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Coalgebraic geometric logic

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Abstract Coalgebras are category-theoretic constructions which can be used to model a wide variety of phenomena. Logic is used as a tool for reasoning about properties of coalgebras. In this thesis, we briefly review some known facts about coalgebraic logic and geometric logic, whereupon we develop coalgebraic geometric logic for coalgebras whose state spaces are topological spaces. We define and investigate notions of equivalence and behaviour: various notions bisimulation, modal equivalence and behavioural equivalence. Furthermore, we give a method of lifting a set functor together with a collection of predicate liftings to a sober functor (and a set of *open* predicate liftings for this new functor). Throughout, we connect results with the guiding examples of monotone logic and conditional logic.

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Contents

1

Introduction

The content of this thesis lies on the intersection of coalgebra, topology and modal logic.

A $\mathbb{T}$-coalgebra is a pair (X, γ) where $\mathbb{T}$ is an endofunctor on some category C, X is an object in C and $\gamma : X \to \mathbb{T}X$ is a morphism in C. The object X is also referred to as the state space, γ as the structure map and the category C is called the base category of the coalgebra (X, γ). Intuitively, the functor $\mathbb{T}$ captures the possible outcomes of the structure map applied to a state, and the structure map γ describes the dynamics of the coalgebra (X, γ).

Coalgebras come with their own generic notions of morphisms, bisimilarity and behaviour and hence are useful to reason about notions related to behaviour and observational indistinguishability. Although the definition of a coalgebra may seem rather abstract, they model a wide variety of structures.

One of the simplest structures which can be described as a coalgebra is a Kripke frame [34, 10]. Kripke frames correspond one-to-one with coalgebras for the powerset functor $\mathbb{P}$ on Set, the category of sets and functions [1]. Moreover, the standard notion of a morphism between Kripke frames corresponds precisely to $\mathbb{P}$-coalgebra morphisms and the coalgebraic definition of a bisimulation coincides with the recognised definition of bisimulation from modal logic. Even in this simple setting where the base category is Set, many other familiar structures can be viewed as coalgebras as well. Among these are transition systems, non-wellfounded sets and deterministic automata [3, 48]. Set-based coalgebras are also called systems and are well researched [49, 28, 48, 27].

The generality of the theory of coalgebra allows for results which are uniform in $\mathbb{T}$. Results on the level of coalgebra can then be applied to structures corresponding to a particular choice of $\mathbb{T}$. Besides, logic can be used as a tool for reasoning about properties of coalgebras, such as bisimilarity. Moss was the first to generalise the concept of modal logic from Kripke frames and models to coalgebraic logic for arbitrary set-based coalgebras [43]. He used so-called relation liftings to define modal operators for propositional logic. This triggered much more research in the area [24, 25, 47, 35, 21, 50, 22]. In [47] Pattinson introduces a different method for defining modal operators, namely via predicate liftings. Coalgebraic logic for set-based coalgebras has been well investigated and is still an active area of research [55, 56, 14, 38].

Coalgebras where the state space is not a set are also useful in modelling various phenomena. For example, trace semantics for non-deterministic automata and context-free grammars can be obtained by modelling these systems as coalgebras over the category of sets and relations [23, 27], and the descriptive frames of modal logic are coalgebras over the category Stone of Stone spaces and continuous functions [37]. Some advances have been made on coalgebraic logic for coalgebras whose underlying spaces are Stone spaces [16, 9]. Research about (logics for) coalgebras over arbitrary topological spaces is more scarce, as observed in [44].

In this thesis we investigate coalgebras whose state space is a topological space, but not necessarily a Stone space. That is, we let the base category be some full subcategory of Top, the category of topological spaces and continuous functions. Firstly, this is motivated by mathematical curiosity. For instance, the aforementioned descriptive frames are coalgebras for the Vietoris functor on Stone. But really, the Vietoris functor is defined on the full category Top and can be restricted to Stone. This raises the question what coalgebras look like for this definition of the Vietoris functor. Of course, there are many more functors on Top whose coalgebras might be of interest. Secondly, in [44] it is suggested that coalgebras over KHaus, the category of compact Hausdorff spaces and continuous functions may be useful in economic theory.

The clopen sets of a Stone space are a subbase for the topology. Moreover, they form a Boolean algebra, hence they are closed under taking complements, finite intersections and finite unions. Therefore, if we use the clopen sets as the interpretants for propositional statements, the logic used to study Stone coalgebras, should contain negation, conjunction and disjunction. Since the empty set serves as a bottom element, this logic is just classical propositional logic. The method of predicate lifting then allows one to define additional modal operators.

However, as soon as one leaves the realm of set- and Stone-based coalgebras, classical propositional logic seizes to be a suitable logic to build upon. This is, in part, due to the fact that the open sets of a topological space, which are a natural choice of the interpretants of propositional statements, are not generally closed under taking complements. Therefore, the coalgebraic logic used to study coalgebras over arbitrary topological spaces must be based on some language without negations. An immediate question which arises is what logic we should use to build *coalgebraic logic* on for these coalgebras.

One of the natural candidates for this logic is *geometric logic*. The language of geometric logic is constructed from a set of propositional statements, arbitrary disjunctions and finite conjunctions [59, 60, 61]. We will see that geometric logic can be viewed as the *logic of finite observations*. Formulas of geometric logic can be interpreted in the frame of open sets of a topological space. There is a duality between the category of spatial frames and homomorphisms and the full subcategory of Top whose objects are sober spaces, which is central to the theory of geometric logic [62].

We modify the method of predicate lifting [47] and use it to define modal operators for geometric logic, which can then be interpreted in models based on coalgebras with a topological space as state space. The duality between sober spaces and spatial frames allows us to view problems from different perspectives. For example, the dual perspective of an enfunctor $\mathbb{T}$ on Sob, the category of sober spaces and continuous functions, gives rise to a concrete construction of a final coalgebra in Coalg($\mathbb{T}$).

The aim of this thesis is to develop a theoretical framework of coalgebraic geometric logic. In particular, we investigate the notions of open predicate liftings, geometric models, modal equivalence, bisimilarity and behavioural equivalence.

Outline of the thesis Chapter 2 sets the stage for the rest of the thesis; we fix notation and introduce formally the structures that will play a role in later chapters. It starts with the definition of a coalgebra, concrete examples of coalgebras, and the notions of coalgebra bisimulation and behavioural equivalence. Subsequently, in section 2.2, we define models over set-based coalgebras and we introduce coalgebraic logic which can be interpreted on these models. Furthermore, a different notion of bisimilarity, Λ-

bisimilarity, is presented. In section 2.3 we change the base category of interest to the category of Stone spaces and continuous functions. We generalise Λ-bisimilarity to a notion of bisimilarity between models over Stone coalgebras and investigate how this relates to modal equivalence and behavioural equivalence.

In chapter 3 we focus on coalgebras whose base category is a full subcategory of Top. We start with the definition of geometric logic and thereafter define so-called open predicate liftings and the *coalgebraic* geometric logic induced by a set of open predicate liftings. The models that we use to interpret this logic are coalgebras with a valuation. We investigate the relation between modal equivalence and behavioural equivalence between such models. Subsequently, we study a concrete example of the functor $\mathbb{D}_{\mathsf{kh}}$, the monotone functor on KHaus, and provide a dual description of this functor in terms of frames, called $\mathbb{M}_{\mathsf{kr}}$. Finally, in section 3.4, we define Λ-bisimulations and see how this relates to modal equivalence and behavioural equivalence.

Chapter 4 is devoted to lifting endofunctors from Set to other categories. In section 4.1 we show how one can lift a set functor together with a set of predicate liftings to a sober functor, i.e. an endofunctor on the category of sober spaces and continuous functions. We show that lifting the powerset functor and the monotone functor on Set together with the usual set of predicate liftings yields the Vietoris functor and the monotone functor on KHaus, respectively. The content of section 4.2 is similar to that of section 4.1, but we lift functors to Stone instead. In section 4.3, a different method for lifting a set functor to a Stone is given. We show that this coincides with the method from section 4.2. This provides a partial solution to a question raised in the conclusion of [36].

The final chapter of this thesis is a case study, where we put the developed theory into practice. In section 5.1 we define so-called *descriptive conditional frames*. These generalise conditional frames in the same manner descriptive general frames generalise Kripke frames. We show that descriptive conditional frames are coalgebras for a certain functor $\mathbb{C}_{\mathsf{st}}$ on Stone, which arises as the lift of the conditional functor on Set. Moreover, we provide an endofunctor on BA, the category of Boolean algebras and homomorphisms, which is dual to $\mathbb{C}_{\mathsf{st}}$. Besides, we define the notion of descriptive conditional bisimilarity, which turns out the be the equivalent to Λ-bisimilarity but differs in the fact that its definition is structural, whereas Λ-bisimilarity is defined in a non-structural manner. In section 5.2 we investigate *geometric conditional frames*. These are coalgebras for the functor $\mathbb{C}_{\mathsf{kh}}$ on KHaus, which arises from lifting the conditional functor on Set using the method from section 4.1. We give an endofunctor on Frm and prove that its restriction to KRFrm is dual to $\mathbb{C}_{\mathsf{kh}}$.

In appendix A.2 we give a different proof of the fact that $\mathbb{M}_{\mathsf{kr}}$ from chapter 3 preserves compactness, which does not depend on the duality with $\mathbb{D}_{\mathsf{kh}}$.

Most notable theorems

- In section 2.3 we define so-called Λ-bisimulations between models over Stone-coalgebras. In **propositions 2.29 and 2.34 and lemma 2.32** this notion is related to modal equivalence and behavioural equivalence.

- In section 3.4 we define Λ-bisimulations between models for coalgebraic geometric logic. **Theorem 3.57** describes the relation of Λ-bisimilarity with modal equivalence and behavioural equivalence.

- For a certain class of set functors, there are two ways of lifting a set functor to a Stone functor. **Theorem 4.24** shows that these coincide on objects.

- We generalise the monotone functor (which is an endofunctor on Stone) to an endofunctor on KHaus and give a description of its algebraic dual, which is an endofunctor on Frm, in **theorem 3.41**. Similarly, we generalise the conditional functor (on Set) to an endofunctor on KHaus in section 5.2 and show that it has a dual functor on Frm in **theorem 5.38**.

- Descriptive conditional bisimilarity is a structural notion of bisimilarity between descriptive conditional frames. **Theorem 5.23** shows that descriptive conditional bisimilarity, Λ-bisimilarity (for certain Λ), modal equivalence and behavioural equivalence coincide.

- In chapter 3 we use duality to prove that the monotone functor $\mathbb{M}$ on Frm preserves compactness. In **theorem A.3** we give a proof of this fact which only plays on the frame side and does not use the duality with $\mathbb{D}_{\mathsf{kh}}$.

Prerequisites We assume familiarity with basic topology and category theory. Excellent references for these topics are [5, 45, 40]. Furthermore, familiarity with modal logic will be helpful in providing intuition. We refer to [10] for an outstanding introduction to basic modal logic. Besides, the reader is advised to have a look at section A.1, where notational conventions are explained.

2

Coalgebras and coalgebraic logic

This chapter sets the stage for the rest of the thesis. We present mostly known facts about coalgebra and coalgebraic logic. The main purpose is to fix notation and, through examples, introduce some structures that will play a role in later chapters. We assume familiarity with basic category theory, as standard reference we use [40].

Section 2.1 introduces coalgebras and related concepts of equivalence for an arbitrary category C. Some concrete and some more abstract examples are given. Section 2.2 focusses on the case $\mathsf{C} = \mathsf{Set}$ and introduces coalgebraic logic for set-coalgebras. Section 2.3 concentrates on the case $\mathsf{C} = \mathsf{Stone}$ and coalgebraic logic for Stone-coalgebras. So-called Λ-bisimulations are introduced, which are a straightforward generalisation from the same notion on set coalgebras [6], and various notions of equivalence between coalgebras are compared.

2.1 Coalgebras

In this section we define coalgebras and corresponding notions of equivalence. We also give a number of illustrative examples. For a thorough (yet accessible) introduction to the theory of coalgebras we refer to [27, 49, 28].

2.1 Definition. Let C be a category and $\mathbb{T}$ an endofunctor on C. A **$\mathbb{T}$-coalgebra** is a pair (X, γ) where X is an object in C, also referred to as the **state space**, and $\gamma : X \to \mathbb{T}X$ is a morphism in C, known as the **transition map** or **structure map**. A **$\mathbb{T}$-coalgebra morphism** between two $\mathbb{T}$-coalgebras (X, γ) and (X', γ') is a morphism $f : X \to X'$ in C such that the following diagram commutes:

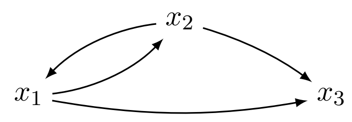
$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X' \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma'} \\ \mathbb{T}X & \xrightarrow[\mathbb{T}f]{} & \mathbb{T}X' \end{array}$$

The collection of $\mathbb{T}$-coalgebras and $\mathbb{T}$-coalgebra morphisms forms a category, which we shall denote by $\mathsf{Coalg}(\mathbb{T})$. The category C is called the **base category** of $\mathsf{Coalg}(\mathbb{T})$. $\lhd$

If the functor $\mathbb{T}$ is clear from the context, we will simply refer to (X, γ) as a coalgebra and to f as a coalgebra morphism or a coalgebra map. In case $\mathbb{T}$ is an endofunctor on Set, $\mathbb{T}$-coalgebras are also known as **systems**. The standard reference for the theory of systems is [49].

Many known structures can be viewed as a coalgebra. We give some examples.

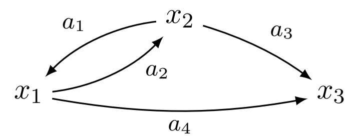
2.2 Example (Transition systems)**.** Consider the following example of a transition system. Here x_1, x_2, x_3 represent states and the arrows describe the relation between these states.



This can be viewed as a $\mathbb{P}$-coalgebra (X, γ) with state space $X = \{x_1, x_2, x_3\}$ and transition map $\gamma : X \to \mathbb{P}X$ given by $\gamma(x_1) = \{x_2, x_3\}$, $\gamma(x_2) = \{x_1, x_3\}$ and $\gamma(x_3) = \varnothing$. The coalgebra (X, γ) then encodes all information of the transition system, that is, given (X, γ) we can recover the given transition system.

In general, transition systems are pairs (X, R) where X is a set and $R \subseteq X \times X$ a relation on X. A transition system (X, R) corresponds to the $\mathbb{P}$-coalgebra (X, γ) where $\gamma : X \to \mathbb{P}X : x \mapsto \{x' \in X \mid xRx'\}$. Conversely, every $\mathbb{P}$-coalgebra (X, γ) gives rise to a transition system (X, R) with $R \subseteq X \times X$ defined by xRx' iff $x' \in \gamma(x)$. These constructions give a one-to-one correspondence between transition systems and $\mathbb{P}$-coalgebras. ◁

2.3 Example (Labelled transition systems)**.** Let us fix a set of labels A and label the transition system from the previous example with $a_1, \ldots, a_4 \in A$.



The result is a labelled transition system (LTS). To make this into a coalgebra we need to adapt our functor. Let $\mathbb{L}\text{ab} : \mathsf{Set} \to \mathsf{Set}$ be the functor defined by

$$\mathbb{L}\text{ab}\, X = \mathbb{P}(A \times X)$$

for a set X and

$$\mathbb{L}\text{ab}(f)(V) = \{(a, f(x)) \mid (a, x) \in V\}$$

for functions $X \to X'$ and $V \in \mathbb{P}(A \times X)$.

The information of the LTS can be encoded as a $\mathbb{L}\text{ab}$-coalgebra as follows: let $X = \{x_1, x_2, x_3\}$ and define $\gamma : X \to \mathbb{L}\text{ab}\, X$ by

$$\gamma : \begin{cases} x_1 \mapsto \{(a_2, x_2), (a_4, x_3)\} \\ x_2 \mapsto \{(a_1, x_1), (a_3, x_3)\} \\ x_3 \mapsto \varnothing \end{cases} .$$

Then (X, γ) corresponds to the LTS above, i.e. we can retrieve the given LTS from (X, γ).

More generally, a labelled transition system with labels in A is a pair $(X, \to)$ where X is a set and $\to \subseteq X \times A \times X$ a labelled transition map. An LTS $(X, \to)$ corresponds to the $\mathbb{L}\text{ab}$-coalgebra (X, γ) where γ is defined by $\gamma(x) = \{(a, x') \in A \times X \mid (x, a, x') \in \to\}$. Conversely, given a $\mathbb{L}\text{ab}$-coalgebra (X, γ) we can retrieve the LTS corresponding to it by defining $\to \subseteq X \times A \times X$ by $(x, a, x') \in \to$ iff $(a, x') \in \gamma(x)$. This gives a 1-1 correspondence between labelled transition systems with labels in A and $\mathbb{L}\text{ab}$-coalgebras. For details see example 2.1 in [49]. ◁

2.4 Example (Weighed transition systems)**.** Suppose we only want to look at labelled transition systems with labels in the set $\mathbb{R}_{\geq 0}$ such that for each state x the sum of the labels of all outgoing arrows is equal to some fixed $n \in \mathbb{R}_{\geq 0}$. Then we have to modify our functor $\mathbb{L}\mathrm{ab}$ to $\mathbb{L}\mathrm{ab}_n : \mathsf{Set} \to \mathsf{Set}$, defined on objects by

$$\mathbb{L}\mathrm{ab}_n X = \big\{ V \subseteq \mathbb{R}_{\geq 0} \times X \mid \sum\{a \mid \exists x \in X \text{ with } (a,x) \in V\} = n \big\}.$$

For a function $f : X \to X'$ and $V \in \mathbb{L}\mathrm{ab}_n X$ let

$$\mathbb{L}\mathrm{ab}_n(f)(V) = \big\{ (b_x, f(x)) \mid (a,x) \in V \text{ and } b_x = \sum\{c \mid (c,x) \in V\} \big\}.$$

It is an easy exercise to see that the aforementioned LTSs correspond precisely to $\mathbb{L}\mathrm{ab}_n$-coalgebras. ◁

2.5 Remark. $\mathbb{L}\mathrm{ab}_1$-coalgebras are precisely discrete time Markov chains (cf. [33, 46]). A slightly different functor to make Markov chains coalgebraic and many more examples of coalgebras of probabilistic systems can be found in [52].

2.6 Example (Kripke frames)**.** In modal logic, the transition systems of example 2.2 are better known as **Kripke frames**. The standard notion of morphisms between Kripke frames (X,R) and (X',R') is that of a bounded morphism: a set-map $f : X \to X'$ is a **bounded morphism** if for all $x,y \in X$ and $z' \in X'$ we have

(i) Rxy implies $R'f(x)f(y)$; and

(ii) $R'f(x)z'$ implies that there exists $z \in X$ with Rxz and $f(z) = z'$.

It is not hard to show that f is a bounded morphism from (X,R) to (X',R') iff it is a coalgebra morphism between the corresponding $\mathbb{P}$-coalgebras. This yields the following isomorphism of categories

$$\mathsf{Krip} \cong \mathsf{Coalg}(\mathbb{P}),$$

where Krip denotes the category of Kripke frames and bounded morphisms. ◁

Kripke frames play an important role in modal logic as they are the structures used to interpret *basic modal logic*. The following two examples are the structures that correspond to *monotone modal logic* and *conditional logic*. These will be key ingredients in guiding examples in subsequent chapters.

2.7 Example (Monotone frames)**.** A **monotone frame** is a pair (X,γ) where X is a set and $\gamma : X \to \breve{\mathbb{P}}(\breve{\mathbb{P}}X)$ is a **monotone neighbourhood function**. That is, γ assigns to each state $x \in X$ a collection of subsets of X, called **neighbourhoods** of x, and whenever $a \in \gamma(x)$ and $a \subseteq b \subseteq X$, we have $b \in \gamma(x)$. A **bounded morphism** between monotone frames (X,γ) and (X',γ') is a map $f : X \to X'$ such that for all $x \in X$ and $a' \subseteq X'$ we have $f^{-1}[a'] \in \gamma(x)$ iff $a' \in \gamma'(f(x))$.

Monotone frames are coalgebras for the functor $\mathbb{D} : \mathsf{Set} \to \mathsf{Set}$ given by

$$\mathbb{D}X = \{W \subseteq \mathbb{P}X \mid \text{ if } a \in W \text{ and } a \subseteq b \text{ then } b \in W\}.$$

For a morphism $f : X \to X'$ define

$$\mathbb{D}f : \mathbb{D}X \to \mathbb{D}X' : W \mapsto \{a' \in \mathbb{P}X' \mid f^{-1}(a') \in W\}.$$

The bounded morphisms correspond precisely to coalgebra morphisms [21, 20, 13]. ◁

In literature, morphisms between Kripke frames and morphisms between monotone frames are both called *bounded morphisms*. It will always be clear from the context which notion of bounded morphisms we mean.

Our next example will be that of conditional frames and conditional frame morphisms. One may find various definitions of conditional frames in the literature [38, 7, 13]. However, conditional frame morphisms have not been defined. In order to avoid confusion let us define the notion of conditional frames, taken from [7], and the corresponding morphisms that we will use here.

2.8 Definition. A **conditional frame** is a pair (X, ν) where X is a set and $\nu : X \times \mathbb{P}X \to \mathbb{P}X$ a function that satisfies for all $x \in X$ and $a, b \in \mathbb{P}X$

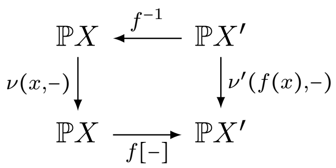
(i) if $a \cap b = \varnothing$, then $\nu(x, a) \cap b = \varnothing$; and

(ii) if $a \subseteq b$ and $\nu(x, b) \subseteq a$ then $\nu(x, a) = \nu(x, b)$.

