The topology of the space of knots

Felix Wierstra

August 22, 2013

Master thesis

Supervisor: prof.dr. Sergey Shadrin

KdV Instituut voor wiskunde
Faculteit der Natuurwetenschappen, Wiskunde en Informatica
Universiteit van Amsterdam
Abstract

In this master thesis we will study the topology of the space of knots i.e. embeddings of $S^1$ in $\mathbb{R}^n$. We will use Vassiliev’s method of constructing a spectral sequence which converges to the 0th cohomology of the spaces knots when $n \geq 4$. Then we will describe the Kontsevich integral, this integral is a universal Vassiliev invariant in the sense that every Vassiliev invariant factors through the Kontsevich integral. After that we will show that Vassiliev’s spectral sequence has an interesting relation with the Hochschild homology of the Poisson operad. We will also look at the cosimplicial models for the space of knots constructed by Sinha and look at the corresponding spectral sequences. We will give a little background for these cosimplicial models by studying the calculus of embeddings and the connection of these spectral sequences with the theory of operads.

Gegevens

Titel: The topology of the space of knots
Auteur: Felix Wierstra, felix.wierstra@gmail.com, 5887364
Supervisor: prof.dr. Sergey Shadrin
Tweede beoordelaar: prof.dr. Eric Opdam
Einddatum: August 22, 2013

Korteweg de Vries Instituut voor Wiskunde
Universiteit van Amsterdam
Science Park 904, 1098 XH Amsterdam
http://www.science.uva.nl/math
## Contents

Introduction

1 Vassiliev’s spectral sequence

1.1 Main idea behind the construction .......................... 9
1.2 Alexander duality ........................................... 11
1.3 Polynomials without multiple roots ......................... 12
1.3.1 The simplicial resolution ............................... 12
1.3.2 The spectral sequence ................................. 13
1.4 The space of all knots .................................... 15
1.4.1 Long knots ............................................. 15
1.4.2 Finite dimensional approximations of the space of long knots ............................................. 16
1.4.3 Singularities, ($A, b$)-configurations and degenerate chord diagrams .......................... 19
1.4.4 The stratification and the simplicial resolution of the discriminant .......................... 22
1.5 Vassiliev’s spectral sequence ............................ 23
1.6 Stabilization .............................................. 24
1.6.1 The affine bundle structure on the resolved discriminant ............................................. 24
1.6.2 The Pontrjagin Thom homomorphism .................. 26
1.7 The diagram complex ..................................... 27
1.7.1 The differential of the diagram complex .............. 28
1.8 The auxiliary spectral sequence .......................... 31
1.8.1 A description of the cells $D$ .......................... 33
1.9 The diagonal of the spectral sequence .................... 36
1.10 Some final remarks about higher degree homology groups and generalizations to higher dimensions .......................... 39
  1.10.1 Higher homology groups ............................ 39
  1.10.2 Generalizations to higher dimensions and arbitrary target manifolds .......................... 39

1
2 Vassiliev invariants

2.1 Vassiliev’s skein relation, chord diagrams and weight systems

2.1.1 Vassiliev’s skein relation

2.1.2 Chord diagrams

2.2 Vassiliev invariants vs other invariants

3 The Kontsevich integral

3.1 The formula for the Kontsevich integral

3.2 The Kontsevich integral and Vassiliev invariants

4 Configuration spaces

4.1 The Fulton-MacPherson completion of a configuration space

4.2 The simplicial completion of a configuration space

5 Operads

5.1 Linear operads

5.2 Examples of linear operads

5.2.1 The associative operad \( \mathcal{ASS} \)

5.2.2 The commutative operad \( \mathcal{COM} \)

5.2.3 The Lie operad \( \mathcal{LIE} \)

5.2.4 The Poisson operad \( \mathcal{POISS}_d \)

5.2.5 The BV-operad \( \mathcal{BV}_d \)

5.3 Topological operads

5.4 Examples of topological operads

5.4.1 The little cubes and little disks operad \( \mathcal{CD}_d \)

5.4.2 The Kontsevich operad \( \mathcal{K}_d \)

5.5 The Hochschild homology of a graded linear operad

5.5.1 A graded Lie algebra structure on a graded linear operad

5.5.2 The Hochschild homology of a graded linear operad

5.5.3 The homology of topological operads

5.6 Results about the Hochschild homology of certain operads

5.7 The space of knots and the Poisson operad

6 Simplicial and cosimplicial models

6.1 Simplicial and cosimplicial objects

6.1.1 The geometric realization, singular and totalization functors

6.2 The nerve of a category

6.3 A model category structure on the category of cosimplicial spaces

6.3.1 The definition of a model category
6.3.2 The model category structure on simplicial spaces ... 89
6.4 The cosimplicial space associated to a multiplicative operad . 93
6.5 A cosimplicial model for the loop space . . . . . . . . . . . . . 94
6.6 The Bousfield Kan spectral sequence associated to a cosimpli-
cial space . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 95
6.6.1 Towers of fibrations . . . . . . . . . . . . . . . . . . . . 96
6.6.2 The homology Bousfield Kan spectral sequence . . . . . . . 97
6.6.3 The homotopy Bousfield Kan spectral sequence . . . . . . . 98
6.6.4 Some final remarks about the Bousfield Kan spectral
sequence . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 99

7 Homotopy limits 100
7.1 Motivation for the concept of homotopy limits . . . . . . . . . 100
7.2 Homotopy colimits . . . . . . . . . . . . . . . . . . . . . . . . 101
7.3 Homotopy limits . . . . . . . . . . . . . . . . . . . . . . . . . 103

8 The calculus of embeddings 106
8.1 Cubical diagrams and polynomial cofunctors . . . . . . . . . . . 107

9 Sinha’s cosimplicial model for the space of knots 112
9.1 The mapping space model . . . . . . . . . . . . . . . . . . . . 113
9.1.1 The Taylor tower of the functor \( \text{Emb}(I,M) \) . . . . . . 114
9.1.2 The Fulton-MacPherson completion . . . . . . . . . . . . . 114
9.1.3 The category of \( f \)-trees . . . . . . . . . . . . . . . . . . 116
9.1.4 A stratification of the completions of configuration spaces117
9.1.5 Modifying the completions of configuration spaces to
take the boundary conditions and the tangent vectors
into account . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 119
9.1.6 Projection and diagonal maps for the simplicial com-
pletion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 121
9.1.7 Projection and diagonal maps for the Fulton-MacPherson
completion . . . . . . . . . . . . . . . . . . . . . . . . . . 123
9.1.8 The relation between the diagonal maps of the simplic-
ial completion and the diagonal maps of the Fulton
MacPherson completion . . . . . . . . . . . . . . . . . . . . 127
9.1.9 A more refined stratification of our modified configu-
ration spaces . . . . . . . . . . . . . . . . . . . . . . . . . 128
9.1.10 The mapping space model . . . . . . . . . . . . . . . . . . 131
9.2 A sketch of the proof of theorem 9.50 . . . . . . . . . . . . . . 132
9.3 The cosimplicial model . . . . . . . . . . . . . . . . . . . . . . 136
9.3.1 A sketch of the proof of theorem 9.64 . . . . . . . . . . . . . 136

3
9.4 The cosimplicial model and operads ........................................... 138
9.5 The spectral sequences corresponding to Sinha’s cosimplicial
model ................................................................. 140

10 Rational homotopy theory ....................................................... 141
10.1 Rational spaces ............................................................. 142
10.2 Commutative differential graded
algebras ................................................................. 145
10.2.1 Polynomial differential forms ................................... 146
10.2.2 Spatial realization .................................................. 147
10.3 Sullivan models ............................................................. 148

11 The collapse of Sinha’s and Vassiliev’s spectral sequences for
$\mathbb{R}^d$ with $d \geq 4$ over the rationals ........................................... 152
In this thesis we will study the topology of the space of knots. The space of knots is the topological space of all embeddings $\mathcal{K}: S^1 \to \mathbb{R}^d$ of the circle in $\mathbb{R}^d$. The main goal is to find the cohomology and homotopy groups of the space of all knots. Specially the zeroth cohomology group is interesting since this is the group of knot invariants. To do this we describe two different methods and there consequences. The first method is Vassiliev’s method and studies the complement of the space of knots in the space of all immersion of the circle in $\mathbb{R}^d$. The second method is due to Sinha and calculates the homology of the space of knots by using the embedding calculus of Goodwillie and Weiss.

Vassiliev’s method works as follows, we start by embedding the space of all knots in the space of all smooth immersions. In this space we can take the complement which we will call the discriminant. The homology of the discriminant and the space of knots can be connected to each other via Alexander duality. Unfortunately Alexander duality only works for finite dimensional space, to get around this difficulty we will approximate the space of knots by polynomials. Then we will construct a resolution of the discriminant which makes it easier to calculate the homology of the discriminant, since it has a natural filtration. With this filtration we can define a spectral sequence that gives us a class of knot invariants which we will call Vassiliev invariants.

The invariants coming from Vassiliev’s spectral sequence are very strong and it is conjectured that they from a complete set of invariants. In chapter 3 we will also describe the Kontsevich integral. The Kontsevich integral is a knot invariant the contains all the information about Vassiliev invariants and it is proven that every Vassiliev invariant can be factored through the Kontsevich integral. The Kontsevich integral is also used to show that Vassiliev’s spectral sequence collapses along the diagonal when we embed our knot in $\mathbb{R}^3$.

Another surprising result about Vassiliev’s spectral sequence is the relation between the Hochschild homology of the Poisson operad and the
first page of the spectral sequence. This result is due to Tourtchine and is partially described in chapter 5, where we also give an introduction into the theory of operads.

The second part of this thesis is about Sinha’s cosimplicial model for the space of knots. This model is based on the calculus of embeddings. The calculus of embeddings is a method to approximate certain functors from some small category to the category of space, by certain simpler functors. For some functors we get using this method a tower of functors whose inverse limit is homotopy equivalent to the original functor. The functors for which this happens are called analytic functors and the corresponding tower will be called the Taylor tower of that functor.

Before we define the calculus of embeddings in chapter 8 we will first need some preliminaries. These preliminaries mainly consist of the basics about simplicial and cosimplicial spaces and homotopy limits. The simplicial and cosimplicial spaces will be defined in 6 and are basic knowledge in algebraic topology. In chapter 7 we will introduce homotopy limits and colimits. Homotopy limits are a modification of the usual categorical limits and colimits to give them better properties. Normal limits and colimits have the disadvantage that whenever two diagrams are objectwise weakly equivalent their limit and colimit are not always weakly equivalent. We will fix this by introducing homotopy limits and colimits.

After we have defined all these preliminaries and the calculus of embeddings we construct in chapter 9 two models for the space of knots. Both models are based on the calculus of embeddings in combination with configuration spaces. The idea is that the easiest way to approximate a knot is by evaluating the knot at a number of points. The first model called the mapping space model and is constructed as the space of all stratum preserving maps between two stratified modifications of some configuration space.

The second model is a cosimplicial model. The advantage of this model is that it is defined as the totalization of a cosimplicial space. In chapter 6 we will define two spectral sequences that converge to the homotopy and homology groups of the totalization of a cosimplicial space. This spectral sequence is called the Bousfield Kan spectral sequence and calculates the homology of the space of knots when embed them in \( \mathbb{R}^d \) when \( d \geq 4 \).

In chapter 9 we will also show that we can obtain a similar model for the space of knots using the McClure–Smith method. This method constructs out of certain operads a cosimplicial space. This is used in chapter 11 where we give a brief explanation of some recent results by Lambrechts Tourtchine and Volić about the collapse of Vassiliev’s and Sinha’s spectral sequences. To do this we first need to make a small digression to rational homotopy theory in chapter 10, in this chapter we will define all the necessary definitions to
at least understand the meaning of the results by Lambrechts, Tourchine and Volić.

Acknowledgements 0.1. I would like to thank my supervisor Sergey Shadrin for supervising me. Further I would like to thank my fellow students for helping me a lot with latex questions and supporting me when necessary. I would like to thank Andrea Barbon for helping me with most of the pictures, they have improved a lot.
Chapter 1

Vassiliev’s spectral sequence

In this chapter we shall describe Vassiliev’s method of studying the space of knots. The idea behind Vassiliev’s method of studying the space of knots is to study the complement of the space of knots seen as a subspace of space of all smooth immersions of the circle in the same target manifold. This complement which is called the discriminant can be used to study the cohomology of the space of knots with the use of Alexander duality. Vassiliev constructed a resolution of the discriminant which gives a filtration on the discriminant. This filtration gives rise to a spectral sequence which we call Vassiliev’s spectral sequence and which gives us the cohomology of the space of knots whenever the target manifold is of dimension 4 or greater. When the dimension of the target manifold is 3 the spectral sequence has some problems with convergence, however it can still be used to obtain cohomology classes of the spaces of knots. It is not known if these cohomology classes generate the whole cohomology ring of the space of knots, but the classes we get can be used to construct a set of knot invariants. The invariants we get in this way are called finite type or Vassiliev invariants and it is conjectured that these invariants form a complete set of invariants.

The chapter is structured as follows, in section 1 we will describe the general idea of the construction. In section 2 we will explain what Alexander duality is and how we are going to use it. In section 3 we use the space of polynomials with multiple roots as the main example of Vassiliev’s method and in section 4 and all the sections after that we will describe Vassiliev’s method in the case of $\mathbb{R}^3$. 
1.1 Main idea behind the construction

In this section we will give a brief overview of Vassiliev’s method. This method can be used in many different situations and we will later give some examples of the situations in which this method is applicable. But before we do that we will first give an impression of how the method works by describing the construction of the spectral sequence step by step. In the next section we will first work out an easier example of Vassiliev’s method by calculating the homology of the space of polynomials of degree $d$ without roots of order greater or equal than $n$. Once we have calculated the homology of spaces of of polynomials of multiple roots we shall study the more interesting but also much more complicated case of the space of all long knots in $\mathbb{R}^3$.

A few examples of situations in which Vassiliev’s method can be applied are:

- The classifying spaces of braid groups,
- classical Lie groups,
- spaces of Morse functions on a manifold $M$,
- spaces of of smooth maps $M \to \mathbb{R}^n$ without complicated singularities,
- loop spaces $\Omega X$,
- polynomials without multiple roots,
- spaces of non singular complex manifolds,
- spaces of knots in $\mathbb{R}^n$.

More information about these examples can be found in [57].

Before we turn to any applications of Vassiliev’s method we shall first describe it step by step. So the problem is that we have a space $X$ and we want to calculate the cohomology of $X$. Then we can do this by following the following steps.

**Step 1: Define the discriminant as a subspace of some bigger space**

The first step is to embed $X$ in a bigger space $Y$. If we choose $Y$ to be a space with an easy homological structure, for example a contractible space, we can look at the complement of $X$ in $Y$. We will call this complement $X - Y$ the discriminant $\Sigma$, the idea is that if we choose $Y$ in the right way we can use Alexander duality to obtain information about $X$. 


Step 2: Approximate all the spaces by finite dimensional spaces

Since the spaces $X$, $Y$ and $\Sigma$ are in general infinite dimensional we are not able to use Alexander duality directly, therefore we need to define finite dimensional approximations $V_i$ of $Y$. In the case when $X$ is some space of functions without singularities and $Y$ is the space of functions that may have some singularities it is often possible to use a series of spaces of polynomials of increasing degree. Within each of these approximation spaces $V_i$ we are able to define a discriminant $\Sigma_i$ as all the functions with at least one singularity.

Step 3: Describe the singularities in more detail

The discriminant $\Sigma_i$ as a subspace of $V_i$ can in general be described rather explicitly. In the case of knots we do this by requiring that the set $\Sigma_i$ consists of all singular knots with self intersections or points where the derivative vanishes. In general, if the approximations are of the form of polynomials, we can describe the singularities as the zero sets of certain polynomials in the coefficients of the approximating polynomials.

Step 4: Construct a simplicial resolution of the discriminant

With the explicit description of the discriminant $\Sigma$ from the previous steps it is now time to start calculating its homology. To do this we will replace $\Sigma$ by a homotopy equivalent space $\sigma$ which we will call simplicial resolution of $\Sigma$. The construction of this space goes roughly as follows, we replace every function $f$ with $n$ singularities by an $n$-simplex and glue these simplices in an appropriate way together. The simplicial resolution has a natural filtration given by the dimension of the simplices.

Step 5: Construct the spectral sequence corresponding to the simplicial resolution

With the filtration from the previous step we are now able to construct a spectral sequence which, if we are lucky, converges to the homology of the discriminant $\Sigma$. It can happen that this spectral sequence does not converge but in even in that case it is often still possible to extract some information from this spectral sequence. For more details in how to construct a spectral sequence corresponding to a filtration see for example [59], [39] or [35].
Step 6: Use Alexander duality to link this spectral sequence to the cohomology of the original space $X$

Now that we are able to calculate the homology of $\Sigma$ it is time to use Alexander duality. Alexander duality gives a link between the homology of the discriminant and the cohomology of the space $X$, as long as the spaces $X$, $Y$ and $\Sigma$ are chosen in the right way.

Step 7: Show that the finite dimensional approximations stabilize

The spectral sequence we have just constructed still depends on the choice and the degree of the approximations, therefore we also have to show that everything becomes independent of the approximations we have chosen. In the case of polynomial approximations we will do this by showing that when the degree of the approximations $V_i$ is high enough the discriminant spaces $\sigma_i \subset V_i$ are affine bundles, i.e. bundles of affine spaces over some base space. We will show that for all $V_i$ with $I > N$ all these affine bundle will have these the same base space. Since all the fibers are contractible all these spaces are homotopy equivalent and have therefore the same homology. Therefore there exists an $N >> 0$ such that all terms of degree smaller or equal to $n$ are independent of the approximation $V_i$ if $i > N$.

1.2 Alexander duality

One of the main ingredients we will use for the construction is Alexander duality. The reason why we need Alexander duality is that it justifies studying the discriminant $\Sigma$ instead of the original space. The statement of Alexander duality is given by the following theorem.

**Theorem 1.1.** Let $K$ be a compact, locally contractible, nonempty, proper subspace of $S^n$, then $\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$ for all $i$.

A proof of Alexander duality can be found in [24]. Please note that Alexander duality only works for finite dimensional spheres. Therefore it is important to choose the ambient space and the discriminant in an appropriate way. In the case of knots we also need to define finite dimensional approximations since the space of knots is infinite dimensional.

**Remark 1.2.** We can give an alternative formulation of the theorem by replacing $S^n$ by an $n$-dimensional real vector space $V$ and $S^n - K$ by $(V - K)^*$, where $(V - K)^*$ is the one point compactification of $V - K$. 

11
1.3 Polynomials without multiple roots

In this section we first show how Vassiliev’s method works for the much simpler case of polynomials of degree \( d \) with no roots of multiplicity greater or equal to \( k \).

**Definition 1.3.** Let \( P_d \) be the space of all monic polynomials of degree \( d \) with coefficients in \( \mathbb{R} \) and let \( \Sigma_k \) be the subspace of all polynomials with at least one root of multiplicity greater or equal to \( k \).

We are now interested in the homology of \( \Sigma_k \) since we can use it to calculate the cohomology of the space of polynomials without roots of multiplicity greater or equal to \( k \). First we observe that the space of all monic polynomials of degree \( d \) is isomorphic to \( \mathbb{R}^n \). We also notice that \( \Sigma_k \) has some complicated self intersections whenever a polynomial has more than 1 different root of multiplicity at least \( k \). To solve this problem we will construct a simplicial resolution of the discriminant. This is a space which is homotopy equivalent to \( \Sigma_k \) but it has a more explicit homological structure because the simplicial resolution has a natural filtration.

1.3.1 The simplicial resolution

In this subsection we will construct the simplicial resolution of \( \Sigma_k \) the discriminant of degree \( d \) polynomials with roots of multiplicity greater or equal to \( k \) which we will denote by \( \bar{\Sigma}_k \). We do this as follows, first we define the tautological normalization of \( \Sigma_k \) by taking the space of all pairs \( \{(x, f) \mid x \in \mathbb{R}, f \in \Sigma, f \text{ has a root of order at least } k \text{ at } x\} \).

The tautological normalization is diffeomorphic to \( \mathbb{R}^{d-k+1} \) and we have a proper surjective finite-to-one map from the tautological normalization to \( \Sigma_k \) by forgetting the \( x \). The idea behind the simplicial resolution is that whenever a point in \( \Sigma_k \) has more than one preimage, i.e. the point is a polynomial with more than one root of multiplicity greater or equal than \( k \), we replace this preimage by a simplex with one vertex for each different root of multiplicity greater or equal than \( k \). To make this idea work we have to fix an embedding

\[
I : \mathbb{R} \to \mathbb{R}^\Omega
\]
where $\Omega$ is a very large but finite number. We want to embed the tautological normalization of $\Sigma_k$ in a larger space in such a way that all the simplices from the simplicial resolution will be non degenerate. We can do this by first fixing an embedding $I : \mathbb{R} \to \mathbb{R}^\Omega$ with the requirement that the image of $\frac{d}{k}$ different points in $\mathbb{R}$ does not lie in the same $\frac{d}{k}$ dimensional subspace. The next step in defining the simplicial resolution is to take the direct product of $\mathbb{R}^\Omega$ and $P_d$, the space of all monic polynomials of degree $d$. In $\mathbb{R}^\Omega \times P_d$ we define for every polynomial $f \in \Sigma_d$ a simplex with vertices $(I(z_1), f), \ldots, (I(z_n))$, where the $z_i$ are all the multiple roots of $f$ of order greater or equal to $d$. So for every polynomial with $n$ different roots of multiplicity greater or equal to $d$ we define a $n-1$ dimensional simplex. The simplicial resolution $\sigma_k$ is the union of all these simplices. The projection $p : \mathbb{R}^\Omega \times P_d \to P_d$ induces a proper map $\sigma_d$ onto $\Sigma_d$, when we extend this map to the one-point compactifications of $\bar{\sigma}$ and $\bar{\Sigma}$ we have a homotopy equivalence.

So to summarize the construction of the simplicial resolution, we started with a discriminant set, then we took the tautological normalization. After that we embedded $\mathbb{R}$ in a particular way in $\mathbb{R}^\Omega$ such that the inverse image of each $f \in \Sigma_d$ defined a non degenerate simplex in $\mathbb{R}^\Omega \times P_d$. To get the simplicial resolution $\bar{\sigma}_d$ of $\Sigma_d$ we took the one point compactification of the union of all the simplices.

### 1.3.2 The spectral sequence

With the simplicial resolution defined in the previous section we are able to construct a spectral sequence converging to the cohomology of the space of polynomials without roots of multiplicity greater or equal than $d$. To do this we first have to define a filtration on $\bar{\sigma}_d$. Fortunately the simplicial resolution has a very natural increasing filtration, where the $i$th term $F_i$ is equal to the union of all simplices of dimension less or equal to $i-1$. Note that the $F_i$ consists of the simplicial resolution of all polynomials $f \in \Sigma_d$ with $i$ or less geometrically distinct multiple roots. We define $F_0$ separately and set it equal to the added point in the one-point compactification.

This filtration gives us a homological spectral sequence with the first page given by $E^1_{p,q} = \bar{H}_{p+q}(F_p \backslash F_{p-1})$. We use here $\bar{H}_*$ which is the Borel-Moore homology and is equal to $H_*(\bar{X}, \bar{X} \backslash X)$, where $\bar{X}$ is the one-point compactification of a topological space $X$.

To calculate the first page of the spectral sequence we have to calculate the homology of the spaces $F_p(\bar{\sigma}) \backslash F_{p-1}(\bar{\sigma})$. We will give an explicit description of this space as a fiber bundle over the space of all subsets of cardinality $p$ of $\mathbb{R}$. Note that if we have a point in $F_p(\bar{\sigma}) \backslash F_{p-1}(\bar{\sigma})$ is of
the following form, \((f, \tau)\) where \(f\) is a polynomial with exactly \(p\) roots of degree \(\geq k\) and \(\tau\) is a point in the added simplex from the simplicial resolution. We can describe the space of all polynomial with exactly \(p\) roots of degree \(\geq k\) as follows. First note that whenever a polynomial \(f\) has \(p\) roots of degree \(\geq k\) which is the set \(\{z_1, ..., z_p\}\) this polynomial is of the form \(g \cdot (x - z_1)^k \cdot ... \cdot (x - z_p)^k\), where \(g\) is a polynomial of degree \(d - p \cdot k\). So this space becomes a fiber bundle over the space of all subsets of cardinality \(p\) of \(\mathbb{R}\), the fiber is the space of all polynomials of degree \(d - p \cdot k\). An explicit description of the space of all cardinality \(p\) subsets of \(\mathbb{R}\) is the subspace of \(\mathbb{R}^p\) such that \(x_1 < x_2 < ... < x_3\), we denote this space by \(B(\mathbb{R}, p)\). Therefore the space \(F_p(\bar{\sigma}) \setminus F_{p-1}(\bar{\sigma})\) can be described as the product of the bundle over \(B(\mathbb{R}, p)\) with the simplex of dimension \(p - 1\). The Borel-Moore homology of this space is \(\bar{H}^* = \mathbb{Z}\) if \(* = d - p \cdot k + (p - 1)\) and zero otherwise. Therefore the \(E^1\) page becomes \(E_{p,q}^1 = \mathbb{Z}\) if \(q = d - p \cdot (k - 1) - 1\) and \(p \leq \frac{d}{k}\) and 0 otherwise. This spectral sequence collapses at the \(E^1\) page and therefore gives the homology of the discriminant immediately.

To calculate the cohomology of the space of polynomials without roots of degree \(\geq k\) we have to use Alexander duality to turn the spectral sequence we have just constructed into a cohomological one. We do this by defining

\[ E_{h}^{p,q} = E_{h-1}^{1-p,d-q-1}. \]

The terms of this spectral sequence are as follows, \(E_{h}^{p,q} = \mathbb{Z}\) if \((k-1) \cdot p + q = 0\) and \(-\frac{d}{k} \leq p \leq 0\) and zero otherwise.

The homology of the space of polynomials without roots of degree \(\geq k\) turns out to be:

\[ H^i(P_d \setminus \Sigma_k) = \begin{cases} \mathbb{Z} & \text{if } i = 2 \mod k \text{ and } 0 \leq i \leq \left\lfloor \frac{d}{k} \right\rfloor (k - 2) \\ 0 & \text{otherwise.} \end{cases} \]

**Remark 1.4.** In the formulas for the cohomology of the spaces of polynomials without multiple roots we can already see an example of stabilization. If we fix the number \(k\) and want to calculate the cohomology of the space of polynomials of degree \(d\) without roots of multiplicity \(\geq k\) we see that \(H^i(P_d \setminus \Sigma_k)\) is independent of \(d\) if \(i\) is small enough compared to \(d\). A topological explanation of this fact is that the number of conditions on the coefficients of the polynomial does not increase as the degree of the polynomial increases. Therefore only the dimension of the fiber of the fiber bundle over the space \(B(\mathbb{R}, p)\) increases. Since the fibers are all contractible all these spaces are homotopy equivalent and therefore the cohomology groups stabilize.
1.4 The space of all knots

In the previous section we have seen an example of Vassiliev’s method in the case of polynomials with multiple roots, in this section we will use Vassiliev’s method in the much more interesting case of knots.

For technical reasons we will work with long knots which will be defined in subsection 1.4.1.

1.4.1 Long knots

For technical reasons we will work with long knots in the rest of this chapter and large parts of the rest of this thesis. We will begin with the definition of a long knot and then motivate why long knots are in this case better than normal knots.

Definition 1.5. A long knot is an embedding of $\mathbb{R}$ into $\mathbb{R}^n$ which converges to a fixed linear embedding of $\mathbb{R}^3$ outside a compact subset.

Remark 1.6. From now on we will refer to long knots as long knots, we will call embeddings of the circle $S^1$ in $\mathbb{R}^3$ compact knots.

Long knots have several technical advantages above compact knots, i.e. embeddings of the circle in $\mathbb{R}^3$. Two examples of these advantages are that the space of long knots is easier to approximate in terms of polynomials and that the space of long knots has a natural homotopy associative commutative product. We will not use this in this thesis but for more details about this product see [8]. It is also important to note that there is a bijection between the connected components of the space of long knots and the space of compact knots. Therefore the 0th cohomology of the spaces coincides but these spaces are not homotopy equivalent. For example the connected component of the unknot in the space of long knots is contractible while the connected component of the unknot in the space of compact knots is not contractible, for a proof and more details see [23] and [8].

Proposition 1.7. There is a one to one correspondence between the equivalence classes of compact knots and the equivalence classes of long knots.

Proof. We can see that the connected components of these spaces are in a one to one correspondence with each other by defining a map from the equivalence classes to long knots to the equivalence classes of compact knots by gluing the ends of the long knot together. We can define another map from compact knots to long knots by cutting a compact knot open. It is clear that these maps are each others inverses, so we have a bijection between the
set of equivalence classes of long knots and the set of equivalence classes of compact knots. We still have to check that these maps are well defined, we will leave this to the reader.

Before we define the space of all long knots we will first recall the definition of the $C^1$ topology. We will give the space of all long knots this topology.

**Definition 1.8.** The space of all long curves $C_d$ is the space of all smooth functions

$$f : \mathbb{R} \to \mathbb{R}^d,$$

such that $f$ coincides with the line

$$\epsilon : \mathbb{R} \to \mathbb{R}^d$$

$$\epsilon(t) = (t, \ldots, t).$$

This space is equipped with the $C^1$ topology given by the following metric

$$d(f, g) = \max(|f - g|) + \max(|f' - g'|).$$

Note that we can take the maximum in this definition instead of the supremum since $f$ and $g$ are equal outside a compact set. As a subspace of the space of all long curves we can find the space of all long knots which is defined in the following definitions.

**Definition 1.9.** The space of all long curves $C_3$ is the space of all functions $f : \mathbb{R} \to \mathbb{R}^3$ such that $f$ is a smooth immersion. The topology of this space is the $C^1$ topology.

**Definition 1.10.** The space of all long knots $K_d$ is the subspace of the space of all long curves $C_3$ given by $K_d = \{f \in C_3 \mid f$ is an embedding$\}$.

**Definition 1.11.** The discriminant $\Sigma$ is the complement of the space of long knots $K_d$ in the space of all long curves $C_d$, i.e. $\Sigma = C_d - K_d$.

### 1.4.2 Finite dimensional approximations of the space of long knots

To construct Vassiliev’s spectral sequence we have to approximate the space of knots by a series of spaces that are finite dimensional, otherwise Alexander duality will not apply to our situation. Therefore we will approximate every
knot by polynomials and later show that when the degree of the polynomial approximation is high enough the contribution of the approximation to the spectral sequence will become independent of the degree.

To define our first approximation we first have to make the linear embedding from the definition of a long knot more explicit. It will be clear that in everything that follows nothing depends on this explicit embedding, we only define it to make life easier. The embedding we choose is the linear subspace spanned by the vector \((1, 1, 1)\). Later we will have to make modifications to these spaces of finite dimensional approximations for some technical reasons. But for the moment the spaces \(\tilde{V}_i\) will demonstrate the whole idea of what is happening.

**Definition 1.12.** Let \(\tilde{V}_i\) be the space of all functions \(f\) of the form

\[
f : \mathbb{R} \to \mathbb{R}^3
\]

\[
t \to \frac{P_x(t), P_y(t), P_z(t)}{(t^2 + 1)^n}
\]

Where \(P_x = t^{2n+1} + a_{2n-1}t^{2n-1} + a_{2n-2}t^{2n-2} + \ldots + a_1t + a_0\).

Please observe that the space \(\tilde{V}_i\) is canonically diffeomorphic to \(\mathbb{R}^{2n+3}\) and that the \(t^{2n}\) term is missing, since it would change the asymptotic behavior of \(f\). By inserting the \(t^{2n}\) term with the right coefficient, we can easily change the approximation to converge to any 1-dimensional linear subspace we want.

It will also be important to see that all the approximations \(\tilde{V}_i\) are included into each other, the inclusion is given by

\[
i : \tilde{V}_i \to \tilde{V}_{i+1}
\]

\[
f \to (t^2 + 1) \cdot f.
\]

So we get the following series of approximations

\[
\tilde{V}_1 \subset \tilde{V}_2 \subset \ldots \subset \tilde{V}_i \subset \tilde{V}_{i+1} \subset \ldots
\]

To show that these approximations are actually useful we have the following proposition proving that the approximations are good enough for our purposes.

**Proposition 1.13.** For every knot \(K\) there is an \(N > 0\) and a polynomial \(f \in \tilde{V}_i\) such that for all \(\epsilon > 0\) we have \(\|K - f\| < \epsilon\).
Proof. The proof is an easy application of the Stone-Weierstrass approximation theorem and is left as an exercise for the reader.

These approximations are still not the approximations we want, if we take the function defined by \( P_x(t) = P_y(t) = P_z(t) \) such that \( P_x(t) \) is of the form \( t^3 - t \) then this approximation fails to be an immersion. It is easy to see that there are many functions for which this goes wrong, for example any non monotone function will have this problem if \( P_x(t) = P_y(t) = P_z(t) \).

We need to solve this problem because we want all our approximations to actually approximate long curves and not that the approximation space contains any functions that are not long curves. In the following definition we will carefully define the conditions we want on our polynomial approximations.

**Definition 1.14.** Let \( V_i \) be a series of approximations of \( C_3 \), then we call a the series of approximations good if it satisfies the following two conditions.

1. Each finite dimensional compact family of long knots can be uniformly approximated by a family of curves in the approximations \( V_i \).

2. For each \( V_j \) the intersection with \( \Sigma_i \) in \( V_i \) is transversal, by \( \Sigma_i \) we denote the discriminant set in \( V_i \), i.e. all functions in \( V_i \) that are not embeddings.

The following lemma show that what happened to the approximation \( \tilde{V}_i \) is not very common and the lemma also guarantees that good approximations exists. The proof can be found in [54].

**Lemma 1.15.** *Almost all series of approximations \( V_i \) are good, in particular good approximations exist.*

A possible way to obtain good approximations from the approximation \( \tilde{V}_i \) is by embedding \( \tilde{V}_i \) in a higher dimensional euclidean space and then to move \( \tilde{V}_i \) a little bit. More details about this can be found in [54].

From now on we shall assume that we are working with a series of good approximations \( V_i \), we shall not be specific about the details of this approximation since they will not be important for the result of the construction.
1.4.3 Singularities, \((A,b)\)-configurations and degenerate chord diagrams

In this subsection we will make some preparations to define a stratification of the discriminant set. First we will define chord diagrams and \((A,b)\)-configurations which will describe the possible singularities of a singular long knot. In this subsection we will argue that there are two types of singularities that are important, the first one are the self intersections of a long knot and the other one is the set of points where the derivative vanishes.

We begin with the definition \(A\)-configurations which contain the information about the self intersections.

**Definition 1.16.** Let \(A\) be an arbitrary finite sequence of integers \((a_1,\ldots,a_n)\) such that \(a_i \geq 2\) for all \(i = 1,\ldots,n\), denote by \(\#A\) the number of terms in \(A\) and let \(|A| = \sum a_i\). An \(A\)-configuration is a family of \(|A|\) pairwise disjoint points on \(\mathbb{R}\) partitioned in \(\#A\) groups with cardinalities \(a_1,\ldots,a_{\#A}\).

If we add to an \(A\)-configuration a set of points \(b\) where we want the derivative to vanish we get the following definition.

**Definition 1.17.** Let \(A\) be an \(A\)-configuration and take a set of \(b\) distinct points in \(\mathbb{R}\) that may coincide with the points of \(A\), we will call such a configuration of points an \((A,b)\)-configuration.

The following definition show how an \((A,b)\)-configuration translates into restrictions on the knots.

**Definition 1.18.** A map \(\phi : \mathbb{R} \to \mathbb{R}^n\) respects an \((A,b)\)-configuration if all the elements of each group are mapped to the same point, and the derivative of \(\phi\) vanishes on all the points of \(b\).

Informally we can think of a function respecting an \((A,b)\)-configuration as a singular long knot with a derivative that vanishes on the points of \(b\) and with self intersection at all the groups of \(A\).

For the constructions we are about to describe we are mainly interested in the type of the singularities and not necessarily in the singularity itself we need to define an equivalence relation on the set of \((A,b)\)-configurations.

**Definition 1.19.** Two \((A,b)\)-configurations are called equivalent if there is an orientation preserving diffeomorphism \(\phi : \mathbb{R} \to \mathbb{R}\) which transforms one of the \((A,b)\)-configurations into the other.
Degenerate chord diagrams

There is a convenient way to subdivide the equivalence classes of $A$-configurations and $(A,b)$-configurations into smaller classes. These classes will be called chord diagrams and degenerate chord diagrams and contain all the information about the singularities of a singular knot. These diagrams will play an important role for the definition of Vassiliev invariants in chapter 2 and in chapter 3 about the Kontsevich integral. We will now only define degenerate chord diagrams, in chapter 2 we will define the non-degenerate version. We do this because since we do not need them yet, as it turns out the degeneration of the diagram is not important for the definition of Vassiliev’s knot invariants. The degenerate chord diagrams have the advantage that they give a more geometric idea of what is happening compared to $(A,b)$-configurations and they will be useful in describing part of the cellular structure of the space of knots.

Definition 1.20. A degenerate chord diagram $D$ is a set of distinct pairs $(x_i, y_i)$ such that $x_i, y_i \in \mathbb{R}$ and $x_i \leq y_i$.

We can draw degenerate chord diagrams by taking the real line and connecting the points of each pair by a line called a chord when $x_i \neq y_i$, and when $x_i = y_i$ we draw a hollow dot. See figure 1.4.3 for an example. We will call the end points of a chord self intersection vertices and we call the points where $x_i = y_i$ and we draw a hollow dot singularity vertices. We can partition the set of all self intersection vertices into groups, we do this by defining each group as the set of self intersection vertices that is connected to each other by chords.
Just like with \((A,b)\)-configurations we can define the set of knots that respects a degenerate chord diagram.

