Frobenius structure of $A_n$ singularities and $(n+1)KdV$ hierarchies

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Abstract

We study the relation between Frobenius structures and integrable hierarchies using a specific well known example. We discuss general Frobenius structures and the specific Frobenius structure related to the singularity of $A_n$ type. Most notably we prove that there is a Frobenius structure, in the axiomatic sense introduced in [1], related to such singularities. We find explicitly flat and canonical coordinates of this Frobenius manifold. We introduce the quantization formalism and use it to derive the vertex operator form of the $(n+1)KdV$ hierarchies. Finally we discuss Givental’s formula for a total descendant potential related to the $A_n$ singularity and follow Givental’s argumentation [2] to prove that this constitutes a tau-function of the $(n+1)KdV$ hierarchy. To do this we use the essential concepts of period vectors (as given in [3]) and the phase form (as introduced by Givental in [2]).
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0 Introduction

Purpose of the Thesis

The theory of Frobenius manifolds has been shown to be both wide spread and deep [1, 3, 4, 5, 6, 7, 8, 9]. Consequently there are a great many different ways to view the general elements of this construction. It has been proposed [4] that the most natural frame for Frobenius manifolds is in the theory of integrable hierarchies. This claim is backed up by the realization that there is a Frobenius manifold related to every integrable hierarchy of a certain type. However it is still an open question what this means for other classes of Frobenius structures [4, 10].

This relation was prompted by the discovery of Witten and Kontsevich of the relation between the topology of the moduli spaces of stable complex curves and the Korteweg-de Vries hierarchy [11, 12]. This relation is seen when we express the integrable hierarchy in terms of Hirota quadratic equations and tau-functions. Which leaves the open question of framing the relation in a way applicable to general hierarchies expressed in Lax form [10], since the HQE-form of integrable hierarchies is only shown for a specific subset. In this case we can also see the direct relation to the theory of Gromow-Witten invariants of a compact Kähler manifold. It is shown that the generating function of such GW-invariants in the case of the simplest manifold (that of a point) is in fact a tau-function of the KdV hierarchy.

At this point we find again the theory of Frobenius manifolds related to the theory of GW-invariants. Namely when we view the canonical moduli space related to a topological field theory, this is a Frobenius manifold [5]. By Givental’s formula we can relate to any semisimple Frobenius manifold a total descendant potential $D$. This is done by acting on this generating function of GW-invariants (or a product of them) with certain (quantized) operators related in an essential way to the Frobenius manifold [13, 2]. Givental conjectures that this total descendant potential is again a generating function of GW-invariants of the corresponding (calibrated) Frobenius structure [13]. This sets up nicely to complete the correspondence by yielding again a tau-function of the corresponding integrable hierarchy.

After this general discussion of the ideas related to Frobenius structures let us get to the point of this thesis. The first examples of Frobenius structures are not found in the theory of topological/cohomological field theories, but rather in the setting of singularity theory [14, 8, 9, 15]. It is shown that the Witten-Kontsevich tau-function mentioned above is in fact related, as the total descendant potential, to the singularity of $A_1$ type [2, 11, 12]. Thus by Givental’s formula we find a natural correspondence of this total descendant potential of the $A_1$ singularity to a total descendant potential related to the $A_n$ singularity. On the other hand there is also the connection of the KdV hierarchy to the $(n+1)$KdV hierarchies (in HQE form) [2], which we will show below.

To study the general relation of Frobenius manifolds, integrable hierarchies and total descendant potentials as tau-functions we will focus on this specific example as laid out in [2]. This will lead to the identification of certain major
players in the construction which would need to be generalized to the case of
an arbitrary Frobenius manifold related to an integrable hierarchy.

**Organization of the Thesis**
The thesis is presented as three parts, divided according to the general nature
of the subjects discussed in them. The first part, entitled Frobenius structures,
starts with 2 sections discussing the basic elements of Frobenius structures, that
we will need. We begin with the axiomatic definitions of Frobenius algebras and
manifolds as presented in [1]. In section 2 the most powerful tools, specifically
the deformed connection and canonical coordinates, will be discussed. This
should give a good picture of the general theory of Frobenius manifolds.

The rest of Part I is devoted to the Frobenius structure related to the
$A_n$ singularity. This will be defined and discussed quite explicitly. All elements
of the first two sections will be shown explicitly. Additionally a few elements
of the structure that are specific to the $A_n$ singularity will be discussed. In
this part the essential concept of oscillatory integrals will be introduced. Most
importantly we prove theorem 7, showing that we actually have to do with a
Frobenius structure as defined in [1] and in fact one related to a solution to the
Witten-Dijkgraaf-Verlinde-Verlinde equations.

Part II will set us up for the discussion on integrable hierarchies and total
descendant potentials by introducing first the quantization formalism. This
formalism is essential for the formulation of Givental’s formula and the vertex
operator form of the $(n+1)$KdV hierarchies. The rest of part II will be used
to derive this form of the $(n+1)$KdV hierarchies starting from a HQE form of
the Kadomtsev-Petviashvili hierarchy. Most important here is the derivation of
the regularity condition (8.18).

Lastly in part III we will define all ingredients of Givental’s formula in a hands
on approach, i.e. in the setting of the Frobenius structure related to the $A_n$
singularity. In this part also the essential concepts including period vectors and
phase forms are introduced. Most importantly we prove theorem 20 following
Givental’s argumentation in [2].

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Part I
Frobenius Structures

1 Definition of a Frobenius Structure

To discuss the Frobenius structure corresponding to the singularity of \(A_n\) type, we will first need to know what a Frobenius structure is exactly. First of all, we shall need to give a definition of Frobenius structures. This involves two main ingredients: Frobenius algebras, which we will define and discuss first, and a complex manifold, which we introduce subsequently. We will follow most of the conventions adopted in [1, 5] and refer to these as well for a more in-depth discussion of the general structure of a Frobenius manifold. We will conclude this section by giving an example of Frobenius structures. Namely those generated by solutions to the Witten-Dijkgraaf-Verlinde-Verlinde (or simply WDVV) equations. This will provide us with a substantial intuition. Moreover it can be shown that a specific type of Frobenius structure is always isomorphic to one generated in this way [1].

Frobenius Algebra

A Frobenius algebra \((A, 1, < \cdot, \cdot >)\) consists of an associative commutative unital algebra over \(\mathbb{C}\) equipped with a symmetric non-degenerate bilinear form \(< \cdot, \cdot >\) such that

\[
<a \cdot b, c> = <a, b \cdot c> \quad \forall a, b, c \in A
\]

(1.1)

this property is called invariance of \(< \cdot, \cdot >\).

A Frobenius algebra \(A\) is called graded if there is a linear operator \(Q: A \to A\) and a number \(d \in \mathbb{C}\) such that

\[
Q(a \cdot b) = Q(a) \cdot b + a \cdot Q(b)
\]

(1.2)

and

\[
<Q(a), b> + <a, Q(b)> = d <a, b>
\]

(1.3)

\(\forall a, b \in A\). Then \(Q\) is called the grading operator and \(d\) is called the charge of the Frobenius algebra.

Furthermore we can consider a graded Frobenius algebra \(A\) over (in the sense of a module) a graded associative commutative ring \(R\). In this case we have two grading operators \(Q_A: A \to A\) and \(Q_R: R \to R\) satisfying

\[
Q_R(\alpha \beta) = Q_R(\alpha)\beta + \alpha Q_R(\beta) \quad \forall \alpha, \beta \in R
\]

(1.4)

\[
Q_A(ab) = Q_A(a)b + aQ_A(b) \quad \forall a, b \in A
\]

(1.5)

\[
Q_A(ab) = Q_R(\alpha)b + \alpha Q_A(b) \quad \forall \alpha \in R, \forall b \in A
\]

(1.6)

and

\[
Q_R(<a, b>) + d <a, b> = <Q_A(a), b> + <a, Q_A(b)> \quad \forall a, b \in A
\]

(1.7)

again we call \(d \in \mathbb{C}\) the charge of the Frobenius algebra.
Frobenius Manifold

A Frobenius manifold (also called a Frobenius structure) consists of a complex manifold \( M \) equipped with a structure of Frobenius algebra \((T_t M, 1_t, <\cdot,\cdot>_t)\) on the tangent spaces \( \forall t \in M \). This Frobenius structure depends analytically on \( t \). Concretely this last comment means that for \((E^1_t,\ldots,E^n_t)\) a local frame, where \( n = \text{Dim}(M) \), the functions \( \gamma^k_{ij}(t) \), defined by

\[
E^i_t \cdot E^j_t = \sum_{k=1}^n \gamma^k_{ij}(t) E^k_t,
\]

and the functions \( <E^i_t,E^j_t>_t \) should be analytic for all \( i,j,k = 1,\ldots,n \). Moreover the following 4 axioms should be satisfied.

1

First of all the bilinear form \( <\cdot,\cdot>_t \), which is already assumed symmetric non-degenerate and invariant in the sense of Frobenius algebra, should be flat. Stated otherwise, there should be locally a system of coordinates \((t_1,\ldots,t_n)\) on \( M \) such that the matrix \( \eta_{ij} = <\partial_{t_i},\partial_{t_j}>_t \) does not depend on the point \( t \in M \). These local coordinates are called flat coordinates.

2

Denoting by \( e \in \mathcal{X}(M) \) the vector field such that \( e_t = 1_t \) for all \( t \in M \) and by \( \nabla \) the Levi-Civita connection corresponding to \( <\cdot,\cdot>_t \), we should have

\[
\nabla e = 0. \tag{1.8}
\]

Note that \( e \) is analytic since we assumed the algebra structure to vary analytically.

3

By the invariance of \( <\cdot,\cdot>_t \) and commutativity of the Frobenius algebras we have naturally the fully symmetric trilinear form \( c \) on \( TM \) defined by

\[
c(u,v,w) := <u \cdot v,w> \quad \forall u,v,w \in TM \tag{1.9}
\]

The corresponding four-linear form

\[
C(z,u,v,w) := (\nabla_z e)(u,v,w) \quad \forall z,u,v,w \in TM \tag{1.10}
\]

should also be fully symmetric.

4

Note that the space \( \mathcal{X}(M) \) of vector fields on \( M \) has acquired the structure of a Frobenius algebra over the ring \( \mathcal{F}(M) \) of functions on \( M \). There should be a fixed linear vector field \( E \in \mathcal{X}(M) \), i.e. \( \nabla \nabla E = 0 \), which we call the Euler field. Then the operators

\[
Q_{\mathcal{F}(M)} := E \tag{1.11}
\]
\[ Q_{\mathcal{M}} := 1d + ad_E \] (1.12)

should introduce the structure of a graded Frobenius algebra over a graded ring of some charge \( d \in \mathbb{C} \) on \( \mathcal{X}(\mathcal{M}) \).

**WDVV Equations**

As an example and general source of intuition, let us consider the Witten-Dijkgraaf-Verlinde-Verlinde equations. The WDVV-equations are firstly a system of equations concerning a function \( \Phi(t^1, \ldots, t^n) \) (shorthand \( \Phi(t) \)) of some complex variables \( (t^1, \ldots, t^n) \), for \( n \in \mathbb{N} \), and a symmetric non-degenerate \( n \times n \) matrix \( \eta^{\alpha\beta} \). Namely

\[ \partial_\alpha \partial_\beta \partial_\delta \Phi(t) \eta^{\delta\mu} \partial_\mu \partial_\lambda \Phi(t) = \partial_\delta \partial_\beta \partial_\lambda \Phi(t) \eta^{\delta\mu} \partial_\mu \partial_\gamma \partial_\alpha \Phi(t) \] (1.13)

for all \( \alpha, \beta, \gamma, \delta = 1, \ldots, n \) where we use the Einstein summation convention (in fact we will use it everywhere in the rest of this section) and we denote \( \partial_\alpha := \frac{\partial}{\partial t^\alpha} \). More elements are involved, but we will introduce these when they are needed.

For any \( \alpha, \beta, \gamma = 1, \ldots, n \) we define the functions \( c^\gamma_{\alpha\beta}(t) \) by

\[ c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\delta} \partial_\delta \partial_\alpha \partial_\beta \Phi(t) \] (1.14)

We can use these functions to define the algebras \( A_t = \langle e_1, \ldots, e_n \rangle \). Where multiplication is given by

\[ e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma, \] (1.15)

for all \( \alpha, \beta = 1, \ldots, n \). Multiplication of the WDVV equation by \( \eta^{\epsilon\alpha} \) (using the summation convention) and substitution of the structure constants yields

\[ c^\lambda_{\delta\epsilon} c^\epsilon_{\alpha\beta} = c^\lambda_{\beta\epsilon} c^\epsilon_{\alpha\gamma}. \] (1.16)

Thus we see that these algebras are associative. Obviously these algebras are also commutative.

We require a marked variable \( t^i \) such that

\[ \partial_\alpha \partial_\beta \partial_i \Phi(t) = \eta_{\alpha\beta} \] (1.17)

where the matrix \( \eta_{\alpha\beta} \) is inverse to \( \eta^{\alpha\beta} \). This matrix defines a symmetric non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) on each algebra \( A_t \) by

\[ \langle e_\alpha, e_\beta \rangle := \eta_{\alpha\beta}. \] (1.18)

This form is clearly invariant in the sense of Frobenius algebra since

\[ \langle e_\alpha \cdot e_\beta, e_\gamma \rangle = \eta^{\delta\epsilon} \partial_\delta \partial_\alpha \partial_\beta \Phi(t) \eta_{\epsilon\gamma} = \partial_\delta \partial_\beta \partial_\gamma \Phi(t), \] (1.19)

for all \( \alpha, \beta, \gamma = 1, \ldots, n \). Note that we now have the vector \( e_i \), which is the unit in every algebra \( A_t \).
Finally we require the quasi-homogeneity condition concerning the constants
\(d_1, \ldots, d_n, d_f \in \mathbb{C}\), which is given by
\[
\Phi(c^{d_1}t^1, \ldots, c^{d_n}t^n) = c^{d_f} \Phi(t).
\] (1.20)

We can restate this condition using a version of Euler’s homogeneous function
theorem. So define the Euler vector field \(E\) (this explains the name) given by
\[
E = \sum_{\alpha=1}^{n} d_\alpha t^\alpha \partial_\alpha.
\] (1.21)

Then we should have
\[
L_E \Phi(t) = d_f \Phi(t) \tag{1.22}
\]
up to second order terms (since this will not have an effect on the algebras,
which only depend on the third derivatives of \(\Phi\)). Also we impose the extra
conditions that \(d_i = 1\) and \(d_\alpha \neq 0\) for all \(\alpha = 1, \ldots, n\).

**Remark 1** In fact the extra condition \(d_\alpha \neq 0\) for all \(\alpha = 1, \ldots, n\) is not
needed, but true for the Frobenius structure corresponding to the \(A_n\) singularity.
Dropping this condition would imply the more complicated Euler vector field:
\[
E = \sum_{\alpha=1}^{n} (d_\alpha t^\alpha + r_\alpha) \partial_\alpha, \text{ with } r_\alpha \neq 0 \text{ only if } d_\alpha = 0. \text{ This case is included in [1, 5].}
\]

Often in the literature it is preferred to use the constants
\(q_\alpha = 1 - d_\alpha\) and \(d = 3 - d_f\). (1.23)

In terms of these constants we have the following relation.

**Lemma 1**
\[
(q_\alpha + q_\beta - d)\eta_{\alpha\beta} = 0, \tag{1.24}
\]
for all \(\alpha, \beta = 1, \ldots, n\). Here no summation is implied.

**Proof:**
The proof follows from equation (1.22) by taking the triple derivative \(\partial_\alpha \partial_\beta \partial_i\)
on both sides. The right hand side then simply reads \((3 - d)\eta_{\alpha\beta}\) by equation
(1.17). For the left hand side we find
\[
\eta_{\alpha\beta} + d_\beta \eta_{\alpha\beta} + d_\alpha \eta_{\alpha\beta} + \sum_{\gamma=1}^{n} d_\gamma t^\gamma \partial_\gamma \eta_{\alpha\beta} = (3 - q_\alpha - q_\beta)\eta_{\alpha\beta}
\]
since \(\eta\) does not depend on \(t\). equating these two expressions then yields the
relation. \(\square\)

Now it is already clear how to relate a structure of Frobenius manifold to these
solutions. The coordinates \(t^\alpha\) are our flat coordinates and they form a complex
manifold $M$. We define the algebra structure on the tangent spaces by the isomorphism $A_t \simeq T_1 M$ given by $e_{\alpha} \mapsto \partial_{\alpha}$. Obviously condition 1 is true and since $e = \partial_t$ corresponds to a flat coordinate we have also condition 2. All that is left is to show the conditions 3 and 4. For condition 3 simply note that

$$\partial_{\mu} c^\lambda_{\alpha\beta} = \partial_{\alpha} c^\lambda_{\mu\beta},$$

(1.25)

for all $\alpha, \beta, \lambda, \mu$.

As for condition 4, we let $Q_R = E$ and $Q_A = \text{Id} + \text{ad}_E$. Now identifying $e_{\alpha} = \partial_{\alpha}$ we have

$$Q_R(fg) = E(fg) = E(f)g + fE(g) = fQ_R(g) + Q_R(f)g,$$

$$Q_A(fe_{\beta}) = fe_{\beta} + [E, fe_{\beta}] = fe_{\beta} + E(f)e_{\beta} - f \sum_{\delta} \partial_{\beta} d_{s} t^{\delta} \partial_{\delta} =
$$

$$= fQ_A(e_{\beta}) + Q_R(f)e_{\beta},$$

$$Q_A(e_{\alpha} \cdot e_{\beta}) = Q_A(c^\alpha_{\gamma\beta} e_{\gamma}) = Q_R(c^\alpha_{\gamma\beta} e_{\gamma}) + c^\alpha_{\gamma\beta} Q_A(e_{\gamma}) =
$$

$$= e_{\alpha} \cdot e_{\beta} + E(c^\alpha_{\gamma\beta} e_{\gamma}) + c^\alpha_{\gamma\beta} [E, \partial_{\gamma}] =
$$

$$= (1 - d_{\alpha} - d_{\beta})(e_{\alpha} \cdot e_{\beta}) + (3 - d) c^\alpha_{\alpha\gamma} e_{\gamma} - \sum_{\epsilon} d_{\epsilon} e^{\gamma} \partial_{\epsilon} \partial_{\alpha} \partial_{\beta} \Phi \partial_{\gamma} - c^\gamma_{\epsilon\alpha\beta} \partial_{\epsilon} \sum_{\delta} d_{s} t^{\delta} \partial_{\delta} =
$$

$$= (2 - d_{\alpha} - d_{\beta})(e_{\alpha} \cdot e_{\beta}) + \sum_{\gamma} (2 - d_{\alpha} - d_{\beta}) e^{\gamma} \partial_{\gamma} \partial_{\alpha} \partial_{\beta} \Phi = (2 - d_{\alpha} - d_{\beta})(e_{\alpha} \cdot e_{\beta}) =
$$

$$= e_{\alpha} \cdot e_{\beta} - e_{\alpha} \cdot \left( \sum_{\delta} \partial_{\beta} d_{s} t^{\delta} \partial_{\delta} \right) + e_{\alpha} \cdot e_{\beta} - \left( \sum_{\delta} \partial_{\alpha} d_{s} t^{\delta} \partial_{\delta} \right) \cdot e_{\beta} =
$$

$$e_{\alpha} \cdot e_{\beta} + e_{\alpha} \cdot [E, e_{\beta}] + e_{\alpha} \cdot e_{\beta} + [E, e_{\alpha}] \cdot e_{\beta} = e_{\alpha} Q_A(e_{\beta}) + Q_A(e_{\alpha}) e_{\beta},$$

$$Q_R(< e_{\alpha}, e_{\beta} >) + d < e_{\alpha}, e_{\beta} > = (q_{\alpha} + q_{\beta}) < e_{\alpha}, e_{\beta} > =
$$

$$= 2 < e_{\alpha}, e_{\beta} > + \sum_{\delta} < - \partial_{\alpha} d_{s} t^{\delta} \partial_{\delta}, e_{\beta} > + \sum_{\delta} < e_{\alpha}, - \partial_{\beta} d_{s} t^{\delta} \partial_{\delta} > =
$$

$$= 2 < e_{\alpha}, e_{\beta} > + < [E, e_{\alpha}], e_{\beta} > + < e_{\alpha}, [E, e_{\beta}] > =
$$

$$= < Q_A(e_{\alpha}), e_{\beta} > + < e_{\alpha}, Q_A(e_{\beta}) >,$$

for all $\alpha, \beta$ and all $f, g \in \mathcal{F}(M)$. Here we used lemma 1 and (1.22).

Now we see that every solution of the WDVV equations defines a Frobenius structure and by lemma 2.1 of [1] also locally every Frobenius manifold with a diagonalizable $\nabla E$ is described by a solution to the WDVV equations.
2 Deformed Connection and Canonical Coordinates

Let us now explore the definition of Frobenius structures above. We will do this by defining two of the most useful tools related to Frobenius structures. Namely the deformed connection $\tilde{\nabla}$ and the canonical coordinates. We will also show some defining qualities of these tools. For a more detailed exposition see [1, 16]. Most notably the deformed connection offers an alternative definition of Frobenius manifolds [17]. Every Frobenius manifold in this section will be assumed to be obtainable from a solution to the WDVV equations, i.e. $\nabla E$ is assumed diagonalizable. This assumption is made merely for convenience and clarity. It should be clear how to adapt statements and proofs made in this section to incorporate the general case. Moreover we will see that the Frobenius structure related to the $A_n$ singularity has diagonalizable $\nabla E$. Thus the results of this section will be perfectly applicable to later sections.

Deformed Connection

From the definition of Frobenius manifold we get automatically the Levi-Civita connection $\nabla$, we will use this and combine it with the multiplication in the tangent bundle to form the deformed connection. In fact we will define a pencil of connections depending on some parameter $z \in \mathbb{C}$.

Suppose $M$ is a Frobenius manifold and $\nabla$ is the corresponding Levi-Civita connection. We define the deformed connection $\tilde{\nabla}$ by

$$\tilde{\nabla}_vw := \nabla_v w + zv \cdot w,$$

(2.1)

where as mentioned $z \in \mathbb{C}, v, w \in \mathcal{X}(M)$. Now we can extend this to a meromorphic connection on the product manifold $M \times \mathbb{C}$ by defining what happens in the added dimension. So on $M \times \mathbb{C}$ we define the deformed connection by

$$0 = \tilde{\nabla}_u \frac{d}{dz} = \tilde{\nabla}_{\frac{d}{dz}} u,$$

(2.2)

and

$$\tilde{\nabla}_{\frac{d}{dz}} v = \partial_z v + E \cdot v - \frac{1}{z} \mu v,$$

(2.3)

where $\mu = \frac{d_f - 1}{2} - \nabla E$, $u, v \in \mathcal{X}(M \times \mathbb{C})$ and $u, v$ have zero component along $\mathbb{C}$. Where we mean that

$$\mu v = \frac{d_f - 1}{2} v - \nabla v E.$$

A natural question is whether this deformed connection is again flat. Although we will not prove it explicitly in the general case, it should become clear that the answer is positive, mainly by virtue of the 4 axioms stated above. In fact, one could alternatively define a Frobenius structure using the metric, algebra structure and deformed connection. Instead of the 4 axioms we would merely ask flatness of the deformed connection [17]. Substantial intuition shall be given to this point. Since, to verify the Frobenius structure of the $A_n$ singularity, we
will show flatness of the deformed connection to check some of the axioms of section 1. In the general case, the deformed connection is shown to be flat exactly by associativity of the algebra structure and these axioms. This is done explicitly in [1]. To show that the deformed connection is flat it is sufficient to prove that a coordinate system exists that gives rise to a frame of horizontal sections. The following lemma gives a set of equations that should hold for such a coordinate system. In the remainder of this section we will denote by $t_i$ the flat coordinates for $i = 1, \ldots, n$, where $n = \text{Dim}(M)$, and we will denote $\partial_i := \frac{\partial}{\partial t_i}$.

Additionally we will define the structure constants $c^i_{jk}(t)$ for $i,j,k = 1, \ldots, n$ by

$$\partial_i \partial_j |_{t} = \sum_{k=1}^{n} c^i_{jk}(t) \partial_k |_{t}, \quad (2.4)$$

for any $t \in M$.

**Lemma 2** For a function $f \in \mathcal{F}(M)$ such that the derivatives $\xi_i := \partial_i f$ satisfy

$$\partial_i \xi_j = z \sum_{k=1}^{n} c^i_{jk} \xi_k \quad (2.5)$$

and

$$\partial_z \xi_j = \sum_{k,l=1}^{n} d_k t_k c^i_{jk} \xi_l - \frac{\xi_j}{z} (q_j - \frac{d}{2}), \quad (2.6)$$

for any $i,j = 1, \ldots, n$, the one-form $df$ is horizontal with respect to the deformed connection, i.e. $\tilde{\nabla} df = 0$.

