Introduction to amoebas and tropical geometry

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Abstract
In this thesis we give an introduction to the theory of tropical geometry and its applications to amoebas. We treat Kapranov’s theorem and Mikhalkin’s limit construction for amoebas. We also compute a number of examples.

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Introduction

In this thesis we give an introduction to the related subjects of amoebas and tropical geometry.

If \( X \) is an algebraic variety in the algebraic torus \((K^*)^n\) over a field \( K \) with a norm \( | \cdot |_K \) the amoeba is the image of \( X \) under the map defined by \( \text{Log}_K(z_1, \ldots, z_n) = (\log(|z_1|_K), \ldots, \log(|z_1|_K)) \). This construction was first used in 1971 by George Bergman to proof a theorem about subgroups of \( \text{GL}(n, \mathbb{Z}) \) in [2]. We briefly explain the relation between amoebas and the group \( \text{GL}(n, \mathbb{Z}) \) in appendix I.

The name 'amoeba' was introduced by Gelfand, Kapranov and Zelevinsky who rediscovered the concept when studying the combinatorics of discriminants of polynomials in [8]. The name is very appropriate considering for example figure [1].

![Figure 1: An example of an amoeba of a curve in \((\mathbb{C}^*)^2\).](image)

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Tropical geometry is a concept from computer science and is named after the Brazilian mathematician and computer scientist Imre Simon. A tropical curve in $\mathbb{R}^2$ is the set of points where a finite collection of linear functions does not have a unique maximum, for example figure 2.

![Figure 2: An example of a tropical curve.](image)

It turns out that many results from algebraic geometry also hold for tropical curves, for example Bezout’s theorem for the number of intersections of two curves. Many results of this type can be found in [21] and [16].

For fields with a non-Archimedean absolute value the relation between amoebas and tropical curves is very direct, the amoeba of a hypersurface is given by a tropical hypersurface. This is Kapranov’s theorem, which we prove in chapter 3.

The relation between curves over the complex numbers and amoebas is more complicated. Using complex analysis Mikhail Passare and Hans Rullgard constructed a tropical curve which is a deformation retract of a given amoeba. This construction can be found in [18].

In the last chapter of this thesis we study Mikhalkin’s theorem on the limit of a sequence of scaled amoebas from the article [14].

A nice application of this theorem is the construction of amoebas with a prescribed number of components. For example the theorem shows that the scaled amoeba of $f = u^3 + w^3 + u^2w^4 + u^4w^3 + ru^3w + ru^2w^2 + ruw^3 + ru^3w^3$ converges to figure 2 for $r \to \infty$. 
If we pick \( r = 4 \) we already get the four bounded components we wanted. We can change the coefficients somewhat without changing the topology of the amoeba and figure 1 is the amoeba of the curve in \((\mathbb{C}^*)^2\) given by 
\[
u^3 + w^3 + u^2w^4 + u^4w^3 + 3.8u^3w + 12.9u^2w^2 + 4.55uw^3 + 4.6u^3w^3.
\]

A much more serious application of amoebas is to study the topology and analytic structure of real and complex varieties as in [20], [14] and [15] by Viro and Mikhalkin.
Chapter 1

Polyhedral complexes and
tropical curves

In this chapter we give a brief introduction to tropical geometry. The goal is to show the connection between tropical hypersurfaces and subdivisions of polytopes, which is theorem 1.28.

1.1 Polyhedral geometry

In this section we state a number of definitions of polyhedral geometry. Most of this section is derived from the excellent reference work [26].

A subset $X$ of $\mathbb{R}^n$ is called convex if for every pair of points $x, y \in X$ the line segment connecting $x$ and $y$ is contained in $X$. The convex hull of a set $A \subset \mathbb{R}^n$ is the intersection of all convex sets which contain $A$ and is denoted by $\text{conv}(A)$.

Definition 1.1. A set $P \subset \mathbb{R}^n$ is called a polytope if it is the convex hull a finite set of points.

The following concrete description of the set of points of a polytope is sometimes useful.

Proposition 1.2. If $A \subset \mathbb{R}^n$ is a finite set then $\text{conv}(A)$ consists of all points of the form $\sum_{v \in B} \lambda_v v$ with $B \subseteq A$, $\lambda_v \in (0, 1]$ and $\sum_{v \in B} \lambda_v = 1$

Sums of this form are called convex combinations of the points in $A$.

We denote the space of linear functionals on $\mathbb{R}^n$ by $(\mathbb{R}^n)^\vee$. If $\varphi$ is a nonzero functional then any level set is a hyperplane and for $c \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n | \varphi(x) \leq c\}$ is called a half space.
Definition 1.3. The intersection of a finite number of half spaces is called a polyhedron.

The previous two definitions are related by the following theorem.

Theorem 1.4. A subset $P \subset \mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedron.

The proof can be found in [26] in chapter 1 and is harder than one would expect. It is now immediately clear that the intersection of two polytopes is again a polytope, which is not easy to prove directly from the definition.

If we would define a polytope as a bounded polyhedron then it would be hard to prove that the Minkowski sum (defined in the following section) of two polytopes would be a polytope, so theorem 1.4 is fundamental.

A subset $F$ of a polyhedron $P$ is called a face of $P$ if $F = \emptyset$ or there is a functional $\varphi \in (\mathbb{R}^n)^\vee$ such that $F = \{x \in P|\varphi(x) = M\}$, where $M$ is the maximum of $\varphi$ on $P$. The face $F$ is called the supporting face of $\varphi$ and is denoted with $P^\varphi$. Because $M$ is the maximum of $\varphi$ on $P$ we have $F = P \cap \{x \in \mathbb{R}^n|\varphi(x) \geq M\}$, so $F$ itself is a polyhedron. If $P$ is a polytope it follows from theorem 1.4 that a face is again a polytope.

A face of dimension 0 is called a vertex and a face of dimension $\dim(P) - 1$ is called a facet. A vertex contains only one point and we can identify the vertex with the point it contains. The set of vertices is denoted with $\text{vert}(P)$.

The set of all faces of a polyhedron has the following structure.

Definition 1.5. A polyhedral complex is a collection $C$ of polyhedra such that

1. If $P \in C$ then every face of $P$ is an element of $C$.
2. If $P, Q \in C$ then $P \cap Q$ is a face of $P$ and of $Q$.

The support $|C|$ of $C$ is the union over all elements of $C$. We call $C$ a polyhedral subdivision of $|C|$. Notice that $|C|$ is not necessarily a polyhedron.

Sometimes we refer to the elements of $C$ as the cells of the complex. The cells are partially ordered by inclusion. If all maximal cells have the same dimension $n$ we say that $C$ is pure of dimension $n$ and denote $\dim(C) = n$. The vertices of the cells of $C$ are cells of $C$ by property 1. and are called the vertices of $C$. If $C$ is pure of dimension $n$ the cells of dimension $n - 1$ are called the facets of $C$.

This version of theorem 1.4 gives no easy proof for the fact that the Minkowski sum of two polyhedra is again polyhedron, which follows directly from a more precise version, see page 30 of [26].
An isomorphism $\varphi$ between complexes $C$ and $D$ is a bijection $\varphi : |C| \to |D|$ which is affine on all elements of $C$ and induces a bijection $C \to D$.

Now we give an example of a polyhedral complex we encounter later.

**Example 1.6.** Let $P \subset \mathbb{R} \times \mathbb{R}^n$ be a polytope and let $P_{\max} \subset P$ be the set of all $(\lambda, x) \in P$ such that if $(t, x) \in P$ for some $t \in \mathbb{R}$ then $t \leq \lambda$. Thus $P_{\max}$ consists of all points of $P$ which are maximal on a line of the form $\mathbb{R} \times \{x\}$ with $x \in \mathbb{R}^n$. The set of all faces of $P$ contained in $P_{\max}$ is called the upper envelope of $P$. We can define the set $P_{\min}$ as the set of points which are minimal on lines of the form $\mathbb{R} \times \{x\}$, then the set of all faces contained in $P_{\min}$ is called the lower envelope of $P$.

**Lemma 1.7.** The upper envelope of a polytope $P$ is a polyhedral subdivision of $P_{\max}$ and the lower envelope is a polyhedral subdivision of $P_{\min}$.

**Proof.** It follows directly from the definitions that the upper envelope is a polyhedral complex. Now let $p = (\lambda, x) \in P_{\max}$ an let $H_1, \ldots, H_k$ be the half spaces defining $P$. By definition of $P_{\max}$ the must be an $i$ such that $(\lambda + \varepsilon, x) \notin H_i$ for every $\varepsilon \in \mathbb{R}_{>0}$. Now $p$ is contained in the face $F$ defined by $H_i$. If $L$ is the boundary of $H_i$ then $F = L \cap P$. If $w \in L$ then $w + (\varepsilon, 0, \ldots, 0) \notin H_i$ for any $\varepsilon \in \mathbb{R}_{>0}$, so $F$ is an upper face of $P$.

The case of the lower envelope is identical.

1.2 The Newton polytope and its subdivisions

We will regularly see the following class of polytopes throughout this thesis. The importance of the Newton polytope for tropical geometry is explained by theorem 1.28 If $K$ is a field the ring of Laurent polynomials is the ring $K[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$. Thus a Laurent polynomial is a finite sum $f = \sum_{v \in \mathbb{Z}^n} a_v z^v$, with $a_v \in K$ and $z^v = z_1^{v_1} \cdots z_n^{v_n}$. This gives the following polytope.

**Definition 1.8.** If $f = \sum_{v \in \mathbb{Z}^n} a_v z^v$ is a Laurent polynomial then the Newton polytope $\text{Newt}(f)$ is the convex hull of the set $\{v \in \mathbb{Z}^n | a_v \neq 0\}$.

We can immediately verify that $\text{Newt}(f)$ behaves nicely with respect to multiplication. The Minkowski sum $A + B$ of two subsets $A, B \subset \mathbb{R}^n$ is defined by $A + B = \{a + b | a \in A, b \in B\}$.

**Lemma 1.9.** If $f$ and $g$ are Laurent polynomials over a field, then $\text{Newt}(fg) = \text{Newt}(f) + \text{Newt}(g)$.
Proof. If \( f = \sum_i a_i z^i \) and \( g = \sum_i b_i z^i \), then \( fg = \sum_i c_i z^i \), where \( c_i = \sum_{j,k: j+k=i} a_j b_k \). If \( c_i \neq 0 \) then there are \( j, k \in \mathbb{Z}^n \) such that \( a_j \neq 0 \) and \( b_k \neq 0 \), so \( i = j + k \) with \( j \in \text{Newt}(f) \) and \( k \in \text{Newt}(g) \). Thus \( \text{Newt}(fg) \subset \text{Newt}(f) + \text{Newt}(g) \).

To show the converse it is enough to show that all vertices of \( \text{Newt}(f) + \text{Newt}(g) \) are contained in \( \text{Newt}(fg) \). If \( k \) is a vertex of \( \text{Newt}(f) + \text{Newt}(g) \), then \( k \) is a sum of unique vertices \( i \in \text{Newt}(f) \) and \( j \in \text{Newt}(g) \). Thus the coefficient \( c_k = a_i b_j \neq 0 \), so \( k \in \text{Newt}(fg) \).

The Newton polytope is an example of a lattice polytope, which is a polytope such that all vertices are integral points. A polyhedral subdivision of a lattice polytope is called a lattice subdivision if all of its elements are lattice polytopes. It is clear that any lattice polytope can be realized as the Newton polytope of a polynomial.

We now describe a way to make lattice subdivisions of a lattice polytope. This method works similarly for arbitrary polytopes and can be found in lecture 5 of [26].

Let \( Q \subset \mathbb{R}^n \) be a lattice polytope and let \( A \subset Q \) be a set of integral points containing all vertices of \( Q \) and let \( v: A \to \mathbb{R} \) be any function. In the applications of this construction \( Q \) will be the Newton polytope of a polynomial \( f \) and \( v \) will depend on the coefficients of \( f \).

We define a polytope \( P \subset \mathbb{R}^{n+1} \) by \( P = \text{conv}\{(v(i), i)| i \in A\} \). We have a natural projection \( \pi: P \to Q \).

Lemma 1.10. The restriction \( \pi: P_{\max} \to Q \) is a bijection.

Proof. If \( x \in P \) then we can write \( x = \sum_{i \in A} \lambda_i (v(i), i) \) with \( \lambda_i \in [0,1] \) and \( \sum_{i \in A} \lambda_i = 1 \), so \( \pi(x) = \sum_{i \in A} \lambda_i i \in Q \). Thus \( \pi(P) \subset Q \). It is clear that \( \pi \) is injective. Now let \( x \in Q \). We have \( Q = \text{conv}(A) \) because \( A \) contains all vertices of \( Q \). Thus we can write \( x = \sum_{i \in A} \lambda_i i \) with \( \lambda_i \in [0,1] \) and \( \sum_{i \in A} \lambda_i = 1 \). Now \( y = \sum_{i \in A} \lambda_i (v(i), i) \in P \). Thus \( P_{\max} \) must contain an element \( \tilde{y} \) such that \( y_1 = \tilde{y}_1, \ldots, y_n = \tilde{y}_n \). Clearly \( \pi(y) = \pi(\tilde{y}) = x \) so \( \pi \) is surjective.

If \( p \) is a vertex of a face contained in \( P_{\max} \) then \( p \) is also a vertex of \( P \) so \( p \) is of the form \( (v(i), i_1, \ldots, i_n) \) for some \( i \in A \), so \( \pi(p) = (i_1, \ldots, i_n) \) is an integral point. Thus the projection of the upper envelope defines a lattice subdivision of \( \text{Newt}(f) \). A subdivision of a polytope which can be constructed in this way is called a regular subdivision.

We can immediately construct an interesting lattice subdivision.
Proposition 1.11. If $Q \subset \mathbb{R}^n$ is a lattice polytope there is a function $v: Q \cap \mathbb{Z}^n \rightarrow \mathbb{Q}$ such that the set of vertices of the regular subdivision induced by $v$ is $Q \cap \mathbb{Z}^n$.

Proof. After a translation we can assume $Q$ is contained in the box $[0, N]^n$ for some $N \in \mathbb{N}$. Let $f: [0, N] \rightarrow \mathbb{R}$ be a concave function which is nowhere linear, for example $f(t) = N^2 - t^2$. Now we define $v: Q \cap \mathbb{Z}^n \rightarrow \mathbb{Q}$ by

$$v(i) = \sum_{j=1}^n f(i_j)$$

and note that the image indeed lies in $Q$.

We must show that $(v(k), k)$ is an upper vertex of the polytope $\text{conv}(\{(v(i), i) | i \in Q \cap \mathbb{Z}^n\})$ in $\mathbb{R} \times \mathbb{R}^n$ for every $k \in Q \cap \mathbb{Z}^n$.

Suppose that $k = \sum_{i \in A} \lambda_i i$ with $A \subset Q \cap \mathbb{Z}^n$ and $\lambda_i \in (0, 1]$.

We get

$$v(k) = \sum_{j=1}^n f(\sum_{i \in A} \lambda_i i_j) \geq \sum_{j=1}^n \sum_{i \in A} \lambda_i f(i_j)$$

$$= \sum_{i \in A} \lambda_i \sum_{j=1}^n f(i_j) = \sum_{i \in A} \lambda_i v(i)$$

by applying Jensen’s inequality to each term of the first sum. This shows $v(k)$ is maximal, so $(v(k), k)$ lies in the upper envelope.

