discussed in detail in [2] and [3] and references therein. To understand this, we first need to discuss a particular limit of the $W_N$ minimal models which is the so called ‘t Hooft limit:

$$N, k \to \infty \quad 0 \leq \lambda = \frac{N}{N + k} \leq 1 \text{ kept fixed},$$ (E.67)

$k$ is as before the level of the affine lie algebra. $\lambda$ is what is known as the ‘t Hooft coupling. Gutperle and Gopakumar showed that in this limit, the $W_{N,k}$ minimal models at central charge $c = c_{N,k}$ given by (E.66) are equivalent to the $W_{\infty}[\mu]$ algebra, at the same value of the central charge and with $\mu = \lambda$ the ‘t Hooft coupling.

On the gravity side, they showed that the asymptotic symmetry algebra of the $\mathfrak{hs}[\mu]$ higher spin theory, corresponded to the large $c$ limit of the $W_{\infty}[\mu]$ algebra. Based on this observation, they conjectured in [2] that:

90The ‘t Hooft coupling makes an entrance in the non-abelian $SU(N)$ Yang-Mills theory. It is then defined in terms of the coupling strength $g_{YM}$ as $\lambda = g_{YM}^2 N$ and $4\pi g_s^2 = g_s$. ‘t Hooft proved that when $N \to \infty$, whilst $\lambda$ is kept fixed, string theory can be well approximated by classical gravity.
• Vasiliev $\mathfrak{hs}[\mu]$ higher spin theory on AdS$_3$, is dual to the above t Hooft limit of the $\mathcal{W}_N$ minimal models, where the 't Hooft coupling coincides with $\mu = \lambda^{91}$.

There is one subtlety though. The symmetry algebras of the $\mathfrak{hs}[\mu]$ higher spin theories and the 't Hooft limit of the $\mathcal{W}_N$ minimal models, are both $\mathcal{W}_\infty$ algebras, but a priori there is no reason why these should be the same $\mathcal{W}_\infty$ algebra. This issue was resolved though in [79], where they used the method described in subsection 4.4.4 to quantise the the asymptotic symmetry algebra of the $\mathfrak{hs}[\mu]$ higher spin theory. They observed a non-trivial agreement between the resulting $\mathcal{W}_\infty[\mu]$ quantum algebra and the $\mathcal{W}_\infty[\lambda]$ algebra arising as the 't Hooft limit of the $\mathcal{W}_N$ minimal model algebras, in favour of the proposed duality. There have been further checks in favour of the duality. For example, it has been shown that the perturbative spectrum of the higher spin AdS theory, matches precisely with the perturbative CFT spectrum. Further checks have included comparison of correlation functions and computations of black hole entropy at the CFT side.

Concerning the matching of the spectra we note lastly, that it is important that the fact that duality holds in the t Hooft limit, in which case we considering the higher spin theory in the semiclassical regime $c \to \infty$. It is only in this regime that the spectrum of the higher spin theory can be computed.

The duality also holds for $\mu = N$ an integer, in which case $\mathfrak{hs}[\mu]$ can be truncated to $\mathfrak{sl}(N)$ and a similar truncation can be accomplished for $\mathcal{W}_\infty(N)$ to $\mathcal{W}_N$.

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91 For $N = 2$, one recover the Virasoro minimal models with $c < 1$, with for example the Ising model, or Potts model as duals.
F Quantum $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra

This section will discuss the $\mathcal{W}(2, \frac{5}{2}, 4)$ algebra. Besides the Virasoro stress-tensor $W^2(z) = T(z)$ of conformal weight $h = 2$, it contains two additional currents: a bosonic current $W^4(z) = U(z)$ and the fermionic current $W^{5/2}(z) = Q(z)$. Their, conformal weights with respect to the stress energy tensor are respectively $h_U = 4$ and $h_Q = 5/2$, c.f. [F.2] and [F.3]. They are both primaries. This algebra exists for the generic central charge $c$ and their currents obey the following operator product expansions [80].