**Proof:**

To find these equations we will simply have to apply the deformed connection to $df$. So let us find the Christoffel symbols $\Gamma$, for any combination of sub- and superscripts, for the deformed connection. Note first of all that by equations (2.2) we have $\Gamma^k_{iz} = 0 = \Gamma^k_{zz}$ for any $i,k = 1, \ldots, n$, i.e. we need not worry about the $dz$ part of $df$. Secondly we have

$$\tilde{\nabla}_{\partial_i} \partial_j = z \sum_{k=1}^{n} c^i_{jk} \partial_k$$

so $\Gamma^k_{ij} = z c^k_{ij}$ and $\Gamma^i_{zk} = 0$, for any $i,j,k = 1, \ldots, n$. Additionally

$$\tilde{\nabla}_{\partial_i} \partial_k = \sum_{i,j=1}^{n} \left( d_i t_i c^j_{ik} - \frac{\delta_{jk}}{z} (q_j - \frac{d}{2}) \right) \partial_j$$

so $\Gamma^j_{zk} = -\frac{\delta_{jk}}{z} (q_k - \frac{d}{2}) + \sum_{i=1}^{n} d_i t_i c^j_{ik}$ and $\Gamma^i_{zk} = 0$ for $k = 1, \ldots, n$. Thus we find

$$\tilde{\nabla}_{\partial_i} df = \sum_{j=1}^{n} \left( \partial_i \xi_j - z \sum_{k=1}^{n} c^i_{jk} \xi_k \right) dt_j \quad (2.7)$$
\[ \nabla \frac{df}{dt} = \sum_{j=1}^{n} \left( \partial_{z} \xi_{j} - \sum_{i,k=1}^{n} d_{k} t_{k} c_{ij}^{k} \xi_{k} + \frac{\xi_{j}}{z} (q_{j} - \frac{d}{2}) \right) dt_{j} \]  

(2.8)

Thus equating all the independent parts to 0 we find exactly equations (2.5) and (2.6). Lemma is proved. \( \square \)

The above equations might seem a bit messy. However they are a lot easier to interpret when put in the following form:

\[ \partial_{i} \xi = z(\partial_{i} \bullet) \xi \]  

(2.9)

and

\[ \partial_{z} \xi = (\mathcal{E} - \frac{1}{z} \mu) \xi, \]  

(2.10)

where \((\partial_{i} \bullet)\) is a matrix with entries \((\partial_{i} \bullet)_{jk} = c_{ij}^{k}\), \(\mathcal{E}_{ij} = \sum_{k=1}^{n} d_{k} t_{k} c_{ij}^{k}\) and \(\xi\) is a vector with entries \(\xi_{i} = \partial_{i} f\) for \(i, j, k = 1, \ldots, n\).

Let us now turn to the new matrix introduced in the definition of the deformed connection. It is diagonalizable by diagonalizability of \(\nabla E\). Furthermore we have the following lemma.

**Lemma 3** \(\mu\) is anti-adjoint, i.e.

\[ \langle \mu v, w \rangle + \langle v, \mu w \rangle = 0 \quad \forall v, w \in TM. \]  

(2.11)

**Proof:**

The proof comes down to the relation given in lemma 1. By linearity it is sufficient to show that

\[ \langle \mu \partial_{i}, \partial_{j} \rangle + \langle \partial_{i}, \mu \partial_{j} \rangle = 0 \quad \forall i, j = 1, \ldots, n. \]

In terms of this basis we know that \(\mu\) is diagonal and the entries on the diagonal are given above so we find that

\[ \langle \mu \partial_{i}, \partial_{j} \rangle + \langle \partial_{i}, \mu \partial_{j} \rangle = (q_{i} + q_{j} - d) \langle \partial_{i}, \partial_{j} \rangle \]

then lemma 1 concludes the proof. \( \square \)

**Canonical Coordinates**

We return to the situation of a Frobenius manifold \(M\) (as opposed to \(M \times \mathbb{C}\)). A Frobenius manifold is said to be semisimple if the tangent spaces are semisimple in the sense of Frobenius algebras. A Frobenius algebra \(A\) is called semisimple if it contains no nilpotent elements, i.e. no elements \(0 \neq a \in A\) such that \(\exists n \in \mathbb{N}\) such that

\[ a^{m} = 0. \]  

(2.12)

In fact the semisimplicity condition is a local one. Thus it makes sense to call a point \(t \in M\) semisimple if \(T_{t}M\) is a semisimple algebra. Then there exists
in fact a semisimple neighborhood of $t$ (i.e. such that all the points in this neighborhood are semisimple). This makes sense since the algebra structure should vary analytically. In a semisimple neighborhood there exists another natural system of coordinates called canonical. Where the flat and deformed flat coordinates are natural in the geometric sense, these canonical coordinates are natural by virtue of the algebraic properties of their generated frame.

The canonical coordinates are a system of coordinates such that the corresponding frame of the tangent bundle is orthogonal and has the structure constants $\delta_{ij} \delta_{ik}$ for multiplication of the vector fields corresponding to the $i$th and $j$th canonical coordinate. To show existence of this coordinate system we will first need to prove the following lemma about semisimple Frobenius algebras. We will follow the path laid out in [1].

**Lemma 4** Any finite-dimensional Frobenius Algebra $A$ is isomorphic to an orthogonal direct sum

$$A \cong A_0 \oplus \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$  \hspace{1cm} (2.13)

where $A_0$ is a nilpotent algebra

**Proof:**

Let $\lambda_0, \lambda_1, \ldots, \lambda_k \in A^*$ be the roots of $A$ where $\lambda_0 \equiv 0$. So the $\lambda_i$ are linear functionals such that for any $a \in A$ the eigenvalues of the multiplication by $a$ are $\lambda_0(a), \ldots, \lambda_k(a)$. Let

$$A_j := \bigcap_{a \in A} \text{Ker}((a \cdot) - \lambda_j(a)I)^{p_j(a)}$$  \hspace{1cm} (2.14)

for $j = 0, 1, \ldots, k$, $I$ the identity matrix and $p_j(a)$ the Riesz-index corresponding to the linear map $(a \cdot)$ and the eigenvalue $\lambda_j(a)$. So in fact $A_j$ is the intersection of all the generalized eigenspaces corresponding to the linear maps $(a \cdot)$ and eigenvalues $\lambda_j(a)$ for all $a \in A$. In particular we see that $A_0$ consists of the nilpotent elements of $A$. So $A$ is semisimple if and only if $A_0 = \{0\}$. It is an established fact of linear algebra that $A_i \cap A_j = \{0\}$ for $i \neq j$ [18]. So clearly

$$A = \bigoplus_{j=0}^k A_j.$$  \hspace{1cm} (2.15)

Moreover $A_i \cdot A_j = \{0\}$ whenever $i \neq j$ since these lie in the intersection. Thus we have $\lambda_j(A_i) = 0$ for $i \neq j$.

Now, since we have invariance of the bilinear form, it is left to prove that the $A_i$ are one-dimensional for $i \neq 0$. For $i \neq 0$ let $0 \neq v_i \in A_i$ be a vector such that

$$vv_i = \lambda_i(v)v_i$$

for all $v \in A$, i.e. an eigenvector. Suppose $\lambda_i(v_i) = 0$ then

$$v_i^2 = 0,$$

which would make $v_i$ nilpotent. However $A_0 \cap A_i = \{0\}$ so $\lambda_i(v_i) \neq 0$. Put then

$$\pi_i = \frac{v_i}{\lambda_i(v_i)}.$$  \hspace{1cm} (2.16)
So we have $\pi_i^2 = \pi_i$. Suppose $w_i \in A_i$ is another eigenvector. Then

$$w_i = \lambda_i(\pi_i)w_i = \pi_i w_i = \lambda_i(w_i),$$

so $w_i$ is proportional to $\pi_i$. Suppose that $u_i \in A_i$, then there is $n \in \mathbb{N}$ such that

$$(\pi_i - \lambda_i(\pi_i))^n u_i = 0.$$ 

Note that $(\pi_i - \lambda_i(\pi_i))^2 = 1 - \pi_i$ so we have either $(\pi_i - \lambda_i(\pi_i))^n = \pi_i - \lambda_i(\pi_i)$ or $(\pi_i - \lambda_i(\pi_i))^n = \lambda_i(\pi_i) - \pi_i$, depending on the parity of $n$. So in either case we have

$$\lambda_i(u_i) \pi_i = \pi_i u_i.$$

So again we find that $u_i$ is proportional to $\pi_i$. Thus in fact all the vectors in $A_i$ are proportional to $\pi_i$. In conclusion we have the subalgebra $A_0$ of nilpotent elements and the subalgebras $A_i$ which are generated by the $\pi_i$ with $\pi_i \pi_j = \delta_{ij} \pi_i$.

Now let us move on to the case of a semisimple Frobenius structure.

**Proposition 5** For $M$ a Frobenius manifold we have, near a semisimple point $t \in M$, a system of local coordinates $(u_1, \ldots, u_n)$, where $n = \text{Dim}(M)$, such that

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_i},$$

(2.17)

for all $i, j = 1, \ldots, n$. These coordinates are called canonical coordinates.

**Proof:**

Since the algebra structure on the tangent spaces varies analytically, we have, by the previous lemma, a frame of idempotents $\pi_1, \ldots, \pi_n$ with $\pi_i \cdot \pi_j = \delta_{ij} \pi_i$, for all $i, j = 1, \ldots, n$. Thus it is sufficient to show that the lie brackets $[\pi_i, \pi_j]$ vanish for all $i, j = 1, \ldots, n$. To show this we will use the deformed connection. Define the coefficients $\Gamma^k_{ij}$ and $f^k_{ij}$ by

$$\nabla_{\pi_i \pi_j} = \sum_{k=1}^n \Gamma^k_{ij} \pi_k$$

(2.18)

and

$$[\pi_i, \pi_j] = \sum_{k=1}^n f^k_{ij} \pi_k,$$

(2.19)

for all $i, j = 1, \ldots, n$. Now from flatness of the connection we find that

$$0 = \tilde{\nabla}_{\pi_i \pi_j} \pi_k - \tilde{\nabla}_{\pi_j \pi_i} \pi_k - \tilde{\nabla}_{[\pi_i, \pi_j]} \pi_k =$$

$$= \left( \sum_{l, p=1}^n \Gamma^l_{jk} \Gamma^p_{il} - \Gamma^l_{ik} \Gamma^p_{jl} - f^l_{ij} \Gamma^p_{lk} \right) \pi_p +$$

$$+ z \sum_{l=1}^n \left( \Gamma^l_{jk} \delta_{il} - \Gamma^l_{ik} \delta_{jl} + \Gamma^l_{ik} \delta_{jk} - \Gamma^l_{jk} \delta_{ik} - f^l_{ij} \delta_{kl} \right) \pi_l +$$

10
\[ z^2 (\delta_{jk} \delta_{ij} - \delta_{ik} \delta_{ij}) \pi_i, \quad (2.20) \]

for any \( i, j, k = 1, \ldots, n \). Thus, equating the linear parts in \( z \), we find that

\[ \Gamma^l_{jk} \delta_{il} + \Gamma^l_{ki} \delta_{kj} - \Gamma^l_{ki} \delta_{jl} - \Gamma^l_{kj} \delta_{ki} = f^l_{ij} \delta_{kl}, \quad (2.21) \]

for any \( i, j, k, l = 1, \ldots, n \). So setting \( l = k \) yields

\[ f^k_{ij} = 0. \]

Lastly we have, of course, the freedom to call these coordinates whatever we want. We call them canonical. \( \square \)

Now we can recast all the elements that make up the Frobenius structure in the following form.

**Theorem 6** Let \( u_1(t), \ldots, u_n(t) \) be the eigenvalues of the operator of multiplication by the Euler vector field. They form a system of local coordinates near a semisimple point \( t \in M \) and in these coordinates we have

\[ \partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_i}, \quad (2.22) \]

\[ e = \sum_{i=1}^{n} \partial_{u_i}, \quad (2.23) \]

\[ E = \sum_{i=1}^{n} u_i \partial_{u_i}, \quad (2.24) \]

and

\[ < \cdot, \cdot > = \sum_{i=1}^{n} h_i u_i^2, \quad (2.25) \]

where \( e \) is the unit vector field, \( E \) is the euler vector field and

\[ h_i = \partial_{u_i} \sum_{j,k=1}^{n} < \partial_{u_k}, \partial_j > t_j, \quad (2.26) \]

with \( (t_1, \ldots, t_n) \) the flat coordinates.

**Proof:**

The lemma takes care of the existence of a coordinate system that satisfies (2.22) and thus also (2.23). So the first thing we should prove is that the coordinates that generate the frame \((\pi_1, \ldots, \pi_n)\) in the proof of the lemma are indeed the eigenvalues of the operator of multiplication by the Euler vector field. Recall the grading operators (1.11) and (1.12). Then (1.5) implies

\[ \mathcal{L}_E(a \cdot b) - \mathcal{L}_E(a) \cdot b - a \cdot \mathcal{L}_E(b) = a \cdot b, \quad (2.27) \]

for all \( a, b \in \mathcal{X}(M) \). With \( a = \pi_i, b = \pi_j \) and \( i \neq j \) this yields

\[ \mathcal{L}_E(\pi_i) \cdot \pi_j + \mathcal{L}_E(\pi_j) \pi_i = 0. \]
Multiplying this through by $\pi_j$ then yields

$$\mathcal{L}_E(\pi_i)\pi_j = 0,$$

which implies $\mathcal{L}_E(\pi_i) = \lambda_i \pi_i$, for some $\lambda_i \in \mathcal{F}(M)$ since it should hold for all $j \neq i$. Now applying (2.27) with $a = b = \pi_i$ yields

$$\lambda_i \pi_i - 2\lambda_i \pi_i = \pi_i,$$

which shows that $\lambda_i = -1$. Recall that the lemma showed that $(\pi_1, \ldots, \pi_n)$ is the frame generated by some coordinates $(u_1, \ldots, u_n)$. In these coordinates we could write $E = \sum_{i=1}^n E_i(u)\partial_{u_i}$, then $\mathcal{L}_E(\pi_i) = -\pi_i$ yields

$$\frac{\partial E_i}{\partial u_j} = \delta_{ij}. \quad (2.28)$$

So we see that after a translation in the coordinates if necessary (which will not affect the corresponding frame), we find (2.24).

The equation (2.25) follows nearly immediately. First of all note that, writing again $\pi_i = \partial_{u_i}$, we find

$$<\pi_i, \pi_j> = <e, \pi_i \cdot \pi_j> = <e, \delta_{ij} \pi_i>$$

which yields

$$<\cdot, \cdot> = \sum_{i=1}^n <e, \pi_i > du_i^2,$$

since $(\pi_1, \ldots, \pi_n)$ forms an orthogonal frame. Then we find

$$<e, \pi_i> = \sum_{k=1}^n <\partial_{u_k}, \partial_{u_i}> = \sum_{k,j=1}^n <\partial_{u_k}, \partial_j > dt_j(\partial_{u_i}) =$$

$$\partial_{u_i} \sum_{k,j=1}^n <\partial_{u_k}, \partial_j > t_j = h_i.$$

□

**Corollary 2** Note that all the points $t \in M$ where $(E(t)\bullet)$ has $n$ distinct eigenvalues are semisimple. Note also that we can find our canonical coordinates simply by finding these eigenvalues. Lastly we can recast the equations related to the deformed connection in terms of the new coordinates. In these coordinates for instance $E$ will just be a diagonal matrix. This is done and the results are studied in for instance [1]. We will however not need this complete analysis.

Note that the frame generated by the canonical coordinates is automatically orthogonal. Then it is clear that to every frame generated by canonical coordinates there is also related an orthonormal frame with structure constants $\sqrt{<\partial_{u_i}, \partial_{u_i}>} \delta_{ij} \delta_{ik}$ where the $u_i$ represent the corresponding canonical coordinates.
3 Frobenius Structure of $A_n$ Singularity

Now we are ready to define and discuss the Frobenius structure associated to a singularity of $A_n$ type. By such a singularity we mean the germ of functions $f: \mathbb{C} \to \mathbb{C}$ that preserve 0 and have an $n$-fold degenerate critical point at 0. To define the Frobenius structure related to this singularity we will be working with a certain representative. We will state in this section all the basic elements that make up the Frobenius structure. However, we will not prove that the conditions 1-4 of section 1 are met. This will be done by first exploring the concepts of flat coordinates, deformed flat coordinates and the Euler field in this explicit case. Finally in section 6 we will prove that we have in fact been discussing a Frobenius structure, i.e. show that the conditions 1-4 are met. We will start by defining the structure at the point 0 in our manifold (in all coordinate systems we will use). This specific structure in the point 0 (which is not semisimple) will be used in the next part to discuss the $(n+1)$KdV hierarchies.

Local Algebra as Frobenius Algebra

Let $[f]: (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function at a singularity of $A_n$ type, for some $n \in \mathbb{N}$. We pick the representative $f(x) = x^{n+1}$. Note that this is a choice of preference.

Let $H := \mathbb{C}[x]/\langle f_x \rangle = \langle 1, x, x^2, \ldots, x^{n-1} \rangle_{\mathbb{C}}$ denote the local algebra of the singularity. We define on $H$ the bilinear form $\langle \cdot, \cdot \rangle$ by

$$\langle v, w \rangle = \text{Res}_{x=\infty} v(x)w(x) \frac{dx}{f_x}$$

(3.1)

for $v, w \in H$, which is clearly well defined. Note that by commutativity of $H$ this form is symmetric and for $u, v, w \in H$ we have $\langle u \cdot v, w \rangle = \langle u, v \cdot w \rangle$.

If $v \in H$ is such that $\forall w \in H$ we have $\langle v, w \rangle = 0$, say $v = \sum_{i=1}^{n} v_i x^{n-i}$ with $v_i \in \mathbb{C}$ $\forall i$, then $\langle v, -(n+1)x^{n-1} \rangle = v_j$ for $j = 1, 2, \ldots, n$. This shows $v = 0$ and thus that the form is non-degenerate. So this gives $(H, 1, \langle \cdot, \cdot \rangle)$ the structure of a Frobenius algebra.

Parameter Space as Frobenius Manifold

Now we would like to extend this structure of Frobenius algebra to the tangent spaces of the parameter space $T$ of the miniversal deformation $F(x, \tau)$ of $f$, thus giving $T$ the structure of a Frobenius manifold. To clarify, the miniversal deformation is given by $n$ complex numbers $\tau = (\tau_1, \ldots, \tau_n)$ as

$$F(x, \tau) = x^{n+1} + \tau_1 x^{n-1} + \ldots + \tau_n.$$

Note that we have $F(x, 0) = f$. By parameter space we mean the space of such complex numbers. So in fact $T \simeq \mathbb{C}^n$ and to every point $\tau = (\tau_1, \ldots, \tau_n) \in T$ we relate a deformation $F(x, \tau)$. So we automatically have a coordinate system on $T$.

We begin at $\tau = 0$ by identifying the basis $\{\partial_{\tau_i}|_0\}_{i=1}^{n}$ of $T_0 T$ with the basis $\{[x^{n-i}]\}_{i=1}^{n}$ of $H$ by sending $\partial_{\tau_i}|_0$ to $[x^{n-i}]$ and pulling back $\langle \cdot, \cdot \rangle$ under
this identification. We denote this pullback by \langle \cdot, \cdot \rangle_0 (as we will in general denote the bilinear form on any \( T_\tau T \) by \( \langle \cdot, \cdot \rangle_\tau \)). In terms of the basis of \( T_0 T \) mentioned above we have for all \( i,j = 1, \ldots, n \)

\[
\langle \partial_{\tau_i}, \partial_{\tau_j} \rangle_0 = -\frac{\delta_{i+j,n+1}}{n+1} = \text{Res}_{x=\infty} \frac{\partial_{\tau_i} F[0]}{\partial_{\tau_j} F[0]} \frac{\partial_{\tau_j} F[0]}{\partial_{x} F[0]} dx,
\]

(3.2)

where \( |_0 \) refers to setting \( \tau = 0 \). We extend this form by defining

\[
\langle \partial_{\tau_i}, \partial_{\tau_j} \rangle_\tau = \text{Res}_{x=\infty} \frac{\partial_{\tau_i} F[\tau]}{\partial_{\tau_j} F[\tau]} \frac{\partial_{\tau_j} F[\tau]}{\partial_{x} F[\tau]} dx,
\]

(3.3)

for all \( i,j = 1, \ldots, n \).

The coordinates \( \tau_1, \ldots, \tau_n \) are natural in the sense of simplicity of the expression for \( F \) in terms of these coordinates. Therefore we will call them the natural coordinates. However they are obviously not generally flat and we will show momentarily that they are also not canonical. If \( (\tilde{a}_1, \ldots, \tilde{a}_n) \) is another (local) system of coordinates on \( T \) then, since \( \partial_{\tilde{a}_i} = \sum_{j=1}^n \partial_{a_j} \tau_j \partial_{\tau_j} \), we have

\[
\langle \partial_{a_i}, \partial_{a_j} \rangle_\tau = \sum_{p=1}^n \langle \partial_{a_i}, \partial_{\tau_p} \rangle_\tau \sum_{p=1}^n \langle \partial_{a_j}, \partial_{\tau_p} \rangle_\tau = \sum_{p,q=1}^n (a_i \tau_p | \tau) (\partial_{\tau_p} \tau_q | \tau) < \partial_{\tau_p}, \partial_{\tau_q} >_\tau = \\
= \sum_{p,q=1}^n (a_i \tau_p | \tau) (\partial_{\tau_p} \tau_q | \tau) \text{Res}_{x=\infty} \frac{\partial_{\tau_p} F[\tau]}{\partial_{\tau_q} F[\tau]} \frac{\partial_{\tau_q} F[\tau]}{\partial_{x} F[\tau]} dx = \\
= \text{Res}_{x=\infty} \frac{\partial_{a_i} F[\tau]}{\partial_{a_j} F[\tau]} dx,
\]

(3.4)

for all \( i,j = 1, \ldots, n \). So we actually have an expression for the metric in any coordinate system.

Now we also have naturally an algebra structure on the tangent spaces to \( T \), which agrees with the algebra structure on \( H \), by pulling back the algebra structure on \( \mathbb{C}[x]/\langle \partial_x F[\tau] \rangle \) under the isomorphism \( \partial_{\tau_i} |_\tau \mapsto [\partial_{\tau_i} F[\tau]] \). Note that

\[
\langle \partial_{\tau_i}, \partial_{\tau_j}, \partial_{\tau_k} \rangle_\tau = \text{Res}_{x=\infty} \frac{\partial_{\tau_i} F[\tau]}{\partial_{x} F[\tau]} \frac{\partial_{\tau_j} F[\tau]}{\partial_{x} F[\tau]} \frac{\partial_{\tau_k} F[\tau]}{\partial_{x} F[\tau]} dx,
\]

(3.5)

for all \( i,j,k = 1, \ldots, n \), which shows that this metric is invariant in the sense of Frobenius algebras. Also clearly the metric is symmetric.

To summarize we have now the manifold \( T \) of parameters \( \tau \) of the deformation \( F(x, \tau) = \sum_{n+1} + \tau_i x^{n+1} + \cdots + \tau_n \), i.e. every point is a specific deformation of \( f \). We define an algebra structure on the tangent spaces \( T_\tau T \) by pulling back the algebra structure on \( \mathbb{C}[x]/\langle \partial_x F[\tau] \rangle \) under the isomorphism of vector spaces given by sending \( v_\tau \in T_\tau T \) to \( [v_\tau F[\tau]] \in \mathbb{C}[x]/\langle \partial_x F[\tau] \rangle \). Then we define the bilinear form \( \langle \cdot, \cdot \rangle_\tau \) on \( T \) by \( \langle v_\tau, w_\tau \rangle_\tau = \text{Res}_{x=\infty} \frac{\langle v_\tau F[\tau], w_\tau F[\tau] \rangle}{\partial_{x} F[\tau]} dx \). Now we want
to show that this defines a Frobenius manifold with diagonalizable \( \nabla E \). Then there is an underlying solution of the WDVV equations. To do this we will need to show that the form \( \langle \cdot, \cdot \rangle_\tau \) (which we will call metric from now on) is flat. The flat coordinates will then be a lot more convenient to show that \( T \) satisfies the other axioms to be a Frobenius manifold. The following theorem will be proved at the end of section 6.

**Theorem 7** The parameter space \( T \) of miniversal deformations of the singularity \( f(x) = x^{n+1} \) for \( n \in \mathbb{N} \) together with the algebra structure on the tangent spaces given by \( v_\tau \cdot w_\tau = z_\tau \) for some \( v_\tau, w_\tau \in T_\tau T \) and \( z_\tau \in T_\tau T \) such that \( (v_\tau(F))(w_\tau(F)) = z_\tau(F) \) \( + G(x, \tau) \partial_x F|_\tau \) (where \( G \) is just some polynomial in \( x \) and \( \tau \) and \( z_\tau(F) \) is not divisible by \( (\partial_x F)|_\tau \) in \( \mathbb{C}[x] \)) and the bilinear form \( \langle \cdot, \cdot \rangle_\tau \) given by \( \langle v_\tau, w_\tau \rangle = \text{Res}_{x=\infty} \frac{(v_\tau(F))(w_\tau(F))}{\partial_x F|_\tau} \) is a Frobenius manifold.

In other words the construction given in this section defines a Frobenius manifold. First we will devote a section to the flat coordinates.

### 4 Flat coordinates of \( A_n \) Singularity

By now we have all the main ingredients of the Frobenius structure. Namely the manifold, metric and algebra structure. However we have not yet shown that these satisfy all the requirements. Even though the natural coordinates are natural, they are not the easiest to work with to explore the other aspects of the Frobenius structure. In sections 1 and 2 most discussions were framed in the flat coordinate system. This system is arguably the most useful to work with on any Frobenius manifold. Finding the flat coordinates in the case of the \( A_n \) singularity is not the easiest task however. In this section we will give an abstract definition of the coordinates and show that we can always find them, at least recursively, expressed in terms of the natural coordinates. Subsequently we will explore the relation between the flat and natural coordinates. Of course we will also show flatness of the coordinates.

**Definition of the Flat Coordinates**

Consider again the \( A_n \) singularity \( f(x) = x^{n+1} \). This function has the miniversal deformation \( F(x, \tau) = x^{n+1} + \tau_1 x^{n-1} + \ldots + \tau_n \) for any \( (\tau_1, \ldots, \tau_n) \). To define the flat coordinates \( (t_1, \ldots, t_n) \) on the parameter space of these deformations we first consider the Laurent series \( k = (F(x, \tau))^{\frac{1}{n+1}} \). Since at any point in the parameter space of \( F \), i.e. for any choice of \( \tau \), \( F(x, \tau) \) is a polynomial of order \( n + 1 \) in \( x \) we know that

\[
(F(x, \tau))^{\frac{1}{n+1}} = x + b_1 x^{-1} + b_2 x^{-2} \ldots =: k \tag{4.1}
\]

for some infinite sequence \( b_i \) in \( \mathbb{C} \) such that

\[
k^{n+1} = F(x, \tau) = x^{n+1} + \tau_1 x^{n-1} \ldots + \tau_n. \tag{4.2}
\]

We will show how to express the \( \tau_i \) in the \( b_i \) by finding \( n \) sets \( S_{(n,i)} \) for \( 1 \leq i \leq n \), which we define in the next section.
Finally we define the flat coordinates $t_i$ as solutions of the equation

\[ x = k + \frac{1}{n+1} \left( \frac{t_1}{k} + \frac{t_2}{k^2} + \ldots + \frac{t_n}{k^n} \right) + \frac{1}{O(k^{n+1})} \]  

(4.3)

(So we have some part of order smaller than $-n$ in $k$, we denote this fact in the following by adding $O(k^{-n-1})$ at any point in the parameter space of $F(x, \tau)$.)