Because $f$ is not linear we have equality if and only if $i_j = k_j$ for all $i \in A$, which can only happen if $A = \{k\}$. This shows $(v(k), k)$ is not contained in $\text{conv}(\{(v(i), i) | i \in Q \cap \mathbb{Z}^n - \{k\}\})$, so it is a vertex. \hfill \Box

The function used in the proof of the previous lemma is generally not the most convenient when constructing a subdivision. In the following example we use a simpler one.

Example 1.12. Consider the ring $\mathbb{C}(t)[u, w]$ of polynomials in two variables over the field of rational functions over $\mathbb{C}$ and let

$$f = u^3 + w^3 + 1 + u^2w^3 + u^3w^2 + tu + tu^2 + tw + tw^2 + tw^3 + tuw^3 + t^2uw + t^2u^2w + t^2uw^2 + t^2u^2w^2,$$

then $\text{Newt}(f) = \text{conv}(\{(0, 0), (0, 3), (3, 0), (3, 2), (2, 3)\})$. We define $v: \text{Newt}(f) \cap \mathbb{Z}^2 \rightarrow \mathbb{R}$ by $v(i) = \text{val}_i(a_i)$ where $a_i$ is the coefficient of $f$ at $u^{i_1}w^{i_2}$. It is easy to check that figure 1.1 is the associated subdivision of $\text{Newt}(f)$.
Not every lattice subdivision of a lattice polytope is regular, see [26] page 132 for a simple counterexample. Many results about regular subdivisions can be found in [8]. More counterexamples can be found in [12].

1.3 Tropical algebra

Tropical varieties are also derived from algebra, so we give the necessary definitions of tropical algebra in this section. Similarly to algebraic geometry there are two ways to define tropical polynomials, as abstract sums of monomials or as polynomial functions. As in the case of finite fields a distinct formal sums can define the same functions.

In this thesis we only consider tropical hypersurfaces which are easy to define directly using a function on $\mathbb{R}^n$ and in this chapter this concrete definition would be sufficient. In chapter 3.2 briefly consider the algebra of tropical polynomials.

Definition 1.13. A semiring is a set $R$ with addition $\oplus$ and multiplication $\odot$ such that

- $(R, \oplus)$ is a commutative monoid with identity element $e_{\oplus}$
- $(R, \odot)$ is a monoid with identity element $e_{\odot}$

and for all $a, b, c \in R$ the operations must satisfy

- $a \odot (b \oplus c) = a \odot b \oplus a \odot c$
- $(b \odot c) \odot a = b \odot a \oplus c \odot a$
• $e_\oplus \odot a = e_\oplus$.

The axioms are the same as those of a ring except for the lack of an
additive inverse and the fact that $e_\oplus \odot a = e_\oplus$ does not follow from the other
axioms.

A standard example is $\mathbb{R}_{\geq 0}$ with addition and multiplication. The follow-
ing example is less intuitive.

**Example 1.14.** For $t \in \mathbb{R}_{> 1}$ let $\mathbb{R}_t = \mathbb{R} \cup \{-\infty\}$, let $x \oplus_t y = \log_t(t^x + t^y)$
and $x \odot_t y = x + y$, where we conveniently put $t^{-\infty} = 0$. We have $\log_t(x) \oplus_t \log_t(y) = \log_t(x + y)$ and $\log_t(x) \odot_t \log_t(y) = \log_t(xy)$ for every $x, y \in \mathbb{R}_{\geq 0}$,
so the operation on $\mathbb{R}_t$ is just the operation of $\mathbb{R}_{\geq 0}$ transfered by the map $\log_t$. Thus $\mathbb{R}_t$ is a semiring and $\mathbb{R}_t \cong \mathbb{R}_{\geq 0}$.

**Definition 1.15.** The tropical semiring is the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with
operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

It is not hard to check that $\mathbb{T}$ satisfies all the axioms of a semiring.

We can also get $\mathbb{T}$ as a limit of the semirings $\mathbb{R}_t$ given in the previous
example. We have $\max\{x, y\} = \lim_{t \to \infty} \log_t(t^x + t^y)$. To see this let $x = \max\{x, y\}$, so $\log(1 + t^{y-x}) \leq \log(2)$ is bounded. Now $\lim_{t \to \infty}(t^x + t^y) = x + \lim_{t \to \infty} \frac{\log(1 + e^{t-y})}{\log(t)} = x$. This gives an immediate proof that $\mathbb{T}$ is a semiring.

The process of taking the limit of the semirings $\mathbb{R}_t$ is called Maslov de-
quantization. It is explained briefly on page 28 of [14].

**Definition 1.16.** A tropical polynomial is a formal sum $\bigoplus_{i \in I} a_i \odot x_i^1 \odot ... \odot x_i^n$
where $I \subset \mathbb{Z}^n$ is a finite index set, $a_i \in \mathbb{T}$ and $x_1, ..., x_n$ are real variables.

If $x = (x_1, ..., x_n)$ and $i = (i_1, ..., i_n) \in \mathbb{Z}^n$ we use $x^i$ instead of $x_1^{i_1} \odot ... \odot x_n^{i_n}$
and $i \cdot x = i_1 x_1 + ... + i_n x_n$, where $\cdot$ is the standard inner product.

Using the definition of $\oplus$ and $\odot$ a tropical polynomial $f = \bigoplus_{i \in I} a_i \odot x^i$
defines a function on $\mathbb{R}^n$ given by $f(x_1, ..., x_n) = \max_{i \in I} \{a_i + i \cdot x\}$. We say
two tropical polynomials are equivalent if they define the same function.

**Example 1.17.** Two tropical polynomials with different coefficients can be
equal. For example $0 \oplus x \odot x^2$ and $0 \oplus x^2$ both define the function $f(x) = \max\{0, 2x\}$ on $\mathbb{R}$.

For every equivalence class of tropical polynomials we can define a pick
canonical polynomial with the least possible number of terms as follows. If $f$
is given by $\max_{i \in I} \{a_i + i \cdot x\}$ and $k \in I$ define $C_k = \{x \in \mathbb{R}^n | a_k + k \cdot x = f(x)\}$. On $C_k$ the function $f$ is given by $ak + k \cdot x$.

Put $J = \{i \in I | C_i \neq \emptyset\}$ and $f_{\text{red}} = \max_{i \in J} \{a_i + i \cdot x\}$ then $f$ and $f_{\text{red}}$ define
the same function on $\mathbb{R}^n$. It is clear that all tropical polynomials equivalent
to $f$ have the same reduced polynomial.
1.4 Tropical hypersurfaces

Analogous to algebraic geometry we want to assign a hypersurface to a tropical polynomial. The following definition is natural.

**Definition 1.18.** If $f$ is a tropical polynomial in $n$ variables with at least 2 terms then the hypersurface $V(f)$ of $f$ is the set of all points of $\mathbb{R}^n$ where $f$ is not locally linear.

It is clear that equivalent tropical polynomials define the same tropical hypersurface. A tropical hypersurface in $\mathbb{R}^2$ is called a tropical curve.

**Lemma 1.19.** For a tropical polynomial $f = \max_{i \in I} \{ a_i + i \cdot x \}$ the following sets are equal

1. The hypersurface $V(f)$.
2. The set of all $p \in \mathbb{R}^n$ where $f$ achieves its maximum at least twice.
3. The intersection $\bigcap_{k \in I} (C_k^o)^c$.

**Proof.** Suppose $f$ is not locally linear in $p \in \mathbb{R}^n$, then there are distinct $k_1, k_2 \in I$ such that $f$ is equal to $a_i + k_i \cdot x$ on points arbitrarily close to $p$. Because $f$ is continuous it follows that $f(p) = a_1 + k_1 \cdot p = a_2 + k_2 \cdot p$, so the first set is contained in the second.

If $f$ achieves its maximum for both $k_1, k_2 \in I$ it is clear that $p$ cannot lie in $C_k^o$ for any $k \in I$, so the second set is contained in the third.

Suppose $p \in \bigcap_{k \in I} (C_k^o)^c$, then there must be distinct $k_1, k_2 \in I$ such that $a_1 + k_1 \cdot p = a_2 + k_2 \cdot p = f(p)$. If $p \notin V(f)$ then $a_1 + k_1 \cdot x = a_2 + k_2 \cdot x$ on an open neighborhood of $p$, so $k_1 = k_2$. This shows the third set is contained in the first.

**Example 1.20.** Figure [1.20] is the tropical curve defined by $\max \{ 0, 3y, 3x, 3x + 2y, 2x + 3y, x + 1, 2x + 1, y + 1, 2y + 1, 3x + y + 1, x + 3y + 1, x + y + 2, x + 2y + 2, 2x + y + 2, 2x + 2y + 2 \}$. The edges which cross the axes extend to infinity. If we consider figure [1.20] and [1.1] as graphs they are dual. We will see that when we put the structure of a polyhedral complex on $V(f)$ they can be seen as dual polyhedral complexes.

We now construct a polyhedral subdivision of $\mathbb{R}^n$ such that the union over all facets is $V(f)$. In the case of figure [1.2] it is easy to see that this is possible. This way we also get a polyhedral subdivision of $V(f)$.

For $A \subseteq I$ let $C_A = \{ x \in \mathbb{R}^n | a_i + i \cdot x = f(x) \text{ for all } i \in A \}$. Notice that $C_{\{k\}} = C_k$ and $C_A = \cap_{k \in A} C_k$. Let $V(f) = \{ C_A | A \subseteq I \text{ and } \# A \geq 1 \} \cup \{ \emptyset \}$. 

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Proposition 1.21. For a tropical polynomial \( f = \max_{i \in I} \{a_i + i \cdot x\} \) the collection \( \mathcal{V}(f) \) is a polyhedral complex and \( \mathcal{V}(f) = \mathcal{V}(f_{\text{red}}) \).

Proof. If \( f_{\text{red}} = \max_{i \in J} \{a_i + i \cdot x\} \), then for any \( A \subseteq J \) the set \( C_A \) is the same in \( \mathcal{V}(f_{\text{red}}) \) as in \( \mathcal{V}(f_{\text{red}}) \) as \( f \) and \( f_{\text{red}} \) define the same function on \( \mathbb{R}^n \). Thus \( \mathcal{V}(f_{\text{red}}) \subseteq \mathcal{V}(f) \).

For all \( A \subseteq I \) have \( C_A = \bigcap_{k \in A} C_k \) and \( C_k = \{x \in \mathbb{R}^n | a_k + k \cdot x = f(x)\} \) is a polyhedron, so all sets in \( \mathcal{V}(f_{\text{red}}) \) are polyhedra.

We now want to show that \( C_A \cap C_B = C_{A \cup B} \) is a face of \( C_A \) and \( C_B \) for any nonempty \( A, B \subset I \). Let \( k, j \) be distinct elements of \( I \) then the equation \( a_k + k \cdot x = a_j + j \cdot x \) defines a hypersurface \( H \) such that \( C_k \) and \( C_j \) lie in different half spaces defined by \( H \). By definition of \( H \) we have \( H \cap C_k = H \cap C_j \) and \( C_j \cap C_k \subset H \), so \( C_j \cap C_k = H \cap C_k = H \cap C_j \) is a face of both \( C_j \) and \( C_k \). For any nonempty \( A \subset I \) and \( k \in A \) we have \( C_A \cap C_k = \bigcap_{i \in A} C_i \cap C_k \), so \( C_A \) is an intersection of faces of \( C_k \), so \( C_A \) is also a face of \( C_k \). By the same argument \( C_A \cap C_B = C_{A \cup B} \) is a face of \( C_k \) for some \( k \in A \), so \( C_A \cap C_B \) is also a face of \( C_A \).

Now let \( j \in J \), so \( C_k \) \( \cap \) \( C_j = C_\cap \neq \emptyset \) and \( \dim(C_j) = n \). The boundary of \( C_j \) is contained in the union of the hypersurfaces \( H_i \) defined by \( a_j + j \cdot x = a_i + i \cdot x \) for \( j \neq i \). Each \( H_j \) defines a face \( H_j \cap C_j = C_{\{i,j\}} \) of \( C_j \) of dimension at most \( n = 1 \) so these faces are proper.

If \( F \) is a facet of \( C_j \) then \( \dim(F) = n - 1 \) and \( F = \bigcup_{i \neq j} H_i \cap F \). We have \( \dim(H_i \cap F) \leq \dim(F) \) and the union is finite so there is a \( k \neq j \) with \( \dim(H_k \cap F) = \dim(F) \). As \( H_k \cap F = (H_k \cap C_j) \cap F \) it follows that
$H_j \cap F$ is a face of $F$ so $F = F \cap H_j$. Therefore $F \subseteq H_k \cap C_j$, so $F$ is also a face of $H_k \cap C_j = C_{\{k,j\}}$. As $\dim(F) = \dim(H_k \cap F)$ this implies $F = C_{\{k,j\}} \in \mathcal{V}(f_{\text{red}})$.

It now follows directly that any face of $C_j$ is also in $\mathcal{V}(f_{\text{red}})$ because it is the intersection of all facets containing it. The same statement also follows for faces of general $C_A \in \mathcal{V}(f_{\text{red}})$, because a face of $C_A$ is also a face of $C_k$ for any $k \in A$. This shows $\mathcal{V}(f_{\text{red}})$ is a polyhedral complex.

Now pick a general $C_A \in \mathcal{V}(f)$ then $C_A = \bigcup_{j \in J} C_A \cap C_j$. As $C_A \cap C_j$ is a face of $C_A$ it follows by comparing the dimensions that there is a $j \in J$ such that $C_A = C_A \cap C_j$. As $C_A \cap C_j$ is a face of $C_j$ and $\mathcal{V}(f_{\text{red}})$ is a polyhedral complex it follows that $C_A \in \mathcal{V}(f_{\text{red}})$, so it follows that $\mathcal{V}(f) = \mathcal{V}(f_{\text{red}})$.

\[\square\]

**Corollary.** The complex $\mathcal{V}(f)$ is pure of dimension $n$ and any nonempty element is the intersection of maximal elements. The tropical curve $V(f)$ is the union over all facets of $\mathcal{V}(f)$.

**Proof.** The first statement is clear for $\mathcal{V}(f_{\text{red}})$, so it also holds for $\mathcal{V}(f)$. The second part follows directly from lemma 1.19. \[\square\]

### 1.5 The Newton polytope of a tropical curve

Now we give a different construction of $\mathcal{V}(f)$. This construction allows us to make the connection with subdivisions mentioned in example 1.20. To make this subdivision we need to define what the Newton polytope of a tropical polynomial is and we need to introduce normal cones.

**Definition 1.22.** For a polytope $P$ with face $F$ the normal cone $N_F(P)$ of $P$ at $F$ is the subset of $(\mathbb{R}^n)^\vee$ defined by

$$N_F(P) = \{ \varphi \in (\mathbb{R}^n)^\vee | \varphi(z) \leq \varphi(y) \text{ for all } z \in P \text{ and } y \in F \}.$$ 

This means that $N_F(P)$ consists of all functions $\varphi$ such that the maximum of $\varphi$ on $P$ is attained on all of $F$. If $\varphi$ is a functional such that $P^\varphi = F$, then $\varphi \in N_G(P)$ for any face $G$ with $G \subseteq F$ and $F$ is the maximal face for which $\varphi$ is contained in the normal cone.