**Definition 1.21.** We say that a singular long knot \(K: \mathbb{R} \rightarrow \mathbb{R}^3\) respects a chord diagram if \(K(x_i) = K(y_i)\) for all pairs \((x_i, y_i)\) such that \(x_i \neq y_i\) and \(K'(x_i) = 0\) for all pairs \((x_i, y_i)\) with \(x_i = y_i\).

With this definition we can define a stratification on the the discriminant set with strata indexed by the degenerate chord diagrams. To each degenerate chord diagram we associate the set of all functions \(f\) that respect this chord diagram.

**Remark 1.22.** Note that a singular knot \(K\) can respect more than one different chord diagram.

**Definition 1.23.**
- The *complexity* of an \((A,b)\)-configuration is defined to be the number \(|A| - \#A + b\).
- The *complexity* of a degenerate chord diagram \(D\) is defined by
  \[
  c(D) = \#\text{geometrically distinct points} - \# \text{groups of self intersection vertices}.
  \]

**Equivalences of chord diagrams**

Since we are mainly interested in the nature of the singularities and position of the singularity on the not is less important we can define certain equivalences between the degenerate chord diagrams. There are two kinds of equivalences we are interested in. The first kind of equivalence is called combinatorial equivalence and corresponds to two diagrams having the same shape. The second equivalence is just called equivalent and corresponds to two diagrams giving the same conditions to a singular knot. The following definitions will make these ideas more formal.

**Definition 1.24.** Let \(D\) and \(D'\) be two degenerate chord diagrams, we call \(D\) and \(D'\) *combinatorially equivalent* if there is an orientation preserving diffeomorphism \(f: \mathbb{R} \rightarrow \mathbb{R}\) such that \(f\) maps the chords and the singularity vertices of \(D\) to the chord and singularity vertices of \(D'\).

**Definition 1.25.** Two degenerate chord diagrams \(D\) and \(D'\) are called *equivalent* if \(D\) is combinatorially equivalent to a diagram with the same groups of self intersections and the same set of singularity vertices as \(D'\).
Another more informal way to think about this last equivalence is to say that two degenerate chord diagrams are equivalent if the set of singular knots corresponding to the combinatorial equivalence classes of these diagrams are the same.

**Remark 1.26.** Note that the equivalence classes of the degenerate chord diagrams are exactly all the equivalence classes of the \((A,b)\)-configurations.

**Remark 1.27.** In this section we have defined both degenerate chord diagrams and \((A,b)\)-configurations although the information both definitions contain is essentially the same. We did this because for the construction of Vassiliev’s spectral sequence it is more convenient to work with degenerate chord diagrams, but the link with the Hochschild homology of the Poisson operad is much clearer when we consider \((A,b)\)-configurations (see section 5.7).

### 1.4.4 The stratification and the simplicial resolution of the discriminant

With the degenerate chord diagrams and \((A,b)\)-configurations we can define a stratification on the discriminant set. To each equivalence class of degenerate chord diagrams, i.e. an \((A,b)\)-configuration, we assign the set of singular knot that respect this chord diagram. Unfortunately even with this stratification the discriminant set is still difficult to describe, therefore we will now construct the simplicial resolution from 1.3.1 in the case of the discriminant set.

We will do this by taking the same steps again. First we will construct the tautological resolution, in this case we will define this as

\[
\Sigma \times \text{Sym}^2(\mathbb{R}),
\]

where \(\text{Sym}^2(\mathbb{R})\) is the space \(\{(x,y) \in \mathbb{R}^2 \mid x \leq y\}\). In this space we consider the subspace \(\bar{\Sigma}\) of all pairs \((f,(x,y))\) such that \(f(x) = f(y)\) if \(x \neq y\) or
\( f'(x) = 0 \) if \( x = y \). From \( \Sigma \) we can define the projection map on \( \Sigma \) by forgetting the pair \( (x, y) \), this projection map is a proper finite-to-one map. From this map we will construct the simplicial resolution by first embedding the space \( \bar{\Sigma} \) into a higher dimensional euclidean space such that all the points are insufficient general position. Then we can take for every \( f \in \Sigma \) the inverse image of the projection map and use this inverse image to define the vertices of a simplex. The resolved discriminant is the union of all simplices and is denoted by \( \sigma \).

It is important to note here that the discriminant is infinite dimensional and to embed the space \( \bar{\Sigma} \) into a higher dimensional space we need to take another infinite dimensional space. For the finite dimensional approximations we could do the same but with using finite dimensional spaces instead. As we will see the homology groups of these spaces will stabilize and therefore approximate the homology of the resolved discriminant \( \sigma \).

We will define a filtration on the simplicial resolution of the discriminant by the complexity of the diagrams. More concretely if we define the coordinates on the space \( \sigma \) as \((f, D, \tau)\), where \( f \) is a singular knot, \( D \) a degenerate chord diagram such that \( f \) respects \( D \) and \( \tau \) a point in the simplex of the simplicial resolution. Then we define the space \( \sigma_i \) as the space of all points \((f, D, \tau)\) such that \( D \) has complexity less or equal than \( i \).

### 1.5 Vassiliev’s spectral sequence

The next step we will take to construct the spectral sequence is to define a filtration on the discriminant of the space of all long knots. We have seen in the previous section that we have a stratification on the discriminant given by the degenerate chord diagrams. In this section we will combine the simplicial resolution from section 1.4.4 and the polynomial approximations from section 1.4.2 to define a filtration on the polynomial approximations. This filtration will give us a spectral sequence that is still dependent on the degree of the approximation, but in the next section we will show that this spectral sequence becomes independent of the degree of the approximation is high enough. We will do this by observing that the approximations to each space are isomorphic to an affine bundle over a certain base space and that this base space, called the diagram complex, is independent of the degree of the approximation once the degree is high enough.

First we will define Vassiliev’s spectral sequence as the spectral sequence associated to the resolved discriminant after applying Alexander duality. To do this we need to use the polynomial approximations, otherwise we cannot use Alexander duality. For every \( V_i \) of dimension \( N \) we can define
a discriminant which we by abuse of notation also denote by Σ. Of this Σ we can construct the simplicial resolution which we denote by σ, this σ comes also with a filtration defined by the complexity of the diagrams. We will now define Vassiliev’s spectral sequence as follows.

**Definition 1.28.** Let \( E^1_{p,q} \) be the spectral sequence associated to the filtration of σ defined by the complexity of the degenerate chord diagrams. Then we define Vassiliev’s main spectral sequence \( E^p_{1,q} \) as the spectral sequence obtained from this spectral sequence by applying Alexander duality. More concrete:

\[
E^p_{1,q} = E^1_{-p,N-q-1} = H_{N-(p+q+1)}(\sigma_p^*, \sigma_{p-1}^*) = H_{N-(p+q+1)}(\sigma_p^*/\sigma_{p-1}^*).
\]

Where \( \sigma_i \) is the set of all diagrams of complexity \( \leq i \) in an approximation space \( V_k \) of dimension \( N \)

In the next section we will show that the \( H_{N-(p+q+1)}(\sigma_p^*, \sigma_{p-1}^*) \) is independent of \( N \) for \( N \) large enough.

### 1.6 Stabilization

With the construction from the previous section we were able to construct a spectral sequence that in some cases calculates the homology of the space of knots, or at least of an approximation of the space of knots. In this section we will show that once the degree of the approximation is high enough the terms in the spectral sequence become independent of the degree of the approximation. We will do this by first giving an explicit affine bundle structure on the space \( V_i \). After that we will take a small detour and define the Pontrjagin Thom homomorphism, this homomorphism will give a relation between the discriminant \( \Sigma_N \) in an approximation \( V_N \) and the discriminant \( \Sigma_M \) in another approximation \( V_M \). It will turn out that this map is an isomorphism whenever the degree is high enough. Using this isomorphism and using Alexander duality to turn the spectral sequence from a homological spectral sequence into a cohomological one we will lose the dependence on \( N \).

#### 1.6.1 The affine bundle structure on the resolved discriminant

The first step in showing that the homology groups of the approximations of the discriminant stabilize is to show that the resolved discriminant of all
degenerate chord diagrams of complexity less or equal to $i$ has the structure of an affine bundle. This bundle has as base space the simplicial resolution of the space of diagrams of complexity less or equal to $i$ and as fiber $\mathbb{R}^n$ for some $n$. We will prove that when $N$, the dimension of the approximation, $V_k$ becomes large enough the base space does not change and only the dimension of the fiber increases as $N$ goes to infinity.

Let $D$ be a degenerate chord diagram if we want to approximate the space of all singular long knots that respect this degenerate chord diagram we can use the polynomial approximations from section 1.4.2. We will first use the approximation $\tilde{V}$ to demonstrate the idea and later modify this to apply this to the technical correct approximation $V$. Whenever a polynomial approximation given by $\frac{\phi(t)}{(1+t^2)^N}$, with $\phi$ a polynomial of degree $2N+1$ of the form $t^{2N+1} + a_{2N-1}t^{2N-1} + \ldots + a_0$, respects a degenerate chord diagram, we can write down the following equations

$$\frac{\phi_{xl}(a_{ij})}{(1 + a_{ij}^2)^N} = \frac{\phi_{xl}(a_{ik})}{(1 + a_{ik}^2)^N}$$

for all coordinates $x_l$ and for all groups $a_i$ and all elements of these groups $a_{ij}$ and $a_{ik}$. For all elements of $b$ we can write down the following equation

$$\frac{\partial}{\partial x_i} \frac{\phi}{(1 + b^2)^N} = 0$$

for all coordinates $x_i$. These equations give a number of linear conditions on the coefficients of the polynomials, unfortunately these conditions are not necessarily linearly independent. The following proposition fortunately tells us that this is not a real problem since, if the degree of the approximation is high enough the equations will be linearly independent.

**Proposition 1.29.** Let $D$ be a degenerate chord diagram of complexity $p$, then there exists a number $N_0$ such that for all $N > N_0$ the number of linear conditions on the coefficients of the polynomial approximations of degree $N$ is equal to $3p$.

The proof of this proposition is left to the reader. If we define $\pi$ to be the map from $\sigma_p - \sigma_{p-1}$ to the space $W_p$ that sends a triple $(f, D, \tau)$ to the pair $(D, \tau)$. Here we define $W_p$ as the base space of this bundle. So we can describe the space $\sigma_p - \sigma_{p-1}$ as an affine bundle over some base space $W_p$ with an fiber of dimension $N - 3p$, since $W_p$ is defined by $3p$ linearly independent equations.

The next step in the stabilization process is to show that $\tilde{H}_{N-s}(\sigma_p - \sigma_{p-1}) = H_{3p-s}(W_p^*)$. We will do this by using a version of the Pontrjagin Thom isomorphism which we will define in the next subsection.
1.6.2 The Pontrjagin Thom homomorphism

To show that our approximations will stabilize we need maps between the homology of the approximations. In this section we will define a homomorphism from the homology of one approximation to the homology of another approximations, to do this we will use a version of a homomorphism that is called the Pontrjagin Thom homomorphism. We will show that this homomorphism in combination with filtration we have defined in section 1.4.4. We begin with a description of the situation we are working with.

What is going on is that we have one space of approximations $V_1$ as a linear subspace of a bigger space of approximations $V_2$, within these two spaces we also have discriminant sets $\Sigma_1$ and $\Sigma_2$. Because $V_1$ is a linear subspace of $V_2$ we know that $\Sigma_1$ is also a subspace of $V_2$. What we want is to define an isomorphism between the homology groups of a low degree of $\Sigma_1$ and the homology groups of low degree of $\Sigma_2$ that also respects the filtrations of $\Sigma_1$ and $\Sigma_2$. Therefore we will first define a homomorphism called the Pontrjagin Thom homomorphism. Later in this section we will state that if we modify this homomorphism and give some mild restrictions we obtain an isomorphism that respects the filtration.

**Proposition 1.30.** Let $V_1$ and $V_2$ be Euclidean spaces such that $V_1 \subset V_2$. Define in $V_1$ and $V_2$ discriminant sets $\Sigma_1$ and $\Sigma_2$, such that $\Sigma_1 = V_1 \cap \Sigma_2$. Let $s = \dim V_2 - \dim V_1$ be the difference in dimensions between $V_2$ and $V_1$. If $\Sigma_2$ and $V_1$ intersect transversal, i.e. there exist an $\epsilon > 0$ and a neighborhood $V_\epsilon$ of $V_1$ such that $V_\epsilon \cap \Sigma_2$ is diffeomorphic to $\Sigma_1 \times \mathbb{R}^s$, then there exists a homomorphism of reduced homology groups $\tilde{H}_i(\Sigma_2\cdot) = \tilde{H}_{i-s}(\Sigma_1\cdot)$. We will call this homomorphism the Pontrjagin Thom homomorphism.

**Proof.** The homomorphism is constructed as follows, first we collapse everything in $\Sigma_2$ that does not lie in the space $\Sigma_1 \times \mathbb{R}^s$ to a point. Then we see that this space is equal to the $s$-fold suspension of $\Sigma_1$, so we can use the suspension isomorphism $H_n(\Sigma_1\cdot) = H_{n-s}(S^s\Sigma_1\cdot)$, $S^s\Sigma_1\cdot$ is here the $s$-fold suspension.

There is also a relative version of the Pontrjagin Thom isomorphism which is as follows.

**Proposition 1.31.** Under the assumptions of 1.30 with as extra assumption that both $\Sigma_1$ and $\Sigma_2$ have a filtration $(\Sigma_1)_j$ and $(\Sigma_2)_j$ such that $(\Sigma_2)_j$ intersects $V_1$ transversal, i.e. $(\Sigma_2)_j \cup V_\epsilon = (\Sigma_1)_j \times \mathbb{R}^s$. Then we have a relative Pontrjagin Thom homomorphism given

$$H_i((\Sigma_2\cdot)_j, (\Sigma_2\cdot)_{j-1}) H_{i-s}((\Sigma_1\cdot)_j, (\Sigma_1\cdot)_{j-1}).$$
We can use this homomorphism to show that the groups in the main spectral sequence stabilize in the region $p \geq -P$ and $q \leq Q$ for some $P$ and $Q$. We will state this in the following proposition, we will not prove this proposition since it uses the construction used in the proof of Alexander duality.

**Proposition 1.32.** Under the assumptions of proposition 1.30 and proposition 1.31 there exist numbers $P$ and $Q$, such that the the Pontrjagin Thom homomorphisms are isomorphisms for $j \leq P$ and $i > \text{dim}(V) - Q + j$.

Using this homomorphism we get the following proposition which show that the terms of the discriminant are independent of of the degree of the approximation, if the degree of the approximation is high enough compared to the term of the spectral sequence.

**Proposition 1.33.** If $N$ is large enough then we have the following equality

$$
\tilde{H}_{N-s}(\sigma_p - \sigma_{p-1}) = H_{3p-s}(W_p^\bullet).
$$

Or more general, if we assume that $V_1$ and $V_2$ are approximations of a high enough degree then there exist numbers $P$ and $Q$, such that the $E^{p,q}_1$ terms of the spectral sequences for the cohomology of $V_1 - \Sigma_1$ and $V_2 - \Sigma_2$ coincide for $p \leq -P$ and $q \leq Q$.

### 1.7 The diagram complex

In the previous sections we have seen that the only missing piece in our construction of the spectral sequence is the homology of the diagram complex $W_p^\bullet$. In this section we will give a cellular structure for $W_p^\bullet$ which we will use calculate the homology of the diagram complex. This turns out to be rather difficult, therefore we need to construct another spectral sequence in the next section, we will call this spectral sequence the auxiliary spectral sequence.

The diagram complex $W_p^\bullet$ is the space $(\sigma_p - \sigma_{p-1})^\bullet$ consisting of all degenerate chord diagrams of complexity $p$. We will now describe a cell decomposition of this space in which each cell is indexed by an equivalence class of a degenerate chord diagram.

The problem with the diagram complex $W_p^\bullet$ is that it has complicated self intersections, by this we mean that there are polynomials that respect several degenerate chord diagrams. Whenever a polynomial $f$ respects two different degenerate chord diagrams $D$ and $D'$ this means that there are two sets of linear equations constraining the coefficients of $f$. These two sets are not necessarily completely independent but we know that these sets of
constraints are not equal since this would imply that the degenerate chord diagrams are equal.

Therefore there is a one-to-one correspondence between the polynomials that respect only one chord diagram of complexity $p$ and the interior of certain cells which are labeled by the equivalence classes of chord diagrams of complexity $p$. For each equivalence class of a degenerate chord diagram $[D]$ we obtain a cell which is the product of an open simplex $\Delta_D$ whose vertices are labeled by the chords of $D$ and another open simplex $E_D$ which consists of all diagrams combinatorially equivalent to $D$. The space $\Delta_D$ is the simplex coming from the simplicial resolution. The space $E_D$ is the space of all diagrams combinatorially equivalent to $D$, if we recall that two diagrams $D$ and $D'$ are combinatorially equivalent if there is an orientation preserving diffeomorphism of $\mathbb{R}$ to $\mathbb{R}$ mapping the vertices of $D$ to the vertices of $D'$. So this space $E_D$ is diffeomorphic to the configuration space of $k$ ordered points in $\mathbb{R}$, where $k$ is the number of distinct vertices of $D$. Therefore $E_D$ is diffeomorphic to an open simplex of dimension $k$.

1.7.1 The differential of the diagram complex

With this description of the cells of $W_p^\bullet$ it is time to see how these cells are attached to each other. To do this we need to define the boundary map in the corresponding cellular chain complex. To construct this map it is important to first study the boundary of a single cell. Let $[D]$ be a cell, then we know that $[D]$ is the product of two simplices $[D] = \Delta_D \times E_D$,

therefore its boundary is given by the union of $\partial \Delta_D \times E_D$ and $\Delta_D \times \partial E_D$. So we need to describe the boundaries of $\Delta_D$ and $E_D$.

If $[D']$ is a part of the boundary of $[D]$ such that the cell $[D']$ is indexed by a chord diagram $D'$ then we want the complexity of $D'$ to be the same as the complexity of $D$, otherwise we would no longer be in the space $(\sigma_i - \sigma_{i_1})^\bullet$. Therefore there is only a limited number of diagrams allowed in the boundary. We will describe the situations that are possible after we first look at the possibilities we have to decrease the dimension of a cell without changing the complexity of the cell.

Recall that the complexity of diagram $D$ is defined as

\[ c(D) = \# \text{geometrically distinct points} + \# \text{groups of chords} \]

and the dimension of a cell is given by

\[ \text{Dim}([D]) = \# \text{number of vertices} + \# \text{chords} - 1. \]
Figure 1.3: The collapse of two points from different groups. On the left we see what happens when two vertices that are not singularity vertices collapse. On the right we see what happens when one of the vertices is a singularity vertex.

Then we see that there are several ways to decrease the dimension without changing the complexity of the diagram. These ways are given by removing chords such that the number of groups does not change, this corresponds to the boundary of $\Delta_D$. The other way is by moving to vertices together until they become one vertex, this can only be done in a limited number of ways which we will describe later. Collapsing two vertices corresponds to the boundary of $E_D$. Whenever a part of the boundary of a cell $[D]$ is of complexity smaller than $c(D)$ we collapse this part to a point which we identify with the base point.

The boundary of $\Delta_D$ is given by spaces of the form $\Delta_{D'}$ where $D'$ is a degenerate chord diagram obtained from $D$ by removing some chords, such that the complexity of the diagram stays the same. Since we are only interested in the part of the boundary that is of dimension $\text{Dim}([D]) - 1$ we only obtain diagrams with one chord less than $D$.

The $E_D$ boundary part is given by moving two distinct adjacent vertices of the degenerate chord diagram $D$ close to each other and collapsing them to one point. There are 3 possible situations in which this can happen.

1. The two vertices are the end points of different chords such that both chords belong to different groups. We also require that at most one of the points is also a singularity vertex where the derivative vanishes. Then we collapse these points into one point, joining the two groups into one group. If one of the points is a singularity vertex the the resulting point after collapsing is also a singularity vertex, see also figure 1.

2. Both points are the end points of the same chord and this chord is not connected to any other groups. In this case we collapse the end points and the chord to a singularity vertex, see also figure 2.

3. One of the points is the end point of a chord but not a singularity vertex and the other point is a singularity vertex. In this case we
Figure 1.4: The collapse of the two end points of an isolated chord becomes a singularity vertex.

Figure 1.5: The collapse of the end point of a chord with a singularity vertex is given by end point of the chord which is now turned into a singularity vertex.

contract both points to the end point of the chord and turn this point into a singularity vertex. See figure 3 for more details.

To fix the signs of the differential we need to define an orientation on the space \([D]\). Since \([D]\) is the product of two simplices we need to define an order on the vertices of the simplices. We order them as follows, we begin by ordering the vertices of \(E_D\). Since \(E_D\) is the configuration space of \(k\) points we can order them in a natural way. Since the simplex is open the boundary does not belong to \(E_D\) and corresponds to degenerate configurations where the configurations are allowed to have multiple points. The vertices correspond to configurations where one part of the points is on the left end of the interval and the other part is on the right end of the space. We order them by defining the \(i\)th vertex as the vertex corresponding to the degenerate configuration with \(k - i\) points on the left end and \(i\) points on the right end of the interval.

The orientation on \(\Delta_D\) is defined as follows. Because \(\Delta_D\) is the open simplex with vertices labeled by chords of \(D\), we need to order these chords. We will do this in two steps, first we will order the chords in each group of self intersections and then we will order the groups. Let \(c\) and \(d\) be two chords belonging to the same group then we say that \(c\) is smaller than \(d\) if the left end point of the chord \(c\) is smaller than the left end point of \(d\). If
Figure 1.6: A few examples of the differential of the diagram complex $W_p^\bullet$.

the left end points coincide then we call $c$ smaller if the right end point of $c$ is smaller then the right endpoint of $d$. The order of the groups is also given by the left end point. A group is smaller than another group if the left end point of the first group is smaller than the left end point of the right group. We will now define the following order on the vertices by saying that a chord is smaller than another chord if the group of the first chord is smaller than the group of the second chord. If the chords belong to the same group we say that the first chord is smaller if it is smaller than the other chord within that group.

With this orientation on the cell $[D]$ we can now define the differential.

**Definition 1.34.** Let $[D]$ be a the cell corresponding to a degenerate chord diagram $D$, then we define the differential $d([D])$ as the sum over all chord diagrams $D'$ such that $D'$ is obtained for $D$ by either removing a chords such that the number of groups of self intersections does not change. Or $D'$ is obtained from $D$ by merging two vertices together in one of the ways described above. The signs are described according to the orientation defined above.

In figure 1.7.1 we see an example of the differential in the case $p = 2$.

**Remark 1.35.** We have a different convention about the sign of the differential than [54], we follow the conventions of [10].

### 1.8 The auxiliary spectral sequence

The spectral sequence we have defined so far is not yet useful since we do not yet know what the homology of the diagram complex $W_p^\bullet$ is. To calculate
Figure 1.7: On the left is the diagram of complexity $k$ with a minimal number of vertices. On the right the degenerate chord diagram with a maximal number of distinct vertices.

this homology we have described the cellular chain complex on $W^\bullet_p$ but we still don’t know its homology. In this section we will define a second spectral sequence that will calculate the homology of the diagram complex, we will call this spectral sequence the *auxiliary spectral sequence*. In section 5.7 we will also see that this spectral sequence has some interesting relations with the Poisson operad, but first we will construct this spectral sequence.

First we will define a filtration on $W^\bullet_p$ given by the number of vertices, since $W^\bullet_p$ consists of cells indexed by equivalence classes of degenerate chord diagrams of complexity $p$ we can define a filtration given by the number of vertices. Let $W^\bullet_p(k)$ be the union of the space of all diagrams with at most $k$ geometrically distinct vertices and the base point. If we denote by $[i]$ be the integer part of $i$, then we have that for all $k$ smaller than $[\frac{p}{2}] + 1$ the space $W^\bullet_p$ is equal to the base point since there are no diagrams of complexity $p$ with less than $[\frac{p}{2}] + 1$ distinct vertices. When $k$ is greater or equal to $2p$ we know that the space $W^\bullet_p$ is equal tot he whole space $W^\bullet_p$, since there are no diagrams with more than $2p$ geometrically distinct vertices and that the only diagram of complexity $p$ with $2p$ vertices is the diagram with $p$ chords and no chords ending at the same point, see also figure 1.8. With this filtration we are going to construct another spectral sequence which we will call the auxiliary spectral sequence.

To construct the auxiliary spectral sequence we have to describe the spaces $W^\bullet_p(k)/W^\bullet_p(k-1)$. To do this we give a description of $W^\bullet_p(k)/W^\bullet_p(k-1)$ as the wedge of certain spaces indexed by the equivalence classes of degenerate chord diagrams, note that the equivalence classes of degenerate chord diagrams are given by $(A,b)$ configurations. So we get

$$W^\bullet_p(k)/W^\bullet_p(k-1) = \bigvee_{\mathcal{D}} [\mathcal{D}].$$
The wedge runs over all equivalence classes of degenerate chord diagrams with exactly \( k \) vertices. The spaces \([\mathcal{D}]\) consist of the union of the base point with all cells \([D]\), where \( D \) is a degenerate chord diagram equivalent to \( \mathcal{D} \). In the next section we will give an explicit description of these spaces \([\mathcal{D}]\).

### 1.8.1 A description of the cells \([\mathcal{D}]\)

The spaces \([\mathcal{D}]\) are the union of the base point with all cells indexed by a diagram equivalent to \( \mathcal{D} \). We could of course write down all the diagrams equivalent to \( \mathcal{D} \) and do all our calculations in this way, but this is almost impossible to do in practice since the number of diagrams is growing very rapidly as the complexity of the diagrams increases. Therefore we have to come up with a better description of the spaces \([\mathcal{D}]\).

The plan is to define a series of simpler spaces \( \Delta^1(a_i) \), which we will call the complex of connected diagrams. These spaces \( \Delta^1(a_i) \) are indexed by the groups of self intersections. After defining these spaces we will prove that the space \([\mathcal{D}]\) is equal to the smash product of a certain number of these spaces with a sphere of a certain dimension. We begin by carefully defining the spaces \( \Delta^1(a_i) \).

**Definition 1.36.** Let \( a \geq 1 \) be an integer and \( A \) as set of a distinct point. Then we define the complex of connected graphs \( \Delta^1(a) \) as follows. First take the \( \frac{a(a-1)}{2} - 1 \)-dimensional simplex with vertices that are indexed by the chords connecting pairs of points of \( A \). To each point \( x \) of this simplex we can assign a degenerate chord diagram in the following way. If the point \( x \) lies in the interior of the simplex we assign the chord diagram with all points connected by all possible chords. If \( x \) lies in the interior of one of the faces we assign the chord diagram with the set \( A \) as vertices and draw a chord between two points if that chord is one of the vertices of the face. For an arbitrary point \( x \) we take the smallest face that \( x \) contains and draw a chord for each vertex of the smallest face that contains \( x \). The complex of connected graphs \( \Delta^1(a) \) is the space obtained from this simplex by collapsing all the faces whose corresponding chord diagram is not connected to a point.

**Lemma 1.37.** The homology of the space \( \Delta^1(a) \) is as follows.

\[
H_i(\Delta^1(a)) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0, \\
\mathbb{Z}(a-1)! & \text{if } i = a - 2, \\
0 & \text{otherwise}.
\end{cases}
\]

This proposition can be proofed by relatively straightforward calculation, otherwise it can be found in [54].
Lemma 1.38. Let $\mathcal{D}$ be the equivalence class of a degenerate chord diagram and let $[\mathcal{D}]$ be the corresponding space. Let the degenerate chord diagram have $m$ distinct groups of points of which the $i$th-group has $a_i$ different points. If we let $b$ be the number of singularity vertices and $k$ be the number of total vertices. Then the space $[\mathcal{D}]$ can be decomposed as follows

$$[\mathcal{D}] = \Delta^1(a_1) \wedge \ldots \wedge \Delta^1(a_m) \wedge S^{k+m+b-1}.$$ 

Where the $\wedge$ stands for the smash product of based topological spaces, i.e. $X \wedge Y = (X \times Y)/(X \vee Y)$.

Proof. We will first check the proposition for the equivalence class of a degenerate chord diagram with one group consisting of $a$ points and without singularity vertices. In this case the space $[\mathcal{D}]$ is the space of all diagrams equivalent to one point compactification of the space of all diagrams equivalent to $D$. This space is the product of the unordered configuration space of the vertices with at each point the space of all possible graphs equivalent to $D$. The first space is diffeomorphic to $\mathbb{R}^a$ and the second space is $\Delta^1(a)$, the complex of connected graphs without the base points. The one point compactification of the product of these two spaces is equal to the smash product of these spaces. So we get

$$[\mathcal{D}] = S^a \wedge \Delta^1 a.$$ 

For an arbitrary diagram, $[\mathcal{D}]$ can be described as follows. The space of diagrams combinatorially equivalent to $D$ is again parametrized by the configuration space of the geometrically distinct vertices of $D$, this space is diffeomorphic to $\mathbb{R}^{k+m+b-1}$. The other space is the product of $\mathbb{R}^{m+b-1}$ with the spaces $\Delta^1(a_i)*$, i.e. the complex of connected graphs without the base point. The $\mathbb{R}^{m+b-1}$ term comes from the singularity vertices and the order of the groups. The other term comes from all the diagrams that are equivalent to $D$. So we finally end up with the one point compactification of the space:

$$\mathbb{R}^k \times (\mathbb{R}^{m+b-1} \times (\Delta^1(a_1) - *) \times \ldots \times (\Delta^1(a_m) - *))$$

The one point compactification of this space is equal to

$$\Delta^1(a_1) \wedge \ldots \wedge \Delta^1(a_m) \wedge S^{k+m+b-1}$$

which proofs the proposition.

We want to combine the previous two lemmas to calculate the homology of the cells $[\mathcal{D}]$. To do this we will need a special case of the K"unneth theorem in the case of smash products.
Lemma 1.39. Let $X$ and $Y$ be $CW$-complexes and $R$ be a commutative ring, then we have the following short exact sequence in homology:

$$0 \to \bigoplus_i (\tilde{H}_i(X;R) \otimes_R \tilde{H}_{n-i}(Y;R)) \to \tilde{H}_n(X \wedge Y;R) \to$$

$$\bigoplus_i \text{Tor}_R(\tilde{H}_i(X;R), \tilde{H}_{n-i-1}(Y;R)) \to 0.$$  

The proof of the lemma can be found in [24]. From this lemma and the previous lemmas we can conclude that the reduced homology of $\mathfrak{D}$ is only non zero in degree $p + k - 1$. We can see this by repeatedly applying lemma 1.39 to the space $[\mathfrak{D}] = \Delta^1(a_1) \wedge \ldots \wedge \Delta^1(a_m) \wedge S^{k+m+b-1}$. The reduced homology of $\Delta^1(a)$ is $\mathbb{Z}(a-1)!$ in degree $a-2$ and the reduced homology of $S^n$ is $\mathbb{Z}$ in degree $n$, both these groups are free abelian and therefore the $\text{Tor}$ term of the short exact sequence vanishes. Using this we get that the homology of $[\mathfrak{D}]$ is only non zero in degree $(a_1 - 2) + \ldots + (a_m - 2) + (k+m+b-1)$ if we recall that $p = a_1 + \ldots + a_m + b - m$ we get indeed that the only degree in which the reduced homology is non zero is degree $p + k - 1$.

With all this we can write down the auxiliary spectral sequence. Since we are mainly interested in the zeroth homology of the space of knots, we are mainly interested in $H_{3p-1}(W^*_p)$. From the spectral sequence we get the following lemma.

Lemma 1.40. \(H_{3p-1}(W^*_p) = H_{3p-1}(W^*_p/W^*_p(2p-2))\)

Proof. The only terms where the spectral sequence is non zero is part of the $p$th row and some part of the zeroth row that corresponds to the zero homology since the space is connected. The part corresponding to the zeroth homology is not interesting since it will not effect the $3p-1$th homology. The only terms that determine the degree $3p-1$ homology groups are the terms $E^2_{p,2p-1}$ and $E^2_{p,2p-2}$, the first one because $H_{3p-1}$ will be a quotient of this group and the second term because the differential of the spectral sequence might have a non trivial image in this term. Because of the positions all the other non zero terms the differential will vanish after the $E^2$ page. So the group $H_{3p-1}(W^*_p)$ depends only on $W^*_p/W^*_p(2p-2)$ and therefore we have $H_{3p-1}(W^*_p) = H_{3p-1}(W^*_p/W^*_p(2p-2))$. \qed

35
1.9 The diagonal of the spectral sequence

In this section we will finally give a description of the diagonal of the $E^1$ term of Vassiliev’s main spectral sequence. Remember that the zeroth cohomology of the space of knots is given by $\bigoplus_p E_{\infty}^{-p,p}$, the direct sum of the $(-p,p)$ groups of the $E_\infty$ term. Also remember that the $E_\infty^{-p,p}$ term is a quotient of the $E_1^{-p,p}$ term by certain higher terms coming from the spectral sequence.

From the previous sections we have found the following description of the groups $E_1^{-p,p}$, $E_1^{-p,p} = H_{3p-1}(W_p^*) = H_{3p-1}(W^*/W^*(2p-2))$. In this section we will give a concrete description of the spaces $H_{3p-1}(W^*/W^*(2p-2))$. We will do this by describing the cycles and the boundaries in $W^*/W^*(2p-2)$ and then calculate the homology explicitly.

The cells of the cellular chain complex of $W^*/W^*(2p-2)$ are of the following form.

1. A degenerate chord diagram with $2p$ self intersection vertices and $p$ chords such that each group contains exactly one chord. In this case there are no singularity vertices.

2. Degenerate chord diagrams with no singularity vertices and $2p - 1$ distinct self intersection points. The diagram has $p + 1$ chords such that there are $p - 2$ groups with one chord and 1 group consisting of 3 vertices connected by 3 chords.

3. Diagrams with $2p - 1$ vertices such that $2p - 2$ of these vertices form pairs joined by chords, there is one singularity vertex.

4. Degenerate chord diagrams with $2p - 1$ joined together in $p - 1$ groups, $p - 2$ of these groups are groups consisting of two point joined by a chord. The other group consists of 3 points together with 2 chords.

Note that the (1) and (2) correspond to cells of dimension $3p - 1$ and that (3) and (4) correspond to cells of dimension $3p - 2$. To calculate the homology we have to determine the image of the cells of the form (1) and (2) under the differential in the cells of the form (3) and (4). To do this we will need to use a little trick.

This trick goes as follows, instead of looking at the homology we will look at the cohomology. Since all modules we are dealing with are free and finite dimensional we can dualize the chain complex and consider the coboundary map. Then we know that the kernel of the differential is isomorphic to the cokernel of the coboundary map. Since the coboundary is much easier to describe in this situation
To be more concrete, let $d_{3p-1} : C_{3p-1} \to C_{3p-2}$ be the boundary operator in the cellular chain complex of $W_p^*/W_p^*(2p-2)$, then we define the coboundary operator $d^*_{3p-1} : C_{3p-2} \to C_{3p-1}$ as follows.

**Definition 1.41.** The coboundary map is defined as follows

1. On diagrams of the form (3) we define the coboundary map by sending a diagram with $2p - 2$ self intersection vertices forming $p - 1$ groups and one singularity vertex to the diagram which has $2p$ self intersection vertices forming $p$ groups. This is done by turning the singularity vertex into a chord by placing both end next to each other close to where the singularity vertex used to be. In figure 1 for an example of this map.

2. The other situation where we have to define the coboundary map is for diagrams of type (4). In this case we have a diagram $D$ with $2p - 1$ self intersection vertices which are joined into $p - 2$ groups, $p - 1$ of these groups consist of 2 points joined by one chord and one group consists of 3 points joined by 2 chords. The image of $d^*_{3p-1}$ of diagrams of this form is given by the sum of the following 3 diagrams. The first diagram is obtained from $D$ by adding a third chord to the group consisting of 3 vertices. The second and the third diagram is obtained from $D$ by splitting the only point where two chords end into two points and connecting the chords that used to end at this point to the new points in the two possible ways. A more concrete description is given in figure 2.

**Remark 1.42.** The diagrams in the pictures represent functions that are one on this diagram and zero otherwise and not the diagram itself. Since diagrams are easier to draw then function we did it this way since it still makes things clearer.

**Remark 1.43.** To define the signs corresponding to part (2) of the coboundary operator we need to use the orientation of the diagram complex. We will alter this orientation a little by multiplying the orientation of each chord.
Figure 1.9: An example of the coboundary map on diagrams with one group which consists of 3 vertices joined together by 3 chords.

In the future we will refer to (1) of definition 1.41 as the $1T$ relation since it implies that degenerate chord diagrams with an isolated chord are boundaries in this complex, therefore any cohomology class will vanish on these functions. Part (2) of definition 1.41 will be referred to as the $4T$ relation, because it can be shown that this relation generates the kernel in the complex. Using this we can also eliminate all diagrams that have groups consisting of more than two points and one chord. Therefore the only diagrams that are left are degenerate chord diagrams that have $2p$ vertices joined in $p$ groups, such that each group contains two points and one chord connecting these two points. We will call degenerate chord diagrams of this form *non-degenerate chord diagrams* or when it is clear from the context just chord diagrams. From all this we obtain the following description of the diagonal terms of the main spectral sequence.

**Theorem 1.44.** The $E_1^{p,p}$ term of Vassiliev’s main spectral sequence is given by the dual of the space generated non-degenerate chord diagrams modulo the $1T$ and $4T$ relations.
1.10 Some final remarks about higher degree homology groups and generalizations to higher dimensions

We conclude this chapter with some final remarks about the spectral sequence. We will also refer the interested reader to some articles that generalize the construction we have just described.