We will show later that this is in fact well defined since one can express the flat coordinates in terms of natural coordinates recursively and vice versa.

The Sets $S_{(n,i)}$

Consider the following algebraic problem. Fix $n,i \in \mathbb{N}$ with $0 < i \leq n$. We want to find all combinations of numbers $n_j \in \mathbb{N} \cup \{0\}$ for $j = 0, 1, 2, \ldots$ such that

\[ n - i = n_0 - \sum_{j=1}^{\infty} jn_j \]  

and

\[ n + 1 = \sum_{j=0}^{\infty} n_j. \]  

(4.4) \hspace{1cm} (4.5)

Suppose $n_0, n_1, n_2, \ldots$ solves this problem then it is clear that $n_0 \neq 0$ since if $n_0 = 0$ then since $n + 1 = n_1 + n_2 + \ldots$ we know that not all the $n_j$ are 0 for $j > 0$ so $-\sum_{j=1}^{\infty} jn_j < 0 \leq n - i \forall 0 < i \leq n$. So at least $n_0 \neq 0$. Moreover the second equality in the problem tells us that only finitely many of the $n_j$ can be non-zero.

Now if $n_0, n_1, n_2, \ldots$ solves this problem we define the non-zero-tuple form of this solution to be the $(m+1)$-tuple of elements in $\mathbb{Z}^2$

\[ ((n_0, 0), (n_{j_1}, j_1), (n_{j_2}, j_2), \ldots, (n_{j_m}, j_m)) \]

where $m$ is the number of non-zero $n_j$ with $j \neq 0$ and $j_k$ runs over the indices of these non-zero $n_j$. For example, suppose $i = 3$ (and $n$ is higher than 3) then $n_0 = n - 1$ and $n_1 = 2$ solves the problem. Then the non-zero-tuple form of this solution is $((n-1, 0), (2, 1))$. Now we simply define

\[ S_{(n,i)} = \{ \text{non-zero-tuple forms of solutions of the above problem with } n \text{ and } i \}. \]  

(4.6)

Now we see that by equating the coefficients of equal powers of $x$ in $k^{n+1}$ and $F(x, \tau)$ we have

\[ \tau_i = \sum_{((n_0, 0), (n_{j_1}, j_1), \ldots, (n_{j_m}, j_m)) \in S_{(n,i)}} \left( \begin{array}{c} n+1 \\ n_0, n_{j_1}, \ldots, n_{j_m} \end{array} \right) \prod_{k=1}^{m} b_{j_k}^{n_{j_k}}, \]  

for all $i = 1, \ldots, n$.

Finding the Flat Coordinates

Now we can find the flat coordinates recursively and thus show that they are well defined. First of all we need to find the way that the natural coordinates depend on the $b_i$.

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Lemma 8 For all $0 < i \leq n$ we have that $\tau_i$ in $F(x; \tau)$ does not depend on $b_j$ for $j > i$.

Proof:

We will show that in the problem at the beginning of the last section $n_j = 0$ if $j > i$. Suppose there is a solution $n_0, n_1, \ldots$ with $n_j \neq 0$ for a $j > i$, then since

$$n_j \in \mathbb{N} \cup \{0\}$$

we have $n_j \geq 1$. Then

$$n_0 = n + 1 - \sum_{k=1}^{\infty} n_k \leq n + 1 - n_j \leq n.$$

Then we have $n_0 - \sum_{k=1}^{\infty} kn_k \leq n_0 - jn_j \leq n_0 - j \leq n - j < n - i$. This contradiction proves the lemma.

So in fact $\tau_i$ can only depend on the $b_j$ with $j \leq i$, but then we can also express the $b_j$ using $\tau_j$ and $b_k$ with $k < j$. So in fact we can express $b_j$ using the $\tau_k$ with $k \leq j$.

Suppose

$$x = k + \frac{1}{n+1} \left( \frac{t_1}{k} + \frac{t_2}{k^2} + \ldots + \frac{t_n}{k^n} \right) + O(k^{-n-1})$$

then

$$x^{-1} = \frac{1}{k + \frac{1}{n+1} \left( \frac{t_1}{k} + \frac{t_2}{k^2} + \ldots + \frac{t_n}{k^n} \right) + O(k^{-n-1})} =$$

$$= \frac{1}{k} \left( 1 + \left( \frac{t_1}{n+1} \frac{1}{k^2} + \ldots + \frac{t_n}{n+1} \frac{1}{k^n} \right) + O(k^{-n-2}) \right) \hspace{1cm} (4.8)$$

To show that the flat coordinates are well defined we will give a recursive way of finding them. Going by the degree of $k$ in the expression

$$x + b_1 x^{-1} + b_2 x^{-2} \ldots = k$$

after we replace $x = k + \frac{1}{n+1} \left( \frac{t_1}{k} + \frac{t_2}{k^2} + \ldots + \frac{t_n}{k^n} \right) + O(k^{-n-1})$. Then we see that in the term $x^{-j}$ the highest power of $k$ is $-j$. However in the term $x$ we see that the highest degree of $k$ in which $t_j$ appears is $-j$. So we see that we can express first $t_1$ in terms of only $b_1$. Then using this answer we can express $t_2$ in terms of $b_1$ and $b_2$ and so on. Moreover using the above Lemma and its consequence we can express $t_j$ using only the $\tau_i$ with $i \leq j$. This shows the coordinates are well defined and gives us a way to express the natural coordinates in terms of the flat coordinates.

Example 1 Let us set $n = 4$ to do an example. So we have

$$F(x, \tau) = x^5 + \tau_1 x^3 + \tau_2 x^2 + \tau_3 x + \tau_4.$$  
First we determine the sets $S_{(4,i)}$, for $i = 1, 2, 3, 4$. We have

$$S_{(4,1)} = \{ (4,0), (1,1) \};$$
$$S_{(4,2)} = \{ (4,0), (1,2) \};$$

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\[ S_{(4,3)} = \{((4,0),(1,3)),((3,0),(2,1))\}; \]
\[ S_{(4,4)} = \{((4,0),(1,4)),((3,0),(1,1),(1,2))\}. \]

So the formula above gives
\[
\begin{align*}
\tau_1 &= 5b_1; \\
\tau_2 &= 5b_2; \\
\tau_3 &= 5b_3 + 10b_1^2; \\
\tau_4 &= 5b_4 + 20b_1b_2.
\end{align*}
\]

Now let us express the flat coordinates in terms of the \( b_j \). So first we look at the coefficient of \( k^{-1} \) this gives \( \frac{1}{5} + b_1 = 0 \). Going through the degrees \(-1\) through \(-4\) yields
\[
\begin{align*}
\frac{t_1}{5} + b_1 &= 0; \\
\frac{t_2}{5} + b_2 &= 0; \\
\frac{t_3}{5} - b_1 \frac{t_1}{5} + b_3 &= 0; \\
\frac{t_4}{5} - b_1 \frac{t_2}{5} - b_2 \frac{t_1}{5} + b_4 &= 0.
\end{align*}
\]

Now we can solve starting from the top to get an expression of the \( \tau_i \) in terms of the flat coordinates. This yields
\[
\begin{align*}
\tau_1 &= -t_1; \\
\tau_2 &= -t_2; \\
\tau_3 &= \frac{t_1^2}{5} - t_3; \\
\tau_4 &= \frac{2}{5} t_1 t_2 - t_4.
\end{align*}
\]

Relation between Natural and Flat Coordinates

The example shows a very easy relation between \( \tau_1 \) and \( t_1 \). It shows the same relation between \( \tau_2 \) and \( t_2 \). In fact the dependence of \( \tau_i \) on \( t_i \) seems rather straightforward so we can guess the following lemma from the example.

**Lemma 9** \[ \partial_{t_i} \tau_i = -1 \] for all \( i = 1, \ldots, n \).

**Proof:**

First of all we will prove that the relation for the power \(-i\) of \( k \) in the procedure of writing \( t \) in terms of \( b \) gives an equation of the following form
\[ \frac{t_i}{n+1} + b_i + G(b_1, \ldots, b_{i-1}) = 0 \]
for some function \( G \). Again we get this equation by substituting the expression for \( x \) in terms of \( t \) and \( k \) in \( k = x + \sum_{i=1}^{\infty} b_i x^{-i} \) and equating like powers of \( k \).
Since the only power of $k$ present on the left hand side is 1 this basically means that we find the coefficient of $k$ to the power $-i$ on the right hand side and equate it to 0. Indeed the first term $k^{-i}$ we encounter is in $x$ and has coefficient $\frac{t_i}{n+i}$. In the other powers of $x$ the order of $k$ that has coefficient involving $t_i$ will always be lower then $-i$ so the only term involving $t_i$ is $\frac{t_i}{n+i}$. The highest order of $k$ in $x^{-j}$ is $k^{-j}$ so indeed no $b_j$ with $j > i$ appear in the expression and $b_i$ only appears by adding it once. So the above expression holds.

Secondly we want to show that a similar expression holds for $\tau$ namely

$$\tau_i = H(b_1, \ldots, b_{i-1}) + (n + 1)b_i$$

for some function $H$. We already know that $\tau_i$ does not depend on $b_j$ with $j > i$. So what is left is to show this specific dependence on $b_i$. Now if we look at the expression (4.7) we see that we need to prove that there is no non-zero-tuple in $S_{(n,i)}$ that has a term $(n_i,i)$ with $n_i > 1$ and apart from the non-zero-tuple $((n,0),(1,i))$ there is no non-zero-tuple with a term $(1,i)$. This is quite easy to prove.

Suppose we have a solution with $n_i > 1$ then

$$n + 1 = \sum_{j=0}^{\infty} n_j \geq n_0 + n_i > n_0 + 1$$

so $n > n_0$, but then

$$n - i = n_0 + \sum_{j=1}^{\infty} jn_j \leq n_0 - in_i < n - i,$$

which is absurd. Thus such a solution does not exist. Suppose we have a solution with $n_i = 1$ then we have $n \geq n_0$ and $n - i \leq n_0 - i$ which shows that $n_0 = n$ and thus yields the non-zero-tuple $((n,0),(1,i))$.

Thus when we define $\tau$ in terms of $t$ we get $\tau_i = -t_i + J(t_1, \ldots, t_{i-1})$ where $J(t_1, \ldots, t_{i-1}) = H(b_1, \ldots, b_{i-1}) - (n + 1)G(b_1, \ldots, b_{i-1})$ with the $b$'s in terms of the $t$'s and we remember that $b_j$ does not depend on $t_p$ if $p > j$. Then the lemma is proved.

□

**Corollary 3** This shows also that $\tau_1 = -t_1$ in general.

**Flatness**

Now, to justify calling these new coordinates flat, let us explicitly calculate $<\partial_i, \partial_j>$ for $1 \leq i, j \leq n$, where from now on we denote $\partial_k = \partial_{t_k}$ and we suppress the subscript $\tau$ if it is immaterial. Recall that the metric was given by the formula

$$<\partial_i, \partial_j> = \text{Res}_{x=\infty} \frac{(\partial_i F)(\partial_j F)}{\partial_x F} dx$$

To calculate this we switch to the variable $k$ and use the identities

$$\partial_j F = \partial_j k^{n+1} = (\partial_j \partial_k) k^{n+1}$$

and

$$\partial_i F dx = (\partial_i x) k^{n+1} dx = (\partial_i x) dx^{n+1}$$


to obtain
\[
<\partial_i,\partial_j> = (n + 1) \Res_{k=\infty} \left( \frac{1}{(n+1)k^p} + O(k^{-n-1}) \right) k^n dk =
\]
\[
= \Res_{k=\infty} \left( \frac{1}{(n+1)k^{p+q}} + O(k^{-n-1-\min(p,q)}) \right) k^n dk.
\]
Now we know that 1 \leq i, j so \( n - n - 1 - \min(i, j) < -1 \) thus only the first part can contribute to the residue. This yields finally
\[
<\partial_i,\partial_j> = \frac{-\delta_{i+j,n+1}}{n + 1}, \quad (4.9)
\]
which is indeed independent of the point \( t \in \mathcal{T} \).

5 Multiplication and Euler Vector Field of \( A_n \) Singularity

We would like to prove that \( \mathcal{T} \) is a Frobenius manifold and in fact one that can be defined by a solution of the WDVV equations. To do this we will have to define the Euler vector field \( E \) and look a little closer at the structure constants \( c^k_{ij} \) for \( \mathcal{T} \) with the basis given by the flat coordinates.

The Polynomials \( K_{ij} \) and Relations

We defined the multiplication on \( T \mathcal{T} \) by identifying the basis \{\( \partial \tau_i \}_{i=1}^n \} in \( T \mathcal{T} \) with the basis \{\([\partial \tau_i F]|_t\)\} in \( \mathbb{C}[x]/\langle \partial_x F \rangle \). We can easily adapt this construction to find the structure constants corresponding to the frame generated by the flat coordinates, as pointed out above. We will now elaborate on this construction and find a useful relation in the following lemma.

Lemma 10 Denoting by \( \partial_i \) the derivative with respect to the flat coordinate \( t_i \), there are, for all \( i, j = 1, \ldots, n \), polynomials \( K_{ij}(x, t) \) satisfying
\[
(\partial_i F)(\partial_j F) = \sum_{p=1}^n c^p_{ij}(t) \partial_p F + K_{ij} \partial_x F \quad (5.1)
\]
and
\[
\partial_i \partial_j F = \partial_x K_{ij} \quad (5.2)
\]
where \( \partial_i \cdot \partial_j = \sum_{p=1}^n c^p_{ij}(t) \partial_p \).

Proof:

The coefficients \( c^p_{ij}(t) \) that show up in these equations come from the multiplication \( \partial_i \cdot \partial_j = \sum_{p=1}^n c^p_{ij}(t) \partial_p \) in each of the algebras \( T \mathcal{T} \). They are defined by identifying \( \partial_i \) with \([\partial_i F]|_t\) in the algebra \( \mathbb{C}[x]/\langle \partial_x F |_t \rangle \), going from \( \partial_t \), as in
(3.4) shows this easily. So in fact the first equation follows by definition of this multiplication since
\[ \partial_i \cdot \partial_j - \sum_{p=1}^{n} c_{ij}^p(t) \partial_p = 0 \] (5.3)
so
\[ \left[ (\partial_i F)(\partial_j F) - \sum_{p=1}^{n} c_{ij}^p(t) \partial_p F \right] = [0] \] (5.4)
i.e. there is a uniquely defined polynomial \( K_{ij} \) satisfying the first of the above equations. So the real trick is in the second equation. By looking at the possible degrees in \( x \) of the polynomials in the first equations we see already that
\[ K_{ij} = n - 1 \sum_{p=0}^{n-1} K_{ij}^{(p)}(t)x^p. \]
We can also write \( F(x, \tau) = x^{n+1} + \tau_1 x^{n-1} + \ldots + \tau_n \), plugging these into the second equation and equating the coefficients of different degrees of \( x \) we see that we need to prove that
\[ \partial_i \partial_j \tau_a = (n+1-a)K_{ij}^{(n+1-a)} \quad \forall a = 2, \ldots, n \] (5.5)
and that \( \partial_i \partial_j \tau_1 = 0 \). However since \( \tau_1 = t_1 \) by corollary 3 we already know that \( \partial_i \partial_j \tau_1 = 0 \quad \forall i, j = 1, \ldots, n \). To prove the rest we first express \( \tau_a \) as a residue and switch to the variable \( k \)
\[ \tau_a = - \text{Res}_{x=\infty} F(x, \tau)x^{(a-n-1)}dx = - \frac{1}{a-n} \text{Res}_{k=\infty} k^{n+1}dx^{a-n} = \frac{n+1}{a-n} \text{Res}_{k=\infty} k^n x^{a-n}dk \] (5.6)
where \( k^{n+1} = F(x, t) \) as before, i.e. \( x = k + \frac{t_1}{n+1}(\frac{t_2}{k} + \ldots + \frac{t_n}{k^n}) + O(k^{-n-1}) \).

The next step is to consider the second derivatives \( \partial_i \partial_j \tau_a \). This yields
\[ \partial_i \partial_j \tau_a = (n+1) \text{Res}_{k=\infty} k^n((a-n-1)x^{a-n-2}(\partial_i x)(\partial_j x) + x^{a-n-1}\partial_i \partial_j x)dk, \] (5.7)
since we have expressed \( x \) as a function of \( k \) and the flat coordinates. Now consider the second summand in this residue. From the above expression of \( x \) we see that the highest power of \( k \) in \( x^{a-n-1} \) is \( a-n-1 \) and the highest possible power of \( k \) in \( \partial_i \partial_j x \) is \( -n-1 \) so the highest possible power of \( k \) in this second summand is \( n + a - n - 1 - n - 1 = a - n - 2 \leq -2 < -1 \) so it does not contribute to the residue and we have
\[ \partial_i \partial_j \tau_a = \text{Res}_{k=\infty} (a-n-1)x^{a-n-2}(\partial_i x)(\partial_j x)dk^{n+1} = \text{Res}_{x=\infty} (a-n-1)x^{a-n-2}(\partial_i x)(\partial_j x)dF. \] (5.8)
Now somehow we will need to relate this to $K_{ij}$ i.e. we need to relate it to derivatives of $F(x,t)$ with respect to the flat coordinates. Luckily we have $\partial_t x dF = \partial_t x \partial_x F dx = \partial_t F dx$. So

$$\partial_i \partial_j \tau_n = \text{Res}_{x=\infty} (a - n - 1) x^{a-n-2} \frac{\partial_t F}{\partial_x F} dx =$$

$$= \text{Res}_{x=\infty} (a - n - 1) x^{a-n-2} \left( K_{ij} + \sum_{p=1}^n c_{ij}^{(p)} \frac{\partial_p F}{\partial_x F} \right) dx. \quad (5.9)$$

Let us consider this second summand again, the highest degree of $x$ in $\partial_t F$ is $n$; in the sum the highest degree of $x$ is $n - 1$ (namely in $\partial_t F$). So the highest degree of $x$ in the second summand is $a - n - 2 + n - 1 - n = a - n - 3 \leq -3 < -1$ so again the second summand doesn’t contribute to the residue and we have

$$\partial_i \partial_j \tau_n = \text{Res}_{x=\infty} (a - n - 1) x^{a-n-2} \left( \sum_{p=1}^{n-1} K_{ij}^{(p)} x^p \right) dx =$$

$$= -(a-n-1) K_{ij}^{(n+1-a)} = (n+1-a) K_{ij}^{(n+1-a)}, \quad (5.10)$$

so the lemma is proved. $\square$

**Constants of Homogeneity**

To show that $\mathcal{T}$ is a Frobenius manifold we will have to fix a linear vector field $E$ on it, such that (1.11) and (1.12) define a grading on the Frobenius algebra $\mathcal{A}(\mathcal{T})$. Since the coordinates $(t_1, \ldots, t_n)$ are flat we see that for any choice $(d_1, \ldots, d_n) \in \mathbb{C}^n$

$$E = \sum_{i=1}^n d_i t_i \partial_i \quad (5.11)$$

has $\nabla \nabla E = 0$. Since for flat coordinates we have $\nabla \partial_k = \partial_k$ for $\nabla$ the Levi-Civita connection. By which is meant that if $v = \sum_{i=1}^n v_i \partial_i$ and $w = \sum_{i=1}^n w_i \partial_i$ then

$$\nabla_w v = \sum_{i,j=1}^n w_i (\partial_i v_j) \partial_j.$$

Moreover $\nabla \partial_k E = d_k \partial_k$ so in the basis given by the flat coordinates $\nabla E = \text{diag}(d_1, \ldots, d_n)$ thus $\nabla E$ is indeed diagonalized. Now let us imagine for a moment that $\mathcal{T}$ is a Frobenius manifold that is induced by a solution $\Phi$ to the WDVV equations. Then, since our function $F$ is related to $\Phi$ by equation (5.1), we would like to figure out consistent constants $d_i$ and $d_f$ of homogeneity. Such that equation (1.22) holds up to second order terms in the flat coordinates $t_i$. We might then just as well find the degrees of homogeneity for the third derivatives $c_{ijk} = \partial_i \partial_j \partial_k \Phi$. These offer us the relation to $F$ by the equation defining multiplication. So, if we assign the degrees $d_i$ to $t_i$ for all $i = 1, \ldots, n$ and $d_f$ to $\Phi$, then we should assign to $c_{ijk}$ the degree $d_f - d_i - d_j - d_k$. Actually in the definition of multiplication (5.1) we find $c_{ij}^k = \sum_{l=1}^n \eta^{kl} c_{ijl}$, where $\eta^{kl}$ is
inverse to $\eta_{kl} = \langle \partial_k, \partial_l \rangle = -(n+1)\delta_{k+l,n+1}$. Then we see that
\[ c^k_{ij} = -(n+1)\partial_i \partial_j \partial_{n+1-k} \Phi \]
and thus $c^k_{ij}$ is of degree $d_f - d_i - d_j - d_{n+1-k}$.

Then we can find the constants $d_1, \ldots, d_n, d_f$ by requiring
\[ F(c^x x, c^{d_1} t_1, \ldots, c^{d_n} t_n) = c F(x, t). \]
(5.13)
The expression (4.2) for $F$ shows that we need to assign to $x$ the degree $d_x = \frac{1}{n+1}$ and equation (4.3) shows we need to assign the same degree to $k$. This equation also shows that $d_i - id_x = d_x$ and thus that $d_i = (i+1)d_x = \frac{i+1}{n+1}$, for all $i = 1, \ldots, n$. Since the degree of the total $\frac{1}{x}$ should equal the degree of $x$ and we have already deduced that the degree of $k$ equals that of $x$.

Using equations (5.1) and (5.2) we can now do a check and find out the degree $d_f$. First of all since $F$ has degree 1 we know that $\partial_i F$ has degree $1 - d_i = \frac{n+1}{n+1}$ so the term on the left hand side has total degree $2n - i - j + 1 - d_k = \frac{2n - i - j}{n+1}$ for $d_f$. This yields $d_f = \frac{2n+4}{n+1}$. So
\[ d_i = \frac{i+1}{n+1} \text{ and } d_f = \frac{2n+4}{n+1} \quad \forall i = 1, \ldots, n. \]
(5.14)

**Remark 4** Note that we have also obtained the homogeneity condition
\[ \frac{x}{n+1} \partial_x F(x, t) + \sum_{i=1}^n i+1 n+1 t_i \partial_i F(x, t) = F(x, t) \]
(5.15)
and we have found that explicitly
\[ E = \sum_{i=1}^n i+1 n+1 t_i \partial_i \]
(5.16)
and
\[ \mathcal{L}_E \Phi = \frac{2n+4}{n+1} \Phi \]
(5.17)
up to second order terms. In terms of the alternative constants given in (1.23) we find $\beta_i = \frac{n+1}{n+1}$ and $d = \frac{n+1}{n+1}$ as in [1] (after renumbering the flat coordinates).

**Remark 5** In fact finding these constants results from assigning to $F$ the degree 1. We will see that this yields an $E$ that satisfies condition 4 in section 1.
6 Deformed Flat and Canonical Coordinates of $A_n$ Singularity

In this section we will show explicitly that the deformed connection defined in section 2 is flat for the Frobenius structure corresponding to the $A_n$ singularity. We will do this by finding a solution to the system of equations (2.9) and (2.10). This will subsequently allow us to prove that our structure does in fact satisfy the definition of a Frobenius structure, i.e. we will prove theorem 7. At this point we will have 3 natural coordinate systems: the natural coordinates defined in the deformation, the flat coordinates and the deformed flat coordinates. So in this section we will also finish this list by finding the canonical coordinates for the $A_n$ singularity.

Deformed Flat Coordinates

Consider the oscillatory integrals

\[ J_B = (-2\pi z)^{-\frac{1}{2}} \int_B e^{zF(x,t)} dx \] (6.1)

where $B$ is a cycle in some relative homology group, which ensures that the integrand is integrable over $B$. We will specify the relative homology later. For now, we just want to prove that the exterior derivatives of these functions are horizontal with respect to the deformed connection $\tilde{\nabla}$, whenever the cycles are such that the integrals make sense. To show this it is sufficient to show that the equations (2.5) and (2.6) hold with $\xi_j = \partial_j (-2\pi)^{\frac{1}{2}} J_B$.

We can use the polynomials $K_{ij}$ and Stokes’ theorem to show the first equality

\[
\partial_k \xi_j = \partial_k \left( z^{\frac{1}{2}} \int_B (\partial_j F) e^{zF} dx \right) =
\]

\[ = z^{\frac{1}{2}} \int_B (\partial_k \partial_j F) e^{zF} dx + z^{\frac{1}{2}} \int_B (\partial_k F) (\partial_j F) e^{zF} dx =
\]

\[ = z^{\frac{1}{2}} \int_B (\partial_k K_{ij}) e^{zF} dx + z^{\frac{1}{2}} \int_B (K_{ij} \partial_k F + \sum_{k=1}^n c_{ij}^k(t) \partial_k F) e^{zF} dx =
\]

\[ = z \sum_{k=1}^n c_{ij}^k(t) \partial_k(z^{\frac{1}{2}} \int_B e^{zF} dx) + z^{\frac{1}{2}} \int_B \partial_x (K_{ij} e^{zF}) dx = z \sum_{k=1}^n c_{ij}^k(t) \xi_k \] (6.2)

so indeed the oscillatory integrals satisfy the first equation. In the last equality we used the fact that the second integral vanishes by virtue of the behaviour of the exponent near the end points of the cycle $B$. In other words we used the properties of the relative homology which will be defined below.