A map $f : X \to X'$ is a **conditional frame morphism** between conditional frames (X, ν) and (X', ν') if for all $x \in X$ and $a' \subseteq X'$,

$$f[\nu(x, f^{-1}(a'))] = \nu'(f(x), a'). \tag{2.1}$$

This definition is motivated by the fact that f is a conditional frame morphism iff the following diagram commutes,



We write CF for the category of conditional frames and conditional frame morphisms. $\lhd$

2.9 Remark. Condition (i) in the previous definition can be reformulated as $\nu(x, a) \subseteq a$. Therefore our conditions are equivalent to the ones in definition 1 of [7]. We have chosen this slightly altered formulation in view of chapter 5, where they make a difference when dealing with *geometric* conditional frames.

Let us adopt a coalgebraic perspective on conditional frames and their morphisms.

2.10 Example (Conditional frames)**.** Conditional frames are coalgebras for the functor $\mathbb{C}$. For a set X, $\mathbb{C}X$ is the collection of functions $h : \mathbb{P}X \to \mathbb{P}X$ that satisfy

(C1) if $a \subseteq X$ then $h(a) \subseteq a$; and

(C2) if $a \subseteq b \subseteq X$ and $h(b) \subseteq a$ then $h(a) = h(b)$.

For a function $f : X \to X'$ define $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}X'$ by $\mathbb{C}f(h)(a) = f[h(f^{-1}(a))]$. We need to check that this is a well-defined functor, i.e. $\mathbb{C}f(h) \in \mathbb{C}X'$ for $h \in \mathbb{C}$. For (C1), let $a \subseteq X'$, then

$$\mathbb{C}f(h)(a) = f[h(f^{-1}(a))] \subseteq f[f^{-1}(a)] \subseteq a.$$

For (C2), assume $a \subseteq b$ and $\mathbb{C}f(h)(b) \subseteq a$. We need to show that $\mathbb{C}f(h)(a) = \mathbb{C}f(h)(b)$. Since $a \subseteq b$ we know $f^{-1}(a) \subseteq f^{-1}(b)$ and because $\mathbb{C}f(h)(b) = f[h(f^{-1}(b))] \subseteq a$ we know $h(f^{-1}(b)) \subseteq f^{-1}(a)$. Now we may apply (C2) to $\mathbb{C}X$ to find $h(f^{-1}(a)) = h(f^{-1}(b))$, from which it follows that

$$\mathbb{C}f(h)(a) = f[h(f^{-1}(a))] = f[h(f^{-1}(b))] = \mathbb{C}f(h)(b).$$

So $\mathbb{C}$ is well defined on morphisms.

There is an isomorphism $\mathsf{CF} \cong \mathsf{Coalg}(\mathbb{C})$ which is given on objects by observing that functions $X \times \mathbb{P}X \to \mathbb{P}X$ satisfying (i) and (ii) from definition 2.8 correspond one-to-one with elements of the set $\mathbb{C}X$. Furthermore, conditional frame morphisms are tailored to coincide with $\mathbb{C}$-coalgebra morphisms. It is routine to check the details. $\lhd$

2.11 Remark. If we omit (i) and (ii) from definition 2.8 we get the definition of a **selection function frame**. These are known to be coalgebraic [38]. Although the authors only consider the frames, not the morphisms, it is easy to see that the above notion of a morphism is precisely a coalgebra morphism for the functor given in [38].

More examples showcasing the wide scope of coalgebra can be found in a variety of areas such as biology [65], economics [44] and quantum computing [2, 26].

There are two standard notions of equivalence for coalgebras: coalgebra bisimilarity and behavioural equivalence. It is a well-known fact that these two notions coincide for many choices of the functor $\mathbb{T}$, namely if $\mathbb{T}$ preserves weak pullbacks. These notions will be used throughout this thesis.

2.12 Definition. Let C be a category which has products and a forgetful functor $\mathbb{Y} : \mathsf{C} \to \mathsf{Set}$. Let $\mathbb{T}$ be an endofunctor on C, let (X, γ) and (X', γ') be $\mathbb{T}$-coalgebras, and let $x \in \mathbb{Y}X$ and $x' \in \mathbb{Y}X'$. The states x and x' are called **behaviourally equivalent**, $x \simeq x'$, if there exist a coalgebra (Y, δ) and coalgebra morphisms $f : (X, \gamma) \to (Y, \delta)$ and $f' : (X', \gamma') \to (Y, \delta)$ such that $f(x) = f'(x')$.

Let B be an object in C such that $\mathbb{Y}B \subseteq \mathbb{Y}X \times \mathbb{Y}X'$, with projections $\pi : B \to X$ and $\pi' : B \to X'$. B is called a **coalgebra bisimulation** or **Aczel-Mendler bisimulation** between (X, γ) and (X', γ') if there exists a transition map $\beta : B \to \mathbb{T}B$ that makes π and π' coalgebra morphisms. That is, β is such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{\ \pi\ } & B & \xrightarrow{\ \pi'\ } & X' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma'} \\ \mathbb{T}X & \xleftarrow[\mathbb{T}\pi]{} & \mathbb{T}B & \xrightarrow[\mathbb{T}\pi']{} & X' \end{array}$$

Two states $x \in \mathbb{U}X, x' \in \mathbb{U}X'$ are called **bisimilar**, notation $x \leftrightarroweq x'$, if they are linked by a coalgebra bisimulation. $\lhd$

Finally, recall the definition of a final object in a category.

2.13 Definition. An object X in a category C is called **final** if for all objects X' in C there exists a unique morphism $f : X' \to X$. A $\mathbb{T}$-coalgebra is called final if it is a final object in $\mathsf{Coalg}(\mathbb{T})$. $\lhd$

Let $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ be a functor and suppose the category $\mathsf{Coalg}(\mathbb{T})$ has a final object (Z, ζ). For each object $\mathfrak{X} = (X, \gamma)$ in $\mathsf{Coalg}(\mathbb{T})$ let $f_{\mathfrak{X}} : X \to Z$ be the unique coalgebra map to (Z, ζ). Then it is an easy consequence of finality of (Z, ζ) that two states x and x' in two coalgebras (X, γ) and (X', γ') are behaviourally equivalent if and only if $f_{\mathfrak{X}}(x) = f_{\mathfrak{X}'}(x')$. In fact, originally, behavioural equivalence was only defined for categories with a final object; two states were called behaviourally equivalent if they were mapped to the same element under the unique maps to the final object. Definition 2.13 is equivalent to this one in case the category has a final object, but can also be used for categories without a final object.

2.2 Set-based coalgebraic logic

In this section we briefly describe how set-based coalgebras relate to logic. Much more information can be found in e.g. [38, 55, 39, 14]. We start by introducing the idea of a so-called predicate lifting, corresponding language, and models for interpreting this language. Thereafter a different notion of bisimulation is given and some examples are examined.

2.14 Definition. Let $\mathbb{T}$ be an endofunctor Set. A **predicate lifting** for $\mathbb{T}$ of arity n is a natural transformation

$$\lambda : \breve{\mathbb{P}}^n \to \breve{\mathbb{P}} \circ \mathbb{T}.$$

The **dual** of an n-ary predicate lifting λ is given by

$$\lambda^{\partial}_X : \breve{\mathbb{P}}^n X \to \breve{\mathbb{P}} X : (a_1, \dots, a_n) \mapsto \mathbb{T}X \setminus \lambda(X \setminus a_1, \dots, X \setminus a_n).$$

A collection Λ of predicate liftings for $\mathbb{T}$ is called a **similarity type** (for $\mathbb{T}$), and is said to be **closed under duals** if $\lambda \in \Lambda$ implies $\lambda^{\partial} \in \Lambda$. A similarity type Λ is **separating** for $\mathbb{T}$ if for all sets X and all distinct $x, x' \in \mathbb{T}X$ there exists a $\lambda \in \Lambda$ and $a_1, \dots, a_n \in \breve{\mathbb{P}}X$ such that precisely one of x, x' belongs to the set $\lambda_X(a_1, \dots, a_n)$. ◁

Fix a set Φ of proposition letters.

2.15 Definition. Let $\mathbb{T}$ be a functor on Set. A **$\mathbb{T}$-model** is a triple $\mathfrak{X} = (X, \gamma, V)$ where (X, γ) is a $\mathbb{T}$-coalgebra and $V : \Phi \to \mathbb{P}X$ is a valuation. A **$\mathbb{T}$-model morphism** f from (X, γ, V) to (X', γ', V') is a $\mathbb{T}$-coalgebra morphism $f : (X, \gamma) \to (X', \gamma')$ such that $f^{-1} \circ V' = V$. The collection of $\mathbb{T}$-models and $\mathbb{T}$-model morphisms forms a category, $\mathsf{Mod}(\mathbb{T})$.

An **Aczel-Mendler bisimulation** between two $\mathbb{T}$-models is an Aczel-Mendler bisimulation between the underlying $\mathbb{T}$-coalgebras such that for all $(x, x') \in B$ and $p \in \Phi$, $x \in V(p)$ iff $x' \in V'(p)$. ◁

Every similarity type induces a modal language that we can interpret on $\mathbb{T}$-models.

2.16 Definition. The **language** induced by the similarity type Λ is the set $\mathcal{L}(\Lambda)$ of formulas defined by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. The symbols $\top$, $\vee$, $\to$ and $\leftrightarrow$ denote the usual abbreviations. The **semantics** of $\varphi \in \mathcal{L}(\Lambda)$ on a $\mathbb{T}$-model $\mathfrak{X} = (X, \gamma, V)$ is given inductively by

$$\llbracket p \rrbracket^{\mathfrak{X}} = V(p), \quad \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^{\mathfrak{X}} = \llbracket \varphi_1 \rrbracket^{\mathfrak{X}} \cap \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}, \quad \llbracket \neg\varphi \rrbracket^{\mathfrak{X}} = X \setminus \llbracket \varphi \rrbracket^{\mathfrak{X}},$$
$$\llbracket \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \rrbracket^{\mathfrak{X}} = \gamma^{-1}(\lambda(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \dots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})),$$

where $p \in \Phi$ and λ ranges over Λ. ◁

Besides Aczel-Mendler bisimulations, other notions of bisimulations between $\mathbb{T}$-coalgebras and $\mathbb{T}$-models have been proposed and linked to modal equivalence and behavioural equivalence. We content ourselves with stating the definition of a so-called Λ-bisimulation for future reference, and refer to [17, 6, 22] for more information.

2.17 Definition. Let $B \subseteq X \times X'$ be a relation. A pair $(a, a') \in \mathbb{P}X \times \mathbb{P}X'$ is called B-coherent if $B[a] \subseteq a'$ and $B^{-1}[a'] \subseteq a$. $\lhd$

Properties of coherent pairs of sets may be found in [22]. We prove one property which is not in [22] for future reference.

2.18 Lemma. *Let $B \subseteq X \times X'$ be a relation and (a, a') a B-coherent pair. Then $(X \setminus a, X' \setminus a')$ is B-coherent.*

Proof. Assume towards a contradiction that $B[X \setminus a] \nsubseteq X' \setminus a'$, then $B[X \setminus a] \cap a' \neq \varnothing$, so some element in $X \setminus a$ is related to an element in a'. But then $B^{-1}[a'] \nsubseteq a$, a contradiction. Therefore we must have $B[X \setminus a] \subseteq X' \setminus a'$. In a similar way it can be shown that $B^{-1}[X' \setminus a'] \subseteq X \setminus a$. $\square$

2.19 Definition. Let $\mathbb{T}$ be a set functor, i.e. an endofunctor on Set, Λ a collection of predicate liftings for $\mathbb{T}$ and (X, γ, V) and (X', γ', V') two $\mathbb{T}$-models. A relation $B \subseteq X \times X'$ is called a **Λ-bisimulation** if for all $\lambda \in \Lambda$, $(x, x') \in B$ and B-coherent pairs (a_i, a_i') we have

- $x \in V(p)$ iff $x' \in V'(p)$;
- $\gamma(x) \in \lambda_X(a_1, \ldots, a_n)$ iff $\gamma'(x') \in \lambda_{X'}(a_1', \ldots, a_n')$.

Two states are called Λ-bisimilar if they are linked by a Λ-bisimulation. $\lhd$

The remainder of this section is devoted to examples.

2.20 Example (Normal modal logic)**.** In example 2.6 we showed that $\mathbb{P}$-coalgebras correspond precisely to Kripke frames. Define $\lambda^{\Box} : \breve{\mathbb{P}} \to \breve{\mathbb{P}} \circ \mathbb{P}$ by $\lambda^{\Box}_X(a) = \{b \in \mathbb{P}X \mid b \subseteq a\}$ and set $\Lambda = \{\lambda^{\Box}\}$. Then $\mathcal{L}(\Lambda)$ is the standard relational semantics for modal logic. We write $\Box$ instead of $\heartsuit^{\lambda^{\Box}}$. If $\mathfrak{X} = (X, \gamma, V)$ is a Kripke model (a $\mathbb{P}$-model) and $\varphi \in \mathcal{L}(\Lambda)$ is a formula, then

$$[\![\Box\varphi]\!]^{\mathfrak{X}} = \{x \in X \mid \gamma(x) \subseteq [\![\varphi]\!]^{\mathfrak{X}}\},$$

so $\mathfrak{X}, x \Vdash \Box\varphi$ iff for all $y \in \gamma(x)$ we have $\mathfrak{X}, y \Vdash \varphi$. This yields the usual Kripke semantics of modal logic [10].

It is an easy exercise to show that $\{\lambda^{\Box}\}$-bisimilar states satisfy precisely the same formulas. Moreover, every Kripke bisimulation is also a $\{\lambda^{\Box}\}$-bisimulation [6, Example 3.3]. $\lhd$

2.21 Example (Monotone modal logic)**.** Example 2.7 shows that monotone frames are $\mathbb{D}$-coalgebras. Define $\lambda^{\Box} : \breve{\mathbb{P}} \to \breve{\mathbb{P}} \circ \mathbb{D}$ by $\lambda^{\Box}_X(a) = \{W \in \mathbb{D}X \mid a \in W\}$ and set $\Lambda = \{\lambda^{\Box}\}$. Then $\mathcal{L}(\Lambda)$ is the standard semantics of modal logic. Write $\Box$ for $\heartsuit^{\lambda^{\Box}}$. If $\mathfrak{X} = (X, \gamma, V)$ is a neighbourhood model and φ a formula in $\mathcal{L}(\Lambda)$ then, similar to the previous example, we have $\mathfrak{X}, x \Vdash \Box\varphi$ iff $[\![\varphi]\!]^{\mathfrak{M}} \in \gamma(x)$. This yields the usual monotone semantics of modal logic [20, 21, 13]. As in the previous example, $\{\lambda^{\Box}\}$-bisimilar states satisfy precisely the same formulas. $\lhd$

Next, we will look at conditional logic. Conditional logic provides an example of a non-monotone modality: the conditional implication, $\Rightarrow$. The modality $\Rightarrow$ is meant to express a notion of conditionality which in general is different from the usual implication $\to$. A formula $\varphi_1 \Rightarrow \varphi_2$ should be read as "If φ_1 is the case, then usually φ_2 is the case."

For an example, suppose Morty usually visits his grandpa Rick on Fridays. This can be formalised as

$$\text{Friday} \Rightarrow \text{visit Rick}.$$

However, if Morty is ill he will not visit his grandpa, so

$$\text{Friday} \wedge \text{ill} \Rightarrow \neg(\text{visit Rick}).$$

The non-monotonicity shows itself in the fact that the conclusion is not maintained if more information becomes available. For more information on conditional logic, see [13, 4, 42, 51, 7, 11].

2.22 Example (Conditional logic)**.** The language of conditional logic is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \Rightarrow \varphi_2,$$

with $p \in \Phi$. We abbreviate $\vee$ and $\top$ as usual, and let $\varphi_1 \Downarrow \varphi_2 := \neg(\varphi_1 \Rightarrow \neg\varphi_2)$. A possible way to read $\varphi_1 \Rightarrow \varphi_2$ is as: "If φ_1 holds, then usually φ_2 holds as well." On a conditional model (viewed as $\mathbb{C}$-model, cf example 2.10) $\mathfrak{X} = (X, \gamma, V)$, truth of the proposition letters and of the Boolean cases is treated as usual. Truth of the implication is given by

$$\mathfrak{X}, x \Vdash \varphi_1 \Rightarrow \varphi_2 \quad \text{iff} \quad \gamma(x)(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}) \subseteq \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}.$$

and consequently $\mathfrak{X}, x \Vdash \varphi_1 \Downarrow \varphi_2$ iff $\gamma(x)(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}) \cap \llbracket \varphi_2 \rrbracket^{\mathfrak{X}} \neq \varnothing$. The intuition behind this is that the function $\gamma(x) : \mathbb{P}X \to \mathbb{P}X$ indicates for each set $a \subseteq X$ the relevant states in a. We say that a state x satisfies $\varphi_1 \Rightarrow \varphi_2$ if the relevant states of $\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}$ as seen from x, are all contained in $\llbracket \varphi_2 \rrbracket^{\mathfrak{X}}$.

Define $\lambda^{\Rightarrow} : \breve{\mathbb{P}}^2 \to \breve{\mathbb{P}} \circ \mathbb{C}$ by $\lambda^{\Rightarrow}(a, b) = \{h : \mathbb{P}X \to \mathbb{P}X \mid h(a) \subseteq b\}$. Then

$$\mathfrak{X}, x \Vdash \varphi_1 \Rightarrow \varphi_2 \quad \text{iff} \quad \gamma(x) \in \lambda^{\Rightarrow}(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}).$$

This yields conditional semantics [7, 13]. Additionally we may define $\lambda^{\Downarrow} : \breve{\mathbb{P}}^2 \to \breve{\mathbb{P}} \circ \mathbb{C}$ by $\lambda^{\Downarrow}(a, b) = \{h : \mathbb{P}X \to \mathbb{P}X \mid h(a) \cap a \neq \varnothing\}$. Then we have $\mathfrak{X}, x \Vdash \varphi_1 \Downarrow \varphi_2$ iff $\gamma(x) \in \lambda^{\Downarrow}(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \llbracket \varphi_2 \rrbracket^{\mathfrak{X}})$. ◁

2.23 Remark. The introduction of the modality $\Downarrow$ may seem superfluous at this point. Indeed, it is only an abbreviation so we don't really need a predicate lifting to describe its truth. However, it turns out to be useful when generalising conditional logic for Stone-coalgebras in section 5.1. Besides, when dealing with geometric conditional logic (section 5.2) the modalities $\Rightarrow$ and $\Downarrow$ will no longer be mutually expressible, but relate via a weaker relational.

More examples of the interplay between logic and set-coalgebras can be found in [38].

2.3 Stone-based coalgebraic logic

The final section of this chapter is devoted to logic on Stone-coalgebras. Analogously to the set case, we define so-called *clopen* predicate liftings, a language, and models for interpreting this language. We then show that, provided Λ is a *characteristic* set of predicate liftings, the notions of modal equivalence and behavioural equivalence coincide. Thereafter, Λ-bisimulations for models on Stone coalgebras are introduced. The section closes with two examples of well-known Stone-coalgebras which occur in modal logic.

2.24 Definition. Let $\mathbb{T}$ be an endofunctor on Stone. A **clopen predicate lifting** of arity n is a natural transformation

$$\lambda : \mathbb{C}\mathrm{lp}^n \to \mathbb{C}\mathrm{lp} \circ \mathbb{T}.$$

A clopen predicate lifting is said to be **monotone** if for all topological spaces $\mathfrak{X}$ and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}\mathrm{lp}(\mathfrak{X})$, if $a_i \subseteq b_i$ for all $1 \leq i \leq n$, then $\lambda_{\mathfrak{X}}(a_1, \ldots, a_n) \subseteq \lambda_{\mathfrak{X}}(b_1, \ldots, b_n)$. The **dual** of a clopen predicate lifting λ is given by $\lambda^{\partial}_{\mathfrak{X}}(a_1, \ldots, a_n) := \mathbb{T}\mathfrak{X} \setminus \lambda(X \setminus a_1, \ldots, X \setminus a_n)$. A collection Λ of predicate liftings for $\mathbb{T}$ is said to be **characteristic** if for every Stone space $\mathfrak{X}$ the collection

$$\{\lambda_{\mathfrak{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_i \in \mathbb{C}\mathrm{lp}\,\mathfrak{X}\}$$

forms a subbase for the topology on $\mathbb{T}\mathfrak{X}$. ◁

The condition for a collection of predicate liftings to be characteristic can be regarded as the topological counterpart of being separated.

2.25 Remark. In [16] the authors define a *topological* predicate lifting as the Stone space variation of a predicate lifting. A **topological predicate lifting** for a Stone functor $\mathbb{T}$ is a natural transformation

$$\lambda : \breve{\mathbb{P}}^n \circ \mathbb{U} \to \breve{\mathbb{P}} \circ \mathbb{U} \circ \mathbb{T}$$

such that for all Stone spaces $\mathfrak{X}$ and $a_1, \ldots, a_n \in \mathbb{C}\mathrm{lp}\,\mathfrak{X}$ the set $\lambda_{\mathfrak{X}}(a_1, \ldots, a_n)$ is clopen in $\mathbb{T}\mathfrak{X}$. Although $\lambda_{\mathfrak{X}}(a)$ is defined for all subsets $a \subseteq \mathfrak{X}$, the only information that is used in the semantics of the language is the action of $\lambda_{\mathfrak{X}}$ on the clopens of $\mathfrak{X}$.

If $\lambda : \breve{\mathbb{P}} \circ \mathbb{U} \to \breve{\mathbb{P}} \circ \mathbb{U} \circ \mathbb{T}$ is a (unary) topological predicate lifting then we can obtain a clopen predicate lifting λ^r by restricting for each Stone space $\mathfrak{X}$ the map $\lambda_{\mathfrak{X}}$ to $\mathbb{C}\mathrm{lp}\,\mathfrak{X}$. By definition of a topological predicate lifting we have $\lambda^r_{\mathfrak{X}}(a) \in \mathbb{C}\mathrm{lp}(\mathbb{T}\mathfrak{X})$ for all $a \in \mathbb{C}\mathrm{lp}\,\mathfrak{X}$, so $\lambda^r_{\mathfrak{X}}$ is indeed a map to $\mathbb{C}\mathrm{lp}(\mathbb{T}\mathfrak{X})$. Naturality of λ^r follows immediately from the naturality of λ. For every Stone space $\mathfrak{X}$ the action of $\lambda_{\mathfrak{X}}$ and $\lambda^r_{\mathfrak{X}}$ on clopens of $\mathfrak{X}$ is the same.