1.10.1 Higher homology groups

From the previous discussion we can also calculate the higher homology groups by calculating the other homology groups of the diagram complex. This can be done by using the auxiliary spectral sequence and by calculating the full $E^2$ term instead of just the groups corresponding to the $H_{3p-1}$ group. We will not do this in this thesis but an interested reader could read Vassiliev’s original article [54]. As will turn out the spectral sequence constructed in this way will not converge nicely in the three dimensional case. Although it can be shown that the diagonal of this spectral sequence collapses over the rational numbers, see also chapter 2 and chapter 3.

1.10.2 Generalizations to higher dimensions and arbitrary target manifolds

We finish this chapter with some remarks about generalizations of the spectral sequence when we embed our knot in $\mathbb{R}^n$ instead of $\mathbb{R}^3$ or even in an arbitrary manifold $M$. The main difference between $\mathbb{R}^3$ and $\mathbb{R}^n$ is that for $n \geq 4$ all knots are isotopic to the unknot and that this makes the zeroth cohomology less interesting. Vassiliev’s method can still be used to calculate the other cohomology groups of the space of embeddings of $\mathbb{R}$ in $\mathbb{R}^n$ with fixed behavior at infinity. The main difference in this case is that the number of conditions on the coefficients of the approximating polynomials will increase. Since the possible singularities a knot can have are still the same singularities as in $\mathbb{R}^3$ the diagram complex stays the same we only need to take different groups corresponding to the different cohomology classes of $\mathcal{K}_n$. For more information about generalizations to higher dimension and arbitrary manifolds we refer the reader to [56] and [55].
Chapter 2

Vassiliev invariants

Although Vassiliev’s spectral sequence does not converge well in the case of \( \mathbb{R}^3 \), it still gives us a class of knot invariants called Vassiliev or finite-type invariants. In the previous chapter we have seen how these invariants are constructed from Vassiliev’s spectral sequence. In this chapter we will describe them from a more combinatorial point of view, which can be used to study Vassiliev invariants without knowing about Vassiliev’s spectral sequence. We will begin by giving an alternative definition of Vassiliev invariants and give a couple of theorems that compare Vassiliev invariants to certain other invariants. In this chapter we will work with compact knots instead of the long knots from the previous chapter. We do this because most other knot invariants are only defined for compact knots and it is good to compare Vassiliev invariants with other knot invariants. In this chapter we will also try to give a geometric idea how Vassiliev invariants work. The Vassiliev invariants for compact knots can be defined in the same way by constructing a spectral sequence as we did for the long knots.

Remark 2.1. For compact knots it is also possible to define the space of all knots and the discriminant. The space of all knots is defined as the space of all embeddings of \( S^1 \) in \( \mathbb{R}^3 \) and is denoted by \( K \). The discriminant is defined as the complement of the space of all knots \( K \) in the space of all immersions of \( S^1 \) in \( \mathbb{R}^3 \). We denote the space of all immersions by \( \text{Imm}(S^1, \mathbb{R}^3) \) and the discriminant by \( \Sigma \).

Convention 2.2. In this chapter we assume that all our knots are oriented.

Convention 2.3. In this chapter we will define a singular knot as an immersion of the circle in \( \mathbb{R}^3 \), such that the only singularities that are allowed are double points, higher order self intersections and points where the derivative vanishes are no longer allowed.
2.1 Vassiliev’s skein relation, chord diagrams and weight systems

In section 1.4.3 of chapter 1 we have already seen the definition of degenerate chord diagrams, in this chapter we will be only concerned with non-degenerate chord diagrams which correspond to singular knots with only double points as singular points, i.e. we no longer allow triple points or points with an even higher intersection multiplicity and points where the derivative vanishes. As we will see these diagrams contain all the information we need to define the Vassiliev invariants. In this section we will first define how we can extend a knot invariant to the discriminant and define what it means to be a Vassiliev invariant. Then we will show how we can use chord diagrams to define Vassiliev invariants.

2.1.1 Vassiliev’s skein relation

In the construction of Vassiliev’s spectral sequence we mainly looked at the discriminant set, i.e. knots with singularities. This idea has also another more explicit interpretation in terms of knots and knot invariants. We will first explain this idea and then give Vassiliev’s skein relation, this relation allows us to extend normal knot invariants to singular knots.

The idea is that every two knots can be connected with each other by a path through the space of all immersions of \( S^1 \) in \( \mathbb{R}^3 \), such that the path crosses the discriminant set only transversal. A knot invariant is a locally constant functions on the connected components of the space of all knots. So this path gives us a function from the space of all immersions to some abelian group \( G \), such that this function is constant on each connected component of the space of knots. Every time this path crosses the discriminant set its value changes from the value of one connected component to the value on the other connected component. This idea leads to a definition similar to the derivative in calculus. We will use this idea of a derivative to define Vassiliev’s skein relation, this skein relation allows us to extend knot invariant defined on the space of knot to the space of immersions.

Definition 2.4. Let \( \Phi \) be a knot invariant, i.e. a function from \( \mathcal{K} \) to some abelian group \( G \) which is constant on the the connected components of \( \mathcal{K} \). Then we extend \( \Phi \) to the parts of the discriminant where the singularities are only double points in the following way. Let \( K \) be a singular knot with singularities \( \epsilon_1, \ldots, \epsilon_n \). Then we can resolve each double point of \( K \) in two possible ways, one is by changing the double point in an over crossing and one is by changing the double point in an under crossing. By \( K_{\epsilon_i}^{+} \) we denote
Figure 2.1: Vassiliev’s skein relation is given by resolving the double point by the difference of the values of $\Phi$ of the positive resolution and the value of $\Phi$ of the negative resolution. Outside the circles the knots are the same.

the resolution of $\epsilon_i$ which replaces the double point $\epsilon_i$ by an over crossing. By $K_{\epsilon_i}$ we denote the resolution of $\epsilon_i$ which replaces the double point $\epsilon_i$ by an under crossing. The singular knot can be resolved in $2^n$ possible ways. We extend $\Phi$ to $\Sigma$ by setting

$$\Phi(K) = \sum_{(a_1, \ldots, a_n) \in \{-1,1\}^n} a_1 \cdot \ldots \cdot a_n \Phi(K_{\epsilon_1} \cdot \ldots \cdot \epsilon_n).$$

The sum runs here over all possible resolutions.

**Remark 2.5.** The idea is to resolve each singularity by the skein relation from figure 2.1.1. By doing this repeatedly we obtain the formula from definition 2.4. We call the skein relation from figure 2.1.1 **Vassiliev’s skein relation**.

**Remark 2.6.** The Vassiliev invariants are now only defined for singular knots with only double points, we only need to do this since it is not necessary for the path connecting two knots to cross the discriminant at triple points or at points where the derivative vanishes.

We can interpret a Vassiliev invariant as something similar to a derivative in the following way. The idea is that whenever two knot are separated by the discriminant then a knot invariant with different values at each side of the discriminant makes a "jump" from one value to another. So the difference between these values is a bit like a derivative.

**Definition 2.7.** A knot invariant $\Phi$ is called a **Vassiliev invariant** of degree (sometimes also called order) $\leq n$ if $\Phi(K) = 0$ for all singular knots $K$ with more than $n$ double points.
We can interpret a Vassiliev invariant as something similar to a derivative in the following way. The idea is that whenever two knot are separated by the discriminant then a knot invariant with different values at each side of the discriminant makes a "jump" from one value to another. So the difference between these values is a bit like a derivative.

**Example 2.8.** There are many examples of Vassiliev invariants, as we will see in the next section is that all of the classical knot polynomials can be written as a (possibly infinite) sum of Vassiliev invariants. An example of a Vassiliev invariant of order $\leq n$ is the coefficient of the $n$th term of the Conway polynomial.

**The algebraic structure of the Vassiliev invariants**

Vassiliev invariants can be multiplied and add to each other and therefore form an algebra. We will now describe this algebraic structure on the set of all Vassiliev invariants with values in a commutative ring $R$.

**Definition 2.9.** Let $V_n$ be the set of all Vassiliev invariants of order $\leq n$ with values in a commutative ring $R$. Note that we have an inclusion of spaces given by $V_n \subseteq V_{n+1}$. If we denote the union of all these spaces by $V$ we define this as the space of all Vassiliev invariants and see that this space is an filtered $R$-module.

**Theorem 2.10.** The space of all Vassiliev invariants $V$ is a commutative filtered algebra with the product defined by the pointwise multiplication of two invariants. Moreover the product of two Vassiliev invariants $f$ and $g$ of order $\leq m$ and $\leq n$ is of degree $\leq n + m$.

### 2.1.2 Chord diagrams

The easiest way to describe Vassiliev invariants is by using chord diagrams. A chord diagram with $n$ chords contains all the information about the singularities of the knot and therefore determines the possible values of the Vassiliev invariants of order $\leq n$. In this subsection we will first introduce chord diagrams and there algebraic structure. Then we show how chord diagrams are related to singular knots and explain how to associate a chord diagram to a singular knot. After that we will introduce weight systems, these are functions from chord diagrams to some abelian group of commutative ring and give us the order $n$ Vassiliev invariants.

**Definition 2.11.** A chord diagram is an oriented circle with finitely many pairs of different points marked on it. Such that these pairs of points are connected by a line called a chord.
Remark 2.12. Note that the difference between degenerate chord diagrams and non degenerate chord diagrams lies in the fact that degenerate chord diagrams can have singularity vertices which were denoted by a hollow dot and that they can have multiple intersections and not only double points.

To every singular knot we can associate a chord diagram in the following way. Recall that a singular knot is an oriented circle embedded in $\mathbb{R}^3$ with some singularities, we will now on the circle connect those points with a chord if the image of those points is the same. So if $K: S^1 \to \mathbb{R}^3$ is a singular knot then we connect two points $x$ and $y$ on $S^1$ with a chord if and only if $K(x) = K(y)$.

The set of chord diagrams has also some nice algebraic structures which we will now describe.

Definition 2.13. Define $A_n$ to be the set of all chord diagrams with $n$ chords. If $\mathcal{R}$ is a commutative ring we define $\mathcal{R}A_n$ as the $\mathcal{R}$-module generated by all chord diagrams with $n$ chords modulo the $4T$ relation from figure 2.1.2 and the $1T$ relation, which is defined by setting all chord diagrams with an isolated chord to zero, an isolated chord is a chord that does not intersect any other chord.

Remark 2.14. The reason that we want the $1T$ relation is that whenever we resolve the singularity corresponding to the isolated chord we get two possibly singular knots which are isotopic, therefore if we take there difference we will always get zero. The $4T$ relation can be explained by figure 2.1.2.

Remark 2.15. It is also possible to define the space $\mathcal{R}A_n$ as the space of all chord diagrams of degree $n$ only modulo the $4T$ relation. By forgetting the $1T$ relation we are looking at framed knots instead of unframed knots.
Figure 2.3: To go from one knot two another there are two possible ways. One is by first crossing the walls $S$ and $E$, the other way is by first crossing $W$ and then $N$. Since we want these paths to give us the same knot invariant we obtain the $4T$ relation.

Figure 2.4: The $4T$ relation for chord diagrams. We have only drawn the chords where the diagrams are different.
Definition 2.16. Let $C \in \mathcal{R}A_n$ and $D \in \mathcal{R}A_m$ be two chord diagrams, then we define a product $\mathcal{R}A_n \times \mathcal{R}A_m \to \mathcal{R}A_{n+m}$ called the connected sum, by defining the connected sum of the chord diagrams $C$ and $D$ as the chord diagram obtained by cutting $C$ and $D$ open at an arbitrary point and then connecting the two circles such that the orientations match. In figure 2.1.2 we see an example of the connected sum of two diagrams.

The connected sum of chord diagrams is a priori not well defined since it depends on the points where we attach the circles. The following lemma states that the connected sum is well defined modulo the $4T$ relation from figure 2.1.2.

Lemma 2.17. The connected sum of chord diagrams is well defined modulo the $4T$ relation defined in figure 2.1.2.

On the set of chord diagrams we can also define a comultiplication, this is done in the following definition.

Definition 2.18. Let $D \in \mathcal{R}A_n$ be a chord diagram then we define the coproduct

$$\delta : \mathcal{R}A_n \to \bigoplus_{k+l=n} \mathcal{R}A_k \otimes \mathcal{R}A_l$$

as

$$\delta(D) = \sum_{J \subseteq [D]} D_J \otimes D_{\overline{J}}$$

In this expression $[D]$ is defined as the set of chords of the diagram $D$. The sum runs here over all subsets $J$ of $[D]$ and $\overline{J}$ is defined as the complement of $J$ in $[D]$. The diagram $D_J$ is the diagram obtained by removing all the chords of the set $J$ from the diagram $D$. So the sum runs over all possible ways to split the set of chords of the diagram $D$. See figure 2.1.2 for an example of this comultiplication.

Lemma 2.19. The comultiplication is well defined modulo the $4T$ relation.
Figure 2.6: An example of the comultiplication of a chord diagram.

**Theorem 2.20.** Let \( \mathcal{R} \) be a commutative ring, then the algebra of chord diagrams \( \bigoplus_{n \geq 0} \mathcal{R} A_n \) is a graded commutative, co-commutative bialgebra.

**Remark 2.21.** The algebra of chord diagrams is not graded commutative but commutative in the sense that \( a \cdot b = b \cdot a \).

**Remark 2.22.** The bialgebra of chord diagrams is even a Hopf algebra. Since we will not use this structure we will not describe it, more details can be found in [10].

The proof of lemma 2.17 and 2.19 and theorem 2.20 can be found in [10].

**Chord diagrams for long knots**

We can also define the algebra we have define above in the case of long knots. We do this by defining our chord diagrams as follows.

**Definition 2.23.** The chord diagram of degree \( n \) for a long knot is defined as \( \mathbb{R} \) together with a set of \( n \) pairs of points such that all the points are disjoint. We connect the points of each pair by a line which we call a chord.

**Weight systems**

We have seen that we can represent the equivalence classes of singular knots by chord diagrams, since we are mainly interested in knot invariants we will now look at the dual of the bialgebra of chord diagrams.
Definition 2.24. A weight system of order $n$ with values in a commutative ring $\mathcal{R}$ is a linear functional from $\mathcal{R}A_n$, the degree $n$ part of the algebra of chord diagrams, to the ring $\mathcal{R}$. We denote the set of weight systems of order $n$ by $\mathcal{W}_n$ and the union of all $\mathcal{W}_n$ by $\mathcal{W}$.

The set of weight systems is the dual algebra of the bialgebra of chord diagrams, therefore is the set of weight systems again a bialgebra, the exact structure is described in the following theorem. The proof of the theorem can be found in [10].

Theorem 2.25. On the set of weight systems we define the following multiplication and comultiplication. Let $w_1$ and $w_2$ be weight systems of order $m$ and $n$ and $D_1$ and $D_2$ be chord diagrams of order $k$ and $l$, then we define the multiplication $\cdot : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ as

$$(w_1 \cdot w_2)(D) = (w_1 \otimes w_2)(\delta(D)).$$

The comultiplication $\delta : \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$ is defined as

$$\delta(w_1)(D_1 \otimes D_2) = w_1(D_1 \cdot D_2).$$

The counit is defined by evaluating the weight system on the chord diagram without chords.

We will now explain the relation between Vassiliev invariants and weight systems by defining a map from the algebra of Vassiliev invariants to the bialgebra of weight systems. We will call this map the symbol of a Vassiliev invariant. The definition of this map is very similar to the symbol of a differential operator. It is good to keep this analogy in mind, if conjecture 2.32 turns out to be true and every knot invariant can be written as a "Taylor series" of Vassiliev invariants this map will play the exact same role as the symbol for differential operators.

Definition 2.26. Let $v$ be a Vassiliev invariant of order $n$, then we define the map $\alpha_n : \mathcal{V} \to \mathcal{R}A_n$ by

$$\alpha_n(v)(D) = v(K)$$

where $K$ is a singular knot such that the corresponding chord diagram is $D$.

Note that the map $\alpha_n$ is a map to the set of chord diagrams with exactly $n$ chords. Therefore it kills all Vassiliev invariants of degree $\leq n - 1$. So the map $\alpha_n$ only keeps the degree $n$ part of the Vassiliev invariant $v$. 

48
Definition 2.27. Let \( v \) be a Vassiliev invariant of order \( n \) then we define the symbol of \( v \), denoted by \( \text{symb}(v) \), as the image of \( \alpha_n \), i.e.

\[
\text{symb}(v) = \alpha_n(v).
\]

Remark 2.28. Every Vassiliev invariant \( v \) of order \( n \) defines a weight system \( \text{symb}(v) \) of order \( n \) given by the symbol of \( v \), it can be shown that the map \( \text{symb} \) commutes with the multiplications of \( \mathcal{V} \) and \( \mathcal{W} \). It is however not an algebra homomorphism since it does not commute with the addition. To see this look at two Vassiliev invariants \( v_1 \) and \( v_2 \) of order \( n \) and \( m \) with \( n < m \), then we have \( \text{symb}(v_1 + v_2) = \text{symb}(v_2) \), this is in general not equal to \( \text{symb}(v_1) + \text{symb}(v_2) \).

2.2 Vassiliev invariants vs other invariants

In this section we will compare Vassiliev invariants with some of the other known invariants. We will do this relatively briefly since the only purpose of this section is to motivate why Vassiliev invariants are important. As we will see many known knot invariants turn out to be "Taylor series" in terms of Vassiliev invariants. We will not give any proofs but refer the reader to the original sources for proofs and more details about the relationship between other knot invariants and Vassiliev invariants.

Theorem 2.29. The coefficient of \( n \)th term of the Conway polynomial is a Vassiliev invariant of degree \( n \).

This theorem is proofed in [1].

Theorem 2.30. After a suitable change of variables each coefficient of the Jones polynomial, the HOMFLY polynomial and the Kauffman polynomial is a Vassiliev invariant.

The proof of this theorem can be found in [4]. We also have a more general statement about the Reshetikhin-Turaev invariants coming from the Yang-Baxter equation.

Theorem 2.31. After a change of variables the coefficients of the Reshetikhin-Turaev quantum group invariants are Vassiliev invariants.

The proof of this theorem can be found in [34]. Since all known polynomial invariants can be expressed as the infinite sum of Vassiliev invariants we have the following conjecture.

Conjecture 2.32. The Vassiliev invariants form a complete system of invariants for knots.
Chapter 3

The Kontsevich integral

In the previous chapter we have seen how Vassiliev invariants can be described by chord diagrams and weight systems. In this chapter we will prove that the dual of the algebra of chord diagrams is isomorphic to the algebra of Vassiliev invariants. We do this by using the Kontsevich integral, this is a knot invariant which, as the name suggests, is given by an integral over the knot. The remarkable property of this invariant is that realizes all Vassiliev invariants in one formula, in a way we will make precise later in this chapter. We will also use this integral to show that Vassiliev’s spectral sequence collapses along the diagonal in the case when the target manifold is \( \mathbb{R}^3 \). It is conjectured that this integral is a complete knot invariant, but it is not yet known if this is the case. In the case of braids and string links Bar-Natan and Kohno proved in [3] and [28] that this is indeed the case but for knots the problem is still open. The references for this chapter are [29], [2], [10], [9].

**Convention 3.1.** In this chapter we will only consider embeddings of the circle \( S^1 \) in \( \mathbb{R}^3 \), and we will interpret \( \mathbb{R}^3 \) as the product \( \mathbb{C} \times \mathbb{R} \). In this way we can interpret a knot as a path through this space. On the product \( \mathbb{C} \times \mathbb{R} \) we choose coordinates \((z, t)\).

### 3.1 The formula for the Kontsevich integral

We will begin with the formula for the Kontsevich integral, we will later in this chapter make precise and prove that all Vassiliev invariants factor through the Kontsevich integral. We will first give a couple of similar integral formulas to see where the formula for the Kontsevich integral comes from.
The braiding number of two braids

As a warm up we will first consider a braid with 2 strands. If we want to calculate the braiding number, i.e. the number of times the braids are twisted around each other, of this braid we can first parametrize the strand of this braid by two functions $z(t) : \mathbb{R} \to \mathbb{C}$ and $w(t) : \mathbb{R} \to \mathbb{C}$. We can now calculate the braiding number by integrating over the difference of $z(t)$ and $w(t)$. The formula looks as follows:

$$\frac{1}{2\pi i} \int_0^1 \frac{dz(t) - dw(t)}{z(t) - w(t)}.$$

Since complex integration does not depend on the number of twists and not on the particular paths, this integral calculates the number of twists.

The Gauss integral formula for the linking number

We can generalize this formula to a link invariant which calculates the linking number of two linked knots. To give the definition of this invariant we first have to put a very mild restriction on the knots we are working with.

**Definition 3.2.** A **Morse knot** $K$ is an embedding of the circle $S^1$ in to $\mathbb{R}^3$ such that the projection of the function $K : S^1 \to \mathbb{C} \times \mathbb{R}$ on the $\mathbb{R}$ is a Morse function, i.e. the projection of the function $K$ has only non degenerate critical points. Similarly we define a **Morse link** as a link such that the projection on $\mathbb{R}$ is a Morse function. We call a Morse knot or link **strict** if all the critical points are distinct.

**Convention 3.3.** From now on we will assume that all the knots and links we consider are strict Morse knots and strict Morse links.

**Remark 3.4.** The condition on our knots and links that they have to be strict Morse knots and links can be relaxed, we will not do this since all our constructions will be interesting enough for only Morse knots.

**Definition 3.5.** Let $K$ and $L$ be the two components of a two component link, the we define the linking number $lk(K, L)$ as:

$$lk(K, L) = \frac{1}{2} \sum_{c \in V(K, L)} \epsilon(c).$$

In this formula $V(K, L)$ is the set of all crossings that both involve a strand from $K$ and a strand from $L$. The function $\epsilon$ is the function that is 1 if $c$ is a positive crossing and $-1$ is $c$ is a negative crossing. Recall that a crossing
Figure 3.1: On the left we see a positive crossing and on the right a negative crossing.

is positive if the upper strand points to the right when crossing the lower strand and $c$ is a negative crossing if the upper strand points to the left (see also figure 3.1).

See for more details definition 2.0.4 in week 37 of [44].

**Definition 3.6.** Let $L$ and $K$ be the components of a link that consist of two components. Then we define the *Gauss integral formula for the linking number* $lk(K, L)$ as the following integral:

$$lk(K, L) = \frac{1}{2\pi i} \int \sum_j (-1)^{\downarrow_j} \frac{d(z_j(t) - w_j(t))}{z_j(t) - w_j(t)}$$

We should interpret this formula as follows. The boundaries of the integral are the lowest and highest critical point. The sum runs over all possible pairs of strands of which one strand is from $K$ and other one from $L$. The $\downarrow_j$ is the number of strands oriented downwards in the $j$th pair. See also figure 3.1

**Theorem 3.7.** The Gauss integral formula for the linking number calculates the linking number as defined in definition 3.5.

The proof of this theorem can be found in [10].
Figure 3.2: In the piece of the knot between the second and the third critical point the sum runs over the following pairs (1, 2), (1, 4) and (2, 3).

The Kontsevich integral

We will now define the Kontsevich integral is now a generalization of the Gauss integral formula for the linking number. We will do this by replacing the differential form by another differential form and by replacing the algebra in which the integral takes its values in into the algebra of chord diagrams. We will first define the unnormalized version of the Kontsevich integral. This integral is invariant under small deformations but is not invariant if the number of critical points of the Morse knot is changed. After that we will normalize the Kontsevich integral and make it into a knot invariant.

**Definition 3.8.** Let \( K \) be a strict Morse knot with critical points \( t_{\min} < t_1 < \ldots < t_m < t_{m-1} < \ldots < t_2 < t_1 < t_{\max} \). Then we define the unnormalized Kontsevich integral \( Z(K) \) by the following formula:

\[
Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{\min} < t_m < \ldots < t_1 < t_{\max}} \sum_{P=\{(z_j, z'_j)\}} (-1)^{|P|} D_P \prod_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j}.
\]

This formula should be interpreted as follows. The domain of integration is a \( m \)-dimensional simplex defined by \( t_{\min} < t_m < \ldots < t_1 < t_{\max} \) without the subspaces where one of the \( t_i \) coincides with one of the critical points.
points, the domain of integration therefore is the disjoint union of several different connected components. The second sum runs over all possible sets of \( m \) pairs of points on the lines of the knot in the connected component, i.e. \( P = \{(z_1, z'_1), \ldots, (z_m, z'_m)\} \). Note that the number of pairs may differ from connected component to connected component. The \( \downarrow_P \) is defined as the number of lines of the knot in that connected component pointing downwards. We define \( D_P \) to be the chord diagram obtained by taking the chord diagram corresponding to the knot and then connect all the points of \( P \) by chords.

**Example 3.9.** To make the definition of the Kontsevich integral clearer we will now calculate coefficient of the diagram from figure 3.1 in the the second term of the Kontsevich integral of the hump knot from figure 3.1. We first note that the domain of integration of the hump knot is the simplex on the right side of figure 3.1 and that it consists of six connected components. We now need to define for each connected component the corresponding chord diagram. We do this by connecting the two points of each pair by a chord and then draw the corresponding chord diagram. This diagram is only dependent on the connected component of the domain of integration. So to calculate the coefficient of the diagram from figure 3.1 we need to integrate the sum over the integral of the connected component labeled by the diagram from figure 3.1 in figure 3.1.

**Definition 3.10.** Let \( T \) be a tangle (see [10]), the we define the Kontsevich integral for tangles

**Theorem 3.11.** The Kontsevich integral \( Z(T) \) has the following properties.

1. The Kontsevich integral converges for all Morse tangles and in particular Morse knots.
Figure 3.4: On the left we see the hump knot with its critical points. On the right we see how the domain of integration looks like and which chord diagram corresponds to which connected component of the integration domain.

Figure 3.5: An example of the integration domain for the Kontsevich integral for the second term of the hump knot.
2. The Kontsevich integral is multiplicative with respect to the composition of tangles, i.e. let $S$ and $T$ be tangles such that the composition $ST$ is defined then we have $Z(ST) = Z(S)Z(T)$.

3. The Kontsevich integral is invariant under deformations that does not change the number of critical points.

We will not proof this theorem here and refer to [10] for the proof. We will however make a few remarks about this theorem. The Kontsevich integral is multiplicative for tangles but this is unfortunately not the case for the disjoint union of tangles. For the disjoint union it can be shown that the Kontsevich integral is asymptotically multiplicative with respect to the disjoint union, this is done by using parametrized tensor products, see also [10] for more details. The second point from theorem 3.11 gives us also an easier way to calculate the Kontsevich integral by cutting our knot up into some basic tangles and then take their product, note however that it is still very difficult to calculate the Kontsevich integral.

**Definition 3.12.** Let $H$ be the hump knot from figure 3.1 and $Z(H)$ its Kontsevich integral Then we define the normalized Kontsevich integral $I(K)$ of a knot $K$ with $c$ critical points as

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}}.$$  

**Remark 3.13.** There are several ways to normalize the Kontsevich integral which have all there own advantages and disadvantages. We took one of the normalizations from [10].

**Theorem 3.14.** The normalized Kontsevich integral is a knot invariant.

The proof can be found in [2].

**The Kontsevich integral for long knots**

Since we mainly worked with long knots in this thesis it is important to also have a definition of the Kontsevich integral for long knots. There are two ways to define the Kontsevich integral for a long knot, we can just view the long knot as a tangle with one incoming and one outgoing strand and then calculate the Kontsevich integral of this tangle. The other way is by first turning the long knot into a compact knot by connecting the ingoing and outgoing strand and then calculate the Kontsevich integral of this compact knot. Proposition 3.16 tells us that these methods are essentially the same.
Figure 3.6: The long knot on the left is turned into a compact knot by connecting the points outside the compact subset in which the long knot does not coincide with the linear embedding of $\mathbb{R}$.

**Definition 3.15.** Let $T$ be long knot such that $T$ coincides with the linear embedding $\epsilon : \mathbb{R} \to \mathbb{R}^3$ outside a compact subset $C \subset \mathbb{R}^3$. Then we define $K_T$ the closure of the long knot $T$ as the knot obtained by joining the endpoints of $K \cup C$ by a line such that intersection of this line with $C$ are only the endpoints of the line.

**Proposition 3.16.** The Kontsevich integral for a long knot $T$ seen as tangle and the Kontsevich integral of the closure of a long knot $K_T$ coincide if we close the tangle chord diagrams by identifying the end points of the tangle cord diagram and therefore turning it into a normal chord diagram.

The proof of this proposition can be found in [10].

### 3.2 The Kontsevich integral and Vassiliev invariants

In this section we will use the formulas from the previous section and explain how the Kontsevich integral is related to Vassiliev’s spectral sequence. The main result is that the Kontsevich integral contains exactly as much information as the knot invariants constructed out of Vassiliev’s spectral sequence.

**Theorem 3.17.** Let $V_k$ be the space of all Vassiliev invariants of order $\leq k$, then we have the following isomorphism

$$V_k / V_{k-1} \cong W_k.$$
Where $\mathcal{W}_k$ is the space of all weight systems of degree $k$. Moreover for every weight system $w$ of order $k$ there exists a Vassiliev invariant $v$ such that $\text{symb}(v) = w$.

**Proof.** We will only give the key ingredients of the proof, a full proof can be found in [10] or [29]. We will first show that for every weight system $w$ there exist a Vassiliev invariant $v$ such that $\text{symb}(v) = w$. We do this by defining the knot invariant $v$ by

$$v(K) = w(I(K)).$$

The proof that this is a Vassiliev invariant with symbol $w$ follows from the following property of the Kontsevich integral. Let $K_D$ be a singular knot with corresponding chord diagram $D$, then the Kontsevich integral $I(K_D)$ is of the form $D + \text{higher order terms}$. We will not prove this property of the Kontsevich integral and the proof can be found in [10]. From this property it follows that the $n$th graded component of the Kontsevich integral is a Vassiliev invariant of order $n$. So we can factor each Vassiliev invariant through the Kontsevich integral.

So finally we conclude that every weight system corresponds to a Vassiliev invariant and that for every Vassiliev invariant $v$ there exists a weight system $w$ such that $w = \text{symb}(v)$. So there is an isomorphism between the spaces $V_n/V_{n-1}$ and $W_n$.

**Remark 3.18.** We can define the Kontsevich integral for singular knots by extending the Kontsevich integral to singular knot by using Vassiliev’s skein relation.

We have so far seen that the Kontsevich integral is at least as strong as the Vassiliev invariants, in the next theorem we will see that they are exactly as strong.

**Theorem 3.19.** The Kontsevich integral is exactly as strong as the set of all Vassiliev invariants, i.e. for any two knots $K_1$ and $K_2$ we have

$$I(K_1) = I(K_2)$$

if and only if

$$v(K_1) = v(K_2)$$

for all Vassiliev invariants $v$.

As a consequence of theorem 3.17 we can show that Vassiliev’s spectral sequence collapses along the diagonal over the rational numbers. This is explained in the following corollary.
Corollary 3.20. Vassiliev’s main spectral sequence collapses along the diagonal \((-p, p)\) over the rationals.

Proof. The proof follows from the fact that, over the rational numbers, \(V_n/V_{n-1}\) is isomorphic to the \(n\)th graded component of the algebra of weight systems. Because the elements of the algebra of weight systems are zero cohomology classes of the space of knots they are elements of the \(E_\infty^{p,p}\) term of the spectral sequence. So all the Vassiliev invariants have to survive till the \(E_\infty\) page of the spectral sequence, therefore the spectral sequence collapses at along the diagonal. \(\Box\)
Chapter 4

Configuration spaces

In everything that will follow configuration spaces will play an important role, since one of the easiest ways of approximating a knot is by evaluating it at \( n \) points. Therefore we can define a map \( ev_n : K \to C_n(M) \) called the evaluation map from the knot to the configuration space of \( n \) ordered points in a manifold \( M \). Unfortunately this map has not all the properties we want it to have (see chapter 9 for more details). To fix this we need to add boundary conditions to the configuration space, we will do this by giving certain completions of the configuration space. In this chapter we will introduce the two completions we will need in the rest of this thesis. We will begin with what we will call the Fulton MacPherson completions, this completion is also known as the Axelrod Singer completion or as the canonical completion. After that we will also define the simplicial completion, this completion looks a lot like the Fulton MacPherson completion but has the advantage that it will give a simplicial space. We will begin with a precise definition of the configuration space of \( n \) distinct ordered points. In chapter 9 we will study these completions into more depth, the main reason that we already introduce them now is that we will need them to define some of the operads in the next chapter. The main article this section is based on is [48] and therefore we will refer the reader to this article for more details.

Definition 4.1. Let \( M \) be a manifold, the configuration space \( C_n(M) \) of \( n \) distinct ordered points in \( M \) is the following set \( C_n(M) = \{ (x_1,...,x_n) \in M^n \mid x_i \neq x_j \forall i,j \in \{1,...,n\} \} \).

We will call the space \( \{ (x_1,...,x_n) \in M^n \mid x_i = x_j \text{ for some } i,j \in \{1,...,n\} \} \) the fat diagonal.

For a lot of applications it is necessary to define compactifications and completions of the configuration space \( C_n(M) \), for the rest of this paper we will need two different completions. We will call the first one the Fulton
MacPherson completion and the second the simplicial completion, for reasons we will explain below. Before we give the definitions of the completions we will first need to introduce some ambient spaces and some notation.

**Convention 4.2.** For all the constructions that will follow in this section we will assume that there is an embedding of our manifold \( M \) in \( \mathbb{R}^{N+1} \) and that \( M \) has a Riemannian metric \( d(\cdot, \cdot) \). All the constructions will be independent of the choice of the embedding and the metric, but it will be convenient for all the definitions to have them. We will assume that all manifolds in the rest of this chapter and the rest of this thesis are equipped with a metric and an embedding in \( \mathbb{R}^{N+1} \) for some \( N \), unless stated otherwise.

**Definition 4.3.** Let \( \mathbb{N} \) be the set \( \{1, ..., n\} \)

**Remark 4.4.** Note that the definition of \( C_n(M) \) also makes sense when \( M \) is a finite set, specially the space \( C_m(\mathbb{N}) \) will be used a lot, this is just the finite set of all subsets of \( \mathbb{N} \) of cardinality \( m \).

For all that follows we also need the following maps.

**Definition 4.5.** Let \( M \) be a manifold, \( S^N \) be the \( n \)-dimensional unit sphere and \( I = [0, \infty) \) the one point compactification of the non negative real numbers \( \mathbb{R}_{\geq 0} \). Then we can define the following maps.

- Let \( \pi_{ij} : C_n(M) \to S^N \) be the map that sends \((x_1, ..., x_n)\) to \( \frac{x_i - x_j}{|x_i - x_j|} \), i.e. the unit vector pointing from \( x_2 \) to \( x_1 \).

- Let \( s_{ijk} : C_n(M) \to I \) be the map defined by \( s_{ijk}(x_1, ..., x_n) = \frac{|x_i - x_j|}{|x_i - x_k|} \), i.e. the map that measures the relative distance between the vectors \( x_i - x_j \) and \( x_i - x_k \). This is important to keep track of the order in which the points collide when approaching the diagonal.

We will define the completions as the closure of the image of certain products of maps in some ambient spaces. In the following definition we will define these ambient spaces.

**Definition 4.6.** Let \( A_n[M] \) be the space \( M^n \times (S^N)^{C_2(\mathbb{N})} \times I^{C_3(\mathbb{N})} \), this will be the ambient space for the Fulton-MacPherson completion. The ambient space for the simplicial completion will be the space \( A(\{[M]\}) = M^n \times (S^N)^{C_2(\mathbb{N})} \).

**Notation 4.7.** To make notation simpler and to avoid writing down expressions like \((x_1, ..., x_n) \times \prod_{\{i,j\} \in C_2(\mathbb{N})} (u_{ij}) \times \prod_{\{i,j,k\} \in C_3(\mathbb{N})} (d_{ijk}) \in M^n \times (S^N)^{C_2(\mathbb{N})} \times I^{C_3(\mathbb{N})} \), we will use the following conventions in the notation. On \( A_n[M] \) and \( C_n[M] \) we denote by \((x_1) \times (u_{ij}) \times (d_{ijk})\) the coordinate \((x_1, ..., x_n) \times \)
\[\prod_{\{i,j\} \in C_2} (u_{ij}) \times \prod_{\{i,j,k\} \in C_3} (d_{ijk})\]. On \(A_n \langle [M] \rangle\) and \(C_n \langle [M] \rangle\) we denote the coordinates by \((x_i) \times (u_{ij})\) instead of \((x_1, \ldots, x_n) \times \prod_{\{i,j\} \in C_2} (u_{ij})\). On maps we will use the convention that \(\prod_{\{i,j\} \in C_2} (u_{ij}) \times \prod_{\{i,j,k\} \in C_3} (d_{ijk})\). There is a similar convention for maps involving the simplicial completion.

### 4.1 The Fulton-MacPherson completion of a configuration space

The Fulton-MacPherson completion was originally defined by Fulton and MacPherson in the case of varieties, the defined their completion as the blow up of the configuration space along the fat diagonal. In the case of manifolds this completion was originally defined by Axelrod and Singer, but the definition we will give here comes from [48].

**Definition 4.8.** Let \(M\) be a manifold and \(A_n[M] = M^n \times (S^{n-1})^C_{2^n} \times I^{C_3(2^n)}\) as in definition 4.6, then the Fulton-MacPherson completion is the closure of image of the following map

\[
\alpha_n : C_n(M) \to A_n[M]
\]

\[
\alpha_n = \iota \times (\pi_{ij}) \times ((s_{ijk})).
\]

**Remark 4.9.** If we define a metric \(d\) on \(M\) then there exists an \(\epsilon > 0\) such that \(C_n[M]\) is diffeomorphic to the subspace \(\{(x_1, \ldots, x_n) \in C_n(M) \mid d(x_i, x_j) \geq \epsilon \forall i, j \in \mathbb{N}\}\) i.e. the space of all points that lie at least a distance \(\epsilon\) away from the fat diagonal. The disadvantage of this definition is that \(C_n(M)\) is not a subspace of the space \(C_n[M]\) in a canonical way and that certain other properties are far from clear from this definition.