To show the second equation we will first compute $z \partial_z J_B$ so using equation (5.15) we get

\[ z \partial_z J_B = -\frac{1}{2} J_B + (-2\pi)^{-\frac{1}{2}} z^{\frac{1}{2}} \int_B Fe^{zF} dx =
\]
\[
\int_B \left( \frac{x}{1 + n} \partial_x F + \sum_{i=1}^n d_i t_i \partial_i F \right) e^{zF} dx =
\]

\[
= -\frac{1}{2} J_B + \sum_{i=1}^n d_i t_i \partial_i J_B + (-2\pi z)^{-\frac{1}{2}} \int_B \partial_x \left( \frac{x}{1 + n} e^{zF} \right) dx - \frac{1}{n+1} J_B =
\]

\[
= \frac{d - \frac{2}{2}}{2} J_B + \sum_{i=1}^n d_i t_i \partial_i J_B \quad (6.3)
\]

since \(\frac{1}{2} + \frac{1}{1+n} = \frac{n+1}{2n+2} = \frac{2-d}{2}\). In the last equality we used again the properties of the relative homology defined below, to show that the third summand vanishes.

Then we see that indeed

\[
\partial_x \xi_j = \frac{1}{z} \left( \frac{d - \frac{2}{2}}{2} \xi_j + d_j \xi_j + \sum_{i=1}^n d_i t_i \partial_i \xi_j \right) =
\]

\[
= \sum_{i,k=1}^n d_i t_i c_{ij} \xi_k - \frac{\xi_j}{z} \left( q_j - \frac{d}{2} \right), \quad (6.4)
\]

which is the second equation.

Now let us turn to the cycle \(B\). We need to choose it such that the exponent in the definition (6.1) oscillates at the end points. So let

\[
\mathcal{Y} := \lim_{M \to \infty} H_1(\mathbb{C}, \{x | \text{Re } zF(x,t) < -M\}). \quad (6.5)
\]

Then for any cycle \(B \in \mathcal{Y} \simeq \mathbb{Z}^n\) we can assure that indeed the integrals with derivatives of \(x\) as integrands vanish and thus that \(\partial_j J_B\) satisfies the equations (2.5) and (2.6).

So let us construct once and for all a basis \(\{B_i\}_{i=1}^n\) of \(\mathcal{Y}\). This then yields a basis \(\{dJ_B\}_{i=1}^n \cup \{dz\}\) of horizontal one-forms which shows that \(\nabla\) is flat. With the deformed flat coordinates \(J_B\).

Denote by \(V_{\lambda,t}\) the Milnor fiber

\[
V_{\lambda,t} := \{x \in \mathbb{C} | F(x,t) = \lambda\}, \quad (6.6)
\]

for \(\lambda \in \mathbb{C}\) and \(t \in \mathcal{T}\). Then, for \(i = 1, \ldots, n\), we define the cycle \(B_i\) as follows. For any semisimple \(t \in \mathcal{T}\) we have, as will be shown below, \(n\) critical values \((u_1, \ldots, u_n)\) of the function \(F(x,t)\), with some ordering. First we pick a path from \(\lambda = u_i\) to \(z\lambda \to -\infty\), where all the other (isolated) critical values are avoided. In every Milnor fiber along this path we pick a cycle that vanishes as \(\lambda\) approaches \(u_i\). We call their union \(B_i\). For an in-depth discussion of these constructions we refer to [14].
Proof of Theorem 7

In this section we will complete the proof of theorem seven. We have now constructed explicitly the flat coordinates and thus already shown that the metric is flat. Also non-degeneracy of the metric shows from the matrix
\[ \langle \partial_i, \partial_j \rangle = -\delta_{i+n+1} \] which is clearly not singular (and anti-diagonal). We had already shown invariance by equation (3.5) and symmetry is clear. Thus the first condition of section 1 is satisfied.

Now
\[ \partial_n F = \partial_n (x^{n+1} + \tau_1 x^{n-1} + \ldots + \tau_n) = \partial_n \tau_n = -1, \] since \( \tau_i \) does not depend on \( t_j \) with \( j > i \) and we had already deduced that \( \partial_i \tau_i = -1 \) for all \( i = 1, \ldots, n \).

So we find that for all \( i = 1, \ldots, n \) we have
\[ (\partial_n F)(\partial_i F) = -\partial_i F \] so we see from (5.1) that
\[ c^k_{ni} = -\delta_{ik} = c^k_{in}, \] for all \( i, k = 1, \ldots, n \). So
\[ (-\partial_n) \cdot v = v = v \cdot (-\partial_n) \quad \forall v \in T_t \mathcal{T} \quad \text{for all} \quad t \in \mathcal{T}, \quad (6.7) \]
i.e. \((-\partial_n)\) is the unit vector field. Now it is clear that indeed \( \nabla (-\partial_n) = 0 \), since \( t_n \) is a flat coordinate, and thus the second condition of section 1 is satisfied.

Next we need to prove symmetry of the four-linear form \( C \) defined in equation (1.10). By linearity it is sufficient to show symmetry on a basis and since we already have symmetry of the three-linear form \( c \) defined in equation (1.9) we only need to show that
\[ \partial_m c^k_{ij} = \partial_i c^k_{mj}. \] (6.8)
This makes sense since if \( \mathcal{T} \) is a Frobenius manifold it is one that has an underlying solution to the WDVV equations and in that case the structure constants are just third derivatives and should definitely satisfy this equation. We have not yet proved that \( \mathcal{T} \) is a Frobenius manifold however, but this is where the deformed connection comes in. Since we have shown that the deformed connection is flat, we have in general that
\[ \tilde{\nabla}_v \tilde{\nabla}_w u - \tilde{\nabla}_w \tilde{\nabla}_v u - \tilde{\nabla}_{[v,w]} u = 0 \quad \forall v, w, u \in \mathcal{X}(\mathcal{T} \times \mathbb{C}). \] (6.9)
Substituting \( v = \partial_i, w = \partial_j \) and \( u = \partial_k \) then yields
\[ \tilde{\nabla}_{\partial_i} (z \partial_j \cdot \partial_k) - \tilde{\nabla}_{\partial_j} (z \partial_i \cdot \partial_k) = 0, \]
since \([\partial_i, \partial_j] = 0\). This then yields
\[ z \sum_{l=1}^{n} (\partial_l c^l_{jk} - \partial_j c^l_{ik}) \partial_l + z^2 \sum_{l,m=1}^{n} (c^l_{jk} c^m_{il} - c^l_{ik} c^m_{jl}) \partial_m = 0. \] (6.10)
So equating the coefficients of every basis vector and like powers of \( z \) to 0 we find exactly the equations (6.8) and the equations of associativity of the algebras.
(1.16), which we already knew. In any case this proves that the four-linear term C is indeed symmetric for the parameter space of the $A_n$ singularity. Thus the third condition of section 1 is satisfied.

The last axiom asks for a linear vector field and, since we want to show also that there is an underlying solution of the WDVV equations, we also want it to be diagonalizable. Of course we have already shown that $E$ in equation (5.11) has both these features. All that is left to be shown is that the equations (1.4)-(1.7) hold for the grading operators defined in (1.11) and (1.12). The first equality (1.4) holds. It is simply equivalent to the Leibniz rule for the vector field $E$. Similarly we have

$$Q_A(fv) = fv + [E, fv] = fv + E(fv) + f[E, v] = Q_R(fv) + fQ_A(v),$$

immediately for all $f \in \mathcal{F}(\mathcal{T})$ and $v \in \mathcal{X}(\mathcal{T})$. So (1.5) holds as well.

Now note that, in the setting of lemma 2, we should have $\partial_i \partial_z \xi_j = \partial_z \partial_i \xi_j$, for all $i, j = 1, \ldots, n$. Equating the different powers in $z$ we find again the condition 3 and we see that $\mu$ should not depend on the point $t \in \mathcal{T}$. The relevant information concerning the grading operators comes for the terms independent of $z$ however. This yields

$$\sum_{k=1}^{n} c_{ij}^k \xi_k - \sum_{k,l=1}^{n} c_{ij}^k \delta_{kl} \left( \frac{2 - d_i}{2} - d_k \right) \xi_l =$$

$$\sum_{k,l=1}^{n} (\partial_i d_k t_k) c_{ij}^l \xi_l + \sum_{k,l=1}^{n} d_k t_k (\partial_i c_{ij}^l) \xi_l - \sum_{k,l=1}^{n} \delta_{kl} c_{ij}^l \xi_l \left( \frac{2 - d_i}{2} - d_j \right), \quad (6.11)$$

for any $i, j = 1, \ldots, n$. Then using the symmetry of the four-linear form $C$, i.e. condition 3, we find

$$\sum_{k,l=1}^{n} d_k t_k (\partial_k c_{ij}^l) = \sum_{k=1}^{n} (1 - d_i - d_j + d_k) c_{ij}^k, \quad (6.12)$$

So we find that indeed

$$Q_A(\partial_i \cdot \partial_j) = \sum_{k=1}^{n} Q_R(c_{ij}^k) \partial_k + c_{ij}^k Q_A(\partial_k) =$$

$$= \sum_{l,k=1}^{n} d_l t_l \partial_i c_{ij}^l \partial_k + \sum_{k=1}^{n} c_{ij}^k \partial_k + c_{ij}^k [E, \partial_k] =$$

$$= (2 - d_i - d_j) \partial_i \cdot \partial_j = 2\partial_i \cdot \partial_j + \partial_i \cdot [E, \partial_j] + [E, \partial_i] \cdot \partial_j =$$

$$= \partial_i \cdot Q_A(\partial_j) + Q_A(\partial_i) \cdot \partial_j,$$

so (1.6) holds. Finally we have

$$< Q_A(\partial_i), \partial_j > + < \partial_i, Q_A(\partial_j) > = (2 - d_i - d_j) \frac{-\delta_{i+j,n+1}}{n+1} =$$
\[ \left( 2 - \frac{i + 1}{n + 1} - \frac{n + 1 - i + 1}{n + 1} \right) \frac{-\delta_{i+j,n+1}}{n + 1} = d < \partial_i, \partial_j > = Q_R(\partial_i, \partial_j) + d < \partial_i, \partial_j > \]

So we see that all 4 conditions are met and thus \( T \) is indeed a Frobenius manifold with an underlying solution to the WDVV equations. Theorem 7 is proved. \( \square \)

**Remark 6** To prove theorem 7 we actually used only the metric and the flat deformed connection. Thus it is possible to give an equivalent definition of Frobenius manifold. In this sense a Frobenius manifold is a manifold with a flat metric and a pencil of connections that extends in the sense of equations (2.1)-(2.3) to a flat connection on the direct product \( M \times \mathbb{C} \). This definition is used in [17].

**Canonical Coordinates of \( A_n \) Singularity**

As was noted in section 2 the eigenvalues of the operation of multiplication by the Euler vector field serve as canonical coordinates. Now we claim that the semisimple points of \( T \) are the points in which \( F \) has no degenerate critical values. So the points in which \( F \) has \( n \) distinct non-degenerate critical values. This fact is expressed in the following lemma.

**Lemma 11** The eigenvalues of \( E \) in a semisimple point \( t \in T \) are the critical values of \( F(x,t) \).

**Proof:**

Suppose \( v \) is an eigenvector at the point \( t \in T \) then

\[ \mathcal{E}v = \lambda v \quad \text{for some} \quad \lambda \in \mathbb{C}. \]

In other words we have \( E \cdot v = \lambda v \). We can translate this to the condition

\[ [\mathcal{L}_E F|_t] \cdot [\mathcal{L}_x F|_t] = \lambda [\mathcal{L}_x F|_t] \]

in the space \( \mathbb{C}[x]/(\partial_x F|_t) \). Now in general we have the homogeneity condition (5.15) which yields

\[ [F|_t - \frac{x}{n + 1} \partial_x F|_t] \cdot [\mathcal{L}_x F|_t] = \lambda [\mathcal{L}_x F|_t], \]

but \([\partial_x F|_t] = 0\). So we arrive at the equation

\[ [F|_t - \lambda] \cdot [\mathcal{L}_x F|_t] = [0]. \quad (6.13) \]

So in fact

\[ (F|_t - \lambda) \mathcal{L}_x F|_t = K(x) \partial_x F|_t, \]

for some \( K(x) \in \mathbb{C}[x] \). At a critical point \( x_i \) with \( i = 1, \ldots, n \) the right hand side of this equation vanishes. Thus we have either \( \lambda = F(x_i, t) \) or \( \mathcal{L}_x F|_t = 0 \). However the degree of the right hand side in \( x \) is \( n \) while the degree of \( \mathcal{L}_x F|_t \) in \( x \) is not higher than \( n - 1 \). Thus there is at least 1 critical point such that \( \mathcal{L}_x F|_t \neq 0 \).
So we see that every eigenvector of $\mathcal{E}$ has a critical value as eigenvalue. Now suppose $v \neq w$ are both eigenvectors of $\mathcal{E}$. Suppose the critical points of $F|_t$ are $x_1, \ldots, x_n$ and the critical value corresponding to $v$ is $F(x_i, t)$. Then we need to show that the above argument does not fail for $w$. So we need to show that $\mathcal{L}_v F|_t$ does not vanish for a different critical value from $x_i$. But if $(\mathcal{L}_w F|_t)|_{x_j} = 0$ for all $x_j$ but $x_i$ and the same goes for $\mathcal{L}_v F|_t$ then we find $v = w$. This contradiction shows that for every one of the distinct eigenvectors $v_i$ of $\mathcal{E}$ there is also a distinct critical point that is not a root of $\mathcal{L}_v F|_t$. □

This shows that the coordinate system $(u_1, \ldots, u_n)$ of critical values is the canonical coordinate system. So let us go on to calculate the metric for the frame $(\partial u_1, \ldots, \partial u_n)$ generated by the canonical coordinates. Note that we can expand $\partial x F|_t$ for any semisimple $t \in \mathcal{T}$ around the critical point $x_i$ corresponding to the critical value $u_i$. In fact since $F|_t$ is a polynomial we will again find a polynomial. We have (dropping for now the $|_t$ since it is immaterial)

$$\partial_x F = \partial_x F|_{x_i} + \Delta_i(x - x_i) + O((x - x_i)^2), \quad (6.14)$$

where $\Delta_i$ denotes the Hessian of $F$ in the point $x_i$. However we can drop the first term since $x_i$ is a critical point. On the other hand we can expand $F$ itself in the same way, which yields

$$F = F|_{x_i} + \partial_x F|_{x_i}(x - x_i) + \frac{1}{2}\Delta_i(x - x_i)^2 + O((x - x_i)^3). \quad (6.15)$$

Note that this time the second summand vanishes while we have $F|_{x_i} = u_i$. Now since the $u_i$ are the canonical coordinate system we have the multiplication

$$\partial u_i : \partial u_j = \delta_{ij}\partial u_i.$$

So we have by the definition of the multiplication that

$$(\partial u_i F)(\partial u_j F) = \delta_{ij}\partial u_i F + N_{ij}\partial_x F, \quad (6.16)$$

for some polynomial $N_{ij} \in \mathbb{C}[x]$. Finally we have

$$< \partial u_i, \partial u_j > = \text{Res}_{x=\infty} \frac{\delta_{ij}\partial u_i F + N_{ij}\partial_x F}{\partial_x F} \, dx = \delta_{ij} \text{Res}_{x=\infty} \frac{1 + \partial u_i \left(\frac{1}{2}\Delta_i(x - x_i)^2 + O((x - x_i)^3)\right)}{\Delta_i(x - x_i) + O((x - x_i)^2)} \, dx = -\delta_{ij} \Delta_i^{-1} \quad (6.17)$$

Thus we can define the normalized frame $\{\sqrt{-\Delta_i}\partial u_i|_{i=1}\}^n$. Then let $\Psi(\tau)$ denote the transition matrix from the frame of flat coordinates to the orthonormal frame, i.e.

$$\sum_{j=1}^n |\Psi|_{ij} \partial_j = \sqrt{-\Delta_i}\partial u_i. \quad (6.18)$$

Obviously this transformation matrix could be defined for any (semisimple) Frobenius structure. However we will not need many properties for the rest. For a complete exposition of $\Psi$ in the general case see [1].
Part II
KdV hierarchies

7 Quantization Formalism and Operators on the Fock Space

We will need the quantization formalism, in order to give a formulation of the \((n+1)\)KdV hierarchies and Givental’s formula for the total descendant potential. This will allow us to define the Fock space and quantized operators on it. The descendant potential will be found as an element in the Fock space, which is obtained from another element by action of certain operators. In this section we will first give a definition of the Fock space and subsequently we will show how to quantize linear and quadratic Hamiltonians and certain operators to act on this Fock space. For this section we will return to the situation in the beginning of section 3, i.e. we restrict ourselves to the point \(0 \in \mathcal{T}\).

Quantization Formalism

As before let \(H = \mathbb{C}[x]/(f_x)\) denote the local algebra of the singularity. We know that the residue pairing \(\langle \cdot, \cdot \rangle_0\) is a non-degenerate symmetric bilinear form on \(H\). Now let \(\mathcal{H} := H((z^{-1}))\) denote the space of Laurent series in one indeterminate \(z^{-1}\) with coefficients in \(H\). Let \(\Omega\) denote the bilinear form on \(H\) defined by

\[
\Omega(f, g) = \frac{1}{2\pi i} \oint f(-z), g(z) >_0 dz \quad \forall f, g \in \mathcal{H}.
\] (7.1)

Note that, since \(\langle \cdot, \cdot \rangle_0\) is symmetric, the change of variable \(z \mapsto -z\) in the integral shows that for all \(f, g \in \mathcal{H}\) we have

\[
\Omega(f, g) = -\Omega(g, f).
\]

Moreover if \(\Omega(h, \cdot) = 0\) for some \(h \in \mathcal{H}\) then we would need \(\langle h(-z), \cdot \rangle_0 = 0\), thus by non-degeneracy of this last form we see that \(\Omega\) is non-degenerate and thus that \(\Omega\) is a symplectic form on \(H\).

Consider the polarization \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\) of \(\mathcal{H}\) into the Lagrangian subspaces \(\mathcal{H}_+ = H[z]\) and \(\mathcal{H}_- = z^{-1}H[[z^{-1}]]\). This identifies \((\mathcal{H}, \Omega)\) with the cotangent bundle \(T^*\mathcal{H}_+\). Then let \(\{q_i\}\) be a coordinate system on \(\mathcal{H}_+\) and \(\{p_i\}\) the dual coordinate system on \(\mathcal{H}_-\) such that \(\Omega = \sum_i p_i \wedge q_i\), i.e. they form a Darboux coordinate system on \(T^*\mathcal{H}_+\). Suppose we have the orthonormal basis \(e_1, \ldots, e_n\) for \(H\), then we have for a general element \(f \in \mathcal{H}\) that

\[
f = \sum_{k \geq 0} \sum_{a=1}^n e_a q_{a,k} z^k + \sum_{k \geq 0} \sum_{a=1}^n e_a p_{a,k} (z)^{k-1}.
\] (7.2)

The quantization formalism will associate to any constant, linear and quadratic function \(G\) on \((\mathcal{H}, \Omega)\) a differential operator \(\hat{G}\) of order smaller than 3. This operator will act on functions on \(\mathcal{H}_+\). We will call the space of such functions the Fock space.
The quantization of a constant, linear or quadratic function on \((\mathcal{H}, \Omega)\) is carried out by the recipe
\[
q_i := \frac{q_i}{\sqrt{\hbar}} \quad p_i := \sqrt{\hbar} \partial q_i,
\]
and
\[
\{q_i q_j\} := \frac{q_i q_j}{\hbar} \quad \{q_i p_j\} := q_i \partial q_j, \quad \{p_i p_j\} := \hbar \partial q_i \partial q_j,
\]
where \(\sqrt{\hbar}\) is just some formal variable. The quantization is thus a representation of the Heisenberg algebra of constant and linear Hamiltonians. However it is only a projective representation of the Lie algebra of quadratic Hamiltonians, i.e. we have
\[
\{F, G\} = [\hat{F}, \hat{G}] \quad \forall \text{ constant and linear Hamiltonians } F, G
\]
\[
\{F, G\} = [\hat{F}, \hat{G}] + \mathcal{C}(F, G) \quad \forall \text{ quadratic Hamiltonians } F, G,
\]
where \(\{\cdot, \cdot\}\) denotes the Poisson bracket corresponding to \(\Omega\), \([\cdot, \cdot]\) denotes the commutator bracket and \(\mathcal{C}\) is a cocycle characterized by the properties
\[
\mathcal{C}(p_i p_j, q_i q_j) = 1 \text{ if } i \neq j, \quad \mathcal{C}(p_i^2, q_i^2) = 2
\]
and \(\mathcal{C} = 0\) on all other pairs of quadratic Darboux monomials [2]. To reiterate, these differential operators act on formal functions (with coefficients depending on \((\sqrt{\hbar})^{\pm 1}\)) on \(\mathcal{H}_+\). The space of such functions is called the Fock space. Most notably the total descendant potential will be viewed as an element of the Fock space.

**Twisted Loop Group**

Let us now turn to quantizations of different (rather specific) operators. Let us define the loop group \(L\text{GL}(\mathcal{H})\) as the space of \(\text{GL}(\mathcal{H})\) valued formal functions in \(z\). These then constitute transformations on the space \(\mathcal{H}\) and it is natural to restrict our attention to those elements of the loop group that preserve \(\Omega\). This yields the following definition of the twisted loop group
\[
\mathcal{L}^{(2)}\text{GL}(\mathcal{H}) := \{U \in \mathcal{L}\text{GL}(\mathcal{H})\mid U^T(-z) U(z) = 1\},
\]
where the transposition is with respect to \(<\cdot, \cdot>_0\) and the 1 represents the identity matrix in \(\text{GL}(\mathcal{H})\). This space has corresponding Lie algebra \(\mathcal{L}^{(2)}\text{End}(\mathcal{H})\).

We can make a split of this space into the subalgebras of upper and lower triangular trannformations, \(g_+\) respectively \(g_-\), given by
\[
g_{\pm} := \{u(z) = \sum_{k>0} u_k z^{\pm k} \mid u_k \in \text{End}(\mathcal{H}), u^T(-z) + u(z) = 0\}
\]

We can quantize the elements of the twisted loop group as follows. For \(U \in \mathcal{L}^{(2)}\text{GL}(\mathcal{H})\) we define \(\hat{U} = \exp(\ln \hat{M})\). So we express first the operator \(U\) as \(e^u\) for some \(u \in \mathcal{L}^{(2)}\text{End}(\mathcal{H})\). Subsequently we express this as a linear or quadratic hamiltonian \(H_u\) given by
\[
H_u(f) = \frac{1}{2} \Omega(f, uf),
\]
for $f \in \mathcal{H}$. Then we exponentiate the result of the procedure set out in (7.3) and (7.4). For future reference let us set out the general quantization of elements of the lower, respectively upper, triangular matrices. We do this by giving the quantization of the corresponding elements of the Lie subalgebras $\mathfrak{g}_{\pm}$.

Explicit calculation yields for $s = \sum_{k>0} s_k z^{-k}$ that

$$
\hat{s} = \sum_{p>0} \sum_{\alpha,\beta=1}^{n} (s_p)_{\alpha\beta} \left( (-1)^{p-1} \sum_{i \geq 0} q_{\alpha,i+p} \partial_{\beta,i} + \frac{1}{2\hbar} \sum_{i+j=p-1} (-1)^i q_{\alpha,i} q_{\beta,j} \right),
$$

where $(s_p)_{\alpha\beta} = \langle e_\alpha, s_p e_\beta \rangle_0$ for the orthonormal basis and $\partial_{\beta,i} = \partial \partial q_{\beta,i}$. Similarly we have for $r = \sum_{k>0} r_k z^k$ that

$$
\hat{r} = \sum_{p>0} \sum_{\alpha,\beta=1}^{n} (r_p)_{\alpha\beta} \left( (-1)^{p-1} \sum_{i \geq 0} q_{\alpha,i+p} \partial_{\beta,i} + \frac{\hbar}{2} \sum_{i+j=p-1} (-1)^{i+1} q_{\alpha,i} q_{\beta,j} \right),
$$

where again $(r_p)_{\alpha\beta} = \langle e_\alpha, r_p e_\beta \rangle_0$.

**Action on the Fock Space**

We will need to know how the quantized symplectic operators given by such upper and lower triangular elements act on the Fock space. Thus we need to find a way to split the operators into a multiplication by elements in $\mathcal{H}_+$ and derivations. For the following discussion see also [13, 19]. Let us start by describing this procedure for the lower triangular operators. So suppose $S \in \mathcal{L}^{(2)} GL(H)$ such that

$$S(z) = e^{s(z)} \quad \text{where} \quad s \in \mathfrak{g}_-,$$

the above discussion shows that

$$
\hat{\ln S} = \sum_{p>0} \sum_{\alpha,\beta=1}^{n} (s_p)_{\alpha\beta} \left( (-1)^{p-1} \sum_{i \geq 0} q_{\alpha,i+p} \partial_{\beta,i} + \frac{1}{2\hbar} \sum_{i+j=p-1} (-1)^i q_{\alpha,i} q_{\beta,j} \right),
$$

and

$$
\hat{\ln S^{-1}} = \sum_{p>0} \sum_{\alpha,\beta=1}^{n} (s_p)_{\alpha\beta} \left( (-1)^{p} \sum_{i \geq 0} q_{\alpha,i+p} \partial_{\beta,i} + \frac{1}{2\hbar} \sum_{i+j=p-1} (-1)^{i+1} q_{\alpha,i} q_{\beta,j} \right),
$$

where $s = \sum_{k>0} s_k z^{-k}$ as before. First we will write this as the product of two exponents using the Campbell-Baker-Hausdorff formula. So denoting

$$Y = \frac{1}{2\hbar} \sum_{i,j \geq 0} \sum_{\alpha,\beta=1}^{n} (s_{i+j+1})_{\alpha\beta} (-1)^{i+1} q_{\alpha,i} q_{\beta,j}$$

(7.15)
and

$$X = \sum_{p>0} \sum_{i>0} \sum_{\alpha,\beta=1}^n (-1)^{s_p}(s_p)_{\alpha,\beta}q_{\alpha,\beta}i + p\partial_{\alpha,i}, \quad (7.16)$$

were we used the fact that $s^T_p = (-1)^{p-1}s_p$ since $s \in g_-$, we require $Z$ such that

$$e^{X+Y} = e^X e^Z.$$ 

Now note that since

$$[q_{\alpha,i} + p\partial_{\beta,i}, q_{\gamma,s}q_{\epsilon,r}] = q_{\alpha,i} + p\partial_{\beta,i} \delta_{\gamma,\epsilon} \delta_{s,r} q_{\gamma,s}, \quad (7.17)$$

for all $\alpha,\beta,\gamma,\epsilon = 1, \ldots, n$ and $i, r, s, p \geq 0$, we see that $[X,Y] \in \text{Ker } Y$. Where we view $Y$ as a linear operator on the Heisenberg Lie algebra given by the quantization formalism. In this case the CBH-formula yields

$$Z = -e^{-\text{ad}_X} + 1 + Y = \sum_{p \geq 0} (-1)^{p\text{ad}_X}Y = \frac{1}{2\hbar} \sum_{p \geq 0} \sum_{(\alpha,\beta,i,j)\in\mathbb{Z}^n} \sum_{\gamma,s} \sum_{\epsilon,r} \sum_{(f_1,\ldots,f_n)\in\mathbb{Z}^n} \sum_{(g_1,\ldots,g_n)\in\mathbb{Z}^n} \sum_{i,j} \sum_{\alpha,\beta=1}^n \frac{p}{(p+1)!} (-1)^{i+1+f_1+\ldots+f_n+b} \cdot (s_{f_1} \ldots s_{f_n} s_{i+1} q_{\alpha,i} + f_1 + \ldots + f_n q_{\beta,j} + g_1 + \ldots + g_n), \quad (7.18)$$

where we set $s_0 = 0$. In most respects a horrible formula to work with of course. However note that we could give $Z$ in the form

$$2\hbar Z = W(q,q) = \sum_{k,l \geq 0} <W_{kl}q_k,q_l>.$$ 

Now the quadratic form $W$ gives us a description for $Z$ and we see that $W$ is defined by

$$\sum_{k,l} W_{kl} = \frac{S'(w)S(z) - 1}{w^{-1} + z^{-1}}, \quad (7.20)$$

by expanding the right hand side and comparing to (7.18). Here 1 signifies the identity matrix.