The n-ary case is similar. So every topological predicate lifting yields a clopen predicate lifting which gives the same language and semantics.

We have not found a converse, i.e., a way to turn each open predicate lifting into a topological predicate lifting. Nor have we found a counterexample that this is not possible. We leave this as an interesting open question.

2.26 Definition. Let $\mathbb{T}$ be a functor on Stone. A **$\mathbb{T}$-model** is a triple $\mathfrak{X} = (\mathfrak{X}, \gamma, V)$ where $(\mathfrak{X}, \gamma)$ is a $\mathbb{T}$-coalgebra and $V : \Phi \to \mathbb{C}\mathrm{lp}\,\mathfrak{X}$ is an admissible valuation of the proposition letters. A **$\mathbb{T}$-model morphism** from $(\mathfrak{X}, \gamma, V)$ to $(\mathfrak{X}', \gamma', V')$ is a $\mathbb{T}$-coalgebra morphism $f : (\mathfrak{X}, \gamma) \to (\mathfrak{X}', \gamma')$ such that $f^{-1} \circ V' = V$. The collection of $\mathbb{T}$-models and $\mathbb{T}$-model morphisms forms a category, called $\mathsf{Mod}(\mathbb{T})$.

An **Aczel-Mendler bisimulation** between two $\mathbb{T}$-models is an Aczel-Mendler bisimulation between the underlying $\mathbb{T}$-coalgebras such that for all $(x, x') \in B$ and $p \in \Phi$, $x \in V(p)$ iff $x' \in V'(p)$. ◁

2.27 Definition. The **language** induced by a collection of clopen predicate liftings Λ is the set $\mathcal{L}(\Lambda)$ of formulas

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. The symbols $\top$, $\vee$, $\to$ and $\leftrightarrow$ denote the usual abbreviations. The **semantics** of $\varphi \in \mathcal{L}(\Lambda)$ on a $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ is given inductively by

$$\llbracket p \rrbracket^{\mathfrak{X}} = V(p), \quad \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^{\mathfrak{X}} = \llbracket \varphi_1 \rrbracket^{\mathfrak{X}} \cap \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}, \quad \llbracket \neg \varphi \rrbracket^{\mathfrak{X}} = X \setminus \llbracket \varphi \rrbracket^{\mathfrak{X}},$$
$$\llbracket \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \rrbracket^{\mathfrak{X}} = \gamma^{-1}(\lambda(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})).$$

A formula φ is called **valid** on $\mathsf{Mod}(\mathbb{T})$ if for every $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and all $x \in \mathcal{X}$ we have $\mathfrak{X}, x \Vdash \varphi$. Denote the collection of valid formulas of $\mathcal{L}(\Lambda)$ by $\mathbf{Log}(\mathbb{T}, \Lambda)$.

Two states x and x' in two $\mathbb{T}$-models $\mathfrak{X}$ and $\mathfrak{X}'$ are called **modally equivalent**, notation $x \equiv_{\Lambda} x'$, if for all $\varphi \in \mathcal{L}(\Lambda)$, $\mathfrak{X}, x \Vdash \varphi \Leftrightarrow \mathfrak{X}', x' \Vdash \varphi$. ◁

2.28 Proposition. *Let $\mathbb{T}$ be an endofunctor on* Stone*, Λ a set of predicate liftings for $\mathbb{T}$ and f a $\mathbb{T}$-model morphism from $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ to $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$. Then*

$$\mathfrak{X}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{X}', f(x) \Vdash \varphi.$$

Proof. The proof of this lemma is similar to the proof of proposition 3.20, which is in turn similar to the proof of theorem 6.17 in [56]. □

The next theorem connects behavioural equivalence to modal equivalence. The proof is inspired by theorem 4.1 in [16].

2.29 Proposition. *Let $\mathbb{T}$ be an endofunctor on* Stone *and Λ a characteristic set of predicate liftings for $\mathbb{T}$. Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be two $\mathbb{T}$-models and $x \in \mathcal{X}$, $x' \in \mathcal{X}'$ states in these models. Then x and x' are modally equivalent if and only if they are behaviourally equivalent.*

Proof. Our strategy to prove this proposition is to construct a final coalgebra of theories and then exploit that two states are behaviourally equivalent if and only if their theories are the same.

Let $\mathcal{Z}$ be the collection of maximal satisfiable sets of formulas of $\mathcal{L}(\Lambda)$, with a topology generated by the clopen subbase $\{\widetilde{\varphi} \mid \varphi \in \mathrm{CL}\}$, where $\widetilde{\varphi} = \{\Gamma \in \mathcal{Z} \mid \varphi \in \Gamma\}$. By definition $\mathcal{Z}$ is (homeomorphic to) the dual Stone space of the Lindenbaum-Tarski algebra of $\mathbf{Log}(\mathbb{T}, \Lambda)$, so every clopen set is of the form $\widetilde{\varphi}$ for some $\varphi \in \mathcal{L}(\Lambda)$.

For every $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ define a map

$$\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Z} : x \mapsto \{\varphi \mid \mathfrak{X}, x \Vdash \varphi\}.$$

2.29.A Claim. *Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be two $\mathbb{T}$-models, $x \in \mathcal{X}, x' \in \mathcal{X}'$. If $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ then $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) = \mathbb{T}\,\mathrm{th}_{\mathfrak{X}'}(\gamma'(x'))$.*

Proof of claim. Suppose $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \neq \mathbb{T}\,\mathrm{th}_{\mathfrak{X}'}(\gamma'(x'))$, then there exists a clopen set $c \in \mathrm{Clp}(\mathbb{T}\mathcal{Z})$ such that $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in c$ and $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}'}(\gamma'(x')) \notin c$. Since Λ is characteristic for $\mathbb{T}$ and every clopen set of $\mathcal{Z}$ is of the form $\widetilde{\varphi}$, there exist $\lambda \in \Lambda$ and $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n$ such that $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \subseteq c$.

Observe $\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_i) = \llbracket \varphi_i \rrbracket^{\mathfrak{X}}$ for $1 \leq i \leq n$. The fact that $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ and naturality of λ yield

$$\gamma(x) \in (\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}(\lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)) = \lambda_{\mathcal{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_1), \ldots, \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_n)) = \lambda_{\mathcal{X}}(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})$$

and similarly $\gamma'(x') \notin \lambda_{\mathcal{X}'}(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})$. Therfore $\mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)$ and $\mathfrak{X}', x' \nVdash \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)$, so $\mathrm{th}_{\mathfrak{X}}(x) \neq \mathrm{th}_{\mathfrak{X}'}(x')$. This proves the claim. ◇

Let $\Gamma \in \mathfrak{Z}$ and let $(\mathfrak{X}, x)$ be a pointed $\mathbb{T}$-model such that $\mathrm{th}_{\mathfrak{X}}(x) = \Gamma$. Such a pointed model always exists because the elements of $\mathfrak{Z}$ are assumed to be satisfiable. Define $\zeta(\Gamma) := \mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x))$. This gives rise to a map

$$\zeta : \mathfrak{Z} \to \mathbb{T}\mathfrak{Z}$$

which by the previous claim is well-defined, because it does not depend on the choice of the pointed model $(\mathfrak{X}, x)$. Moreover, we argue that ζ is continuous:

2.29.B Claim. *The map $\zeta : \mathfrak{Z} \to \mathbb{T}\mathfrak{Z}$ is continuous.*

Proof of claim. Since Λ is characteristic it suffices to show that $\zeta^{-1}(\lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n))$ is open in $\mathfrak{Z}$ for $\lambda \in \Lambda$ and $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n \in \mathrm{Clop}\,\mathfrak{Z}$. Fix such a λ and $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n$. We will show that

$$\zeta^{-1}(\lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)) = \widetilde{\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)}.$$

For $\Gamma \in \mathfrak{Z}$, let $(\mathfrak{X}, x)$ be a pointed $\mathbb{T}$-model with $\mathrm{th}_{\mathfrak{X}}(x) = \Gamma$. Then

$$\begin{aligned}
\zeta(\Gamma) \in \lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) &\Leftrightarrow \mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \\
&\Leftrightarrow \gamma(x) \in \lambda_{\mathcal{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_1), \dots, \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_n)) \\
&\Leftrightarrow \gamma(x) \in \lambda_{\mathcal{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}}) \\
&\Leftrightarrow \mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \\
&\Leftrightarrow \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \in \mathrm{th}_{\mathfrak{X}}(x) \\
&\Leftrightarrow \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \in \Gamma \\
&\Leftrightarrow \Gamma \in \widetilde{\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)}.
\end{aligned}$$

This proves continuity of ζ. ◇

We have established that $(\mathfrak{Z}, \zeta)$ is a $\mathbb{T}$-coalgebra. Endow $(\mathfrak{Z}, \zeta)$ with the valuation $V_{\mathfrak{Z}} : \Phi \to \mathrm{Clop}\,\mathfrak{Z} : p \mapsto \widetilde{p}$. By construction each map $\mathrm{th}_{\mathfrak{X}}$ is a $\mathbb{T}$-model morphism. It then follows that $\mathfrak{Z}, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$: the case $\varphi = p$ holds by definition of $V_{\mathfrak{Z}}$, the Boolean cases follow by an easy induction, and the case $\varphi = \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)$ follows from the proof of the previous claim. In addition, $\mathfrak{Z}$ is final.

2.29.C Claim. *The $\mathbb{T}$-model $\mathfrak{Z} = (\mathfrak{Z}, \zeta, V_{\mathfrak{Z}})$ is final in* $\mathsf{Mod}(\mathbb{T})$.

Proof of claim. Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ be any $\mathbb{T}$-model and $f : \mathfrak{X} \to \mathfrak{Z}$ a $\mathbb{T}$-model morphism. It follows from proposition 2.28 that for all $x \in \mathcal{X}$ we have $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{Z}, f(x) \Vdash \varphi$ iff $\varphi \in f(x)$, so $f(x) = \mathrm{th}_{\mathfrak{X}}(x)$ hence $f = \mathrm{th}_{\mathfrak{X}}$. ◇

The proposition now follows: suppose x and x' are modally equivalent, then $\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathfrak{Z}$ and $\mathrm{th}_{\mathfrak{X}'} : \mathcal{X}' \to \mathfrak{Z}'$ are $\mathbb{T}$-model morphisms such that $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$, so x and x' are behaviourally equivalent. Conversely, if x and x' are behaviourally equivalent, then we must have $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ so by proposition 2.28 $x \equiv_{\Lambda} x'$. □

We will now define a notion of bisimulation between models and relate this to modal equivalence and behavioural equivalence. The following definition of Λ-bisimulation is an adaptation of ideas in [6, 17].

2.30 Definition. Let $\mathbb{T}$ be an endofunctor on Stone and Λ a collection of predicate liftings for $\mathbb{T}$. Let $(\mathfrak{X}, \gamma)$ and $(\mathfrak{X}', \gamma')$ be $\mathbb{T}$-coalgebras and $B \subseteq \mathfrak{X} \times \mathfrak{X}'$ be a subspace with projections $\pi : B \to \mathfrak{X}$ and $\pi' : B \to \mathfrak{X}'$. Let

$$\begin{array}{ccc} pb(\mathfrak{clp}(\pi), \mathfrak{clp}(\pi')) & \xrightarrow{\overline{\pi}'} & \mathfrak{clp}(\mathfrak{X}') \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \mathfrak{clp}(\pi')} \\ \mathfrak{clp}(\mathfrak{X}) & \xrightarrow[\mathfrak{clp}(\pi)]{} & \mathfrak{clp}(B) \end{array}$$

be the pullback diagram of the cospan $(\mathfrak{clp}(\pi), \mathfrak{clp}(\pi'))$ in BA. We say that B is a Λ-bisimulation if for all $(x, x') \in B$ and $\lambda \in \Lambda$ we have

$$\mathfrak{clp}(\pi) \circ \mathfrak{clp}(\gamma) \circ \lambda_{\mathfrak{X}} \circ \overline{\pi}^n = \mathfrak{clp}(\pi') \circ \mathfrak{clp}(\gamma') \circ \lambda_{\mathfrak{X}'} \circ \overline{\pi}'^n.$$

A relation B between two $\mathbb{T}$-models $(\mathfrak{X}, \gamma, V)$ and $(\mathfrak{X}', \gamma', V')$ is a Λ-bisimulation if it is a Λ-bisimulation between the underlying $\mathbb{T}$-coalgebras and for all $(x, x') \in B$ and $p \in \Phi$ we have $x \in V(p)$ iff $x' \in V'(p)$. Two states $x \in \mathfrak{X}$ and $x' \in \mathfrak{X}'$ are called **Λ-bisimilar**, notation $x \leftrightarrows_{\Lambda} x'$, if there is a Λ-bisimulation linking them. $\lhd$

2.31 Remark. Observe that $(a, a') \in \mathrm{Clp}\,\mathfrak{X} \times \mathrm{Clp}\,\mathfrak{X}'$ is B-coherent, i.e. $B[a] \subseteq a'$ and $B^{-1}[a'] \subseteq a$, if and only if it is in $pb(\mathfrak{clp}(\pi), \mathfrak{clp}(\pi'))$. It follows from unraveling that B is a Λ-bisimulation if and only if for all $\lambda \in \Lambda$ and all B-coherent pairs of clopens $(a_i, a_i') \in \mathrm{Clp}\,\mathfrak{X} \times \mathrm{Clp}\,\mathfrak{X}'$ we have

$$\gamma(x) \in \lambda_{\mathfrak{X}}(a_1, \ldots, a_n) \quad \text{iff} \quad \gamma'(x') \in \lambda_{\mathfrak{X}'}(a_1', \ldots, a_n').$$

The following statements are easy to verify.

2.32 Lemma. *Let $\mathbb{T}$ be an endofunctor on* Stone, *Λ a set of predicate liftings for $\mathbb{T}$ and $(\mathfrak{X}, \gamma, V)$ and $(\mathfrak{X}', \gamma', V')$ $\mathbb{T}$-models. If two states $x \in \mathfrak{X}$ and $x' \in \mathfrak{X}'$ are Λ-bisimilar, then they are modally equivalent.*

2.33 Proposition. *Let $\mathbb{T}$ be an endofunctor on* Stone *and Λ set of predicate liftings for $\mathbb{T}$. Every Aczel-Mendler bisimulation between $\mathbb{T}$-models is a Λ-bisimulation.*

If Λ is characteristic, it follows from the previous lemma and proposition combined with proposition 2.29 that Aczel-Mendler bisimilarity implies behavioural equivalence. If moreover $\mathbb{T}$ preserves weak pullbacks, the converse holds as well. The proof of this is similar to theorem 4.3 and the preceding discussion in [49].

However, we do not wish to make this assumption. For example, the Vietoris functor does not preserve weak pullbacks [9, Corollary 4.3]. The next proposition shows that for monotone Λ, behavioural equivalence implies Λ-bisimilarity, without assuming $\mathbb{T}$ to preserve weak pullbacks.

2.34 Proposition. *Let Λ be a monotone characteristic set of predicate liftings for a functor $\mathbb{T}$ and suppose two states x and x' in $\mathbb{T}$-models $\mathfrak{X} = (\mathfrak{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathfrak{X}', \gamma', V')$ are behaviourally equivalent. Then x and x' are Λ-bisimilar.*

Proof. Since x and x' are behaviourally equivalent, there must be some $\mathbb{T}$-coalgebra $(\mathfrak{Y}, \delta)$ and some coalgebra morphisms $f : \mathfrak{X} \to \mathfrak{Y}, f' : \mathfrak{X}' \to \mathfrak{Y}$ such that $f(x) = f'(x')$. Let

$$B = \{(u, u') \in \mathfrak{X} \times \mathfrak{X}' \mid f(u) = f'(u')\},$$

then clearly xBx'. We claim that B is a Λ-bisimulation.

In order to show that B is a Stone space, it suffices to show that B is closed. To see this, suppose $(u,u') \notin B$. Then $f(u) \neq f'(u')$ and since $\mathfrak{Y}$ is Hausdorff there exist disjoint clopens $a, a' \in \mathrm{Clp}\,\mathfrak{Y}$ that contain $f(u)$ and $f'(u')$ respectively. Now $f^{-1}(a) \times (f')^{-1}(a')$ contains (u,u'), is open in $\mathfrak{X} \times \mathfrak{X}'$ and is disjoint from B. Therefore B is closed in $\mathfrak{X} \times \mathfrak{X}'$. It follows from proposition 2.29 that for all $(x,x') \in B$ we have $x \in V(p)$ iff $x' \in V'(p)$.

Let $\lambda \in \Lambda$ be n-ary and for $1 \leq i \leq n$ let (a_i, a_i') be a B-coherent pair of clopens. Suppose uBu' and $\gamma(u) \in \lambda_{\mathfrak{X}}(a_1,\ldots,a_n)$. We will show that $\gamma'(u') \in \lambda_{\mathfrak{X}'}(a_1',\ldots,a_n')$, the converse direction is similar.

Let us construct for each pair (a_i, a_i') a clopen set $b_i \in \mathrm{Clp}\,\mathfrak{Y}$ such that $f[a_i] \subseteq b_i$ and $(f')^{-1}(b_i) \subseteq a_i'$. Since a_i is clopen, $f[a_i]$ is closed in $\mathfrak{Y}$, so we may write $f[a_i] = \bigcap\{c \in \mathrm{Clp}\,\mathfrak{Y} \mid f[a_i] \subseteq c\}$. Because continuous maps preserve arbitrary meets, we have

$$\bigcap\{(f')^{-1}(c) \mid f[a_i] \subseteq c \in \mathrm{Clp}\,\mathfrak{Y}\} = (f')^{-1}(f[a_i]) \subseteq a'.$$

The collection $\{\mathfrak{X}' \setminus (f')^{-1}(c) \mid f[a_i] \subseteq c \in \mathrm{Clp}\,\mathfrak{Y}\}$ is an open cover of the (closed hence) compact set $\mathfrak{X}' \setminus a'$, so there exists a finite number $c_1,\ldots,c_m \in \mathrm{Clp}\,\mathfrak{Y}$ such that $\bigcup_{j=1}^m \mathfrak{X}' \setminus (f')^{-1}(c_j)$ covers $\mathfrak{X}' \setminus a'$. Set $b_i = c_1 \cap \cdots \cap c_m$, then $b_i \in \mathrm{Clp}\,\mathfrak{Y}$ and $(f')^{-1}(b_i) \subseteq a_i$. Moreover $f[a_i] \subseteq b_i$, hence $a_i \subseteq f^{-1}(b_i)$.

By monotonicity and naturality of λ we find

$$\gamma(u) \in \lambda_{\mathfrak{X}}(a_1,\ldots,a_n) \subseteq \lambda_{\mathfrak{X}}(f^{-1}(b_1),\ldots,f^{-1}(b_n)) = (\mathbb{T}f)^{-1}(\lambda_{\mathfrak{Y}}(b_1,\ldots,b_n)),$$

so $(\mathbb{T}f)(\gamma(u)) \in \lambda_{\mathfrak{Y}}(b_1,\ldots,b_n)$. Since f and f' are coalgebra morphisms we have $(\mathbb{T}f)(\gamma(u)) = \delta(f(u)) = \delta(f'(u')) = (\mathbb{T}f')(\gamma'(u'))$ and by monotonicity and naturality of λ again we find

$$\gamma'(u') \in (\mathbb{T}f')^{-1}(\lambda_{\mathfrak{Y}}(b_1,\ldots,b_n)) = \lambda_{\mathfrak{X}'}((f')^{-1}(b_1),\ldots,(f')^{-1}(b_n)) \subseteq \lambda_{\mathfrak{X}'}(a_1',\ldots,a_n').$$

This proves the proposition. □

2.35 Example (Descriptive frames)**.** As a first example of logic on a Stone-coalgebra, we mention descriptive frames for modal logic [37]. Descriptive frames turn out to be coalgebras for the Vietoris functor:

For a topological space $\mathfrak{X}$ let $\mathbb{V}\mathfrak{X}$ be the set of closed subsets of $\mathfrak{X}$ topologised by the subbase

$$\boxplus a := \{b \in \mathbb{V}\mathfrak{X} \mid b \subseteq a\}, \qquad \boxplus a := \{b \in \mathbb{V}\mathfrak{X} \mid a \cap b \neq \varnothing\},$$

where a ranges over the opens of $\mathfrak{X}$. This assignment can be extended to a functor on Top by defining $\mathbb{V}f : \mathbb{V}\mathfrak{X} \to \mathbb{V}\mathfrak{X}'$ to be the direct image of f, for continuous functions $f : \mathfrak{X} \to \mathfrak{X}'$. It is well known that the Vietoris functor restricts to KTop, KHaus and Stone, and that the category of descriptive frames and its morphisms is isomorphic to $\mathsf{Coalg}(\mathbb{V}_{\mathsf{Stone}})$ [37]. For a thorough survey of properties of the Vietoris functor, see [58]. ◁

Descriptive monotone frames are another important example of Stone-coalgebras. Below we give a way to view these as Stone coalgebras which is slightly different from, but equivalent to [16, 21]. The example will also play a role in the next chapter.

2.36 Definition ([20], Definition 7.30)**.** A **general monotone frame** is a triple (X,μ,A) where (X,μ) is a monotone frame and $A \subseteq \mathbb{P}X$ is a collection of **admissible subsets** of

X which contains $\varnothing$ and X and is closed under finite intersection, finite union, taking complements and the map

$$m_\mu : \mathbb{P}X \to \mathbb{P}X : a \mapsto \{x \in X \mid a \in \mu(x)\}.$$

A **general monotone frame morphism** from (X, μ, A) to (X', μ', A') is a bounded morphism $f : (X, \mu) \to (X', \mu')$ between the underlying monotone frames such that $f^{-1}(a') \in A$ for all $a' \in A'$.

Let $\mathfrak{X}$ denote the topological space with underlying set X topologised by the clopen subbase A. A general monotone frame is called **differentiated** if $x \in a \Leftrightarrow x' \in a$ for all $a \in A$ implies $x = x'$. It is called **tight** if for all $x \in X$, $c \in K\mathfrak{X}$ and $u \subseteq X$ we have

- $c \in \nu(x)$ iff every admissible superset $a \supseteq c$ is in $\nu(x)$; and
- $u \in \nu(x)$ iff there exists a closed subset $c \subseteq u$ that is in $\nu(x)$.