**Theorem 4.10.** The Fulton-MacPherson completion has the following properties:

1. The Fulton-MacPherson completion of \(M\) is compact if \(M\) is compact.
2. The inclusion map \(i : C_n(M) \to C_n[M]\) is a homotopy equivalence.
3. The Fulton-MacPherson completion is functorial, i.e. if \(f : M \to N\) is a map of manifolds then we have a map \(ev_n(f) : C_n[M] \to C_n[N]\).
4. \(C_n(M)\) is the interior of \(C_n[M]\).
5. $C_n[M]$ is independent of the embedding of $M$ in $\mathbb{R}^{N+1}$.

A proof can be found in [48]

### 4.2 The simplicial completion of a configuration space

The other completion we will use is the simplicial completion. This completion has the advantage that it has a cosimplicial structure, in chapter 9 we will explicitly state and show what this means.

**Definition 4.11.** Let $A_n([M])$ be as in definition 4.6, then we define the simplicial completion of $C_n(M)$ as the closure of the image of the map

$$\alpha_n : C_n(M) \to A_n([M])$$

$$\alpha_n = \iota \times (\pi_{ij}).$$

**Theorem 4.12.** The simplicial completion has the following properties:

1. The simplicial completion is compact if $M$ compact is.

2. The simplicial completion is independent of the embedding of $M$ in $\mathbb{R}^{N+1}$.

3. The simplicial completion is functorial, i.e. If $f : M \to N$ is a smooth map then there is an induced map $ev_n(f) : C_n([M]) \to C_n([N])$.

4. The inclusion map is a homotopy equivalence.

The proof of the theorem can be found in [48]. We will end this chapter with a proposition that show the relation between the two completions. The proof of the proposition can be found in [48].

**Proposition 4.13.** Let $M$ be a manifold, then the simplicial completion $C_n([M])$ and the Fulton MacPherson completion $C_n[M]$ are homotopy equivalent.
Chapter 5
Operads

In this chapter we will briefly recall the definition of an operad and introduce the operads we will need for the rest of the paper. People familiar with operads can skip this chapter or only read the definitions of the relevant operads like the Poisson, BV an Kontsevich operad. For readers unfamiliar with operads an introduction into the theory of operads can be found in for example [16], [37] or [52]. No proofs or details will be given.

5.1 Linear operads

We will begin by defining linear operads. A linear operad is an algebraic structure that is used to encode the structure of an algebra on a vector space. The formal definition is given as follows. The reader who is interested in the intuitive ideas behind operads is referred to [16].

Definition 5.1. A symmetric operad is a collection \( \{P(n)|n \geq 1\} \) of \( \mathbb{K} \)-vector spaces equipped with the following set of data:

- An equivariant action of the symmetric group \( S_n \) on \( P(n) \) for every \( n \).
- Linear maps called compositions:

\[
\gamma_{m_1,...,m_l} : P(l) \otimes P(m_1) \otimes ... \otimes P(m_l) \rightarrow P(m_1 + ... + m_l)
\]

for all \( m_1, ..., m_l \geq 1 \). Such that the maps \( \gamma \) satisfy the following associativity conditions

\[
\gamma(\gamma(\lambda, \mu_1, ..., \mu_n), a_1, ..., a_i) = \gamma(\lambda, \gamma(\mu_1, a_1, ..., a_j), ..., \gamma(\mu_n, a_{s-1}, a_l))
\]

With \( a_i \in P(i) \), \( \mu \in P(j) \) and \( \lambda \in P(k) \). We write \( \mu(\nu_1, ..., \nu_l) \) instead of \( \gamma_{m_1,...,m_l}(\mu \otimes \nu_1 \otimes ... \otimes \nu_l) \).
An element $1 \in P(1)$ called the unit such that $1(\mu) = \mu(1,...,1) = \mu$ for all $\mu \in P(n)$.

For details about the equivariance of the symmetric group action see [37]. We will now give the first example of an operad. This operad is called the endomorphism operad and motivates the idea of how an operad encodes the structure of an algebra on a vector space.

**Example 5.2.** Let $V$ be a vector space, then we define the endomorphism operad $\mathcal{END}(V)$ as the operad whose $n$th space is $\mathcal{END}(V)(n) = \text{Hom}(V^\otimes n, V)$. The composition maps are given by composing two homomorphisms $\mu \in \mathcal{END}(V)(n)$ and $\lambda \in \mathcal{END}(V)(m)$ via the $i$th input of $\mu$. The symmetric group action is defined by permuting the input of the elements of $\mathcal{END}(V)$.

To demonstrate how operads and algebras are related we will first define morphisms of operads and representations of operads.

**Definition 5.3.** Let $\mathcal{O}$ and $\mathcal{P}$ be linear operads, a morphism of operads $f : \mathcal{O} \to \mathcal{P}$ is a sequence of linear maps $f_n : \mathcal{O}(n) \to \mathcal{P}(n)$ commuting with the composition maps and in the case of symmetric operads also commuting with the $S_n$ action.

**Definition 5.4.** Let $V$ be a vector space and $\mathcal{P}$ be a linear operad. Then we define a representation of $\mathcal{P}$ as a morphism of operads from $\mathcal{P}$ to the endomorphism operad of $V$, i.e. a morphism of operads

$$\Pi : \mathcal{P} \to \mathcal{END}(V).$$

So a representation of an operad $\mathcal{P}$ defines a collection of multilinear maps on a vector space $V$. These maps can be seen as products and compositions of products, and therefore define an algebra structure on $V$, with algebra structure we mean a vector space with some multilinear operations.

There are also non-symmetric operads. These operads are almost the same as symmetric operads except for the action of the symmetric group.

**Definition 5.5.** A non-symmetric operad is a collection of vector spaces $\{\mathcal{P}(n)\}_{n \geq 0}$ that satisfy all the axioms of a symmetric operad except those that involve the action of the symmetric group.

Note that we can turn every symmetric operad in a non-symmetric one by forgetting the symmetric group action. It is often convenient to represent the elements of a linear operad by decorated trees. We will not give all the formal definitions but see a few informal words about this point of view, a formal treatment can be found in [16]. Let $\mathcal{P}$ be an operad, then
we see the elements of $\mathcal{P}(n)$ as $n$-ary operations which have $n$ inputs and 1 output. We can represent this operation by a rooted decorated tree with $n$ leaves corresponding to the input and 1 root corresponding to the output. The vertices of the tree correspond to the operations in the operad. In this way we can also define free operads and generators and relations for operads, for more details about this see [16].

5.2 Examples of linear operads

Now that we know the definition of a linear operad it is time to give some examples. We will give the operad structures from the most common algebras and some other algebras we will need for the rest of this thesis.

5.2.1 The associative operad ASS

The associative operad $ASS$ is the operad that corresponds to associative algebras. Recall that an associative algebra is a vector space $A$ together with a bilinear map $m : A \otimes A \to A$ such that $m(m(x_1, x_2), x_3) = m(x_1, m(x_2, x_3))$ for all $x_1, x_2, x_3 \in A$. To construct the corresponding operad we want to encode these relations in the form of an operad. This is done as follows.

**Definition 5.6.** The associative operad $ASS$ is the operad generated by a single element $m \in ASS(2)$ with the relation

$$m(m(x_1, x_2), x_3) = m(x_1, m(x_2, x_3))$$

for all $x_1, x_2, x_3 \in ASS$. The $S_n$ action on $O(n)$ is defined by $\sigma(y(x_1, \ldots, x_n)) = y(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for $\sigma \in S_n$, $y \in ASS(n)$.

It is easy to check that the dimension of the space $ASS(n)$ is $n!$.

5.2.2 The commutative operad COM

The next example of an operad is the operad corresponding to commutative algebras. This operad can be seen as a quotient of the associative operad by the ideal generated by $\langle m(x_1, x_2) - m(x_2, x_1) \rangle$. We can also describe it directly by generators and relations.

**Definition 5.7.** The commutative operad $COM$ is the operad generated by one element $m \in COM(2)$ subject to the relations $m(x_1, x_2) = m(x_2, x_1)$ and $m(m(x_1, x_2), x_3) = m(x_1, m(x_2, x_3))$. The action of $S_n$ is the trivial action.

In this case the spaces $COM(n)$ all turn out to be 1 dimensional.
5.2.3 The Lie operad $\mathcal{LIE}$

Another example of an operad is the operad $\mathcal{LIE}$ corresponding to Lie algebras. In this case we want an anti-commutative bracket that satisfies the Jacobi identity. In the language of generators and relations we obtain the following definition.

**Definition 5.8.** The operad $\mathcal{LIE}$ is the operad generated by a single element $[\cdot, \cdot] \in \mathcal{LIE}(2)$, satisfying the following relations $[x_1, x_2] = -[x_2, x_1]$ and $[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0$. The action of $S_n$ on $\mathcal{LIE}(n)$ is given by the sign representation of $S_n$ on $\mathcal{LIE}_n$.

5.2.4 The Poisson operad $\mathcal{POISS}_d$

The Poisson operad is the operad that corresponds to Poisson algebras, recall that a Poisson algebra of degree $d$ is a graded commutative algebra with a Lie bracket of degree $-d$.

**Definition 5.9.** The Poisson operad of degree $d$ $\mathcal{POISS}_d$ is the operad generated by a two binary operations $\cdot, [\cdot, \cdot] \in \mathcal{POISS}_d(2)$ with $\cdot$ of degree 0 and $[\cdot, \cdot]$ of degree $-d$, with the following relation

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{\delta(x)-d} y \cdot [x, z]$$

for all $x, y, z \in \mathcal{POISS}_d$.

**Remark 5.10.** Poisson algebras with a bracket of degree 0 are normally just called Poisson algebras. If the bracket is of degree $-1$ the algebras are also known as Gerstenhaber algebras.

A decomposition of the Poisson operad

The Poisson operad $\mathcal{POISS}_d$ has an interesting decomposition which we will use later in the thesis to calculate part of its homology and to link the homology of the Poisson operad to the auxiliary spectral sequence constructed in section 1.8. As we will see there are some similarities between the decomposition we are going to construct and the degenerate chord diagrams and $(A, b)$-configurations from section 1.4.3.

We will first decompose the spaces $\mathcal{POISS}_d(n)$. Remember that these spaces correspond to the space of all possible operations in a Poisson algebra with $n$ inputs and 1 output. We can represent these operations by trees with only trivalent internal vertices of which each vertex is labeled by either the bracket $[\cdot, \cdot]$ or by the commutative product $\cdot$. The leaves except for
the root are labeled by the set \{x_1, ..., x_n\} such that each element appears exactly once. We will decompose the Poisson operad in the following way

\[ \mathcal{POISS}_d(n) = \bigoplus_A \mathcal{POISS}_d(A, n). \]

The direct sum runs over all possible partitions of the set \( n = \{1, ..., n\} \). The space \( \mathcal{POISS}_d(A, n) \) is defined in the following way. Let \( A = \{A_1, ..., A_{\#A}\} \) be a partition of \( n \), i.e. \( \bigcup_{i=1}^{\#A} A_i = n \). To each partition \( A \) we assign all the basis elements of \( \mu \in \mathcal{POISS}_d(n) \) such that \( \mu \) is of the product of \( \#A \) brackets with the \( i \)-th bracket containing containing all the elements of \( A_i \).

**Example 5.11.** The decomposition of \( \mathcal{POISS}_d(3) \) is constructed as follows. First we have to determine all the partitions of the set \( \{x_1, x_2, x_3\} \), these are given by:

1. \( \{x_1, x_2, x_3\} \),
2. \( \{x_1, x_2\}, x_3 \),
3. \( \{x_1, x_3\}, x_2 \),
4. \( \{x_1, \{x_2, x_3\}\} \),
5. \( \{x_1, x_2, x_3\} \).

The basis elements corresponding to partition (1) is given by the orbit under the symmetric group of the following element \([x_1, x_2 \cdot x_3]\), i.e. the elements \([x_1, x_2 \cdot x_3]\) and \([x_2, x_1 \cdot x_3]\). The basis element corresponding to partition (2) is \([x_1, x_2] \cdot x_3 \). For partitions (3) and (4) the exact same thing happens. Partition (5) gives us the elements \( x_1 \cdot x_2 \cdot x_3 \). It is left as an exercise to the reader to check that the direct sum of these elements is indeed \( \mathcal{POISS}_d(3) \).

Now we will give a little lemma on the dimension of the spaces \( \mathcal{POISS}_d(A, n) \).

**Lemma 5.12.** Let \( A = \{A_1, ..., A_{\#A}\} \) be a partition of \( n \) and each \( A_i \) of cardinality \( a_i \), then we have the following isomorphism

\[ \mathcal{POISS}_d(A, n) \cong \bigotimes_{i=1}^{\#A} K^{(a_i - 1)!} \]
5.2.5 The BV-operad $\mathcal{B}V_d$

**Definition 5.13.** A BV-algebra is a Gerstenhaber algebra $A$ (i.e. a Poisson algebra with bracket of degree $-1$) together with a linear map $\delta : A \rightarrow A$ of degree $-1$ such that it satisfies the following relations

1. $\delta^2 = 0$,
2. $\delta(ab) = \delta(a) + (-1)^{\hat{a}}a\delta(b) + (-1)^{\hat{b}}[a,b],$
3. $\delta([a,b]) = [\delta(a),b] + (-1)^{\hat{a}+1}[a,\delta(b)].$

Please note that conditions (1) and (2) imply condition (3), we still state condition (3) here since it can be useful to have an expression involving only the bracket and $\delta$. There is also the generalization of a BV-algebra by replacing the Poisson algebra by a Poisson algebra with a bracket of degree $-d$. We do this in the following definition.

**Definition 5.14.** A $d$-Batalin-Vilkovisky algebra, or $d$-BV algebra for short, is a $d$-Poisson algebra with a linear map $\delta : A \rightarrow A$ of degree $-d$, such that $\delta$ satisfies the relations from definition 5.13.

From the definitions of a BV-algebra we can construct the following operad.

**Definition 5.15.** The BV-operad $\mathcal{B}V_d$ of degree $d$ is the operad generated by a commutative product $\cdot \in \mathcal{B}V_d(2)$ of degree 0, a Lie bracket $[,] \in \mathcal{B}V_d(2)$ of degree $-d$ and a differential $\delta \in \mathcal{B}V_d(1)$ of degree $-d$. Satisfying the relations from definition 5.13.

**A decomposition of the BV-operad**

We want to construct a decomposition of the BV-operad in a similar way as we did for the Poisson operad. Since the BV-operad is a Poisson operad together with a differential we can use the decomposition of the Poisson operad we only need to take care of the differential.

Similar to the Poisson operad we represent every element of $\mathcal{B}V_d(n)$ as a tree with $n$ leaves and one root. This tree has only trivalent vertices corresponding to the bracket $[,]$ and the product $\cdot$ and bivalent vertices corresponding to the differential $d$. Because of the relations of the $\mathcal{B}V_d$ operad we can move the bivalent vertices such that the only places where the bivalent vertices labeled by the differential may occur are the vertices exactly above the leaves. We will now decompose the $\mathcal{B}V_d$ operad by using star partitions.
Definition 5.16. Let $\underline{n}$ be the set $1, \ldots, n$, then we define a star partition of $\underline{n}$ as follows. Let $A$ be a partition of $\underline{n}$ and let $S \subset \underline{n}$. Then we call the pair $(A, S)$ a star partition of $\underline{n}$.

With these star partitions we will decompose $BV_d$ as follows. To each star partition of $\underline{n}$ we associate the elements of $BV_d(\underline{n})$ that correspond to the partition $A$ as in the decomposition of the Poisson operad and such that the elements have a bivalent vertex at a leave if and only if the label of this leave is an element of $S$. So we get the following decomposition,

$$BV_d = \bigoplus (A, S) BV_d(A, S, n)$$

Where $BV_d(A, S, n)$ is the space of all elements corresponding to the star partition $(A, S)$ and the direct sums runs over all star partitions.

5.3 Topological operads

Besides linear operads there is also another kind of operad, this other type of operad is called a topological operad and will play an important role in the rest of this thesis. A topological operad is kind of similar to linear operads except that instead of taking a collection of vector spaces together with the operad maps and $S_n$ action we will now take a collection of topological spaces together with operad maps and $S_n$ actions.

Definition 5.17. A topological operad $\mathcal{O}$ is a collection of topological spaces $\mathcal{O}(n)$ satisfying the following conditions.

1. There is an $S_n$ action on each space $\mathcal{O}(n)$.

2. There are composition maps

$$\gamma : \mathcal{O}(l) \times \mathcal{O}(m_1) \times \ldots \times \mathcal{O}(m_l) \to \mathcal{O}(m_1 + \ldots + m_l)$$

That are compatible with the $S_n$ action.

3. There is a unit element $id \in \mathcal{O}(1)$.

5.4 Examples of topological operads

We will now give some examples of topological operads. The little disks and cubes operad was historically one of the first topological operads and plays an important role in recognizing loop spaces. The other example we will give is the Kontsevich operad, this operad will be used to define another version of the cosimplicial model for the space of knots.
5.4.1 The little cubes and little disks operad $\mathcal{LD}_d$

The first examples of topological operads where historically the little disks and the little cubes operads. These operads where used by May to determine if a topological space is weakly equivalent to a loop space or not. In this subsection we will briefly describe these operads and for completeness we will also state the recognition principle. We will begin with the little disks operad.

**Definition 5.18.** The little $d$-disks operad $\mathcal{LD}_d$ is the topological operad with $\mathcal{LD}_d(n)$ given by configuration space of $n$ ordered disjoint $d$-dimensional disks in the $d$-dimensional unit disk. The composition maps $\circ_j : \mathcal{LD}_d(n) \otimes \mathcal{LD}_d(m) \to \mathcal{LD}_d(n+m-1)$ are given by inserting the unit disk into the $j$-th disk. See also figure 5.4.1.

**Definition 5.19.** The little $d$-cubes operad $\mathcal{LC}_d$ is the topological operad with $\mathcal{LC}_d(n)$ given by configuration space of $n$ ordered disjoint $d$-dimensional cubes in the $d$-dimensional unit cube such that the sides of the cubes are parallel to the sides of the unit cube. The composition maps are similar to the composition maps of $\mathcal{LD}_d$ and are given by inserting the unit cube into the $j$-th cube.
The next thing we will do is state the recognition principle, this theorem originally due to May (see [37]) gives a criterion to determine whether a topological space is weakly equivalent to a loop space or not. It is also historically one of the first applications of operads and therefore at least worth mentioning. It goes roughly as follows.

**Theorem 5.20.** Let $X$ be a topological space with an action of the little $n$-cubes operad then under some conditions $X$ is weakly equivalent to $\Omega^n Z$, the $n$-fold loop space of some topological space $Z$.

The proof and the exact statement of this theorem can be found in [37].

### 5.4.2 The Kontsevich operad $K_d$

The following example of a topological operad is the Kontsevich operad. This operad will play an important role in the construction of the cosimplicial model in chapter 9. The Kontsevich operad is similar to the little disks and little cubes operads and in sections 5.4.1, we will state that these operads are also weakly equivalent. First we will give the general definition of the Kontsevich operad as a certain completion of $C_n(\mathbb{R}^d)$, the configuration space of $n$ points in $\mathbb{R}^d$, up to scaling and translations.

**Definition 5.21.** Let $d \geq 1$, the Kontsevich operad $K_d$ is the topological operad with $K_d(n)$ given by the closure of the image of the following maps:

$$\alpha_s = (\alpha_{ij})_{1 \leq i < j \leq n} : C_n(\mathbb{R}^d) \to \prod_{1 \leq i < j \leq n} S^{d-1}$$

Where the maps $\alpha_{ij}$ are defined by

$$\alpha_{ij}(x_1, \ldots, x_n) = \frac{(x_i - x_j)}{\|x_i - x_j\|}.$$

The composition is defined by inserting the configurations spaces into each other, an example of this composition is given in figure 5.4.2.

For more details about the Kontsevich operad see [49]. Note that $K_1$ is the operad with a discrete set of points in graded component. This set of points is homeomorphic to the symmetric group seen as a topological space, we pick out the connected component corresponding to the ordering of the points $1 < 2 < \ldots < n$ and denote it by $K_1^{(0)}$. Then we see that $K_1^{(0)}$ is equal to the non symmetric operad $\mathcal{ASS}$ in the category of topological spaces. Since there is an inclusion of $K_n$ in $K_m$ if $n < M$, we can define a morphism of operads from the associative operad $\mathcal{ASS}$ to the Kontsevich operad. We will need this map in section 9.4.
5.5 The Hochschild homology of a graded linear operad

In this section we will construct the Hochschild complex of a graded linear operad. We will do this since there are some surprising connections between the Hochschild homology of the Poisson operad of degree \(d-1\) and the space of long knots in \(\mathbb{R}^d\), we will say more about this connection in section 5.7. We will construct the Hochschild complex in the following steps, first we will define a graded Lie algebra structure on an arbitrary graded linear operad. Then we will use this graded Lie algebra structure to define the Hochschild complex of a multiplicative operad, i.e. an operad \(O\) with a morphism \(\Pi: ASS \to O\). This section is mainly based on [52] and follows all the sign conventions from this paper.

5.5.1 A graded Lie algebra structure on a graded linear operad

To define the differential of the Hochschild complex we first need to define a graded Lie algebra structure on the graded linear operad \(O\).

**Definition 5.22.** Let \(O = \{O(n), n \geq 0\}\) be a graded linear operad and let \(x \in O(n)\) be an element of \(O(n)\), we denote the degree of \(x\) by a tilde, i.e. \(\text{deg}(x) = \tilde{x}\). We also define an alternative grading by putting \(n_x = n - 1\) for \(x \in O(n)\), please note that this grading is just the number of inputs minus the number of outputs.
On $\mathcal{O}$ we are able to define a new grading $| \cdot |$ by setting the degree of $x$ to be $| x | = \tilde{x} + n_x = \tilde{x} + n - 1$. The space $\mathcal{O}$ is now a graded Lie algebra with respect to the new grading.

We will define the bracket in the following way. First we will define a collection of multi linear operations on $\mathcal{O}$ by taking certain sums of compositions. Then we will define show that these sums form a Pre-Lie algebra structure and turn this into a graded Lie algebra structure. In the next section we will show how we can use this to define the differential of the Hochschild complex.

First note that the operad composition of $\mathcal{O}$ respects new grading $| \cdot |$, because for $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$ the degree of $x \circ y$ in the new grading is $| x \circ y | = \tilde{x} + \tilde{y} + n + m - 1$. This is equal to the sum of $| x |$ and $| y |$ minus 1, which is exactly the grading $x \circ y$ should have to respect the grading. We will use this to define the following multi linear operations on $\mathcal{O}$.

**Definition 5.23.** Let $x, x_1, x_2, ..., x_n \in \mathcal{O}$ define

$$x \{ x_1, ..., x_n \} = \sum (-1)^{\gamma}(x; id, ..., id, x_1, id, ..., id, x_{n-1}, id, ..., id).$$

Where the sum runs over all possible substitutions of $x_1, ..., x_n$ into $x$ into $x$ with the prescribed order, $\gamma = \sum \frac{n_x}{r_p} + \sum_{p=1}^n \tilde{x}_p + \sum_{p < q} n_{r_p} \tilde{x}_q$ and $r_p$ is defined to be the total number of inputs in $x$ after $x_p$. We also adopt the convention that $x \{ \} := x$.

The next step in defining the Hochschild complex is to recall the definition of a Pre-Lie algebra.

**Definition 5.24.** Let $A$ be a graded vector space with a bilinear map $\circ : A \otimes A \to A$, we call $A$ a Pre-Lie algebra if for all $x, y, z \in A$ the following identity holds:

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{|y||z|}((x \circ z) \circ y - x \circ (z \circ y)).$$

We will now show that every Pre-Lie algebra can be turned into a Lie algebra.

**Proposition 5.25.** Let $A$ be a Pre-Lie algebra and $x, y \in A$, then the following bracket

$$[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$$

defines a Lie bracket on the vector space $A$.

**Proof.** The proof of this proposition is an easy check of the definition of a Lie algebra and is left as an exercise for the reader. □
Lemma 5.26. A graded linear operad \( O \) equipped with the grading \(| \cdot |\) and the Lie bracket from proposition 5.25 form a graded Lie algebra.

The proof is again an easy check of the definition and is therefore left as an exercise for the reader.

5.5.2 The Hochschild homology of a graded linear operad

With the graded Lie algebra structure from the previous section we are now able to define the Hochschild complex of a linear operad.

**Definition 5.27.** Let \( O \) be a graded linear operad, we call \( O \) a multiplicative operad if there exists a morphism of operads

\[ \Pi : \ASS \to O \]

For each multiplicative operad \( O \) we can define an element \( m = \Pi(m_2) \in O(2) \) as the image of the multiplication of \( \ASS \), this element \( m \) is called the multiplication. Will now define the Hochschild complex using the bracket from the previous section and this element \( m \).

**Definition 5.28.** Let \( O \) be a multiplicative operad, then the Hochschild complex of \( O \) is the complex with as graded vector space \( O \) and the differential \( \partial \) is defined to be:

\[ \partial x = [m, x] = m \circ x - (-1)^{|x|} x \circ m \]

Where \( m \) is the multiplication and \([,] \) is the Lie bracket from definition 5.25.

**Proposition 5.29.** The Hochschild complex is a complex, i.e. \( \partial^2 = 0 \).

The proof is again just writing down the definitions and will therefore be omitted. We will now give some examples of Hochschild complexes of which we will calculate the homology in section 5.6. The first example will show that the complex we have just defined is a generalization of the Hochschild complex of an associative algebra.

**Example 5.30.** Let \( O \) be \( \End(V) \), the endomorphism operad of a vector space \( V \), together with an associative algebra structure on \( V \) given by a map \( \Pi : \ASS \to \End(V) \). Then the Hochschild complex of \( \End(V) \) is the standard Hochschild complex of the associative algebra \( V \).
5.5.3 The homology of topological operads

For topological operads we can also define the concept of homology. For a topological operad $\mathcal{O}$ we can consider the singular homology of the topological spaces $\mathcal{O}(n)$. Let $\mathbb{K}$ be a field, then the spaces $H_*(\mathcal{O}(n), \mathbb{K})$ will from a linear operad, on which the compositions maps are the induced maps on homology.

5.6 Results about the Hochschild homology of certain operads

In this section we will state some important results on the Hochschild homology of certain operads. Some of these results will be needed in section 5.7 to see the connection between Vassiliev’s spectral sequence and the Poisson operad.

**Theorem 5.31.** The operads $\mathcal{ASS}$ and $\mathcal{COM}$ are acyclic, i.e. all the homology groups vanish.

**Theorem 5.32.** The little discs operad, the little cubes operad and the Kontsevich operad are homotopy equivalent.

A proof can be found in [31]. As a consequence of this theorem the homology of these operads is isomorphic.

**Theorem 5.33.** The homology groups of $\mathcal{LC}_d$ have no torsion and are equal to

- If $d = 1 \{H_*(\mathcal{LC}_d(n), \mathbb{K}), n \geq 0\}$ is isomorphic to $\mathcal{ASS}$.
- If $d \geq 1 \{H_*(\mathcal{LC}_d(n), \mathbb{K}), n \geq 0\}$ is isomorphic to $\mathcal{POISS}_{d-1}$.

A proof of this theorem can be found in [11].

5.7 The space of knots and the Poisson operad

With all the definitions from the previous sections we are finally in a position to see how the Poisson operad and Vassiliev’s spectral sequence are related. The following theorem by Tourtchine show that the auxiliary spectral sequence and the Hochschild homology of the $\mathcal{BV}_d$ operad are isomorphic. We will not proof the theorem completely and only give a motivation why they
are isomorphic. There is also another explanation of this result which is due to Sinha, we will not give this explanation and refer the reader to [49]. For a more detailed explanation of Tourchines results we refer the reader to [52] and [53].

**Theorem 5.34.** The first term of Vassiliev’s auxiliary spectral sequence for \( \mathbb{R}^d \), with \( d \) even is isomorphic to a subcomplex of the Hochschild homology complex of the operad \( BV_{d-1} \). This subcomplex is the subcomplex spanned by all star partitions \((A,S)\), such that \( A \) does not contain subsets of a single element that is not contained in \( S \).

It is easy to see that there is a relation between the star partitions from section 5.2.5 and the \((A,b)\)-configurations from section 1.4.3. In particular there is a one to one correspondence between the \((A,b)\)-configurations and the star partitions \((A,S)\) such that \( A \) does not contain singletons that are not contained in \( S \). To proof the theorem we will split the Hochschild complex \((BV_{d-1}, \partial)\) in two pieces, one of these pieces corresponds to the homology and the other part is acyclic. Let \( E_j \) be the subspace of \((BV_{d-1}, \partial)\) spanned by all star partitions \((A,S)\) containing exactly \( i \) singletons that do not belong to \( S \). Then we define a filtration on \((BV_{d-1}, \partial)\) by defining \( F_i = \bigoplus_{j \geq i} E_j \). The following proposition shows that all the star partitions that do not correspond to \((A,b)\) configurations are acyclic in the homology of \((BV_{d-1}, \partial)\).

**Proposition 5.35.** The Hochschild complex \((BV_{d-1}, \partial)\) is equal to the direct sum of \( E_0 \) and \( F_1 \), i.e. the differential maps \( E_0 \) into \( E_0 \) and \( F_1 \) into \( F_1 \). Furthermore the subcomplex \( E_0 \) is homology equivalent to \((BV_{d-1}, \partial)\) and the subcomplex \( F_1 \) is acyclic.

So the homology of the \( BV \) operad corresponds to the \((A,b)\)-configurations defined in section 1.4.3. Therefore we have a link between the Hochschild homology of the \( BV \) operad and Vassiliev’s spectral sequence. Since the \( b \) in an \((A,b)\) configuration turns out to vanish in homology we can reduce this result to Poisson operads. Since there are no stars in the Poisson operad the elements in the decomposition only correspond to \((A,b)\)-configurations where \( b \) is empty. If we define the normalized Hochschild complex of the Poisson operad \((POISS^{Norm}_{d-1}, \partial)\) as the subcomplex of the Poisson operad spanned by all \( A \)-configurations not containing singletons then we get the following theorem.

**Theorem 5.36.** Let \( d \geq 3 \), then the subcomplex of the first page of the auxiliary spectral sequence spanned by all \( A \)-configurations is isomorphic to the normalized Hochschild complex \((POISS^{Norm}_{d-1}, \partial)\) with inverse grading.
From this theorem we get the main result of this section. To formulate the main result we shall first define a quotient of the normalized Hochschild complex. We will take the quotient of \((\text{POISS}^{\text{Norm}}_{d-1}, \partial)\) modulo the “neighboring commutativity relation”. This relation is defined by setting the bracket \([,]\) to zero if it contains two elements that are neighbors, i.e. all the elements containing a bracket of the form \([x_n, x_{n+1}]\) are set to zero. We denote this quotient by \((\text{POISS}^{\text{Zero}}_{d-1}, \partial)\). Note that this relation is similar to the 1T relation on the algebra of chord diagrams. Then we define the main theorem of this chapter as follows.

**Theorem 5.37.** The homology of the complex \((\text{POISS}^{\text{Zero}}_{d-1}, \partial)\) is isomorphic to the first page of Vassiliev’s main spectral sequence.
Chapter 6

Simplicial and cosimplicial models

In this chapter we will define the basics about simplicial and cosimplicial spaces which we will need in the next chapter to construct a cosimplicial model for the space of knots. The advantage of simplicial and cosimplicial spaces and objects is that they have a very rigid combinatorial structure which makes it relatively easy to use them for calculations. We will all so see that every simplicial or cosimplicial space has a natural filtration which we can use to construct a spectral sequence. In the next chapter we will use the techniques described in this chapter to construct another spectral sequence for the space of knots.

The chapter is structured as follows, in section 6.1 we will give the definition of simplicial and cosimplicial spaces and define morphisms of simplicial and cosimplicial spaces so we can talk about the categories of simplicial and cosimplicial spaces. In section 6.3 we recall the definition of a model category and define a model category structure on the category of cosimplicial spaces, i.e. define the notion of a weak equivalence, fibration and cofibration. In section 6.4 we will recall the definition of an operad and show that non-symmetric multiplicative operads give rise to a cosimplicial space. In section 6.5 we give an explicit example of the cosimplicial model for the loop space. This example is similar to the cosimplicial model for the space of knots so we will use it as a guiding example in this chapter. In section 6.6 we will construct the Bousfield Kan spectral sequence associated to a cosimplicial space. Under certain conditions this spectral sequence will converge to the homology of a cosimplicial space.
6.1 Simplicial and cosimplicial objects

The purpose of simplicial and cosimplicial objects is to generalize the idea of simplices in geometry. Due to their combinatorial structure, these simplicial methods are very powerful and show up in a great variety of applications. In this section, we will recall the basic notions of simplicial and cosimplicial objects and fix the notation. For more details and proofs see [7], [17] and [59] on which this chapter is mainly based, for a more elementary introduction we strongly recommend [14].

**Definition 6.1.** Let $\Delta$ be the category whose objects are finite ordered sets $[n] = \{0 < 1 < \ldots < n\}$ and with morphisms given by non-decreasing monotone functions.

This category $\Delta$ will be very important for the definition of simplicial and cosimplicial spaces, therefore we shall first study its structure a little better. It is convenient to introduce two classes of morphisms called the face and degeneration maps, the main advantage of these morphisms is that every morphism can be written as a composition of these morphisms.

**Definition 6.2.** Let $C$ be a category, a simplicial object $F$ in a category $C$ is a contravariant functor $F : \Delta \to C$, a cosimplicial object $G$ in an arbitrary category $C$ is a covariant functor $G : \Delta \to C$.

We can make this definition more explicit by saying that a simplicial object $X$ in a category $C$ is a collection of objects $X_n$, $n = 1, 2, \ldots$, together with for all $1 \leq i \leq n$ two kinds of maps. The first kind of maps are called the face maps $d_i : X_n \to X_{n-1}$ and the second kind of maps are called the degeneracy maps $s_i : X_n \to X_{n+1}$ such that they satisfy the simplicial identities. The simplicial identities are given by

- $d_id_j = d_jd_i$ if $i < j$
- $d_is_j = s_{j-1}d_i$ if $i < j$
- $= id$ if $i = j$ or $i = j + 1$
- $= s_jd_{i-1}$ if $i > j + 1$
- $s_is_j = s_js_{i-1}$ if $i > j$.

Similarly to a simplicial object we also have a more explicit definition of a cosimplicial object in a category $C$. A cosimplicial object in a category $C$ is a sequence of objects $X_n$, $n = 1, 2, \ldots$, in $C$ together with coface maps $d^i : X_{n-1} \to X_n$ and codegeneracy maps $s^i : X_{n+1} \to X_n$ such that they satisfy the cosimplicial identities. The cosimplicial identities are given by the dual of the simplicial identities, or more explicitly, we have the following,
$$d^j d^i = d^i d^{j-1} \text{ if } i < j$$

$$s^j d^i = d^i s^{j-1} \text{ if } i < j$$

$$= id \text{ if } i = j \text{ or } i = j + 1$$

$$= d^{i-1} s^j \text{ if } i > j + 1$$

$$s^j s^i = s^{i-1} s^j \text{ if } i > j.$$

Definition 6.3. A simplicial (respectively cosimplicial) space is a simplicial (respectively cosimplicial) object in the category of topological spaces.

Now we will give some examples of simplicial spaces and objects. The first example is an example of a cosimplicial space, after that we will several examples of simplicial spaces and an example of an simplicial abelian group, i.e. a simplicial object in the category of abelian groups. The first example is the standard topological simplex and the other example gives one of the possible simplicial structures of the circle $S^1$. A third example will be given in section 6.1.1 where we show set of all singular simplices of a topological space is a simplicial space.

Example 6.4. The topological standard simplex $\Delta$ is the cosimplicial space defined as follows, let $\Delta_n$ be the subset of $\mathbb{R}^{n+1}$ given by

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1 \text{ and } 0 \leq x_i \leq 1 \text{ for all } x_i\}.$$ 

The coface maps

$$d^i : \Delta_{n-1} \to \Delta_n$$

are given by

$$d^i (x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

and the codegeneracy maps

$$s^i : \Delta_{n+1} \to \Delta_n$$

are given by

$$s^i (x_1, \ldots, x_n) = (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n).$$

It is an easy exercise to check that these maps satisfy the cosimplicial identities.

Example 6.5. A simplicial model for the circle is given by the simplicial set with one 0 simplex which we call $\ast$ and one non degenerate 1 simplex which we call $\tau$. The face maps of $\tau$ are defined by $d_0(\tau) = \ast$ and $d_1(\tau) = \ast$. To make this space into a simplicial set we also need to define degeneracy maps and degenerate simplices. We need to define one degenerate 1 simplex as the
Figure 6.1: On the left we see a simplicial model given by one vertex with one non-degenerate 1-simplex. On the right we see another simplicial structure on the circle with two vertices and non-degenerate two 1-simplices.

image of $s_0(*)$. So in degree 0 we have just one point and in degree 1 we have $\tau$ and $s_0(*)$. In degree 2 we need to add the image of the degeneracy maps $s_0$ and $s_1$. So we have to add the degenerate simplices $s_0(s_0(*))$, $s_0(\tau)$ and $s_1(\tau)$, we do not have to add $s_1(s_0(*))$ since this is the same as $s_0(s_0(*))$ by the cosimplicial identities. In general we have in degree $n$ the following $n$ degenerate simplices $(s_0)^n(*)$ and for all $1 \leq i \leq n$ we have the degenerate simplex $s_{n-1}s_{n-2}...\hat{s}_i...s_0(\tau)$. Where $\hat{s}_i$ means that we have to omit the $i$th degeneracy map. It is important to note that even though the model for the circle is the model for a finite dimensional space we still have infinitely many simplices. It is true however that we only have non degenerate simplices in finitely many dimensions.