Suppose that $G(q)$ is an element of the Fock space. Then we see now that

$$\hat{S}^{-1}G(q) = e^X e^Z G(q) = e^{\frac{w(q,q)}{2\hbar}} e^X G(q) = e^{\frac{w(q,q)}{2\hbar}} G([Sq]_+), \quad (7.21)$$

where we denote by $[Sq]_+$ the truncation of negative powers in $z$ of $S(z)q(z)$.

We can describe the upper triangular operators in a similar fashion. So suppose $R \in L^{(2)}GL(H)$ such that

$$R(z) = e^{\tau(z)} \quad \text{where } \tau \in g_+.$$ 

Denoting this time
\[ Y = \frac{\hbar}{2} \sum_{i,j \geq 0} \sum_{\alpha, \beta = 1}^n (r_{i+j+1})_{\alpha\beta} (-1)^j \partial_{\alpha,i} \partial_{\beta,j} \]  

(7.22)

and

\[ X = \sum_{p > 0} \sum_{i \geq 0} \sum_{\alpha, \beta = 1}^n (-1)(r_p)_{\alpha\beta} q_{\beta,i} \partial_{\alpha,i+p}, \]  

(7.23)

where we used again that \( r^T_p = (-1)^{p-1}r_p \) since \( r \in g_+ \). As before we have

\[ [X,Y] \in \text{Ker}[Y, \cdot], \]  

since

\[ [q_{\alpha,i} \partial_{\beta,i+p}, \partial_{\gamma,s} \partial_{\epsilon,r}] = (-\delta_{\alpha,\gamma}\delta_{i,s} \partial_{\epsilon,r} + \delta_{\beta,\epsilon} \delta_{i,r} \partial_{\gamma,s}) \partial_{\beta,i+p}, \]  

(7.24)

for all \( \alpha, \beta, \gamma, \epsilon = 1, \ldots, n \) and \( i, r, s, p \geq 0 \). Thus we find this time that \( Z \) equals

\[ \frac{\hbar}{2} \sum_{(p \geq 0)} \sum_{(a+b=p)} \sum_{(f_1, \ldots, f_a \geq 0)} \sum_{(g_1, \ldots, g_b \geq 0)} \sum_{(r_{f_1} \cdot \cdot \cdot r_{f_a} \cdot \cdot \cdot r_{g_b} \cdot \cdot \cdot r_{g_b})} \frac{p}{(p+1)!} (-1)^{j+f_1+\cdot\cdot\cdot+f_a+a} \]  

(7.25)

where we set \( r_0 = 0 \). Again we express \( Z \) using a quadratic form such that

\[ \frac{2}{\hbar} Z = V(\partial, \partial) = \sum_{k,l \geq 0} <q_k, V_{kl} q_l>. \]  

(7.26)

This time \( V \) is defined by

\[ \sum_{k,l} V_{kl} w^k z^l = \frac{I - R(w)R^T(z)}{w + z}. \]  

(7.27)

So for \( G \) as before we have this time

\[ \hat{R}^{-1} G(q) = e^{\lambda V(0,0)} G(R q). \]  

(7.28)

From the action of these inverse operators we can infer the action of \( \hat{S} \) and \( \hat{R} \).

8 \( (n+1) \text{KdV Hierarchy and Vertex Operators} \)

To show how the Frobenius structure related to the \( A_n \) singularity allows one to turn a solution to the KdV hierarchy into a solution of the \((n+1)\text{KdV hierarchy} \) we will need to put these hierarchies in a specific form. In fact there are many ways to view integrable hierarchies in general [10, 4]. We will start from the ansatz of the Kadomtsev-Petviashvili (or simply KP) hierarchy given by Hirota quadratic equations in vertex operator form [20]. We will subsequently show how to translate these equations to the frame of the quantization formalism and the Fock space in the case of the KdV and \((n+1)\text{KdV hierarchies} \). To do this we will also use certain elements of the Frobenius structure in the point \( 0 \in \mathcal{T} \), as we did in the last section.
From KP to \((n+1)\)KdV Hierarchy

We start from the KP hierarchy in the form given in [2]. Thus a function \(\Phi(x)\), which is assumed to be of the form

\[\Phi(x) = \exp\left\{\sum_{g=0}^{\infty} \hbar^{g-1} \phi^{(g)}(x)\right\}, \quad (8.1)\]

is a solution to the KP hierarchy if

\[\text{Res}_{\xi = \infty} \frac{d}{d\xi} \exp\left\{\sum_{j>0}^{\infty} \xi_j \sqrt{\hbar} (x_j - x_j') \exp\left\{-\sum_{j>0}^{\infty} \frac{\xi^{-j}}{j} \sqrt{\hbar} (\partial x_j' - \partial x_j'')\right\} \Phi(x')\Phi(x'') = 0 \quad (8.2)\]

where \(x = (x_1, x_2, \ldots)\) is a vector variable. By definition, solutions of the \((n+1)\)KdV hierarchy are those solutions of the KP hierarchy that do not depend on the variables \(x_j\) with \(j \equiv 0 \mod (n+1)\). A more convenient interpretation is available however. First of all the change of variables

\[x_j = \frac{x_j' + x_j''}{2}, \quad y_j = \frac{x_j' - x_j''}{2}, \quad \partial_{x_j} = \partial_{x_j'} + \partial_{x_j''} \quad \text{and} \quad \partial_{y_j} = \partial_{x_j'} - \partial_{x_j''} \quad (8.3)\]

yields the alternative version of the KP hierarchy

\[\text{Res}_{\xi = \infty} \frac{d}{d\xi} \exp\left\{2 \sum_{j>0}^{\infty} \frac{\xi_j}{\sqrt{\hbar}} y_j \right\} \exp\left\{-\sum_{j>0}^{\infty} \frac{\xi^{-j}}{j} \sqrt{\hbar} \partial_{y_j}\right\} \Phi(x+y)\Phi(x-y) = 0. \quad (8.4)\]

Expanding this equation in \(y\) then yields an infinite system of partial differential equations which is another form of the KP hierarchy. This would yield the residue of some infinite series in \(y\) where the coefficients of monomials \(y^m\) are Laurent series in \(\xi^{-1}\). So for each of these Laurent series there is a highest power of \(\xi\) depending on \(m\).

2KdV

Let us look specifically at the 2KdV hierarchy (also just called KdV hierarchy). So we are looking at solutions of KP that do not depend on \(x_j\) for \(j \equiv 0 \mod 2\). Thus we can omit the derivatives \(\partial_{x_{2k}}\), but note that we cannot omit the multiplications by \(y_{2k}\). This yields the following form of the KdV hierarchy

\[\text{Res}_{\xi = \infty} \frac{d}{d\xi} S(\xi^2) \exp\left\{2 \sum_{j>0}^{\infty} \frac{\xi_j}{\sqrt{\hbar}} y_j \right\} \exp\left\{-\sum_{j>0}^{\infty} \frac{\xi^{-j}}{j} \sqrt{\hbar} \partial_{y_j}\right\} \Phi(x+y)\Phi(x-y) = 0, \quad (8.2)\]

where the sums run only over positive odd values of \(j\) and \(S(\xi^2) = e^{2 \sum_{k>0}^{\infty} \frac{\xi^{2k}}{\sqrt{k}} y_{2k}}\).

Now since \(\Phi\) does not depend on \(x_j\) for \(j\) odd we can treat \(S\) as an arbitrary function in \(\xi^2\) (and \(\sqrt{\hbar}\)). This shows that vanishing of the residue above is equivalent to the coefficients of odd (positive) powers of \(\xi^{-1}\) in the coefficients of \(y^m\) in the expansion of the one-form

\[d\xi \exp\left\{2 \sum_{j>0}^{\infty} \frac{\xi_j}{\sqrt{\hbar}} y_j \right\} \exp\left\{-\sum_{j>0}^{\infty} \frac{\xi^{-j}}{j} \sqrt{\hbar} \partial_{y_j}\right\} \Phi(x+y)\Phi(x-y)\]
vanishing. Since every one of these coefficients can be made to equal the coefficient of $y^n$ in the residue by picking a suitable $S$. We can then state that $\Phi(x_{rat})$ is a solution to the KdV hierarchy if and only if the one-form above has no odd non-positive powers of $\xi$. By substracting the one-form with the opposite sign in the exponents we see that the resulting form cannot have any negative even powers of $\xi$. This gives us the final form of the KdV hierarchy. Namely, a function $\Phi(x_{rat})$ solves the KdV hierarchy if and only if the form

$$
\sum_{\pm} \pm d\xi \exp\left\{ \pm \sum_j \frac{\xi^j}{\sqrt{h}} (x'_j - x''_j) \right\} \exp\left\{ \pm \sum_j \frac{\xi^{-j}}{\sqrt{h}} (\partial x'_j - \partial x''_j) \right\} \Phi(x')\Phi(x'')
$$

is regular in $\xi^2$. By regular in $\xi^2$ we mean exactly that in the expansion of the form in $y$ the Laurent series that form coefficients to the different powers of $y$ have no negative powers in $\xi^2$. So they should be polynomials in $\xi^2$.

Now let $T(t)$ denote the Witten-Kontsevich tau-function, where $t = t(z) = t_0 + t_1 z + t_2 z^2 + \ldots$. Then we define

$$
D_{A_1}(q) := T(t), \quad \text{where } q(z) := t(z) - z.
$$

Witten’s conjecture which was proved by Kontsevich [11, 12] now says that $D_{A_1}$ satisfies the condition (8.5) after the substitution

$$
q_k = \frac{2k + 1}{2^k \sqrt{2}^{2k+1}}
$$

for $k \in \mathbb{N} \cup \{0\}$, which yields also $\partial_{x_{2k+1}} = \frac{(2k+1)!}{2^k \sqrt{2}^{2k+1}} \partial_{q_k}$. Substituting this in (8.5) we see that the exponents are quantized linear Hamiltonians in the Heisenberg algebra as given in section 7. Clearly we can seperate the parts depending on $q'$ and $q''$. Moreover we can use the notation where $\lambda = \xi^2$ such that we can sum over the roots of $\lambda$ instead of over $\pm$, this yields

$$
\sum_{\sqrt{\lambda}=\pm \xi} \frac{d\lambda}{\sqrt{\lambda}} [\tilde{\Gamma}^-(\lambda)D_{A_1}(q')] [\tilde{\Gamma}^+ (\lambda)D_{A_1}(q'')]
$$

is regular in $\lambda$, where

$$
\tilde{\Gamma}^\pm (\lambda) = \exp\left\{ \pm \sum_{k<0} \frac{1}{\sqrt{2}} \left( \frac{d}{d\lambda} \right)^k \lambda^{-\frac{1}{2}} q_{-k-1} \right\} \exp\left\{ \pm \sum_{k \geq 0} \frac{(-1)^k}{\sqrt{2}} \left( \frac{d}{d\lambda} \right)^k \lambda^{-\frac{1}{2}} p_k \right\}.
$$

Here we have omitted the accents on $q$ and $p$, it should be clear where the accents are meant to be. In any case we will soon get rid of the accents. The term $\left( \frac{d}{d\lambda} \right)^k \lambda^{-\frac{1}{2}}$ for $k < 0$ signifies a function $f_k(\lambda)$ such that $\left( \frac{d}{d\lambda} \right)^{-k} f_k(\lambda) = \lambda^{-\frac{1}{2}}$, i.e. $f_k(\lambda) = \frac{1}{2} \left( \frac{\lambda^{k+\frac{1}{2}}}{(k+\frac{1}{2})!} \right)$.

Now we can get rid of the accents by realizing that the linear Hamiltonians that appear in the exponents $\Gamma^\pm$ can be encoded by their Hamiltonian vector fields. Then by the standard relation $\dot{q} = \partial p f$ and $\dot{p} = -\partial q f$ for the Hamiltonian vector fields of the Hamiltonian $f$ in Darboux coordinates, we see in general that $q_k \mapsto -\sqrt{2}(-z)^{1-k}$ and $p_k \mapsto \sqrt{2} z^k$. Thus the following formal expression

$$
\sum_{\sqrt{\lambda}=\pm \xi} \frac{d\lambda}{\sqrt{\lambda}} [\tilde{\Gamma}^-(\lambda)D_{A_1}(q')] [\tilde{\Gamma}^+ (\lambda)D_{A_1}(q'')],
$$

36
should be regular in $\lambda$. Here we have

$$\Gamma^\pm(\lambda) = \exp\left\{\pm \sum_{k < 0} (\frac{d}{d\lambda})^k \lambda^{-\frac{1}{2}} (-z)^k \right\} \exp\left\{\pm \sum_{k \geq 0} (\frac{d}{d\lambda})^k \lambda^{-\frac{1}{2}} (-z)^k \right\}. \quad (8.9)$$

This is what we will call the vertex operator form of the KdV hierarchy. The $\Gamma^\pm$ are what is called vertex operators in this context. Note that $\mathcal{D}_{A_1}$ is an element in the Fock space and the vertex operators act on it.

$(n + 1)\text{KdV}$

Now we would like to find vertex operators for the $(n + 1)\text{KdV}$ hierarchy as we have found for the KdV hierarchy. It is apparent that we could simply write

$$\Gamma^\pm = \exp\left\{\pm \sum_{k \in \mathbb{Z}} (\frac{d}{d\lambda})^k \lambda^{-\frac{1}{2}} (-z)^k \right\}.$$ 

Thus we should generalize the role of $\lambda^{-\frac{1}{2}}$ and its (anti-)derivatives. We can write this as the integral

$$\lambda^{-\frac{1}{2}} = \int_\alpha \frac{1}{x} = 2 \int_\alpha \frac{dx}{dx^2}$$

over the one-point cycle $\alpha = [x_\alpha]$ such that $\lambda = x_\alpha^2$ and $\eta := \frac{dx_\alpha}{dx}$ is a 0-form such that $\eta \wedge dx^2 = dx$. This corresponds to $\lambda = f(x_\alpha)$ and $\eta = \frac{df}{dx}$ for $f$ the $A_1$ singularity. This leads to the following definition of the vertex operators for the $(n + 1)\text{KdV}$ hierarchy

$$\Gamma^\alpha = \exp\left\{\sum_{k \in \mathbb{Z}} I^{(k)}(\lambda)(-z)^k \right\} \quad (8.10)$$

where the $I^{(k)}_\alpha$ are functions with values in the local algebra of the $A_n$ singularity, which are called period vectors, and $\alpha \in f^{-1}(\lambda)$ is a one-point cycle. Just as for their analogue in the 2KdV hierarchy we will define the period vectors as consecutive (anti-)derivatives of each other

$$\frac{d}{d\lambda} I^{(k)}(\lambda) = I^{(k+1)}(\lambda). \quad (8.11)$$

In the next part we will need a generalization of the period vectors and there we will specify how exactly to fix the anti-derivatives. In this case it should be clear from the expression (8.14) below what has been done. Otherwise we refer to appendix A.

From now on we will omit the subscript $\alpha$ for the period vectors when it is warranted. We define $I^{(0)}$ using the non-degenerate form $\langle \cdot,\cdot,\cdot \rangle_0$ by

$$\langle I^{(0)}_\beta(\lambda), [\phi] \rangle_0 = - \int_\beta \phi(x) \frac{dx}{df(x)}, \quad (8.12)$$

where $[\phi] \in H$, $\beta \subset f^{-1}(\lambda)$. Since the form is non-degenerate this defines the period vector uniquely up to monodromy of the cycle.
Now let us apply this definition to the $A_n$ singularity. So we pick a one-point cycle $\alpha = [x_\alpha]$ such that $\lambda = x_\alpha^{n+1}$. This yields

$$< I_{\alpha}^{(0)}(x^{n-1}), x^{n-1}>_0 = - \int_{\alpha} x^{n-i} \frac{dx}{dx^{n+1}} = -\frac{\lambda^{n+1}}{n+1},$$

(8.13)

so we have $I_{\alpha}^{(0)} = \sum_{i=1}^{n} [x^{i-1}] \lambda^{n+1}$. This follows from the bilinear form

$$< [x^i], [x^j] >_0 = \frac{-\delta_{i+j,n+1}}{n+1}.$$

So we find

$$\sum_{k \in \mathbb{Z}} I^{(k)}(-z)^k =$$

$$\sum_{k \in \mathbb{Z}} \sum_{i=1}^{n} [x^{i-1}] \frac{z^k}{(n+1)^k} \prod_{r=0}^{\infty} [i + r(n+1)]^{\lambda^{-\frac{i+k(n+1)}{n+1}}}.$$

(8.14)

Now we can apply the same procedure in the general case as we did in the case of the KdV hierarchy. First we reverse the encoding sending the operators $p, q$ to their Hamiltonian vector fields. We see that

$$\Omega(\sqrt{n+1} [x^{n-i}], [x^{i-1}]) = (-1)^k,$$

(8.15)

while the other elements in this sum are orthogonal. So we encode $q_{i,k} \mapsto -\sqrt{n+1}(-z)^{k-1} [x^{n-i}], [x^{i-1}] z^k$ and $p_{i,k} \mapsto \sqrt{n+1} [x^{i-1}] z^k$. By substituting this and subsequently making the substitution

$$q_{i,k} = \frac{i(i+(n+1)(i+2(n+1) \cdots (1+k(n+1))}{(n+1)^k} \frac{1}{\sqrt{n+1}} x_{k(n+1)+i}$$

(8.16)

and the consequence for $p$ we find

$$\sum_{k \in \mathbb{Z}} I^{(k)}(-z)^k =$$

$$= - \sum_{k \geq 0} \sum_{i=1}^{n} \frac{\lambda^{-\frac{i+k(n+1)}{n+1}}}{\sqrt{n+1}} x_{k(n+1)+i} + \sum_{j \geq 0} \sum_{i=1}^{n} \frac{\lambda^{-\frac{i+k(n+1)}{n+1}}}{\sqrt{n+1}} \partial x_{k(n+1)+i}.$$

(8.17)

So in fact we have

$$\sum_{k \in \mathbb{Z}} I^{(k)}(-z)^k = - \sum_{j \in (\mathbb{Z}-n\mathbb{Z})_{\geq 0}} \frac{\xi^j}{\sqrt{n+1}} x_j + \sum_{j \in (\mathbb{Z}-n\mathbb{Z})_{\geq 0}} \frac{\xi^{-j}}{\sqrt{n+1}} \partial x_j,$$

where $\xi^{n+1} = \lambda$. Comparing this to the KP hierarchy (8.2) we can construct the $(n+1)$KdV hierarchy in vertex operator form in the same way as we did for the case $n = 2$ (so it is related to the $A_1$ singularity). This yields the following statement. A function $\mathcal{D}(q)$ satisfies the $(n+1)$KdV hierarchy if and only if

$$\sum_{\alpha} \lambda^{-\frac{\pi^2}{n^2}} d\lambda(\Gamma^{-\alpha}\mathcal{D})(q)(\Gamma\mathcal{D})(q^n),$$

(8.18)
is regular in $\lambda$. Here the sum is over the different one-point cycles $\alpha = [x_\alpha(\lambda)]$ such that $\lambda = x_\alpha^{n+1}$. So the sum is taken over the $n + 1$ different roots $\lambda^{\frac{1}{n+1}}$ differing by roots of unity and

$$
\Gamma^\alpha = \exp\left\{ \sum_{k \in \mathbb{Z}} I^{(k)}_\alpha(\lambda)(-z)^k \right\}.
$$

(8.19)

Equation (8.18) is the form of the KdV hierarchy that we will be working with. It has been shown by Kontsevich that $D_{A_1}$ satisfies the equation for $n = 1$. The rest of this thesis will be dedicated to showing that in fact the $(n + 1)$KdV hierarchy splits up in a particular way in $n$ copies of the KdV hierarchy.
Part III
Solving the \((n + 1)\)KdV Hierarchy

9 Givental’s Formula

In this last part we will show how the solution to the KdV hierarchy together with Givental’s formula for the total descendant potential corresponding to a semisimple Frobenius structure yields a solution to the \((n + 1)\)KdV hierarchy [2, 7, 13]. We will do this in a hands on approach, i.e. we will give the formula and its ingredients strictly in the setting of the \(A_n\) singularity. Subsequently we will show why this indeed yields a solution to the \((n + 1)\)KdV hierarchy following the reasoning made in [2] apart from a few minor adjustments. Let us begin by stating Givental’s formula

\[
D(q) := C(t) \hat{S}^{-1} \hat{R} \exp \left( \frac{U}{\hat{z}} \right) \prod_{i=1}^{n} \mathcal{D}_{A_i}(q_i).
\]

(9.1)

Here \(C(t)\) signifies an invertible matrix which in our case will be reduced to a constant matrix, \(\hat{S}\) and \(\hat{R}\) are certain lower respectively upper triangular quantized symplectic operators, which will be defined below. We have the matrix \(U\) which is diagonal with the eigenvalues of multiplication by the Euler vector field on the diagonal. Recall that \(\Psi\) is the basis transformation matrix from the frame of flat coordinates to the frame of normalized canonical coordinates. Furthermore we have

\[
(q_1, \ldots, q_n) = \Psi^{-1} q \in \mathbb{C}^n[z].
\]

(9.2)

Here we are referring to the coordinates of \(q \in H[z]\) in terms of the normalized canonical coordinates in \(T_1 T\) identified with \(H = T_0 T\) via the metric. The quantization \(\hat{\psi}\) is viewed in the sense that for \(G(q)\) an arbitrary element of the Fock space we have \(\hat{\psi}G(q) = G(\Psi^{-1} q)\). Lastly \(t \in T\) is a semisimple point. Givental proves in [13] that this total descendant potential \(D\) is independent of the point \(t \in T\) (in fact this is the reason for \(C(t)\)). We will first spend a few sections in defining and discussing the ingredients of this formula.

**Important Remark**

We need to remark here the nature of the operators \(\hat{S}\) and \(\hat{R}\). Above it is mentioned that these are lower respectively upper triangular operators. So we have

\[
S_t(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \ldots
\]

(9.3)

and

\[
R_t(z) = 1 + R_1 z + R_2 z^2 + \ldots,
\]

(9.4)

where 1 denotes the identity matrix and \(\hat{S}\) and \(\hat{R}\) are the quantized versions. The parameter \(z\) is a coordinate on \(\mathbb{C}\) and we could switch to the coordinate system \(w = \frac{1}{z}\). This would seem to change the nature of these operators. However, we need to consider the action on functions in the Fock space. Here we have a clear description of the action of lower and upper triangular operators.
given by (7.21) and (7.28). In discussing the elements corresponding to the Frobenius structure of the $A_n$ singularity, for instance the deformed connection, we have used the conventions of [1], which are clearest in this setting. However when discussing the quantization and symplectic operators it is more useful to use the convention of [2]. Practically this means that we will switch from one coordinate system on $\mathbb{C}$ to the other when discussing these parts. This will ensure that the nature of the symplectic operators is always expressed clearly. Whenever such a change of coordinate systems is implied we will also note this directly.

Note then that in Givental’s formula above the term in the exponent is $U_{\beta}$ in the coordinate system that expresses the nature of the symplectic operators. Furthermore the coordinate system for the vertex operators in (8.19) also matches that of (9.3) and (9.4). On the other hand the coordinate used in (6.1) is opposite to that used in (9.3) and (9.4).

10 Period Vectors

To show the solution of the $(n+1)$KdV hierarchy that arises from the $A_n$ singularity and the Witten-Kontsevich tau-function we shall need to use the Frobenius structure considered in part I. First of all we will want to generalize the definition of vertex operators to the rest of the parameter space of the singularity. To do this we simply need to generalize the definition of the period vectors along any cycles in the Milnor fibers of the singularity [14]. In this section we will define the period vectors and vector fields related to the oscillating integrals. We will show that, in the case that the cycle in the definition of the period vector is of a certain type we can write the vector fields related to the oscillating integrals as a version of Laplace transform of the period vectors. We will derive some useful relations for both the oscillating integrals and the period vectors. In this entire section we might suppress the indices referring to cycles when it is appropriate.

Period Vectors on $\mathbb{T}$

We want to expand the definitions given in the previous part to find period vectors as vectorvalued functions that assign to each point $t \in \mathbb{T}$ and $\lambda \in \mathbb{C}$ a vector $I_{\beta}^{(k)}(\lambda, t) \in T_T \mathbb{T}$, such that $I_{\beta}^{(k)}(\lambda, 0) = I_{\beta}^{(k)}(\lambda)$. They appear in the definition of the vertex operators and related objects like the phase factors which will be introduced in another section. Here $\beta$ is some cycle in the minlor fiber $V_{\lambda,t}$ that appears in the vertex operator $\Gamma^\beta$ and $k \in \mathbb{Z}$. We will specify the cycle below.