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots$$

$$T(z)Q(w) = \frac{(5/2)Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{(z-w)} + \cdots$$

$$T(z)U(w) = \frac{4U(w)}{(z-w)^2} + \frac{\partial U(w)}{(z-w)} + \cdots$$

$$Q(z)Q(w) = \frac{(2c/5)Q(w)}{(z-w)^2} + \frac{2T(w)}{(z-w)^3} + \frac{\partial T(w)}{(z-w)}$$

$$+ \frac{1}{(z-w)} \left[ \frac{3}{10} \partial^2 T(w) + \frac{27}{5c+22} \Lambda(w) + C_{ \frac{5}{2} }^4 U(w) \right] + \cdots$$

$$U(z)Q(w) = \frac{aQ(w)}{(z-w)^4} + \frac{b\partial Q(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \left[ d\partial^2 Q(w) + eTQ(w) \right]$$

$$+ \left( \frac{1}{(z-w)} \right) \left[ f\partial^3 Q(w) + g(\partial TQ)(w) + h\partial(TQ)(w) \right] + \cdots$$

$$U(z)U(w) = \frac{1}{(z-w)^8} + A \left[ \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)^3} - \frac{1}{12} \frac{\partial^3 T(w)}{(z-w)^4} + \frac{1}{120} \frac{\partial^5 T(w)}{(z-w)^6} \right]$$

$$+ B \left[ \frac{2\partial^2 T(w)}{(z-w)^2} + \frac{\partial^3 T(w)}{(z-w)^3} - \frac{1}{12} \frac{\partial^5 T(w)}{(z-w)^4} \right]$$

$$+ C \left[ \frac{2\partial^3 T(w)}{(z-w)^3} - \frac{1}{12} \frac{\partial^5 T(w)}{(z-w)^4} \right]$$

$$+ D \left[ \frac{2\Lambda(w)}{(z-w)^4} + \frac{\partial \Lambda(w)}{(z-w)^3} + \frac{1}{12} \frac{\partial^3 \Lambda(w)}{(z-w)^4} \right]$$

$$+ E \left[ \frac{2\partial^2 \Lambda(w)}{(z-w)^2} + \frac{\partial^3 \Lambda(w)}{z-w} \right]$$

$$+ F \left[ \frac{2\partial \Lambda(z)}{(z-w)^3} + \frac{\partial \Lambda(z)}{(z-w)^2} \right] + G \left[ \frac{2\Delta(w)}{(z-w)^2} + \frac{\partial \Delta(w)}{z-w} \right]$$

$$+ H \left[ \frac{2U(w)}{(z-w)^2} + \frac{\partial U(w)}{(z-w)^3} - \frac{1}{12} \frac{\partial^3 U(w)}{(z-w)^4} \right]$$

$$+ I \left[ \frac{1}{(z-w)^2} \frac{2\partial^2 U(w)}{(z-w)^2} + \frac{1}{(z-w)} \frac{\partial^3 U(w)}{(z-w)^2} \right]$$

$$+ J \left[ \frac{1}{(z-w)^2} \frac{2\partial W(w)}{(z-w)^2} + \frac{1}{(z-w)} \frac{\partial \Omega(w)}{(z-w)^2} \right]$$

Here the + $\cdots$ as before denote regular terms, which we are not interested in for deriving the mode algebra. The structure constants are all fixed in terms of the central charge $c$. As explained in the previous section this can be done by either imposing the graded Jacobi identity on the algebra of the Laurent modes [80] or by imposing associativity on the operator algebra [81].
The expressions for the composite operators are explicitly given by:

\[ :\Lambda : (z) = : TT : (z) - \frac{3}{10} : \partial^2 T : (z) \]  
(F.7)

\[ :\Xi : (z) = : T\partial T : (z) - \frac{3}{70} : \partial^4 T : (z) - \frac{31}{7(5c + 22)} : \partial^2 \Lambda : (z) \]  
(F.8)

\[ :\Delta : (z) = : TL : (z) - \frac{3}{10} : \partial^2 \Lambda : (z) \]  
(F.9)

\[ :\Omega : (z) = : TU : (z) \]  
(F.10)

\[ :\Gamma : (z) = : \partial QQ : (z) - \frac{5}{18} : \partial T\partial T : (z) \]  
(F.11)

with : : denoting normal ordering. All of these composite operators are quasi-primary.