**Example 6.6.** Another simplicial model for the circle is constructed as follows. In degree 0 we define two 0 vertices which we will call $p$ and $q$. Then we define in degree 1 two non degenerate 1 simplices which we will call $m$ and $n$. We define the face maps by $d_0(m) = p = d_1(n)$ and $d_1(m) = q = d_0(n)$. The degenerate simplices and the degeneracy maps are defined as in example 6.5. In figure 6.1 we see a picture of both simplicial models and how they correspond to the circle.

**Remark 6.7.** Note that there are in general many different simplicial structures for the same topological space. Another simplicial model for the circle would be for example the simplicial set with two non degenerate zero simplices and two non degenerate 1-simplices. See also figure 6.1.

**Example 6.8.** A simplicial model for the n-sphere $S^n$ is given by a simplicial set with one non-degenerate 0-simplex $*$ and one non degenerate n-simplex $\sigma$, the face maps are defined by $d_i(\sigma) = *$ for $i = 0, ..., n$. The degeneracy maps and the degenerate simplices are defined as in example 6.5.
The next logical step in our definition of simplicial spaces is to define the notion of a morphism between simplicial objects over a category $C$.

**Definition 6.9.** Let $G = \{G_n, n \geq 0\}$ and $H = \{H_n, n \geq 0\}$ be simplicial objects in a category $C$, a morphism of simplicial objects $f : G \to H$ is a collection of order preserving maps $f_n : G_n \to H_n$ such that each $f_n$ commutes with the face and degeneracy maps. In the case of cosimplicial objects $C$ and $D$ in a category $C$ we define a morphism of cosimplicial spaces $f : C \to D$ as a collection of order preserving maps $f : C_n \to D_n$ such that each $f_n$ has to commute with the coface and codegeneracy maps.

### 6.1.1 The geometric realization, singular and totalization functors

In this subsection we will define the geometric realization functor from the category of simplicial sets to the category of topological spaces and his adjoint the singular functor which assigns to every topological space a simplicial set. We will also define the dual construction of the geometric realization which is called the totalization of a cosimplicial space, the totalization of a cosimplicial object is the space of all maps from the standard simplex $\Delta$ to the cosimplicial space.

**Definition 6.10.** Let $X_\bullet = \{X_n, n \geq 0\}$ be a simplicial set, i.e. a simplicial object in the category of sets, the geometric realization of a simplicial space $X$ is the topological space constructed as follows. First we equip $X_\bullet$ with the discrete topology and let $|\Delta^n|$ be the standard simplex equipped with the standard topology, the geometric realization $|X|$ of $X$ is then the following space

$$|X| = \prod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim,$$

where we have the following two relations $(d_i x, p) \sim (x, d^i p)$ for all $x \in X_{n+1}$ and $p \in \Delta^n$ and $(s_i x, p) \sim (x, s^i p)$ for all $x \in X_{n-1}$ and $p \in \Delta^n$.

**Remark 6.11.** This definition applies to more simplicial objects than just simplicial sets, in general we can apply this definition to an arbitrary simplicial object by forgetting all the extra structure and only considering the underlying set. It is of course necessary that the object has an underlying set. So for example we can define the geometric realization of a simplicial abelian group by just forgetting the group structure and only considering the underlying simplicial set.
Lemma 6.12. The geometric realization of a simplicial set $X$ is a CW-complex with one $n$-cell for each non degenerate $n$-simplex of $X$.

A proof of this lemma can be found in [38].

Example 6.13. The geometric realizations of the two simplicial sets corresponding to the two simplicial models of the circle given in figure 6.1 are the circle.

Definition 6.14. The singular functor $\text{Sin}$ assigns to every topological space $X$ a simplicial set $\text{Sin}(X)$ in the following way. The $n$-simplices are defined to be the maps of topological spaces

$$\alpha : \Delta[n] \rightarrow X$$

from the standard simplex to $X$. The face and degeneracy maps of $\text{Sin}(X)$ are defined by first composing $f$ with the face and degeneracy maps of $\Delta[n]$. More explicitly the face maps $d_i$ are given by

$$d_i : \text{Sin}(X)_{n-1} \rightarrow \text{Sin}(X)_n$$

$$d_i(\alpha) = \alpha(d^i)$$

The degeneracy maps $s_i$ are defined as

$$s_i : \text{Sin}(X)_{n+1} \rightarrow \text{Sin}(X)_n$$

$$s_i(\alpha) = \alpha(s^i)$$

Now that we have defined what $\text{Sin}$ does on objects we only have to define what it does to morphisms. Let $f : X \rightarrow Y$ be a continuous map of topological spaces $X$ and $Y$, then we define the map

$$\text{Sin}(f) : \text{Sin}(X) \rightarrow \text{Sin}(Y)$$

as the map that sends a singular simplex $\alpha : \Delta[n] \rightarrow X$ to the singular simplex

$$f \circ \alpha : \Delta[n] \rightarrow Y.$$ 

Proposition 6.15. The geometric realization is left adjoint to the singular functor, i.e. there is a bijective correspondence between the following sets

$$\text{Hom}(\mid \text{Sin}(X), Y \mid) \cong \text{Hom}(X, \text{Sin}(Y)).$$

Where $\text{Hom}(\mid \text{X} \mid, Y)$ is the space of continuous maps between the topological spaces $\mid \text{X} \mid$ and $Y$ and $\text{Hom}(X, \text{Sin}(Y))$ is the space of simplicial maps between $X$ and $\text{Sin}(Y)$. 

84
The proof can be found in [17] or [7].

**Definition 6.16.** Let $X^•$ be a cosimplicial space, then we define $\text{Tot}(X^•)$, the *totalization* of $X^•$ as the space

$$\text{Tot}(X^•) = \text{Hom}_\Delta(\Delta, X^•),$$

of all cosimplicial maps from the standard simplex $\Delta$ to $X^•$. The totalization defines a functor from cosimplicial spaces to topological spaces.

A more concrete description of the elements of the totalization of $X^•$ is as follows. An element of $\text{Tot}(X^•)$ is a sequence of maps $\{\alpha_n\}$,

$$\alpha_n : \Delta_n \rightarrow X^n,$$

for $n \geq 0$ such that $\alpha_n$ commutes with all the coface and codegeneracy maps, i.e.

$$d^i \circ \alpha_n = \alpha_{n+1} \circ d^i$$

and

$$s^i \circ \alpha_n = \alpha_{n-1} \circ s^i.$$  

### 6.2 The nerve of a category

To every small category $\mathcal{C}$ we are able to associate a simplicial object $N(\mathcal{C})$ in the following way. First we define the set of vertices of the simplicial set $N(\mathcal{C})$ as the set of objects in $\mathcal{C}$. We define a one 1-simplex between two vertices for each morphism connecting the corresponding objects. The 2-simplices are defined to be the commutative triangles in this category, so whenever we have two morphisms $f$ and $g$ we can define a 2-simplex by defining the vertices to be the source and target of $f$ and the target of $g$, the edges are $f$, $g$ and $fg$ and the center is the 2-cell of the simplex. In general we can define a, possibly degenerate, $n$-simplex for each sequence of $n+1$ objects $C_0, ..., C_n$ together with $n$ morphisms $f_i : C_{i-1} \rightarrow C_i$ such that we have the following sequence

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n.$$  

We can do this in the same fashion as we did with the 2-simplex by defining the edges to be the morphisms $f_1, ..., f_n$ and $f_n \circ ... \circ f_1$ and then all the corresponding higher simplices. The face and degeneracy maps are given as follows. The face maps act on a sequence of composable morphisms by composing the $i$th and $i+1$th morphism. So we get the following formula

$$d_i(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} ... C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} ... \xrightarrow{f_n} C_n) =$$
The degeneracy maps are defined by inserting the identity morphism at the ith place in the sequence, i.e.

\[ s_0(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} C_n) = C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{id} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} C_n \]

The following two examples will be important for the construction of the cosimplicial model in chapter 9 and also in general for the calculus of embeddings.

**Example 6.17.** Let \( n = \{1, \ldots, n\} \) be the set of the number 1 till \( n \) and let \( \mathcal{P}(n) \) be the poset consisting of all subsets of \( n \) including the empty set ordered by inclusion. We can consider this poset as a category by setting the all the subsets of \( n \) to be the objects and say that there is one morphism from an object \( C_1 \) to an object \( C_2 \) if \( C_1 \subseteq C_2 \). The geometric realization of the nerve of this category is an \( n \) dimensional cube divided into simplices.

**Example 6.18.** For this example we will consider again the poset of all subsets of \( n \) but this time we will not include the empty set in this poset, we will denote this set by \( \mathcal{P}_\nu(n) \). We can consider this poset as a category in an analogous way as in the previous example. The nerve of this category is the barycentric subdivision of a \( n-1 \) simplex.

**Proposition 6.19.** Let \( \mathcal{C} \) be a small category with a final object then the geometric realization of the nerve of \( \mathcal{C} \) is contractible.
6.3 A model category structure on the category of cosimplicial spaces

In this section we will briefly recall the definition of a model category and define the notions of a fibration, cofibration and weak equivalence in the categories of simplicial and cosimplicial spaces. We will begin by recalling the definition of a model category.

6.3.1 The definition of a model category

Before we give the definition of a model category we will first need some preliminary definitions.

**Definition 6.20.** Let $C$ be a category, then we define a functorial factorization $(\alpha, \beta)$ as an ordered pair of functors $\alpha : Mor(C) \rightarrow Mor(C)$ and $\beta : Mor(C) \rightarrow Mor(C)$ such that $f = \beta(f) \circ \alpha(f)$ for all morphisms $f$ in $(C)$.

**Definition 6.21.** Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be morphisms in $C$, then we call $f$ a *retract* of $g$ if there exists a diagram of the form,

\[
\begin{array}{ccc}
A & \rightarrow & C \\
f \downarrow & & g \downarrow \\
B & \rightarrow & D \\
& f \downarrow & \\
& & B
\end{array}
\]

such that the diagram commutes and the composition of the horizontal arrows is the identity.
Definition 6.22. Let $i : A \to B$ and $p : X \to Y$ be morphisms in a category $C$. Then we say that $i$ has the left lifting property with respect to $p$ and that $p$ has the right lifting property with respect to $i$, if for every morphisms $f : A \to X$ and $g : B \to Y$ such that the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{g} & Y
\end{array}
\]

there exists a map $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

With these preliminary definitions we define a model category as a category satisfying the conditions from the following definition.

Definition 6.23. Let $C$ be a category with three classes of distinguished morphisms called fibrations, cofibrations and weak equivalences. These morphisms together with two functorial factorization $(\alpha, \beta)$ and $(\gamma, \delta)$ should satisfy the following axioms.

1. In category $C$ all small limits and small colimits exist.

2. (2-out-of-3) If $f$ and $g$ are weak equivalences and the composition $gf$ is defined, then if two of the morphisms $f$, $g$ and $gf$ are weak equivalences then so is the third.

3. (Retracts) Let $f$ and $g$ be morphism in $C$ such that $f$ is a retract of $g$, if $g$ is a weak equivalence, fibration or cofibration then so is $f$.

4. (Lifting) Every acyclic cofibration, i.e. a cofibration that is also a weak equivalence, has the left lifting property with respect to all fibrations in $C$. Similarly we also want that every trivial fibration, i.e. a fibration that is also weak equivalence, has the right lifting property with respect to all cofibrations.

5. (Factorization) Let $f$ be morphism in $C$ then there exist factorizations $(\alpha, \beta)$ and $(\gamma, \delta)$ of $f$ such that $\alpha(f)$ is a cofibration and $\beta(f)$ a weak equivalence.

Remark 6.24. The definition of a model category we gave here comes from [27]. There are other definitions which are similar but differ at some small details.

From each model category we can construct a new category by ”inverting” the weak equivalences of the category. We call this new category the homotopy category. More formally this is done as follows.
**Definition 6.25.** Let $C$ be a model category and let $W$ be the subcategory of weak equivalences. Then we define the category $F(C, W^-)$ as the free category on the morphisms of $C$ and the inverses of the morphisms of $W$. The we define the homotopy category $C[W^-]$ as the quotient of $F(C, W^-)$ by the relation that two strings of composable morphisms $(f_1, ..., f_n)$ and $(g_1, ..., g_m)$ are equivalent if $f_n \circ ... \circ f_1 = g_m \circ ... \circ g_1$.

### 6.3.2 The model category structure on simplicial spaces

We will now define the model category structure on the category of simplicial spaces. To do this we will first define another space called the $k$th-horn which will make all definitions clearer. Then we will define fibrations, cofibrations and weak equivalences. After that we will state a theorem that says that the category of simplicial spaces is a model category.

**Definition 6.26.** The $k$th-horn $| \Lambda^k_n |$ of the standard $n$-simplex $| \Delta^n |$ is the of $\Delta^n$ obtained by removing the face $d_k \Delta^n$ and the interior of $| \Delta^n |$. Or similarly we can define the $k$th horn as the subsimplicial set of $\Delta^n$ generated by the image of the maps $d_0 i_n, ..., d_{k-1} i_n, d_{k+1} i_n, ..., d_n i_n$.

**Definition 6.27.** Let $p : X \to Y$ be a map of simplicial sets, then we call $p$ a fibration of simplicial spaces if $p$ has the right lifting property with respect to the following diagram,

$$
\begin{array}{ccc}
\Lambda^k_n & \xrightarrow{f} & X \\
\downarrow i & & \downarrow p \\
\Delta^n & \xrightarrow{g} & Y.
\end{array}
$$

for all morphisms $f$ and $g$.

There are some relations between the Serre fibrations in the category of topological spaces and the fibrations in the category of simplicial spaces. The proof can be found in [7].

**Lemma 6.28.** 1. A map $f : X \to Y$ of topological spaces is a Serre fibration if and only if the map $\text{Sin}(f) : \text{Sin}(X) \to \text{Sin}(Y)$ is a fibration of simplicial spaces.

2. If $f : X \to Y$ is a fibration of simplicial spaces then the map $| f | : | X | \to | Y |$ is a Serre fibration.

Similar to the fibrations we can also define the cofibrations in the category of simplicial spaces.
Definition 6.29. A cofibration of simplicial spaces is a map of simplicial spaces \( i : X \rightarrow Y \) which is injective.

Similar to lemma 6.28 we have the following lemma which gives the relation between cofibrations in the category of simplicial spaces and topological spaces.

Lemma 6.30.  
1. A map \( i : X \rightarrow Y \) is a cofibration of simplicial spaces if and only if the map \( |i| : |X| \rightarrow |Y| \) is a cofibration of topological spaces.

2. If \( i : X \rightarrow Y \) is a cofibration of simplicial spaces then so is the map \( Sin(i) : Sin(X) \rightarrow Sin(Y) \).

The following definitions are important technical conditions which are used a lot in the theory of simplicial spaces.

Definition 6.31. An object \( X \) in a model category is called fibrant if the unique map \( p : X \rightarrow \{\ast\} \) from \( X \) to the one point space \( \{\ast\} \) is a fibration. Dually an object \( X \) is called cofibrant if the inclusion of the empty set \( i : \emptyset \rightarrow X \) is a cofibration.

Remark 6.32. This definition makes sense in arbitrary model categories although we will mainly use it in the context of simplicial and cosimplicial spaces. Please note that this condition can be trivial in some categories, so is for example every topological space fibrant and every simplicial set cofibrant.

Example 6.33. Before we will give examples of fibrant simplicial spaces we will first give an example of a space that is not fibrant. The standard \( n \)-simplex \( \Delta^n \) is not fibrant because we can define a map that cannot be extended. We will only give this map in the case of the \( \Delta^1 \), the reader can easily extend this map to higher dimensional simplices. Let \( f : \Lambda^2_0 \rightarrow \Delta^1 \) be the map that sends the edge \([0,2]\) to the point 0 and the edge \([0,1]\) to the edge \([0,1]\) of \( \Delta^1 \). This map cannot be extended since the morphism that sends the edge \([1,2]\) to the edge \([1,0]\), i.e. 1 goes to 1 and 2 goes to 0, is not order preserving and therefore not a map of simplicial sets.

Example 6.34. Let \( X \) be a topological space, then \( Sin_4(X) \) the space of all singular simplices of \( X \) is a fibrant simplicial space. We can easily see this since we can retract each \( n \)-simplex to the \( k \)-th-horn of this simplex and in this way we can extend the map. See also figure 6.3.2.

We will give two definitions of a simplicial homotopy which at first sight might look different, but lemma 6.37. The first definition will look
Figure 6.4: The set of singular simplices of a topological space is fibrant since we can contract every singular simplex to a horn.

similar to the usual definition of homotopy in the category of topological spaces. The second definition is more simplicial in the sense that it does not have an analog in the category of topological spaces.

**Definition 6.35.** Let $X$ and $Y$ be simplicial sets and $f : X \to Y$ and $g : X \to Y$ be simplicial maps, a homotopy of a simplicial maps is a map $H : I \times X \to Y$ such that $H |_{X \times \{0\}} = f$ and $H |_{X \times \{1\}} = g$.

**Definition 6.36.** Let $f : X \to Y$ and $g : X \to Y$ be simplicial morphisms between simplicial sets $X$ and $Y$, then we define a homotopy between $f$ and $g$ to be a series of maps $h^p_i : X_p \to Y_{p+1}$ such that for each $i$ such that $0 \leq i \leq p$ the following identities hold.

$$d_0 h_0 = f$$
$$d_{p+1} h_p = g$$
$$d_i h_i = h_{j-1} d_i \quad \text{if} \ i < j$$
$$d_{j+1} h_{j+1} = d_{j+1} h_j \quad \text{if} \ i = j$$
$$d_i h_j = h_j d_{i-1} \quad \text{if} \ i > j + 1$$
$$s_i h_j = h_{j+1} s_i \quad \text{if} \ i \leq j$$
$$s_i h_j = h_j s_{i-1} \quad \text{if} \ i > j$$
Lemma 6.37. The two definitions of a simplicial homotopy are equivalent.

For a proof of the lemma see [14].

Lemma 6.38. Let $X$ and $Y$ be simplicial sets, and $\text{Hom}(X, Y)$ be the space of simplicial maps between $X$ and $Y$ then homotopy is an equivalence relation on $\text{Hom}(X, Y)$ if $Y$ is fibrant.

The proof of this lemma can again be found in [14]. The important thing to notice here is that in general homotopy is not and equivalence relation on the category of simplicial sets. But as the lemma states it is an equivalence relation if $Y$ is fibrant. In general we will consider always consider fibrant or cofibrant replacements of our simplicial sets such that homotopy becomes an equivalence relation. We will now define the simplicial homotopy groups and the notion of a weak equivalence. There are several possible definitions to do this, we will give the one that is analog to the definition of the homotopy groups in the category of simplicial spaces.

Definition 6.39. Let $X$ be a fibrant simplicial space then we define the simplicial homotopy groups $\pi_n(X, \ast)$ as the set of equivalence classes of maps $(\partial \Delta^{n+1}, \ast) \to (X, \ast)$. The basepoint of $\partial \Delta^{n+1}$ is here given by the vertex $[0]$ and all homotopies are relative to the basepoint. The groups structure is given by the usual composition.

With this definition we can define the weak equivalences in the category of simplicial sets. Which is done in the following definition.

Definition 6.40. Let $X$ and $Y$ be simplicial spaces and $f : X \to Y$ be a map between $X$ and $Y$, we call $f$ a weak equivalence if $f$ induces an isomorphism on all homotopy groups $f_* : \pi(X) \to \pi(Y)$ for all choices of basepoint.

Similar to the fibrations and cofibrations we have a lemma that gives us a relation between the weak equivalences of the category of simplicial spaces and the category of topological spaces.

Lemma 6.41. 1. A map $f : X \to Y$ of topological spaces is a weak equivalence of topological spaces if and only if the map $\text{Sin}(f) : \text{Sin}(X) \to \text{Sin}(Y)$ is a weak equivalence of simplicial sets. 2. A map $f : X \to Y$ of simplicial sets is a weak equivalence of simplicial set if and only if the map $\mid f \mid : \mid X \mid \to \mid Y \mid$ is a weak equivalence of topological spaces.
3. Let $X$ be a simplicial set then the map $X \to \text{Sin}(\mid X \mid)$ is a weak equivalence in the category of simplicial sets and for a topological space $Y$ the map $Y \to \mid \text{Sin}(Y) \mid$ is a weak equivalences in the category of topological spaces.

Since the definition of the simplicial homotopy groups and simplicial homotopy in general require the existence of fibrant replacements, we have the following definition which guarantees there existence.

Proposition 6.42. Let $X$ be a simplicial space, then there exists a fibrant simplicial space $\tilde{X}$ such that $\tilde{X}$ and $X$ are homotopy equivalent. Dually we also have that for every cosimplicial space $Y$ there exists a cofibrant cosimplicial space $\tilde{Y}$ such that $\tilde{Y}$ and $Y$ are homotopy equivalent. We call the space $\tilde{X}$ (respectively $\tilde{Y}$) the fibrant replacement of $X$ (respectively cofibrant replacement of $Y$).

Proof. The proof of the proposition follows from the factorization axiom of a model category. This axiom states that we can factor every map $f : X \to Z$ as two maps $X \xrightarrow{w} \tilde{X} \xrightarrow{p} Y$ where $w : X \to \tilde{X}$ is a weak equivalence and $p : \tilde{X} \to Y$ a fibration. If we take for the space $Y$ the one point space $\{\ast\}$ then we see that $\tilde{X}$ is weakly equivalent to $X$ and that $\tilde{X}$ is fibrant. In the cofibrant case we can do a similar construction of a cofibrant replacement.

Now that we have defined the notions of fibrations, cofibrations and weak equivalences we see in the next theorem that they indeed form a model category.

Theorem 6.43. The category of simplicial is a model category.

The proof can be found in [7].

6.4 The cosimplicial space associated to a multiplicative operad

In this section we will describe a construction to associate to every multiplicative operad a cosimplicial object. This construction is due to McClure and Smith and more details can be found in [40].

Definition 6.44. Let $O$ be a non-symmetric operad over a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, then we call $O$ multiplicative if there exists a morphism of operads $\Pi : \text{ASS} \to O$ from the non-symmetric associative operad $\text{ASS}$ to the operad $O$. 

93
To each multiplicative operad we associate a cosimplicial object as follows.

**Definition 6.45.** Let $\mathcal{O}$ be a multiplicative operad and let $m \in \mathcal{O}(2)$ be the multiplication and $e \in \mathcal{O}(0)$ be the image of $\text{ASS}(0)$. Then we define a cosimplicial object $\mathcal{O}^\bullet$ by defining $\mathcal{O}^n$ as the object $\mathcal{O}(n)$. The coface maps are given by:

$$d^i : \mathcal{O}^n \to \mathcal{O}^{n+1}$$

$$d^i(x) = \begin{cases} 
m \circ_2 x & \text{if } i = 0 \\
x \circ_i m & \text{if } 1 \leq i \leq n \\
m \circ_1 x & \text{if } i = n + 1 
\end{cases}$$

And the codegeneracy maps are given by

$$s^j : \mathcal{O}^n \to \mathcal{O}^{n-1}$$

$$s^j(x) = m \circ_j e.$$  

**Theorem 6.46.** The cosimplicial object defined in definition 6.45 is indeed a cosimplicial object.

The proof is relatively straightforward and can be found in [41] and [40].

### 6.5 A cosimplicial model for the loop space

In this section we will describe a cosimplicial model for the loop space of a topological space. This is an important example since the cosimplicial model for the space of knots is based on this model.

**Definition 6.47.** Let $Z$ be a topological space with basepoint $*$, the geometric cobar construction $F^\bullet Z$ of $Z$ is the following cosimplicial space. Let $F^m Z$ be $Z^m$ and define $F^0 Z$ to be the basepoint. To make it into a cosimplicial space we define the following coface and codegeneracy maps. The codegeneracy maps are

$$s^i : F^m Z \to F^{m-1}$$

$$(z_1, ..., z_m) \to (z_1, ..., \hat{z}_i, ..., z_m)$$

where $\hat{z}_i$ means that we omit the $i$-th entry. The coface maps are

$$d^i : F^m \to F^{m+1}$$

$$d^i(z_1, ..., z_m) = \begin{cases} 
(*, z_1, ..., z_m) & \text{if } i = 0 \\
(z_1, ..., z_i, z_{i+1}, ..., z_m) & \text{if } 1 \leq i \leq m \\
(z_1, ..., z_m, *) & \text{if } i = m + 1
\end{cases}$$
Theorem 6.48. $\text{Tot}(F^*Z)$, the totalization of $F^*Z$, is homeomorphic to $\Omega(Z)$ the loop space of $Z$.

Proof. The key ingredient of the proof is that an element of the totalization $\{f_n\} \in \text{Tot}(F^*Z)$ is completely determined by the image of $f_1$. This is because the coface and codegeneracy maps determine the image of $f_2$ and then inductively the image of $f_n$. So each map is determined by the image of the map the map of the interval to $Z$, which gives us a homeomorphism with the loop space of $Z$. 

6.6 The Bousfield Kan spectral sequence associated to a cosimplicial space

In this section we will describe two spectral sequences which we will call the homotopy Bousfield Kan spectral sequence and the homology Bousfield Kan spectral sequence. The first spectral sequence has the property that, under some conditions, it converges to the homotopy groups of the totalization of a cosimplicial space $X^*$. The second spectral sequence converges under some conditions to the homology groups of a cosimplicial space. The homology Bousfield Kan spectral sequence can also be seen as a generalization of the Eilenberg Moore spectral sequence. A more complete description of both spectral sequences can be found in the original articles [7] for the homotopy version and [5] for the homological version or in the following books [17] and [51]

The idea behind the spectral sequences is as follows. First observe that we can see the standard n-dimensional simplex $\Delta(n)$ as the total space of an fibration with base space $\Delta(n-1)$. By continuing like this we can describe the standard simplex $\Delta$ as the inverse limit of the following tower of fibrations.

$$ \cdots \rightarrow \Delta(n) \rightarrow \Delta(n-1) \rightarrow \cdots \rightarrow \Delta(2) \rightarrow \Delta(1) \rightarrow \Delta(0) = * $$

Since the totalization of a cosimplicial space is the space of all cosimplicial maps from $\Delta$ to $X^*$, we can also try to describe it as a tower of fibrations in a similar fashion. To do this we will first study towers of fibrations in a more general context and after that we will look at the homotopy Bousfield Kan spectral sequence and finally the homological version of this spectral sequence.
6.6.1 Towers of fibrations

To a tower of fibrations of topological spaces $X_s$ we can associate a spectral sequence converging to the homotopy groups of the spaces $X_s$. We will show how to associate an exact couple to the homotopy groups of a tower of fibrations. We will do this by using the Serre long exact sequence for the homotopy groups of a fibration. Therefore we will begin by recalling the Serre long exact sequence for the homotopy groups of a topological space.

Since the goal of this section is only to give a heuristic understanding of the concept of spectral sequences associated to towers of fibrations we might ignore some details now and then, more details can be found in [17], [7] and [22].

**Theorem 6.49.** Let $p : (E,e_0) \to (b_0)B$ be a Serre fibration of pointed spaces with fiber $i : (F,e_0) \to (E,e_0)$, then this induces a map $\partial : \pi_n(B,b_0) \to \pi_n(F,e_0)$ which gives the following long exact sequence for the homotopy groups of the spaces $(B,b_0)$, $(E,e_0)$ and $(F,e_0)$.

$$
\cdots \xrightarrow{p_{n+1}} \pi_{n+1}(B,b_0) \xrightarrow{\partial} \pi_n(F,e_0) \xrightarrow{i} \pi_n(E,e_0) \xrightarrow{p_n} \pi_n(B,b_0) \xrightarrow{\partial} \pi_{n-1}(F,e_0) \xrightarrow{i} \cdots
$$

The proof of this theorem can be found in [22].

**Remark 6.50.** Note that since the $\pi_1(B)$, $\pi_1(E)$ and $\pi_1(F)$ are not necessarily abelian groups and $\pi_0(B)$, $\pi_0(E)$ and $\pi_0(F)$ are only pointed sets the Serre long exact sequence for the homotopy groups is only a long exact sequence of groups and pointed sets at the end. Will not go into detail about this, but this has the consequence that corresponding spectral sequence also has pointed sets and non abelian groups at the lowest terms.

With the Serre long exact sequence and a tower of fibrations we are now going to construct a spectral sequence. Let $X_s$ be a tower of fibrations of topological spaces:

$$
\cdots \xrightarrow{p_{s+1}} X_s \xrightarrow{p_s} X_{s-1} \xrightarrow{p_{s-1}} X_{s-2} \xrightarrow{p_{s-2}} X_{s-3} \xrightarrow{p_{s-3}} \cdots
$$

If take the homotopy groups of the spaces $X_s$ and $F_s$ we get the following diagram.

$$
\cdots \xrightarrow{p_{s+1}^*} \pi_*(X_s) \xrightarrow{p_s^*} \pi_*(X_{s-1}) \xrightarrow{p_{s-1}^*} \pi_*(X_{s-2}) \xrightarrow{p_{s-2}^*} \cdots
$$
Where $\partial$ is the map from the Serre long exact sequence. This map $\partial$ gives us an exact couple and therefore induces a spectral sequence, which we will call the homotopy Bousfield Kan spectral sequence. In the homological case we can do something similar, we will not describe how that spectral sequence is constructed we will only define it.

### 6.6.2 The homology Bousfield Kan spectral sequence

Let $x^*$ be cosimplicial simplicial space, i.e. a cosimplicial object in the category of simplicial spaces, for example the loop space model from section 6.5 for the circle. In the this section we define a spectral sequence that under some conditions converges to the homology of the totalization of a cosimplicial simplicial space. This spectral sequence is called the homology Bousfield Kan spectral sequence and is constructed similar to the homotopy Bousfield Kan spectral sequence from the previous section. We will just give the definitions here and do not care about the convergence of the spectral sequence. In [5] the reader can find some criteria for the convergence of the spectral sequence.

**Definition 6.51.** Let $L = \{L_n, n \geq 0\}$ be a simplicial abelian group and $M = \{M^n, n \geq 0\}$ be a cosimplicial abelian group then we can define the following complexes. The normalized complex corresponding to $L$ is the complex given by

$$N_n L = L_n = L_n/ \text{im} s_0 + \ldots + \text{im} s_{n-1},$$

together with the following differential:

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : N_n L \to N_{n-1} L.$$

The normalized complex of a cosimplicial abelian group $M$ is the following:

$$N^n M = M^n \cap \text{kers}^0 \cap \ldots \cap \text{kers}^{n-1},$$

with the following differential:

$$\delta = \sum_{i=0}^{n+1} (-1)^i d^i : N^n M \to N^{n-1} M.$$

**Definition 6.52.** Let $L$ be a simplicial abelian group and $M$ be a cosimplicial abelian group. Then we define the simplicial homotopy groups as $\pi_n(L) = H_n(N \cdot L)$ and the cohomotopy groups as $\pi^n(M) = H^n(N^\bullet M).$
For a cosimplicial simplicial abelian group $B$ we are able to construct a double complex by applying first $N^\bullet$ and then $N_\bullet$. From this double complex we can take the total complex $(TB)$, the $n$th term of this total complex is given by

$$(TB)_n = \prod_{m \geq 0} N^m N_{n+m} B,$$

together with a differential given by

$$\partial_T = \partial + (-1)^{n+1} \delta : (TB)_n \to (TB)_{n-1}.$$ 

This double complex comes with a filtration given by $T_mB = TB/F_{m+1}TB$, where the space $(F_{m+1}TB)_n = \prod_{k \geq m+1} N^k N_{k+n} B$. So $T_mB$ is the truncation of $TB$ by $F_{m+1}TB$. From this filtration we can now construct the corresponding spectral sequence. For a general cosimplicial simplicial space we can do the following.

**Definition 6.53.** Let $X$ be a cosimplicial simplicial space and $A$ an abelian group then we can define cosimplicial simplicial abelian group $A \otimes X$ as follows. If we define $A \otimes S = \oplus_{a \in S}$ for a set $S$, then we define $(A \otimes X)^m_t$, the $m$, $t$th component of the cosimplicial simplicial abelian groups $A \otimes X$ as $A \otimes (X_t^m)$.

The homology Bousfield Kan spectral sequence is then defined as follows. Let $X$ be a cosimplicial simplicial space then we define the $E^1$ page of the homology Bousfield Kan spectral sequence with coefficients in an abelian group $A$ as

$$E^1_{m,t} = H_{t-m}(F^m/F^{m+1})T(A \otimes X) \cong N^m H_t(X; A).$$

The $E^2$ page of the spectral sequence then looks like

$$E^2_{m,t} = H^m(N^\bullet H_t(X; A) = \pi^m(H_t(X; A)).$$

**Proposition 6.54.** The spectral sequence described above converges, under some conditions, to the homology of the totalization of a cosimplicial space $X$.

The proof and the conditions of the is proposition are given in [5].

**6.6.3 The homotopy Bousfield Kan spectral sequence**

The $E_1$ page of the homotopy Bousfield Kan spectral sequence of a cosimplicial space $X$ is given by

$$E_1^{s,t} = \pi_{t-s} \Omega X \cong \pi_t N^s X.$$
Theorem 6.55. Under some conditions this spectral sequence converges to the homotopy groups of the totalization of the cosimplicial space $X$.

The proof and the conditions can be found in [17] or [7].

6.6.4 Some final remarks about the Bousfield Kan spectral sequence

We conclude this section about the Bousfield Kan spectral sequences with some final remarks. First of all it is important to notice that not all the homotopy groups are not abelian or even groups. Therefore the spectral sequence is fringed, because of this it is important to take some care in defining what happens at the lowest rows of the spectral sequence. We did not do this because we have only briefly described what the spectral sequences look like. The interested reader can find more details about this in [17] and [7].

Another important issue about these spectral sequences is the convergence. There are several criteria that guarantee convergence of the spectral sequence. These criteria can be found in [5], [7], [17].
Chapter 7  
Homotopy limits

Before we begin with the overview of the calculus of embeddings we need some preliminaries. Therefore we will give a short explanation about homotopy limits in this section. Homotopy limits and colimits are a replacement of the usual limits and colimits in category theory. The problem with the usual limits and colimits is that in the category of spaces, they do not map homotopy equivalences to homotopy equivalences. Or in other words if two diagrams $I$ and $J$ are of the same shape and there is a natural transformation $\eta$ from $I$ to $J$ such that the maps $\eta$ are homotopy equivalences, then the limits or colimits of $I$ and $J$ are not necessarily homotopy equivalent (see also example 7.3). To fix this we introduce homotopy limits and colimits which we will use throughout the rest of this thesis. We assume that the reader is familiar with categorical limits and colimits, otherwise [36] is a standard reference.

7.1 Motivation for the concept of homotopy limits

First we will give a simple example of a situation where normal limits and colimits fail to preserve homotopy equivalences.

Recall that the definition of a diagram of spaces is as follows.

**Definition 7.1.** Let $I$ be a small category and $\text{Top}$ be the category of topological spaces, then we define a diagram of spaces as a functor $D : I \rightarrow \text{Top}$. We call the category $I$ the indexing category of the diagram or the shape of the diagram.

**Remark 7.2.** Sometimes we just write $D$ for a diagram $D : I \rightarrow \text{Top}$ in this case we assume that there exists some indexing category $I$ which is the
Example 7.3. Let $I$ be the diagram

$$\{\ast\} \leftarrow S^n \rightarrow D^{n+1},$$

where $\{\ast\}$ is the one point space, $S^n$ is the $n$-sphere and $D^{n+1}$ the $n+1$ disk. The only non trivial map is given by the inclusion of $S^n$ in $D^{n+1}$. Let $J$ be the diagram

$$\{\ast\} \leftarrow S^n \rightarrow \{\ast\},$$

There is an obvious homotopy equivalence between these diagrams given by contracting the $D^{n+1}$ to a point. But the colimit of $I$ is equal to $S^n$, while the limit of $J$ is equal to $\{\ast\}$. These spaces are clearly not homotopy equivalent.

7.2 Homotopy colimits

The definition of the homotopy colimit we will give now is based on [12]. The plan is to construct an analog of the colimit functor which is homotopy invariant. We will do this by constructing a space that is similar to the mapping cylinder.

Definition 7.4. Let $f : X \rightarrow Y$ be a map of topological spaces, then we define the mapping cylinder of $f$ as the space

$$X \times I \coprod Y / .$$

Where we identify the points $(x, 1) \in X \times I$ with $f(x) \in Y$.

If we apply the mapping cylinder construction to the diagrams in example 7.3 we get the following. We replace every every map of the diagram by the corresponding mapping cylinder and then glue the mapping cylinders together along $S^n$. The result of these constructions is shown in figure 7.2 both diagrams we now get a space that is homotopy equivalent to $S^n$.

In general we want to define a similar construction, but before we do this we first have to look at the following example.

Example 7.5. Let $D$ be the diagram:

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$
Figure 7.1: On the left we see the result of replacing the morphisms by mapping cylinders of the diagram $\ast \leftarrow S^n \rightarrow D^{n+1}$ and on the right for the diagram $\ast \leftarrow S^n \rightarrow \ast$. Both spaces are clearly homotopy equivalent to $S^n$.

In this diagram we have 3 maps, $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and a third map $gf : X \rightarrow Z$. So if we now want to construct our mapping cylinder analog we could do this in two ways. By only constructing the mapping cylinders of $f$ and $g$ or by also gluing the map $fg$ to this space. The result of both constructions is in general not the same and also not homotopy equivalent.