Let us define then the period vectors for two cases which we will both use. First suppose $\alpha$ is a one point cycle in the minlor fiber $V_{\lambda,t}$ (thus $\alpha$ depends on the point $(t,\lambda) \in \mathbb{T} \times \mathbb{C}$). Then we have the following definition of the period vectors, which matches the one given in [3].
Definition 1 The period vectors are defined by

\[ <I_{\alpha}^{(-p)}(\lambda, t), \partial_i > = -\partial_i \int_{\alpha} d^{-1} \left( \frac{\lambda - F(x, t)}{p!} \right) dx \]  \hspace{1cm} (10.1)

and

\[ I_{\alpha}^{(p)}(\lambda, t) = \partial_\lambda p I_{\alpha}^{(0)}(\lambda, t) \]  \hspace{1cm} (10.2)

for all \( p \geq 0 \).

Here \( d^{-1} \) signifies the linear operator acting on forms on \( C \) by the rule

\[ d^{-1}(x^k dx) = \frac{x^{k+1}}{k+1}. \]

As one would expect given the notation. Note that the period vectors might be multiple valued depending on the monodromy of \( \alpha \).

Note that by non-degeneracy of the form \( <\cdot, \cdot> \) the period vectors are indeed defined above. Moreover

\[ <I_{\alpha}^{(0)}, \partial_i > = -\partial_i \int_{\alpha} d^{-1} dx = -\int_{\alpha} \partial_i x = -\int_{\alpha} \frac{\partial_i F}{\partial x} dx, \]

since we have \( \partial_i x \partial_j F = \partial_j F \). Now, since \( \partial_i F|_0 = x^{n-i} \) and \( \frac{1}{\sigma_i F} \wedge dF = dx \), this shows that the definition agrees with (8.12).

We expand this definition to include period vectors corresponding to 1-cycles (relative to milnor fibers) \( \beta(t, \lambda) \in H_1(C, \mathbb{V}_\lambda, t; \mathbb{Z}) \) as follows.

Definition 2 These period vectors are defined by

\[ <I_{\beta}^{(-p)}(\lambda, t), \partial_i > = -\partial_i \int_{\beta} (\lambda - F(x, t))^p \frac{dx}{p!} \]  \hspace{1cm} (10.3)

and

\[ I_{\beta}^{(p)}(\lambda, t) = \partial_\lambda p I_{\beta}^{(0)}(\lambda, t) \]  \hspace{1cm} (10.4)

for all \( p \geq 0 \).

Note that by Stokes’ theorem we have obviously

\[ I_{\beta}^{(k)} = I_{\alpha_2}^{(k)} - I_{\alpha_1}^{(k)}, \]  \hspace{1cm} (10.5)

where \( \alpha_1 \) and \( \alpha_2 \) are the endpoints of \( \beta \), such that \( \beta \) starts at \( \alpha_1 \).

The following lemma summarizes the most valuable properties of the period vectors.

Lemma 12 We have for \( I \) the following useful relations:

\[ \partial_\lambda I^{(k)} = I^{(k+1)}, \quad \partial_i I^{(k)} = - (\partial_i \bullet) \partial_\lambda I^{(k)} \]

\[ \partial_\lambda I^{(k)} = \partial_\lambda I^{(k)} \quad (\lambda \partial_\lambda + E) I^{(k)} = (-\mu - k - \frac{1}{2}) I^{(k)}, \]  \hspace{1cm} (10.6)

for any \( k \in \mathbb{Z} \).
Proof:

Note that we have omitted the cycles in the above statements. In fact they are valid for any appropriate 0- or 1-cycle. To prove this, note that we will only have to prove it explicitly for the one-point cycles, by virtue of equation (10.6).

Suppose then that \( \alpha = [x, \lambda] \) such that \( F(x, t) = \lambda \). Note that, by definition, we have the upper left identity for \( k \geq 0 \). So suppose that \( p > 0 \). To prove the upper left identity we need to show that

\[
\partial_\lambda I_\alpha^{(-p)} = I_\alpha^{(-p+1)}.
\]

Let \( G_p(\lambda, t, x) \) be the function such that

\[
< I_\alpha^{(-p)}, \partial_i > = -\partial_i \int_\alpha G_p,
\]

for all \( i = 1, \ldots, n \). So we have

\[
dG_p = \frac{(\lambda - F)^p}{p!} dx.
\]

Then we have, for all \( i = 1, \ldots, n \),

\[
< \partial_\lambda I_\alpha^{(-p)}, \partial_i > = -\partial_\lambda \partial_i \int_\alpha G_p = -\partial_i \partial_\lambda (G_p|_{x, \lambda}) =
\]

\[
= -\partial_i (\partial_\lambda G_p)|_{x, \lambda} - \partial_i (\partial_\lambda G_p)|_{x, \partial_\lambda x} =
\]

\[
= -\partial_i \int_\alpha \partial_\lambda G_p = -\partial_i \int_\alpha d^{-1} (\partial_\lambda G_p dx) =
\]

\[
= -\partial_i \int_\alpha d^{-1} \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} dx \right) = < I_\alpha^{(-p+1)}, \partial_i >,
\]

where we used the chain rule and, in the fourth equality, the fact that

\[
(\partial_\lambda G_p)|_{x, \lambda} = \frac{(\lambda - F(x, t))^p}{p!} = 0,
\]

since \( F(x, t) = \lambda \). So by non-degeneracy of the bilinear form this proves the upper left identity.

Similarly, for the lower left identity we have

\[
-\partial_i \partial_\lambda \int_\alpha d^{-1} \left( \frac{(\lambda - F)^p}{p!} dx \right) = -\partial_\lambda \int_\alpha d^{-1} \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} dx \right),
\]

for all \( i = 1, \ldots, n \). Where we used that \( -\partial_\lambda F = 1 \). Thus we have

\[
< \partial_\lambda I_\alpha^{(-p)}, \partial_i > = < I_\alpha^{(1-p)}, \partial_i >.
\]

Thus, in light of the upper left identity we see that the lower left identity holds for any \( k \in \mathbb{Z} \).

Then the upper right identity can be restated as

\[
\partial_k I^{(k)} = -(\partial_k \partial_\lambda) I^{(k)}.
\]
Note that, by virtue of the upper left identity, it is sufficient to prove it for $k \leq 2$. Now we have

$$\left(-\partial_i\partial_j - \sum_{k=1}^{n} c_{ij}^k \partial_k \partial_k \right) \int_{\alpha} d^{-1} \left( \frac{(\lambda - F)^p}{p!} dx \right) =$$

$$\int_{\alpha} d^{-1} \left[ \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} \partial_i \partial_j F - \frac{(\lambda - F)^{p-2}}{(p-2)!} \partial_i F \partial_j F + \frac{(\lambda - F)^{p-2}}{(p-2)!} \sum_{k=1}^{n} c_{ij}^k \partial_k F \right) dx \right] =$$

$$\int_{\alpha} d^{-1} \left[ \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} \partial_x K_{ij} - \frac{(\lambda - F)^{p-2}}{(p-2)!} K_{ij} \partial_x F \right) dx \right] =$$

$$\int_{\alpha} \frac{(\lambda - F)^{p-1}}{(p-1)!} K_{ij} = 0,$$

for all $i, j = 1, \ldots, n$ and $p \geq 2$. So we see that, for all $i, j = 1, \ldots, n$ and $p \geq 2$, we have

$$0 = \partial_i I_{\alpha}^{(-p)}(\partial_j) + \partial_j I_{\alpha}^{(-p)}(\partial_i) =$$

$$\partial_i I_{\alpha}^{(-p)}(\partial_j) + \partial_j (\partial_\lambda I_{\alpha}^{(-p)}(\partial_i)) > \partial_i + (\partial_\lambda \partial_\lambda) I_{\alpha}^{(-p)}(\partial_j),$$

which proves the upper right identity.

Finally let us turn to the lower right identity. Note that it is yet again enough to prove the identity for $k < 0$, since, by virtue of the upper left identity, it then follows for all higher superscripts. So suppose again $p > 0$. We have for all $i = 1, \ldots, n$

$$\lambda \partial_\lambda I_{\alpha}^{(-p)}(\partial_i) = -\partial_i \lambda \int_{\alpha} d^{-1} \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} dx \right) =$$

$$= p I_{\alpha}^{(-p)}(\partial_i) - \partial_i \int_{\alpha} d^{-1} \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} F dx \right) =$$

$$= p I_{\alpha}^{(-p)}(\partial_i) - \partial_i \int_{\alpha} d^{-1} \left( \frac{(\lambda - F)^{p-1}}{(p-1)!} x \partial_x F + EF \right) dx =$$

$$= p I_{\alpha}^{(-p)}(\partial_i) - \partial_i \left[ \int_{\alpha} d^{-1} \left( \frac{(\lambda - F)^{p}}{p!} dx \right) + \right.$$

$$\left. + \frac{1}{n+1} I_{\alpha}^{(-p)}(\partial_i) \right] =$$

$$= < \left( (p - E + \frac{1}{n+1} - d_i) I_{\alpha}^{(-p)}(\partial_i) > ,$$

where we used the upper left identity in (10.7) and (5.15). On the other hand we have for all $i = 1, \ldots, n$

$$< \left( -\mu - \frac{1}{2} \right) I_{\alpha}^{(-p)}(\partial_i) \mu \partial_i > = < I_{\alpha}^{(-p)}(\partial_i) > - \frac{1}{2} < I_{\alpha}^{(-p)}(\partial_i) >$$

$$= (\mu_i - \frac{1}{2}) < I_{\alpha}^{(-p)}(\partial_i) > = \left( \frac{1}{n+1} - d_i \right) < I_{\alpha}^{(-p)}(\partial_i) >.$$
where we used (2.11) and (5.14). Combining these two we see that indeed for all \( i = 1, \ldots, n \) we have

\[
< (\lambda \partial_\lambda + E) I^{(-p)}_\alpha, \partial_i > =< (p + \frac{1}{n+1} - d_i) I^{(-p)}_\alpha, \partial_i > =< (-\mu + p - \frac{1}{2}) I^{(-p)}_\alpha, \partial_i > ,
\]

which concludes the lemma. \( \square \)

Apart from these identities we will need an expansion of certain period vectors \( I^{(0)}_{\beta_i} \), for \( i = 1, \ldots, n \), in what follows. Here the 1-cycle \( \beta_i \) is related to the cycles appearing in the definition of the deformed flat coordinates. Namely \( \beta_i \) is that cycle which vanishes as \( \lambda \) nears \( u_i \) along the path that defines \( B_i \). We will call these cycles the vanishing cycles [2, 14].

For any \( i = 1, \ldots, n \) we see, from expression (6.15), that, in a neighborhood of the critical point \( x_i \), the equation \( F(x,t) = \lambda \) has two solutions (see also [3])

\[
x_{i,\pm} = x_i \pm \frac{1}{\sqrt{\Delta_i}} \sqrt{2(\lambda - u_i)} + O(\sqrt{2(\lambda - u_i)}^2).
\]

(10.11)

So we see that

\[
\int_{\beta_i} dx = x_{i,+} - x_{i,-} = \frac{2}{\sqrt{\Delta_i}} \sqrt{2(\lambda - u_i)} + O(\sqrt{2(\lambda - u_i)}^2),
\]

for all \( i = 1, \ldots, n \). Which yields finally

\[
< I^{(0)}_{\beta_i}, \partial_j >= \frac{\partial_i u_i}{\sqrt{\Delta_i}} \sqrt{2(\lambda - u_i)} (1 + \ldots),
\]

(10.12)

for all \( i, j = 1, \ldots, n \) and the dots signify a power series in \( 2(\lambda - u_i) \).

## 11 The symplectic operators \( R_t \) and \( S_t \)

### The symplectic operator \( R_t \)

The relation between the cycles \( \beta_i \) and \( B_i \) mentioned above warrants a description of certain vector fields \( J_{B_i} \) related to the oscillating integrals \( J_{B_i} \) as a version of Laplace transform of the period vectors \( I^{(0)}_{\beta_i} \). We will need this relation to discuss the quantized symplectic operator \( \hat{R}_t \) in Givental’s formula. Moreover the vector fields \( J_B \) are used to define this operator in the first place. So let us define the vector fields \( J_B \) and derive the main properties.

**Definition 3** For all \( B \in \mathcal{Y} \) we define \( J_B \) by

\[
< J_B, \partial_i > = \partial_i J_B \quad \text{for all} \quad i = 1, \ldots, n
\]

(11.1)

Here \( J_B(z,t) \in T_t T \) is a vector valued function on \( \mathbb{C} \times T \). We will show momentarily how these vector fields are related to the ingredients of Givental’s formula.

The following lemma summarizes the most useful properties of the vector fields \( J \).
Lemma 13 The vector fields $J$ defined above satisfy the following relations:

$$\partial_i J = z(\partial_i \cdot) J \quad (z \partial_z - E)J = \mu J,$$ \hspace{1cm} (11.2)

for all $i = 1, \ldots, n$.

Proof:

To show the above equations it is enough, by non-degeneracy of $< \cdot, \cdot >$, to show that

$$< \partial_i J, \partial_j > = < z(\partial_i \cdot) J, \partial_j > \quad (z \partial_z - E)J, \partial_j >= < \mu J, \partial_j >$$

for all $i, j = 1, \ldots, n$.

Starting with the first equation we have

$$< \partial_i J, \partial_j > = \partial_i < J, \partial_j > = \partial_i \partial_j J = z \sum_{k=1}^n c_{ij}^k \partial_k J =$$

$$z < J, \sum_{k=1}^n c_{ij}^k \partial_k > = z < J, \partial_i \cdot, \partial_j > = < z(\partial_i \cdot) J, \partial_j >,$$

where we used invariance of the bilinear form and (6.2). So this shows the first equation.

To prove the second equation we compute

$$< \mu J, \partial_j > = < J, -\mu \partial_j > = z(-\frac{\mu_j}{z} \partial_j J) = z(\partial_z \partial_j J - \sum_{i,k=1}^n d_i t_i c_{ij}^k \partial_k J) =$$

$$z \partial_z \partial_j J - \sum_{i=1}^n d_i t_i \partial_i \partial_j J = < (z \partial_z - E)J, \partial_j >,$$

where we used the result of lemma 3 and (6.4). This shows the second equation. \(\square\)

Now according to [2, 1] the symplectic operator $R_t$ is uniquely determined by its asymptotics. Moreover this asymptotic behaviour, near the critical points, is given by matrix solutions $J$ to the system of equations (11.1) and (11.2), in the following way

$$(-z)J \sim \Psi R_t e^{zU}$$ \hspace{1cm} (11.3)

So the asymptotics of a matrix with columns given by the vectors fields $J_{B_i}$ will give us a big part of Givental’s formula. So let us explore the asymptotics.

First of all note that we have

$$J_{B_i} = (-2\pi z)^{-\frac{1}{2}} \int_{u_i}^\infty e^{z\lambda} f_{\mu_i}^{(0)}(\lambda) d\lambda,$$ \hspace{1cm} (11.4)

for all $i = 1, \ldots, n$, where the integration may be carried out over any ray in the half plane where Re $z\lambda < 0$. Here we used the correspondence of the cycles
and $\beta_i$, which should be clear from this relation. So when we extend the compact cycle $\beta_i$ from the critical value to infinity we get the non-compact cycle $B_i$. Now to see this we simply calculate

$$(-2\pi z)^{-\frac{1}{2}} < \int_{u_i}^\infty e^{z\lambda} I_{\beta_i}^{(0)} d\lambda, \partial_j >= (-2\pi z)^{-\frac{1}{2}} \int_{u_i}^\infty e^{z\lambda} (\partial_j \int_{\beta_i} dx) d\lambda =$$

$$= (-2\pi z)^{-\frac{1}{2}} \partial_j \int_{u_i}^\infty e^{z\lambda} dx d\lambda = \partial_j J_{B_i},$$

for any $i,j = 1, \ldots, n$. Where we used Fubini’s theorem and the specific definition of $\beta_i$. Since the cycle $\beta_i$ extends to $B_i$ when we integrate from $u_i$ to $\infty$ and since the first cycle is in the Milnor fiber we have $\lambda = F$ in the integral. This shows that equation (11.4) is indeed correct.

This relation allows us to study the asymptotics of $J$ simply by using the expansion (10.12) for $I$. Now we can use this expansion to obtain an expression for the components of $I$ in the canonical frame. Namely

$$[J_{\beta_i}^{(0)}]_j = <J_{\beta_i}^{(0)}, \sqrt{-\Delta_j} \partial_a> = \sum_{a=1}^n \Psi_{ja} <I_{\beta_i}^{(0)}, \partial_a> =$$

$$\sum_{a=1}^n \Psi_{ja} \left( \delta_{ia} + \sum_{k>0} A_{ai}^k [2(\lambda - u_i)]^k \right) \frac{2}{\sqrt{2(\lambda - u_i)}}, \quad (11.5)$$

for all $i,j = 1, \ldots, n$ and some series of matrices $A^k$. Now we simply compute the integral, which yields

$$2(-2\pi z)^{-\frac{1}{2}} \int_{u_i}^\infty e^{z\lambda} [2(\lambda - u_i)]^{k-\frac{1}{2}} d\lambda =$$

$$(-z)^{k-1} e^{zu_i} (2\pi)^{\frac{1}{2}} \int_{-\infty}^\infty e^{-x^2} x^{2k} dx = (-z)^{-1-k} (2k-1)!! e^{zu_i}, \quad (11.6)$$

for all $i = 1, \ldots, n$ and $k \in \mathbb{Z}_{\geq 0}$, where we used the substitution $2(\lambda - u_i) = -\frac{x^2}{2\pi}$. So we see that

$$(-z)[J_{B_i}]_j \sim \sum_{a=1}^n \Psi_{ja} \left( \delta_{ia} + \sum_{k>0} (2k-1)!! A_{ai}^k (-z)^{-k} \right) e^{zu_i}, \quad (11.7)$$

for all $i = 1, \ldots, n$. Thus when we switch to the coordinates in which

$$R = 1 + R_1 z + R_2 z^2 + \ldots$$

we see that

$$R_{ai}^k = (-1)^k (2k-1)!! A_{ai}^k, \quad (11.8)$$

Although this might not be a complete definition of $R$ it will be quite sufficient, as will be shown below.
The symplectic operator $S_t$

Recall the equations (2.9) and (2.10) satisfied by deformed horizontal sections, i.e. those given by the oscillatory integrals. These will give us the definition of the $S_t$ matrix. Namely $S_t$ is the symplectic operator that sends solutions $\phi$ of

$$
\left( \partial_z + \frac{\mu}{z} \right) \phi = 0
$$

(11.9)

to solutions of (2.10) near $z = 0$. So for such solutions $\phi$ we have near $z = 0$

$$
\left( \partial_z - \mathcal{E} + \frac{\mu}{z} \right) S_t \phi = 0.
$$

(11.10)

Note that in this coordinate $S_t(z) = 1 + S_1 z + S_2 z^2 + \ldots$. Existence and uniqueness of this operator is shown in [1, 13]. Dubrovin shows that the gauge transformation $S_t$ can be found in the case that $\mu$ is not resonant. By non-resonance we mean that $\mu_i - \mu_j \notin \mathbb{Z}$, for all $i, j = 1, \ldots, n$. This clearly holds in the case of the Frobenius structure corresponding to our $A_n$ singularity. In the general case there would be a slight adaption of the statements made below [1]. Furthermore a choice of $S_t$ is called a calibration of the Frobenius manifold. In fact the operator $S_t$ can be used to define the deformed connection and thus the Frobenius structure [13].

**Lemma 14** The operator $S_t(z)$ is a fundamental solution to

$$
\partial_i S = z(\partial \bullet) S,
$$

(11.11)

for all $i = 1, \ldots, n$. Moreover we have

$$
[S_t, \mu] = (z \partial_z - \mathcal{E}) S_t.
$$

(11.12)

**Proof:**

We find easily the fundamental matrix solution $z^{-\mu}$ of (11.9). Now we had already found solutions to (2.10) given by the oscillatory integrals. Thus we have here a fundamental matrix solution $P$ with entries $P_{ij} = \partial_i J_{B_j}$. Then we see that near $z = 0$ we have

$$
P = S_t z^{-\mu} C,
$$

(11.13)

for some invertible matrix $C$ determined by the ordering of the bases $\{\partial_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$. This then immediately implies (11.11). To find (11.12) we simply compute

$$
0 = \left( \partial_z + \frac{\mu}{z} - \mathcal{E} \right) S_t z^{-\mu} = (\partial_z S_t) z^{-\mu} - S_t \mu z^{-\mu-1} + \mu S_t z^{-\mu-1} - \mathcal{E} S_t z^{-\mu} =
$$

$$
= ((z \partial_z - \mathcal{E}) S_t + [\mu, S_t]) z^{-\mu-1},
$$

where we used (11.11). This proves the lemma. □
12 Phase Factors

We have now sufficiently defined all ingredients of Givental’s formula apart from $C(t)$, which we shall come to in section 14. To prove that $D$ in (9.1) satisfies the regularity condition of (8.18) we need to commute the operators present in Givental’s formula past the vertex operators (8.19). For any semisimple $t \in T$ this will yield eventually $n$ independent copies of the KdV hierarchy (8.8) (including $D_{A_1}$), which we know is regular in $\lambda$. However, apart from these we will also find some extra factors called phase factors. Givental introduced the phase form [2], which can be used to express most of the phase factors. In the end this will allow us to show that certain phase factors are holomorphic in $\lambda$ and thus regular. In this section we will define and discuss the phase form, mainly to show some properties that we need in the final section.

Suppose $\beta$ is a cycle such that the corresponding period vector is defined.

**Definition 4** The phase form $\tilde{W}_\beta$ corresponding to the cycle $\beta$ is given by

$$\tilde{W}_\beta(\lambda, t) := - < I_\beta^{(0)}, d I_\beta^{(-1)} >,$$

where $d$ is to be taken as on $F(T)$.

Note that we have

$$\tilde{W}_\beta = - < I_\beta^{(0)}, d I_\beta^{(-1)} > = - \sum_{i=1}^n < I_\beta^{(0)}, \partial_i I_\beta^{((-1)} > dt_i = \sum_{i=1}^n < I_\beta^{(0)}, \partial_i I_\beta^{(0)} > dt_i,$$

where we used the upper right identity of (10.6). These phase forms have certain interesting properties as shown in [2, 7]. We will use this section to show the properties that we need. For a more complete discussion see the references above.

Firstly note that the phase form is closed, as a form on $T$, since

$$d \tilde{W}_\beta = \sum_{i,j=1}^n \partial_j < I_\beta^{(0)}, \partial_i I_\beta^{(0)} > dt_j \wedge dt_i =
\sum_{i,j=1}^n \partial_j < I_\beta^{(0)}, \partial_i I_\beta^{(0)} > dt_j \wedge dt_i + \sum_{i,j=1}^n < \partial_j I_\beta^{(0)}, \partial_i I_\beta^{(0)} > dt_j \wedge dt_i =
\sum_{i,j=1}^n (\partial_j < I_\beta^{(0)}, \partial_i I_\beta^{(0)} > + < \partial_i I_\beta^{(0)}, \partial_j I_\beta^{(0)} > + < I_\beta^{(0)}, (\partial_j (\partial_i \bullet)) I_\beta^{(0)} >) dt_j \wedge dt_i = 0$$

where we used the symmetry and invariance of $< \cdot, \cdot >$, the consequence $\partial_j (\partial_i \cdot) = \partial_i (\partial_j \cdot)$ of (6.8) and the fact that $dt_j \wedge dt_i = -dt_i \wedge dt_j$. Since this means that what is inside the brackets is symmetric under interchanging $i$ and $j$ while $dt_j \wedge dt_i$ is anti-symmetric.

Secondly by the lower left equation in lemma 12 we see that the phase form is invariant under $\partial_\lambda - \partial_n$. In other words we have

$$L_{\partial_\lambda - \partial_n} \tilde{W}_\beta = (i_{\partial_\lambda - \partial_n} \circ d + d \circ i_{\partial_\lambda - \partial_n}) \tilde{W}_\beta =$$
\[
\begin{align*}
    & \quad \mathcal{W}_\beta(t) = \tilde{\mathcal{W}}_\beta(0, t) \\
\end{align*}
\]
then we have
\[
\tilde{\mathcal{W}}_\beta(\lambda, t) = \mathcal{W}_\beta(t - \lambda 1),
\]
where \(-1 \in \mathcal{T}\) such that \(F(x, -1) = f(x) - 1\). So \(-1 = (0, \ldots, 0, -1)\) in natural coordinates and \(-1 = (0, \ldots, 0, 1)\) in flat coordinates.

In the next section we will need the relation between the phase forms and intersection indices of the corresponding cycles. First of all we introduce a bilinear form on the cotangent spaces, see also [1, 15]. We define the intersection form \((\cdot, \cdot)\) by
\[
(\omega_1, \omega_2) = \langle I_1 \cdot I_2, E \rangle,
\]
for all \(\omega_1, \omega_2 \in T^*\mathcal{T}\). Here the multiplication is pulled back from the tangent bundle using the (non-degenerate) metric and \(i_E\) signifies the interior product.

So suppose \(I_1, I_2 \in \mathcal{T}\) such that \(\langle I_1, \cdot \rangle = \omega_1\) and \(\langle I_2, \cdot \rangle = \omega_2\), then we have
\[
(\omega_1, \omega_2) = \langle I_1 \cdot I_2, E \rangle,
\]
since \(\omega_1 \cdot \omega_2 = \langle I_1 \cdot I_2, \cdot \rangle\) by definition.

Now it is proven in [15] and [14] that this intersection form is proportional to the intersection index in homology carried over to the cotangent bundle when the self-intersection of vanishing cycles is normalized to +2. For details we refer to the references given above, since most elements of this discussion are not relevant to the task at hand. So we restrict ourselves to the comment that, since \(i_E(\mathcal{W}_\beta) = \langle I_0^0 \cdot I_0^0, E \rangle\), we see that
\[
i_E(\mathcal{W}_\beta) = -(\beta, \beta),
\]
where \((\cdot, \cdot)\) denotes the intersection index. Which in turn implies that
\[
\int_{-1}^{\lambda} \mathcal{W}_\beta = -(\beta, \beta) \int_{1}^{\lambda} \frac{d\xi}{\xi},
\]
for a proof see [2, 7, 15].