A general monotone frame is called **compact** if A is compact. A **descriptive monotone frame** is differentiated, tight and compact general monotone frame. $\lhd$

The following definition is taken from [16] and is equivalent to definition 3.9 in [21].

2.37 Definition. For a Stone space $\mathfrak{X} = (X, \tau)$ define $\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$ to be the collection of sets $W \subseteq \mathbb{P}X$ such that $a \in W$ iff there exists a closed $c \subseteq a$ such that every clopen superset of c is in W. Endow $\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$ with the topology generated by the clopen subbase

$$\boxplus a := \{W \in \mathbb{D}'_{\mathsf{st}}\mathfrak{X} \mid a \in W\}, \quad \diamondplus a := \{W \in \mathbb{D}'_{\mathsf{st}}\mathfrak{X} \mid Xa \notin W\},$$

where a ranges over $\mathbb{C}\mathrm{lp}\,\mathfrak{X}$.

For continuous functions $f : \mathfrak{X} \to \mathfrak{X}'$ define

$$\mathbb{D}'_{\mathsf{st}} f : \mathbb{D}'_{\mathsf{st}}\mathfrak{X} \to \mathbb{D}'_{\mathsf{st}}\mathfrak{X}' : W \mapsto \{a \in \mathbb{P}X \mid f^{-1}(a) \in W\}. \qquad \lhd$$

Descriptive monotone frames are known to be coalgebras for $\mathbb{D}'_{\mathsf{st}}$. In fact, the category of descriptive conditional frames and general monotone frame morphisms, DMF, is isomorphic to the category of $\mathbb{D}'_{\mathsf{st}}$-coalgebras and $\mathbb{D}'$-coalgebra morphisms [21],

$$\mathsf{DMF} \cong \mathsf{Coalg}(\mathbb{D}'_{\mathsf{st}}).$$

The functor $\mathbb{D}_{\mathsf{st}}$ in the next definition arises from definition 2.37 by replacing the use of clopen sets by open sets. This functor will turn out to be equivalent to $\mathbb{D}'_{\mathsf{st}}$, but allows for a generalisation to the category of compact Hausdorff spaces in section 3.3.

2.38 Definition. Let $\mathfrak{X} = (X, \tau)$ be a Stone space. Let $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$ be the collection of sets $W \subseteq \mathbb{P}X$ such that $a \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. Endow $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$ with the topology generated by the subbase

$$\boxplus a := \{W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X} \mid a \in W\}, \quad \diamondplus a := \{W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X} \mid X \setminus a \notin W\},$$

where a ranges over $\Omega\mathfrak{X}$. For continuous functions $f : \mathfrak{X} \to \mathfrak{X}'$ define $\mathbb{D}_{\mathsf{st}} f : \mathbb{D}_{\mathsf{st}}\mathfrak{X} \to \mathbb{D}_{\mathsf{st}}\mathfrak{X}' : W \mapsto \{a' \in \mathbb{P}X \mid f^{-1}(a') \in W\}$. $\lhd$

2.39 Lemma. *If $f : \mathfrak{X} \to \mathfrak{X}'$ is a morphism in* Stone*, then $\mathbb{D}_{\mathsf{st}} f$ is a well-defined continuous function from $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$ to $\mathbb{D}_{\mathsf{st}}\mathfrak{X}'$.*

Proof. $\mathbb{D}_{\mathsf{st}}f$ *is well-defined.* Let $W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}$. We need to show that $\mathbb{D}_{\mathsf{st}}f(W) \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}'$. Suppose $a' \in \mathbb{D}_{\mathsf{st}}f(W)$. Then $f^{-1}[a'] \in W$ and so there exists a closed $c \subseteq f^{-1}[a']$ such that $c \in W$. Since $\mathfrak{X}$ is compact and $\mathfrak{X}'$ is Hausdorff, $f[c]$ is a closed set in $\mathfrak{X}'$. Besides $f[c] \subseteq a'$. Suppose $f[c] \subseteq b$ for some open $b \in \Omega\mathfrak{X}'$, then $c \subseteq f^{-1}[b]$ so $f^{-1}[b] \in W$ and hence $b \in \mathbb{D}_{\mathsf{st}}f(W)$. So all open supersets of $f[c]$ are in $\mathbb{D}_{\mathsf{st}}f(W)$, and therefore $f[c] \in \mathbb{D}_{\mathsf{st}}f(W)$.

$\mathbb{D}_{\mathsf{st}}f$ *is continuous.* For continuity we need to show that both $(\mathbb{D}_{\mathsf{st}}f)^{-1}[\boxplus a']$ and $(\mathbb{D}_{\mathsf{st}}f)^{-1}[\diamondplus a']$ are open in $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$, whenever $a' \in \Omega(\mathfrak{X}')$. It follows for a straightforward computation that $(\mathbb{D}_{\mathsf{st}}f)^{-1}(\boxplus a') = \boxplus f^{-1}(a')$, which is open in $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$ by definition. In a similar way we find $(\mathbb{D}_{\mathsf{st}}f)^{-1}(\diamondplus a') = \diamondplus f^{-1}[a'] \in \Omega\mathbb{D}_{\mathsf{st}}\mathfrak{X}$. □

Note that the first part of the previous lemma makes use of the fact that $\mathfrak{X}$ is compact and $\mathfrak{X}'$ is Hausdorff.

For an element $W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}$ its upward closure is defined by $\uparrow(W) := \{u \subseteq X \mid \exists u' \in W \text{ s.t. } u' \subseteq u\}$. The following lemma gives a more intuitive characterisation of the action of $\mathbb{D}_{\mathsf{st}}$ on morphisms. The proof is straightforward.

2.40 Lemma. *Let $f : \mathfrak{X} \to \mathfrak{X}'$ be a continuous map between compact Hausdorff spaces and suppose $W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}$. Then*

$$\mathbb{D}_{\mathsf{st}}f(W) = \uparrow(\{f[u] \mid u \in W\}).$$

Finally, let us show that definition 2.38 is equivalent to definition 2.37 when restricted to Stone. It will follow as a corollary that $\mathbb{D}_{\mathsf{st}}\mathfrak{X}$ is a Stone space whenever $\mathfrak{X}$ is a Stone space.

2.41 Theorem. *Let $\mathfrak{X} = (X, \tau)$ be a Stone space. Then $\mathbb{D}_{\mathsf{st}}\mathfrak{X} \cong \mathbb{D}'_{\mathsf{st}}\mathfrak{X}$.*

Proof. We first show that the sets underlying both topological spaces are the same. It is obvious that $\mathbb{D}_{\mathsf{st}}\mathfrak{X} \subseteq \mathbb{D}'_{\mathsf{st}}\mathfrak{X}$. Conversely, take $W \in \mathbb{D}'_{\mathsf{st}}\mathfrak{X}$. To show that $W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}$ take an arbitrary $a \in W$. By definition of $\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$ there exists a closed set $k \subseteq a$ such that all clopen supersets of k are in W. Let b be any open superset of k. Since the clopen sets form a basis for $\mathfrak{X}$, for each $x \in a$ we can find a clopen c_x such that $x \in c_x \subseteq b$. The set k is covered by a finite amount of such sets because it is closed and $\mathfrak{X}$ is compact. Therefore there is a clopen set c such that $k \subseteq c \subseteq b$. By assumption we have $c \in W$, hence $b \in W$. This shows that for all $a \in W$ there is a closed subset k of a such that every open superset of k is in W, so $W \in \mathbb{D}_{\mathsf{st}}\mathfrak{X}$.

Next let us compare the topologies. It follows immediately form the definitions that $\Omega\mathbb{D}'_{\mathsf{st}}\mathfrak{X} \subseteq \Omega\mathbb{D}_{\mathsf{st}}\mathfrak{X}$. For the converse, it suffices to show that $\boxplus a, \diamondplus a \in \Omega\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$ for $a \in \Omega\mathfrak{X}$. Suppose $W \in \boxplus a$. Then $a \in W$ hence there is a closed $k \subseteq a$ such that all open supersets of k are in W. Since $\mathfrak{X}$ is a Stone space there exists a clopen c such that $k \subseteq c \subseteq a$. By assumption $c \in W$, so $W \in \boxplus c$. Since $\boxplus c \subseteq \boxplus a$, this proves that every element in $\boxplus a$ has an open neighbourhood in $\Omega\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$ contained in $\boxplus a$, hence $\boxplus a \in \Omega\mathbb{D}'_{\mathsf{st}}\mathfrak{X}$. The case of $\diamondplus$ can be treated similarly. □

2.42 Corollary. *The functor $\mathbb{D}_{\mathsf{st}}$ is an endofunctor on* Stone.

Another guiding example of logic on Stone-coalgebras is that of descriptive conditional frames, which will be developed in chapter 5.

3

Coalgebraic geometric logic

In this chapter we investigate how one can extend geometric logic (i.e. logic with finite conjunctions and infinite disjunctions) with extra modalities. Some advances have been made in this field: Johnstone [29] defines a point-free, syntactic version of the Vietoris functor, using an extension of geometric logic with two unary operators, $\Box$ and $\Diamond$. Furthermore, in [57] the authors define the so-called Vietoris powerlocale functor $V_{\mathbb{T}}$: $\mathsf{Frm} \to \mathsf{Frm}$ for a given set functor $\mathbb{T}$ which satisfies some categorical properties, and take steps towards developing a logic with finite conjunctions, infinite disjunctions and a single modality.

Whereas the authors of [57] use the method of relation lifting to define the new modality, we use a modified form of predicate liftings. Besides, where they take an algebraic point of view, we adopt a topological approach. A category of (certain) topological spaces will form the base category of the coalgebras that we use, and the open sets serve as the interpretants of proposition letters.

In the Stone case, there is a dual equivalence between Stone and BA; the clopen sets in a Stone space, which are the interpretants of the proposition letters, form a Boolean algebra. This allows one to take both a topological and an algebraic point of view, i.e., every endofunctor on Stone gives rise, via this duality, to an endofunctor on BA and vice versa. In the new setting for coalgebraic geometric logic a similar duality is desirable. The open sets of a topological space also form an algebraic structure: a frame. In order to have a dual equivalence between topological spaces and frames, we have to restrict both categories (the category of topological spaces and continuous maps and the category of frames and frame homomorphisms) to suitable full subcategories. We will see in section 3.1 that there are several possibilities for this restriction. It is not *a priori* clear which of these is the right one. Throughout the chapter we will encounter pros and cons of each of these possibilities.

This chapter is structured as follows: In section 3.1 we lay the foundations for this chapter by investigating geometric logic and dualities. We find three candidates for the base category of coalgebraic geometric logic: the (full) subcategories of Top whose objects are sober spaces, compact sober spaces and compact Hausdorff spaces respectively. In the subsequent sections we develop coalgebraic geometric logic (section 3.2), examine two examples (section 3.3) and investigate bisimulations between the models for coalgebraic logic (section 3.4). The choice of base category will be continually remarked upon; where possible we will give definitions and results for all choices and whenever this is not possible we will indicate the problem.

3.1 Geometric logic and duality

Before defining geometric logic, we recall some definitions concerning frames.

3.1 Definition. A **frame** is a complete lattice F in which for all $a \in F$ and $S \subseteq F$ the infinite distributive law holds:

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A **frame homomorphism** is a function between frames that preserves finite meets and arbitrary joins. ◁

3.2 Definition. A **presentation** is a pair $\langle G, R \rangle$ where G is a set of generators and R is a collection of relations between expressions constructed from the generators using arbitrary joints and finite meets.

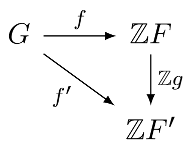
Let F be a frame. Recall that $\mathbb{Z}F$ is the underlying frame. We say that $\langle G, R \rangle$ **presents** F if there is an assignment $f : G \to \mathbb{Z}F$ of the generators such that (i), (ii) and (iii) hold:

(i) The set $\{f(g) \mid g \in G\}$ generates F.

The assignment f can be extended to an assignement $\widetilde{f}$ for any expression x build from the generators in G using $\wedge$ and $\bigvee$. We require

(ii) If $x = x'$ is a relation in R, then $\widetilde{f}(x) = \widetilde{f}(x')$ in F.

(iii) For any F' and assignment $f' : G \to \mathbb{Z}F'$ satisfying property (ii) there exists a frame homomorphism $h : F \to F'$ such that the diagram



commutes. ◁

The frame homomorphism from (iii) is necessarily unique, because the image of the generating set $\{f(g) \mid g \in G\}$ under h is determined by the diagram. A detailed account of frame presentations may be found in chapter 4 of [59].

3.3 Remark. We will regularly want to define a frame homomorphism $F \to F'$ from a frame F presented by $\langle G, R \rangle$ to some frame F'. By definition 3.2 it suffices to give an assignment $f' : G \to F'$ such that (ii) holds, because this yields a unique frame homomorphism $F \to F'$. By abuse of notation, we will denote the unique frame homomorphism $F \to F'$ such that the diagram in (iii) commutes with f' as well.

The next propositions allows us to define a frame by generators and relations. A proof can be found in [29, Proposition II2.11].

3.4 Proposition. *Any presentation by generators and relations presents a frame.*

3.5 Definition. A set B of elements in a frame F is called **directed** if for all $a, b \in B$ there is a $c \in B$ such that $a \leq c$ and $b \leq c$. We denote the disjunction $\bigvee B$ of a directed set B by $\bigvee^{\uparrow} B$, that is, the symbol $\bigvee^{\uparrow}$ indicates that the set B is a directed set.

A collection $B \subseteq \mathbb{P}X$ of subsets of a set X is called directed if for all $a, b \in B$ there is a $c \in B$ such that $a \subseteq c$ and $b \subseteq c$. We write $\bigcup^{\uparrow} B$ for the union $\bigcup B$ of such a directed set B. ◁

The opens of a topological space with the inclusion order provide an example of a frame. In this case the meet and join are simply set-theoretic intersection and union. Indeed, a finite meet of open sets is again open, as is an arbitrary union of opens. In fact, there is a contravariant functor $\mathsf{Top} \to \mathsf{Frm}$ sending a topological space to its frame of open sets.

3.6 Definition. Let $\mathfrak{X}$ be a topological space. Define $\mathfrak{opn}\,\mathfrak{X}$ to be the frame of open sets of $\mathfrak{X}$ (that is, the collection of open sets ordered by inclusion; it is routine to check that this is indeed a lattice). For a continuous function $f : \mathfrak{X} \to \mathfrak{X}'$ let $\mathfrak{opn}\, f = f^{-1} : \mathfrak{opn}\,\mathfrak{X}' \to \mathfrak{opn}\,\mathfrak{X}$. The map $\mathfrak{opn} : \mathsf{Top} \to \mathsf{Frm}$ is a contravariant functor.

A frame isomorphic to $\mathfrak{opn}\,\mathfrak{X}$ for some topological space $\mathfrak{X}$ is called **spatial**. ◁

An equivalent definition of spatiality is given in [29, II1.5]. The following definition is stated for future reference.

3.7 Definition. Let F be a frame. A filter in F is a nonempty upwards closed set J such that $a, b \in J$ implies $a \wedge b \in J$. A filter is called **prime** if $a \vee b \in J$ implies $a \in J$ or $b \in J$. A **completely prime filter** is a filter such that for all $S \subseteq A$, $\bigvee S \in J$ implies there is $a \in S$ with $a \in J$.

For $a, b \in F$ we say that a is **well inside** b, notation: $a \lessdot b$, if there is a $c \in F$ such that $c \wedge a = \bot$ and $c \vee b = \top$. An element $a \in F$ is called **regular** if $a = \bigvee\{b \in F \mid b \lessdot a\}$ and a frame is called **regular** if all of its elements are regular. The **negation** of $a \in F$ is defined as ${\sim}a = \bigvee\{b \in F \mid a \wedge b = \bot\}$.

A frame F is **compact** if for all directed sets S, $\bigvee^{\uparrow} S = \top$ implies $\top \in S$. ◁

3.8 Lemma. *For all elements a, b in a frame F we have $a \lessdot b$ iff ${\sim}a \vee b = \top$.*

Proof. See III1.1 in [29]. □

3.9 Lemma. *Finite meets and arbitrary joins of regular elements are regular.*

Proof. It is known that $d \leq c \lessdot a \leq b$ implies $d \lessdot b$. We first show that $c \lessdot a$ and $d \lessdot b$ implies $c \wedge d \lessdot a \wedge b$. It is clear that $c \wedge d \lessdot a$ and $c \wedge d \lessdot b$. Since ${\sim}(c \wedge d) \vee (a \wedge b) = ({\sim}(c \wedge d) \vee a) \wedge ({\sim}(c \wedge d) \vee b) = \top \wedge \top = \top$ we know $c \wedge d \lessdot a \wedge b$.

Now suppose a and b are regular elements, then

$$a \wedge b = \bigvee\{c \mid c \lessdot a\} \wedge \bigvee\{d \mid d \lessdot b\} = \bigvee\{c \wedge d \mid c \lessdot a, d \lessdot b\} \leq \bigvee\{c \mid c \lessdot a \wedge b\} \leq a \wedge b,$$

so $a \wedge b$ is regular. If a_i is regular for all i in some index set I, then

$$\bigvee_{i \in I} a_i = \bigvee_{i \in I} \Big(\bigvee\{c \mid c \lessdot a_i\} \Big) \leq \bigvee \Big\{ c \mid c \lessdot \bigvee_{i \in I} a_i \Big\} \leq \bigvee_{i \in I} a_i,$$

so an arbitrary join of regular elements is regular. □

Now let us proceed to geometric logic. As stated in the introduction, geometric logic can be viewed as the logic of finitely observable statements. A finitely observable statement is a statement which can be verified in a finite amount of time. For example, the statement

> "There exist glow-in-the-dark turtles."

To verify this statement, it suffices to find a single glow-in-the-dark turtle. Therefore the statement is finitely observable. On the other hand, to refute the statement, one would have to find all turtles in the world and check that they do not glow in the dark. To be complete, one should also check all past and future turtles. In a practical sense, this statement can never be refuted. Thus, the statement

> "Glow-in-the-dark turtles do not exist"

is *not* finitely observable.[1]

The previous discussion shows that finitely observable statements are not closed under taking negations. The reader can easily convince himself that the collection of finitely observable statements is not closed under implications either. However, finitely observable statements *are* closed under taking arbitrary disjunctions and finite conjunctions. This intuition leads to the following definition of geometric logic.

3.10 Definition. Let Φ be a set of proposition letters. The geometric formulae over Φ are given by

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee_{i \in I} \varphi_i,$$

where $p \in \Phi$. We abbreviate $\bot = \bigvee \varnothing$. Write $\mathcal{GL}$ for the collection of geometric formulas.

A **sequent** is a pair of $\mathcal{GL}$-formulas. We write $\varphi \vdash \psi$ if (φ, ψ) is a sequent. Intuitively, this should be thought of as "φ implies ψ". A **geometric theory** over Φ is a collection of sequents that contains the axioms $\varphi \vdash \varphi$ and is closed under the following rules: **cut**

$$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi},$$

the **conjuction rules**

$$\varphi \vdash \top, \quad \varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi, \quad \frac{\varphi \vdash \psi \qquad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi},$$

the **disjunction rules**

$$\varphi \vdash \bigvee S \quad (\varphi \in S), \quad \frac{\varphi \vdash \psi \text{ (for all } \varphi \in S)}{\bigvee S \vdash \psi}$$

and **frame distributivity**

$$\varphi \wedge \bigvee S \vdash \bigvee \{\varphi \wedge \psi \mid \psi \in S\}.$$

Let T be a theory. If T contains $\varphi \vdash \psi$ and $\psi \vdash \varphi$ we say that φ and ψ are **equivalent with respect to** T. We call φ and ψ **equivalent** if they are equivalent with respect to every theory. ◁

3.11 Remark. The collection $\mathcal{GL}$ is not generally a set; it may be a proper class. However, frame distributivity allows us to reduce every formula to an equivalent disjunction of finite conjunctions of symbols in Φ. Therefore the collection of formulas modulo equivalence forms a set. Let T be a geometric theory. The rules imply that the Lindenbaum-Tarski algebra, i.e. the set of geometric formulas modulo equivalence with respect to T of a theory is a frame [62]. Accordingly, we shall call it the Lindenbaum-Tarski frame.

[1] The existence of glow in the dark turtles has never been refuted. In fact, they have been observed recently. See [19] for a scientific article and [41] for a video.

For more information about the connection between geometric logic and frames we refer to [62]. Topological spaces with a valuation form models for geometric logic.

3.12 Definition. A valuation of a topological space $\mathcal{X}$ is a map $V : \Phi \to \Omega\mathcal{X}$. One may define truth of $\mathcal{GL}$-formulas in $\mathfrak{X} = (\mathcal{X}, V)$ inductively by

$$[\![\top]\!]^{\mathfrak{X}} = \mathbb{U}\mathcal{X}, \quad [\![p]\!]^{\mathfrak{X}} = V(p), \quad [\![\varphi_1 \wedge \varphi_2]\!]^{\mathfrak{X}} = [\![\varphi_1]\!]^{\mathfrak{X}} \cap [\![\varphi_2]\!]^{\mathfrak{X}}, [\![\bigvee_{i \in I} \varphi_i]\!]^{\mathfrak{X}} = \bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}}.$$

We write $\mathfrak{X}, x \Vdash \varphi$ iff $x \in [\![\varphi]\!]^{\mathfrak{X}}$. ◁

In definition 3.6 we have seen the functor $\mathfrak{opn} : \mathsf{Top} \to \mathsf{Frm}$. We will now define a functor in the opposite direction that is right adjoint to $\mathfrak{opn}$.

3.13 Definition. A **point** of a frame F is a frame homomorphism $p : F \to 2$, with $2 = \{\top, \bot\}$ the two-element frame. Let $\mathfrak{pt} F$ be the collection of points of F endowed with the topology $\{\widetilde{a} \mid a \in F\}$, where $\widetilde{a} = \{p \in \mathfrak{pt} F \mid p(a) = \top\}$. For a frame homomorphism $f : F \to F'$ define $\mathfrak{pt} f : \mathfrak{pt} F' \to \mathfrak{pt} F$ by $p \mapsto p \circ f$. The assignment $\mathfrak{pt}$ defines a functor $\mathsf{Frm} \to \mathsf{Top}$.