In example 7.5 we saw that we have to be more careful in defining the homotopy colimit if there are composable arrow in our diagram. To fix this we will use a construction very similar to the nerve of a category. The plan is to insert a another space for every sequence of composable morphisms.

**Definition 7.6.** Let $D$ be a diagram of topological spaces of shape $I$, then we define the simplicial replacement of $D$ as the simplicial space whose $n$th space is

$$\text{srep}(D)_n = \coprod_{i_0 \rightarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n),$$

i.e. the disjoint union of the spaces $D(i_j)$ together with strings of composable morphisms $i_0 \rightarrow \cdots \rightarrow i_n$. The face and degeneracy maps are defined similar as in the definition of the nerve by composing maps and inserting the identity map, see section 6.2 for more details about these maps. We denote the simplicial replacement of a diagram $D$ by $\text{srep}(D)$.

**Remark 7.7.** Note that the difference with the nerve of a small category is that we now have the disjoint union of spaces instead of a vertex for each object.

**Example 7.8.** The nerve of a small category $I$ is an example of the simplicial replacement of the diagram given by the functor $\mathcal{F} : I \rightarrow \text{Top}$ which sends every object in $I$ to the one point space $\{\ast\}$. 

102
Definition 7.9. Let $D$ be a diagram of topological spaces, then we define the homotopy colimit of $D$ as the geometric realization of the simplicial replacement of $D$. We denote this by

$$hocolim(D) = \left| \text{srep}(D) \right|.$$ 

Remark 7.10. There are several constructions of several different homotopy colimits, it can be shown that all these different constructions are however homotopy equivalent. Since the homotopy colimit is only unique up to homotopy it is not the case that the homotopy colimit satisfies any universal properties.

Proposition 7.11. Let $D$ and $D'$ be diagrams of spaces such that all the objects of the diagram are cofibrant, If there is an object wise weak equivalence between the diagrams, then the homotopy colimits $hocolim(D)$ and $hocolim(D')$ are also weakly equivalent.

The proof can be found in [12].

From the homotopy colimit we can also construct a canonical map to the normal colimit. This map will turn out to be important for the calculus of embeddings and therefore we will define it here. To define this map we first need to see that the colimit of a diagram $D$ is equal to a certain coequalizer in the simplicial replacement of $D$. To see this we first recall the definition of the coequalizer.

Definition 7.12. Let $f : X \to Y$ and $g : X \to Y$ be maps between topological spaces $X$ and $Y$ then we define the coequalizer of $f$ and $g$ as the space $Y/\sim$, where the relation $\sim$ is defined by $y_1 \sim y_2$ if there exists an $x \in X$ such that $f(x) = y_1$ and $g(x) = y_2$.

Note that the equalizer of the diagram with two arrows $f : X \xrightarrow{f} Y$ is equal to the colimit of that diagram. If we now note that the colimit of a diagram $D$ is just the coequalizer of the maps $d_0$ and $d_1$ in $\text{srep}(D)$, the simplicial replacement of $D$.

Proposition 7.13. There exists a canonical map $\alpha : hocolim(D) \to \text{colim}(D)$ defined by the map induced by the coequalizer.

7.3 Homotopy limits

We will now define homotopy limits, these are the dual construction of the homotopy colimit so many this will look similar. The main difference is that
we will not use the geometric realization but the totalization of a cosimplicial set. This makes everything more difficult to picture. The rest of the definitions are very similar.

**Definition 7.14.** Let $C$ be a small category and $c$ an object in $C$. We can define a new category $C \downarrow c$, this category has as objects the morphisms of $C$ which have $c$ as target. Let $c, d \in C$ be objects in $C$ and $f : b \to c$ and $g : d \to c$ be two objects of $C \downarrow c$, we define a morphism between these to objects to be the morphisms from $C$ which commute with $f$ and $g$.

With this category $C \downarrow c$ we have just constructed we can define a functor from $C$ to spaces. We do this in the following way, we send each object $d \in C$ to the geometric realization of the nerve of the category $C \downarrow d$. We denote the geometric realization of the nerve of a small category $C$ by $|C|$. On morphisms we have the following, let $f : c \to d$ be a morphism in $C$. This morphism $f$ induces a map between the category $C \downarrow c$ and the category $C \downarrow d$. This map will induce a morphism on the realization of the nerve of $C \downarrow c$ and the realization of the nerve of $C \downarrow d$.

**Definition 7.15.** Let $F$ be a functor from a small category $C$ to the category of space, then the homotopy limit of $F$ is $\text{Nat}(|C \downarrow -|, F)$, the space of all natural transformations from $|C \downarrow -|$ to $F$.

To end this section we will briefly summarize the properties of the homotopy colimit functor, these properties will be used extensively in the rest of this thesis.

**Proposition 7.16.** The homotopy limit has the following properties.

1. The homotopy limit is functorial.

2. If $F : D \to D'$ is a natural transformation of diagrams $D$ and $D'$ such that for each object $F$ is a weak equivalence, then the homotopy limits are also weakly equivalent.

3. There is a canonical map between the homotopy limit and the ordinary limit.

4. If a functor $F : D \to D'$ is left cofinal (see definition 7.17 then composition with $F$ induces a weak equivalence on the homotopy limits.

It often happens that we want to compare the homotopy limits of two diagrams of different shape. To compare the homotopy limits between those diagrams we introduce cofinal functors. For a functor being cofinal implies that the homotopy limits are weakly equivalent.
Definition 7.17. A functor $\mathcal{F} : D \to D'$ between two diagrams $D$ and $D'$ is called left cofinal if for each object $c \in D'$ the over category $(c \downarrow \mathcal{F})$ is non empty and connected.

In proposition 7.16 we see that a cofinal functor induces weak equivalences on the homotopy limits of the diagrams.
Chapter 8

The calculus of embeddings

In this chapter we will give a brief overview of the calculus of embeddings. The calculus of embeddings is a method of studying contravariant functors from some category $\mathcal{O}$ to topological spaces and was mainly developed by Goodwillie, Klein and Weiss. Since the calculus of embeddings is especially effective in the case of The plan is to first fix a manifold $M$ and then consider the functor $\text{Emb}(-, M)$, which sends a manifold $N$ to $\text{Emb}(N, M)$ the space of embeddings of $N$ in $M$. To do this we will do the following, let $F$ be a functor from some category $\mathcal{O}$ to spaces. Then we will try to construct a series of functors that approximate the functor $F$. This construction is similar to the Taylor approximation for analytic functions in calculus. Therefore we will call these simpler functors polynomial functors of degree $\leq k$. For every functor $F$ we can construct a tower of functors which will approximate $F$. In some case the inverse limit of this tower is equal to $F$ and then we can extract lots of information from this tower. The embedding functor will turn out to be analytic if the difference between the dimensions of the manifolds $M$ and $N$ is large enough.

This chapter is mainly focused on the construction of the Taylor tower of the embedding functor and is constructed as follows. In section 8.1 we will introduce cubical diagrams and state some of there properties. Then we will use these cubical diagrams to

This chapter is mainly based on [60], [61] and [21], for more details we refer to those papers. Another series of papers that might be interesting is [18], [19] and [20].

**Convention 8.1.** Since we will use the words contravariant functor a lot we will abbreviate them to *cofunctor*.
8.1 Cubical diagrams and polynomial cofunctors

Before we begin with defining polynomial we will first give some restrictions on the cofunctors we will work with. We will do this because restricting the cofunctors we work with, will have some technical advantages. To restrict the functors we work with we will first need some extra definitions which will make it possible to define what the good cofunctors are.

Definition 8.2. Let $M$ be a manifold and let $\mathcal{O}(M)$ be the poset of all open subsets of the manifold $M$. We can see this as a category with the open subsets of $M$ as objects and by defining one morphism between two objects $U$ and $V$ if $U \subseteq V$. We denote by $\mathcal{F}$ the space of all functors from the category $\mathcal{O}(M)$ to the category of spaces.

Definition 8.3. Let $i_1 : V \rightarrow W$ be codimension zero embeddings of smooth manifolds, then we call $i_1$ an isotopy equivalence if there exists a smooth embedding $i_2 : W \rightarrow V$ such that $i_1i_2$ is isotopic to $Id_W$ and $i_2i_1$ is isotopic to $Id_V$.

Definition 8.4. Let $F$ and $G$ be cofunctors from some small category $I$ to spaces, then we will call $F$ and $G$ equivalent if there there exists a natural transformation $\eta : F \rightarrow G$ such that $\eta_a : F(a) \rightarrow G(a)$ is a weak equivalence for all objects $a \in I$.

We will now restrict our functors to good functors which we will define in the following definition.

Definition 8.5. A cofunctor $F$ is called good if:
1. $F$ takes isotopy equivalences to homotopy equivalences,
2. for any sequence $\{V_i, i \geq 0\}$ of objects in $\mathcal{O}$ such that $V_i \subset V_{i+1}$ the canonical map $F(\cup_i V_i) \rightarrow \text{holim}_i F(V_i)$ is a homotopy equivalence.

Definition 8.6. Let $\mathcal{F}$ be the category of all good cofunctors $F$ from $\mathcal{O}$ to spaces. The morphisms in this category are the natural transformations between the good cofunctors. We will call two objects $F$ and $G$ equivalent in $\mathcal{F}$ if there is a chain of equivalences connecting $F$ and $G$.

Two examples of good cofunctors are given by the immersion and the embedding functors $\text{Imm}(\cdot, M)$ and $\text{Emb}(\cdot, M)$ for some manifold $M$. The functor $\text{Imm}(\cdot M)$ is defined as the functor from spaces to spaces that sends a space $N$ to $\text{Imm}(N, M)$, the space of immersions from $N$ to $M$. The embedding functor is also a functor from spaces to spaces and sends a space $N$ to the space $\text{Emb}(N, M)$, the space embeddings of $N$ into $M$. 
Proposition 8.7. The cofunctors $Imm(-, M)$ and $Emb(-, M)$ are good.

The proof of this proposition can be found in [61]. We will now define what it means for a cofunctor to be polynomial. We will do this by using cubical diagrams. We can think about a cubical diagram as a diagram of spaces in the form of a cube, a more formal definition is given as follows.

**Definition 8.8.** Let $S$ be a finite set of $k$ elements and let $\mathcal{P}(S)$ be the category with as objects all subsets of $S$, including the empty set, and inclusions as morphisms. Denote by $\mathcal{P}_\nu(S)$ the full subcategory of $\mathcal{P}$ consisting of all nonempty subsets.

**Definition 8.9.** A cubical diagram is a cofunctor $F$ from $\mathcal{P}(n)$ to the category of spaces, where $n$ is the set $1, \ldots, n$.

**Definition 8.10.** A cubical diagram is called homotopy Cartesian (or Cartesian for short) if the map from $F(\emptyset)$ to $\text{holim}_{S \in \mathcal{P}_\nu(n)} F(S)$ is a homotopy equivalence. With $\text{holim}_{S \in \mathcal{P}_\nu(n)} F(S)$ we mean the restriction of $F$ to the the subcategory $\mathcal{P}_\nu(n)$.

**Definition 8.11.** A cofunctor $F$ is called polynomial of degree $\leq k$ if $F(\mathcal{P}(n))$ is homotopy Cartesian for all $n \geq k + 1$.

Informally we can think of a polynomial cofunctor $F$ as a cofunctor from which we can reconstruct $F(\emptyset)$, from knowing its restriction to $\mathcal{P}_\nu$. We will use this in the following way for studying the embedding functor $Emb(M, N)$. Let $M$ and $N$ be manifolds and let $V$ be an open subset in $M$. If we then pick a set $S = \{A_1, \ldots, A_n\}$ of $n$ pairwise disjoint closed subsets $A_i$ of a manifold $V$. Then we can define a cubical diagram, i.e. a functor from $\mathcal{P}(n+1)$ to spaces, by sending a subset $s \subset S$ given by $s = \{V_{s_1}, \ldots, V_{s_i}\}$ to the space $Emb(V - \bigcup_s A_s, N)$. So if a functor is polynomial of degree $\leq n$ we can reconstruct, up to homotopy, the value of the functor on the empty set from all the other sets. We will now give some examples of polynomial functors.

**Example 8.12.** The constant functor that send everything to the same object is polynomial of degree 0. This can be seen as follows, the 1 cube has two vertices. If you the value on one of the vertices and also know that the functor assumes the same value on every object then you indeed know what the value on the other vertex is. It is easy to see that this is the only cofunctor that is polynomial of degree 1

**Proposition 8.13.** The immersion functor $Imm(-, M)$ is polynomial of degree $\leq 1$. 

108
The proof of this proposition can be found in [61]. The next lemma may sound rather trivial but the proof which can be found in [19] is far from trivial.

Lemma 8.14. If a functor $F$ is polynomial of degree $\leq k$ it is also polynomial of degree $\leq k + 1$.

The plan is to try to approximate functors that are not polynomial by polynomial functors. This approximation has certain similarities with the concept of the Taylor series of an analytic function and will therefore be called the Taylor tower of a functor $F$. To approximate the embedding functor we will first introduce a large class of polynomial functors which will be useful in what follows.

Definition 8.15. Recall that $O(M)$ is the category whose objects are open subset of $M$ with inclusions as morphism. Then we define $O_k(M)$ as the category of open subsets of $M$ that are diffeomorphic to the disjoint union of at most $k$ open balls.

Proposition 8.16. Let $M$ be an $m$-dimensional manifold with no compact components then the cofunctor $M \rightarrow \text{Emb(}\coprod_k \mathbb{R}^m, N\text{)}$ is polynomial of degree $\leq k$.

The proof of this proposition can be found in [61]. In general we can restrict polynomial cofunctors of degree $\leq k$ to the category $O_k(M)$, it turns out that a polynomial cofunctor of degree $\leq k$ is ”determined” by its values on $O_k(M)$. This will be made precise in the following theorem, the proof of the theorem can be found in [61].

Theorem 8.17. Let $w : E \rightarrow F$ be a natural transformation between polynomial cofunctors $E : M \rightarrow N$ and $F : M \rightarrow N$ of degree $\leq k$. If the restriction of $w$, $E$ and $F$ to $O_k(M)$, $w \mid_{O_k(M)} : E \mid_{O_k(M)} \rightarrow F \mid_{O_k(M)}$ is an equivalence then $w$ is an equivalence.

Remember that the plan was to approximate general functors by simpler polynomial functors, we will now begin with explaining how we can approximate cofunctors by simpler ones and how we can build Taylor towers. The next theorem gives us a method to extend polynomial cofunctors of degree $\leq k$ from $O_k(M)$ to spaces to cofunctors from $O(M)$ to spaces. This is an important result since it implies that once you know what a polynomial cofunctor of degree $\leq k$ does on $O_k$ it is completely determined.

Theorem 8.18. Let $E$ be a cofunctor from $O_k(M)$ to spaces such that $E$ takes isotopy equivalences to homotopy equivalences. Then we can extend $E$ to a cofunctor $E^l$ from $O(M)$ to spaces by defining
The cofunctor $E'$ is polynomial of degree $\leq k$ and $E'|_{\mathcal{O}_k}$ is equivalent to $E$.

See [61] for a proof. By using the previous theorem we can define the $k$th-Taylor approximation of a cofunctor in the following way.

**Definition 8.19.** Let $F$ be a cofunctor from $\mathcal{O}$ to spaces, the $k$th-Taylor approximation of $F$ denoted by $T_kF$ is the cofunctor

$$T_kF = (F|_{\mathcal{O}_k})^!.$$

This functor $F$ is polynomial of degree $\leq k$ and comes together with a canonical natural transformation $\eta_k : F \to T_kF$.

**Theorem 8.20.** Let $\mathcal{F}$ be the category of all cofunctors from $\mathcal{O}$ to spaces, the Taylor approximation functor $T_k : \mathcal{F} \to \mathcal{F}$ and the corresponding natural transformation have the following properties.

1. The functor $T_k$ takes equivalences to equivalences.
2. The functor $T_k$ is polynomial of degree $\leq k$ for all $F \in \mathcal{F}$.
3. If a cofunctor $F$ is polynomial of degree $\leq k$ then $\eta : F \to T_kF$ is an equivalence.
4. The map $T_k(\eta) : T_kF \to T_kT_kF$ is an equivalence for all $F \in \mathcal{F}$.

The proof of this theorem can be found in [61]. So for every functor $F$ we can define a series of functors $T_kF$, note that there is a restriction map $r_k : T_kF \to T_{k-1}F$. This restriction map turns out to be a fibration. So for every functor we can define a tower of fibrations which we shall call the Taylor tower of $F$.

**Definition 8.21.** We call the tower of fibrations $T_kF$ the Taylor tower for a cofunctor $F$. If $F$ is weakly equivalent to the homotopy limit of this tower then we call the cofunctor $F$ analytic.

So an analytic cofunctor is a functor that can be approximated by the simpler polynomial functors. An example of an analytic functor is the embedding functor $Emb(M, N)$ if the difference of dimension between $M$ and $N$ is large enough. It is now time to understand the fibers of the Taylor tower a little better. To do this we define the layers of the Taylor tower and homogeneous cofunctors. These layers and homogeneous cofunctors are the homotopy fibers of the Taylor tower.
Definition 8.22. We define $L_k F$ the $k$th layer of the Taylor tower as

$$L_k F = hfiber[T_k F \xrightarrow{r_k} T_{k-1} F].$$

These layers are examples of the more general concept of homogeneous cofunctors.

Definition 8.23. Let $F$ be a polynomial cofunctor of degree $\leq k$, we call $F$ homogeneous of degree $K$ if $T_{k-1} F(V)$ is contractible for all $V \in \mathcal{O}$.

We can classify all homogeneous cofunctors by the cofunctors in the following example.

Example 8.24. Let $\binom{M}{k}$ be the space of all subsets of $M$ of order $k$, i.e. $C_k(M)/S_k$ the configuration space of $k$ points in $M$ modulo the action the symmetric group $S_k$. Let $p : Z \to \binom{M}{k}$ be a fibration together with a (partial) section $s : \binom{M}{k} \cap Q \to Z$ for some neighborhood of the fat diagonal $Q$. Then we define a functor $F$ by defining $F(V)$ as the space of all sections $t : \binom{M}{k} \to Z$ of $p : Z \to \binom{M}{k}$, such that $t$ agrees with $s$ on $\binom{M}{k} \cap Q'$ for some $Q' \subset Q$. This functor is a homogeneous polynomial cofunctor of degree $\leq k$.

Theorem 8.25. Let $F \in \mathcal{F}$ be a homogeneous cofunctor of degree $k$, then there is an equivalence between $F$ and one of the functors described in the previous example.

The proof of this theorem can be found in [61]. We will conclude this chapter with the main theorem of this chapter which will have as consequence that the embedding functor is analytic if the difference between $M$ and $N$ is large enough. The proof of the theorem can be found in [16].

Theorem 8.26. Let $M$ be an $m$ dimensional manifold and $N$ be an $n$ dimensional manifold, if $V \in \mathcal{O}(M)$ and $n - m \geq 3$ then the cofunctor $Emb(V, N)$ is analytic.
Chapter 9

Sinha’s cosimplicial model for the space of knots

In chapter 1 we have seen a method to calculate the homology of the space of knots. In this chapter we will present another approach for calculating the homology and homotopy groups of the space of knots. This approach will construct a cosimplicial model for the space of knots, this cosimplicial model was constructed by Sinha and is therefore called Sinha’s cosimplicial model. The concept behind his construction is to alter the cosimplicial model for the loop space from section 6.5 in chapter 6 in such a way that double points are no longer allowed. Recall that the loop space model for a topological space $M$ is the totalization of the cosimplicial space which as $n$-th space $M^n$, the coface maps $s_i$ are forgetting the $i$th-point and the face maps $d_i$ are doubling the $i$th point or inserting the base point. A logical modification of this model in the case of knots would be to replace $M^n$ by the configuration space of $n$ distinct ordered points in $M$. There are several problems with this approach, first of all there are no obvious codegeneracy maps since doubling a point is no longer possible. The second problem is that in the homotopy category the map $ev_n$ that evaluates the knot at $n$ points is equal to the map that evaluates the knot at one point, so it does not contain more information than evaluating the knot at one point. Both these problems can be solved by inserting boundary conditions on our configuration spaces by completing them in a certain way. The first problem of the diagonal maps is solved by allowing double points in the configuration. Since we use the Fulton MacPherson and simplicial completion this now becomes possible and we have an evaluation map that is in the homotopy category no longer equivalent to the map that evaluates the knot at one point.

In this chapter we will not work with the usual long knots but with a small modification of the space of long knots.
Definition 9.1. In this chapter a knot will mean an embedding of the unit interval $I$ in the cube $I^n$ with some fixed boundary points with fixed tangent vectors. In other words a knot is a function $f : I \to I^n$ such that $f(0)$, $f(1)$, $f'(0)$ and $f'(1)$ are fixed.

Lemma 9.2. The space of long knots in $\mathbb{R}^n$ and the space of long knots in the cube are homotopy equivalent.

Proof. A possible way to see that these spaces are homotopic is too first include the unit cube in $\mathbb{R}^n$ and then extend the long knot in the cube to a long knot in $R^3$ by adding two lines one that starts at the end point of $f$ and goes to infinity coinciding with the fixed linear embedding $\epsilon$. The other line goes from minus infinity along $\epsilon$ and goes to the starting point of our long knot $f$. So we have an inclusion of the space of long knots in the cube and the normal long knots. We can see that these spaces are homotopic by scaling $\mathbb{R}^n$ to the interior of the cube which will scale all the long knots in $\mathbb{R}^n$ to a long knot in the cube. So this is a homotopy between the space of long knots in $\mathbb{R}^3$ and the space of long knots in the cube.

9.1 The mapping space model

Before we will define the cosimplicial model for the space of knots we will first define another model for the space of knots that will be closely related to the cosimplicial model, we will call this model the mapping space model. Recall from chapter 8 the definition of the Taylor tower of the embedding functor. In this section we will construct a series of spaces $AM_n(M)$ which will be weakly homotopy equivalent to the $n$th component of the Taylor tower of the space of knots $P_nEmb(I, M)$. We will define the space $AM_n(M)$ as the space of a special kind of maps between $C'_n[I, \partial]$ and $C'_n[M, \partial]$, where $C'_n[I, \partial]$ and $C'_n[M, \partial]$ are certain completions of the configuration spaces of $I$ and $M$. In this section we will first define the spaces $P_nEmb(I, M)$ after that we will recall the Fulton-MacPherson and the simplicial completion of a configuration space. The cosimplicial models will be based on these completions combined with the cosimplicial model for the loop space. After we have defined the right spaces we will define a stratification on these spaces, which will help us to understand these spaces better. Then we will modify these models to impose some boundary conditions. Finally we will define the special kind of maps, which we will call aligned maps, that will give us Sinha’s mapping space model.
9.1.1 The Taylor tower of the functor $\text{Emb}(I, M)$

We will now give an explicit construction of the Taylor tower of the space of long knots. Remember that the idea behind the calculus of embeddings is to approximate the embedding functor by functors that are easier to understand. These easier functors are the embedding functors of $k$ disjoint open $n$-disks in the target manifold $M$. Note that in the case of knots the open disks are just disjoint open intervals. Since all sets of disjoint open intervals of the interval are diffeomorphic we can just fix a set of open intervals of the unit interval $I$ and study these.

**Definition 9.3.** Let $I = [0, 1]$ be the interval and define the following collection of subsets $J_i = \left(\frac{1}{2^i}, \frac{1}{2^i + \frac{1}{10^i}}\right)$.

With these fixed subsets we are able to define the polynomial approximations of the embedding functor $\text{Emb}(I, M)$. We will begin with constructing the cubical diagrams from definition 8.9.

**Definition 9.4.** Let $S \subseteq n$ then we define $E_S(M)$ to be the space of all embeddings of $I - (\cup s \in SJ_i)$ in $M$ such that the speed of is constant on each component of $I - (\cup s \in SJ_i)$. We give this space the $C^1$ topology defined in definition 1.8.

Recall from definition 8.9 that $\mathcal{P}(n)$ denotes the cube and is the poset of subset of $n$ and that $\mathcal{P}_\nu(n)$ is the cube without the empty set.

**Definition 9.5.** Let $E_n(M)$ be the cubical diagram with the space $E_S(M)$ at the vertex corresponding to the set $S$. Let $E$ be the diagram corresponding to $\mathcal{P}_\nu(\setminus)$, the cube without the empty set.

So we can define the $n$th Taylor approximation of the space of knots as follows.

**Definition 9.6.** The $n$th Taylor approximation $P_n \text{Emb}(I, M)$ for $\text{Emb}(I, M)$, the space of long knots in a manifold $M$ is the homotopy limit of the diagram $E_n(M)$.

9.1.2 The Fulton-MacPherson completion

We begin to recall the definitions of the Fulton-MacPherson and simplicial completions from chapter 4. Recall that $C_n(M)$ denotes the configuration space of $n$ distinct points in a manifold $M$. If $S$ is a finite set we define $C_n(S)$ to be the set of all ordered subsets $S$ of pairwise disjoint points of cardinality $n$. Also recall that $I$ denotes both the interval and the one point
compactification of the set $[0, \infty)$ since these spaces are diffeomorphic we will denote them by the same letter, from the context it will be clear to which one we refer.

**Definition 9.7.** Let $M$ be a manifold, such that $M$ is isometrically embedded into $\mathbb{R}^{N+1}$. Then we define the following spaces.

1. Let $C_n(M)$ be the configuration space of $n$ distinct points in $M$, i.e.
   
   $$C_n(M) = M^n - \{(x_1, ..., x_n) \in M^n \mid x_i \neq x_j \forall i \neq j\}$$

2. Let $A_n[M]$ be the space $M^2 \times (S^N)^{C_2(\mathbb{R})} \times I^{C_3(\mathbb{R})}$.

3. Let $A_n([M])$ be the space $M^2 \times (S^N)^{C_2(\mathbb{R})}$.

4. Let $\iota: C_n(M) \to A_n[M]$ and $\iota: C_n(M) \to A_n([M])$ be the inclusion maps of $C_n(M)$ in the spaces $A_n[M]$ and $A_n([M])$.

5. Let $\pi_{ij}: C_n(M) \to S^N$ be the map given by $\pi_{ij}(x_1, ..., x_n) = \frac{x_i-x_j}{|x_i-x_j|}$.

6. Let $s_{ijk}: C_n(M) \to I$ be the map that sends $(x_1, ..., x_n)$ to $\frac{|x_i-x_j|}{|x_i-x_k|}$.

7. The Fulton-MacPherson completion of $C_n(M)$ is the closure of the image of the map $\alpha_n = \iota \times \prod_{(i,j)} \pi_{ij}(\pi_{ij}) \times \prod_{(i,j,k)} (s_{ijk})$.

8. The simplicial completion of the configuration space $C_n(M)$ is the closure of the map $\beta_n = \iota \times \prod_{(i,j)} (\pi_{ij})$.

Remember the conventions from 4 about the notation of maps.

**Notation 9.8.** To make notation simpler and to avoid writing down expressions like $(x_1, ..., x_n) \times \prod_{(i,j)} C_2(\mathbb{R}) (u_{ij}) \times \prod_{(i,j,k)} C_3(\mathbb{R}) (d_{ijk}) \in M^2 \times (S^N)^{C_2(\mathbb{R})} \times I^{C_3(\mathbb{R})}$, we will use the following conventions in the notation. On $A_n[M]$ and $C_n[M]$ we denote by $(x_i \times (u_{ij} \times (d_{ijk})$ the coordinate $(x_1, ..., x_n) \times \prod_{(i,j)} C_2(\mathbb{R}) (u_{ij}) \times \prod_{(i,j,k)} C_3(\mathbb{R}) (d_{ijk})$. On $A_n([M])$ and $C_n([M])$ we denote the coordinates by $(x_i \times (u_{ij})$ instead of $(x_1, ..., x_n) \times \prod_{(i,j)} C_2(\mathbb{R}) (u_{ij})$. On maps we will use the notation that $(f_i) \times (g_{ij}) \times (h_{ijk}) : M^2 \times (S^N)^{C_2(\mathbb{R})} \times I^{C_3(\mathbb{R})} \to M^2 \times (S^N)^{C_2(\mathbb{R})} \times I^{C_3(\mathbb{R})}$ is the map which would look like $(f_1, ..., f_n) \times \prod_{(i,j)} C_2(\mathbb{R}) (g_{ij}) \times \prod_{(i,j,k)} C_3(\mathbb{R}) (h_{ijk})$. There is a similar convention for maps involving the simplicial completion.

**Proposition 9.9.** The forgetful map $p: A_n[M] \to A_n([M])$ that forgets the $I^{C_3(\mathbb{R})}$ part is a homotopy equivalence. This map extends to a homotopy equivalence between the Fulton-MacPherson and the simplicial completion of a configuration space.

The proof of this proposition can be found in [48].
9.1.3 The category of $f$-trees

To make the completions of configuration spaces we are working with more understandable we will define a stratification on the completions of the configuration spaces. To do this it is first necessary to introduce the category of $f$-trees. In the next section we will see that every $f$-tree corresponds to a stratum of our stratification. We will begin by recalling the definition of a stratification.

**Definition 9.10.** A *stratification* of a space $M$ is a collection of disjoint subspaces $\{M_e\}$, which we will call the strata, such that the intersection of the closure of any two strata is the closure of some stratum. To every stratification we associate a poset by defining $M_e \leq M_d$ if $M_e$ is contained in the closure of $M_d$.

We will now define $f$-trees, these will correspond to the strata of the stratification.

**Definition 9.11.** An $f$-tree is an acyclic connected graph with one special vertex $v_r$ called the root, such that there are no bivalent vertices except for possibly the root. The univalent non-root vertices are called the leaves of the $f$-tree. We also require that the leaves, except for the root if it is univalent, are labeled by the numbers $\{1, \ldots, n\}$.

The next thing we do is define a category $\Psi_n$ whose objects are $f$-trees with exactly $n$ non root leaves.

**Definition 9.12.** Let $\Psi_n$ be the category whose objects are all $f$-trees with $n$ non root leaves and with the following set of morphisms. We define a morphism between two $f$-trees $T$ and $T'$ if $T'$ is obtained by contracting a set edges of $T$. The contraction of an edge $e$ is defined by removing the edge and identifying the end vertices of that edge.

In figure 9.1.3 we see an example of the category $\Psi_3$. The next thing we need for the stratification is some terminology about trees.

**Definition 9.13.** Let $T$ be an $f$-tree with $n$ leaves, then we have the following definitions about the $f$-tree $T$.

- The root path of a leaf is the unique path to the root.
- A leaf $l$ lies over a vertex $v$ if $v$ is contained in the root path of $l$, i.e. the path from $l$ to the root passes through $v$.
- Two leaves $i$ and $j$ exclude a leave $k$ if there exists a vertex $v$ such that $i$ and $j$ lie over $v$ but $k$ does not lie over $v.$

116
For every \( f \)-tree \( T \) we define \( \text{Ext}(T) \) to be the subset of \( n^3 \) of all triples \((i, j, k)\) such that \( i \) and \( j \) exclude \( k \).

Please note that if \( \text{Ext}(T) = \text{Ext}(T') \) for some \( f \)-trees \( T \) and \( T' \) then either \( T \) and \( T' \) are isomorphic or one is obtained from the other by adding an edge to the root vertex. The category \( \Psi_n \) has a subcategory which we will define below which will be useful in the future.

**Definition 9.14.**

1. We call a set of leaves \( S \in \Sigma \) consecutive if for all \( i, j \in S \) we have that all \( k \) such that \( i < k < j \) we have that \( k \) is an element of \( S \).

2. Let \( \Psi_n^0 \) be the full subcategory of \( \Psi_n \) consisting of all \( f \)-trees \( T \) such that set of leaves over any vertex is consecutive and the root is not univalent.

Note that an \( f \)-tree \( T \in \Psi_n^0 \) can be embedded in the upper half plane, such that the root is at the

**9.1.4 A stratification of the completions of configuration spaces**

We will use the trees from the previous section to define a stratification on the Fulton-MacPherson completion. Since The Fulton-MacPherson completion
is a manifold with corners we can also define another stratification. This stratification coincides with the stratification by trees which we will prove in proposition 9.19

**Definition 9.15.** Let $M$ be a manifold with corners, i.e. a manifold for which the boundary is locally diffeomorphic to the space $(\mathbb{R}_{\geq})^{d_1} \times \mathbb{R}^{d_2}$ ($d_1$ and $d_2$ may vary from point to point). In this case $M$ is stratified in the following way. A stratum consists of one of the connected components of the set of all the points $x \in M$ such that the neighborhoods of $x$ locally looks like $(\mathbb{R}_{\geq})^{d_1} \times \mathbb{R}^{d_2}$ for a fixed $d_1$ and $d_2$.

**Example 9.16.** Let $M$ be $|\Delta_n|$ the $n$-dimensional standard simplex, this is a manifold with corners and the stratification is given as follows. The first strata are the vertices, the next set of strata is the set of all lines connecting the vertices. In general we define the set of the $i$th strata as the set of all $i$-dimensional faces.

Next we will define the stratification by trees. The idea behind this stratification is that we look at in the set $C_n[M] - C_n(M)$, i.e. the set of all points we have added to complete the configuration space. A point in this set can be seen as several points of our original configuration space colliding and collapsing onto each other. We can use trees to see how many points are colliding and in which order they collide.

**Definition 9.17.** To every point $(x_i) \times (u_{ij}) \times (s_{ijk}) \in C_n[M]$ we are able to associate an $f$-tree $T(x)$ Let $Ex(T(x)) \subset n^3$ be the set of all leaves $i$ and
that exclude \( k \). Then to a point \( x \) we associate the \( f \)-tree that has as set \( \text{Ex}(T) \) given by \((i,j,k) \in \text{Ex}(T)\) if \( d_{ijk} = 0 \). This defines an \( f \)-tree which is not unique, but if we define the \( f \)-tree \( T \) to be univalent if an only if all the \( x_i \) are equal, this assigns to every point an \( f \)-tree. Let \( C_T(M) \) be the subspace of \( C_n[M] \) of all points \( x \in C_n[M] \) such that \( T(x) = T \). Let \( C_T[M] \) be the closure of \( C_T(M) \), this defines a stratification on \( C_n[M] \) with the strata given by the subspaces \( C_T[M] \).

**Lemma 9.18.** The stratification defined in definition 9.17 is indeed a stratification.

We will omit the proof of this lemma, but the interested reader can find the details in [48]. The next proposition will show that the stratifications we have just defined are actually the same.

**Proposition 9.19.** The stratifications defined in definition 9.15 and definition 9.17 coincide.

For the proof see [48]. We will now see in the following definition and proposition how the category \( \Psi_n \) is related to the stratification of \( C_n[M] \).

**Definition 9.20.** To a stratification we can associate a poset by defining a stratum \( \alpha \) to be smaller than a stratum \( \beta \) is \( \alpha \subset \overline{\beta} \), i.e. if \( \alpha \) is contained in the closure of \( \beta \).

**Proposition 9.21.** The poset associated to the stratification of \( C_n[M] \) by trees is isomorphic to the category of \( f \)-trees.

The proof of this proposition can be found in [48].

### 9.1.5 Modifying the completions of configuration spaces to take the boundary conditions and the tangent vectors into account

The completions from the previous section are a good start for our model of the space of knots, but we need to modify these spaces to also take care of the boundary conditions and the tangent vectors. Remember that we have assumed that a knot is an embedding of the interval \( I = [0,1] \) into the cube with constant velocity and some fixed boundary points and tangent vectors. Therefore we only have to consider the unit tangent bundle.

We will first take care of the tangent bundle and then we shall explain how we take care of the boundary conditions. We will first give a more general definition of how to add a tangent bundle to a space.
Definition 9.22. Let \( X(M) \) be a space together with a map \( f : X(M) \to M^n \) for some \( n \geq 1 \). Then we define the space \( X'(M) \) to be the pull-back of the following square:

\[
\begin{array}{ccc}
X'(M) & \to & (STM)^n \\
\downarrow & & \downarrow p \\
X(M) & \xrightarrow{f} & M^n
\end{array}
\]

Let \( STM \) be the unit tangent bundle of \( M \), i.e. the following subspace of the tangent bundle \( STM = \{ x \in TM \mid d(x, x) = 1 \} \). Then we define the map \( p : (STM)^n \to M^n \) as the projection map from \( STM^n \) to \( M^n \). If \( g : X(M) \to Y(M) \) is a map of two spaces \( X(M) \) and \( Y(M) \) of the form described above then we denote induced map on pull-backs by \( g' : X'(M) \to Y'(M) \).

Note that this definition specially makes sense for configuration spaces, their completions and the spaces \( A_n[M] \) and \( A\langle [M] \rangle \). With this definition we can now define the spaces \( C_n'[M] \) and \( C_n'\langle [M] \rangle \) as the Fulton-MacPherson completed and simplicial completed configurations spaces with tangent vectors. We also have to take care of the boundary conditions. We will do this by adding two points to our configuration spaces. The first point will be added at the first place of the configuration and the second point will be added as the last point of the configuration. Formally we will do this as follows.

Definition 9.23. Let \([n] = \{0, ..., n\}\) and let \( y_0 \) and \( y_1 \) be fixed points on the boundary of \( M \) together with unit tangent vectors \( v_0 \) pointing inwards \( M \) and \( v_1 \) pointing outwards of \( M \). Then we will construct the spaces \( C_n[M, \partial] \), \( C_n[M, \partial] \langle [M] \rangle \), \( C_n'[M, \partial] \) and \( C_n'[M, \partial] \langle [M] \rangle \) as follows.

1. Let \( A_n[M, \partial] \) be the space \( M^{[n+1]} \times (S^N)^{C_n([n+1])} \times I^{C_n([n+1])} \) and let \( A_n[M] = M^{[n+1]} \times (S^N)^{C_n([n+1])} \), i.e. these spaces are just the ambient spaces of the completions of \( n + 2 \) points.