Note also that we could in fact define the phase form for two different cycles. So we introduce the polarized phase form \(\tilde{\mathcal{W}}_{\alpha, \beta}\) in the obvious sense. In this
case we also have the identities $d\tilde{W}_{\alpha,\beta} = 0$ and $L_{\partial_{\alpha} - \partial_{\beta}} \tilde{W}_{\alpha,\beta} = 0$. So we define again $W_{\alpha,\beta}(t) = \tilde{W}_{\alpha,\beta}(0, t)$. Then we have again the identities

$$\tilde{W}_{\alpha,\beta}(\lambda, t) = W_{\alpha,\beta}(t - \lambda 1) \quad \text{and} \quad i_E(W_{\alpha,\beta}) = -(\alpha, \beta).$$

Now note that the period vectors are not necessarily single valued, thus the same thing goes for the phase form. In fact this depends on the monodromy of the cycles corresponding to the phase form. However integrating the phase forms over certain paths will lead to a well defined number. More precisely, we define the discriminant $\Sigma$ corresponding to the $A_n$ singularity as the subset of points $(t, \lambda) \in T \times \mathbb{C}$ where $F(x, t) = \lambda$ has less than $n$ solutions. Then if $\alpha$ is a cycle that is invariant under the monodromy around a loop in $T \times \mathbb{C} - \Sigma$, this means the phase form $W_{\alpha}$ is single valued along this loop so we can define (in this case) the period $\oint W_{\alpha}[2]$.

We will need one more fact about the phase form. Namely that the period $\oint_{\gamma} W_{\alpha}$, for $\alpha$ a cycle that has integer intersection indices with the vanishing cycles, is an integer multiple of $2\pi \sqrt{-1}$. For a proof see [2]. Note that this only makes sense in the case that $\alpha$ is invariant under the monodromy around $\gamma$.

### 13 Commutation with Vertex Operators

In this section we will finally show how to commute the ingredients of Givental’s formula past the vertex operators. This will set us up completely to complete the proof of regularity of $D$ in the sense of (8.18).

**Commutation of $\hat{S}_t$**

We have already made the beginning to being able to commute the operator $\hat{S}_t$ in the previous part. Recall the action of quantized lower triangular symplectic operators as given in (7.21). Suppose $\mathcal{G}(q)$ is an arbitrary element of the Fock space. Then (7.21) implies easily that

$$\hat{S}_t \mathcal{G}(q) = e^{-\frac{\pi}{\hbar} W([S_t^{-1}q]_+, [S_t^{-1}q]_+)} \mathcal{G}([S_t^{-1}q]_+),$$

with $W$ and the notation $[S_t^{-1}q]_+$ as in part II. Note however that this $W$ is related to $S_t$ by (7.20) and thus depends also on the point $t \in T$.

Now suppose $f \in \mathcal{H}$ is an arbitrary element. Then we have

$$f = \sum_{k \in \mathbb{Z}} f_k z^k,$$

where $f_k \in \mathcal{H}$ for all $k \in \mathbb{Z}$. Note that, in the case that $f_k = (-1)^k I_\alpha^{(k)}(\lambda, 0)$ for all $k \in \mathbb{Z}$ we have $e^f = \Gamma^\alpha$.

However, let us first inspect the general case. Denote by $f_+$ and $f_-$ the non-negative respectively the negative powers of $f$ in $z$, i.e. $f_+ = \sum_{k \geq 0} f_k z^k$ and $f_- = f - f_+$. Then we can split the quantization of $e^f$ by the rule

$$\hat{e}^f = e^{\hat{f}_+} e^{\hat{f}_-}.$$
Then we can use (7.21) and (13.1) to see that
\[ \hat{S}e^{\frac{1}{2}W(f_+, f_+)}e^{\hat{S}}G, \]
by a computation similar to that leading to equation (7.21), we use again the CBH-formula.

Then to find out how we can commute \( \hat{S} \) past the vertex operators we have two tasks. We should compute \( W(f_+, f_+) \) and the quantization of \( Sf \), when \( f = \ln \Gamma_\alpha = \sum_{k \in \mathbb{Z}} I_{\alpha}(k)(\lambda, 0)(-z)^k. \)

Both these computations will follow in some way from the following important theorem. We will prove it by combining the proofs of Givental and Milanov [2, 3].

**Theorem 15** Let \( f(\lambda, t) = \sum_{k \in \mathbb{Z}} I_{\alpha}(k)(\lambda, t)(-z)^k \), for some cycle such that the corresponding period vectors are defined. Then we have
\[ f(\lambda, t) = S_t f(\lambda, 0). \] (13.5)

**Remark 7** We mean this in the following way. As shown above (for the \( k = 0 \) case), the period vectors expand as Laurent series in \( \lambda \) (with fractional exponents) near \( \lambda = \infty \). The maximum exponent in \( I^{(k)} \) tends to \(-\infty\) as \( k \) tends to \( \infty \). We can expand \( Sf \) in \( z \). The coefficients of such an expansion, which are infinite sums themselves, converge in the \( \lambda \)-adic sense.

**Proof:**

In fact most of the work to prove this theorem has already been done in lemma’s 12 and 14. Note that by lemma 12 we have
\[ \partial_i f = z^{-1}(\partial_i \cdot f), \quad \partial_\lambda f = \partial_\lambda f \quad \text{and} \quad (z \partial_z + \lambda \partial_\lambda + E)f = (-\mu - \frac{1}{2})f, \] (13.6)
for all \( i = 1, \ldots, n \). Now, denoting \( f_0(\lambda) := f(\lambda, 0) \), we have
\[ \partial_i f_0 = 0, \quad \partial_\lambda f_0 = -z^{-1}f_0 \quad \text{and} \quad (z \partial_z + \lambda \partial_\lambda) f_0 = (-\mu - \frac{1}{2})f_0. \] (13.7)

Now we can combine this with the identities in lemma 14 to see that \( f = S_t f_0 \) satisfies (13.6). Note that when we use the identities of lemma 14 we need to switch to the coordinate that expresses the nature of \( S_t \) again.

Note from (5.16), and in fact the general theory, that \( E|_{t=0} = 0 \). Then we see from the defining property for \( S_t \), given in (11.9) and (11.10), that \( S|_{t=0} = 1 \), the identity matrix. This shows that
\[ S_t f_0|_{t=0} = f_0 = f|_{t=0}. \]

Thus since they are equal in \( t = 0 \) and they are both solutions to (13.6) the lemma is proved. □
So we have found $S_t f$. Then let us compute the quadratic form $W(f_+, f_+)$. To do this we shall first need to study the quadratic form $W$ more closely. Recall the defining equations (7.19) and (7.20). Recall that since $W$ is related to $S$, we can view $W(q, q)$ as dependent on $t \in T$. Now since $S_t|_{t=0} = 1$ we see from equation (7.20) that $W|_{t=0} = 0$. Thus we are able to express the quadratic form as

$$W(q, q) = \int_0^t \sum_{i=1}^n \partial_i W(q, q) dt_i.$$

To make use of this we need to determine $\partial_i W$ for $i = 1, \ldots, n$.

Let us start by differentiating (7.20), this yields

$$\partial_i \left( \frac{S_t^T(w) S_t(z) - I}{w^{-1} + z^{-1}} \right) = \frac{(\partial_i S_t(w))^T S_t(z) + S_t^T(w) \partial_i S_t(z)}{w^{-1} + z^{-1}} =$$

$$w^{-1}(\partial_i \cdot S_t(w))^T S_t(z) + z^{-1} S_t^T(w) \partial_i \cdot S_t(z) = S_t^T(w) \partial_i \cdot S_t(z),$$

for all $i = 1, \ldots, n$. Here we used the identity (11.11), which reads as

$$\partial_i S = z^{-1} \partial_i \cdot S,$$

for all $i = 1, \ldots, n$, in the coordinate that shows the nature of $S$ (which is used in part II and thus in (7.20)). We also used the fact that, by invariance of $<\cdot, \cdot>$, we have that $(\partial_i \cdot)^T = (\partial_i \cdot)$ for all $i = 1, \ldots, n$. So we see that

$$\partial_i W(q, q) = <[Sq]_0, \partial_i \cdot [Sq]_0>, \tag{13.8}$$

where $[Sq]_0 = [Sq]_+|_{z=0}$.

Now by theorem 15 we see that for $f = \sum_{k \in \mathbb{Z}} I_{\beta}^{(k)}(\lambda, 0)(-z)^k$ we have

$$[S_t f]_0 = [S_t f_0] = I_{\beta}^{(0)}(\lambda, t).$$

So we see that

$$W(f_+, f_+) = \int_0^t \sum_{i=1}^n <I_{\beta}^{(0)}, \partial_i \cdot I_{\beta}^{(0)}> dt_i = \int_0^t \tilde{W}_\beta, \tag{13.9}$$

for any cycle $\beta$ such that the period vectors are defined. Note that the phase form is not necessarily singlevalued and the integral may depend on the path from $0$ to $t$. We will however not specify the path, instead we will assume that whenever the ending points are the same in an integral the path is also the same. Now note that we can use the properties of the phase form in (12.6) and (12.9) to write

$$W(f_+, f_+) = \int_{(\lambda, 0)}^{(\lambda, t)} \tilde{W}_\beta = \int_{-\lambda}^{t - \lambda} W_\beta =$$

$$= \int_{-1}^{-\lambda} W_\beta - \int_{-1}^{-\lambda} W_\beta = \int_{-1}^{-\lambda} W_\beta + (\beta, \beta) \int_{-1}^{\lambda} \frac{d\xi}{\xi}. \tag{13.10}$$

Now in the interest of clarity in what follows we give the following definition, which we have already been hinting at.
Definition 5

\[ \Gamma_\alpha^\beta (\lambda) := \exp\{\sum_{k < 0} I^{(k)}_\alpha (\lambda, t)(-z)^k\} \exp\{\sum_{k \geq 0} I^{(k)}_\alpha (\lambda, t)(-z)^k\}. \] (13.11)

Note then that \( \Gamma^\alpha = \Gamma^\alpha_0 \). Using this notation we have the following proposition which determines the commutation of \( S_t \) with the vertex operators. We will in general call any \( \Gamma^\alpha \) a vertex operator.

Proposition 16

\[ \hat{S}_t \Gamma^\alpha_0 \hat{S}_t^{-1} = e^{\int_{t-u}^{t-\lambda} W_{\beta_2} \frac{d\lambda}{\lambda} + (\beta, \beta) \int_{t-u}^{t-\lambda} d\xi^2 \Gamma^\alpha_\xi}. \] (13.12)

Proof:

The proof is given by substituting \( f = \sum_{k \in \mathbb{Z}} f^{(k)}_\alpha (\lambda, 0)(-z)^k \) in (13.4) and using theorem 15 to calculate \( S_t f \) and \( W(f_+, f_+) \). This is done above. □

Commutation of \( \hat{R}_t \)

Now let us turn our attention to the other ingredients of Givental’s formula, most importantly the quantized symplectic operator \( \hat{R}_t \). Proposition 16 shows that we will have to learn how to commute this operator past the vertex operators \( \Gamma^\alpha \) at a semi-simple point \( t \in T \). Now we have a description of the relation between the period vectors and \( \hat{R}_t \), given in (11.5) and (11.8). However this relates only the period vectors corresponding to the \( n \) vanishing cycles \( \beta_i \). Thus we will first need to write the vertex operators corresponding to the one-point cycles \( \alpha \) in terms of the vertex operators corresponding to the vanishing cycles. We will do this explicitly in the next section. In this section we will stick to the general case of a cycle that can be written as the sum of a vanishing cycle and a monodromy invariant cycle. We will also discuss the commutation of \( \hat{R}_t \) with vertex operators corresponding to vanishing cycles.

So, let \( \beta \) be a vanishing cycle vanishing at the point \((u, t)\) and let \( \alpha' \) be a cycle that is invariant under the monodromy group around this point. Now define the cycle \( \alpha \) by

\[ \alpha := (\alpha, \beta)_{\frac{\beta}{2}} + \alpha'. \] (13.13)

Note that we have actually just defined \( \alpha = c_{\frac{\beta}{2}} + \alpha' \) and by virtue of the normalization of \( (\beta, \beta) \) this yields automatically \( c = (\alpha, \beta) \). Then we have the following proposition.

Proposition 17

\[ \Gamma_\alpha^\beta = e^{K} \Gamma_{\alpha'}^{\frac{\beta}{2}} \Gamma^\frac{\beta}{2}, \] (13.14)

where \( K = (\alpha, \beta) \int_{t-\lambda}^{t-u} W_{\frac{\beta}{2}, \alpha'}. \)

Proof:
We will follow the proof set out in [2]. Denote \( h = \sum_{k \in \mathbb{Z}} I^{(k)}(z) \), \( f = (\alpha, \beta) \sum_{k \in \mathbb{Z}} I^{(k)}(-z)^k \), \( g = \sum_{k \in \mathbb{Z}} I^{(k)}_{\alpha'}(-z)^k \) and define \( f_\pm, h_\pm \) and \( g_\pm \) as in (13.3). Then we have

\[
\Gamma_f^\alpha = e^{h - h_+} = e^{g - g_+} = e^{[\hat{f}_-, g_+]} e^{\hat{f}_+} = e^{\Omega(f, g_+)} \Gamma_f^\alpha, 
\]

since the quantization is a representation of the Heisenberg Lie algebra for linear Hamiltonians as mentioned before. So all that is left is to show that \( K = \Omega(f_-, g_+) \). Clearly we have

\[
\Omega(f_-, g_+) = (\alpha, \beta) \sum_{k \geq 0} (-1)^k < I^{(-1-k)}_{\alpha}, I^{(k)}_{\alpha'} >.
\]

Meanwhile we find by repeated integration by parts that

\[
\int_u^\Lambda < I^{(0)}_{\alpha}, I^{(0)}_{\alpha'} > d\xi = \int_u^\Lambda \frac{1}{2} I^{(-1)}_{\alpha}, I^{(0)}_{\alpha'} > d\xi = \ldots = \sum_{k=0}^{m-1} (-1)^k < I^{(-1-k)}_{\alpha}, I^{(k)}_{\alpha'} > |_{\alpha'}^\Lambda - \int_u^\Lambda < I^{(-1-k)}_{\alpha}, I^{(k)}_{\alpha'} > d\xi, 
\]

for all \( m \in \mathbb{N} \). Now note that, since \( \alpha' \) is invariant under the monodromy at the point \( u \), \( I^{(k)}_{\alpha'} \) is holomorphic at this point for all \( k \in \mathbb{Z} \). Additionally we see from equation (10.12) that \( [I^{(-1-k)}_{\alpha}]^k \sim (\lambda - u)^{k+1/2}(1 + \ldots) \). These two observations show that the integral in (13.15) is \( o((\lambda - u)^{m-1/2}) \) so it tends to 0 as \( m \) tends to \( \infty \). Now note that the second observation also shows that \( I^{(-1-k)}_{\alpha}(u,t) = 0 \). So we find that

\[
K = (\alpha, \beta) \int_u^\Lambda < I^{(0)}_{\alpha}, I^{(0)}_{\alpha'} > d\xi = (\alpha, \beta) \int_{(u,t)}^{(\Lambda,t)} < I^{(0)}_{\alpha}, I^{(-1)}_{\alpha'} > -\tilde{W}^{a}_{\alpha, \alpha'}, 
\]

which proves the proposition.

Now let us turn our attention to commuting \( \hat{\Psi} R_t e^{\hat{f}} \) past the vertex operators corresponding to the vanishing cycles. As in the case of the lower triangular operator we have for \( N = \Psi R_t e^{\hat{f}} \) that \( \hat{N}^{-1} e^{\hat{f}} \hat{N} \) is proportional to \( e^{N^{-1} t} \), where \( f \) is as in (13.2). So let us start by finding this exponent.

**Theorem 18** Near \( \lambda = u \), we have

\[
\sum_{k \in \mathbb{Z}} (-z)^k I^{(k)}_{\alpha'} = \Psi R_t e^{\hat{f}} \sqrt{-\Delta} \partial_\alpha \sqrt{2} \sum_{k \in \mathbb{Z}} (-z)^k (\frac{d}{d\lambda})^k \lambda^{-1/2}, 
\]

for all \( i = 1, \ldots, n \). In the coordinate that expresses the nature of \( R_t \).

**Remark 8** Similar to theorem 15 we need an explanation of how to view these infinite series. In fact this remark is the same as in theorem 15 except that the convergence is now in the \( \sqrt{\lambda - u} \)-adic sense.
Proof:

Every statement in this proof will hold for \( i = 1, \ldots, n \). We have in fact done most of the work to prove this theorem already in deriving (11.5) and (11.8). This will be apparent once we deal with the right most part of equation (13.17).

Consider the series \( \sum_{k \in \mathbb{Z}} z^k f^{(k)}_{\beta_i} (\lambda + u_i) \), where we omit the other coordinate \( t \).

Taylor expanding this around \( \lambda \) near \( u_i \) yields

\[
\sum_{k \in \mathbb{Z}} z^k f^{(k)}_{\beta_i} (\lambda + u_i) = \sum_{k \in \mathbb{Z}} z^k \left( f^{(k)}_{\beta_i} (\lambda) + f^{(k+1)}_{\beta_i} (\lambda) u_i + \frac{f^{(k+2)}_{\beta_i}}{2!} u_i^2 + \ldots \right)
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{u_i^n}{n!} z^n \right) \left( \sum_{k \in \mathbb{Z}} z^k f^{(k)}_{\beta_i} (\lambda) \right) = e^{u_i} \sum_{k \in \mathbb{Z}} z^k f^{(k)}_{\beta_i} (\lambda). \tag{13.18}
\]

Now the equations (11.5), (11.8) and (13.18) simply amount to (13.17). \( \square \)

Now let us turn to the constant of proportionality. Note that we need not worry about the \( \Psi \) part in this case, since it is simply a basis transformation. We will treat first the constant of proportionality corresponding to \( \check{\hat{R}}_t \). Just as in (13.4) we find from (7.28) that

\[
\check{\hat{R}}_t^{-1} e^{t \check{\hat{R}}_t} = e^{-\frac{1}{4} V(f_-, f_-)} e^{t \check{R}_t}, \tag{13.19}
\]

where \( f \) and \( f_- \) are as in (13.3). So we will need to compute \( V(f_-, f_-) \) for

\[
f_- = \sum_{k \geq 0} f^{(-k-1)}_{\beta_i} \partial_{q_k}.
\]

We will use a method comparable to the one which was used to determine \( W(f_+, f_+) \). First of all note that

\[
\partial_{\lambda} V(f_-, f_-) = \sum_{k, l \geq 0} < f^{(-k)}_{\beta_i} \partial_{q_k}, [V_{(k-1)} t + V_{(l-1)}] f^{(-l)}_{\beta_i} \partial_{q_l}> = < f^{(0)}_{\beta_i}, f^{(0)}_{\beta_i} > - \sum_{k \geq 0} R_k^T f^{(-k)}_{\beta_i} \sum_{l \geq 0} R_l^T f^{(-l)}_{\beta_i}, \tag{13.20}
\]

where we used the upper left identity of lemma 12 and equation (7.27). Now let us fix a vanishing cycle \( \beta_i \). Then we can use theorem 18 to see that

\[
\sum_{k \geq 0} R_k^T f^{(-k)}_{\beta_i} = \sum_{k \geq 0} R_k^T \sum_{l \geq 0} (-1)^l R_l \left( \frac{d}{d\lambda} \right)^{-l-k} 2 \sqrt{-\Delta} \partial_{u_i} \frac{2 \sqrt{-\Delta} \partial_{u_i}}{\sqrt{2(\lambda - u_i)}} = \frac{2 \sqrt{-\Delta} \partial_{u_i}}{\sqrt{2(\lambda - u_i)}}, \tag{13.21}
\]

where we also used the condition in (7.8).

Now we can expand the vector coefficients \( f_{-k} = f^{(-1-k)}_{\beta_i} \) using (10.12) to find that \( |f_{-k}| \sim (\lambda - u_i)^{k+\frac{1}{2}} (1 + \ldots) \). So we see that at \( \lambda = u_i \) we have \( V(f_-, f_-) = 0 \). Thus we find finally that

\[
V(f_-, f_-) = \int_{u_i}^{\lambda} \left( < f^{(0)}_{\beta_i}, f^{(0)}_{\beta_i} > - \frac{2}{\xi - u_i} \right) d\xi =
\]

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\[
\int_{t-u_1}^{t-u_i} \left( W_{\beta_i} - \frac{2dt_n}{\tau_n - u_i - t_n} \right).
\]

(13.22)

Now we have found all the tools to commute \( \hat{\Psi} \hat{R}_t e^{\hat{U}} \) past the vertex operators corresponding to vanishing cycles.

**Proposition 19** For all vanishing cycles \( \beta_i \), with \( i = 1, \ldots, n \), we have

\[
(\hat{\Psi} \hat{R}_t e^{\hat{U}})^{-1} e^{-\frac{1}{4} \sum_i \Gamma_+^i (\hat{\Psi} \hat{R}_t e^{\hat{U}})} = e^{-\frac{1}{4} \epsilon_i (I \otimes (\Gamma_+^i)_{(i)} \otimes I \ldots)},
\]

(13.23)

where we use the notation of [2]. Thus the \((i)\) means the \( i \)th position in the tensor product. Meanwhile every \( j \)th position only acts on those function dependent on \( q_j \), for all \( j = 1, \ldots, n \). Furthermore

\[
V_i = \int_{t-u_i}^{t-u_1} \left( W_{\beta_i} + \frac{dt_n}{2(\tau_n - u_i - t_n)} \right)
\]

(13.24)

and

\[
v_i = \int_{\lambda - u_i}^{\lambda} \frac{d\xi}{2\sqrt{\xi}}.
\]

(13.25)

**Proof:**

Note that, as with proposition 16, most of the work has already been done. This time we still need to figure out the commutation of the quantized operator \( e^{\hat{U}} \). Note however that this is just a special case of proposition 16. In this case we have the effect that \( \sqrt{-\Delta_i/\partial u_i} \) is transformed into \( \sqrt{-\Delta_i/\partial u_i} \). While the corresponding phase factor is equal to \( \frac{\lambda - u_1}{\lambda} = e^{-2v_i} \). Now the branch of the square root is specified by the path taken in \( v_i \). Then combining this with (13.17), (13.19) and (13.22) yields (13.23).

\[\square\]

14 Ancestor Potential and Regularity

We have assembled all the ingredients of Givental’s formula and have figured out the commutation with vertex operators. All that is left to do is show the regularity of (8.18) with \( D \) as in (9.1). To show this we will introduce the ancestor potential as in [13]. This will show us how to deal with the invertible matrix \( C(t) \) and how to implement the propositions 16 and 19. The proof of regularity then consists merely of dealing with the phase factors.

**Ancestor Potential**

We have assumed the solution to the \((n + 1)\)KdV hierarchy to be of the form (8.1). Combining this with Givental’s formula Kontsevich and Manin proved [21, 13] that we can rewrite the total descendant potential \( D \) in terms of the ancestor potential \( \mathcal{A}_t \) at a point \( t \in \mathcal{T} \) as follows

\[
D = e^{F^t(\delta)} \hat{S}_t^{-1} \mathcal{A}_t.
\]

(14.1)
Here $A_t$ is of the form

$$A_t(h, t) := \exp\left\{\sum_{g=0}^{\infty} \hbar^g - 1 F^g_t\right\}, \quad (14.2)$$

where $F^g_t$ is the genus $g$ ancestor potential [13]. Now we see this ancestor potential as an element of the Fock space by using again the construction in (8.6). Furthermore $F^1(t)$ is the genus 1 Gromov-Witten potential. In staying with our hands on approach let us merely state what these concepts mean in our specific case. We can see from (9.1) that (14.1) is a reasonable restatement and the exponent is related to the invertible matrix $C(t)$.

To be completely clear of all the ingredients of Givental’s formula note that this invertible matrix is related to the $R_t$ matrix. Up to a constant it is defined by

$$C(t) := \exp\left\{\frac{1}{2} \int \sum_{i=1}^{n} R^i_i(u) du\right\}. \quad (14.3)$$

In fact this factor is needed to compensate for the cocyle $C$ arising from the quantization procedure to make the entire expression $D$ independent of the point $t$ used in the definition [13]. Now we can simply define

$$F^1(t) := \frac{1}{48} \sum_{i=1}^{n} \ln \Delta_i(t) + \ln C(t), \quad (14.4)$$

so that we have the expression (14.1) with

$$A_t(q) = \hat{\Psi}(t) \hat{R}_t e^{\hat{U}_z n \prod_{i=1}^{n} \Delta A_1(q_i) \Delta_t^{-\frac{1}{12}}} \hat{U}_z n. \quad (14.5)$$

This is desirable since it was shown in [22, 15] that the function $F^1$ is constant, thus we won’t have to worry about it at all.

This means that we can rewrite the condition (8.18) in terms of the ancestor potential using the commutation of $\hat{S}_t$ past the vertex operators. Before we do so however let us note one more fact about the phase factors this produces. Note that

$$\int_{-\lambda}^{-\lambda} W_{\alpha} = \int_{1}^{\lambda} \frac{-1}{n+1} \sum_{i=1}^{n} x^{i-1} x^{n-i} dx^n dx^{n+1} = -n \frac{n}{n+1} \ln \lambda, \quad (14.6)$$

for $\alpha$ in the level where $x^{n+1} = -\tau_n$. Here we simply used the definition of the phase form and the period vectors. Now we see by the remark about proportionality made in section 12 that for these cycles $\frac{n}{n+1} = (\alpha, \alpha)$. Thus in the expression (8.18)

$$\lambda^{-\frac{n}{n+1}} = \lambda^{-(\alpha, \alpha)}.$$  

Using the Ancestor potential we can rephrase (8.18) in the following way

$$\sum_{\alpha} \lambda^{-(\alpha, \alpha)} d\lambda (G^{-\alpha} D)(q')(G^\alpha D)(q^n) = \ldots$$
\[ e^{2F(t)} \hat{S}_t^{-1} \sum_{\alpha} \lambda^{-(\alpha,\alpha)} e^{\int_{t}^{1-\lambda} W_{\alpha} + (\alpha,\alpha) f_1^t \frac{d\xi}{\xi} \lambda} \, d\lambda (\Gamma_t^{-\alpha} A_t)(q')(\Gamma_t^\alpha A_t)(q'') = \] 

note that the quantized symplectic operator \( \hat{S}^{-1} \) acts on both terms (in Givental’s notation one could write \( \hat{S}^{-1} \otimes \hat{S}^{-1} \)).