A topological space that arises as the space of points of a lattice is called **sober**. The **sobrification** of a topological space $\mathcal{X}$ is $\mathfrak{pt}(\mathfrak{opn}\,\mathcal{X})$. ◁

There is a 1-1 correspondence between the points of a frame and the completely prime filters of the frame: For a completely prime filter F the map $p_F : A \to 2$ defined by $p_F(a) = \top$ iff $a \in F$ and $p_F(a) = \bot$ if $a \notin F$ is a point. Conversely, for a point p the set $p^{-1}(\top)$ is a completely prime filter.

Write SFrm, KSFrm and KRFrm for the full subcategories of Frm whose objects are spatial frames, compact spatial frames and compact regular frames, respectively. For topological spaces, write Sob, KSob and KHaus for the full subcategories of Top whose objects are sober spaces, compact sober spaces and compact Hausdorff spaces respectively. Furthermore, we write $\equiv$ for an equivalence between categories.

3.14 Proposition. *The functor $\mathfrak{pt}$ is a right adjoint to $\mathfrak{opn}$. This adjunction restricts to a duality between the category of spatial frames and the category of sober spaces,*

$$\mathsf{SFrm} \equiv \mathsf{Sob}^{\mathrm{op}}.$$

This duality restricts to the dualities

$$\mathsf{KSFrm} \equiv \mathsf{KSob}^{\mathrm{op}}$$

and

$$\mathsf{KRFrm} \equiv \mathsf{KHaus}^{\mathrm{op}}.$$

For a more thorough exposition of frames and spaces, and a proof of the statements in proposition 3.14 we refer to section C1.2 of [31]. We explicitly mention one isomorphism which is part of this duality, because we will encounter it later on in this thesis.

3.15 Remark. Let $\mathcal{X}$ be a sober space. Then proposition 3.14 entails that there is an isomorphism $\mathcal{X} \to \mathfrak{pt}(\mathfrak{opn}\,\mathcal{X})$. This isomorphism is given by $x \mapsto p_x$, where p_x is the point given by

$$p_x : \mathfrak{opn}\,\mathcal{X} \to 2 : \begin{cases} a \mapsto \top & \text{if } x \in a \\ a \mapsto \bot & \text{if } x \notin a \end{cases}$$

for all $x \in \mathcal{X}$ and $a \in \Omega\mathcal{X}$.

3.2 Coalgebraic geometric logic

As stated in the introduction of this chapter, it is not clear what base space we should use for the models of coalgebraic geometric logic. We have seen at the end of the previous section three candidates, Sob, KSob and KHaus, which all have a dually equivalent algebraic counterpart. We will start this section with defining the language and semantics for coalgebraic geometric logic and thereafter relate the notions of modal equivalence and behavioural equivalence.

The definitions of predicate liftings, geometric $\mathbb{T}$-models, the language and its semantics are the same whether we are working on Sob, KSob or KHaus. We write C for any of these categories.

3.16 Definition. Let $\mathbb{T}$ be an endofunctor on C. An **open predicate lifting for** $\mathbb{T}$ is a natural transformation

$$\lambda : \Omega^n \to \Omega \circ \mathbb{T}.$$

A collection of open predicate liftings for $\mathbb{T}$ is called a **geometric modal signature** for $\mathbb{T}$. An open predicate lifting is called **monotone in its i-th argument** if for all $a_1, \dots, a_n, b$ we have $\lambda_{\mathfrak{X}}(a_1, \dots, a_i, \dots, a_n) \subseteq \lambda_{\mathfrak{X}}(a_1, \dots, a_i \cup b, \dots, a_n)$ and **monotone** if it is monotone in every argument. A geometric modal signature for a functor $\mathbb{T}$ is called **monotone** if every open predicate lifting in it is monotone, and **characteristic** if for every object $\mathfrak{X}$ in C the collection

$$\{\lambda_{\mathfrak{X}}(a_1, \dots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}\ , a_i \in \Omega\mathfrak{X}\}$$

is a sub-base for the topology on $\mathbb{T}\mathfrak{X}$. ◁

Note that if $\mathbb{T}$ is an endofunctor on Sob that restricts to KSob or KHaus, then an open predicate lifting λ for $\mathbb{T}$ restricts to an open predicate lifting for KSob or KHaus respectively.

The next two definitions are the analogs of definitions 2.26 and 2.27. Recall that Φ is some fixed set of proposition letters.

3.17 Definition. A **geometric $\mathbb{T}$-model** for a functor $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ is a triple $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ where $(\mathcal{X}, \gamma)$ is a $\mathbb{T}$-coalgebra and $V : \Phi \to \Omega\mathcal{X}$ is a valuation of the proposition letters. A map $f : \mathcal{X} \to \mathcal{X}'$ is a **geometric $\mathbb{T}$-model morphism** from $(\mathcal{X}, \gamma, V)$ to $(\mathcal{X}', \gamma', V')$ if f is a coalgebra morphism between the underlying coalgebras and $f^{-1} \circ V' = V$. The collection of geometric $\mathbb{T}$-models and geometric $\mathbb{T}$-model morphisms forms a category, which we denote by $\mathsf{Mod}(\mathbb{T})$. ◁

3.18 Definition. The **language** induced by a geometric modal signature is the collection GML(Λ) of formulas defined by the grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee \varphi_i \mid \heartsuit^\lambda(\varphi_1, \dots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. Abbreviate $\bot := \bigvee \emptyset$. ◁

3.19 Definition. The **semantics** of $\varphi \in \text{GML}(\Lambda)$ on a geometric $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$, where $\mathbb{T}$ is an endofunctor on C, is given recursively by

$$[\![\top]\!]^{\mathfrak{X}} = X, \quad [\![p]\!]^{\mathfrak{X}} = V(p), \quad [\![\varphi \wedge \psi]\!]^{\mathfrak{X}} = [\![\varphi]\!]^{\mathfrak{X}} \cap [\![\psi]\!]^{\mathfrak{X}}, \quad [\![\bigvee_{i \in I} \varphi_i]\!]^{\mathfrak{X}} = \bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}},$$

$$[\![\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)]\!]^{\mathfrak{X}} = \gamma^{-1}(\lambda_{\mathcal{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}})).$$

We write $\mathfrak{X}, x \Vdash \varphi$ iff $x \in [\![\varphi]\!]^{\mathfrak{X}}$. Two states x and x' are called **modally equivalent** if they satisfy the same formulas, notation $x \equiv_\Lambda x'$. ◁

The following proposition shows that morphisms preserve truth. Its proof is similar to the proof of theorem 6.17 in [56]. We give it here for the sake of completeness.

3.20 Proposition. *Let $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ be a functor and Λ a geometric modal signature for $\mathbb{T}$. Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be geometric $\mathbb{T}$-models and let $f : \mathfrak{X} \to \mathfrak{X}'$ be a geometric $\mathbb{T}$-model morphism. Then for all $\varphi \in \mathrm{GML}(\Lambda)$ and $x \in \mathcal{X}$ we have*

$$\mathfrak{X}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{X}', f(x) \Vdash \varphi.$$

Proof. We will prove that $[\![\varphi]\!]^{\mathfrak{X}} = (\Omega f)[\![\varphi]\!]^{\mathfrak{X}'}$ for all formulas φ using induction on the complexity of the formula. The propositional case follows from the definition of a geometric $\mathbb{T}$-model morphism. The cases $\wedge$ and $\bigvee$ are routine, so the only case left is $[\![\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{X}} = (\Omega f)[\![\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{X}'}$.

Since f is a coalgebra morphism the left diagram below commutes. Applying Ω to the diagram yields the right commutative diagram.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ \mathbb{T}\mathcal{X} & \xrightarrow[\mathbb{T}f]{} & \mathbb{T}\mathcal{X}' \end{array} \qquad \begin{array}{ccc} \Omega\mathcal{X} & \xleftarrow{\Omega f} & \Omega\mathcal{X}' \\ {\scriptstyle\Omega\gamma}\uparrow & & \uparrow{\scriptstyle\Omega\gamma'} \\ \Omega(\mathbb{T}\mathcal{X}) & \xleftarrow[\Omega(\mathbb{T}f)]{} & \Omega(\mathbb{T}\mathcal{X}') \end{array} \tag{3.1}$$

We observe

$$\begin{aligned} [\![\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{X}} &= (\Omega\gamma)(\lambda_{\mathcal{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}})) && \text{(definition 3.19)} \\ &= (\Omega\gamma)(\lambda_{\mathcal{X}}((\Omega f)[\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, (\Omega f)[\![\varphi_n]\!]^{\mathfrak{X}'})) && \text{(induction)} \\ &= (\Omega\gamma)(\Omega(\mathbb{T}f)(\lambda_{\mathcal{X}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}'}))) && \text{(naturality of } \lambda) \\ &= (\Omega f)(\Omega\gamma')\lambda_{\mathcal{X}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}'}) && \text{(by (3.1))} \\ &= (\Omega f)[\![\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{X}'}. && \text{(definition 3.19)} \end{aligned}$$

This proves the proposition. □

We will now relate modal equivalence to behavioural equivalence. It is necessary to restrict our attention to a single base category, namely Sob, rather than the base category C we have been working with until now. The following remark is very important.

3.21 Remark. The notion of behavioural equivalence depends on the base category we are working over! Suppose $\mathbb{T}$ is a functor on Sob, Λ is a geometric modal signature for $\mathbb{T}$ and $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ are two geometric $\mathbb{T}$-models. If $x \in \mathcal{X}$ and $x' \in \mathcal{X}'$ are behaviourally equivalent, then there exists a $\mathfrak{Z} \in \mathsf{Mod}(\mathbb{T})$ and model morphisms $f : \mathfrak{X} \to \mathfrak{Z}$ and $f' : \mathfrak{X}' \to \mathfrak{Z}$ such that $f(x) = f'(x')$.

Now suppose that $\mathbb{T}$ restricts to KHaus (write $\mathbb{T}_{\mathsf{KHaus}}$ for this restriction) and $\mathcal{X}$ and $\mathcal{X}'$ happen to be compact Hausdorff spaces. If we view $\mathcal{X}$ and $\mathcal{X}'$ as geometric $\mathbb{T}_{\mathsf{KHaus}}$-models, the elements x and x' are not necessarily behaviourally equivalent. Indeed, the model $\mathfrak{Z}$ need not be compact Hausdorff. Therefore, we should be careful with the notion of behavioural equivalence and, if ambiguity arises, specify with respect to which base category the elements are behaviourally equivalent.

It turns out that modal equivalence and behavioural equivalence coincide when $\mathsf{C} = \mathsf{Sob}$ and the geometric modal signature for a functor is characteristic. The remainder of this section is devoted to proving the following theorem, and a variation thereof.

3.22 Theorem. *Let* $\mathbb{T}$ *be an endofunctor on* Sob*,* Λ *a characteristic geometric modal signature for* $\mathbb{T}$ *and* $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ *and* $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ *two geometric* $\mathbb{T}$*-models. Then* $x \in \mathcal{X}$ *and* $x' \in \mathcal{X}'$ *are modally equivalent if and only if they are behaviourally equivalent.*

In order to prove this theorem, we first investigate the necessary constructions.

3.23 Definition. Let $\mathbb{T}$ be an endofunctor on C and Λ a geometric modal signature for $\mathbb{T}$. We call two formulas φ and ψ equivalent on $\mathsf{Mod}(\mathbb{T})$ with respect to Λ, notation $\varphi \equiv_{\mathbb{T},\Lambda} \psi$ if $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \psi$ for all geometric $\mathbb{T}$-models $\mathfrak{X}$ and states $x \in \mathfrak{X}$. Denote the equivalence class of φ by $[\varphi]$. Let $\mathbf{Equiv}(\mathbb{T}, \Lambda, \Phi)$ be the collection of formulas modulo $\equiv_{\mathbb{T},\Lambda}$ and define disjunction and arbitrary conjunction by

$$[\varphi] \wedge [\psi] \coloneqq [\varphi \wedge \psi]$$
$$\bigvee_{i \in I} [\varphi_i] \coloneqq \Big[\bigvee_{i \in I} \varphi_i \Big].$$

(This is well defined by lemma 3.24.) We call this the **equivalence frame** for $\mathbb{T}$ with respect to Λ. ◁

3.24 Lemma. *Let* $\mathbb{T}$ *be an endofunctor on* C *and* Λ *a geometric modal signature for* $\mathbb{T}$*. Then* $\mathbf{Equiv}(\mathbb{T}, \Lambda, \Phi)$ *is indeed a frame.*

Proof. We need to show the conjunction and disjunction from definition 3.23 are well defined, that is, they do not rely on the choice of the representatives. Suppose $\varphi_i \equiv_{\mathbb{T},\Lambda} \psi_i$ for all i in some index set I. Then

$$\mathfrak{X}, x \Vdash \bigvee_{i \in I} \varphi_i \quad \text{iff} \quad \mathfrak{X}, x \Vdash \varphi_i \text{ for some } i \in I \quad \text{iff} \quad \mathfrak{X}, x \Vdash \psi_i \quad \text{iff} \quad \mathfrak{X}, x \Vdash \bigvee_{i \in I} \psi_i.$$

The case for the conjunction is similar. □

The theory of a point x in a geometric $\mathbb{T}$-model $\mathfrak{X}$ is the collection of formulas that are true at x. It is easy to show that the theory of x is a completely prime filter in $\mathbf{Equiv}(\mathbb{T}, \Lambda, \Phi)$. This motivate the next definition.

3.25 Definition. Let $\mathbb{T}$ be an endofunctor on C and Λ a geometric modal signature for $\mathbb{T}$. Define $\mathcal{Z} = \mathfrak{pt}(\mathbf{Equiv}(\mathbb{T}, \Lambda, \Phi))$. For every geometric $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ define

$$\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Z} : x \mapsto \{\varphi \in \mathrm{GML}(\Lambda) \mid \mathfrak{X}, x \Vdash \varphi\}. \quad ◁$$

The space $\mathcal{Z}$ will turn out to be the state space of a final coalgebra in $\mathsf{Mod}(\mathbb{T})$. We use the letter $\mathcal{Z}$ because it is the final letter of the roman alphabet. The functor $\mathbb{T}$ has a dual on SFrm.

3.26 Definition. Let $\mathbb{T}$ be a functor on Sob and Λ a characteristic geometric modal signature for $\mathbb{T}$. Define

$$\mathbb{L} : \mathsf{Frm} \to \mathsf{Frm}$$

by

$$\mathbb{L} = \mathfrak{opn} \circ \mathbb{T} \circ \mathfrak{pt}.$$

Obviously, this functor restricts to an endofunctor on SFrm, where it is (trivially) dual to $\mathbb{T}$. ◁

We will now endow $\mathbf{Equiv}(\mathbb{T},\Lambda,\Phi)$ with an $\mathbb{L}$-algebra structure. For tidiness, and because no confusion is likely to arise, we will write $\mathcal{E}$ for $\mathbf{Equiv}(\mathbb{T},\Lambda,\Phi)$.

Since Λ is characteristic, the frame $\mathbb{L}\mathcal{E}$ is generated by the sets $\{\lambda_{\mathfrak{X}}(a_1,\dots,a_n) \mid \lambda \in \Lambda, a_i \in \Omega\mathfrak{Z}\}$. Therefore we can define an assignment on these generators and use remark 3.3 to extend this to a frame homomorphism $\mathbb{L}\mathcal{E} \to \mathcal{E}$. Recall that by definition all open sets of $\mathfrak{pt}\,\mathcal{E}$ are of the form $\widetilde{\varphi}$ for some formula $\varphi \in \mathrm{GML}(\Lambda)$.

3.27 Definition. Endow $\mathcal{E}$ with an $\mathbb{L}$-algebra structure $\delta : \mathbb{L}\mathcal{E} \to \mathcal{E}$, where δ is defined on generators by

$$\delta : \mathbb{L}\mathcal{E} \to \mathcal{E} : \lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) \mapsto [\heartsuit^{\lambda}(\varphi_1,\dots,\varphi_n)]. \qquad \lhd$$

3.28 Lemma. *δ is well defined.*

Proof. In order to prove that this is well-defined we need to show that the images of the generators of $\mathcal{E}$ satisfy the same relations that they satisfy in $\mathbb{L}\mathcal{E}$. Recall $\mathfrak{Z} = \mathfrak{pt}\,\mathcal{E}$, then $\mathbb{L}\mathcal{E} = \mathfrak{opn}(\mathbb{T}\mathfrak{Z})$. We need to show that

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathfrak{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathfrak{Z}}(\widetilde{\varphi}^{k,\ell}_1,\dots,\widetilde{\varphi}^{k,\ell}_{n_{k,\ell}})\Big) \tag{3.2}$$

implies

$$\bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}\heartsuit^{\lambda^{i,j}}(\varphi^{i,j}_1,\dots,\varphi^{i,j}_{n_{i,j}})\Big) \equiv_{\mathbb{T},\Lambda} \bigvee_{k\in K}\Big(\bigwedge_{\ell\in L_k}\heartsuit^{\lambda^{k,\ell}}(\varphi^{k,\ell}_1,\dots,\varphi^{k,\ell}_{n_{k,\ell}})\Big), \tag{3.3}$$

where the J_i and L_k are finite index sets. We will see that this follows from naturality of λ. Our strategy is to show that the truth sets of the right hand-side and left hand-side of (3.3) coincide in every geometric $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X},\gamma,V)$.

Observe that the map $\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathfrak{Z}$, which sends a point to its theory, is continuous because

$$\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}) = [\![\varphi]\!]^{\mathfrak{X}}, \tag{3.4}$$

which is open in $\mathcal{X}$ for all formulas φ. Compute

$$\begin{aligned}
&\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{X}}([\![\varphi^{i,j}_1]\!]^{\mathfrak{X}},\dots,[\![\varphi^{i,j}_{n_{i,j}}]\!]^{\mathfrak{X}})\Big)\\
&\qquad = \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{X}}(\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}^{i,j}_1),\dots,\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}^{i,j}_{n_{i,j}}))\Big) && \text{(by (3.4))}\\
&\qquad = \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}(\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{i,j}_{\mathfrak{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\big)\Big) && \text{(naturality of } \lambda)\\
&\qquad = (\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathfrak{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\Big)\Big) && (\star)\\
&\qquad = (\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathfrak{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\Big)\Big) && \text{(assumption (3.2))}\\
&\qquad = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}(\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{k,\ell}_{\mathfrak{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\big)\Big) && (\star)\\
&\qquad = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{X}}(\mathrm{th}^{-1}_{\mathfrak{X}}(\psi^{k,\ell}_1),\dots,\mathrm{th}^{-1}_{\mathfrak{X}}(\psi^{k,\ell}_{n_{k,\ell}}))\Big) && \text{(naturality of } \lambda)\\
&\qquad = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{X}}([\![\psi^{k,\ell}_1]\!]^{\mathfrak{X}},\dots,[\![\psi^{k,\ell}_{n_{k,\ell}}]\!]^{\mathfrak{X}})\Big). && \text{(by (3.4))}
\end{aligned}$$

The steps with $(\star)$ hold because inverse images of maps preserve all unions and intersections. This entails that for all geometric $\mathbb{T}$-models and all states x in $\mathfrak{X}$ we have

$$\mathfrak{X}, x \Vdash \bigvee_{i\in I}\Big(\bigwedge_{j\in J_i} \heartsuit^{\lambda^{i,j}}(\varphi_1^{i,j},\dots,\varphi_{n_{i,j}}^{i,j})\Big) \quad \text{iff} \quad \mathfrak{X}, x \Vdash \bigvee_{k\in K}\Big(\bigwedge_{\ell\in L_k} \heartsuit^{\lambda^{k,\ell}}(\varphi_1^{k,\ell},\dots,\varphi_{n_{k,\ell}}^{k,\ell})\Big),$$

and hence (3.3) holds. Therefore δ is well defined. □

Since we defined δ on generators of $\mathbb{L}\mathcal{E}$, by remark 3.3 it extends to a frame homomorphism which, by abuse of notation, we shall also denote by δ. The algebra structure on $\mathcal{E}$ entails a coalgebra structure on $\mathfrak{Z}$.

3.29 Definition. Let $\zeta : \mathfrak{Z} \to \mathbb{T}\mathfrak{Z}$ be the composition

$$\mathfrak{pt}\,\mathcal{E} \xrightarrow{\mathfrak{pt}\,\delta} \mathfrak{pt}(\mathbb{L}\mathcal{E}) =\!=\!=\!= \mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E}))) \xrightarrow{k^{-1}_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E})}} \mathbb{T}(\mathfrak{pt}\,\mathcal{E})$$

Here $k_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E})} : \mathbb{T}(\mathfrak{pt}\,\mathcal{E}) \to \mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E})))$ is the isomorphism given in remark 3.15. Since $\mathfrak{Z} = \mathfrak{pt}\,\mathcal{E}$ this indeed defines a map $\mathfrak{Z} \to \mathbb{T}\mathfrak{Z}$. ◁

For an object $\Gamma \in \mathfrak{Z}$, $(\mathfrak{pt}\,\delta)(\Gamma)$ is the completely prime filter

$$F = \{\lambda(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) \in \mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E}))) \mid [\heartsuit^\lambda(\varphi_1,\dots,\varphi_n)] \in \Gamma\}$$

in $\mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E})))$. The element $\zeta(\Gamma)$ is the unique element in $\mathbb{T}(\mathfrak{pt}\,\mathcal{E})$ that corresponds to F under the isomorphism $k_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E})}$. By definition of $k_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E})}$, this is the unique element in the intersection of

$$\{\lambda(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) \mid [\heartsuit^\lambda(\varphi_1,\dots,\varphi_n)] \in \Gamma\}.$$

Moreover, it follows from the definition of $k_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E})}$ that $[\heartsuit^\lambda(\varphi_1,\dots,\varphi_n)] \notin \Gamma$ implies $\zeta(\Gamma) \notin \lambda(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)$.