It is usual to identify $\mathbb{R}^n$ with its dual $(\mathbb{R}^n)^\vee$ via the map given by $x \mapsto (y \mapsto x \cdot y)$, where $\cdot$ is the usual inner product. With this identification the normal cone is given by

$$N_F(P) = \{ x \in \mathbb{R}^n | x \cdot z \leq x \cdot y \text{ for all } z \in P \text{ and } y \in F \}.$$
We only need to check maximality of $\varphi$ on vertices so we have

$$N_F(P) = \{ x \in \mathbb{R}^n | x \cdot z \leq x \cdot y \text{ for all } z \in A \text{ and } y \in \text{vert}(F) \},$$

where $A$ is any subset of $P$ which contains $\text{vert}(P)$.

The following properties of normal cones can be found in [26]. A polyhedron $C$ is a cone if $\lambda x \in C$ for all $x \in C$ and $\lambda \in \mathbb{R}_{\geq 0}$.

**Proposition 1.23.** Let $P \subset \mathbb{R}^n$ be a polytope and let $F$ and $G$ be nonempty faces of $P$.

1. The normal cone $N_F(P)$ is a polyhedron and a cone.
2. We have $N_F(P) \subset N_G(P)$ if and only if $G \subset F$ and equality if and only if $F = G$.
3. The normal cone satisfies $\dim(F) + \dim(N_F(P)) = n$.
4. If $F_1, ..., F_n$ are faces of $P$ then there is a face $K$ such that $\cap_{i=1}^n N_{F_i}(P) = N_K(P)$.

We can use normal cones to find the upper faces of a polytope. Let $P \subset \mathbb{R} \times \mathbb{R}^n$ be a polytope and let $H_1 = \{ 1 \} \times \mathbb{R}^n$.

**Proposition 1.24.** A face $F$ of $P$ is an upper face if and only if $N_F(P) \cap H_1 \neq \emptyset$.

**Proof.** If we consider $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ as a functional it defines a face $P$ by $F = \{ (s, y) \in \mathbb{R} \times \mathbb{R}^n | ts + x \cdot y = M \}$, where $M$ is the maximum of $(t, x)$ on $P$. If $t > 0$ and $(s, y) \in P^{(t,x)}$ then $(s, y)$ is a maximal point of $P$ on the line $\mathbb{R} \times y$ as $M$ is the maximum of $(t, x)$ on $P$. Thus $P^{(t,x)}$ is an upper face.

If $N_F(P) \cap H_1 \neq \emptyset$ pick $(1, x) \in N_F(P) \cap H_1$, then $F \subseteq P^{(1,x)}$ which is a upper face, so $F$ is an upper face as well. This shows one direction of the equivalence.

Let $H_1, ..., H_n$ be the half spaces defining $P$ and let $L_1, ..., L_n$ be the corresponding hyperspaces, then $P \cap L_i$ is a face of $P$ for $i = 1, ..., n$ and the boundary of $P$ is contained in $\cup_{i=1}^n L_i$.

Let $F$ be a nonempty upper face of $P$ then $F = \cup_{i=1}^n F \cap L_i$ and $F \cap L_i$ is a face of $F$. We reorder the half spaces such that $L_1 \cap F, ..., L_k \cap F$ are proper faces of $F$ and $L_k \cap F, ..., L_n \cap F$ are equal to $F$. As the dimension of a proper face of $F$ is smaller that the dimension of $F$ it follows that $k < n$.

Let $z$ be a point of $F$ which is not contained in any proper face of $F$. If $L_i \cap F$ is a proper face of $F$ then $z \in H^0_i$. Thus $z$ is contained in the
interior of $\cap_{i=1}^k L_i$, so there is an $\varepsilon > 0$ such that $z + (\varepsilon, 0, ..., 0) \in \cap_{i=1}^k L_i$. As $z + (\varepsilon, 0, ..., 0) \notin P$ there must be a $j > k$ such that $z + (\varepsilon, 0, ..., 0) \notin H_j$.

The half space $H_j$ is defined by a functional $(t, x)$ and a constant $M \in \mathbb{R}$ via $H_j = \{(s, y) \in \mathbb{R} \times \mathbb{R}^n | ts + x \cdot y \leq M\}$. As $w \cdot (t, x) = M$ and $(z + (\varepsilon, 0, ..., 0)) \cdot (t, x) > M$ it follows that $t > 0$. Now $(1, \frac{1}{t}x)$ also defines $H_j$, so $(1, \frac{1}{t}x)$ is maximal on $F$. Thus $(1, \frac{1}{t}x)$ is the required element of $N_F(P) \cap H_1$.

We will later combine the previous proposition with the following general fact about cones.

**Lemma 1.25.** Let $C \subset \mathbb{R}^n$ be a polyhedron which is a cone of dimension $d$ and let $H$ be an affine hyperspace which does not contain the origin. If $C \cap H \neq \emptyset$ then $\dim(C \cap H) = d - 1$.

**Proof.** The smallest linear subspace $L$ of $\mathbb{R}^n$ which contains $C$ has dimension $d$ and $H \cap L$ is a hyperspace in $L$, so without loss of generality we can assume $\dim(C) = n$.

After a linear transformation we can assume $H$ is defined by $x_1 = 1$. If all interior points of $C$ satisfy $x_1 \leq 0$ then all points of $C$ satisfy $x_1 \leq 0$, which is a contradiction because $C \cap H = \emptyset$. Thus we can pick a a point $x \in C^\circ$. After multiplication with a positive constant it follows that $H$ must contain an interior point of $C$, so $C \cap H$ contains a disc of dimension $d - 1$, which proves the lemma.

Now we can define the Newton polytope of a tropical polynomial and check that this definition makes sense.

**Definition 1.26.** If $f = \max_{i \in I} \{a_i + i \cdot x\}$ is a tropical polynomial, $\text{Newt}(f) = \text{conv}(I)$.

**Lemma 1.27.** The Newton polytope of $f$ only depends on the function defined by $f$ on $\mathbb{R}^n$.

**Proof.** It is enough to show $\text{Newt}(f) = \text{Newt}(f_{\text{red}})$ and it is clear that $\text{Newt}(f_{\text{red}}) \subseteq \text{Newt}(f)$.

Let $k$ be a vertex of $\text{Newt}(f)$, then $N_k(\text{Newt}(f))$ has an interior point $x$. This point satisfies $i \cdot x < k \cdot x$ for every $i \in I - \{k\}$, so for a sufficiently large $\lambda \in \mathbb{R}$ we have $i \cdot \lambda x + a_i < k \cdot \lambda x + a_k$ for every $i \in I - \{k\}$. Thus $f$ is equal to $k \cdot x + a_k$ on $\lambda x + N_k(\text{Newt}(f))$. As $\lambda x + N_k(\text{Newt}(f))$ is a translated cone of dimension $n$ it follows that $C^\circ_k \neq \emptyset$, so $k \cdot x + a_k$ is a term of $f_{\text{red}}$ and $k \in \text{Newt}(f_{\text{red}})$. This shows $\text{Newt}(f) = \text{Newt}(f_{\text{red}})$. 

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We can now use the construction from section 2 to define a lattice subdivision of Newt(\(f\)). We have seen that the function \(v : I \to \mathbb{R}\) given by \(v(i) = a_i\) defines a polytope \(P = \text{conv}\{(a_i, i) | i \in I\} \subset \mathbb{R} \times \mathbb{R}^n\) and the projection of the upper hull of \(P\) is lattice subdivision of Newt(\(f\)). We denote this subdivision with \(S(f)\).

The following elementary proof is based on lecture notes by Bernd Sturmfels which can be found in [13].

**Theorem 1.28.** If \(f\) is a tropical polynomial there is an inclusion reversing bijection between the nonempty cells of \(\mathcal{V}(f)\) and the nonempty cells of \(S(f)\).

**Proof.** By definition we have an inclusion preserving bijection between \(S(f)\) and the upper hull of \(P\). We can include \(\mathcal{V}(f)\) in \(\mathbb{R} \times \mathbb{R}^n\) as \(\{1\} \times \mathcal{V}(f)\), so it suffices to give an inclusion reversing bijection \(\Phi\) from the upper hull of \(P\) to \(\{1\} \times \mathcal{V}(f)\).

For an upper face \(F\) of \(P\) define \(\Phi(F) = N_F(P) \cap H_1\), where \(H_1 = \{1\} \times \mathcal{V}(f)\) is the set of all vertices. Thus \(N_F(P) \cap H_1 = \{1\} \times C_A\) where \(C_A\) is a cell of \(\{1\} \times \mathcal{V}(f)\). In particular \(N_{(a_i, i)} = \{1\} \times C_i\) for all \(i \in I\) such that \((a_i, i)\) is a vertex.

The function \(\Phi\) is inclusion reversing by lemma 1.23.

Let \(k \in I\) such that \(C_k^o \neq \emptyset\), pick \(x \in C_k^o\) and let \(F = P(1, x) \in \mathcal{V}(f)\) then \((1, x) \in N_F(P)\) so \(F\) is a face of \(P\) by lemma 1.24. Because \(N_F(P) \cap H_1 = \{1\} \times C_k\) is maximal it follows that \(N_F(P) \cap H_1 = \{1\} \times C_k\). Now lemma 1.25 shows that \(F\) is a vertex of \(P\) and it is clear that \(F\) can only be \((a_k, k)\).

Let \(\{1\} \times C_A\) be any nonempty cell of \(\{1\} \times \mathcal{V}(f)\) then we can choose \(A\) such that \(C_k^o \neq \emptyset\) for all \(k \in A\) by corollary 1.4. Now \(\{1\} \times C_A = \cap_{k \in A} N_{(a_k, k)}(P) \cap H_1 = (\cap_{k \in A} N_{(a_k, k)}(P)) \cap H_1\). By lemma 1.23 there is a face \(F\) of \(P\) such that \(N_F(P) = \cap_{k \in A} N_{(a_k, k)}(P)\) and \(F\) is an upper face by lemma 1.24. Thus \(\Phi\) is surjective.

If \(F\) and \(G\) are upper faces of \(P\) and \(N_F(P) \cap H_1 = N_G(P) \cap H_1\) then \((N_F(P) \cap N_G(P)) \cap H_1 = N_F(P) \cap H_1\). By proposition 1.23 there is a face \(K\) of \(P\) with \(N_F(P) \cap N_G(P) = N_K(P)\) which contains \(N_F(P)\) and \(N_G(P)\) and by lemma 1.24 this is an upper face. By lemma 1.25 the dimensions of \(N_F(P)\), \(N_G(P)\) and \(N_K(P)\) are equal, so by proposition 1.23 the dimensions of \(F\), \(G\) and \(K\) are equal. As \(F\) and \(G\) are faces of \(K\) it follows that \(F = G = K\).

This shows the assignment \(\Phi\) is an inclusion reversing bijection. \(\square\)
Chapter 2

The amoeba of a planar curve

In this chapter we give the definition of an amoeba which was first introduced in [8]. The idea of applying a logarithmic function to an algebraic variety is not a completely unexpected one. Applying the logarithm to a real algebraic curve was explored by Oleg Viro, see for example [19].

Also a basic result of multivariable complex analysis is that there is a convergent Laurent series on a set of complex numbers if and only if the logarithm of the absolute values of this set is a convex set. This result is used to derive basic results about amoebas and we cite it as proposition 2.5.

We will give some elementary properties and examples of amoebas. The picture are drawn with the algorithm in appendix 6.

2.1 Definitions and examples

In this chapter we work with Laurent polynomials, which are elements of $K[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$, where $K$ is a field. In most of the section $\mathbb{C}$ is the base field, $n = 2$ and we use $u$ and $w$ instead of $z_1$ and $z_2$. Throughout this chapter $f(z) = f(z_1, \ldots, z_n)$ denotes a Laurent polynomial, and $C_f$ denotes the hypersurface defined by $f$ in $(\mathbb{C}^*)^n$.

If $K$ is a field then an absolute value on $K$ is a function $|\cdot|_K : K \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in K$ we have

1. $|a|_K = 0$ if and only if $a = 0$
2. $|ab|_K = |a|_K |b|_K$
3. $|a + b|_K \leq |a|_K + |b|_K$

If we have the stronger condition
\[ |a + b|_K \leq \max\{|a|_K, |b|_K\} \]

then \(|\cdot|_K\) is called non-archimedean.

If \(K\) has an absolute value then we have a map \(\log_K : (K^*)^n \to \mathbb{R}^n\) given by \(\log_K(x_1, ..., x_n) = (\log(|x_1|_K), ..., \log(|x_n|_K))\). If \(K = \mathbb{C}\) we use \(|\cdot|\) instead of \(|\cdot|_\mathbb{C}\) and \(\log\) instead of \(\log_{\mathbb{C}}\). In this chapter we only consider amoebas over the complex numbers and in the next chapter we look at amoebas over a field with a non-archimedean absolute value.

**Definition 2.1.** If \(X \subset (K^*)^n\) is an algebraic variety then the amoeba of \(X\) is the set \(\log_K(X)\).

We can immediately note the following basic properties.

**Proposition 2.2.** If \(X\) is an irreducible complex variety then the amoeba of \(X\) is closed and connected.

**Proof.** The absolute value map is a proper map by the Heine-Borel theorem and \((\mathbb{C}^*)^n\) is locally compact so the absolute value map is a closed map. Thus the \(\log_{\mathbb{C}}\) map is also closed, so the amoeba of \(X\) is closed.

An irreducible complex variety is always connected (see for example section 7.2 of [22]), so the amoeba is connected as well.

It is not immediately clear that this definition gives an interesting set. However the following picture, which shows the amoeba of the curve given by \(f(u, w) = u^3 + w^3 + 2uw + 1\) clearly shows that the amoeba is an object with non-trivial geometry.

![Figure 2.1: The amoeba of the curve defined by \(f(u, w) = u^3 + w^3 + 2uw + 1\).](image)
Example 2.3. The amoeba of an affine hyperplane. In $(\mathbb{C}^*)^2$ such a hyperplane is given by an equation of the form $f(u, w) = au + bw + c$ with $a, b, c \in \mathbb{C}$ where at least two of $a, b, c$ are not zero.

If $a = 0$ then $w = \frac{-c}{b}$ so the amoeba is the set $\{(x, \log(|c|) - \log(|b|)) : x \in \mathbb{R}\}$, which is the line defined by $y = \log(|c|) - \log(|b|)$ in $\mathbb{R}^2$. The case $b = 0$ is similar.

If $c = 0$ then the amoeba of $f$ is the line given by $y = x + \log(|a|) - \log(|b|)$.

![Figure 2.2: The boundary of the amoeba given by $f(u, w) = u + w + 1$.](image)

In the case $a, b, c$ are not zero, then we can consider the case $a = b = c = 1$, as other values give a translation of this amoeba by $(\log(c) - \log(a), \log(c) - \log(b))$. In this case $C_f = \{(u, -u - 1) : u \in \mathbb{C}^*\}$. If $|a| > 1$ then $|a - b + c| = |u + 1|$ can assume any value in $[|a| - 1, |a| + 1]$, if $|a| = 1$, then $|a - b + c|$ can assume any value in $[1 - |a|, |a| + 1]$. Thus the absolute values of the points in $C_f$ is the part of $\mathbb{R}^2_{>0}$ bounded by the lines $\{(t, t + 1) : t \in (0, \infty)\}$, $\{(t, 1 - t) : t \in (0, 1)\}$ and $\{(t, t - 1) : t \in (1, \infty)\}$. We get the amoeba by applying the logarithm.