**Regularity**

Now we are ready to prove the regularity of \( D \) in the sense of (8.18). So this section will be devoted to proving the following theorem.

**Theorem 20** For \( D \) as in (9.1) the one-form

\[ \left[ \sum_{\alpha} \Gamma_0^\alpha \otimes \Gamma_0^{-\alpha} \lambda^{-(\alpha,\alpha)} d\lambda \right] D \otimes D \]  

is regular in \( \lambda \). Here the tensor product is meant as in [2], i.e. the left slot corresponds to the variable \( q' \) and the right one to \( q'' \). Furthermore the one-point cycles \( \alpha \) are as in (8.18).

**Proof:**

We will follow the proof set out in [2]. Note that due to the previous discussion we can rephrase the theorem as follows. The expression (14.8) is regular in \( \lambda \) if and only if

\[ \left[ \sum_{\alpha} \Gamma_t^{-\alpha} \otimes \Gamma_t^\alpha e^{\int_{t}^{1-\lambda} W_{\alpha} + (\alpha,\alpha) f_1^t \frac{d\xi}{\xi} \lambda} \lambda^{-(\alpha,\alpha)} d\lambda \right] A_t \otimes A_t \]  

is regular in \( \lambda \) for \( t \in T \) semisimple. Note that this means it is immediately regular for all semisimple points in \( T \). Now since the one-point cycles \( \alpha \) form an orbit under the monodromy group we see that he expression (14.9) is invariant under this monodromy. Then according to the discussion in [2] with regard to the elements of the Fock space (see also Appendix B) we merely have to prove that the meromorphic (in \( \lambda \)) form (14.9) has no poles. The function has possible poles at the critical points. Thus the theorem will we proved by showing that there are in fact no poles at the distinct critical points \( u_1, \ldots, u_n \). These are given by the canonical coordinates for the point \( t \in T \).

So let us fix a critical value \( \lambda = u_i \). There will be two one-point cycles \( \alpha_{\pm} \) such that \( \beta_i = \alpha_++\alpha_- \). Then note that all the other one-point cycles \( \alpha \) in (14.9) are invariant under the monodromy around \( u_i \) and thus the corresponding period vectors and phase factors will be holomorphic at the point \( \lambda = u_i \). So we will only need to worry about the summands concerning the cycles \( \alpha_{\pm} \) in (14.9).

Denote \( \alpha' = \frac{\alpha_{+} - \alpha_-}{2} \), then \( \alpha' \) is clearly invariant under monodromy around \( u_i \). Note that

\[ \alpha_{\pm} = \pm \frac{\beta_i}{2} + \alpha', \]  

(14.10)
which shows that $I^{(k)}_{n,k} = I^{(k)}_{n,k} \pm I^{(k)}_{n,k}$ for all $k \in \mathbb{Z}$. Where again $I^{(k)}_{n,k}$ is holomorphic at $\lambda = u_i$ for all $k \in \mathbb{Z}$. Overmore we can use proposition 17 to write

$$
\Gamma^{\alpha'}_t = e^{\pm K_i \Gamma^{\alpha'}_t \frac{1}{2}},
$$

(14.11)

where $K_i = \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\alpha'}$. Then the summands in (14.9), that are not necessarily holomorphic at $\lambda = u_i$, add up to

$$
\Gamma^{-\alpha'}_t \otimes \Gamma^{\alpha'}_t \left[ \left( \sum_{\pm} L_{\pm}(\lambda) \Gamma^{\pm \frac{1}{2}}_t \otimes \Gamma^{\pm \frac{1}{2}}_t \right) \mathcal{A}_2 \otimes \mathcal{A}_t \right] d\lambda,
$$

(14.12)

where the functions $L_{\pm}$ are the combination of phase factors from (14.9) and (14.11).

Now we are in a position to apply proposition 19, by virtue of (14.5). Note that in the tensor product notation of proposition 19 we now get simply some factor of holomorphic functions $\Gamma^{(i)}_t \mathcal{D}_A_i(q_j^i) \mathcal{D}_A_i(q_j^i)$ for $j \neq i$ and the possibly non-holomorphic factor is that which corresponds to the variable $q_i$. Namely

$$
\sum_{\sqrt{\lambda} = \pm \xi} l_{\pm}(\lambda) \frac{d\lambda}{\sqrt{\lambda}} [\Gamma^{-\mathcal{D}_A_i(q_j^i)}] [\Gamma^{+\mathcal{D}_A_i(q_j^i)}],
$$

(14.13)

where the functions $l_{\pm}$ are given by the phase factors $L_{\pm}$, those coming from proposition 19 and the extraction of $\frac{1}{\sqrt{\lambda}}$. Thus the theorem will be proved if we can show that the phase factors $l_{\pm}$ are identical and holomorphic, since in that case (14.13) is regular by Witten’s conjecture.

So finally we turn our attention to the accumulated phase factors. We have

$$
\ln l_{\pm} = \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} + (\alpha_{\pm}, \alpha_{\pm}) \int_1^{\lambda} \frac{d\xi}{\xi} - \ln \lambda^{(\alpha_{\pm}, \alpha_{\pm})} + \ln \sqrt{\lambda} - \int_1^{\lambda} \frac{d\xi}{2\xi} \pm 2 \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm} - \frac{dt_n}{2(\tau_n - u_i - t_n)} - \int_1^{\lambda_{-u_i}} \frac{d\xi}{2\xi},
$$

(14.14)

Here the first two summands come from the commutation of $\hat{S}^{-1}_t$, the third summand comes from equation (8.18) and (14.6), the fourth summand comes from the extraction of $\frac{1}{\sqrt{\lambda}}$ to get (14.13), the fifth and last two summands come from the commutation of $\hat{\Psi}_R e^{\frac{\xi}{2}}$ and finally the sixth summand comes from (14.11). Note that

$$
\mathcal{W}_{a,\pm} = \mathcal{W}_{a,\pm} + \mathcal{W}_{a,\pm} \pm 2 \mathcal{W}_{a,\pm},
$$

(14.15)

by (14.10). So we have

$$
\ln l_{\pm} = \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm} + \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm} + \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm} + \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm} + \int_{t_{-1}}^{t_{-1}+1} \mathcal{W}_{a,\pm} \mathcal{W}_{a,\pm},
$$

(14.16)
where the contours $\gamma_{\pm}$ and $\gamma'_{\pm}$ come from combining the second and third respectively the fourth and fifth summands in (14.14). Apart from the last term the other terms are caused by applying (14.15) and changing the endpoints. The constant $u_1 + 1$ is chosen exactly such that the last two integrals cancel each other. Now note that the contours $\gamma_{\pm}$ and $\gamma'_{\pm}$ as well as the fourth through sixth summands may depend on the cycle $\alpha_{\pm}$, but they do not depend on $\lambda$. Thus all the $\lambda$ dependence comes from the first summand. Note however that here the cycle $\alpha'$ is involved, which is invariant under the monodromy around $u_1$, and therefore the first term is holomorphic at $\lambda = u_i$. Thus we see that the phase factors $l_{\pm}$ are at least holomorphic. So all that is left to show is that they coincide near $\lambda = u_i$.

So let us find the constant of proportionality between $l_+$ and $l_-$. We have

\[
\ln l_+ - \ln l_- = (\alpha_-, \alpha_-) \oint_{\gamma_{\pm} - \gamma_-} \frac{d\xi}{\xi} + 4 \int_{-1}^{1} W_{\frac{2\lambda}{\pi}, \alpha'} + \oint_{\gamma_{\pm} - \gamma_-} \frac{d\xi}{2\xi} =
\]

\[
= (\alpha_-, \alpha_-) \oint_{\gamma_{\pm} - \gamma_-} \frac{d\xi}{\xi} + \int_{-1}^{1} (W_{\alpha_+} - W_{\alpha_-}) + \oint_{\gamma_{\pm} - \gamma_-} \frac{d\xi}{2\xi},
\]

where we used that $4W_{\frac{2\lambda}{\pi}, \alpha'} = W_{\alpha_+} - W_{\alpha_-}$, since $\alpha' = \frac{\alpha + \alpha_-}{2}$ and $\beta = \frac{\alpha + \alpha_-}{2}$.

Suppose $\gamma_3$ is a loop around $\lambda = 0$ that makes exactly enough turns inside the line $-\lambda \mathbf{1}$ such that $\alpha_-$ transported along this loop becomes $\alpha_+$. This loop exists since $\alpha_+$ and $\alpha_-$ belong to the same orbit of the classical monodromy group.

Then by (12.9) the first summand in (14.17) equals $\oint_{\gamma_3} W_{\alpha_-}$.

For $\epsilon \in \mathbb{R}_{>0}$ let $\gamma_2(\epsilon)$ be the path starting at $-\mathbf{1}$ and approaching $t - u_i \mathbf{1}$ along the same path as in the second summand, but stopping a (small) distance $\epsilon$ away. Then let $\gamma_3(\epsilon)$ be a loop of size $\epsilon$ that goes around the discriminant $\Sigma$ (along which the middle integral in (14.17) is taken) near $t - u_i \mathbf{1}$. So the cycle $\alpha_+$ become $\alpha_-$ when it is transported along $\gamma_3(\epsilon)$. Thus we see that

\[
\lim_{\epsilon \to 0} \oint_{\gamma_2(\epsilon)\gamma_3(\epsilon)\gamma_1(\epsilon)} W_{\alpha_+} = \int_{-1}^{1} (W_{\alpha_+} - W_{\alpha_-}) + \lim_{\epsilon \to 0} \oint_{\gamma_3(\epsilon)} W_{\alpha_-}.
\]

Now let us write again $W_{\alpha_+} = W_{\frac{2\lambda}{\pi}, \alpha'} + 2W_{\frac{2\lambda}{\pi}, \alpha'} + W_{\alpha'}$. Then near the point $t - u_i \mathbf{1}$ we see from the expansion (10.12) that the first summand contains the term $\frac{d\lambda}{2(\lambda - u_i)}$, while the rest is either analytic at $\lambda = u_i$ or has a singularity of the form $\frac{d\lambda}{\sqrt{\lambda - u_i}}$. So we find that

\[
\lim_{\epsilon \to 0} \oint_{\gamma_3(\epsilon)} W_{\alpha_+} = \oint \frac{d\lambda}{2(\lambda - u_i)} = \pi \sqrt{-1},
\]

where the last contour is around $\lambda = u_i$. So this part coincides with the last summand of (14.17).

Combining these last two paragraphs we see that

\[
\ln l_+ - \ln l_- = \oint_{\gamma_1} W_{\alpha_-} + \oint_{\gamma_2\gamma_3\gamma_1^{-1}} W_{\alpha_+} + 2\pi \sqrt{-1} = 2\pi \sqrt{-1} + \oint_{\gamma_1\gamma_2\gamma_3\gamma_1^{-1}} W_{\alpha_-} =
\]

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for some $a \in \mathbb{Z}$. Where the last equality follows from the fact that $\alpha_-$ is invariant under the monodromy along the loop $\gamma_1 \gamma_2 \gamma_3 \gamma_2^{-1}$ and the remark at the end of section 12, which is proven in proposition 1 of [2]. Note that we have simply chosen some $\epsilon > 0$ since the argument doesn’t actually depend on $\epsilon$. Then we see that by (14.18) we have $l_+ = l_-$ and the theorem is proved. □

15 Conclusion and Discussion

Now we have seen a (reasonably) complete exposition of the relation between integrable hierarchies and Frobenius structures in the case of the $A_n$ singularity. The most interesting question now is which elements of this exposition will be easily translated to other pairs of Frobenius structures and integrable hierarchies. To give an idea of the efforts in answering this question and the further scope of the subject involving Frobenius structures and integrable hierarchies we will quickly summarize the main points made above and point out the $A_n$ singularity specific elements and possible generalizations.

Quick Summary

Let us begin in the situation after section 7. So we already have the quantization formalism in place. Then we see that naively the hierarchy is defined simply by the vertex operators $\Gamma^\alpha$ and the cycles $\alpha$. Certainly if we look at the form (14.8). The vertex operators are simply explained using the period vectors, which are readily available from the Frobenius structure.

To do this however we also need to specify the allowed cycles $\alpha$ and note that in (14.8) we sum over a full orbit of the monodromy group. So we see that we immediately need the miniversal deformation of the singularity. Actually at this point we only need it in the point $0 \in T$, so specifically we need the singularity itself. Then we simply define the cycles using the Milnor fibers $V_{\lambda,0}$.

Note that this in fact constitutes a choice of representative for the singularity. For instance in [2] a different representative is chosen which leads to slightly different definitions considering the vertex operators and quantization, since $<\cdot, \cdot>$ and thus $\Omega$ are affected. Note also that the period vectors are immediately multiple valued and ramified along the discriminant by definition of the one-point cycles $\alpha$.

Next we need to find the operators $R_t$ and $S_t$ and their commutation past the vertex operators. Here we depend heavily on the deformed connection. For a definition of the $R_t$ operator we depend on the oscillatory integrals, which satisfy the system of equations (11.1) and (11.2). Thus we need a specific type of deformed flat coordinates. In fact if we consider the commutation of the $R_t$ matrix past the vertex operators corresponding to vanishing cycles, we see that the crucial fact is given in the Laplace transform (11.4) and the expansion (10.12). In fact most of the argument of theorem 20 comes down to these two relations, since this allows us to split the $(n+1)$KdV hierarchy into $n$ copies of the KdV hierarchy.
Note that the two main ingredients defining the oscillatory integrals (and thus the asymptotical approximation which yields \( R_t \) and its commutation) are the miniversal deformation \( F \) and the basis of cycles \( \{ B_i \}_{i=1}^n \). Where we use the miniversal deformation both explicitly in the definition (6.1) and in the definition of the basis of cycles \( \{ B_i \}_{i=1}^n \).

Now the commutation of the quantized symplectic operators is evident considering the remarks above. Although for the commutation of \( \hat{R}_t \) we also rely on the split (14.10) and thus the vanishing cycles. These are defined using the basis of cycles \( \{ B_i \}_{i=1}^n \). The most important factors at this point are then the phase factors and thus the phase form. Here we use again the definition of the period vectors. However we also use the specific properties of the one-point and vanishing cycles.

**Specific and General Elements**

Let us try and pinpoint which elements used to prove theorem 20 are specific to the \( A_n \) singularity. First of all we should note that, although we have consequently used the specific \( A_n \) singularity, the construction is easily generalized to include the other simple singularities [3, 7, 9, 15]. The Frobenius structure arising from these singularities was shown by K. Saito [9] and relies upon the existence of a primitive form \( \omega \) (proved by M. Saito [15]). In the case of the \( A_n \) singularity this primitive form is simply \( dx \).

However let us see which elements can be defined also in the setting of an arbitrary Frobenius structure. Obviously everything that was discussed in the first two sections falls into this category. When we look at the definition of the \( S_t \) operator we see that this merely depends on the deformed connection. Thus it is also readily available in the general structure. Moreover the \( R_t \) operator is available in this sense. However not immediately in the way that we defined, or used, it. For this we would need an analogue of the miniversal deformation to define the oscillatory integrals and the basis of cycles \( \{ B_i \}_{i=1}^n \).

Dubrovin shows in [1] that such an analogue, which he calls superpotential, exists for a certain type of semisimple Frobenius manifold. It is also shown there that the analogue of the period vectors \( I^{(0)} \) can be defined, in the sense of a solution to the equations in lemma 12. It seems then that all the elements are available in such a Frobenius structure, if a suitable primitive form can be found. It would need to be checked whether sufficient cycles \( B_i \) are available and whether the phase form has the same nice properties in the general case. Another object that was used quite a lot in the construction of the vertex operators form of the integrable hierarchies and the proof of theorem 20 is the monodromy group. In the case of the \( A_n \) singularity this simply corresponds to the different solutions to \( F(x, t) = \lambda \). However the monodromy group can also be defined in the case of an arbitrary Frobenius manifold [5].

It seems indeed that the discussion of Forbenius manifolds is deeply intertwined with the discussion of integrable hierarchies. Both theories have many interesting aspects in their own right. However this case of the \( A_n \) singularity and of simple singularities in general does show that there is a very natural and immediate connection between both subjects.
Appendix A

In section 8 it was mentioned that there is some freedom in the way to define the period vectors in the point \(0 \in \mathcal{T}\). In fact this freedom is available at an arbitrary point as well. Mainly we are concerned with the period vectors satisfying the upper right identity in lemma 12. Since we need to define \(I^{(k)}\) for any \(k \in \mathbb{Z}\) we have two options. Either we define a specific superscript, \(I^{(0)}\) seems to make most sense, and a consistent way of taking the anti-derivative \(\partial_{\lambda}^{p}\) for \(p < 0\). Or we proceed as we did in section 10, i.e. by defining explicitly all period vectors with negative superscripts and proving the relation. We have chosen to follow the definitions of [3], in [2] the first option is used. In this Appendix we will briefly summarize this method. This method uses the concepts of stabilization of the singularity [14].

Stabilization

The process of stabilization is in particular a way to fix certain anti-derivates of the period vectors. However it is also a meaningful construction in its own right. The idea is to increase the dimension of the domain of the singularity \(f\) while minimally affecting the structure. For a detailed explanation of stabilization see also [14].

To reiterate let \(f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)\) be the germ of a holomorphic function with an isolated critical point at the origin. A stabilization of \(f\) is the germ of the holomorphic function \(\tilde{f}\) defined as

\[
\tilde{f}(x, y_1, \ldots, y_m) = f(x) + \sum_{i=1}^{m} y_i^2 \quad \text{for some} \quad m \in \mathbb{N},
\]

thus \(\tilde{f} : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)\) has again an isolated singularity at the origin. Moreover the multiplicity of the singularity of \(\tilde{f}\) is equal to that of \(f\). It turns out that most properties of the singularity are either equal or very similar to those properties of the stabilization.

If \(F(x, \tau)\) is the deformation of \(f\) then \(F(x, \tau) + \sum_{i=1}^{m} y_i^2\) is the corresponding deformation of the stabilization. If the degenerate isolated singularity splits into multiple non-degenerate singularities in the deformation then the same will happen in the deformation of the stabilization. Note also that the structure of the local algebra

\[
\mathbb{C}[x, y_1, \ldots, y_n]/(f_x, f_{y_1}, \ldots, f_{y_n})
\]

of the stabilization is essentially the same as the structure of the local algebra of the singularity. The primitive form and residue metric of the stabilization should be obvious.

Anti-derivatives

Now we need to slightly adapt the definition in (8.12) to account for the stabilization. Mainly to take care of the anti-derivatives. To do this note that we can move to a stabilization of the singularity \(f\). So suppose \(k < 0\) then we will
pass to the stabilization with dimension \( m = 2l + 1 \), i.e. we add 2\( l \) variables, such that \(-l < k\). The main reason for considering only an odd dimension is that this means certain intersection forms are equal for the stabilization and the original singularity [14]. Then we have in that case

\[
< I_{\beta}^{(0)}(\lambda), [\phi] > = -\left( \frac{1}{2\pi} \frac{d}{d\lambda} \right)^l \int_{\beta} \phi(x) \frac{dx_1 \wedge \ldots \wedge dx_m}{df(x)},
\]

(15.2)

thus to define the \( I^{(k)} \) for \( k < 0 \) we just pass to the dimension \( m = 2l + 1 \) with \(-l < k\). Now the derivatives will cancel and thus the anti-derivatives are fixed. We include the factor \( \frac{1}{(2\pi)^l} \) to make the left hand side of (15.2) independent of \( l \). Since for \( \beta \) in \( f^{-1}(\lambda) \cap C^m \) and \( f = x_1^2 + x_2^2 + \ldots + x_m^2 \) the \( A_1 \) singularity we get that

\[
\int_{\beta} \frac{dx}{df} = (2\pi)^l (2l - 1)!! \lambda^{l - \frac{1}{2}} [2].
\]

**Appendix B**

In proving theorem 20 and in fact in most arguments made in that section there is an open question. Since we have been discussing a lot of infinite series we need now to adress the meaning of the formulas. This will in fact complete the proof of theorem 20 and as mentioned before the discussion is also seen in section 8 of [2]. We will omit most proofs in this section, they can be found in the references however.

We start by defining the nature of functions such as the Witten-Kontsevich tau-function.

**Definition 6** We define an asymptotical function as an expression of the form

\[
\exp \left\{ \sum_{g \geq 0} h^{g-1} \mathcal{F}^{(g)}(t) \right\},
\]

(15.3)

where \( \mathcal{F}^{(g)} \) is a formal function on the space of polynomials \( t = t_0 + t_1 z + \ldots \) with vector coefficients \( t_k = \sum_{\alpha=1}^{n_k} t^\alpha_k e_\alpha \in H \) for all \( k \geq 0 \). Here \( \{e_\alpha\}_{i=1}^{n} \) forms an orthonormal basis of \( H \).

Then we will call such an asymptotical function *tame* if

\[
\partial_{t_{k_1}} \ldots \partial_{t_{k_r}} \mathcal{F}^{(g)}|_{t=0} = 0,
\]

(15.4)

whenever \( k_1 + \ldots + k_r > 3g - 3 + r \). Particularly we note that the Witten-Kontsevich tau-function is tame. This can be seen by the definition (not included above)

\[
\mathcal{T}(t) = \exp \left\{ \sum_{g \geq 0} h^{g-1} \sum_{m \geq 0} \frac{1}{m!} \int_{\mathcal{M}_{g,m}} t(\psi_1) \wedge \ldots \wedge t(\psi_m) \right\},
\]

where \( \mathcal{M}_{g,m} \) is the (compact) Mumford-Deligne moduli space of genus \( g \) stable complex curves with \( m \) marked points and \( \psi_i \) for \( i = 1, \ldots, m \) are the psi-classes, i.e. the first Chern classes of the universal cotangent line bundles. Since we have \( \text{Dim}_C \mathcal{M}_{g,m} = 3g - 3 + m \). To such an asymptotical function we associate an
(asymptotical) element in the Fock space via the dilaton shift $q(z) = t(z) - z$ as we did before. This is then an asymptotical function of $q$ around the shifted origin $q = -z1$ for $1 \in \mathbb{H}$ the unit.

Now to use this tameness of functions in the proof of theorem 20 we will have to see what it has to do with the vertex operator form of the $(n+1)$KdV hierarchies. So we refer to the vertex operators form (8.18) of the $(n+1)$KdV hierarchies. Suppose $\mathcal{G}$ is an asymptotical function. Then we see that, by virtue of the change of coordinates as given in (8.3) (after switching back to $x'$ and $x''$) we have

$$\exp \left\{ 2 \sum_{k \geq 0} < \sigma_{\beta}^{-1-k}, \frac{q_k}{\sqrt{\hbar}} > \right\} \prod_{k \geq 0} \left( -1 \right)^k \left( \frac{q_k}{\sqrt{\hbar}} \right)^{\beta} \mathcal{G}(x+q)\mathcal{G}(x-q),$$

for suitable $\beta$ (and we favor the notation $q$ over $y$). Here we follow roughly the reasoning which arrived us at (8.4). Using the genus expansion (15.3) we can rewrite (15.5) as

$$\exp \left\{ 2 \sum_{k \geq 0} < \sigma_{\beta}^{-1-k}, \frac{q_k}{\sqrt{\hbar}} > + \sum_{g \geq 0} h^{g-1} \sum_{\chi} \mathcal{F}^{(g)}(\chi) \right\},$$

(15.6)

where $$\chi = x \pm q \mp \sqrt{\hbar} \sum_{k \geq 0} \sigma_{\beta}^{(k)}(-z)^k.$$

Now the crucial proposition, which is proved by Givental in [2] using the above expression and the tameness condition, is the following.

**Proposition 21** Suppose $\mathcal{G}$ as in (15.3) is a tame asymptotical function of $x$. Then the expression (15.6) divided by $\exp \left\{ 2 \mathcal{F}^{(0)}(x) \right\}$ expands into power series in $\hbar, x$ and $\frac{q}{\sqrt{\hbar}}$ whose coefficients depend polynomially on finitely many period vectors.

This allows us to view the regularity condition of (8.18) as a condition of single-valuedness of the terms involving the period vectors (including those that are in the phase forms). This is what is meant by the comments in the beginning of the proof of theorem 20.

This would conclude the proof as long as $D$ is in fact a tame asymptotical element of the Fock space. Or in fact we need merely tameness of the asymptotical element $\mathcal{A}_t$ for some semisimple $t \in \mathcal{T}$. Using a construction based on summing over connected graphs [2, 19] it is proved that for tame asymptotical functions $\mathcal{G}$ we have that $\hat{R}^{-1}\mathcal{G}$ is again tame asymptotical for symplectic upper triangular operators $R$.

Since we need only the tameness of $\mathcal{A}_t$ we need not worry about the lower triangular operator $S_t$. However there is another lower triangular operator included in the definition of $\mathcal{A}_t$. Namely the operator $\exp \left\{ \frac{\hat{U}}{\sqrt{\hbar}} \right\}$. In general operators of this form do not behave as nicely as the upper triangular operators. Thus we will need the $n$ copies of the (dilaton shifted) Witten-Kontsevich tau-function
to be \( \exp\left(\frac{\hat{U}}{z}\right) \)-stable. Here a tame asymptotical function \( \mathcal{G} \) is called \( T \)-stable if \( T\mathcal{G} \) is again tame. We have this however since we have

\[
\frac{1}{z} \mathcal{D}_{A_1} = 0
\]

by virtue of the Witten-Kontsevich tau-function satisfying the string equation [11]. This shows that, since every one of the \( n \) factors of \( \mathcal{D}_{A_1} \) get acted on by \( \exp\left(\frac{\hat{U}}{z}\right) \) for \( u \) the corresponding critical value, each of the \( n \) factors of \( \mathcal{D}_{A_1} \) in (9.1) gets preserved and thus \( \mathcal{A}_t \) is tame.
References