Notation. If no confusion is likely to occur we will omit the square brackets that indicate equivalence classes of formulas in $\mathcal{E}$. That is, we shall write $\varphi \in \mathcal{E}$ instead of $[\varphi] \in \mathcal{E}$.

3.30 Definition. Let $V_{\mathfrak{Z}} : \Phi \to \Omega\mathfrak{Z}$ be the valuation $p \mapsto \widetilde{p}$. ◁

The triple $\mathfrak{Z} = (\mathfrak{Z}, \zeta, V_{\mathfrak{Z}})$ is a geometric $\mathbb{T}$-model, because it is a $\mathbb{T}$-coalgebra with a valuation. We can prove a truth lemma for $\mathfrak{Z}$:

3.31 Lemma (Truth lemma)**.** *We have* $\mathfrak{Z}, \Gamma \Vdash \varphi$ *iff* $\varphi \in \Gamma$.

Proof. Use induction on the complexity of the formula. The propositional case follows immediately from the definition of $V_{\mathfrak{Z}}$. The cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \bigvee_{i\in i} \varphi_i$ are routine. So suppose $\varphi = \heartsuit^\lambda(\varphi_1,\dots,\varphi_n)$. We have

$$\begin{aligned} \mathfrak{Z}, \Gamma \Vdash \heartsuit^\lambda(\varphi_1,\dots,\varphi_n) \quad &\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathfrak{Z}}([\![\varphi_1]\!]^{\mathfrak{Z}},\dots,[\![\varphi_n]\!]^{\mathfrak{Z}}) && (\text{definition of } \Vdash) \\ &\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathfrak{Z}}(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) && (\text{induction}) \\ &\text{iff} \quad \heartsuit^\lambda(\varphi_1,\dots,\varphi_n) \in \Gamma. && (\text{definition of } \zeta) \end{aligned}$$

This proves the lemma. □

3.32 Proposition. *For every geometric* $\mathbb{T}$*-model* $\mathfrak{X} = (\mathfrak{X}, \gamma, V)$ *the map* $\mathrm{th}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{Z}$ *is a* $\mathbb{T}$*-coalgebra morphism.*

Proof. We need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathrm{th}_{\mathfrak{X}}} & \mathcal{Z} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\zeta} \\ \mathbb{T}\mathcal{X} & \xrightarrow[\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}]{} & \mathbb{T}\mathcal{Z} \end{array}$$

Let $x \in \mathcal{X}$. Since $\mathbb{T}\mathcal{Z}$ is sober, hence T_0, it suffices to show that $\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x))$ and $\zeta(\mathrm{th}_{\mathfrak{X}}(x))$ are in precisely the same opens of $\mathbb{T}\mathcal{Z}$. Moreover, we know that the open sets of $\mathbb{T}\mathcal{Z}$ are generated by the sets $\lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)$, so it suffices to show that for all $\lambda \in \Lambda$ and $\varphi_i \in \mathrm{GML}(\Lambda)$ we have

$$\mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \quad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n).$$

This follows from the following computation,

$$\begin{array}{lll} \mathbb{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) & & \\ \quad \text{iff} \quad \gamma(x) \in (\mathbb{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}(\lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)) & & \\ \quad \text{iff} \quad \gamma(x) \in \lambda_{\mathcal{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_1), \dots, \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_n)) & & \text{(naturality of } \lambda) \\ \quad \text{iff} \quad \gamma(x) \in \lambda_{\mathcal{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}}) & & \text{(by (3.4))} \\ \quad \text{iff} \quad \mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) & & \text{(definition of } \Vdash) \\ \quad \text{iff} \quad \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \in \mathrm{th}_{\mathfrak{X}}(x) & & \text{(definition of } \mathrm{th}_{\mathfrak{X}}) \\ \quad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) & & \text{(definition of } \zeta) \end{array}$$

This proves the proposition. □

3.33 Theorem. *The geometric $\mathbb{T}$-model $\mathfrak{Z} = (\mathcal{Z}, \zeta, V_{\mathcal{Z}})$ is final in* $\mathsf{Mod}(\mathbb{T})$.

Proof. Proposition 3.32 states that for every geometric $\mathbb{T}$-model $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ there exists a $\mathbb{T}$-coalgebra morphism $\mathrm{th}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{Z}$, so we only need to show that this morphism is unique.

Let $f : \mathfrak{X} \to \mathfrak{Z}$ be any coalgebra morphism. We know from proposition 3.20 coalgebra morphisms preserve truth, so for all $x \in \mathcal{X}$ we have $\varphi \in f(x)$ iff $\mathfrak{Z}, f(x) \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \varphi$. Therefore we must have $f(x) = \mathrm{th}_{\mathfrak{X}}(x)$. □

We now have all the tools to prove theorem 3.22.

3.22 Theorem. *Let $\mathbb{T}$ be an endofunctor on* Sob, *Λ a characteristic geometric modal signature for $\mathbb{T}$ and $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ two geometric $\mathbb{T}$-models. Then $x \in \mathcal{X}$ and $x' \in \mathcal{X}'$ are modally equivalent if and only if they are behaviourally equivalent.*

Proof of theorem 3.22. It follow from proposition 3.32 and theorem 3.33 that $\mathfrak{Z} = (\mathcal{Z}, \zeta, V_{\mathcal{Z}})$ is final in $\mathsf{Mod}(\mathbb{T})$. Suppose x and x' are modally equivalent, then $\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Z}$ and $\mathrm{th}_{\mathfrak{X}'} : \mathcal{X}' \to \mathcal{Z}'$ are $\mathbb{T}$-model morphisms such that $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$, so x and x' are behaviourally equivalent. Conversely, if x and x' are behaviourally equivalent, then we must have $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ so $x \equiv_{\Lambda} x'$. □

Theorem 3.22 does not simply carry over to coalgebras on compact Hausdorff spaces. The reason is that the equivalence frame is not necessarily compact or regular. Therefore its dual, which is the prime candidate of the state space of a final coalgebra, need not be compact Hausdorff. This hinders the construction of a final coalgebra of theories. However, with a little work we can achieve a similar result for the base category KSob of compact sober spaces and continuous maps.

3.34 Theorem. *Let $\mathbb{T}$ be an endofunctor on* KSob, Λ *a characteristic geometric modal signature for* $\mathbb{T}$ *and* $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ *and* $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ *two geometric* $\mathbb{T}$*-models. Then* $x \in \mathcal{X}$ *and* $x' \in \mathcal{X}'$ *are modally equivalent if and only if they are behaviourally equivalent.*

Proof. If x and x' are behaviourally equivalent, then there exists a geometric $\mathbb{T}_{\mathsf{KSob}}$-model $\mathfrak{U} = (\mathcal{U}, \nu, V_{\mathcal{U}})$ and $\mathbb{T}$-model morphisms $f : \mathfrak{X} \to \mathfrak{U}$ and $f' : \mathfrak{X}' \to \mathfrak{U}$ such that $f(x) = f'(x')$. It follows from proposition 3.20 that x and x' are modally equivalent.

For the converse, we use a procedure similar to the one in between the statement of theorem 3.22 and its proof.

Suppose x and x' are modally equivalent. Define the equivalence relation $\equiv_2$ on $\mathrm{GML}(\Lambda)$ by $\varphi \equiv_2 \psi$ iff $[\![\varphi]\!]^{\mathfrak{X}} = [\![\psi]\!]^{\mathfrak{X}}$ and $[\![\varphi]\!]^{\mathfrak{X}'} = [\![\psi]\!]^{\mathfrak{X}'}$. Let $\mathscr{E}_2$ be the collection of formulae modulo $\equiv_2$ and denote by $[\varphi]$ the equivalence class of φ in $\mathscr{E}_2$. (Note these square brackets mean something different than the ones in definition 3.23.) Define disjunction and conjunction by

$$[\varphi] \wedge [\psi] := [\varphi \wedge \psi] \quad \text{and} \quad \bigvee_{i \in I} [\varphi_i] := \Big[\bigvee_{i \in I} \varphi_i \Big].$$

It is routine to check that $\mathscr{E}_2$ is indeed a frame. We claim that $\mathscr{E}_2$ is compact.

3.34.A Claim. *The frame* $\mathscr{E}_2$ *is compact.*

Proof of claim. Suppose $\bigvee_{i \in I} [\varphi_i] = [\top]$ in $\mathscr{E}$. Then $\bigvee_{i \in I} \varphi_i \equiv_2 \top$, so by definition

$$\bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}} = \mathcal{X} \quad \text{and} \quad \bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}'} = \mathcal{X}'.$$

By compactness of $\mathcal{X}$ and $\mathcal{X}'$ there exist finite sets $J, J' \subseteq I$ such that

$$\bigcup_{i \in J} [\![\varphi_i]\!]^{\mathfrak{X}} = \mathcal{X} \quad \text{and} \quad \bigcup_{i \in J'} [\![\varphi_i]\!]^{\mathfrak{X}'} = \mathcal{X}'.$$

It follows that

$$\bigvee_{i \in J \cup J'} \varphi_i \equiv_2 \top.$$

This proves compactness of $\mathscr{E}_2$. ◇

Let $\mathcal{Y} := \mathfrak{pt}(\mathscr{E}_2)$. Then $\mathcal{Y}$ is a compact sober space. It is easy to see that the theory of a point in $\mathfrak{X}$ or $\mathfrak{X}'$, i.e. the collection of formulas it satisfies, is a completely prime filter in $\mathscr{E}_2$. Therefore we may define

$$\mathrm{th}'_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Y} : x \mapsto \{\varphi \in \mathrm{GML}(\Lambda) \mid \mathfrak{X}, x \Vdash \varphi\}.$$

For every open set $\widetilde{[\varphi]}$ in $\mathcal{Y}$ we have $\mathrm{th}'_{\mathfrak{X}}(\widetilde{[\varphi]}) = \{x \in \mathcal{X} \mid \mathfrak{X}, x \Vdash \varphi\} = [\![\varphi]\!]^{\mathfrak{X}}$. Since $[\![\varphi]\!]^{\mathfrak{X}}$ is open in $\mathcal{X}$ this shows that $\mathrm{th}'_{\mathfrak{X}}$ is continuous. Similarly, we may define a continuous map $\mathrm{th}'_{\mathfrak{X}'} : \mathcal{X}' \to \mathcal{Y}$.

Let $\mathbb{L}$ be the functor $\mathbb{L} = \mathfrak{opn} \circ \mathbb{T} \circ \mathfrak{pt}$ on KSFrm. Then $\mathbb{L}$ is dual to $\mathbb{T}$. Since Λ is characteristic, the frame $\mathbb{L}\mathscr{E}_2$ is generated by the sets

$$\{\lambda_{\mathcal{Y}}(\widetilde{[\varphi_1]}, \ldots, \widetilde{[\varphi_n]}) \mid \lambda \in \Lambda, \widetilde{[\varphi_i]} \in \Omega\mathcal{Y}\}.$$

Endow $\mathscr{E}_2$ with an $\mathbb{L}$-algebra structure $\delta_2 : \mathbb{L}\mathscr{E}_2 \to \mathscr{E}_2$ defined on generators by

$$\delta : \mathbb{L}\mathscr{E}_2 \to \mathscr{E}_2 : \lambda_{\mathcal{Y}}(\widetilde{[\varphi_1]}, \ldots, \widetilde{[\varphi_n]}) \mapsto [\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)].$$

One can proof that δ_2 is well defined in a manner similar to the proof of 3.28. It then follows from remark 3.3 that the assignment δ_2 indeed defines a homomorphism, which we shall also call δ_2.

We use the algebra structure on $\mathcal{E}_2$ to define a coalgebra on $\boldsymbol{\mathcal{Y}}$. Let $\beta : \boldsymbol{\mathcal{Y}} \to \mathbb{T}\boldsymbol{\mathcal{Y}}$ be the composition

$$\mathfrak{pt}\,\mathcal{E}_2 \xrightarrow{\mathfrak{pt}\,\delta} \mathfrak{pt}(\mathbb{L}\mathcal{E}_2) = \mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2))) \xrightarrow{k^{-1}_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2)}} \mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2).$$

Here $k_{\mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2)} : \mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2) \to \mathfrak{pt}(\mathfrak{opn}(\mathbb{T}(\mathfrak{pt}\,\mathcal{E}_2)))$ is the isomorphism given in remark 3.15. Since $\boldsymbol{\mathcal{Y}} = \mathfrak{pt}\,\mathcal{E}$ this indeed defines a continuous map $\boldsymbol{\mathcal{Y}} \to \mathbb{T}\boldsymbol{\mathcal{Y}}$. It follows from unravelling the definitions that for any $\Gamma \in \boldsymbol{\mathcal{Y}}$, the image $\beta(\Gamma)$ is the unique element x in $\mathbb{T}\boldsymbol{\mathcal{Y}}$ satisfying

$$\beta(\Gamma) \in \lambda_{\boldsymbol{\mathcal{Y}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \quad \text{iff} \quad [\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \in \Gamma.$$

Endow the coalgebra $(\boldsymbol{\mathcal{Y}}, \beta)$ with the valuation $V_{\boldsymbol{\mathcal{Y}}} : \Phi \to \Omega\boldsymbol{\mathcal{Y}} : p \mapsto \widetilde{[p]}$ and set

$$\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \beta, V_{\boldsymbol{\mathcal{Y}}}).$$

3.34.B Claim. *The maps* $\mathrm{th}'_{\mathfrak{X}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *and* $\mathrm{th}'_{\mathfrak{X}'} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *are* $\mathbb{T}$*-coalgebra morphism.*

Proof of claim. We show this for $\mathrm{th}'_{\mathfrak{X}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, the case for $\mathrm{th}'_{\mathfrak{X}'}$ being similar. We need to show that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \xrightarrow{\mathrm{th}'_{\mathfrak{X}}} & \boldsymbol{\mathcal{Y}} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{T}\boldsymbol{\mathcal{X}} & \xrightarrow[\mathbb{T}\,\mathrm{th}'_{\mathfrak{X}}]{} & \mathbb{T}\boldsymbol{\mathcal{Y}} \end{array}$$

Let $x \in \boldsymbol{\mathcal{X}}$. By the reasoning from proposition 3.32 it suffices to show that for all $\lambda \in \Lambda$ and $\varphi_i \in \mathrm{GML}(\Lambda)$ we have

$$\mathbb{T}\,\mathrm{th}'_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\boldsymbol{\mathcal{Y}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \quad \text{iff} \quad \beta(\mathrm{th}'_{\mathfrak{X}}(x)) \in \lambda_{\boldsymbol{\mathcal{Y}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}).$$

This follows from the following computation,

$$\begin{aligned} &\mathbb{T}\,\mathrm{th}'_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\boldsymbol{\mathcal{Y}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \\ &\qquad \text{iff} \quad \gamma(x) \in (\mathbb{T}\,\mathrm{th}'_{\mathfrak{X}})^{-1}(\lambda_{\boldsymbol{\mathcal{Y}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]})) \\ &\qquad \text{iff} \quad \gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}((\mathrm{th}'_{\mathfrak{X}})^{-1}(\widetilde{[\varphi_1]}), \dots, (\mathrm{th}'_{\mathfrak{X}})^{-1}(\widetilde{[\varphi_n]})) && \text{(naturality of } \lambda) \\ &\qquad \text{iff} \quad \gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}}) && \text{(by (3.4))} \\ &\qquad \text{iff} \quad \mathfrak{X}, x \Vdash \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) && \text{(definition of } \Vdash) \\ &\qquad \text{iff} \quad \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) \in \mathrm{th}'_{\mathfrak{X}}(x) && \text{(definition of } \mathrm{th}'_{\mathfrak{X}}) \\ &\qquad \text{iff} \quad \beta(\mathrm{th}'_{\mathfrak{X}}(x)) \in \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) && \text{(definition of } \beta) \end{aligned}$$

This proves the claim. ◇

One can easily see that if $x \in \boldsymbol{\mathcal{X}}$ and $x' \in \boldsymbol{\mathcal{X}}'$ are modally equivalent, then $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$. Since $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \beta, V_{\boldsymbol{\mathcal{Y}}})$ is a geometric $\mathbb{T}$-model and $\boldsymbol{\mathcal{Y}}$ is a compact sober space, this shows that x and x' are behaviourally equivalent in $\mathsf{Mod}(\mathbb{T})$. □

3.3 Examples

The Vietoris functor, introduced (in a different form) in [64], has been well investigated in the literature [30, 63, 9, 58]. It is defined on the category of all topological spaces and is known to preserve compactness, the Hausdorff property, and zero-dimensionality [37], among others. Let $\mathbb{V}_{\mathsf{kh}}$ denote the restriction of the Vietoris functor to KHaus There is an Isbell dual functor $\mathbb{N}$ for $\mathbb{V}_{\mathsf{kh}}$ on the category of compact regular frames, that is, there is a functor $\mathbb{N} : \mathsf{KRFrm} \to \mathsf{KRFrm}$ and a natural isomorphism $\mathfrak{opn} \circ \mathbb{V}_{\mathsf{kh}} \cong \mathbb{N} \circ \mathfrak{opn}$. Coalgebras for $\mathbb{V}_{\mathsf{kh}}$ are modal compact Hausdorff spaces. These spaces have been extensively examined in [8].

3.35 Definition. The map $\mathbb{N} : \mathsf{KRFrm} \to \mathsf{KRFrm}$ is defined on an object F as the free frame generated by $\Box a, \Diamond a$, $(a \in F)$ subject to the relations

(N1) $\Box \top = \top$

(N2) $\Diamond \bot = \bot$

(N3) $\Box(a \wedge b) = \Box a \wedge \Box b$

(N4) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

(N5) $\Box(a \vee b) \leq \Box a \vee \Diamond b$

(N6) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$

(N7) $\Box \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Box a \mid a \in A\}$

(N8) $\Diamond \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Diamond a \mid a \in A\}$.

For frame morphisms $f : F \to F'$ define $\mathbb{N}f : \mathbb{N}F \to \mathbb{N}F'$ on generators by $(\mathbb{N}f)(\Box a) = \Box(f(a))$ and $(\mathbb{N}f)(\Diamond a) = \Diamond(f(a))$. ◁

In [57] a different but equivalent definition of $\mathbb{N}$ is given, which uses the cover modality.

We state the following proposition for future reference. A proof may be found in [30] or in section III.4 of [29]. An overview about dual functors for $\mathbb{V}$ for different base spaces can be found in [58].

3.36 Proposition. *There is a natural isomorphism* $\mathfrak{opn} \circ \mathbb{V}_{\mathsf{KHaus}} \cong \mathbb{N} \circ \mathfrak{opn}$.

In the remainder of this section we focus on a generalisation of the monotone functor from definition 2.38 and an Isbell dual for it. Observe that the assignment from definition 2.38 in fact defines an endofunctor on KHaus.

3.37 Definition. Let $\mathfrak{X} = (X, \tau)$ be a compact Hausdorff space. Let $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ be the collection of sets $W \subseteq \mathbb{P}X$ such that $a \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. Endow $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ with the topology generated by the subbase

$$\boxplus a := \{W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \mid a \in W\}, \quad \diamondtimes a := \{W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \mid X \setminus a \notin W\},$$

where a ranges over $\Omega\mathfrak{X}$. For continuous functions $f : \mathfrak{X} \to \mathfrak{X}'$ define $\mathbb{D}_{\mathsf{kh}}f : \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \to \mathbb{D}_{\mathsf{kh}}\mathfrak{X}' : W \mapsto \{a \in \mathbb{P}X \mid f^{-1}(a) \in W\}$. ◁

The proof that $\mathbb{D}_{\mathsf{kh}}f$ is indeed a well-defined continuous function is the same as in lemma 2.39. We can easily show that $\mathbb{D}_{\mathsf{kh}}$ preserves compactness.

3.38 Lemma. *If $\mathfrak{X}$ is a compact space then so is $\mathbb{D}\mathfrak{X}$.*

Proof. By the Alexander subbasis theorem it suffices to show that any cover of the form

$$\bigcup_{i \in I} \boxplus a_i \cup \bigcup_{j \in J} \diamondtimes b_j$$

has a finite subcover. So suppose the above covers $\mathbb{D}\mathfrak{X}$. Since $\varnothing \in \mathbb{D}\mathfrak{X}$ and $\varnothing \notin \boxplus a$ for any open set $a \in \Omega\mathfrak{X}$, we must have $|J| \geq 1$. Furthermore, we must have $k \in I$ such that a_k is a superset of $X \setminus b_j$ for some $j \in J$, because otherwise the up-set $\uparrow\{X \setminus b_j \mid j \in J\}$, where each member is a superset of at least one of the b_j, is not in the cover.

Let j be such that $X \setminus b_j \subseteq a_k$. Let $W \in \mathbb{D}\mathfrak{X}$. If $X \setminus b_j \notin W$ then $W \in \otimes b_j$ and if $X \setminus b_j \in W$ then $W \in \boxplus a_k$. This shows that $\boxplus a_k \cup \otimes b_j$ is a finite subcover. □

We know now that $\mathbb{D}_{\mathsf{kh}}$ preserves compactness. However, we do not yet know whether $\mathbb{D}_{\mathsf{kh}}$ preserves the Hausdorff property as well. We will not prove this directly, but first give an Isbell dual $\mathbb{M}$ of $\mathbb{D}_{\mathsf{kh}}$, i.e. a functor on Frm such that for all compact Hausdorff spaces $\mathfrak{X}$ we have

$$\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}) \cong \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}).$$

We then show that $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$ is compact regular, and it follows from proposition 3.14 that $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ is compact Hausdorff.

3.39 Definition. Let F be a frame. Let $\mathbb{M}F$ be the frame generated by the set $\mathbb{M}F = \{\Box a, \Diamond a \ (a \in F)\}$ subject to the relations

(M1) $\Box(a \wedge b) \leq \Box a$

(M2) $\Diamond a \leq \Diamond(a \vee b)$

(M3) $\Box a \wedge \Diamond b \leq \bot$ whenever $a \wedge b \leq \bot$

(M4) $\Box a \vee \Diamond b \geq \top$ whenever $a \vee b \geq \top$

(M5) $\Box \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Box a \mid a \in A\}$

(M6) $\Diamond \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Diamond a \mid a \in A\}$,

where $a, b \in F$ and A is a directed subset of F. For a homomorphism $f : F \to F'$ define $\mathbb{M}f : \mathbb{M}F \to \mathbb{M}F'$ on generators by $\Box a \mapsto \Box f(a)$ and $\Diamond a \mapsto \Diamond f(a)$. The assignment $\mathbb{M}$ defines a functor on Frm. ◁

The following proposition closely resembles that of proposition III4.3 in [29].