Notice that in the cases where the area of the amoeba of a line is 0 the Newton polytope of $f$ is not of full dimension.

### 2.2 Laurent series

From now on we only consider the case of the amoeba of a hypersurface defined over the complex numbers by a Laurent polynomial $f$. To understand
the complement of the amoeba we need to look at the Laurent series expansion of \( \frac{1}{f} \). A Laurent series \( S \) centered at 0 is a formal sum \( \sum_{v \in \mathbb{Z}^n} a_v z^v \), where \( a_v \in \mathbb{C} \) and \( z^v = z_1^{v_1} \cdots z_n^{v_n} \). Because \( \mathbb{Z}^n \) has no natural order a Laurent series is said to converge in a point if it converges absolutely. The domain of convergence is the interior of the set of all point where \( S \) converges.

Example 2.4. Let \( f(u, w) = u + w + 1 \) be the polynomial form example 2.3 and consider the Laurent series \( S = \sum_{i, j \in \mathbb{Z}^2} (-1)^{i+j} u^i w^j \), where we note that \( \binom{i+j}{j} = 0 \) if \( i < 0 \) or \( j < 0 \). If \( S \) converges in \( (u_0, w_0) \) then we have \( S(u_0, w_0) = \sum_{k=0}^{\infty} (-1)^k (u_0 + w_0)^k \) by the binomial theorem. This is a geometric series so it follows that \( S \) is a Laurent expansion of \( \frac{1}{f} \).

Applying the binomial theorem to the absolute value gives

\[
\sum_{i, j \in \mathbb{Z}^2} |(-1)^{i+j} \binom{i+j}{j} u^i w^j| = \sum_{i, j \in \mathbb{Z}^2} \left| \binom{i+j}{j} \right| |u|^i |w|^j = \sum_{n=0}^{\infty} (|u| + |w|)^n,
\]

so this series converges absolutely if and only if \( |u| + |w| < 1 \). Thus the domain of convergence of \( S \) is the set \( \{(u, w) \in (\mathbb{C}^*)^2 ||u| + |w| < 1\} \) and applying the Log function to this domain gives one component of the complement of the amoeba.

Using Pascals identity for binomial coefficients and similar arguments to the previous example one can show that

\[
\sum_{i, j \in \mathbb{Z}^2} (-1)^{-i+1} \binom{-i+1}{j} u^i w^j \quad \text{and} \quad \sum_{i, j \in \mathbb{Z}^2} (-1)^{-j+1} \binom{-j+1}{i} u^i w^j
\]

are Laurent expansions of \( \frac{1}{f} \) and that the domains of convergence are \( \{(u, w) \in (\mathbb{C}^*)^2 ||w| < |u| - 1\} \) and \( \{(u, w) \in (\mathbb{C}^*)^2 ||u| < |w| - 1\} \). Thus the domains of convergence give the other components of the complement of the amoeba. A consistent way to find certain Laurent expansions of \( \frac{1}{f} \) is given in the remark at the end of this section.

The correspondence between components of amoeas and Laurent series follows from a general result in multivariable complex analysis. A slightly different version of this proposition is cited in \([8]\). See \([17]\), theorem 2, for a proof of part (b) and \([11]\) theorem 2.3.2 for part (a).

**Proposition 2.5.** (a) If \( S(z) = \sum_{v \in \mathbb{Z}^n} a_v z^v \) is a Laurent series centered at 0 the domain of convergence is of the form \( \text{Log}^{-1}(B) \) where \( B \subset \mathbb{R}^n \) is a convex open subset.\(^1\)

\( ^1 \)In multivariable complex analysis such a domain is referred to as a logarithmically convex Reinhardt domain.
(b) If $\varphi(z)$ is a holomorphic function on a domain $D$ of the form $\text{Log}^{-1}(B)$ with $B \subset \mathbb{R}^n$ open and connected then there is a unique Laurent series centered at 0 which converges to $\varphi(z)$ on $D$.

We can directly apply this to amoebas.

**Corollary.** If $f$ is a Laurent polynomial then the components of $\mathbb{R}^n - \text{Log}(C_f)$ are convex and the components correspond bijectively to the Laurent series expansions of $\frac{1}{f}$ centered at 0.

**Proof.** Let $A$ be a component of $\mathbb{R}^n - \text{Log}(C_f)$ then $\frac{1}{f}$ is a holomorphic function defined on $\text{Log}^{-1}(A)$. By part (b) of proposition 2.5 there is a unique Laurent series $S$ which converges to $\frac{1}{f}$ on $\text{Log}^{-1}(A)$. By part (a) of the same proposition the domain of convergence of $S$ is of the form $\text{Log}^{-1}(B)$ with $B \subset \mathbb{R}^n$ open and convex. We clearly have $A \subset B$ and as $B$ is convex it is also connected, so $A = B$.

Thus the components of the complement of the amoeba are convex and we can associate a unique Laurent series to each.

If $S'$ is another Laurent series of $\frac{1}{f}$ centered at 0, then it has a domain of convergence of the form $\text{Log}^{-1}(B)$ where $B \subset \mathbb{R}^n$ is a convex open subset. As $B$ cannot intersect the amoeba there is a component $A$ of the complement such that $A \cap B \neq \emptyset$. If $S$ is the power series expansion corresponding to $A$ by the first part of the proof then $S' = S$ by part (b) of proposition 2.5.

**Remark.** It is possible compute some of the Laurent series of $\frac{1}{f}$ directly. If $\gamma$ is a vertex of $\text{Newt}(f)$ then we can write $f(z) = a_\gamma z^\gamma (1 + g(z))$, where $g(z) = \sum_{v \in \mathbb{Z}^k, v \neq \gamma} a_v z^v$. Using the geometric series we get a formal identity

$$\frac{1}{f(z)} = a_\gamma^{-1} z^{-\gamma} \sum_{i=0}^{\infty} (-1)^i g(z)^i.$$  

(2.2.1)

Expanding the terms gives a Laurent series which converges on a component of the complement of the amoeba. The details can be found in [8] or [7].

### 2.3 The amoeba and the Newton polytope

In this section we show some elementary properties of amoebas. In particular all amoebas will look somewhat similar to figure 2.1 with a finite number of tentacles. Furthermore the directions of the tentacles are prescribed by the Newton polytope.
For proposition 2.7 we only consider curves in $(\mathbb{C}^*)^2$. This eliminates the need to use Laurent series. The proofs in [S] use the Laurent series of $\frac{1}{f}$ and work in all dimensions.

For a vertex $\gamma$ of $Q = \text{Newt}(f)$ the normal cone $N_\gamma(Q)$ can be identified with a subset of $\mathbb{R}^n$ by formula [1.5.1] so $N_\gamma(Q) = \{a\in\mathbb{R}^n | a \cdot (v - \gamma) \leq 0 \text{ for every } v \in Q\}$.

**Proposition 2.6.** There is a vector $b \in N_\gamma(Q)$ such that $b + N_\gamma(Q)$ does not intersect the amoeba of $f$.

**Proof.** For any $M \in \mathbb{R}$ we can find a vector $b \in N_\gamma(Q)$ such that $b \cdot (v-\gamma) < M$ for every $v \in Q \cap \mathbb{Z}^n$ different from $\gamma$. Now this condition also holds for every element of $b + N_\gamma(Q)$.

For any $v \in (Q - \{\gamma\}) \cap \mathbb{Z}^n$ and $z \in (\mathbb{C}^*)^n$ such that $\log(z) \in b + N_\gamma(Q)$, we have $\log(|z|^{v-\gamma}) = \log(z) \cdot (v - \gamma) < M$. This gives $|z^v| < |z|^e^M$. Let $k+1$ be the number of nonzero coefficients of $f$, then

$$|\sum_{v \neq \gamma} a_v z^v| < \sum_{v \neq \gamma} |a_v| |z|^e^M \leq k \max\{|a_v| : v \neq 0\} e^M |z|^\gamma.$$  

This shows that if we pick $M$ such that $k \max\{|a_v| : v \neq 0\} e^M \leq |a_\gamma|$ we get $|\sum_{v \neq \gamma} a_v z^v| < |a_\gamma| |z|^\gamma$. Thus $f$ cannot have a zero in $z$, so $a$ does not lie in the amoeba. \[\square\]

**Remark.** As stated in the proof it is enough to find $b$ such that $(b, w - \gamma) < M$ for every integral $w \in Q$ different from $\gamma$, where $M \in \mathbb{R}$ is such that $k \max\{|a_v| : v \neq 0\} e^M \leq |a_\gamma|$ where $k+1$ is the number of nonzero coefficients of $f$.

**Remark.** If we compute the cones in the example 2.3 we can use $M = \log(\frac{1}{2})$ so we can pick $b_{(0,0)} = (\log(\frac{1}{2}), \log(\frac{1}{2})), b_{(1,0)} = 0 \text{ and } b_{(0,1)} = (0, \log(\frac{1}{2}))$.

**Proposition 2.7.** The translated cones $b + N_\gamma(Q)$ in the previous proposition lie in distinct components of the complement of the amoeba.

**Proof.** Let $\gamma$ and $\delta$ be vertices of $\text{Newt}(f)$ and let $b_\gamma + N_\gamma(Q)$ and $b_\delta + N_\delta(Q)$ be the cones which do not intersect the amoeba of $f$. Also let $A_\gamma$ and $A_\delta$ be the components containing $b_\gamma + N_\gamma(Q)$ and $b_\delta + N_\delta(Q)$.

First we assume $\gamma$ and $\delta$ are adjacent vertices. We can assume that $\gamma$ and $\delta$ lie on the $y$-axis, $Q$ lies in $\mathbb{R}^2_{\geq 0}$ and $Q$ meets the $x$-axis. By proposition 5.6 this can be achieved by an automorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ defined in equation 5.1.1 and by noting that $Q$ can be moved by integer vectors without changing the curve in $(\mathbb{C}^*)^2$. Now $f$ does not contain negative exponents and is not divisible by $u$ and $w$ in $\mathbb{C}[u, w]$. In this case $f$ also defines a curve in
Figure 2.3: The boundary of the amoeba given by \( f(u, w) = u + w + 1 \) with the translated normal cones.

\( \mathbb{C}^2 \) which restricts to \( C_f \) in \((\mathbb{C}^*)^2\). On the \( w \)-axis of \( \mathbb{C}^2 \) the polynomial \( f \) is given by \( f(0, w) \) which has more than one term, so \( f \) has a zero \( a = (0, a_2) \) with \( a_2 \neq 0 \).

As curves have no isolated points there are points \( \{(a_{j1}, a_{j2})\}_{j \in \mathbb{N}} \in C_f \) such that \( \lim_{j \to \infty} (a_{j1}, a_{j2}) = a \). Now \( \lim_{j \to \infty} \log(||(a_{j1})||) = -\infty \) and \( \lim_{j \to \infty} \log(||(a_{j2})||) = \log(||(a_2)||) \). Thus for sufficiently large \( j \) the point \( (a_{j1}, a_{j2}) \) lies between \( b_\gamma + N_\gamma(Q) \) and \( b_\delta + N_\delta(Q) \). As components are convex this shows \( b_\gamma + N_\gamma(Q) \) and \( b_\delta + N_\delta(Q) \) cannot lie in the same component so \( A_\gamma \neq A_\delta \).

Now we assume \( \gamma \) and \( \delta \) are arbitrary vertices with \( A_\gamma = A_\delta \) and we show that we can find an adjacent vertex \( \eta \) of \( \gamma \) with \( A_\gamma = A_\eta \). We note that by convexity \( A_\gamma \) contains \( b_\gamma + b_\delta + N_\gamma(Q) + N_\delta(Q) \).

We can write \( N_\gamma(Q) = H_1 \cap H_2 \) for open half planes \( H_1 \) and \( H_2 \). If \( N_\delta(Q) \cap H_1 = \emptyset = N_\delta(Q) \cap H_2 \) then \( N_\gamma(Q) + N_\delta(Q) = \mathbb{R}^2 \), which is impossible, so we can assume \( N_\delta(Q) \cap H_1 \neq \emptyset \). If \( \eta \) is the adjacent vertex of \( \gamma \) for which \( N_\eta(Q) \) lies in \( H_1 \) then any translation of \( N_\eta(Q) \) intersects \( b_\gamma + b_\delta + N_\gamma(Q) + N_\delta(Q) \).

Thus \( A_\gamma \cap A_\eta \neq \emptyset \) and therefore \( A_\gamma = A_\eta \), which completes the proof. \( \square \)
2.4 The components of the complement of the amoeba

Using the integral formula for the coefficients of the Laurent expansion of $\frac{1}{f}$ it is possible to derive more information about the complement of the amoeba. In [7] the authors construct a distinct point in $\text{Newt}(f) \cap \mathbb{Z}^n$ for every component of $A^c_f$. The same function was constructed by Mikhalkin in [15] using the topological linking number.

The results can be stated as follows.

**Theorem 2.8** (Fosberg, Passare, Tsikh). *The order of a component of $A^c_f$ is an element of $\text{Newt}(f)$ with integer coefficients and the orders of distinct components are distinct.*

**Corollary.** For a Laurent polynomial $f$ the number of distinct Laurent expansions of $\frac{1}{f}$ is at most the number points in $\text{Newt}(f) \cap \mathbb{Z}^n$ and at least the number of vertices.

The following property of the order will be useful later on.

**Proposition 2.9.** Let $k \in \mathbb{Z}^n \cap \text{Newt}(f)$ and let $z \in (\mathbb{C}^*)^n$ such that $\log(z) \notin A_f$ and $|a_k z_k| > |\sum_{j \neq k} a_j z_j|$, then the order of the complement of $A^c_f$ that contains $\log(z)$ is $k$.

\[\text{See for example [11] theorem 2.7.1}\]
Chapter 3

Puiseux series and Kapranov’s theorem

The aim of this chapter is to give a proof of Kapranov’s theorem. This theorem describes the amoeba of a hypersurface over an algebraically closed field with a non archimedean absolute value in terms of tropical geometry. It was proved by Mikhail Kapranov in an unpublished manuscript [10]. Kapranov, Einslieder and Lind later published a significantly more abstract version of the theorem in [4].

The original result is quoted in [14] and a more general result is proved in the forthcoming book [13] by Maclagan and Sturmfels. The proof in this chapter is based on course notes [24] by Bernd Stumfels.

3.1 Valuations and Puiseux series

Let $K$ be a field with absolute value $|\cdot|_K$. The absolute value $|\cdot|_K$ is called non-Archimedean if we have $|a + b|_K \leq \max\{|a|_K, |b|_K\}$ for every $a, b \in K$. This is a stronger version of the normal triangle inequality. It is often intuitive to define non-Archimedean absolute values in terms of valuations.

Definition 3.1. A valuation on a field $K$ is a function $\text{val}: K \to \mathbb{R} \cup \{\infty\}$ such that for all $a, b \in K$ we have

1. $\text{val}(a) = \infty$ if and only if $a = 0$
2. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
3. $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$

It follows directly from the seconds axiom that $\text{val}(1) = 0$ and $\text{val}(a^{-1}) = -\text{val}(a)$. We also note the following.
Lemma 3.2. If \( \text{val}(a) \neq \text{val}(b) \) then \( \text{val}(a + b) = \min\{\text{val}(a), \text{val}(b)\} \).