### F.1 Mode algebra

In terms of the composite operators we find the following mode algebra:

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \]  
(F.12)

\[ [L_n, Q_r] = \left(\frac{3n}{2} - r\right)Q_{n+r} \]  
(F.13)

\[ [L_n, U_m] = (3n - m)U_{n+m} \]  
(F.14)

\[ \{Q_r, Q_s\} = \frac{c}{960}(4r^2 - 1)(4r^2 - 9)\delta_{r+s,0} + \frac{1}{20}(6r^2 - 8rs + 6s^2 - 9) \]  
(F.15)

\[ [U_n, Q_r] = \left[a\left(\frac{6}{6}(n + 3)(n + 2)(n + 1) - \frac{b}{2}(n + 2)(n + 3)\left(n + r + \frac{5}{2}\right)\right] + d(n + 3)(n + r + \frac{9}{2}) + f(n + r + \frac{9}{2})(n + r + \frac{7}{2})(n + r + \frac{5}{2}) + \Delta_{r+s} \right] \]  
(F.16)

\[ [U_n, U_m] = \left[c\left(\frac{1}{4}n(n^2 - 1)(n^2 - 4)(n^2 - 9)\delta_{n+m,0} \right) + (n - m) \right] \]  
(F.17)

where again all constants depend on the central charge only and are given in tables 5 and 6.

**Remark:** Recall from our discussion about the asymptotic $\mathcal{W}_3$ algebra that when one is interested in obtaining the quantum algebra from the asymptotic algebra, one has to implement an improved normal ordering prescription consequently changing the Jacobi identities. Thus if one starts with the $\psi$ anti-commutation relations in [11], and demands that the Jacobi identities be still satisfied, this leads to a new
quantum anti-commutator with new structure constants:
\[
\{Q_r, Q_s\} = \frac{c}{960} (9 - 40r^2 + 16r^4) \delta_{r+s,0} \\
+ \left[ \frac{1}{20} (6r^2 - 8rs + 6s^2 - 9) - \frac{81(r+s+2)(r+s+3)}{50c+220} \right] L_{r+s} \\
+ \frac{27}{5c+22} : TT \ r_{r+s} + C_4^1 \ L_{r+s}
\]
If one then absorbs the additional \( L_n \) modes through \( : TT : \Lambda : \), the algebra is expressed in the same form as the asymptotic symmetry algebra and reduces to (F.15). Furthermore, via explicit verification, we checked that the above quantum algebra matches the asymptotic symmetry algebra of the bulk charges in [11].

F.2 Mode Expansions

Next we will compute some of the explicit mode expansions that have been used for the computations in the main text.

\[ : \Lambda :_n = \sum_{p \leq -1} L_p L_{n-p} + \sum_{p \geq 1} L_{n-p} L_p + L_n L_0 + (n+2) L_n - \frac{3}{10} (n+3)(n+2) L_n. \] (F.18)

\[ : \Xi :_n = \partial T \partial T :_n - \frac{3}{70} (n+5)(n+3)(n+2) L_n - \frac{31}{7(5c+22)} (n+5)(n+4) : \Lambda :_n. \] (F.19)

\[ : \Omega :_n = (n+4) U_n + U_n L_0. \] (F.20)

\[ : \Delta :_n = : TA :_n = : TTT :_n - \frac{3}{10} : T \partial^2 T :_n - \frac{3}{10} : \partial^2 \Lambda :_n. \] (F.21)

\[ : \Gamma :_n = : \partial QQ :_n - \frac{5}{18} : \partial T \partial T :_n. \] (F.22)

with
\[ : \partial T \partial T :_n = (p+2)(n-p+2) \left[ \sum_{p \leq -1} L_p L_{n-p} + \sum_{p \geq 1} L_{n-p} L_p \right] \]
\[ + (n+3)(n+2) L_n + 2(n+2) L_n L_0. \]

\[ : T \partial^2 T :_n = (n-p+3)(n-p+2) \left[ \sum_{p \leq -1} L_p L_{n-p} + \sum_{p \geq 1} L_{n-p} L_p \right] \]
\[ + (n+4)(n+3)(n+2) L_n + (n+3)(n+2) L_n L_0. \]

\[ : \partial QQ :_n = - \left( p + \frac{5}{2} \right) \sum_{p \leq -1/2} Q_p Q_{n-p} + \left( p + \frac{5}{2} \right) \sum_{p \geq 1/2} Q_{n-p} Q_p \]
\[ + 2 \{ Q_{-1/2}, Q_{n+1/2} \} + \{ Q_{-3/2}, Q_{n+3/2} \}. \]