2. Let \( i^0 : C_n(M - (y_0 \cup y_1)) \to M^{[n+1]} \) be the inclusion map of \( C_n(M - (y_0 \cup y_1)) \) in \( M^{[n+1]} \) given by the following formula:

\[
i^0((x_1, ..., x_n)) = (y_0, x_1, ..., x_n, y_1).
\]

3. Let the maps \( \pi^\partial_{ij} \) and \( s^\partial_{ijk} \) be the modified maps from definition 9.7, given by \( \pi^\partial_{ij} = \frac{x_i - x_j}{|x_i - x_j|} \) and \( s^\partial_{ijk} = \frac{|x_i - x_j|}{|x_i - x_k|} \).

4. Let \( \gamma : C_n(M - (y_0 \cup y_1)) \to A_n[M, \partial] \) be the map given by \( \gamma = i^0 \times \pi^\partial_{ij} \times s^\partial_{ijk} \), we define \( C[M, \partial] \) as the closure of the image of this map.
5. Let $\delta : C_n(M - (y_0 \cup y_1)) \to A_n([M, \partial])$ be the map $\delta = \iota^0 \times \pi_{ij}$, the space $C_n([M, \partial])$ is the closure of the image of the map $\delta$.

6. Let $(STM)^n_0$ be the subspace of the $[n+1]$-fold tangent bundle $(STM)^{[n+1]}$ such that the first tangent vector is $v_0$ and the last tangent vector is $v_1$.

7. The Fulton-MacPherson completion of the tangent space of $n$ distinct ordered points in a manifold $M$ including boundary conditions and tangent vectors is the pull-back of the following square:

$$
\begin{align*}
C'_n[M, \partial] & \to (STM)^n_0 \\
\downarrow & \downarrow p \\
C_n[M, \partial] & \xrightarrow{\iota^0} M^{[n+1]}.
\end{align*}
$$

8. The simplicial completion including boundary conditions and tangent vectors is the pull-back of this square:

$$
\begin{align*}
C'_n([M, \partial]) & \to (STM)^n_0 \\
\downarrow & \downarrow p \\
C_n([M, \partial]) & \xrightarrow{\iota^0} M^{[n+1]}.
\end{align*}
$$

9.1.6 Projection and diagonal maps for the simplicial completion

For defining a cosimplicial model of the space of knots we need a series of spaces together with coface and codegeneracy maps. For the cosimplicial model of the loop space these maps where defined to be diagonal maps, which double one of the points, and the projection maps, which forget one of the points. In the case of configuration spaces $C_n(M)$ we would like to define similar maps, unfortunately it is not possible to double a point, since this double point does not belong to the configuration space $C_{n+1}(M)$. Because we have completed our configuration spaces we will find a way to get around this problem and we will show how we can define a map similar to the diagonal map in the cosimplicial model of the loop space. We begin with defining the projection maps since these are much easier then the diagonal maps.

**Definition 9.24.** Let $S$ and $R$ be finite sets and let $\sigma : S \to R$ be a map of sets. Let $\text{Map}(S, X)$ be the contravariant functor from sets to spaces.
which sends $S$ to the space of all maps from $S$ to $X$. Another notation for $\text{Map}(S, X)$ is $X^S$. Then we define the map $p_\sigma^X : \text{Map}(R, X) \to \text{Map}(S, X)$ given by $p_\sigma^X((x_i)_{i\in R}) = (x_{\sigma(j)})_{j\in S}$. If the space $X$ is clear from the context we sometimes omit the $X$ in $p_\sigma^X$ and just write $p_\sigma$.

We would like to extend this definition to the configuration spaces $C_n^\prime [M, \partial]$ and $C_n^\prime ([M, \partial])$. We will do this by first extending the maps from definition 9.24 to the spaces $A_n [M]$ and $A_n ([M])$ and there possible modifications. Since these spaces are just products of spaces of the form $\text{Map}(n, X)$ this extension is easy to define, we only need to show that these maps behave well enough with respect to "inclusion" maps of the space $C_n(M)$. This leads to the following definitions. We will treat the case of the simplicial completion $C_n^\prime ([M, \partial])$.

First observe that the space $A_n^\prime ([\mathbb{R}^{N+1}])$ is equal to the space $(\mathbb{R}^{N+1} \times S^N) \times (S^N)^{C_2(\omega)}$. We will identify this space with the space $(\mathbb{R}^{N+1}) \times (S^N)^{2^N}$. Note that this space is also equal to the space $\text{Map}(n, \mathbb{R}^{N+1}) \times \text{Map}(n^2, S^N)$. In this space we can define the maps from definition 9.24.

**Definition 9.25.**  
- Let $\sigma : n \to m$ be a map of sets. In this case we define the following map between $A_n^\prime ([\mathbb{R}^{N+1}])$ and $A_m^\prime ([\mathbb{R}^{N+1}])$. In the case without boundary we get $A_{\sigma} : A_n^\prime ([\mathbb{R}^{N+1}]) \to A_m^\prime ([\mathbb{R}^{N+1}])$ as $A_{\sigma} = p_{\sigma}^{\mathbb{R}^{N+1}} \times p_{\sigma}^{S^N}$.

- In the case when we consider $A_n^\prime ([\mathbb{R}^{N+1}, \partial])$ we do the following. Let $\sigma : [n+1] \to [m+1]$ be a map that sends 0 to 0 and $n+1$ to $m+1$. Then we define the map $A_{\sigma}^\partial : A_m^\prime ([\mathbb{R}^{N+1}, \partial]) \to A_n^\prime ([\mathbb{R}^{N+1}, \partial])$ as $A_{\sigma}^\partial = p_{\sigma}^{\mathbb{R}^{N+1}} \times p_{\sigma}^{S^N}$.

**Remark 9.26.** We will call a map $\sigma : [n+1] \to [m+1]$ such that 0 is send to 0 and $n+1$ to $m+1$ boundary preserving.

The obvious restriction of these maps is given in the following definition.

**Definition 9.27.**  
- Let $F_{\sigma} : C_n^\prime ([\mathbb{R}^{N+1}]) \to C_n^\prime ([\mathbb{R}^{N+1}])$ be the restriction of the map $A_{\sigma}$ from $A_n^\prime ([\mathbb{R}^{N+1}])$ to $A_n^\prime ([\mathbb{R}^{N+1}])$ to a map between $C_n^\prime ([\mathbb{R}^{N+1}])$ and $C_n^\prime ([\mathbb{R}^{N+1}])$.

- Let $F_{\sigma}^\partial : C_n^\prime ([\mathbb{R}^{N+1}, \partial]) \to C_n^\prime ([\mathbb{R}^{N+1}, \partial])$ be the restriction of the map $A_{\sigma}^\partial$ from $A_n^\prime ([\mathbb{R}^{N+1}, \partial])$ to $A_n^\prime ([\mathbb{R}^{N+1}, \partial])$ to a map between the spaces $C_n^\prime ([\mathbb{R}^{N+1}, \partial])$ and $C_n^\prime ([\mathbb{R}^{N+1}, \partial])$. 

122
Remark 9.28. Since our manifold $M$ is embedded in $\mathbb{R}^{N+1}$ we can easily extend all the definitions we just made to the case of an arbitrary manifold by restricting the maps to the subspace $M$ of $\mathbb{R}^{N+1}$.

Lemma 9.29.

With these diagonal maps we will now define a cosimplicial space. In section 9.3 we will show that the totalization of this cosimplicial space is weakly equivalent to the space of knots.

Proposition 9.30. Let $\Delta$ be the category of all finite set of the form $n$ with the maps $d^i$ and $s^i$ from definition 6.1. Then we define a covariant functor $C^\bullet \langle [M] \rangle$ from $\Delta$ to spaces In the following way.

1. On objects the functor $C^\bullet \langle [M] \rangle$ sends $n$ to the space $C_n^\prime \langle [M, \partial] \rangle$ and $\sigma$ to $F_{\sigma}$.

2. On morphisms the following happens. Let $\sigma : [n] \to [m]$ be a morphism in $\Delta$, i.e. it is a non decreasing map. Then we define $\sigma^* : [n+1] \to [m+1]$ to be the extension of the map $\sigma$ defined by $\sigma^*(j)$ is the number of $i \in [n]$ such that $\sigma(i) > m - j$. Then we define the image of the map $\sigma : [n] \to [m]$ under the functor $C^\bullet \langle [M] \rangle$ as the map $F^0_{\sigma^*}$.

The functor $C^\bullet \langle [M] \rangle$ defines a cosimplicial space.

Proof. To prove this proposition there are several things to check. First we have to check that the functors we have defined above are well defined. Then we have to show that these functors commute with the coface and codegeneracy maps. Both these checks are relatively straightforward and are left to the reader.

9.1.7 Projection and diagonal maps for the Fulton-MacPherson completion

For the Fulton-MacPherson completion life is slightly more difficult than in the case of the simplicial completion. The projection maps are in this case similar to the projection maps in the simplicial case and is just forgetting the $i$th-point. The diagonal maps are more difficult and will unfortunately never satisfy the cosimplicial identities (see [48] for a detailed explanation of this fact). Therefore we will try to define diagonal maps $\delta^i$ that satisfy the identity $\delta^i \circ \delta^j = \delta^{i+1} \circ \delta^j$ up to homotopy. We will do this by first defining the associahedron $K_n$ and some other tools which will turn out to be important for defining the diagonal maps.
The problem with the Fulton MacPherson completion is that it is not possible to define diagonal maps that satisfy the cosimplicial identities because of the $I^{CS(2)}$ term in the completion. A possible way to get around this difficulty is to define diagonal maps that satisfy the cosimplicial identities up to homotopy. We will do this by defining maps which are parametrized by a modification of the Fulton-MacPherson completion of the space $I$. In this way we describe what happens to the points in the $I^{CS(2)}$ term where it was unclear what should happen.

The associahedron

We will now describe the associahedron, this is the space that will parametrize the diagonal maps. The Fulton-MacPherson completion of the interval $I$ consists of several connected components, each of these connected components is labeled by a possible ordering of the points in the configuration. Let $\tilde{C}_n[I, \partial]$ be the connected component of $C_n[M, \partial]$ corresponding to the ordering $1 < 2 < ... < n - 1 < n$. If we want to define diagonal maps we can use this connected component to parametrize the points in the $I^{CS(2)}$ term of the Fulton-MacPherson completion. We use this connected component since it corresponds to the order of the points on a knot under the evaluation map. We will now define the associahedron as follows.

**Definition 9.31.** The associahedron is defined as the space $\tilde{C}_n[M, m\partial]$ such that all the unit tangent vectors are positive.

So we define the associahedron as the connected component of the space $C'_n[M, \partial]$ corresponding to the ordering $1 < ... < n$ together with a choice of tangent vectors. Since $STI$ is just $S^0$ we only have to choose between 1 and $-1$ we chose 1 every where. In figure 9.1.7 and figure 9.1.7 we see some examples of the associahedron. In the next proposition we will see that this space is diffeomorphic to Stasheff’s associahedron, the convex polytope of which the vertices correspond to all possible ways to parenthesize of the word $x_0x_1...x_{n+1}$. We will denote Stasheff’s associahedron by $K_n$.

The faces of this polytope can be stratified by trees in the same way as we stratified the $C'_n[M, \partial]$.

**Proposition 9.32.** 1. Let $C''_n[I, \partial]$ be the connected component corresponding to the set $x_0 < x_1 < ... < x_n < x_{n+1}$, where $x_0$ and $x_{n+1}$ are the boundary points, then $C''_n[I, \partial]$ is diffeomorphic to the associahedron $K_{n+2}$.

2. The barycentric subdivision of $K_n$ (and therefore also the barycentric
subdivision of $C_0^0[I, \partial]$ is isomorphic to the realization of the nerve of the category $\Psi_0^{n+2}$.

The proof can be found in [48]. Because of this proposition we will in the rest of this thesis write $K_{n+n}$ instead of $C_0^0[M, \partial]$.

The functor $F_n$.

We will now make a small digression, instead of continuing with defining the diagonal maps we will define a functor that gives us a way to go from the category $\Psi_0^{n+1}$ to the category $P_\nu(n)$. This functor will be important since it will allow us to related manifolds stratified by trees to $P_nEmb(I, M)$, the $n$th Taylor approximation of $Emb(I, M)$.

**Definition 9.33.** Let $F_n : \Psi_0^{n+1} \to P_n$ be the functor that sends a tree $T$ to the set of all adjacent pairs of root joined vertices, i.e. $T$ is send to the set $S$ with $i \in S$ if and only if the $i$th and the $i+1$th vertex are root joined.

In figure 9.1.7 the functor $F_n$ is given in the case of $n = 3$.

**Lemma 9.34.** The functors $F_n : \Psi_0^{n+1} \to P_{\nu}(n)$ are left cofinal.

The proof of this lemma can be found in [50].
The diagonal maps

With all the previous definitions and digressions we are finally ready to define the diagonal maps for the Fulton-MacPherson completion. For a definition of the diagonal maps we will start with the diagonal maps we have defined for the simplicial completion. Since the Fulton-MacPherson completion is the closure of \( C_n(M) \) in the space \( A_n[M] = M^n \times (S^N)^{C_2(2)} \times I^{C_3(2)} \) and the simplicial completion is the closure of \( C_n(M) \) in the space \( A_n([M]) = M^n \times (S^N)^{C_2(2)} \) we only need some diagonal maps on the space \( I^{C_3(2)} \). We will define these maps using the associahedron \( K_{n+2} \) as some sort of parameter space.

We will now construct the map \( \delta^i(k) : C'_n[M, \partial] \times K_{n+1} \rightarrow C'_{n+k}[M, \partial] \). We will do this in three steps. First we will define what happens on the level of finite sets, then we will define what happens in the \( I^{C_3(n)} \) term and then we will combine this to define the diagonal maps.

**Definition 9.35.** Let \( [k] + i \) be defined as the set \( \{k, k + 1, \ldots, k + i\} \), then we define \( \phi_{i,k} : [n + k + 1] \rightarrow [n + 1] \) as follows

\[
\phi_{i,k}(j) = \begin{cases} 
  0 & \text{if } j \leq k \\
  j - k & \text{if } k \leq j \leq k + i \\
  k & \text{if } j \leq k.
\end{cases}
\]

We will now define the map that takes care of the \( I^{C_3(n)} \) term.

**Definition 9.36.** Let \( \iota_{i,k} : I^{C_3([n+1])} \times K_{k+1} \rightarrow I^{C_3([n+k+1])} \) be the map between \( I^{C_3([n+1])} \times K_{k+1} \) and \( I^{C_3([n+k+1])} \). This map is defined as follows, let
Let \((d_{n,0}) \times a\) be the coordinates on \(I_{C_3}^{(n+1)} \times K_{k+1}\) and let \((f_{jlm})\) be the coordinates on \(I_{C_3}^{(n+k+1)}\). Then we define the map \(i_{i,k}(d_{n,0}) \times a) = (f_{jlm})\) as follows

\[
  f_{jlm} = \begin{cases} 
    d_{\phi(j),\phi(l),\phi(m)} & \text{if at most one of the } j, l, m \in [k] + i \\
    0 & \text{if } j, l \in [k] + i \text{ but } m \notin [k] + i \\
    1 & \text{if } l, m \in [k] + i \text{ but } j \notin [k] + i \\
    \infty & \text{if } j, m \in [k] + i \text{ but } l \notin [k] + i \\
    e_{j-i,l-i,m-1}(a) & \text{if } j, l, m \in [k] + i.
  \end{cases}
\]

If we put all these maps together we obtain the following diagonal maps.

**Definition 9.37.** Let \(D_{i,k} : A_n'[M, \partial] \times K_{n+1} \rightarrow A_{n+k}[M, \partial]\) be the map given by the product of the simplicial diagonal map \(A_{\phi_{i,k}}\) from definition 9.27 with the map \(i_{i,k}\) from definition 9.36, i.e. \(D_{i,k} = A_{\phi_{i,k}} \times i_{i,k}\). The map \(\delta^i(k)\) is the restriction of the map \(D_{i,k}\) to the space \(C_n'[m, \partial] \times K_{n+1}\).

The map defined in definition 9.37 is a priori a map from \(C_n'[m, \partial] \times K_{n+1}\) to \(A_{n+k}[M, \partial]\) and not necessarily to the configuration space of \(n + k\) points. The next proposition will show us that this is indeed the case.

**Proposition 9.38.** The map \(\delta^i(k)\) is a map between configuration spaces, i.e. the image of \(\delta^i(k)(C_n'[M, \partial] \times K_{n+1})\) is contained in \(C_{n+k}[M, \partial]\).

The proof of this proposition can be found in [50] and [48]. The map \(\delta^i(k)\) can be seen as some sort of diagonal map. Geometrically what is going on is the following, the map \(\delta^i(k)\) turns the \(i\)th point in the configuration space \(C_n'[M]\) into \(k\) points which lie all infinitesimally close to each other. To do this we still have to specify the tangent vectors and what happens to the \(I_{C_3}^{(k)}\) term. We deal with the tangent vectors by giving all the \(k\) points the same tangent vector as our original point \(i\). The \(I_{C_3}^{(k)}\) term is dealt with by the associahedron \(K_{k+1}\), this associahedron gives us the configuration of points in the interval \(I\) and therefore we have the map \(\delta^i(k)\).

### 9.1.8 The relation between the diagonal maps of the simplicial completion and the diagonal maps of the Fulton MacPherson completion

The diagonal maps of the simplicial completion and the Fulton-MacPherson completion are closely related since they coincide on the first two space in the product of the Fulton-MacPherson completion. We will now describe projection maps between these two spaces and see how they are related.
**Definition 9.39.** Let $A'_n[M]$ and $A'_n\langle [M] \rangle$ be the ambient spaces of the completions. Then we define the projection map

$$Q' : C'_n[M, \partial] \to C'_n\langle [M, \partial] \rangle$$

from the Fulton-MacPherson completion to the simplicial completion as the restriction of the projection map from $C'_n[M]$ to $A'_n\langle [M] \rangle$.

This map will show us that the simplicial and the Fulton MacPherson diagonal maps commute in the following way.

**Proposition 9.40.** Let $Q' : C'_n[M, \partial] \to C'_n\langle [M, \partial] \rangle$ be the projection map from definition 9.39 and let $p_1 : C'_n[M, \partial] \times K_{n+1} \to C'_n[M, \partial]$ be the projection on the $C'_n[M, \partial]$ part of the product. Then the following diagram commutes,

$$\begin{array}{ccc}
C'_n[M, \partial] \times K_{n+1} & \xrightarrow{\delta^i)^k} & C'_{n+k}[M, \partial] \\
\downarrow Q'_\text{proj} & & \downarrow Q' \\
C'_n\langle [M, \partial] \rangle & \xrightarrow{(\delta_i)^{\gamma k}} & C'_{n+k}\langle [M, \partial] \rangle .
\end{array}$$

The proof of the proposition can be found in [50].

### 9.1.9 A more refined stratification of our modified configuration spaces

We will now use the diagonal maps for the Fulton MacPherson completion from the previous section to define a more refined stratification of $C'_n[M, \partial]$. To do this we will introduce substrata of the original strata from definition 9.17 and definition 9.15. Remember that the strata from these definitions can be seen as the points of the completed configuration space where several points where colliding and that the strata kept track in which order they where colliding. In this section we will refine this stratification by also keeping track of the directions in which points are colliding. If $k$ points are colliding in the same direction, i.e. they have the same tangent vectors, this means that they are of the form $d^i(k)$ for some point $i$. We will use this to define substrata as the image of the maps $\delta^i(k)$. To make this idea more formal we will first need some definitions.

**Definition 9.41.** Let $\Psi^0_0 = \cup \Psi^0_n$ be the category of $f$-trees such that all the leaves are consecutive and the root vertex is not univalent. Then we define the functor $K$ from $\Psi^0_0$ to spaces in the following way.
On objects of $\Psi^0$ we have that $K$ sends an $f$-tree $T$ to the space $KT$ which is defined as $\prod_v K_{|v|}$ where the product runs over all internal vertices of $T$. The space $K_{|v|}$ is the space $K_n$ where $|v|$ is the number of outgoing edges of the vertex $v$.

To define what $K$ does on morphisms we will first define what $K$ does on the basic morphisms of $\Psi^0$, then it is easy to extend this to arbitrary morphisms since we can factor them into basic morphisms. Recall that a basic morphism of $\Psi^0$ is defined as the contraction of an edge of a tree $T$. Let $e$ be the $i$th edge of $T$ and let $v$ and $v'$ be the endpoints of this edge, let $w$ be the image of $v$ and $v'$ under the map that contracts the edge $e$. To the morphism that contracts the edge $e$ we assign the map $\delta_i(\ |v'| - 1) : K_{|v|} \times K_{|v'|} \rightarrow K_{|w|}$, on all the other vertices this map is the identity.

From [48] we get the following theorem.

**Theorem 9.42.** The restriction of the functor $K$ to $\Psi_{n+2}^0$ gives the stratification of $C_n'[M, \partial]$ or the connected component $K_{n+2}$ as a manifold with corners and we have that $C_T[I, \partial] \cong K_T$.

The proof can be found in [48]. We will now define a substratification and together with the functor $K$ this will give us the mapping space model of the space of knots.

**Definition 9.43.** Let $T$ be an $f$-tree and $C_T'[M, \partial]$ be the corresponding stratum. A point $x \in C_n[M, \partial]$ can be described as $x = (x_i, v_i) \times (u_{ij}) \times (s_{ijk})$, the $x_i$ describe the points and the $v_i$ describe the tangent vectors, the rest is all the data that is added by the Fulton MacPherson completion. Note that whenever $x \in C_T'[M, \partial]$ this implies that $x_i = x_j$ whenever $i$ and $j$ are not root joined in $T$. We call two points $x = (x_i, v_i) \times (u_{ij}) \times (s_{ijk})$ and $y = (y_i, w_i) \times (u'_{ij}) \times (s'_{ijk})$ aligned if $x_i = y_i$ and $v_i = v_j$, i.e. if the points coincide and have the same tangent vectors.

For each $f$-tree $T$ we can stratify the space $C_T'[M, \partial]$ by defining the strata as the set of all points such that the points coincide and the tangent vectors also coincide. This idea is made more formal in the following definition.

**Definition 9.44.** Let $T$ be an $f$-tree and $C_T'[M, \partial]$ be the corresponding stratum. Then we define the aligned substratum to be the subspace of $C_T'(M, \partial)$ of all points $x$ such that if $i$ and $j$ are not root joined in $T$ they are aligned. We denote this space by $C_T''(M, \partial)$ and the closure of $C_T''(M, \partial)$ in $C_T'[M, \partial]$ is denoted by $C_T''[M, \partial]$.
In figure 9.1.9 we see an illustration of the concept of aligned points. We will now give a proposition that confirms the intuitive idea that aligned points can be obtained from some configuration by repeatedly applying the diagonal maps $\delta^i(k)$. First we will have to give a decomposition of a tree.

**Definition 9.45.** Let $T$ be an $f$-tree, then we can decompose $T$ in a non-unique way as $T_0 \subset T_1 \subset \ldots \subset T_{l-1} \subset T_l = T$, where all the $T_j$ are subtrees of $T$. We can give this decomposition in the following way, we begin by defining $T_0$ to be the root together with all its outgoing edges. Then we define $T_j$ to be the tree $T_{j-1}$ and a leave $i_j$ of this tree, at the end of this leave we add a new vertex $v_j$ together with $m_j$ edges. The leave $i_j$ is now an internal vertex of the tree $T_j$.

In figure 9.1.9 we see an example of the decomposition of an $f$-tree.

**Proposition 9.46.** Let $T$ be an $f$-tree together with a decomposition $T_0 \subset T_1 \subset \ldots \subset T_{l-1} \subset T_l = T$, and define the following map:

$$f : C_n^\prime[M, \partial] \times \prod_{v \in T} K_{\#v} \to C_n^\prime[M, \partial]$$

$$f = (\delta^u(m_l - 1) \times \text{id}) \circ \ldots \circ (\delta^v(m_j - 1) \times \text{id}) \circ \ldots \circ (\delta^1(m_1 - 1) \times \text{id}) .$$
The \textit{id} term in this map corresponds to the identity map on all the factors of $K_{\#v}$ for all $v$ different than $v_j$. The aligned stratum $C_{T'}^v[M, \partial]$ is the image of $C_{\#v-2}^v[M, \partial] \times \prod_{v \in T, v \neq v} K_{\#v}$ under this map.

The proof is a straightforward check of all the definitions and can be found in [50].

\section*{9.1.10 The mapping space model}

We will now define the mapping space model as the space of all aligned maps from the associahedron to the modified completed configuration space of the target manifold.

\textbf{Definition 9.47.} Let $\tilde{C}_n^v[I, \partial]$ be the connected component of $C_n^v[I, \partial]$ corresponding to the points ordered by $1 < 2 < ... < n$. We call a map from $\tilde{C}_n^v[I, \partial]$ to $C_n^v[M, \partial]$ aligned if it respects the stratification from definition 9.44. Denote the space of all aligned maps from $\tilde{C}_n^v[I, \partial] = K_{n+2}$ to $C_n^v[M, \partial]$ by $AM_n(M)$. The spaces $AM_n(M)$ are called the mapping space model for the space of knots.

\textbf{Remark 9.48.} From the spaces $AM_n(M)$ we can define restriction maps to $AM_{n-1}(M)$ by forgetting a point. The restriction of a map $f \in AM_n(M)$ is again aligned since it will again preserve the stratification. It is straightforward to show that all the restriction maps are fibrations. As theorem 9.51 will show is that whenever the difference in dimension between $I$ and $M$ is large enough, $AM_{infty}(M)$ the inverse limit of the tower $AM_n(M)$ is weakly homotopy equivalent to $Emb(I, M)$.

\textbf{Remark 9.49.} Remember that we have a map $ev_n : Emb(I, M) \to AM_n(M)$ given by evaluating the knot in $n$ points with the corresponding tangent
vectors. This map commutes with the restriction maps from $AM_n(M)$ to $AM_{n-1}(M)$ and by the universal property of the inverse limit we have a map $ev_\infty : Emb(I, M) \to AM_{\text{infty}}(M)$.

**Theorem 9.50.** The space $AM_n(M)$ is weakly homotopy equivalent to the space $P_nEmb(I, M)$ for all $n$ including $n = \infty$ and the maps $ev_n : Emb(I, M) \to AM_n(M)$ and $\alpha_n : P_nEmb(I, M) \to P_{n-1}Emb(I, M)$ are equal in the homotopy category.

From this theorem we get the following important theorem which states that Sinha’s mapping space model is indeed homotopy equivalent to the space of long knots.

**Theorem 9.51.** The map $ev_\infty : Emb(I, M) \to AM_\infty$ is a weak homotopy equivalence when the dimension of $M$ is greater or equal to 4.

*Proof.* This theorem follows from theorem 9.50 and theorem 8.26. Theorem 8.26 states that the embedding functor $Emb(N, M)$ is analytic whenever the difference between the dimensions of $N$ and $M$ is greater or equal than 3. Therefore the Taylor tower converges to the embedding functor. This theorem is just a special case in which the manifold $I$ is the interval, therefore $P_nEmb(I, M)$ converges to $Emb(I, M)$ whenever the dimension of $M$ is greater or equal to 4. Since the spaces $AM_n(M)$ are weakly equivalent to $P_nEmb(I, M)$, they converge to weakly equivalent spaces and therefore is the map $ev_\infty : Emb(I, M) \to AM_\infty$ a weak equivalence. \(\square\)

### 9.2 A sketch of the proof of theorem 9.50

We will now give a sketch of the proof of theorem 9.50, since the proof is long and complicated we will first give a sketch of the sketch of the proof. The plan is to connect the space $AM_n$ via a chain of weak equivalences to the space $P_nEmb(I, M)$. Recall that $P_nEmb(I, M)$ is defined as the homotopy limit of the embedding functor applied to the diagram $P_n(n + 1)$, the cube without the empty set. To prove the equivalence of $AM_n$ and $P_nEmb(I, M)$ we will therefore first identify the space $AM_n$ as the homotopy limit of some diagram of space which we will denote by $D_n[M]$. If we can show that the diagrams over which we take the homotopy limits have the same shape and all the components of the diagrams are weakly equivalent we know from one of the elementary properties of the homotopy limit that their homotopy limits also coincide. Unfortunately the diagram of $D_n[M]$ is not the same as the diagram $P_n(n + 1)$, but it can be shown that there homotopy limits coincide since the main difference between the diagrams is that the diagram...
for $D_n[M]$ contains much more identity maps. So we can change the shape of the diagram to the same shape as $P_n(n+1)$. The only two things we have to do to finish the proof is to show that all the components of the diagrams are equivalent and then put all the equivalences together to conclude that $AM_n$ and $P_nEmb(I, M)$ are weakly equivalent.

Since we will only give a sketch of the proof we will omit many details. These details can be found in [50].

### Identifying $AM_n$ as a homotopy limit

Recall that $\Psi^0_n$ is the category of trees such that the set of leaves over any vertex is consecutive. Also recall that functor $K$ from $\Psi^0_n$ to spaces sends an $f$-tree $T$ to $K_T = \prod_v K[v]$, the product of one associahedron for each vertex. We define $K^{nr}$ the restriction of this functor which maps a tree $T$ to $K_T^{nr} = \prod_v v$ is not the root $K[v]$, the product of all non root vertices.

**Definition 9.52.** We define the functor $D_n[M]$ from $\Psi^0_{n+2}$ to topological spaces as follows.

1. On objects $D_n[M]$ sends a tree $T$ to the space $C'_{[v_r]-2}[M, \partial] \times K_T^{nr}$, where $v_r$ is the root vertex of $T$.

2. On basic morphisms that do not contract a root edge $D_n[M]$ does the following, a basic morphism $T \to T'$ that contracts a non root edge $e$ is send to the product of the identity on $C'_n[M, \partial]$ with $i_{T', T}$.

3. On basic morphisms that do contract a root edge we have the following. Let $T \to T'$ be the morphism that contracts the $i$th root edge, such that $v_r$ and $v_t$ be the end vertices of this edge, then this morphism is send to

$$\delta^i(v_r - 1) \times id : \left( (C'_{[v_r]-2}[M, \partial] \times K[v]) \times \prod_{v \in T, v \neq v_r, v_t} K[v] \right)$$

$$\to \left( (C'_{[v'_r]-2}[M, \partial] \times \prod_{v \in T', v \neq v_r} K[v]) \right).$$

The vertex $v'_r$ is the root vertex of the tree $T'$, i.e. $v_r$ is the contraction of $v_r$ and $v_t$.

The following lemma motivates why we need this functor $D_n[M]$.

**Lemma 9.53.** The space $AM_n(M)$ is homeomorphic to the homotopy limit of the diagram $D_n[M]$. 

133
Changing the Fulton-MacPherson completion into the simplicial completion

The next step in proving theorem 9.50 is to change the spaces of the diagram \( \mathcal{D}_n[M] \) into the spaces of the simplicial completion. Since these Fulton-MacPherson completion and the simplicial completion are weakly homotopy equivalent this will not change the homotopy limits of the diagrams. We will do this as follows.

**Definition 9.54.** Let \( \tilde{\mathcal{D}}_n < [M] > \) be the functor from \( \Psi_{n+2}^0 \) to spaces defined as follows:

1. On objects \( \tilde{\mathcal{D}}_n < [M] > \) is defined by sending a tree \( T \) to the space \( \mathcal{C}'_{\langle v, r \rangle} < [M, \partial] > \).

2. On basic morphisms \( \tilde{\mathcal{D}}_n < [M] > \) is define by sending the morphism \( T \to T' \) which contracts the \( i \)-th root edge \( e \) to the the map \((\delta^i)_{\langle [v, r] \rangle}^0\).

3. To basic morphisms that contract a non root edge we assign the identity map.

We can connect the functors \( \mathcal{D}_n[M] \) and \( \tilde{\mathcal{D}}_n < [M] > \) by the following natural transformation

**Definition 9.55.** Let \( Q^\mathcal{D}_n : \mathcal{D}_n[M] \to \tilde{\mathcal{D}}_n < [M] > \) be the natural transformation that sends \( \mathcal{C}'_{\langle v, r \rangle=2}[M, \partial] \times K^r_T \) to \( \mathcal{C}'_{\langle v, r \rangle=2} < [M, \partial] > \) by forgetting the \( K^r_T \) term and projecting \( \mathcal{C}'_{\langle v, r \rangle=2}[M, \partial] \) onto \( \mathcal{C}'_{\langle v, r \rangle=2} < [M, \partial] > \) by the map defined in definition 9.39.

**Lemma 9.56.** The homotopy limit of the diagram \( \mathcal{D}_n[M] \) and the homotopy limit of the diagram \( \tilde{\mathcal{D}}_n ([M]) \) are weakly homotopy equivalent. The equivalence is given by the map homotopy limit of the map \( Q^\mathcal{D}_n \).

Changing the shape of the diagram

The functor \( \tilde{\mathcal{D}}_n < [M] > \) gives us a diagram whose homotopy limit is weakly equivalent to the space \( AM_n \). We want to show that the limit of this diagram and \( P_n Emb(I, M) \) are weakly equivalent. From the previous lemmas we know that both spaces are homotopy limits over certain diagrams. Unfortunately these diagrams have a different shape, therefore we need to change the diagram defined by \( \tilde{\mathcal{D}}_n ([M]) \) into the diagram that defines \( P_n Emb(I, M) \). Note that the shape of \( \tilde{\mathcal{D}}_n ([M]) \) is the category \( \Psi_{n+2}^0 \) and the shape of \( P_n Emb(I, M) \) is \( \mathcal{P}_n(n+1) \). So we have to change the diagram defined by \( \tilde{\mathcal{D}}_n ([M]) \) into another diagram with shape \( \mathcal{P}_n(n+1) \). Since the diagram
\( \mathcal{D}_n ([M]) \) contains many identity maps we can eliminate these maps since they have no influence on the homotopy limit of the diagram. We will do this by using the functor \( F_n \) defined in section 9.1.7.

**Definition 9.57.** Let \( \mathcal{D}_n < [M] > \) be the functor from \( \mathcal{P}_n(n+1) \) to spaces that is defined as follows.

1. On objects we define \( \mathcal{D}_n < [M] > \) by sending a subset \( S \subseteq n+1 \) to \( \mathcal{C}'_{S-1} [M, \partial] >. \)

2. We define \( \mathcal{D}_n < [M] > \) on morphisms by as follows. Let \( S' = S \cup k \), then we associate to the inclusion of \( S \) in \( S' \) the morphism \( \delta^i \) where \( i \) is the number of elements of \( S \) less than \( k \).

From this definition we get the following two lemmas.

**Lemma 9.58.** The functor \( \tilde{\mathcal{D}}_n ([M]) \) is equal to the composition of the functor \( \mathcal{D}_n ([M]) \) and the functor \( F_n \).

**Lemma 9.59.** The homotopy limits of \( \mathcal{D}_n ([M]) \) and \( \tilde{\mathcal{D}}_n ([M]) \) are weakly equivalent.

**Proving that the diagrams are equivalent**

Now that we have two diagrams of the same shape we only need to prove that the components of the diagrams are equivalent. Recall that the space \( E_S(M) \) is defined as the space of embeddings of \( I - \cup_{s \in S} J_s \) with constant speed (see definition 9.4). Then we want to show that \( E_S(M) \) is weakly equivalent to the space \( \mathcal{C}'_{S+1} [M, \partial] >. \) We will do this by embedding these two spaces in a bigger space and show that both the inclusion maps are weak equivalences and therefore that \( E_S(M) \) and \( \mathcal{C}'_{S+1} [M, \partial] > \) are weakly equivalent. We will begin by defining the bigger space in which we will embed \( E_S(M) \) and \( \mathcal{C}'_{S+1} [M, \partial] >.

**Definition 9.60.** Let \( X \) be a topological space with a metric \( d(\cdot, \cdot) \), then we define the space \( \mathcal{H}(X) \) as the set of all compact subspaces of \( X \) together with a metric defined by \( d(A, B) = max(\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)) \) for \( A, B \in \mathcal{H}(X) \).

Note that all the spaces we have considered so far are metrizable so we can apply this construction to the spaces we need. Also note that \( \mathcal{C}'_{n-1} [M, \partial] > \) can be seen as a subspace of \( \mathcal{C}'_{n+1} [M] > \) by relabeling the coordinates. We will now construct a space such that \( E_S(M) \) and \( \mathcal{C}'_{S+1} [M, \partial] > \) are both subspaces of this space.
Definition 9.61. Let $S \in n+1$, let $\{I_k\}$ be the set of connected components of the set $I - \cup_{s \in S} J_s$ and let $f : I - \cup_{s \in S} J_s \to M$ be an embedding, then we denote by $f_k$ the restriction of $f$ to $I_k$. Define the map $ev_S(f) : \prod_k I_k \to C'_#S+1$ as the map that evaluates the embedding at a point. We can see the space $E_S(M)$ as a subspace of $\mathcal{H}(C'_#S+1 < [M])$ by identifying an embedding $f$ with the image of the map $ev_S(f)$. We can also see $C'_#S-1 < [M, \partial]$ as a subspace of $\mathcal{H}(C'_#S+1 < [M])$ by seeing it as the subspace of all compact subsets that are points. We will now define the space $E_S < [M, \partial]$ as the union of the subspaces associated to $E_S(M)$ and $C'_#S-1 < [M, \partial]$ in $\mathcal{H}(C'_#S+1 < [M])$.

The following proposition will state that $E_S(M)$ and $C'_#S+1 < [M, \partial]$ are both weakly equivalent to $E_S < [M, \partial]$ and therefore weakly equivalent to each other.

**Proposition 9.62.** The inclusions of $E_S(M)$ and $C'_#S+1 < [M, \partial]$ in $E_S < [M, \partial]$ are weak homotopy equivalences.

From this proposition we can conclude that the spaces in the diagrams are weakly equivalent and therefore we can conclude that there homotopy limits are weakly equivalent.