Proof. We can assume \( \text{val}(a) < \text{val}(b) \), then \( \text{val}(a + b) \geq \text{val}(a) \). Suppose that \( \text{val}(a + b) > \text{val}(a) \), then \( \text{val}(a) = \text{val}((a + b) - b) \geq \min\{\text{val}(a + b), \text{val}(b)\} > \text{val}(a) \) which is impossible, so \( \text{val}(a + b) \leq \text{val}(a) \).

We can switch between non-Archimedean absolute values and valuations by the following proposition.

Proposition 3.3. If \( | \cdot |_K \) is a non-Archimedean absolute value on a field \( K \) then \( \text{val}(a) = -\log |a|_K \) defines a valuation on \( K \). Conversely if \( \text{val} \) is a valuation on \( K \) the function defined by \( |a|_K = e^{-\text{val}(a)} \) defines a non-Archimedean absolute value on \( K \).

Now we construct the field of Puiseux series, which is an extension of the field of Laurent series in which more exponents are allowed.

The valuation on the field \( \mathbb{C}((t)) \) of formal Laurent series over \( \mathbb{C} \) defined by for \( p = \sum_{i \in I} a_i t^i \) by \( \text{val}(p) = \min\{i \in I | a_i \neq 0\} \) is an example of a non-Archimedean valuation. For \( k > 0 \) the field \( \mathbb{C}((t^{1/k})) \) is an extension of \( \mathbb{C}((t)) \) of degree \( k \) and the valuation can be extended by the same definition.

A Puiseux series over \( \mathbb{C} \) is a series of the form \( \sum_{i \in I} a_i t^i \), where \( a_i \in \mathbb{C} \) and \( I \subset \mathbb{Q} \) is a set of fractions with a common denominator and a minimal element. We denote the set of Puiseux series over \( \mathbb{C} \) with \( \mathbb{C}\{\{t\}\} \). If \( k \) is a common denominator for \( I \) then \( \sum_{i \in I} a_i t^i \in \mathbb{C}((t^{1/k})) \). This shows we have \( \mathbb{C}((t^{1/k})) \subset \mathbb{C}\{\{t\}\} \) for all \( k \in \mathbb{Z}_{>0} \) and \( \mathbb{C}\{\{t\}\} = \bigcup_{k=1}^{\infty} \mathbb{C}((t^{1/k})) \). Thus \( \mathbb{C}\{\{t\}\} \) is a field extending \( \mathbb{C}((t)) \).

If \( p = \sum_{i \in I} a_i t^i \) is a Puiseux series we put \( \text{val}(p) = \min\{J\} \). It is easy to verify that \( \text{val} \) is a non-Archimedean valuation on \( \mathbb{C}\{\{t\}\} \).

The following theorem is due to Puiseux but was also known to Newton. In the language of modern algebra this theorem can be proved easily by applying Hensel’s lemma to the ring \( \mathbb{C}[[t]] \) as is done in [6].

Theorem 3.4 (Puiseux). The field \( \mathbb{C}\{\{t\}\} \) is the algebraic closure of \( \mathbb{C}((t)) \).

We can now take a look at amoebas defined over \( K = \mathbb{C}\{\{t\}\} \). The absolute value on \( K \) is given by \( |p|_K = e^{-\text{val}(p)} \), so the \( \log_K \) function on \( (K^*)^n \) is given by \( \log_K(p_1,...,p_n) = (-\text{val}(p_1),...,-\text{val}(p_n)) \).

We now compute a very simple example of an amoeba which is similar to the general case.

Example 3.5. Let \( C_f \) be the curve in \( (K^*)^2 \) defined by \( f = u + w + t \) and let \( T \) be the tropical curve \( V(x \oplus y \oplus -1) \). We claim \( \text{Log}_K(C_f) = T \cap \mathbb{Q}^2 \).

A point of \( C_f \) is of the form \( (p, -p - t) \) for some \( p \in K^* \). If \( -\text{val}(p) \neq -1 \) then \( \text{val}(-p-t) = \min\{\text{val}(p), 1\} \), so \( (-\text{val}(p), -\text{val}(-p-t)) = (-\text{val}(p), -1) \)
Example 3.6. Let \( k / f \) it is the union of the tropical curves of \( f \) tropical hypersurface. We have delayed this section because it is necessary to use the notion of a

3.2 Tropical polynomials

Thus \( \log \) and \( \log \) to the proof that a polynomial ring is a ring so we can note the following

Proposition 3.7. Let \( f = \bigoplus_{i \in I} a_i \odot x_1^{i_1} \odot \cdots \odot x_n^{i_n} \) and \( g = \bigoplus_{j \in J} b_j \odot x_1^{j_1} \odot \cdots \odot x_n^{j_n} \) we put \( f \odot g = \bigoplus_{k \in K} (a_k \odot b_k) \odot x_1^{k_1} \odot \cdots \odot x_n^{k_n} \) where \( K = I \cup J \) and \( a_k = -\infty \) if \( k \notin I \). We also put \( f \odot g = \bigoplus_{k \in K} c_k \odot x_1^{k_1} \odot \cdots \odot x_n^{k_n} \) where \( K = I + J \) and \( c_k = \bigoplus_{i,j,i+j=k} a_i \odot b_j \).

Example 3.6. Let \( f = x \oplus -y \oplus -1 \) and \( g = -x \oplus -y \oplus -1 \), then
\[
f \odot g = -1 \odot -x \odot -1 \odot -y \oplus 0 \odot x \odot -y \oplus y \odot -x \oplus -1 \odot x \oplus -1 \odot y.
\]

The tropical curve associated to \( f \odot g \) can be seen in figure 3.1. As expected it is the union of the tropical curves of \( f \) and \( g \).

The proof that a polynomial ring over a semiring is a semiring is identical to the proof that a polynomial ring is a ring so we can note the following fact.

Proposition 3.7. The set \( \mathbb{T}[x_1, \ldots, x_n] \) is a commutative semiring for the operations \( \oplus \) and \( \odot \).

In order to use addition and multiplication of tropical polynomials we need to look at the functions the define on \( \mathbb{R}^n \). In case all coefficients are zero the statement about the union of the tropical hypersurfaces is essentially proposition 7.12 in \[26\].
Proposition 3.8. If \( f \) and \( g \) are tropical polynomials and \( \mathbf{v} \in \mathbb{R}^n \) then 
\[
    f \oplus g(\mathbf{v}) = \max\{f(\mathbf{v}), g(\mathbf{v})\} \quad \text{and} \quad f \odot g(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v}).
\]
Furthermore 
\[
    V(f \odot g) = V(f) \cup V(g).
\]

Proof. Pick \( \mathbf{v} \in \mathbb{R}^n \) then 
\[
    \max\{f(\mathbf{v}), g(\mathbf{v})\} = \max_{i \in I} \{a_i + i \cdot \mathbf{v}\}, \max_{j \in J} \{a_j + j \cdot \mathbf{v}\} = \max_{k \in I \cup J} \{a_k, b_k\} + k \cdot \mathbf{v} = (f \odot g)(\mathbf{v}),
\]
which proves the first assertion.

Suppose \( a_k + k \cdot \mathbf{v} \) is a maximal term in \( f(\mathbf{v}) \) and \( a_l + l \cdot \mathbf{v} \) is a maximal term in \( g(\mathbf{v}) \). Thus we have 
\[
    a_i + i \cdot \mathbf{v} + b_j + j \cdot \mathbf{v} \leq a_k + k \cdot \mathbf{v} + b_l + l \cdot \mathbf{v}
\]
for all \( i \in I \) and \( j \in J \). If \( i + j = k + j \) this gives \( a_i + b_j \leq a_k + b_l \) so the coefficient 
\[
    c_{k+j} = \max_{i,j:i+j=k+j} \{a_i + b_j\}
\]
of \( f \odot g \) at \( k + j \) is equal to \( a_k + b_l \). From the same inequality it follows that \( a_k + b_l + (k + l) \cdot \mathbf{v} \) is a maximal term of \( f \odot g(\mathbf{v}) \). This proves the second assertion.

If \( \mathbf{v} \in V(f) \) then there are distinct \( k, k' \in I \) such that \( a_k + k \cdot \mathbf{v} \) and \( a_{k'} + k' \cdot \mathbf{v} \) are maximal. If \( b_l + l \cdot \mathbf{v} \) is a maximal term of \( g \) at \( \mathbf{v} \) then 
\[
    a_k + b_l + (k + l)\mathbf{v} \quad \text{and} \quad a_{k'} + b_l + (k' + l)\mathbf{v}
\]
are maximal terms of \( f \odot g \) at \( \mathbf{v} \) so \( \mathbf{v} \in V(f \odot g) \). The case \( \mathbf{v} \in V(g) \) is identical. If \( \mathbf{v} \notin V(f) \cup V(g) \) then \( f \) and \( g \) are linear on a neighborhood of \( \mathbf{v} \) so \( f \oplus g \) is as well, so we have proved the third assertion. \( \Box \)
The following fact will be used together with lemma 3.2.

**Lemma 3.9.** Let \( f \) and \( g \) be tropical polynomials and let \( c_k = \max_{i,j;i+j=k} \{ a_i + b_j \} \) be the coefficient of \( f \circ g \) at \( x^k \). If \( c_k + k \cdot x \) is maximal at a point \( v \notin V(f \circ g) \) then the maximum in \( c_k = \max_{i,j;i+j=k} \{ a_i + b_j \} \) is unique.

**Proof.** By lemma 3.8 we have \( v \notin V(f) \) and \( v \notin V(g) \) so we can pick \( i \in I \) and \( j \in J \) such that \( a_i + i \cdot x < a_i + j \cdot x \) for every \( i' \in I \) distinct from \( i \) and \( a_{i'} + j' \cdot x < a_j + j \cdot x \) for every \( j' \in J \) distinct from \( j \). As in the proof of lemma 3.8 it follows that \( i + j = k \) and \( c_k = a_i + b_j \). If \( c_k = a_{i'} + b_{j'} \) for \( i' \in I - \{ i \} \) and \( j' \in J - \{ j \} \) then

\[
c_k + k \cdot v = a_{i'} + b_{j'} + (i' + j') \cdot v < a_i + b_j + (i + j) \cdot v = c_k + k \cdot v,
\]

so \( c_k \) is indeed unique. \( \square \)

**Definition 3.10.** If \( K \) is a field with a valuation, \( f = \sum_{i \in I} a_i z^i \) with \( I \subset \mathbb{Z}^n \) a Laurent polynomial then

\[
trop(f) = \bigoplus_{i \in I} -\text{val}(a_i) \odot x^i.
\]

It is necessary to use \( -\text{val} \) because if \( a_i = 0 \) then \( -\text{val}(a_i) = -\infty \), so the term \( -\text{val}(a_i) \odot x^i \) does not contribute to \( trop(f) \) either.

The following example shows that we have to look at functions on \( \mathbb{R}^n \) rather than tropical polynomials.

**Example 3.11.** Let \( f = z + t \) and \( g = z + (-t + t^2) \), then \( trop(fg) = x^2 \oplus -2 \odot x \oplus -2 \) and \( trop(f) \circ trop(g) = x^2 \oplus -1 \odot x \oplus -2 \). However both \( trop(fg) \) and \( trop(f) \circ trop(g) \) define the function \( \max \{ -2, 2x \} \) on \( \mathbb{R}^2 \).

The following proposition explains why tropicalization is a meaningful construction.

**Proposition 3.12.** If \( f \) and \( g \) are polynomials over a field with a valuation then \( trop(fg) \) defines the same function on \( \mathbb{R}^n \) as \( \text{trop}(f) \circ \text{trop}(g) \). In particular we have \( V(\text{trop}(fg)) = V(\text{trop}(f)) \cup V(\text{trop}(g)) \).

**Proof.** Let \( f = \sum_{i \in I} a_i z^i \) and \( g = \sum_{j \in J} b_j z^j \) with \( I, J \subset \mathbb{Z}^n \), then \( fg = \sum_{k \in K} \hat{c}_k z^k \) with \( K = I + J \) and \( \hat{c}_k = \sum_{i,j;i+j=k} a_i b_j \). Thus \( trop(fg) = \bigoplus_{k \in K} -\text{val}(\hat{c}_k) \odot x^k \). We also have \( trop(f) \circ trop(g) = \bigoplus_{k \in K} c_k \odot x^k \), where

\[
c_k = \max_{i,j;i+j=k} \{ -\text{val}(a_i) - \text{val}(b_j) \} = \max_{i,j;i+j=k} \{ -\text{val}(a_i b_j) \}.
\]
By lemma 3.2 we get
\[
- \text{val}(\tilde{c}_k) \leq - \min_{i,j: i+j=k} \{\text{val}(a_i b_j)\} = \max_{i,j: i+j=k} \{\text{val}(a_i b_j)\} = c_k.
\]

Thus \(\text{trop}(fg)(v) \leq \text{trop}(f) \circ \text{trop}(g)(v)\) for every \(v \in \mathbb{R}^n\).

Let \(v \notin V(\text{trop}(f) \circ \text{trop}(g))\) and let \(c_k + k \cdot v\) be the maximal term of \(\text{trop}(f) \circ \text{trop}(g)\) at \(v\) then the maximum in \(c_k = \max_{i,j: i+j=k} \{-\text{val}(a_i) - \text{val}(b_j)\}\) is unique by lemma 3.9. Thus by lemma 3.2 we have \(\text{trop}(fg)(v) = \text{trop}(f) \circ \text{trop}(g)(v)\).

This shows \(\text{trop}(fg)\) and \(\text{trop}(f) \circ \text{trop}(g)\) are equal on the complement of \(V(\text{trop}(f) \circ \text{trop}(g))\). As both functions are continuous they must be equal.

This proves the first assertion and the second one follows directly. \(\Box\)

The same type of statement is false for the tropical sum of polynomials.

**Example 3.13.** Let \(f = z + t + t^2\) and \(g = z - t\) then \(\text{trop}(f) \oplus \text{trop}(g) = (x \oplus -1) \oplus (x \oplus -1) = (x \oplus -1)\) while \(\text{trop}(f + g) = x \oplus -2\), so \(V(\text{trop}(f + g)) = \{\{-2\}\}\) and \(V(\text{trop}(f) \oplus \text{trop}(g)) = \{-1\}\).

### 3.3 Kapranov’s theorem

Now we can sharpen Puiseux’s theorem in two steps. The second step will be Kapranov’s theorem. For a polynomial \(f\) over the field of Puiseux series it gives a complete description of the amoeba \(A_f\) in terms of a tropical hypersurface.

The first step is basically Kapranov’s theorem in one dimension. It is an easy consequence of Puiseux’s theorem and lemma 3.12.

**Lemma 3.14.** Let \(f \in \mathbb{C}\{\{t\}\}[z]\) be a polynomial in one variable over the field of Puiseux series and suppose \(v \in V(\text{trop}(f))\) then \(f\) has a root \(u\) such that \(-\text{val}(u) = v\).

**Proof.** By lemma 3.12 we can assume \(f\) is monic. We prove the lemma by induction on \(\deg(f)\). If \(\deg(f) = 1\) then \(f = z - u\) and \(\text{trop}(f) = x \oplus -\text{val}(u)\), so the result is clear, even in the case \(u = 0\).

Suppose the lemma holds for polynomials of degree at most \(n\). Let \(\deg(f) = n + 1\) and let \(v \in V(\text{trop}(f))\). By Puiseux theorem we have a root \(w\) of \(f\). If \(-\text{val}(w) = v\) we are done. Otherwise we write \(f = (z - w)f'\).