\(^{92}\)It is through this check that we found that the constant \( H \) in [80] should in fact be \( H/2 \).
An explicit formula for the mode expansion for :TTT: we find using (E.24) and \[ [AB,C] = A[B,C] + [A,C]B \]:

\[
:TTT: n = \sum_{p \leq -1} L_p \left( \sum_{q \leq -1} L_q L_{n-p-q} + \sum_{q \geq 1} L_{n-p-q} L_q + L_{n-p} L_0 + (n-p+2)L_{n-p} \right) + \sum_{p \geq 1} \left( \sum_{q \leq -1} L_q L_{n-p-q} + \sum_{q \geq 1} L_{n-p-q} L_q + L_{n-p} L_0 + (n-p+2)L_{n-p} \right) L_p
\]

\[
+ \left( \sum_{q \leq -1} L_q L_{n-q} + \sum_{q \geq 1} L_{n-q} L_q + L_n L_0 + (n+2)L_n \right) L_0
\]

\[
+ \sum_{q \geq 1} \left( L_q [L_{n+1-q}, L_{-1}] + [L_q, L_{-1}] L_{n+1-q} \right) + \sum_{q \geq 1} \left( L_{n+1-q} [L_q, L_{-1}] + [L_{n+1-q}, L_{-1}] L_q \right)
\]

\[
+ L_{n+1} [L_0, L_{-1}] + [L_{n+1}, L_{-1}] L_0 + (n+3)[L_{n+1}, L_{-1}]
\] (F.24)

Using these explicit expressions we can write down the actions of the zero modes of the composite operators that have been used in the main text:

\[
\langle : \Lambda : \rangle = h^2 + \frac{1}{5} h.
\] (F.25)

\[
\langle : \Xi : \rangle = 4h^2 + \frac{6}{7} h - \frac{31 \cdot 20}{7(5c+22)} \left( h^2 + \frac{1}{5} h \right).
\] (F.26)

\[
\langle : \Delta : \rangle = h^3 + \frac{1}{5} h^2 - \frac{72}{10} h - \frac{3 \cdot 20}{10} \left( h^2 + \frac{1}{5} h \right).
\] (F.27)

\[
\langle : \Omega : \rangle = 4q + hq.
\] (F.28)

The expectation value for : \( \Gamma_0 : \) differs between the NS and R sector. For the NS sector we have:

\[
\langle : \Gamma : \rangle_{NS} = - \frac{1}{15} h - \frac{20}{18} h^2 + 3 \cdot \frac{27}{5c+22} \left( h^2 + \frac{1}{5} h \right) + 3C_4^1 \frac{1}{2} q.
\] (F.29)

whereas in the Ramond sector the contribution from \( (\partial QQ)_0 \) vanishes whereby

\[
\langle : \Gamma : \rangle_R = - \frac{20}{18} h^2.
\] (F.30)

Two other formulas that have been useful where:

\[
(TQ)_n = \sum_{p \leq -1} L_p Q_{n-p} + \sum_{p \geq 1} Q_{n-p} L_p + Q_n L_0 + \left( n + \frac{5}{2} \right) Q_n.
\] (F.31)

\[
(\partial TQ)_n = -(p+2) \left( \sum_{p \leq -1} L_p Q_{n-p} + \sum_{p \geq 1} Q_{n-p} L_p \right) - 2Q_n L_0 - \left( n + \frac{5}{2} \right) Q_n
\] (F.32)

Finally, all constants appearing in the OPE’s/mode algebra are given in terms of the central charge only:
Table 5: OPE coefficients of the $W(2, \frac{1}{2}, 4)$ algebra.