3.40 Proposition. *If F is a regular frame, then so is $\mathbb{M}F$.*

Proof. We need to show that for all $c \in \mathbb{M}F$ we have $c = \bigvee\{d \in \mathbb{M}F \mid d \lessdot c\}$. It follows from lemma 3.9 that it suffices to focus on the generators of $\mathbb{M}F$. Let $a \in F$, then we know $\bigvee\{d \in \mathbb{M}F \mid d \lessdot \Box a\} \leq \Box a$. Suppose $b \lessdot a$ in F, then by lemma 3.8 $\sim b \vee a = \top$ and hence $\Diamond{\sim b} \vee \Box a \geq \top$. Also $\sim b \wedge b = \bot$ so it follows from (M3) that $\Diamond{\sim b} \wedge \Box b = \bot$. This proves $\Box b \lessdot \Box a$, because the element $\Diamond{\sim b}$ is such that $\Diamond{\sim b} \vee \Box a = \top$ and $\Diamond{\sim b} \wedge \Box b = \bot$. Since F is regular and $\{b \in F \mid b \lessdot a\}$ is directed, it follows that

$$\Box a = \Box \bigvee\{b \in F \mid b \lessdot a\} = \bigvee\{\Box b \in \mathbb{M}F \mid b \lessdot a\} \leq \bigvee\{d \in \mathbb{M}F \mid d \lessdot \Box a\}$$

so $\Box a = \bigvee\{d \in \mathbb{M}F \mid d \lessdot \Box a\}$. In a similar fashion one may show that $\Diamond a = \bigvee\{d \in \mathbb{M}F \mid d \lessdot \Diamond a\}$. This proves the lemma. □

We now focus on a duality between $\mathbb{D}_{\mathsf{kh}}$ and a restriction of $\mathbb{M}$. The proof of the next theorem is similar to the proof of proposition III4.6 in [29]. The main difference with the proof in [29] is the way we define a point of $\mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ from a given element of $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$. This is (of course) due to the fact that $\mathbb{V}_{\mathsf{kh}}$ and $\mathbb{D}_{\mathsf{kh}}$ are different functors.

3.41 Theorem. *If $\mathfrak{X}$ is a compact Hausdorff space then*

$$\mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})) \cong \mathbb{D}_{\mathsf{kh}}\mathfrak{X}.$$

Proof. Define a map

$$\varphi : \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \to \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})) : W \mapsto p_W,$$

where we define p_W on generators by

$$p_W : \mathbb{M}(\mathfrak{opn}\,\mathfrak{X}) \to 2 : \begin{cases} \Box a \mapsto \top & \text{iff } a \in W \\ \Diamond a \mapsto \bot & \text{iff } X \smallsetminus a \in W \end{cases}.$$

Conversely, for a point $p \in \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ let

$$W_p := {\uparrow}\{X \smallsetminus a \mid p(\Diamond a) = \bot\}.$$

This gives rise to a map $\psi : \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})) \to \mathbb{D}_{\mathsf{kh}}\mathfrak{X}$. It is clear that $W_p \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ because it is the up-set of a collection of closed sets; indeed, for each $b \in W_p$ there exists a closed subset $X \smallsetminus a \subseteq b$ with $p(\Diamond a) = \bot$ and by definition all open supersets of $X \smallsetminus a$ are in W_p.

We will show that the p_W are well-defined, that φ is a bijection and that φ is continuous.

3.41.A Claim. *If $W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ then $p_W : \mathbb{M}(\mathfrak{opn}\,\mathfrak{X}) \to 2$ is a point.*

Proof of claim. Since p_W is a frame homomorphisms defined on generators, it suffices to check that the $p_W(\Box a)$ and $p_W(\Diamond a)$ (where the a range over $\Omega\mathfrak{X}$) satisfy (M1) through (M6) from definition 3.39. Let us check (M1), (M3) and (M5), items (M2), (M4) and (M6) being similar.

(M1) If $p_W(\Box(a \cap b)) = \top$ then $a \cap b \in W$. Since W is upward closed $a \in W$, so $p_W(\Box a) = \top$.

(M3) If $a \cap b = \varnothing$ then $a \subseteq X \smallsetminus b$. Suppose $p_W(\Box a) = \top$ then $a \in W$ so $X \smallsetminus b \in W$ so $p_W(\Diamond b) = \bot$ hence $p_W(\Box a) \wedge p_W(\Diamond b) = \bot$.

(M5) We claim that for all $W \in \mathbb{D}\mathfrak{X}$ and directed sets $A \subseteq \Omega\mathfrak{X}$ we have $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$. The direction from right to left follows from the fact that W is upwards closed. Conversely, suppose $\bigcup^{\uparrow} A \in W$, then there is a closed set $k \subseteq \bigcup^{\uparrow} A$ with $k \in W$. The elements of A now cover the closed therefore compact set k, so there is a finite $A' \subseteq A$ with $k \subseteq \bigcup A'$ and since A is directed there is $a \in A$ with $\bigcup A' \subseteq a$. As $k \in W$ and $k \subseteq a$ it follows that $a \in W$.

Now we have $p_W(\Box \bigcup^{\uparrow} A) = 1$ iff $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$ iff $\bigvee^{\uparrow}\{p_W(\Box a) \mid a \in A\} = 1$. ◇

3.41.B Claim. *For all $p \in \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ we have $X \smallsetminus a \in W_p$ iff $p(\Diamond a) = \bot$ and $a \in W$ iff $p(\Box a) = \top$.*

Proof of claim. If $p(\Diamond a) = \bot$ then $X \smallsetminus a \in W_p$. Conversely, Suppose $X \smallsetminus a \in W_p$, then there is some b with $p(\Diamond b) = \bot$ and $X \smallsetminus b \subseteq X \smallsetminus a$. Therefore $a \subseteq b$ and $p(\Diamond a) \leq p(\Diamond b) = \bot$. This proves $X \smallsetminus a \in W_p$ iff $p(\Diamond a) = \bot$.

If $a \in W_p$ then there is $X \smallsetminus b \subseteq a$ in W_p, so $p(\Diamond b) = \bot$. Then $a \cup b = X$, so it follows from (M4) of 3.39 that $p(\Box a) = \top$. If $a \notin W_p$ and $a' \lessdot a$, then there exists b with $b \cap a' = \varnothing$ and $b \cup a = \mathfrak{X}$. Since $X \smallsetminus b \subseteq a$, set set $X \smallsetminus b$ is not in W_p and hence we must have $p(\Diamond b) = \top$. As $a' \cap b = \varnothing$ it follows from (M3) that $p(\Box a') = p(\varnothing) = \bot$. Now we use (M5) and the fact that $a = \bigvee^{\uparrow}\{a' \mid a' \lessdot a\}$ (this is true because $\mathfrak{X}$ is assumed to be compact Hausdorff so $\mathfrak{opn}\,\mathfrak{X}$ is compact regular) to find

$$p(\Box a) = \bigvee^{\uparrow}\{p(\Box a') \mid a' \lessdot a\} = \bigvee^{\uparrow}\{\bot \mid a' \lessdot a\} = \bot.$$

It follows that $a \in W_p$ iff $p(\Box a) = \top$. ◇

3.41.C Claim. *The maps φ and ψ define a bijection between $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ and $\mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$.*

Proof of claim. For $p \in \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ and $W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ we will show that $p_{W_p} = p$ and $W_{p_W} = W$. In order to prove that (the frame homomorphisms) p and p_{W_p} coincide, it suffices to show that they coincide on the generators of $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$. By definition and claim 3.41.B have

$$p(\Box a) = \top \quad \text{iff} \quad a \in W_p \quad \text{iff} \quad p_{W_p}(\Box a) = \top$$

and

$$p(\Diamond a) = \bot \quad \text{iff} \quad X \setminus a \notin W_p \quad \text{iff} \quad p_{W_p}(\Diamond a) = \bot.$$

In order to show that $W = W_{p_W}$ it suffices to show that $X \setminus a \in W$ iff $X \setminus a \in W_{p_W}$ for all open sets a, because elements of $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ are uniquely determined by the closed sets they contain. This follows immediately from the definitions and claim 3.41.B,

$$X \setminus a \in W \quad \text{iff} \quad p_W(\Diamond a) = \bot \quad \text{iff} \quad X \setminus a \in W_{p_W}. \qquad \Diamond$$

3.41.D Claim. *The map $\varphi : \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \to \mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ is continuous.*

Proof of claim. The opens of $\mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}))$ are generated by $\widetilde{\Box a} = \{p \mid p(\Box a) = \top\}$ and $\widetilde{\Diamond a} = \{p \mid p(\Diamond a) = \top\}$, for $a \in \Omega\mathfrak{X}$. We have

$$\varphi^{-1}(\widetilde{\Box a}) = \varphi^{-1}(\{p \mid p(\Box a) = \top\}) = \{W \in \mathbb{D}\mathfrak{X} \mid a \in W\} = \boxplus a$$

and similarly $\varphi^{-1}(\widetilde{\Diamond a}) = \diamondplus a$. Since $\boxplus a$ and $\diamondplus a$ are open in $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$, this proves continuity of φ. $\Diamond$

We showed that φ is a bijective continuous function, hence a homeomorphism. This completes the proof of the theorem. $\Box$

As announced before definition 3.39 we will now prove that $\mathbb{D}_{\mathsf{kh}}$ sends a compact Hausdorff space to a compact Hausdorff space.

3.42 Corollary. *If $\mathfrak{X}$ is a compact Hausdorff space, then so is $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$.*

Proof. Since $\mathfrak{X}$ is compact Hausdorff the frame $\mathfrak{opn}\,\mathfrak{X}$ is compact regular. By proposition 3.40 $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$ is regular. It follows from lemma 3.38 that $\mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ is compact hence by theorem 3.41 $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$ is compact. So $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$ is compact regular. Since $\mathfrak{pt}(\mathbb{M}(\mathfrak{opn}\,\mathfrak{X})) \cong \mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ the latter is compact Hausdorff by proposition 3.14. $\Box$

3.43 Remark. The proof of corollary 3.42 may appear incoherent, because of its many switches between the frame side and the topological side. However, it demonstrates precisely the power of a duality like the one given in theorem 3.41: regularity is easier to prove on the frame side, whereas for compactness the topological setting was more insightful.

It can be proven that $\mathbb{M}$ preserves compactness on the frame side as well. This proof is given in section A.2 in the appendix.

Denote by $\mathbb{M}_{\mathsf{kr}}$ the restriction of $\mathbb{M}$ to KRFrm. Theorem 3.41 yields a map $\mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) \to \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ for a compact Hausdorff space $\mathfrak{X}$ given by

$$\mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) \xrightarrow{\mathfrak{opn}\circ\mathfrak{pt}} \mathfrak{opn}(\mathfrak{pt}(\mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}))) \xrightarrow{\mathfrak{opn}\,\varphi} \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}).$$

Unravelling the definitions shows that, on generators, it is given by $\Box a \mapsto \boxplus a$ and $\Diamond a \mapsto \diamondplus a$.

3.44 Definition. For every compact Hausdorff space $\mathfrak{X}$ define $\eta_{\mathfrak{X}} : \mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) \to \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ on generators by $\eta_{\mathfrak{X}}(\Box a) = \boxplus a$ and $\eta_{\mathfrak{X}}(\Diamond a) = \diamondplus a$. By the preceding discussion $\eta_{\mathfrak{X}}$ is a well-defined frame isomorphism. ◁

It turns out that the maps $\eta_{\mathfrak{X}}$ constitute a natural isomorphism.

3.45 Proposition. *The collection* $\eta = (\eta_{\mathfrak{X}})_{\mathfrak{X} \in \mathsf{KHaus}}$ *is a natural isomorphism.*

Proof. The isomorphism part is proposition 3.41, so we only need to show naturality. That is, for any morphism $f : \mathfrak{X} \to \mathfrak{X}'$ in KHaus, the following diagram commutes,

$$\begin{array}{ccc} \mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) & \xleftarrow{\mathbb{M}(\mathfrak{opn}\,f)} & \mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}') \\ \downarrow \eta_{\mathfrak{X}} & & \downarrow \eta_{\mathfrak{X}'} \\ \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}})\mathfrak{X} & \xleftarrow[\mathfrak{opn}(\mathbb{D}f)]{} & \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}') \end{array}$$

(Since $\mathfrak{opn}$ is a contravariant functor, the horizontal arrows are reversed.) For this, suppose $\Box a'$ is a generator of $\mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}')$. Then

$$\begin{aligned} \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}f) \circ \eta_{\mathfrak{X}'}(\Box a) &= \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}f)(\boxplus a) && \text{By definition 3.44} \\ &= (\mathbb{D}_{\mathsf{kh}}f)^{-1}(\boxplus a) && \text{By definition of } \mathfrak{opn} \\ &= \boxplus f^{-1}(a) && \text{By lemma 2.39} \\ &= \eta_{\mathfrak{X}}(\Box f^{-1}(a)) && \text{By definition 3.44} \\ &= \eta_{\mathfrak{X}} \circ \mathbb{M}_{\mathsf{kr}}(f^{-1}(\Box a)) && \text{By definition of } \mathbb{M} \\ &= \eta_{\mathfrak{X}} \circ \mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,f)(\Box a). && \text{By definition of } \mathfrak{opn} \end{aligned}$$

and by analogous reasoning $\Omega\mathbb{D}_{\mathsf{kh}}f \circ \eta_{\mathfrak{X}'}(\Diamond a) = \eta_{\mathfrak{X}} \circ \mathbb{M}_{\mathsf{kr}}(\mathfrak{opn}\,f)(\Diamond a)$. This proves that the diagram commutes. □

Applying proposition A.6 to the previous proposition yields the following corollary.

3.46 Corollary. *There is a dual equivalence*

$$\mathsf{Alg}(\mathbb{M}_{\mathsf{kr}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathbb{D}_{\mathsf{kh}}).$$

The functor $\mathbb{D}_{\mathsf{kh}}$ is the analog of the monotone functor from example 2.7 and gives rise to monotone logic on compact Hausdorff spaces. The fact that for both $\mathbb{V}_{\mathsf{kh}}$ and $\mathbb{D}_{\mathsf{kh}}$ the functor given on objects by taking a free frame modulo the axioms of the logic, suggests that KHaus is the most suitable base category for coalgebraic geometric logic.

The functor $\mathbb{M}$ whose restriction is dual to $\mathbb{D}_{\mathsf{kh}}$ is defined for arbitrary frames, not just for compact regular frames. Similarly, the action of $\mathbb{D}_{\mathsf{kh}}$ on objects could be applied to any topological space, not just to compact Hausdorff spaces. (As we have seen in lemma 2.39 the morphisms, as they are defined now, do rely on the fact that we work over the category of compact Hausdorff spaces.) We have not been able to extend the duality from theorem 3.41 to either sober spaces and spatial frames, or compact sober spaces and compact spatial frames. Even more important, we have not found a proof that $\mathbb{D}$ preserves sobriety or that $\mathbb{M}$ preserves spatiality. Neither have we found counterexamples showing that it is false. We leave this as an interesting open question.

For the Vietoris functor and its dual more is known, for an overview see [58]. However, we still do not know whether the Vietoris functor preserves sobriety, even less so if there is a dual equivalence with an endofunctor on SFrm that holds in this generality.

A third example of a functor which induces logic on compact Hausdorff-coalgebras is that of the conditional functor. This functor gives rise to a duality similar to the duality from theorem 3.41 and will be treated in chapter 5.

3.4 Bisimulations

In this section we give various notions of bisimulations between models for coalgebraic geometric logic and compare them. As in section 3.2 we use the symbol C to denote either of the categories Sob, KSob and KHaus.

3.47 Definition. Let $\mathbb{T}$ be an endofunctor on C. An **Aczel-Mendler bisimulation** between two geometric $\mathbb{T}$-models is an Aczel-Mendler bisimulation B between the underlying $\mathbb{T}$-coalgebras such that for all $(x, x') \in B$ and $p \in \Phi$, $x \in V(p)$ iff $x' \in V'(p)$. We say that two states x and x' are Aczel-Mendler bisimilar and write $x \leftrightarroweq x'$ if there is an Aczel-Mendler bisimulation linking them. ◁

The following lemma is an immediate consequence of proposition 3.20.

3.48 Lemma. *Let $\mathbb{T}$ be a functor on C, Λ a geometric modal signature for $\mathbb{T}$ and x and x' states in two $\mathfrak{X}, \mathfrak{X}'$ two geometric $\mathbb{T}$-models. Then $x \leftrightarroweq x'$ implies $x \equiv_\Lambda x'$.*

If Λ is characteristic and $\mathsf{C} = \mathsf{Sob}$ or $\mathsf{C} = \mathsf{KSob}$, it follows from the previous lemma combined with theorems 3.22 and 3.34 that Aczel-Mendler bisimilarity implies behavioural equivalence. If moreover $\mathbb{T}$ preserves weak pullbacks, the converse holds as well. The proof of this is similar to theorem 4.3 and the preceding discussion in [49].

However, we do not wish to make this assumption. For example, the Vietoris functor does not preserve weak pullbacks [9, Corollary 4.3]. Similarly to section 2.3 we define Λ-bisimulations for C-coalgebras. They are an adaptation of ideas in [6, 17]. Under some conditions on Λ, Λ-bisimilarity coincides with behavioural equivalence for any $\mathbb{T}$.

3.49 Definition. Let $\mathbb{T}$ be an endofunctor on C, Λ a geometric modal signature for $\mathbb{T}$ and $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ geometric $\mathbb{T}$-models. A **Λ-bisimulation** between $\mathfrak{X}$ and $\mathfrak{X}'$ is a relation $B \subseteq \mathbb{U}\mathcal{X} \times \mathbb{U}\mathcal{X}'$ such that for all $(x, x') \in B$ and $p \in \Phi$ and all B-coherent pairs of opens $(a_i, a_i') \in \Omega\mathcal{X} \times \Omega\mathcal{X}'$ we have

$$x \in V(p) \quad \text{iff} \quad x' \in V'(p)$$

and

$$\gamma(x) \in \lambda_{\mathcal{X}}(a_1, \ldots, a_n) \quad \text{iff} \quad \gamma'(x') \in \lambda_{\mathcal{X}'}(a_1', \ldots, a_n'). \tag{3.5}$$

Two states are called Λ-bisimilar if there is a Λ-bisimulation linking them, notation: $x \leftrightarroweq x'$. ◁

3.50 Remark. Let $\mathbb{T}$ be an endofunctor on C, Λ a geometric modal signature for $\mathbb{T}$ and $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ geometric $\mathbb{T}$-models. Suppose $B \subseteq \mathcal{X} \times \mathcal{X}'$ with the subspace topology is an object in C. Let $\pi : B \to \mathcal{X}$ and $\pi' : B \to \mathcal{X}'$ be the respective projections and

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\overline{\pi}'} & \mathfrak{opn}(\mathcal{X}') \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \mathfrak{opn}(\pi')} \\ \mathfrak{opn}(\mathcal{X}) & \xrightarrow[\mathfrak{opn}(\pi)]{} & \mathfrak{opn}(B) \end{array}$$

the pullback diagram of the cospan $(\mathfrak{opn}(\pi), \mathfrak{opn}(\pi'))$ in Frm. Then a pair $(a, a') \in \Omega\mathcal{X} \times \Omega\mathcal{X}'$ is B-coherent if and only if it is in $\mathcal{P}$. If we unravel the definitions we find that equation 3.5 holds if and only if for all $\lambda \in \Lambda$ the following diagram commutes:

$$\begin{array}{ccccc}
\mathfrak{opn}(\mathcal{X})^n & \xleftarrow{(\overline{\pi})^n} & \mathcal{P}^n & \xrightarrow{(\overline{\pi}')^n} & \mathfrak{opn}(\mathcal{X}')^n \\
\downarrow{\scriptstyle \lambda_{\mathcal{X}}} & & & & \downarrow{\scriptstyle \lambda_{\mathcal{X}'}} \\
\mathfrak{opn}(\mathbb{T}\mathcal{X}) & & & & \mathfrak{opn}(\mathbb{T}\mathcal{X}') \\
\downarrow{\scriptstyle \mathfrak{opn}(\gamma)} & & & & \downarrow{\scriptstyle \mathfrak{opn}(\gamma')} \\
\mathfrak{opn}(\mathcal{X}) & \xrightarrow[\mathfrak{opn}(\pi)]{} & \mathfrak{opn}(B) & \xleftarrow[\mathfrak{opn}(\pi')]{} & \mathfrak{opn}(\mathcal{X}')
\end{array} \tag{3.6}$$

This point of view elucidates the similarity with Λ-bisimilarity for set-based coalgebras in [6].

As is to be expected, Λ-bisimilar states satisfy the same formulas.

3.51 Lemma. *Let $\mathbb{T}$ be an endofunctor on C and Λ a geometric modal signature for $\mathbb{T}$. Then $\leftrightarrows_\Lambda \subseteq \equiv_\Lambda$.*

Proof. Suppose $x \in \mathcal{X}$ and $x' \in \mathcal{X}'$ are Λ-bisimilar and B a Λ-bisimulation with xBx'. We will show that for all formulas φ we have $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}', x' \Vdash \varphi$ using induction on the complexity of the formula. The propositional case is by definition. If φ is a finite meet or an arbitrary join of formulas then the lemma is obvious. Suppose $\mathfrak{X}, x \Vdash \heartsuit^\lambda(\varphi_1, \dots, \varphi_n)$, then $\gamma(x) \in \lambda_{\mathcal{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}})$. We have $B[[\![\varphi_i]\!]^{\mathfrak{X}}] \subseteq [\![\varphi_i]\!]^{\mathfrak{X}'}$ by the induction hypothesis, so by definition of a Λ-bisimulation we find $\gamma'(x') \in \lambda_{\mathcal{X}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}'})$ and hence $\mathfrak{X}', x' \Vdash \heartsuit^\lambda(\varphi_1, \dots, \varphi_n)$. The converse direction is proven symmetrically. □

It turns out that the collection of Λ-bisimulations between two geometric $\mathbb{T}$-models forms a complete lattice.