By the case \(n = 1\) we have \(V(z - w) = \{-\text{val}(w)\}\) and by lemma 3.12 it follows that \(V(\text{trop}(f)) = V(\text{trop}(f')) \cup \{-\text{val}(w)\}\). Thus \(v \in V(\text{trop}(f'))\) so by induction we get a root \(u\) of \(f'\) such that \(-\text{val}(u) = v\). As \(u\) is also a root of \(f\) we are done. \(\Box\)
For products of linear polynomials we could repeat the proof of lemma 3.14 with a suitable version of example 3.5, but this does not work for irreducible polynomials.

The solution from [24] is to use an inclusion of $K^*$ in $(K^*)^n$ and apply lemma 3.14 to get a root of the pullback of the polynomial.

However the inclusion used in [24] does not always work because tropicalization does not always behave nicely with respect to pullbacks as we can see in example 3.16. This can be fixed by requiring an additional assumption on the inclusion.

**Theorem 3.15** (Kapranov). If $f = \sum_{i \in I} a_i z^i$ is a polynomial over the field of Puiseux series in $n$ variables then $A_f = V(\text{trop}(f)) \cap \mathbb{Q}^n$.

**Proof.** Let $f = \sum_{i \in I} a_i z^i$ with $I \subset \mathbb{Z}^n$.

For $p \in C_f$ let $v = \log K(p) = (-\text{val}(p_1), ..., -\text{val}(p_n))$ then $-\text{val}(a_ip^i) = -\text{val}(a_i) + i \cdot v$ for all $i \in I$. We have

$$\text{trop}(f)(v) = \max_{i \in I} \{-\text{val}(a_i) + i \cdot v\}. \quad (3.3.1)$$

and this maximum is attained for only one index in $I$ if and only if the minimum $\min_{i \in I} \{\text{val}(a_ip^i)\}$ is attained by one index in $I$. If that is the case then we have $A_f \subseteq V(\text{trop}(f)) \cap \mathbb{Q}^n$.

Now let $v \in V(\text{trop}(f)) \cap \mathbb{Q}^n$. We construct an inclusion of $K^*$ into $(K^*)^n$ to apply lemma 3.14.

For any nonzero $\alpha \in \mathbb{Z}^n$ with coprime coefficients the function $\varphi(s) = (t^{-\alpha_1}s^{\alpha_1}, ..., t^{-\alpha_n}s^{\alpha_n})$ defines an inclusion $\varphi: K^* \to (K^*)^n$. If we define $\psi: \mathbb{R} \to \mathbb{R}^n$ by $\psi(\lambda) = v + \lambda \alpha$ then the diagram

$$\begin{array}{ccc}
K^* & \xrightarrow{-\text{val}} & \mathbb{R} \\
\varphi \downarrow & & \downarrow \psi \\
(K^*)^n & \xrightarrow{\log K} & \mathbb{R}^n
\end{array} \quad (3.3.2)
$$

is commutative and $\psi(0) = v$. Thus it suffices to find an $\alpha$ such that $\text{trop}(\varphi^* f)$ has a root at 0.

To ensure this we pick an $\alpha \in \mathbb{Z}^n$ such that $\alpha \cdot i \neq \alpha \cdot j$ for every pair of distinct $i, j \in I$. As each of these equations defines the complement of hypersurface it is possible to pick such an $\alpha$. Now the pullback of $f(z)$ is the polynomial $g = \varphi^* f$ given by $g(z) = f(t^{-\alpha_1}z^{\alpha_1}, ..., t^{-\alpha_n}z^{\alpha_n})$ on $K^*$.

To get the tropicalization of $g$ we expand it so $g(z) = \sum_{k \in K} b_k z^k$, where
\[ b_k = \sum_{i \in I, \alpha \cdot i = k} a_i t^{-i \cdot v}. \]

By the choice of \( \alpha \) each sum \( b_k \) has only one term so
\[-\operatorname{val}(b_k) = -\operatorname{val}(a_i) + i \cdot v \]
if there is a \( i \in I \) with \( \alpha \cdot i = k \) and \( b_k = -\infty \) otherwise.

The restriction of \( \operatorname{trop}(f) \) to the image of \( \psi \) is given by
\[
\operatorname{trop}(f)(\mathbf{v} + \lambda \alpha) = \max_{i \in I} \{ -\operatorname{val}(a_i) + i \cdot (\mathbf{v} + \lambda \alpha) \}.
\]

Thus \( \operatorname{trop}(g) \) is equal to the restriction of \( \operatorname{trop}(f) \) and \( \operatorname{trop}(g) \) is not locally linear at 0, so \( g \) has a zero \( s \) with \( -\operatorname{val}(s) = 0 \). Now \( \varphi(s) \) is a root of \( f \) because \( g \) is the pullback of \( f \) and \( \operatorname{Log}_K(\varphi(s)) = \mathbf{v} \) because diagram 3.3.2 commutes.

The following example shows that tropicalization does not always commute with pullbacks.

**Example 3.16.** Let \( f = t u w^2 + (-t + t^2)u^2 w + tu^5 \), then \( \operatorname{trop}(f) = \max\{-1 + x + 2y, -1 + 2x + y, -1 + 5x\} \) and \( V(f) \) is the curve in figure 3.2

![Figure 3.2](image-url)

Figure 3.2: The tropical curve defined by \( \max\{-1 + x + 2y, -1 + 2x + y, -1 + 5x\} \).

Pick \( \alpha = (1, 1) \) and define \( \varphi \) and \( \psi \) as in diagram 3.3.2. Now \( \alpha \) does not satisfy the conditions required in the proof of Kapranov’s theorem.

The pullback \( \varphi^* f \) of \( f \) to \( K^* \) is given by \( \varphi^* f(z) = t^2 z^3 + t z^5 \), so \( \operatorname{trop}(\varphi^* f) = \max\{-2 + 3x, -1 + 5x\} \) and the restriction of \( \operatorname{trop}(f) \) to \( \mathbb{R} \) is given by
\[
\max\{-1 + 3x, -1 + 5x\}.
\]

Thus \( \operatorname{trop}(\varphi^* f) \) is locally linear at 0 and the root is at \( x = -\frac{1}{2} \). Thus with lemma 3.14 we get a root \( p \) of \( f \) with \( \operatorname{Log}_K(p) = (\frac{1}{2}, \frac{1}{2}) \). In figure 3.2 we can see that the root has shifted along one of the branches of \( V(\operatorname{trop}(f)) \).
Even if trop commutes with the pullback lemma \ref{lem:pullback} can still give the wrong root if the inclusion of $\mathbb{R}$ in $\mathbb{R}^n$ does not intersect $V(f)$ transversally.

**Example 3.17.** Let $f = u + w + t$, then $\text{trop}(f) = \max\{x, y, -1\}$ and $\mathbf{v} = (0, 0) \in V(\text{trop}(f))$. Let $\alpha = (1, 1)$ and define $\varphi$ and $\psi$ as in diagram \ref{fig:diagram}. Now $\varphi^* f(z) = 2z + t$, so $\text{trop}(\varphi^* f) = \max\{\lambda, -1\}$. Also $\text{trop}(f) = \max\{x, y, -1\}$, so the restriction to $\mathbb{R}$ is given by $\max\{\lambda, \lambda, -1\}$ which is equal to $\text{trop}(\varphi^* f)$.

The tropical root of $\text{trop}(\varphi^* f)$ is $\lambda = -1$, so the root $\mathbf{p}$ of $f$ given by lemma \ref{lem:pullback} satisfies $\text{Log}_K(\mathbf{p}) = (-1, -1)$, so it is not the root we were looking for.

**Remark.** Kapranov’s theorem holds for any algebraically closed field with a non-Archimedean absolute value. The proof given here also works in the general case. We proved \ref{lem:pullback} for arbitrary fields with a valuation, so the proof of lemma \ref{lem:pullback} is identical in the general case. In the proof of Kapranov’s theorem the elements $t^{-v_1}, ..., t^{-v_n}$ have to be replaced with elements of the right absolute value. This is possible in an algebraically closed field.
Chapter 4

A limit of amoebas

In this chapter we study the limit of a collection of amoebas. We follow an article [14] by Grigory Mikhalkin. This approach makes it possible to construct algebraic curves which have certain topological properties.

This kind of construction was first used by Oleg Viro to construct real algebraic curves with a particular topology in [19]. In [14] it is used to find the topology of an arbitrary smooth hypersurface in $(K^*)^n$.

4.1 The construction of the limit

Let $f = \sum_{j \in Q} a_j z^j$ be a Laurent polynomial, where $Q \subset \mathbb{Z}^n$ is the set $\{j \in \mathbb{Z}^n | a_j \neq 0\}$. We make a limit of amoebas by changing both the base of the logarithm and the coefficients of $f$. Let $C_f$ be the curve defined by $f$ and let $A_f$ be the amoeba of $f$. For $r \in \mathbb{R}_{>1}$ we define $\operatorname{Log}_r : \mathbb{C}^n \to \mathbb{R}^n$ by $\operatorname{Log}_r(z_1, ..., z_n) = (\log_r |z_1|, ..., \log_r |z_n|)$, then $\operatorname{Log}_r(C_f) = A_f \cdot M$, where $M$ is the diagonal matrix with entries $\frac{1}{\log(r)}$. Thus $\operatorname{Log}_r(C_f)$ is homeomorphic with $A_f$. The coefficients of $f$ are changed by the formula $f_r = \sum_{j \in Q} a_j r^{v(j)} z^j$ for some function $v : Q \to \mathbb{Q}$. If $X_r$ is the hypersurface defined by $f_r$, then we define $A_r = \operatorname{Log}_r(X_r)$.

We can also look at the tropical polynomial $f_t(z) = \sum_{j \in Q} a_j t^{-v(j)} z^j \in \mathbb{C}\{\{t\}\}[z^\pm]$. The fact that the minus sign appears in the exponent is an unfortunate consequence of our conventions in the Log map and the definition of the tropicalization of a polynomial.

The polynomial $f_t$ is called the patchworking polynomial. As $\mathbb{C}\{\{t\}\}$ is a field with a non-archimedean norm, the polynomial $f_t$ defines a non-archimedean amoeba. By Kapranov’s theorem the closure of this amoeba is the tropical curve defined by $\operatorname{trop}(f_t) = \max_{j \in Q} \{v(j) + j \cdot \mathbf{x}\}$, where $\mathbf{x} = (x_1, ..., x_n)$ is a real variable. We denote this tropical curve with $A_K$. 

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The goal of this chapter is to prove that for $r \to \infty$ the amoebas $\mathcal{A}_r$ converge to $\mathcal{A}_K$.

**Example 4.1.** We can already compute an easy example. Let $f = u + w + 1$ and $v(0, 0) = v(1, 0) = v(0, 1) = 0$, then $\mathcal{A}_r = M \cdot \mathcal{A}_f$, where $M$ is the diagonal matrix with entries $\frac{1}{\log(r)}$. It is easy to see these set converge to the union of the three lines given by $\{((-t, 0)|t \in \mathbb{R}_{\geq 0})\} \cup \{(0, t)|t \in \mathbb{R}_{\geq 0})\} \cup \{((-t, -t)|t \in \mathbb{R}_{\geq 0})\}$, which is the tropical curve given by $x \oplus y \oplus 0 = \text{trop}(f)$.

![Figure 4.1: The set $\mathcal{A}_r$ with $r = e^7$ in blue and the tropical curve of $x \oplus y \oplus 1$ in red.](image)

In case $f$ is a Laurent polynomial in two variables we have $\text{vol}(\mathcal{A}_f) \leq \pi^2 \text{vol}(\text{Newt}(f))$ by [18]. Thus we get $\text{vol}(\text{Log}_r(C_f)) \leq \frac{\pi^2}{\log(r)^n} \text{vol}(\text{Newt}(f))$, so the volume of $\text{Log}_r(C_f)$ tends to zero when $r$ goes to $\infty$. This also happens when we change the coefficients of $f$ as the newton polytope remains the same. Thus we know that if $\lim_{r \to \infty} \mathcal{A}_r$ exists it has zero volume.

Furthermore proposition 2.6 and theorem 1.28 together with the preceding example already suggest a strong relation between the limit and a tropical curve.
4.2 The Hausdorff distance

We need to make precise what it means for a sequence of sets to converge, this is expressed by the Hausdorff distance.

**Definition 4.2.** The Hausdorff distance between two closed sets $A, B \subset \mathbb{R}^n$ is given by

$$\max\{\sup_{a \in A}(d(a, B)), \sup_{b \in B}(d(b, A))\}.$$  

The Hausdorff distance does not give a metric on the set of closed subsets of $\mathbb{R}^n$ because it is not always finite. However it does satisfy the triangle inequality. For a collection $\{A_r\}_{r \in \mathbb{R}^+}$ we can define the limit as the closed set $B \subset \mathbb{R}^n$ such that $\lim_{r \to \infty} d_H(A_r, B) = 0$ if such a set exists. This set is unique by the following lemma.

**Lemma 4.3.** If $\{A_r\}_{r \in \mathbb{R}^+}$ converges to $B$ then $B$ consists of all $b \in \mathbb{R}^n$ for which there are $a_r \in A_r$ such that $\lim_{r \to \infty} a_r = b$.

**Proof.** By definition of $d_H$ for every $b \in B$ we can find a $a_r \in A_r$ with $d(a_r, b) \leq d_H(A_r, B)$, so $\lim_{r \to \infty} a_r = b$.

If $b \notin B$ then $d(b, B) > \varepsilon$ for some $\varepsilon \in \mathbb{R}_{>0}$. By the convergence of the $A_r$ we have an $M \in \mathbb{R}_{>0}$ such that $d_H(A_r, B) \leq \frac{\varepsilon}{2}$ for all $r > M$. Thus also $\sup_{a \in A_r} d(a, B) \leq \frac{\varepsilon}{2}$. By the triangle inequality we get that $d(b, A_r) > \frac{\varepsilon}{2}$ for $r > M$, so $b$ is not the limit of points in the sets $A_r$. \hfill \Box

The converse of the lemma does not hold, for example if $A_r = [0, r]$ then the set of limit points is $[0, \infty)$, but $d_H([0, r], [0, \infty)) = \infty$ for every $r \in \mathbb{R}_{>0}$.

4.3 A neighborhood of the tropical curve

Now we can construct a collection of decreasing neighborhoods of $A_K$ which contain $A_r$. Let $M \in \mathbb{R}_{>1}$ and let $A'_K$ be the subset of $\mathbb{R}^n$ described by the inequalities

$$c_k + k \cdot x \leq \max_{j \in Q \setminus \{k\}} \{c_j + j \cdot x\} + \log_r(M),$$

where $k$ ranges over $Q$ and $c_j = v(j) + \log_r |a_j|$. To ensure both $A_K \subset A'_K$ and $A_r \subset A'_K$ we pick $M = \max\{N, M_f^2\}$, where $N = \#Q - 1$ and $M_f = \max_{j \in Q} \{|a_j|, |a_j|^{-1}\}$.

**Lemma 4.4.** The set $A'_K$ is a neighborhood of $A_K$ for every $r \in \mathbb{R}_{>1}$ and $\lim_{r \to \infty} d_H(A'_K, A_K) = 0$.  