<table>
<thead>
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<th>( \sqrt{6(14c+13)} )</th>
<th>A</th>
<th>1</th>
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<tr>
<td>a</td>
<td>( \frac{15\sqrt{14c+13}}{4\sqrt{6(5c+22)}} )</td>
<td>B</td>
<td>3/20</td>
</tr>
<tr>
<td>b</td>
<td>( \frac{3\sqrt{14c+13}}{\sqrt{6(5c+22)}} )</td>
<td>C</td>
<td>1/168</td>
</tr>
<tr>
<td>d</td>
<td>( \frac{3(c+8)\sqrt{4c+14}}{2(2c+25)\sqrt{6(5c+22)}} )</td>
<td>D</td>
<td>21</td>
</tr>
<tr>
<td>e</td>
<td>( \frac{45\sqrt{14c+13}}{2(2c+25)\sqrt{6(5c+22)}} )</td>
<td>E</td>
<td>( 600c^4+4195c^3+34480 )</td>
</tr>
<tr>
<td>f</td>
<td>( \frac{3(c+5)\sqrt{4c+14}}{2(2c+25)\sqrt{6(14c+13)}} )</td>
<td>F</td>
<td>2158c+21385</td>
</tr>
<tr>
<td>g</td>
<td>( \frac{162c+2025}{2(2c+25)\sqrt{6(5c+22)(14c+13)}} )</td>
<td>G</td>
<td>( 6(2c^2+84c+490) )</td>
</tr>
<tr>
<td>h</td>
<td>( \frac{652c+2151}{2(2c+25)\sqrt{6(5c+22)(14c+13)}} )</td>
<td>H</td>
<td>( 2(2c+25)\sqrt{6(5c+22)(14c+13)} )</td>
</tr>
</tbody>
</table>

F.3 Notes on the $WB_2$ algebra

This algebra can be realised as the Casimir algebra of $B_2$, in which case it goes under the name as $WB_2$. However, as discussed in [81], $B_2$ has two Casimirs, which are the stress energy tensor and a fourth order Casimir, which is a Virasoro primary field of dimension four. To realize $WB_2$ algebra as the Casimir algebra of $B_2$ the additional primary of weight 5/2 has to be introduced. It is thus the bosonic currents that form the Casimirs of $B_2$.

Coset central charge

By applying then the coset construction discussed in section E.4.3 to the coset:

\[
\frac{B_2 \oplus B_2}{B_2},
\]

(F.33)

it can be shown that the unitary minimal models belong to the series:

\[
e(\mathcal{W}B_2) = \frac{5}{2} \left( 1 - \frac{12}{(m+3)(m+4)} \right).
\]

(F.34)

Free field realisation: Coulomb gas

An explicit free field realisation of this algebra has been constructed in [81] using the Coulomb gas formalism. The propagators are:

\[
\langle \phi_i(z)\phi_j(w) \rangle = \delta_{ij} \ln(z-w), \quad \langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}.
\]

\[93\] $B_2$ is notation in the classification of the semi-simple lie algebra. Examples are $A_n = \mathfrak{sl}_{n+1}$, $B_n = \mathfrak{so}_{2n+1}$, $C_n = \mathfrak{sp}_{2n}$ and $D_n = \mathfrak{so}_{2n}$. Here $n$ denotes the rank of the algebra, i.e. the number of Casimir elements. Those of type $A_n$ we have already seen in section E.4.3.

\[94\] This is related to the root structure of the algebra [81]. I will not discuss the details of this here.

\[95\] See e.g. [22] for a good explanation of this formalism.

\[96\] Note that the authors of [81] have chosen a positive sign for the bosonic propagator, contrary to the usual convention in the literature $\langle \phi_i(z)\phi_j(w) \rangle = -2\delta_{ij} \ln(z-w)$. They did this to avoid the appearance of factors of $i$, which can be seen explicitly when comparing to [67]. The overall factor of two makes an appearance in $\hat{F}$.37.
The explicit expression for the stress energy tensor is

\[ T(z) = \frac{1}{2} \partial \phi_1 \partial \phi_1(z) + \frac{1}{2} \partial \phi_2 \partial \phi_2(z) + \alpha_0 \partial^2 \phi_1(z) + \frac{1}{2} \partial \psi \partial \psi(z) \]  

(F.36)

\( \alpha_0 \) is what is known as a background charge in the Coulomb gas formalism. This stress energy tensor satisfies a Virasoro algebra with central charge

\[ c = \frac{5}{2} - 12\alpha_0^2 \]  

(F.37)

as is easily derived by computing \( T(z)T(w) \) using (F.35). This means that the conformal field theory is unitary only for those values of \( \alpha_0 \) for which \( c \) belongs to the minimal series F.34. The expressions for the dimension 4 and \( \frac{5}{2} \) fields can be found in [81] and can be derived using the approach of [67]. I.e. one writes down the most general expressions in terms of the free fields, computes their OPE’s using F.35 and requires that these reduce to the OPE relations of the stress energy tensor, and a spin-4 and \( \frac{5}{2} \) field.