**Proposition 9.63.** The homotopy limit of $D\langle [M] \rangle$ is weakly equivalent to the homotopy limit of $E_n$. Moreover the homotopy limit of $D\langle [M] \rangle$ is weakly equivalent to $P_nEmb(I, M)$.

If we now put all the equivalences together we see that $AM_n$ is weakly equivalent to $P_nEmb(I, M)$. This concludes the proof of theorem 9.50.

### 9.3 The cosimplicial model

In this section we will construct a cosimplicial model for the space of knots. We will do this by showing that the totalization of the space $C^\bullet \langle [M] \rangle$ from definition 9.30 is weakly equivalent to the space of knots.

**Theorem 9.64.** The space $Emb(I, M)$ is weakly equivalent to the totalization of a fibrant replacement of the cosimplicial space $C^\bullet \langle [M] \rangle$ from definition 9.30.

#### 9.3.1 A sketch of the proof of theorem 9.64

We will now give a sketch of theorem 9.64, the is just like the proof of theorem 9.50 rather long and technical so we will only give the key ingredients of the
proof. The plan is to start with the space $D_n\langle [M]\rangle$ from 9.57, we know that the homotopy limit of this space is equivalent to $P_n\Emb(I,M)$, the $n$th Taylor approximation of the space $Emb(I,M)$. So if we can show that $D_n\langle [M]\rangle$ is somehow related to the cosimplicial space $C^n\langle [M]\rangle$ (we will be precise about this later), we can use this to show that the totalization of the space $C^\bullet\langle [M]\rangle$ is weakly equivalent to the space $Emb(I,M)$. We will first define some restrictions of the category $\Delta$ and define what it means for a cosimplicial object to be restricted to a subcategory of $\Delta$. For more details about this proof we refer the reader to [50].

Definition 9.65. Let $\Delta$ be the category whose objects are finite sets $n$ and with order preserving maps as morphisms. Then we denote by $\Delta_n$ the subcategory of $\Delta$ of all set of cardinality $\leq n$. Let $X$ be a cosimplicial space, then we denote by $i_n X$ the restriction of $X$ to $\Delta_n$.

We will begin with defining a functor from $P_\nu(n+1)$ to the category $\Delta_n$. Remember that the geometric realization of the nerve of $P_\nu(n)$ is isomorphic to the barycentric subdivision of the $n$-simplex. We will use this to define a functor from $P_\nu(n)$ to $\Delta_n$.

Definition 9.66. Let $G_n : P_\nu(n) \to \Delta_n$ be the functor that sends a subset $S \in n$ to the set $\#S - 1$ and on morphisms we define $G_n$ as follows. Let $i : S \to S'$ be the inclusion $S$ in $S'$, then we send $i$ to the inclusion of $\#S - 1$ in $\#S' - 1$ according to the inclusion of $S$ in $S'$. Or more concrete if $S = \{s_1, \ldots, s_i\}$ and $S' = \{s_1, \ldots, s_i, s'_1, \ldots, s'_j\}$ then we assign to each element minus one plus its number in the ordered sets $\#S - 1$ and $\#S' - 1$. Then we define the map $G_n(i)$ as the map that sends an element to minus one plus its order in the set $S'$.

Note that the maps in the cubical diagram $P_\nu(n)$ are the coface maps of the category $\Delta_n$. With this functor $G$ we are able to see the relation between the functor $D_n\langle [M]\rangle$ and the functor $C^\bullet\langle [M]\rangle$.

Proposition 9.67. We have the following equality of functors

$$D_n\langle [M]\rangle = i_nC^\bullet\langle [M]\rangle \circ G_n.$$ 

The proof of this proposition follows relatively straightforward from the definitions of $D_n\langle [M]\rangle$, $C^\bullet\langle [M]\rangle$ and $G_n$. The next thing we want to do in proving theorem 9.64 is to show how the totalization behaves with respect to truncating a cosimplicial space.
Theorem 9.68. Let $X$ be a cosimplicial space, then there is a weak equivalence between the homotopy limit of $i_n X \circ G_n$ and the $n$th totalization of a fibrant replacement of $X$, i.e. we have the following

$$i_n X \circ G_n \simeq \text{Tot}_n(\tilde{X}),$$

where $\tilde{X}$ is a fibrant replacement of $X$.

From theorem 9.68 we can already conclude that the homotopy limit of $D_n \langle [M] \rangle$ and the $n$th totalization of a fibrant replacement of $C^\bullet \langle [M] \rangle$ are weakly equivalent. If we now note that $Emb(I, M)$ is weakly equivalent to the inverse limit of the Taylor tower $P_n Emb(I, M) \simeq \text{holim}(D_n \langle [M] \rangle)$ and that $\text{Tot}(C^\bullet \langle [M] \rangle)$ is the inverse limit of the tower $\text{Tot}_n(C^\bullet \langle [M] \rangle)$. Since we use a fibrant replacement these inverse limits are weakly equivalent.

So we are only left with proving theorem 9.68. The proof of this theorem follows immediately from the following two theorems.

Theorem 9.69. Let $X$ be a cosimplicial space and $\tilde{X}$ be a fibrant replacement of $X$. Then we a weak equivalence between the homotopy limit of $i_n X$ and the $n$th totalization of $\tilde{X}$.

Theorem 9.70. The functor $G_n$ is left cofinal.

We will not prove these theorems, the proof of theorem 9.69 can be found in [7] and the proof of theorem 9.70 can be found in [50].

9.4 The cosimplicial model and operads

The cosimplicial model for the space of knots can be constructed in two ways. The first one is by taking certain completions of configuration spaces which we have done in the previous sections and the second one is by using the McClure Smith method to associate a cosimplicial space to the Kontsevich operad. The totalization of this space will be weakly equivalent to the space of knots when the dimension of the target space is greater or equal than 4. We will describe the second method in this section since it is easier to understand and since the Kontsevich operad is weakly equivalent to the little disks operad we can use the formality of the little disks operad to prove theorems about the cosimplicial model.

To construct the cosimplicial model we first have recall the definition of the Kontsevich operad.
**Definition 9.71.** The Kontsevich operad $\mathcal{K}_d$ is the operad with $\mathcal{K}_d(n)$ given by the closure of the image of the following map:

$$\alpha_* = (\alpha_{ij})_{1 \leq i < j \leq n} C_n(\mathbb{R}^d) \to \prod_{1 \leq i < j \leq n} S^{d-1}$$

Where the maps $\alpha_{ij}$ are defined by

$$\alpha_{ij}(x_1, \ldots, x_n) = \frac{(x_i - x_j)}{\|x_i - x_j\|}$$

The operad maps are given by inserting the $i$th space in one of the points, see also section 5.4.2.

The next thing we do is to construct the corresponding cosimplicial object using the McClure Smith method from section 6.4. Please recall that the definition of the cosimplicial object associated to an operad is given by $O^n = O(n)$ and the coface and codegeneracy maps are constructed as follows. The coface maps are given by

$$d^i : O^n \to O^{n+1}$$

$$d^i(x) = \begin{cases} 
  m \circ_2 x & \text{if } i = 0 \\
  x \circ_i m & \text{if } 1 \leq i \leq n \\
  m \circ_1 x & \text{if } i = n + 1 
\end{cases}$$

And the codegeneracy maps are given by

$$s^j : O^n \to O^{n-1}$$

$$s^j(x) = m \circ_j e$$

where $e \in O(0)$ is the image of $\Pi : \mathcal{A}\mathcal{S}\mathcal{S}(0) \to O(0)$ and $m \in O(2)$ is the image of $\Pi : \mathcal{A}\mathcal{S}\mathcal{S}(2) \to O(2)$.

To use this construction we need to define a map from the non symmetric associative operad to the Kontsevich operad. To do this we first observe that the Kontsevich operad $\mathcal{K}_1(n)$ for $d = 1$ is given by the configuration space of $n$ ordered points on a line modulo the action of $\mathbb{R} \rtimes \mathbb{R}$. The corresponding space is a just the space of all possible orderings of the set $1, \ldots, n$. If we denote by $\mathcal{K}_1^0(n)$ the path component corresponding to the ordering $1, \ldots, n$

**Theorem 9.72.** The cosimplicial model defined in section 9.3 and the cosimplicial model coming from the McClure-Smith method and the Kontsevich operad are weakly equivalent.

The proof will be omitted and can be found in [49] uses that all the spaces defining the Kontsevich operad are weakly equivalent to the configuration spaces defining the cosimplicial model.

139
9.5 The spectral sequences corresponding to Sinha’s cosimplicial model

With the models obtained from the previous sections and the spectral sequences from chapter 6 we are able to construct several spectral sequences. In this section we shall briefly state what the $E_1$ pages of these spectral sequences are. We will not calculate anything further or state some results about vanishing lines in this section. For a more detailed description of these spectral sequences we refer to [50], [49] and [47]. In chapter 11 we will however give some recent results about the collapse of these spectral sequences.

The homotopy Bousfield Kan spectral sequence is given in the following theorem.

Theorem 9.73. Let $M$ be a manifold of dimension greater or equal than four, then there is a second quadrant spectral sequence converging to the homotopy groups $\pi_\bullet(Emb(I, M))$. The $E_{p,q}^{−1}$ term is given by

$$E_{p,q}^{−1} \cap K e r s_i^i \subseteq \pi_q(C_p^\prime([M])) .$$

The differential of the $E_1$ page is given by the restriction of the following map to the kernels of $s^i$

$$\sum (-1)^i s_i : \pi_q(C_{p-1}^\prime([M])) \to (C_p^\prime([M])).$$

For the cohomology Bousfield Kan we have a similar result which we will not describe here, details about this spectral sequence can be found in [50], [49] and [47].
Chapter 10

Rational homotopy theory

In the next chapter we want to prove that over the rational numbers the Bousfield Kan spectral sequence corresponding to Sinha’s cosimplicial model collapses at the $E^2$ page, to do this we need some extra technical tools from rational homotopy theory. Rational homotopy theory is the study of spaces with homotopy groups which are vector spaces over the rational numbers $\mathbb{Q}$. Studying these spaces has the advantage that we lose all information about the torsion which makes more calculations possible. The plan is to replace every non rational topological space by a rational space that is rationally equivalent to the original space. By studying this rationalization we can obtain a lot of information about the original space. To obtain this information we will show that there is an equivalence of categories between the category of rational space and a certain category of commutative differential graded algebras. Therefore we can replace the rational spaces by certain differential graded commutative algebras and study them instead. In this chapter we will first define what a rational space is and what the morphisms and equivalences are in this category. After that we will study commutative differential graded algebras over $\mathbb{Q}$ and explain the relation between the category of rational spaces and the category of commutative differential graded algebras over $\mathbb{Q}$. In section 10.3 we will introduce Sullivan models, Sullivan models are special commutative differential graded algebras which make many calculations possible. Another important reason for defining Sullivan models is that there is a bijection between the set of minimal Sullivan algebras and the set of equivalence classes of rational spaces, therefore we have the theorem that two spaces are rationally equivalent if and only if the have isomorphic Sullivan models.

Since this chapter is mainly meant to give an overview of rational homotopy theory we will omit almost all the proofs. More details and proofs can be found in [25] and [13] on which this chapter is mainly based. For a
more complete but still brief introduction of rational homotopy theory we refer to [25], for a complete treatment of the subject [13] is a good reference.

10.1 Rational spaces

To start with our brief overview of rational homotopy theory we start with the definition of a rational space and the process of rationalization, we will also give some examples of rational spaces.

**Definition 10.1.** Let $X$ be a simply connected space, we call $X$ rational if $X$ satisfies one of the following equivalent properties:

1. $\pi_\ast X$ is a $\mathbb{Q}$-vector space
2. $\tilde{H}_\ast(X; \mathbb{Z})$ is a $\mathbb{Q}$-vector space
3. $\tilde{H}_\ast(\Omega X; \mathbb{Z})$ is a $\mathbb{Q}$-vector space

Now that we know what a rational space is we would like to see some examples of rational spaces. We will give the definition of the rational $n$-sphere and the rational $n$-disc, as we will see these space are not exactly the easiest examples of spaces, but they will important since we can use them to build the rational analog of CW complexes.

**Example 10.2.** The rational $n$-sphere $(S^n)_0$ is the following space

$$(S^n)_0 = \left( \bigvee_{k \geq 1} S^n_k \right) \cup \left( \bigcup_{k \geq 2} D^{n+1}_k \right)$$

where we attach the $D^{n+1}_{k+1}$ to $S^n_k \vee S^n_{k+1}$ via the map $\iota_{n,k} - (k+1)\iota_{n,(k+1)}$. Where $\iota_{n,k}$ is the homotopy class of the inclusion of $S^n$ in $\bigvee_{k \geq 1} S^n_k$ as the $k$-th $S^n$. Please note that the reduced homology of $(S^n)_0$ with rational coefficients is given by $\tilde{H}_k((S^n)_0) = \mathbb{Q}$ if $k = n$ and 0 otherwise.

**Example 10.3.** The rational $n + 1$-disk $(D^{n+1})_0$ is the space given by

$$(D^{n+1})_0 := ((S^n)_0 \times I) / ((S^n)_0 \times \{0\}).$$

There is also another way to obtain the rational n-sphere as a direct limit of a series of spaces converging to $(S^n)_0$, the spaces are given by

$$X(r) = \left( \bigvee_{1 \leq k \leq r} S^n_k \right) \cup \left( \bigcup_{2 \leq k \leq r-1} D^{n+1}_k \right).$$
It is left as an exercise to the reader to check that \( \lim_{r \to X} X(r) = (S^n)_0 \).

With these definitions of the rational sphere and disc it is possible to define the notion of a (relative) rational CW-complex, which is usually denoted as a \( CW_0 \)-complex. A (relative) \( CW_0 \)-complex can be constructed in a similar fashion as a (relative) non-rational CW-complex. In the following definition we will see how this is formally done.

**Definition 10.4.** A pair of spaces \((X, A)\) is a **rational relative \( CW_0 \) complex** if:

1. \( X(1) = A \),
2. for all \( n \geq 1 \) the following push out exists
   \[
   \bigsqcup_{\alpha \in J_n}(S^n)_0 \xrightarrow{\bigcup_i f_\alpha} X(n) \xrightarrow{i} \bigsqcup_{\alpha \in J_n}(D^{n+1})_0 \xrightarrow{\bigcup_i f_\alpha} X(n + 1).
   \]
   Where \( i \) is the inclusion of \((S^n)_0\) in \((D^{n+1})_0\) as the boundary of \((D^{n+1})_0\) and the set \( J_n \) is the indexing set of the rational spheres \((S^n)_0\).
3. The space \( X \) is equipped with the weak topology, i.e. a subset \( U \) is open if and only if \( U \cap X(n) \) is open for all \( n \).

The next step in defining rational homotopy theory is to construct a way to turn non-rational spaces into rational spaces. In the following definition we will first define what rationalization exactly means. Then we will theorem 10.7 proof that rationalizations exist and explain how we can construct rationalizations for \( CW \)-complexes.

**Definition 10.5.** Let \( X \) be a simply connected topological space and let \( Y \) be a simply connected rational space, a continuous map \( l : X \to Y \) is called a **rationalization** of \( X \) if the map
\[
\pi_* l \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q} \cong \pi_* Y
\]
is an isomorphism.

**Example 10.6.** The rational n-sphere is the rationalization of the n-sphere and the rational n-disc is the rationalization of the n-disc.

The next theorem will state that for every topological space there always exist a rationalization. Furthermore the proof will give a method to construct this rationalization. We will not give the proof, we will however explain how to construct a rationalization of a \( CW \) complex.
**Theorem 10.7.** Let $X$ be a simply connected space then there exists a relative CW-complex $(X_0, X)$ such that

1. $(X_0, X)$ has no zero and no one cells,
2. the inclusion $i : X \to X_0$ is a rationalization of $X$.

Furthermore let $Y$ simply connected rational space and $f : Y \to X$ be a continuous map, then there exists a continuous map $g : X_0 \to Y$, that is unique up to homotopy, such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{j} & & \uparrow{g} \\
X_0 & & 
\end{array}
\]

**Proof.** For a complete proof we refer the reader to the proof of theorem 9.7 in [13]. We will however give the general idea of the proof in the case of CW-complexes. Let $X$ be a CW-complex, then we construct the rationalization of $X$ by taking all the cells of $X$ and replace them rational cells with the appropriate gluing maps. For a general topological space we can take a CW approximation and rationalize this CW approximation.

**Remark 10.8.** There is also another method to rationalize topological spaces. This method is called localization, the idea is that you add multiplicative inverses to a set of elements of the homotopy groups by attaching cells to the original topological space. The advantage of this method is that it is possible to only localize to a particular set of the homotopy groups and not the whole group. A disadvantage is that it is not as easy as just replacing all the cells by rational cells.

This theorem shows us that every space has a rationalization and that it is unique up to rational homotopy, we will now define what it means to be a rational homotopy equivalence.

**Definition 10.9.** Let $X$ be a simply connected topological space, then we define the *rational homotopy type* of $X$ as the weak homotopy type of the rationalization of $X$. A continuous map $\varphi : X \to Y$ between two simply connected topological spaces $X$ and $Y$ is called a *rational homotopy equivalence* if it satisfies the following equivalent conditions.

- $\pi_*(\varphi) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.
- $H_*(\varphi, \mathbb{Q}) : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ is an isomorphism.
• $H^*(\varphi, \mathbb{Q}) : H_*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$ is an isomorphism.

• The rationalization of $\varphi$, $\varphi_0 : X_0 \rightarrow Y_0$ is a weak homotopy equivalence.

For many definitions and theorems that will follow we need the following definition.

**Definition 10.10.** Let $X$ be topological space and $\varphi : Z \rightarrow X$ be a rationalization of $X$, then we say that $X$ is of **finite rational type** if $H^*(X; \mathbb{Q})$ is of finite type or equivalently if $Z$ is of finite type, i.e. they are finite dimensional or have finitely many cells in each degree.

### 10.2 Commutative differential graded algebras

In this section we will introduce a special category of differential graded commutative algebras over $\mathbb{Q}$. These algebras will play an important role since we will state that there is an equivalence of categories between the category of these algebras and the category of simply connected rational spaces of finite type. We will start this section by giving the definition of these differential graded commutative algebras over $\mathbb{Q}$, after that we will define the functors between the categories of differential graded algebras and rational spaces. The functor from rational spaces to commutative differential graded algebras assigns to every rational space the algebra of piecewise polynomial differential forms. We go the other way around by using spatial realization, this functor assigns to an algebra a simplicial set. The geometric realization of this set gives us a topological space back. In the next section we will define a special subclass of these algebras which we will call Sullivan algebras. These algebras are important because the rational homotopy type of a topological space is completely determined by the corresponding Sullivan minimal model. In this section $\otimes$ will mean the tensor product over $\mathbb{Q}$, unless this is otherwise specified.

**Definition 10.11.** A **commutative differential graded algebra** (CDGA) is a cochain complex $(A^*, d)$ together with a multiplication map $\mu : (A^*, d) \otimes_{\mathbb{Q}} (A^*, d) \rightarrow (A^*, d)$ and a map $\eta : \mathbb{Q} \rightarrow (A^*, d)$ called the unit. These maps should satisfy the following conditions,

1. $\mu$ is graded commutative, i.e. $a \cdot b = (-1)^{pq} \cdot a$ where $p$ and $q$ are the degrees of $a$ and $b$,

2. $\mu$ is associative,
3. $\eta$ is a unit, i.e. $\mu(\eta \otimes \text{Id}_A) = \text{Id}_A = \mu(\text{Id}_A \otimes \eta)$,

4. $\mu$ and $d$ satisfy the Leibniz rule, i.e. for $a \in A^p$ and $b \in A^q$ $d(\mu(a, b)) = \mu(d(a), b) + (-1)^{|a|}\mu(a, d(b))$.

To make the commutative differential graded algebras into a category we also need to define the notion of a morphism, which we will do in the following definition.

**Definition 10.12.** Let $(A, d_A, \mu_A, \eta_A)$ and $(B, d_B, \mu_B, \eta_B)$ be CDGA, a morphism of commutative differential graded algebras is a cochain map

$$f : (A, d_A, \mu_A, \eta_A) \rightarrow (B, d_B, \mu_B, \eta_B)$$

such that $f\mu_A = \mu_B(f \otimes f)$ and $f\eta_A = \eta_B$.

**10.2.1 Polynomial differential forms**

Here we will give a functor that assigns to every topological space a CDGA. We will do this by first defining a functor from simplicial sets to CDGA which assigns to every simplicial set the algebra of polynomial differential forms on this set. We will first define this algebra and then explain how to turn a rational space into a commutative graded algebra.

**Definition 10.13.** The algebra of polynomial differential forms $\mathcal{A}^*_n$ is defined as the simplicial commutative differential graded algebra with the $n$th algebra given by

$$\mathcal{A}^*_n = (\Lambda(t_0, ..., t_n; y_0, ..., y_n)/J_n, d).$$

The $t_i$ all have degree 0 and the $y_i$ have degree 1, such that $dt_i = y_i$. The ideal $J_n$ is generated by $1 - \sum_{i=0}^n$ and $\sum_{j=0}^n y_j$. We define the face and degeneracy maps as:

$$d_i : \mathcal{A}^*_n \rightarrow \mathcal{A}^*_{n-1}$$

$$d_i(t_k) = \begin{cases} t_k & \text{if } k < i \\ 0 & \text{if } k = i \\ t_{k-1} & \text{if } k > i \end{cases}$$

$$s_i : \mathcal{A}^*_n \rightarrow \mathcal{A}^*_{n+1}$$

$$s_i(t_k) = \begin{cases} t_k & \text{if } k < i \\ t_k + t_{k+1} & \text{if } k = i \\ t_{k+1} & \text{if } k > i. \end{cases}$$
We will now define a functor from simplicial sets to $CDGA_{\mathbb{Q}}$ by taking the algebra of polynomial differential forms. We will extend this functor to topological spaces by turning a topological space into a simplicial set by using the singular simplices functor. We will call the functor from topological spaces to $CDGA_{\mathbb{Q}}$ defined by composing the functor of polynomial differential forms with the singular simplices functor the algebra of piecewise-linear de Rham forms on $X$. To make everything more precise we have the following definitions.

**Definition 10.14.** Let $\mathcal{A}^* : sSet \to CDGA_{\mathbb{Q}}$ be the functor from the category of simplicial sets to the category of commutative graded algebras, which sends a simplicial set $K$ to $sSet(K, \mathcal{A}^*)$, the space of all simplicial maps between $K$ and $\mathcal{A}^*$.

Now that we know how to go from simplicial sets to commutative differential graded algebras we can extend this functor to topological spaces in the following definition.

**Definition 10.15.** Let $X$ be a topological space and $\text{Sin}^\bullet(X)$ be the set of all singular simplices of $X$. Then we define the algebra of piecewise-linear de Rham forms as the composition of the functor $\mathcal{A}^*$ with $\text{Sin}$ the singular simplices functor. This functor is denoted by $\mathcal{A}_{PL}(X) = \mathcal{A}^*(\text{Sin}^\bullet(X)) = sSet(\text{Sin}^\bullet(X), \mathcal{A}^*)$.

### 10.2.2 Spatial realization

From the previous section we know how we to go from the category of topological spaces to the category $CDGA_{\mathbb{Q}}$. We also need a way to go back from the category $CDGA_{\mathbb{Q}}$ to the category of topological spaces. The functor from $CDGA_{\mathbb{Q}}$ is called the spatial realization functor and will be defined below. To define this functor we will, as in the case of the polynomial differential forms functor, first define a functor to the category of simplicial sets and then extend this functor to topological spaces by using the geometric realization functor.

**Definition 10.16.** Let $A$ be a commutative differential graded algebra, then we define the functor $K_\bullet : CDGA_{\mathbb{Q}} \to sSet$ by $K_\bullet(A) = CDGA_{\mathbb{Q}}(A, \mathcal{A}^*)$, i.e. the simplicial set of all commutative differential graded algebra maps from $A$ to $\mathcal{A}^*_\bullet$. The face and degeneracy maps are defined object wise by using the face and degeneracy maps of $\mathcal{A}^*_\bullet$.

**Definition 10.17.** Let $\langle \cdot, - \rangle$ be the functor from $CDGA_{\mathbb{Q}}$ to the category of topological spaces defined by $\langle - \rangle = |K(-)|$, the composition of the
geometric realization functor and the functor $\mathcal{K}_\bullet$. This functor is called the spatial realization functor.

**Proposition 10.18.** The functors $\mathcal{A}^* : sSet \rightarrow CDGA_{\mathbb{Q}}^{op}$ and $\mathcal{K}_\bullet : CDGA_{\mathbb{Q}}^{op} \rightarrow sSet$ are adjoint and form a Quillen pair.

A proof of this proposition can be found in [6].

### 10.3 Sullivan models

We will now introduce Sullivan models. These models are an important tool in rational homotopy theory since the minimal Sullivan models are in one to one correspondence with rational homotopy types of topological spaces. We will first introduce Sullivan algebras then we will state some results about Sullivan models and show how to construct them.

**Definition 10.19.** Let $V$ be a graded vector space and denote the subspace of $V$ spanned by all the elements with even grading by $V^{even}$ and the subspace of odd elements by $V^{odd}$. Then we define the **free commutative graded algebra** $\Lambda V = SV^{even} \otimes \wedge V^{odd}$ as the tensor product of the symmetric algebra over all elements of even grading with the exterior algebra on all elements with odd grading.

We will now see that Sullivan algebras are free commutative graded algebras equipped with a differential that has to satisfy some conditions.

**Definition 10.20.** Let $V = \{V(k)\}_{k \geq 0}$ be a graded vector space, then we define a **Sullivan algebra** $(\Lambda V, d)$ as the free commutative graded algebra of the vector space $V$ together with a differential $d : \Lambda V(k) \rightarrow \Lambda V(k + 1)$ such that

1. $d = 0$ in $V(0)$
2. $d : V(k) \rightarrow \Lambda V(k - 1)$.

There is a special class of Sullivan algebras which are called minimal Sullivan algebras. As we will see these are in one to one correspondence with equivalence classes of rational homotopy types.

**Definition 10.21.** We call a Sullivan algebra $(\Lambda V, d)$ **minimal** if

$$\text{Im}(d) \subset \Lambda V^+ \cdot \Lambda V^+.$$ 

Where $\Lambda V^+$ denotes the subspace of $\Lambda V$ of all elements with grading $\geq 1$. 

148
Definition 10.22. Let \((A,d)\) be a commutative differential graded algebra over \(\mathbb{Q}\), then we call a Sullivan algebra \((\Lambda V,d)\) a Sullivan model for \((A,d)\) if there exists a quasi isomorphism

\[ m : (\Lambda V,d) \to (A,d). \]

Let \(X\) be a path connected topological space, then we define a Sullivan model for \(X\) as a Sullivan algebra \((\Lambda V,d)\) such that \((\Lambda V,d)\) is a Sullivan model for the algebra \(A_{PL}(X)\).

Definition 10.23. We call a Sullivan model \((\Lambda V,d)\) for a topological space \(X\) minimal if \((\Lambda V,d)\) is a minimal Sullivan algebra.

Proposition 10.24. Let \((A,d)\) be a commutative differential graded algebra over a field \(K\) of characteristic 0, such that \(H^0(A) = K\). Then there exists a Sullivan model for \((A,d)\).

Proof. We will give a sketch of the proof of this proposition since it will give us a construction for a Sullivan model for \((A,d)\). The proof constructs the graded vector space \(V\) inductively. We first define \(V^0\) together with a map \(m_0 : (\Lambda V^0,0) \to (A,d)\) such that the map \(H(m_0) : V^0 \to H^+(A)\) is an isomorphism. Since the differential of \((\Lambda V^0,0)\) is 0, \(V^0 \cong H^+(A)\), and \(H^0(A) = K\) the map \(m_0\) induces a surjective map on the cohomology. We will now assume that we have the subspaces \(V^0,\ldots,V^k\) of \(V\) we will now construct the space \(V^{k+1}\) out of these spaces. We also need to assume that we have a maps

\[ m_i : \left( \Lambda \left( \bigoplus_{i=0}^{k} V_i \right), d \right) \to (A,d) \]

for all \(i \in \{0,\ldots,k\}\). Now let \([z_\alpha]\) be a basis for the set of cocycles in \(\Lambda \left( \bigoplus_{i=0}^{k} V_i \right)\), then we define \(V^{k+1}\) as the space generated by the set \([v_\alpha]\). The \(v_\alpha\) here are in one to one correspondence with the \(z_\alpha\), the only difference is that we define the degree of \(v_\alpha\) by \(\deg(v_\alpha) = \deg(z_\alpha) - 1\). We extend the differential of \(\Lambda \left( \bigoplus_{i=0}^{k} V_i \right)\) to a differential on \(\Lambda \left( \bigoplus_{i=0}^{k+1} V_i \right)\) by defining \(d(v_\alpha) = z_\alpha\). We define the map \(m_{k+1} : \Lambda \left( \bigoplus_{i=0}^{k+1} V_i \right) \to (A,d)\) as follows. Since \(z_\alpha\) is a cocycle we know that \(m_k z_\alpha = d(a_\alpha)\) for some element \(a_\alpha \in A\), we then extend \(m_{k+1}\) by defining it as follows \(m_{k+1}(v_\alpha) = a_\alpha\). This concludes the construction of the space \(V^{k+1}\), the rest of the spaces can be build inductively from this. There are still quiet a few things to check but we will leave this as an exercise to the reader. \(\square\)
Proposition 10.25. Let \((A, d)\) be a commutative differential graded algebra over a field \(K\) of characteristic 0 such that \(H^0(A) = K\) and \(H^1(A) = 0\) then there exists a unique minimal model up to isomorphism, which we will construct in the proof.

Remark 10.26. There are stronger statements about the existence and construction of minimal models for commutative differential graded algebras than proposition 10.25. It is for example not necessary to have the condition that \(H^1(A) = 0\), but if we drop this condition the constructions will be more complicated. Therefore we have chosen to keep this condition for the sake of simplicity.

Proof. We will only proof the existence, a proof for the uniqueness can be found in [13]. We will just like in the proof of proposition 10.24 construct the model inductively. Since \(H^1(A) = 0\) we know that \(V\) does not contain any elements of degree zero. We begin by defining \(V^2\) by choosing \(V^2\) and a map \(m_2 : (\Lambda V^2, 0) \to (A, d)\) such that \(H^2(m_2) : V^2 \to H^2(A)\) is an isomorphism. From this it follows that \(H^1(m_2)\) is an isomorphism since \(H^1(A) = 0\) and that \(H^3(m_2)\) is injective because \((\Lambda V^2)^3 = 0\). We will now define the rest inductively, we begin by assuming that we have a space \((\Lambda V^{\leq k}, d)\) together with a map \(m_k : (\Lambda V^{\leq k}, d) \to (A, d)\). Then we extend this map to a map \(m_{k+1} : (\Lambda V^{\leq k+1}, d) \to (A, d)\) by choosing cocycles \(a_\alpha \in A^{k+1}\) and cocycles \(z_\beta \in (\Lambda V^{\leq k})^{k+2}\) such that

\[
H^{k+1}(A) = \text{Im}(H^{k+1}(m_k)) \oplus \bigoplus\limits_\alpha K[a_\alpha]
\]

and

\[
\text{ker}(H^{k+2}(m_k)) = \bigoplus\limits_\beta K[z_\beta].
\]

Note that there exist elements \(b_\beta \in A\) such that \(m_k(z_\beta) = d(b_\beta)\) for all \(\beta\). We will now define the space \(V^{k+1}\) as the space with as basis \(\{v'_\alpha, v''_\beta\}\) where the elements \(v'_\alpha\) are in one to one correspondence with the elements \(a_\alpha\) and the elements \(v''_\beta\) are in correspondence with the elements \(b_\beta\). We then define \(\Lambda V^{\leq k+1}\) as \(\Lambda V^{\leq k+1} = \Lambda V^{\leq k} \otimes \Lambda V^{k+1}\). We extend the differential by \(d(v'_\alpha) = 0\) and \(d(v''_\beta) = z_\beta\). The map \(m_{k+1} : (\Lambda V^{\leq k+1}, d) \to (A, d)\) is defined as the extension of the map \(m_k\) by setting \(m_{k+1}(v'_\alpha) = a_\alpha\) and \(m_{k+1}(v''_\beta) = b_\beta\).

We have now constructed a minimal Sullivan model for the commutative differential graded algebra \((A, d)\), the proof that the constructed algebra is indeed a minimal Sullivan model can be found in [13].
**Theorem 10.27.** There is a bijection between the set of rational homotopy types and the set of isomorphism classes of minimal Sullivan models over $\mathbb{Q}$.

**Remark 10.28.** There is also a relative version of Sullivan models which we did not treat here. For more information about relative Sullivan models see [13] and [25].
Chapter 11

The collapse of Sinha’s and Vassiliev’s spectral sequences for \( \mathbb{R}^d \) with \( d \geq 4 \) over the rationals

We conclude this thesis with some recent results about the spectral sequences we have constructed in this thesis. Most of the results we will give here come from the article [33] by Lambrechts, Tourtchine and Volić. The results are mainly about the relation between the spectral sequences and about there collapse when the coefficients are the rationals. This chapter is mainly a summary of results, therefore we shall omit almost all the proofs or just give a very brief sketch.

We have already seen that Vassiliev’s spectral sequence collapses along the diagonal when the target manifold is \( \mathbb{R}^3 \) and the coefficients are the rationals. In [33] they prove that over the rationals both Sinha’s and Vassiliev’s spectral sequence collapse when the target manifold is \( \mathbb{R}^d \) and \( d \geq 4 \). The exact statement of the theorems is as follows.

**Theorem 11.1.** The spectral sequence associated to Vassiliev’s method that calculates the cohomology of the space of long knots \( Emb(\mathbb{R}, \mathbb{R}^d) \) collapse at the \( E^1 \) page if \( d \geq 4 \) and the coefficients are the rational numbers.

The idea behind the proof is as follows. The idea is not to use something like the Kontsevich integral, but to use Sinha’s spectral sequence. If \( d \geq 4 \) then the spectral sequences converge to the same graded module, we can use this to show that if the agree on some page the rest of the spectral sequence should also agree. It is indeed the case that the spectral sequences agree as stated in the following proposition.
Proposition 11.2. If $d \geq 4$, then the $E^1$ page of Vassiliev’s spectral sequence for the space $Emb(\mathbb{R}, \mathbb{R}^d)$ is up to regrading isomorphic to the $E^2$ page of Sinha’s spectral sequence for the space $Emb(\mathbb{R}, \mathbb{R}^d)$.

See [58] for the proof of this proposition. Now that we know that the spectral sequences have isomorphic $E^2$ and $E^1$ pages we will prove that Vassiliev’s spectral sequence collapses by first proving that Sinha’s spectral sequence collapses. This is the statement of the following theorem.

Theorem 11.3. The homology Bousfield-Kan spectral sequence from Sinha’s cosimplicial model associated to the cosimplicial space associated to the Kontsevich operad collapses at the $E^2$ page over the rationals.

To prove this theorem we will use two main ingredients. The first one is proposition 11.5 which will give us a criterion when the Bousfield Kan spectral sequence collapses and the other one is the formality of the little disks and the Kontsevich operad.

Definition 11.4. Let $I$ be a diagram of topological spaces and $\mathbb{K}$ be a field of characteristic 0. Then we call $I\mathbb{K}$-formal if $A_{PL}(I)$ is formal, i.e. if $A_{PL}(I)$ is connected to its homology by a zig zag of quasi-isomorphisms.

Note that a cosimplicial space $X^\bullet$ is formal if the functor $X^\bullet : \Delta \to Top$ formal is. The following proposition gives a criterion for when the Bousfield Kan spectral sequence collapses.

Proposition 11.5. Let $X^\bullet$ be a cosimplicial space and let $\mathbb{K}$ be a field of characteristic 0. The homology Bousfield Kan spectral sequence associated to $X^\bullet$ collapses over $\mathbb{K}$ at the $E^2$ page, if $X^\bullet$ is $\mathbb{K}$-formal.

The only thing that is left to prove that Sinha’s and therefore Vassiliev’s spectral sequence collapse is to show that Sinha’s cosimplicial model is $\mathbb{K}$-formal. This turns out to be difficult and we will not proof this here. The key idea is to use a version the formality of the little disks operad.

Definition 11.6. A morphism of operad $f : \mathcal{O} \to \mathcal{P}$ is relatively formal if it can be connected to its homology by a chain of quasi-isomorphisms. Or equivalently there exists a commutative diagram of the following form.

\[
\begin{array}{cccccc}
\mathcal{O} & \to & \mathcal{O}_1 & \to & \ldots & \to & \mathcal{O}_n & \to & H_*(\mathcal{O}) \\
\downarrow f & & \downarrow & & & & \downarrow & & \downarrow H_*(f) \\
\mathcal{P} & \to & \mathcal{P}_1 & \to & \ldots & \to & \mathcal{P}_n & \to & H_*(\mathcal{P})
\end{array}
\]

Definition 11.7. We call a multiplicative operad $\mathcal{O}$, together with the morphism $\Pi : \mathcal{ASS} \to \mathcal{O}$ multiplicatively formal if it is relatively formal with respect to the morphism $\Pi$. 

153
In [43] is proven that the Kontsevich operad is multiplicatively formal which is formally stated in the following theorem.

**Theorem 11.8.** Let $d \geq$ and let the coefficients be $\mathbb{R}$, then the Kontsevich operad $K_d$ is multiplicatively formal.

The proof of this theorem uses model categories and is relatively simple compared to the proof in [33]. Up to almost all details this concludes the sketch of the proof of the collapse of Sinha’s and therefore also Vassiliev’s spectral sequence.
Bibliography


[34] Xiao-Song Lin. Vertex modules, quantum groups and vassiliev’s knot invariants.