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Proof. The tropical curve $A_K$ is given by the inequalities
\[
v(k) + k \cdot x \leq \max_{j \in Q-\{k\}} \{v(j) + j \cdot x\},
\]
so if $y \in A_K$ we get $c_k + k \cdot y \leq \max_{j \in Q-\{k\}} \{v(j) + j \cdot y\} + \log_r |a_k|$. We have $\log_r |a_k| \leq \log_r (M_f)$ and $\max_{j \in Q-\{k\}} \{v(j) + j \cdot y\} \leq \max_{j \in Q-\{k\}} \{v(j) + \log_r |a_j| + j \cdot y\} + \log_r (M_f)$. This shows $A_K \subseteq A_K'$ for every $r \in \mathbb{R}_{>0}$ and this also implies $\sup_{y \in A_K} (d(y, A_K')) = 0$.

Now we estimate $d(y, A_K)$ for $y \in A_K$. If $y \in A_K$, then $d(y, A_K) = 0$. Thus we can assume $y$ is a the component of the complement of $A_K$ defined by $v(k) + k \cdot x > \max_{j \in Q-\{k\}} \{v(j) + j \cdot x\}$ for some $k \in Q$. Pick $j_1 \in Q - \{k\}$ such that $v(j_1) + j_1 \cdot y = \max_{j \in Q-\{k\}} \{v(j) + j \cdot y\}$.

By definition of $A_K$, we have $v(k) + \log_r |a_k| + k \cdot y \leq \max_{j \in Q-\{k\}} \{v(j) + \log_r |a_j| + j \cdot y\} + \log_r (M)$. Let $y' = y - \frac{3 \log_r (M)}{1 \cdot (k-j)^2} (k-j_1)$. We have $v(k) + k \cdot y' \leq v(j_1) + j_1 \cdot y'$, so $y'$ does not lie in the same component of $A_K$ as $y$. Thus $d(y, A_K) \leq d(y, y') = 3 \log_r (M)$. This shows $\sup_{y \in A_K} d(y, A_K) \leq 3 \log_r (M)$, which proves the lemma.

We only need the sets $A_K'$ to show the convergence of the $A_r$ and the following two properties are used in the proof of theorem 4.7.

\begin{lemma}
For every $r \in \mathbb{R}_{>0}$ we have $A_r \subseteq A_K'$.
\end{lemma}
\begin{proof}
If $x \in A_r$ we can pick $z \in (\mathbb{C}^*)^n$ with $f_r(z) = 0$ and $\Log_r(z) = x$. We have $\sum_{j \in Q} a_j r^{v(j)} z^j = 0$, so $|a_k r^{v(k)} z^k| = |\sum_{j \in Q-\{k\}} a_j r^{v(j)} z^j|$ for every $k \in Q$. Applying $\log_r$ and the triangle inequality gives
\[
c_k + k \cdot \Log_r(z) = \log_r(\left|\sum_{j \in Q-\{k\}} a_j r^{v(j)} z^j\right|) \\
\leq \log_r(\sum_{j \in Q-\{k\}} |a_j r^{v(j)} z^j|) \\
= \log_r(\sum_{j \in Q-\{k\}} |r^{e_j}| |z^j|) \\
\leq \log_r(N \max_{j \in Q-\{k\}} |r^{e_j}| |z^j|) \\
= \max_{j \in Q-\{k\}} \{c_j + j \cdot \Log_r(z)\} + \log_r(N),
\]
so $x = \Log_r(z) \in A_K'$.
\end{proof}

\begin{lemma}
If the components of the complement of $A_K'$ given by $c_k + k \cdot x > \max_{j \in Q-\{k\}} \{c_j + j \cdot x\}$ for $k_1, k_2 \in Q$ and $k_1 \neq k_2$ are nonempty they lie in distinct components of the complement of $A_r$.
\end{lemma}
Proof. Let \( x \) be in the component given by \( c_{k_i} + k_1 \cdot x > \max_{j \in Q - \{k_i\}} \{c_j + j \cdot x\} + \log_r(M) \) an let \( \text{Log}_r(z) = x \). We have \( r^{v_j + j \cdot x} = r^{v(j)} |a_j| |z| \), so we get \( r^{v(k_i)} |a_k| |z^k| > M \max_{j \in Q - \{k_i\}} \{r^{v(j)} |a_j| |z^j|\} > |\sum_{j \in Q - \{k_i\}} r^{v(j)} a_j z^j| \) by the choice of \( M \). Now we can apply theorem 2.9 so \( x \) lies in the component of \( A_r \) corresponding to \( k_i \). By the same theorem the components of \( A_r \) corresponding to \( k_1 \) and \( k_2 \) are distinct which proves the lemma.

4.4 The limit

Now we can prove the statement of the introduction and construct some curves with nice amoebas.

Theorem 4.7. When \( r \to \infty \) the amoebas \( A_r \) converge to \( A_K \) for the Hausdorff distance.

Proof. By lemma 4.4 and 4.5 we have \( \lim_{r \to \infty} \sup_{y \in A_r} d(y, A_K) = 0 \).

Let \( \varepsilon > 0 \). Because \( A_K \) is a finite collection of subsets of hypersurfaces and \( \lim_{r \to \infty} d_H(A_r, A_K) = 0 \) we can find a \( T > 0 \) such that \( A_K \) does not contain any ball of diameter \( \varepsilon \).

Now let \( x \in A_K \). Because the components of the complement of \( A_K \) are convex we can find a ball \( B \) of diameter \( \varepsilon \) which intersects two distinct components \( C_1 \) and \( C_2 \) of the complement of \( A_K \). If \( B \) contains no point of \( A_r \) for some \( r > T \) we get that \( C_1 \) and \( C_2 \) are connected in the complement of \( A_r \). This is impossible by lemma 4.6. Thus \( d(x, A_r) < \varepsilon \).

We now consider an example.

Example 4.8. Consider the curve defined by \( f(u, w) = u^3 + w^3 + \lambda uw + 1 \). If \( \lambda = 0 \) then \( A_f \) is the image of the amoeba \( u + w + 1 \) under a linear map by corollary 5.2 and therefore the complement has no bounded components. In figure 4.2 we can see that the amoeba also has no holes for small values of \( \lambda \). Nu we pick the function \( v \) by \( v(0, 0) = v(3, 0) = v(0, 3) = 1 \) and \( v(1, 1) = 1 \) and note that \( A_r \) is the amoeba of \( u^3 + w^3 + rwu + 1 \) multiplied coordinate wise with \( \frac{1}{\log(\varepsilon)} \). Now theorem 4.7 shows that \( A_f \) has a bounded component for sufficiently large values of \( \lambda \). The case of \( \lambda = 2 \) is figure 2.1.

We can also use theorem 4.7 to show that the maximum number of components given by corollary 2.4 is achieved by some curves.

Theorem 4.9. For every finite set \( Q \subset \mathbb{Z}^n \) there is a Laurent polynomial \( g \) with \( \text{Newt}(g) = \text{conv}(Q) \) such that the number of components of \( A_g \) is \( \# \text{Newt}(g) \cap \mathbb{Z}^n \).

\( \text{This could in principle be proved by looking at the critical points of Log map.} \)
Proof. Let $I = \text{Newt}(g) \cap \mathbb{Z}^n$ and let $f((z)) = \sum_{i \in I} z^i$. By proposition [1.11] there is a function $v: I \to \mathbb{Z}$ such that the subdivision of $\text{Newt}(f) = \text{Newt}(g)$ induced by the tropical curve $V$ given by $\max_{i \in I} \{a_i + i \cdot x\}$ has $I$ as set of vertices. By theorem [1.28] the components of the complement of $V$ correspond bijectively to $I$. Now we apply theorem [4.7] and note $\mathcal{A}_K = V$.

Thus for sufficiently large $r$ it follows that $\mathcal{A}_r$ has at least $\# I$ components. The amoeba of $g((z)) = \sum_{i \in I} r^{v(i)} z^i$ is homeomorphic to $\mathcal{A}_r$ so it follows that $\mathcal{A}_r$ has exactly $\# I$ components.

Now we give an example of an amoeba constructed this way.

**Example 4.10.** We want to construct a polynomial with Newton polytope $N = \text{conv}\{(0,0), (3,0), (0,3), (2,3), (3,2)\}$ and maximal number of components. Let $Q = N \cap \mathbb{Z}^2$ and let $f = \sum_{j \in Q} u^j w^j$. Now we must pick a function $v: Q \to \mathbb{Q}$ such that $\text{trop}(f_t)$ has a maximal number of components. If we pick $v$ as in example example [1.12] then $\text{trop}(f_t)$ will be the tropical polynomial from example [1.20] which indeed has the maximum possible number of components.

The convergence of the modified amoebas $\mathcal{A}_r$ to the tropical curve can be seen in figure [4.3].

For sufficiently large $r$ the complement of $\mathcal{A}_r$ has a component for every component of the complement of the tropical curve of $f_t$ by lemma [4.6].

As $\mathcal{A}_r$ is the amoeba of $f_r$ multiplied with a constant we can use any $f_r$. 
Figure 4.3: Converging amoebas.
with sufficiently large \( r \), for example \( f_{14} \).

\[
f_{14} = u^3 + w^3 + 1 + u^2 w^3 + u^3 w^2 + 14u + 14u^2 + 14w + 14w^2 + 14u^3 w + 14uw^3 + 196uw + 196u^2 w + 196uw^2 + 196u^2 w^2
\]

Figure 4.4: The amoeba of \( f_{14} \).
Chapter 5

Appendix I

In this appendix we consider the effect of an endomorphism of \((\mathbb{C}^*)^2\) on the amoeba of a curve. This is the special case \(n = d = 2\) of section 4 of [25], although the formulation is slightly different.

We are interested in the endomorphisms of \((\mathbb{C}^*)^2\) with the structure of a maximal ideal spectrum and not of an algebraic group, so multiplication with a constant is an endomorphism.

### 5.1 Endomorphisms of \((\mathbb{C}^*)^2\)

The points of \((\mathbb{C}^*)^2\) correspond to the closed points of \(\text{spec}(\mathbb{C}[u, w, u^{-1}, w^{-1}])\), which are by definition the maximal ideals of \(\mathbb{C}[u, w, u^{-1}, w^{-1}]\). By Hilbert’s nullstellensatz and elementary properties of localization the set of maximal ideals of \(\mathbb{C}[u, w, u^{-1}, w^{-1}]\) is \(\{(u - \lambda_1, w - \lambda_2) | \lambda_1, \lambda_2 \in \mathbb{C}\}\). The correspondence between maximal ideals and points is given by mapping a maximal ideal \(m\) to the single element of \(\{(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2 | f(\lambda_1, \lambda_2) = 0 \text{ for all } f \in m\}\), thus \((u - \lambda_1, w - \lambda_2) \mapsto (\lambda_1, \lambda_2)\).

For \(M \in \text{Mat}_2(\mathbb{Z})\) and \(\alpha = (\alpha_1, \alpha_2) \in (\mathbb{C}^*)^2\) we define

\[
(\alpha_1, \alpha_2)^M = (\alpha_1^{m_{11}} \alpha_2^{m_{21}}, \alpha_1^{m_{12}} \alpha_2^{m_{22}}).
\]

By definition the endomorphisms of \((\mathbb{C}^*)^2\) correspond the endomorphisms of \(\mathbb{C}[u, w, u^{-1}, w^{-1}]\) as \(\mathbb{C}\)-algebra and we can use commutative algebra to describe these.

**Lemma 5.1.** The set \(\text{End}(\mathbb{C}[u, w, u^{-1}, w^{-1}])\) consists of all maps defined by

\[
f(z) \mapsto f(\alpha \cdot z^M),
\]

with \(M \in \text{Mat}_2(\mathbb{Z})\) and \(\alpha = (\alpha_1, \alpha_2) \in (\mathbb{C}^*)^2\), where \(f(z) = f(u, w) \in \mathbb{C}[u, w, u^{-1}, w^{-1}]\). Furthermore all these maps are distinct.
Proof. Denote \( A = \mathbb{C}[u, w, u^{-1}, w^{-1}] \). By the universal property of localization, homomorphisms \( \varphi: A \to A \) are the unique extensions of homomorphisms \( \tilde{\varphi}: \mathbb{C}[u, w] \to A \) such that \( \tilde{\varphi}(u) \) and \( \tilde{\varphi}(w) \) are units. The set of units is \( \{ \alpha_{ij}u^i w^j | \alpha_{ij} \in \mathbb{C}^* \text{ and } i, j \in \mathbb{Z} \} \). Thus a homomorphism \( \tilde{\varphi}: \mathbb{C}[u, w] \to A \) extends to an endomorphism of \( A \) if and only if \( \tilde{\varphi}(u) = \alpha_1 u^{m_1} w^{m_{21}} \) and \( \tilde{\varphi}(w) = \alpha_2 u^{m_{12}} w^{m_{22}} \) with \( \alpha_1, \alpha_2 \in \mathbb{C}^* \) and \( m_{11}, m_{21}, m_{12}, m_{22} \in \mathbb{Z} \). In this case \( \tilde{\varphi} \) and its extension \( \varphi: A \to A \) are given by \( f(z) \mapsto f(\alpha \cdot z^M) \). This proves the first statement in the lemma.

If \( \varphi \) is the endomorphism given by \( M \in \mathrm{Mat}_2(\mathbb{Z}) \) and \( \alpha = (\alpha_1, \alpha_2) \in (\mathbb{C}^*)^2 \) then \( \varphi(u) = \alpha_1 u^{m_{11}} w^{m_{21}} \) and \( \varphi(w) = \alpha_2 u^{m_{12}} w^{m_{22}} \), so \( M \) and \( \alpha \) are unique.

We can use this lemma to show the group of automorphisms is given by a semidirect product. It is easy to check \( (\alpha \beta)^M = \alpha^M \beta^M \) and \( \alpha^M \beta^N = (\alpha N)^M \) for all \( \alpha, \beta \in (\mathbb{C}^*)^2 \) and \( M, N \in \mathrm{Mat}_2(\mathbb{Z}) \). This way we get a homomorphism \( \mathrm{GL}(2, \mathbb{Z}) \to \mathrm{Aut}_{\mathrm{Grp}}((\mathbb{C}^*)^2) \) and thus a semidirect product \( (\mathbb{C}^*)^2 \rtimes \mathrm{GL}(2, \mathbb{Z}) \) of groups.

**Proposition 5.2.** The map \( \Psi: (\mathbb{C}^*)^2 \rtimes \mathrm{GL}(2, \mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[u, w, u^{-1}, w^{-1}]) \), defined for \( (\alpha, M) \in (\mathbb{C}^*)^2 \rtimes \mathrm{GL}(2, \mathbb{Z}) \) and \( f(z) = f(u, w) \in \mathbb{C}[u, w, u^{-1}, w^{-1}] \) by

\[
\Psi(\alpha, M)(f(z)) = f(\alpha \cdot z^M),
\]

is an isomorphism.

Proof. By the previous lemma we know that formula \( 5.1.1 \) defines an endomorphism of \( \mathbb{C}[u, w, u^{-1}, w^{-1}] \). It is clear that \( \Psi \) maps the unit element of \( (\mathbb{C}^*)^2 \rtimes \mathrm{GL}(2, \mathbb{Z}) \) to the identity map, so if \( \Psi \) is multiplicative all endomorphisms defined by formula \( 5.1.1 \) are automorphisms.