As discussed already in appendix B one may then use these explicit expressions for the spin-2, 4 and \( \frac{5}{2} \) fields to construct the Ward identities using the method outlined in [46]. Comparing these then to the flatness conditions for the \( osp(1|4) \) black hole connection computed in [13] one will find the map 7.2.

---

\(^97\)Note the difference with [72][67] where \( c = 1 - 24\alpha_0^2 \). This factor of 2 arises due to their convention for the bosonic propagator \( -2\delta_{ij} \ln(z - w) \).
The semiclassical limit

In order to compare the quantum results from the CFT to the bulk results, we need to consider the semiclassical limit of the $W$ algebra. Naively, this involves a large-$c$ expansion in $1/c$, but the correct procedure is a little more subtle. The correct procedure to define the semiclassical limit has been defined in [83]. To take the semiclassical limit, we are instructed to rescale all currents (denoted collectively by $W_s(z)$) and the central charge of the $W$ algebra, by a power of $\hbar$:

$$W_s(z) = \hbar^{-1} \tilde{W}_s(z), \quad c = \hbar^{-1} \tilde{c}$$

whilst the rescaled currents and central charge $\tilde{W}, \tilde{c}$ are held fixed. Note that, it is thus $\tilde{W}$ that we refer to as the quantum definition and $W$ that defines the classical version. This is easiest to see when rewriting the above as $\tilde{W} = h W$. If $\tilde{W}$, is to be held fixed as $\hbar \to 0$, then this means that $W$ has to grow large and is thus the classical definition of the current.

We lastly note that these rescaling imply that when we expand in $\hbar \to 0$, the R.H.S of the OPE and commutation relations become linear in $\hbar$, with corrections of $O(\hbar^2)$. In particular, the semiclassical OPE algebra (which translates in to Poisson brackets) is obtained from the quantum OPE algebra (which translates into commutators) via:

$$\tilde{W}_s \tilde{W}_s' \bigg|_{\text{semi}} = \lim_{\hbar \to 0} \frac{1}{\hbar} \tilde{W}_s \tilde{W}_s'$$

or equivalently:

$$\{ \tilde{W}_s, \tilde{W}_s' \}_{PB} = \lim_{\hbar \to 0} \frac{1}{\hbar} [\tilde{W}_s \tilde{W}_s']$$
H Explicit expressions for the higher level Gram matrix elements.

H.1 NS level-3/2

The matrix elements for the level-3/2 NS Kac determinant are given below. We have denoted the level-1/2 BPS bound by \( \langle \alpha | \alpha \rangle \).

\[
\langle \beta | \beta \rangle = -4 \langle \alpha | \alpha \rangle - \frac{2}{5} h(1 + h) + \frac{27}{5c + 22} \left( (8h^2 + 2h(1 + h)\left( \frac{6}{5} + h \right)) \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} \left( 4q + 2hq \right)
\]

\[
\langle \beta | \chi \rangle = \frac{6h}{5} + \frac{27}{5c + 22} \left( 4h^2 + \frac{4}{5} h \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} q
\]

\[
\langle \beta | \gamma \rangle = \langle \alpha | \alpha \rangle - (2 - 2b + 12d - 24f) - (2c - 4h_{\text{alggebra}})(4h + 5) + 6g(1 + h)
\]

\[
\langle \beta | \delta \rangle = \langle \alpha | \alpha \rangle \left( \frac{2}{5} \left( 6 \left( 4h + 5 \right) + 4(2 - 2b + 12d - 24f) - 6g(1 + h) \right) \right)
\]

\[
\langle \gamma | \chi \rangle = \langle \alpha | \alpha \rangle \left( \frac{27}{5c + 22} \left( 4h^2 + \frac{4}{5} h \right) + \sqrt{\frac{6(14c + 13)}{5c + 22}} q + \frac{9h}{5} \right)
\]