Let \( \psi \) and \( \varphi \) be the endomorphisms defined by \( (\beta, N) \) and \( (\alpha, M) \) with \( \beta, \alpha \in (\mathbb{C}^*)^2 \) and \( N, M \in \mathrm{Mat}_2(\mathbb{Z}) \) then we have

\[
\psi \circ \varphi(u) = \alpha_1 \beta_1^{m_{11}} \beta_2^{m_{21}} u^{m_{11}n_{11} + m_{21}n_{21} + m_{11}n_{12} + m_{21}n_{22}}
\]

and

\[
\psi \circ \varphi(w) = \alpha_2 \beta_1^{m_{12}} \beta_2^{m_{22}} u^{m_{12}n_{11} + m_{22}n_{21} + m_{12}n_{22}}.
\]

This shows \( \psi \circ \varphi \) is the endomorphism defined by \( (\alpha \beta^M, NM) \). In particular we have \( \Psi(\beta, N) \circ \Psi(\alpha, M) = \Psi(\alpha \beta^M, NM) = \Psi((\beta, N)(\alpha, M)) \), which shows that \( \Psi \) is well defined and a homomorphism.

If \( \varphi \) is an endomorphism and \( \psi = \varphi^{-1} \) it follows by equation \( 5.1.2 \) and \( 5.1.3 \) that \( NM \) is the unit matrix, so \( N \in \mathrm{GL}(2, \mathbb{Z}) \) and \( \varphi \in \text{Im}(\Psi) \). Finally \( \Psi \) is surjective by the uniqueness of \( (\alpha, M) \) in the previous lemma. \( \square \)
Equations 5.1.2 and 5.1.3 also show that the composition of endomorphisms is also well behaved with respect to matrix multiplication, so would also be possible to give a description of $\text{End}(\mathbb{C}[u, w, u^{-1}, w^{-1}])$ in terms of the semigroups $(\mathbb{C}^*)^2$ and $\text{Mat}_2(\mathbb{Z})$.

For every endomorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ we have an endomorphism of the maximal spectrum. The following proposition shows how this map behaves with respect to the identification of the maximal spectrum with $(\mathbb{C}^*)^2$.

**Proposition 5.3.** If $\varphi$ is the endomorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ defined by $\alpha \in (\mathbb{C}^*)^2$ and $M \in \text{Mat}_2(\mathbb{Z})$ then the corresponding endomorphism $\varphi^*$ of $(\mathbb{C}^*)^2$ is given by $\varphi^*(\lambda) = \alpha \lambda^M$.

**Proof.** Denote $A = \mathbb{C}[u, w, u^{-1}, w^{-1}]$ and let $\text{msp}(A)$ be the maximal ideal spectrum of $A$. If $\varphi$ in an endomorphism of $A$ the canonical map $\tilde{\varphi} : \text{msp}(A) \to \text{msp}(A)$ is given by $\tilde{\varphi}(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m})$ for maximal ideals $\mathfrak{m}$ of $A$.

We have identified $(\mathbb{C}^*)^2$ with $\text{msp}(A)$ by the map $Z : \text{msp}(A) \to (\mathbb{C}^*)^2$ which maps a maximal ideal to its set of zeros. Thus we need to show that the diagram

$$
\begin{array}{ccc}
\text{msp}(A) & \xrightarrow{Z} & (\mathbb{C}^*)^2 \\
\downarrow{\tilde{\varphi}} & & \downarrow{\varphi^*} \\
\text{msp}(A) & \xrightarrow{Z} & (\mathbb{C}^*)^2 
\end{array}
$$

commutes. Let $\mathfrak{m} \in \text{msp}(A)$, then $\mathfrak{m} = (u - \lambda_1, w - \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{C}^*$, so $Z(\mathfrak{m}) = (\lambda_1, \lambda_2) = \lambda$ and $\varphi^*Z(\mathfrak{m}) = \alpha \lambda^M$. If $f \in \varphi^{-1}(\mathfrak{m})$ then $f(\alpha \lambda^M) = \varphi(f)(\lambda)$, so $\alpha \lambda^M$ is the zero of $\varphi^{-1}(\mathfrak{m})$. \qed

**Proposition 5.4.** Let $\varphi$ be the endomorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ defined by $\alpha \in (\mathbb{C}^*)^2$ and $M \in \text{Mat}_2(\mathbb{Z})$. If $\det(M) \neq 0$ then the corresponding endomorphism $\varphi^*$ of $(\mathbb{C}^*)^2$ is surjective.

**Proof.** It is enough to to prove this with $\alpha = (1, 1)$. By the Smith normal form theorem we can find $A, B, \in \text{GL}(2, \mathbb{Z})$ and $D \in \text{Mat}_2(\mathbb{Z})$ such that $AMB = D$ and $D$ is a diagonal matrix. From the assumptions it follows directly that $\det(D) \neq 0$, so $D$ defines a surjective endomorphism of $(\mathbb{C}^*)^2$.

The equation $AMB = D$ also holds when we consider the maps defined by $A, B, M, D$, so $M$ is also surjective. \qed

The above argument can easily be turned into an equivalence.

**Proposition 5.5.** Let $\varphi^*$ be a surjective endomorphism of $(\mathbb{C}^*)^2$ as in the previous proposition. If $C_f \subset (\mathbb{C}^*)^2$ is the curve defined by $f \in \mathbb{C}[u, w, u^{-1}, w^{-1}]$ then the restriction $\varphi^*[C_f] : C_f \to C_f$ is surjective.

**Proof.** We have $f(\varphi^*(\lambda)) = \varphi(f)(\lambda)$ for every $\lambda \in (\mathbb{C}^*)^2$, so $\lambda \in C_{\varphi(f)}$ if and only if $\varphi^*(\lambda)$ lies in $C_f$. The corollary follows because $\varphi^*$ is surjective. \qed
5.2 The transformation of the amoeba

It turns out that the surjective maps of curves in the previous section give linear bijections between the amoebas of the curves.

First we give the action on the Newton polytope.

**Proposition 5.6.** Let $\varphi$ is the endomorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ defined by $\alpha \in (\mathbb{C}^*)^2$ and $M \in \text{Mat}_2(\mathbb{Z})$, then $\text{Newt}(\varphi(f)) = M \text{Newt}(f)$

**Proof.** The monomial $u^i w^j$ occurs in $f$ if and only if the monomial $\varphi(u^i w^j) = u^{n_{11}i + n_{12}j} w^{n_{21}i + n_{22}j}$ occurs in $\varphi(f)$. As a linear map maps the convex hull of a set to the convex hull of the image of the set, the proposition follows. $\square$

The map $\varphi^*|_{C_{\varphi(f)}} : C_{\varphi(f)} \to C_f$ from corollary 5.5 translates directly to a map between the amoebas of $f$ and $\varphi(f)$. Because $\varphi^*$ is surjective this map becomes a bijection.

**Corollary.** Let $C_f \subset (\mathbb{C}^*)^2$ be the curve defined by $f \in \mathbb{C}[u, w, u^{-1}, w^{-1}]$ and let $\varphi$ be the endomorphism of $\mathbb{C}[u, w, u^{-1}, w^{-1}]$ defined by $\alpha \in (\mathbb{C}^*)^2$ and $M \in \text{Mat}_2(\mathbb{Z})$ with $\det(M) \neq 0$, then

$$A_f = M A_{\varphi(f)} - \log(\alpha).$$

**Proof.** By proposition 5.5 the map $\varphi^*|_{C_{\varphi(f)}} : C_{\varphi(f)} \to C_f$ is surjective and by proposition 5.3 we have $\varphi^*(\lambda) = \alpha \lambda^M$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\begin{pmatrix} x \\ y \end{pmatrix}) = M \cdot \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \log |\alpha_1| \\ \log |\alpha_2| \end{pmatrix}$, then it is easy to check that the diagram

$$\begin{array}{ccc}
C_{\varphi(f)} & \xrightarrow{\varphi^*} & C_f \\
\downarrow \text{Log} & & \downarrow \text{Log} \\
\mathbb{R}^2 & \xrightarrow{T} & \mathbb{R}^2
\end{array}$$

commutes, which shows the sets $\text{Log}(C_f)$ and $T \text{Log}(C_{\varphi(f)})$ are equal. $\square$

**Example 5.7.** Corollary 5.2 makes it possible to draw some amoebas in two different ways. Consider $f = u + w + 1$ and $M = \begin{pmatrix} 3 & 5 \\ 7 & 2 \end{pmatrix}$, then $\varphi(f) = w^3 w^7 + w^5 w^2 + 1$.

In figure 5.1 the amoeba of $f$ has been translated by $M$, however the distribution of the computed points has become rather uneven as only a few points of the left tentacle have been computed.

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Directly computing points of the amoeba of $u^3w^7 + u^5w^2 + 1$ is slower because of the higher degree, but the points are distributed evenly as can be seen in figure 5.2.

The difference in scale is because both the amoeba of $f$ and $\varphi(f)$ were computed in the square defined by $-4 \leq x, y \leq 4$. 
Chapter 6

Appendix II

In this appendix we give the Sage [23] code used to draw the amoebas in this thesis. It is very similar to the matlab code available at [1]. From a practical point of view the use of Sage it convenient because it has a number built in operations on polynomials. Furthermore it is open source and free to use.

6.1 Computing the amoeba of a planar curve

The following code can be put directly into a worksheet, except for an extra linebreak in line 9.

In the first box we declare the box \([\text{xmin}, \text{xmax}] \times [\text{ymin}, \text{ymax}]\) in which we want to draw the amoeba, the variables, the number of points drawn and the polynomial \(f\).

```python
1 var ( 'xmin', 'xmax', 'ymin', 'ymax' )
2 xmin=−3
3 xmax=3
4 ymin=−3
5 ymax=3
6 var ( 'angle_step', 'grid_step' )
7 grid_step=250
8 angle_step=120
9 var ( 'u', 'w' )
10 f=4*u*w−u^2*w−w^2*u−1
```

The following box computes points along vertical lines in the chosen box. Because we only look at polynomials with real coefficients the angle of the possible solutions can be taken in \([0, \pi)\).
\begin{verbatim}
x_grid_step_size=(xmax-xmin)/grid_step
List_x=[]
for j in range(1, grid_step):
    r=xmin+j*x_grid_step_size
    for k in range(0, angle_step):
        theta=(k/angle_step)*pi*I
        z1=e^r*e^(theta)
        g=f.substitute(u=z1)
        S=g.roots(w, ring=ComplexField(prec=10), multiplicities=False)
        for p2 in S:
            logp2=log(abs(p2))
            if logp2>ymin and logp2<ymax:
                List_x.append((r, logp2))

We can now plot the list of points with the following command.
list_plot(Lx, size=2, aspect_ratio=1)
\end{verbatim}

If we only compute points along vertical lines any tentacles of the amoeba which are vertical will be drawn badly because the become thinner than the space between the lines on which the points are computed. This can be seen in the next example.

**Example 6.1.** If we run the algorithm for \( f = 1 + u^3 + w^3 + 2uw \) in the box \([-3, 3] \times [-3, 3]\) with grid_step = 250 and angle_step = 120 we get figure 6.1.
Figure 6.1: Incomplete amoeba for $f = 1 + u^3 + w^3 + 2uw$.

Thus we have to add a second list of points, now computed on horizontal lines. Then we get the full picture. We can plot both list with the following command.

```
list_plot(List_x)+list_plot(List_y)
```

The complete amoeba of example 6.1 is figure 2.1

### 6.2 Critical points

If we compute too few point to fill up the amoeba we get a picture where we can see some indication as to how the curve is folded onto the amoeba. For example in figure 6.2 we can see that the boundary of the hole in the amoeba is the union of three foldlines which continue in the amoeba.
Figure 6.2: Some points on the amoeba of \( f = 1 + u^3 + w^3 + 1.25uw \).

These lines are critical points of the Log map on the curve. In [25] an algorithm to compute these lines is given.

In the following amoeba the boundary of the hole in the amoeba seems to be smooth.
Figure 6.3: Some points on the amoeba of $f = 4uw - u^2w - w^2u - 1$. 
Populaire samenvatting

Een vlakke kromme is de verzameling van punten in het vlak die aan een bepaald soort vergelijking voldoen. In figuur 6.4(a) staan bijvoorbeeld de oplossingen van de vergelijking $y - x^2 - 2 = 0$. In dit geval is de kromme een parabool. In het algemeen bestaat een kromme uit een aantal lijnen en cirkels.

In figuur 6.4(b) staat een geschaalde versie van dezelfde kromme. Vanuit dit perspectief lijkt de kromme uit twee rechte lijnen te bestaan.

Figuur 6.4

Om te zien met welke rechte lijnen we te maken hebben bekijken we een tropische kromme. Deze naam heeft niets met een regenwoud te maken maar met het feit dat dit soort meetkunde vernoemd is naar de Braziliaanse wiskundige Imre Simon.

In figuur 6.5 is aangegeven in welk deel van het vlak de functies $2x$, $y$ en $\log(2)$ maximaal zijn. De punten waar meerdere van deze functies maximaal zijn vormen de drie rode lijnstukken. Dit is de tropische kromme van $2x$, $y$ en $\log(2)$. We zien dat twee van de lijnen figuur 6.4(b) goed benaderen.

1Om technische redenen kijken we alleen naar oplossingen die niet op de coördinaatassen liggen.

2We sturen een punt $(x, y)$ naar $(\log |x|, \log |y|)$. 

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Figure 6.5: Een tropisch kromme.

Nu gaan we uitzoeken waar de derde lijn vandaan komt.

In figuur 6.4(a) hebben we oplossingen van $y - x^2 - 2 = 0$ met reëele getallen bepaald en dit geeft een kromme in het vlak. We kunnen ook zoeken naar oplossingen met complexe getallen. Als we dit doen geven de oplossingen een oppervlak in een vierdimensionale ruimte, waar we dus geen plaatje van kunnen tekenen.

Wel kunnen we weer dezelfde soort schaling toepassen als we afbeelding 6.4(b) gedaan hebben. De vierdimensionale ruimte wordt nu naar een twee-dimensional ruimte afgebeeld. Het beeld van verzameling van oplossingen is nu te zien in figuur 6.6. We kunnen nu opmerken dat de lijn in figuur 6.4(b) de bovenrand van figuur 6.6 is en dat de we de tak gevonden hebben die hoort bij de derde lijn van de tropische kromme.
Figure 6.6: De amoeba van de vergelijking $y - x^2 - 2 = 0$.

De figuur die we krijgen als we de procedure van afbeelding 6.6 toepassen op de complexe oplossingen van een vergelijking heet de amoeba van de vergelijking. De gelijkenis met de eencellige organismes met dezelfde naam wordt duidelijk als we naar een ander voorbeeld van een amoeba kijken, bijvoorbeeld afbeelding 6.7.

Figure 6.7: De amoeba van de vergelijking $x^3 + y^3 + x^2y^4 + x^4y^3 + 3.8x^3y + 12.9x^2y^2 + 4.55xy^3 + 4.6x^3y^3 = 0$.  

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In deze scriptie hebben we een constructie van Grigory Mikhalkin bekeken waar een amoeba steeds verder vervormd wordt tot die steeds meer op een tropische kromme gaat lijken. Het effect van deze constructie is duidelijk te zien in afbeelding 6.8.

Figure 6.8: Het vervormen van een amoeba.

Ook hebben we het geval bekeken waar we niet kijken naar oplossingen van vergelijkingen met complexe getallen, maar met oplossingen in een ander getalsysteem. Dit geval wordt geheel beschreven door de stelling van Kapranov.

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Bibliography


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