\[
\langle \gamma | \delta \rangle = \langle \alpha | \alpha \rangle \left( 4a - 12b + 24d - 24f - 2g(h + 1) + (2 + h)(4c - 4h_{\text{algebra}}) \right)
\]

\[
\langle \gamma | \delta \rangle = \langle \alpha | \alpha \rangle \left( \frac{1}{2} \left( 18a - 4(9b - 24d - e(2h + 7) + 30f + g(h + 2)) - 5(2h + 7)h_{\text{algebra}} \right) \right)
\]

\[
+(4a - 18b + 48d - 60f) \left( \frac{27(2h + 1)(10h + 7)}{20(5c + 22)} - \frac{1}{5} (h + \frac{1}{2}) - \langle \alpha | \alpha \rangle \right)
\]

\[
+ \sqrt{\frac{84c + 78}{5c + 22}} (a - 6b + 18d + 3e(h + 2) - 2(12f + g(h + 1) + 2(h + 2)h_{\text{algebra}}) + q) \right)
\]

\[
+ g \left( (2h - 1) \langle \alpha | \alpha \rangle - \frac{1}{2} h + \frac{1}{5} h(1 + 2h) - \frac{27}{5c + 22} \left( \frac{1}{20} (2h + 1)(2h + 5)(10h + 7) \right) \right)
\]

\[
- \sqrt{\frac{84c + 78}{5c + 22}} (2h + 5)(a - 6b + 18d + 3e(h + 2) - 2(12f + g(h + 1) + 2(h + 2)h_{\text{algebra}}) + q) \right)
\]

\[
+(4e - 5h) \left( \langle \alpha | \alpha \rangle (2 - h) - \frac{2}{5} (1 + 2h) - \frac{1}{10} h(1 + 2h) - \frac{27(h + 1)(2h + 1)(10h + 7)}{20(5c + 22)} \right)
\]

\[
+ \sqrt{\frac{84c + 78}{5c + 22}} (h + 4)(a - 6b + 18d + 3e(h + 2) - 2(12f + g(h + 1) + 2(h + 2)h_{\text{algebra}}) + q) \right)
\]
\[ \langle \gamma | \gamma \rangle = \langle \alpha | \alpha \rangle \left[ \frac{1}{35} (h + \frac{1}{2}) - 2 (3D - 20E) \left( \frac{1}{5} (h + \frac{1}{2}) + (h + \frac{1}{2})^2 \right) 
- 2 (3H - 20I) \left( q + a - 6b + 18d - 24f - 2g(h + 1) + (2 + h)(3e - 4h_{\text{algebra}}) \right) 
+ 2F \left( \frac{-36}{7} (h + \frac{1}{2}) - \frac{12400}{7} \frac{1}{2} (h + \frac{1}{2}) + (h + \frac{1}{2})^2 \right) \right. 
+ 2G \left( -6 \frac{1}{5} (h + \frac{1}{2}) + (h + \frac{1}{2})^2 \right) - \frac{3}{10} (24(h + \frac{1}{2}) + 6(h + \frac{1}{2})^2) + 8(h + \frac{1}{2}) + 4(h + \frac{1}{2})^2 + (h + \frac{1}{2})^3 \right) 
+ 2J \left( h + \frac{9}{2} \right) (a - 6b + 18d + 3e(h + 2) - 2(12f + g(h + 1) + 2(h + 2)h_{\text{algebra}}) + q) 
+ 2K \left( \langle \alpha | \alpha \rangle + \frac{7}{5} (h + \frac{1}{2}) \right) \left[ \frac{5}{18} (6(h + \frac{1}{2}) + 4(h + \frac{1}{2})^2) + \frac{3 \cdot 27}{5c + 22} \left\{ \frac{1}{5} (h + \frac{1}{2}) + (h + \frac{1}{2})^2 \right\} 
+ 3 \left\{ \frac{6(14c + 13)}{5c + 22} (a - 6b + 18d + (h + 2)(3e - 4h_{\text{algebra}}) - 24f - 2g(h + 1) + q) \right\} \right] \]
References


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[59] S. Carlip "What we don’t know about the BTZ black hole entropy" Class. Quantum Grav 15 (1998) 3609-3625