3.52 Proposition. *Let Λ be a geometric modal signature of a functor $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ and let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be two geometric $\mathbb{T}$-models. The collection of Λ-bisimulations between $\mathfrak{X}$ and $\mathfrak{X}'$ forms a complete lattice.*

Proof. It is obvious that the collection of Λ-bisimulations is a poset. We will show that this collection is closed under taking arbitrary unions; the result then follows from theorem 4.2 in [12].

Let I be some index set and for all $j \in J$ let B_j be Λ-bisimulations between $\mathfrak{X}$ and $\mathfrak{X}'$ and set $B = \bigcup_{j \in J} B_j$. We claim that B is a Λ-bisimulation.

Let (a_i, a_i') be a B-coherent pairs of opens. Suppose xBx' and $\gamma(x) \in \lambda_{\mathcal{X}}(a_1, \dots, a_n)$. Then there is $j \in J$ with xB_jx'. As $B_j[a_i] \subseteq B[a_i] \subseteq a_i'$ and $B_j^{-1}[a_i'] \subseteq B^{-1}[a_i'] \subseteq a_i$, all B-coherent pairs (a_i, a_i') are also B_j-coherent. As B_j is a Λ-bisimulation we have $\gamma'(x') \in \lambda_{\mathcal{X}'}(a_1', \dots, a_n')$. □

Every Aczel-Mendler bisimulation is a Λ-bisimulation.

3.53 Proposition. *Let $\mathbb{T}$ be an endofunctor on C and Λ a geometric modal signature for $\mathbb{T}$. Then $\leftrightarrows \subseteq \leftrightarrows_\Lambda$.*

Proof. Suppose B is an Aczel-Mendler bisimulation and let β be the map that turns B into a coalgebra, then the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{X} & \xleftarrow{\pi} & B & \xrightarrow{\pi'} & \mathcal{X}' \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma'} \\
\mathbb{T}\mathcal{X} & \xleftarrow[\mathbb{T}\pi]{} & \mathbb{T}B & \xrightarrow[\mathbb{T}\pi']{} & \mathbb{T}\mathcal{X}'
\end{array} \tag{3.7}$$

We will show that B is a Λ-bisimulation. By definition $x \in V(p)$ iff $x' \in V'(p)$ whenever xBx'. We prove the forth condition from definition 3.49. Let $\lambda \in \Lambda$ and $(x, x') \in B$. Suppose $(a_1, a_1'), \dots, (a_n, a_n')$ are B-coherent pairs of opens and $\gamma(x) \in \lambda_{\mathfrak{X}}(a_1, \dots, a_n)$. Then we have

$$\begin{aligned}
\beta(x, x') &\in (\mathbb{T}\pi)^{-1}(\lambda_{\mathfrak{X}}(a_1, \dots, a_n)) && \text{(follows from (3.7))} \\
&= \lambda_B(\pi^{-1}(a_1), \dots, \pi^{-1}(a_{n_\lambda})) && \text{(naturality of } \lambda) \\
&\subseteq \lambda_B\big((\pi')^{-1} \circ \pi'[\pi^{-1}(a_1)], \dots, (\pi')^{-1} \circ \pi'[\pi^{-1}(a_n)]\big) && \text{(monotonicity of } \lambda) \\
&= \lambda_B\big((\pi')^{-1}(B[a_1]), \dots, (\pi')^{-1}(B[a_n])\big) && (B[a] = \pi_2 \circ \pi_1^{-1}(a)) \\
&\subseteq \lambda_B((\pi')^{-1}(a_1'), \dots, (\pi')^{-1}(a_n')) && \text{(monotonicity of } \lambda) \\
&= (\mathbb{T}\pi')^{-1}(\lambda_{\mathfrak{X}'}(a_1', \dots, a_n')). && \text{(naturality of } \lambda)
\end{aligned}$$

Therefore

$$\gamma'(x') = (\mathbb{T}\pi')(\beta(x, x')) \in \lambda_{\mathfrak{X}'}(a_1', \dots, a_{n_\lambda}'),$$

as desired. □

It follows from lemma 3.51 and theorem 3.22 that, provided Λ is a characteristic geometric modal signature for an endofunctor $\mathbb{T}$ on Sob or KSob, Λ-bisimilarity implies behavioural equivalence. In order to prove a converse statement we need a slight strengthening of the definition of open predicate liftings. We define the notion of a strong open predicate lifting in the next definition and subsequently prove that for strong monotone sets of predicate liftings, behavioural equivalence implies Λ-bisimilarity.

3.54 Definition. Let $\mathbb{T}$ be an endofunctor on C. A **strong open predicate lifting for** $\mathbb{T}$ is a natural transformation

$$\lambda : \breve{\mathbb{P}}^n \circ \mathbb{U} \to \breve{\mathbb{P}} \circ \mathbb{U} \circ \mathbb{T}$$

such that for objects $\mathfrak{X}$ in C and $a_1, \dots, a_n \in \Omega\mathfrak{X}$ the set $\lambda_{\mathfrak{X}}(a_1, \dots, a_n)$ is open in $\mathbb{T}\mathfrak{X}$. A strong open predicate lifting λ is said to be **monotone in its i-th argument** if for all (not necessarily open) subsets $a_1, \dots, a_n, b \in \breve{\mathbb{P}}(\mathbb{U}\mathfrak{X})$ we have $\lambda_{\mathfrak{X}}(a_1, \dots, a_i, \dots, a_n) \subseteq \lambda_{\mathfrak{X}}(a_1, \dots, a_i \cup b, \dots, a_n)$ and it is called **monotone** if it is monotone in every argument. The dual of a strong open predicate lifting λ is λ^∂ defined by

$$\lambda^\partial_{\mathfrak{X}}(a_1, \dots, a_n) = \mathfrak{X} \setminus \lambda_{\mathfrak{X}}(\mathfrak{X} \setminus a_1, \dots, \mathfrak{X} \setminus a_n).$$

A collection of strong open predicate liftings Λ is called a strong geometric modal signature. A strong geometric modal signature is **closed under duals** if $\lambda \in \Lambda$ implies $\lambda^\partial \in \Lambda$ and **characteristic** if for every $\mathfrak{X}$ in C the collection

$$\{\lambda_{\mathfrak{X}}(a_1, \dots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}, a_i \in \Omega\mathfrak{X}\}$$

forms a sub-base for the topology on $\mathbb{T}\mathfrak{X}$. ◁

It follows immediately from the definitions that every strong open predicate lifting restricts to a "normal" open predicate lifting. We call an open predicate lifting λ **strong** if there exists a strong open predicate liftings that restricts to λ and **strong monotone** if there exists a monotone strong open predicate lifting that restricts to it. A geometric modal signature is called **strong monotone** if every open predicate lifting in it is strong monotone.

3.55 Example. The box- and diamond-lifting for both basic modal logic and monotone modal logic are strong monotone open predicate liftings. ◁

The proof of the following proposition is similar to the proof of proposition 2.34 and is inspired by the proof of theorem 4.1 in [16].

3.56 Proposition. *Let $\mathbb{T}$ be an endofunctor on C and Λ a strong monotone geometric modal signature for $\mathbb{T}$. Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be two $\mathbb{T}$-models. Then* $\simeq_{\mathsf{Coalg}(\mathbb{T})} \subseteq \leftrightarroweq_{\Lambda}$.

Proof. Suppose x and x' are behaviourally equivalent. Then there is some geometric $\mathbb{T}$-model $\mathfrak{U} = (\mathcal{U}, \nu, V_{\mathcal{U}})$ and there are $\mathbb{T}$-model morphisms $f : \mathfrak{X} \to \mathfrak{U}$ and $f' : \mathfrak{X}' \to \mathfrak{U}$ such that $f(x) = f'(x')$. We will define a Λ-bisimulation B linking x and x'.

Let B be the pullback of f and f' in Top,

$$B = \{(u, u') \in X \times X' \mid f(u) = f'(u')\}.$$

Then clearly xBx'. It follows from proposition 3.20 that u and u' satisfy precisely the same formulas whenever $(u, u') \in B$.

Suppose $\lambda \in \Lambda$ is n-ary and for $1 \leq i \leq n$ let (a_i, a_i') be a B-coherent pair of opens. Suppose uBu' and $\gamma(u) \in \lambda_{\mathcal{X}}(a_1, \ldots, a_n)$. We will show that $\gamma'(u') \in \lambda_{\mathcal{X}'}(a_1', \ldots, a_n')$, the converse direction is similar. By monotonicity and naturality of λ we obtain

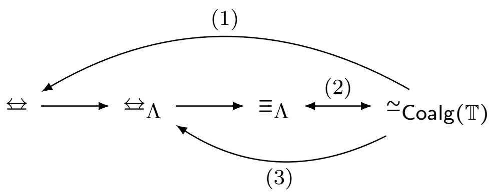
$$\gamma(u) \in \lambda_{\mathcal{X}}(a_1, \ldots, a_n) \subseteq \lambda_{\mathcal{X}}(f^{-1}(f[a_1]), \ldots, f^{-1}(f[a_n])) = (\mathbb{T}f)^{-1}(\lambda_{\mathcal{Y}}(f[a_1], \ldots, f[a_n])),$$

so $(\mathbb{T}f)(\gamma(u)) \in \lambda_{\mathcal{Y}}(f[a_1], \ldots, f[a_n])$. (Note that the $f[a_i]$ need not be open in $\mathcal{Y}$, but $\lambda_{\mathcal{Y}}(f[a_1], \ldots, f[a_n])$ is defined because λ is assumed to be strong.) Since f and f' are coalgebra morphisms and $f(u) = f'(u')$ we have $(\mathbb{T}f)(\gamma(u)) = \delta(f(u)) = \delta(f'(u')) = (\mathbb{T}f')(\gamma'(u'))$. Coherence of (a_i, a_i') and monotonicity and naturality of λ yield

$$\begin{aligned}
\gamma'(u') &\in (\mathbb{T}f')^{-1}(\lambda_{\mathcal{Y}}(f[a_1], \ldots, f[a_n])) \\
&= \lambda_{\mathcal{X}'}((f')^{-1}(f[a_1]), \ldots, (f')^{-1}(f[a_n])) && \text{(naturality of } \lambda) \\
&= \lambda_{\mathcal{X}'}(B[a_1], \ldots, B[a_n]) && \text{(strong monotonicity of } \lambda) \\
&\subseteq \lambda_{\mathcal{X}'}(a_1', \ldots, a_n'). && \text{(coherence of } (a_i, a_i'))
\end{aligned}$$

This proves the proposition. □

Let $\mathbb{T}$ be an endofunctor on C and Λ a geometric modal signature for $\mathbb{T}$. The following diagram summarises the results from theorems 3.22, 3.34 and 3.34, lemma 3.51 and proposition 3.56. The arrows indicate that one form of equivalence implies the other. The numbers indicate that the implication is under a certain condition.



Here (1) holds if $\mathbb{T}$ preserves weak pullbacks, (2) is true when $\mathbb{T}$ is a functor on Sob or a functor on KSob, and Λ is characteristic, and (3) holds when Λ is strong. This yields the following theorem.

3.57 Theorem. *Let* $\mathbb{T}$ *be an endofunctor on* Sob *and* Λ *a characteristic strong monotone geometric modal signature.*

(i) If x *and* x' *are two states in two geometric* $\mathbb{T}$*-models, then*

$$x \leftrightarroows_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x' \quad \textit{iff} \quad x \simeq_{\mathsf{Mod}(\mathbb{T})} x'.$$

(ii) If $\mathbb{T}$ *restricts to the endofunctor* $\mathbb{T}_{\mathsf{KSob}}$ *on* KSob *and* x, x' *are two states in two geometric* $\mathbb{T}_{\mathsf{KSob}}$*-models, then*

$$x \leftrightarroows_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x' \quad \textit{iff} \quad x \simeq_{\mathsf{Mod}(\mathbb{T}_{\mathsf{KSob}})} x'.$$

(iii) If $\mathbb{T}$ *restricts to the endofunctor* $\mathbb{T}_{\mathsf{KHaus}}$ *on* KHaus *and* x *and* x' *are two states in two geometric* $\mathbb{T}_{\mathsf{KHaus}}$*-models, then*

$$x \leftrightarroows_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x'.$$

3.58 Remark. *A priori*, we cannot apply the previous theorem to the monotone functor $\mathbb{D}_{\mathsf{kh}}$. However, we will see in example 4.9 that $\mathbb{D}_{\mathsf{kh}}$ is the restriction of a certain functor on Sob.

From the point of view of bisimulations, the category of sober spaces and continuous maps seems to be an appropriate base category for coalgebraic geometric logic. However, we saw in section 3.3 that looking at concrete examples suggests that the category of compact Hausdorff spaces is a natural choice of base category. We leave it as an interesting direction for further research to investigate how these concrete examples of functors fit in the general theory we have developed.

4

Lifting functors

This chapter is devoted to lifting endofunctors from Set to the categories Sob and Stone. A functor $\mathbb{T}$ together with a set of predicate liftings Λ for $\mathbb{T}$ induces a functor on Sob and on Stone, each accompanied by a set of open respectively clopen predicate liftings. This will give rise to analogs of set-based frames. For example, the functor $\mathbb{D}_{\mathsf{st}}$ (from definition 2.38) corresponding to descriptive monotone frames is the lift of the functor $\mathbb{D}$ (see example 2.7) with respect to a certain set of predicate liftings.

Section 4.1 presents a way of lifting set functors with respect to a set of predicate liftings Λ to functors on Sob. This is inspired by the method of lifting a set functor to a functor on Stone from [36], which is presented in section 4.2.

A different method of lifting a set functor to a functor on Frm has been developed in [57]. This trivially yields a lifted functor $\mathsf{Sob} \to \mathsf{Sob}$ by using the duality between frames and topological spaces (see section 3.1). The two differences of this way of lifting with the method that we present in section 4.1 is that is *does not* depend on a set of predicate liftings, but it *does* put restrictions on the functor $\mathbb{T}$ that is being lifted. We suspect a connection between these two ways of lifting a set functor to a sober functor, and leave this as an interesting open question.

In section 4.2 and 4.3 set functors are lifted to Stone functors in two ways. The first way is a reformulation of the method in [36] and is similar to the procedure in section 4.1. It relies on a given collection of predicate liftings Λ. The second way (lift via pro-completions) only works for a certain class of set functors, but does not depend on a collection of predicate liftings. We prove that for such a $\mathbb{T}$ and a suitable choice of Λ the two ways of lifting coincide on objects (theorem 4.24) thus partially answering a question posed in the conclusion of [36].

4.1 From Set to Sob

In this section we lift a functor $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ together with a set of predicate liftings Λ for $\mathbb{T}$ to a functor $\ddot{\mathbb{T}}_\Lambda : \mathsf{Top} \to \mathsf{Top}$ which restricts to $\ddot{\mathbb{T}}_\Lambda : \mathsf{Sob} \to \mathsf{Sob}$. To define the action of $\ddot{\mathbb{T}}_\Lambda$ on objects, for a topological space $\mathfrak{X}$ we take the following steps:

Step 1. Construct a frame of the images of predicate liftings applied to the open sets of $\mathfrak{X}$ (viewed simply as subsets of $\mathbb{T}(\mathbb{U}\mathfrak{X})$), this is $\dot{\mathbb{F}}_\Lambda \mathfrak{X}$;

Step 2. Quotient $\dot{\mathbb{F}}_\Lambda \mathfrak{X}$ with a suitable relation that ensures $\bigvee^{\uparrow}_{b \in B} \lambda(b) = \lambda(\bigvee^{\uparrow} B)$ whenever λ is monotone;

Step 3. Employ the functor $\mathsf{pt} : \mathsf{Frm} \to \mathsf{Sob}$ to obtain a sober topological space.

This is the content of definitions 4.1, 4.3 and 4.5. It is an adaptation of section 4 in [36].

Recall that $\mathbb{U}$ is the forgetful functor which sends a topological space to its underlying set, and $\mathbb{Q}$ is the contravariant functor sending a set to its powerset Boolean algebra.

4.1 Definition. Let $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ be a functor and Λ a collection of predicate liftings for $\mathbb{T}$. We define a contravariant functor $\dot{\mathbb{F}}_\Lambda : \mathsf{Top} \to \mathsf{Frm}$. For a topological space $\mathfrak{X}$ define $\dot{\mathbb{F}}_\Lambda \mathfrak{X}$ to be the subframe of $\mathbb{Q}(\mathbb{T}(\mathbb{U}\mathfrak{X}))$ generated by the set

$$\{\lambda_{\mathbb{U}\mathfrak{X}}(a_1, \dots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary, } a_1, \dots, a_n \in \Omega\mathfrak{X}\}.$$

That is, we close this set under finite intersections and arbitrary unions in $\mathbb{Q}(\mathbb{T}(\mathbb{U}\mathfrak{X}))$. For a continuous function $f : \mathfrak{X} \to \mathfrak{X}'$ let $\dot{\mathbb{F}}_\Lambda f : \dot{\mathbb{F}}_\Lambda \mathfrak{X}' \to \dot{\mathbb{F}}_\Lambda \mathfrak{X}$ be the restriction of $\mathbb{Q}(\mathbb{T}(\mathbb{U}f))$ to $\dot{\mathbb{F}}_\Lambda \mathfrak{X}'$. ◁

4.2 Lemma. *The map $\dot{\mathbb{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to show that $\dot{\mathbb{F}}_\Lambda$ is well defined on morphisms and that it is functoral. To show that the action of $\dot{\mathbb{F}}_\Lambda$ on morphisms is well-defined, it suffices to show that $(\dot{\mathbb{F}}_\Lambda f)(\lambda_{\mathbb{U}\mathfrak{X}}(a_1', \dots, a_n')) \in \dot{\mathbb{F}}_\Lambda(\mathfrak{X})$ for all generators $\lambda_{\mathbb{U}\mathfrak{X}}(a_1', \dots, a_n')$ of $\dot{\mathbb{F}}_\Lambda \mathfrak{X}'$, because frame homomorphisms preserve finite meets and all joins. This holds by naturality of λ:

$$(\dot{\mathbb{F}}_\Lambda f)(\lambda_{\mathbb{U}\mathfrak{X}'}(a_1, \dots, a_n)) = (\mathbb{T}f)^{-1}(\lambda_{\mathbb{U}\mathfrak{X}'}(a_1, \dots, a_n)) = \lambda_{\mathbb{U}\mathfrak{X}}(f^{-1}(a_1), \dots, f^{-1}(a_n)).$$

By continuity of f we have $f^{-1}(a_i) \in \Omega\mathfrak{X}$ so the latter is indeed in $\dot{\mathbb{F}}_\Lambda \mathfrak{X}$. Functoriality of $\dot{\mathbb{F}}_\Lambda$ follows from functoriality of $\mathbb{Q} \circ \mathbb{T} \circ \mathbb{U}$. □

4.3 Definition. Let Λ be a collection of predicate liftings for a set functor $\mathbb{T}$ and let $\mathfrak{X}$ be a topological space. Let $\ddot{\mathbb{F}}_\Lambda \mathfrak{X}$ be the quotient of $\dot{\mathbb{F}}_\Lambda \mathfrak{X}$ with respect to the congruence $\sim$ generated by

$$\textstyle\bigvee^{\uparrow}_{b \in B} \lambda(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \sim \lambda(a_1, \dots, a_{i-1}, \bigvee^{\uparrow} B, a_{i+1}, \dots, a_n)$$

for all $a_i \in \Omega\mathfrak{X}$, $B \subseteq \Omega\mathfrak{X}$ directed, and $\lambda \in \Lambda$ monotone in its i-th argument. Write $q_{\mathfrak{X}} : \dot{\mathbb{F}}_\Lambda \mathfrak{X} \to \ddot{\mathbb{F}}_\Lambda \mathfrak{X}$ for the quotient map and $[x]$ for the equivalence class in $\ddot{\mathbb{F}}_\Lambda \mathfrak{X}$ of an element $x \in \dot{\mathbb{F}}_\Lambda \mathfrak{X}$. For a continuous function $f : \mathfrak{X} \to \mathfrak{X}'$ define $\ddot{\mathbb{F}}_\Lambda f : \ddot{\mathbb{F}}_\Lambda \mathfrak{X}' \to \ddot{\mathbb{F}}_\Lambda \mathfrak{X} : [\lambda_{\mathbb{U}\mathfrak{X}}(a_1, \dots, a_n)] \mapsto [\dot{\mathbb{F}}_\Lambda(\lambda_{\mathbb{U}\mathfrak{X}}(a_1, \dots, a_n))]$. ◁

4.4 Lemma. *The assignment $\ddot{\mathbb{F}}_\Lambda$ defines a functor.*

Proof. We need to prove functoriality of $\ddot{\mathbb{F}}_\Lambda$ and that $\ddot{\mathbb{F}}_\Lambda f$ is well defined for every continuous map $f : \mathfrak{X} \to \mathfrak{X}'$.

In order to show that $\ddot{\mathbb{F}}_\Lambda$ is well defined, it suffices to show that $\ddot{\mathbb{F}}_\Lambda f$ is invariant under the congruence $\sim$. If $f : \mathfrak{X} \to \mathfrak{X}'$ is a continuous, then

$$\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{b \in B} (\dot{\mathbb{F}}_\Lambda f)(\lambda_{\mathbb{U}\mathfrak{X}'}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b \in B} (\mathbb{T}f)^{-1}(\lambda_{\mathbb{U}\mathfrak{X}'}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b \in B} \lambda_{\mathbb{U}\mathfrak{X}'}(f^{-1}(a_1), \dots, f^{-1}(a_{i-1}), f^{-1}(b), f^{-1}(a_{i+1}), \dots, f^{-1}(a_n)) \\
&\qquad \sim \lambda_{\mathbb{U}\mathfrak{X}'}(f^{-1}(a_1), \dots, f^{-1}(a_{i-1}), f^{-1}(\textstyle\bigvee^{\uparrow} B), f^{-1}(a_{i+1}), \dots, f^{-1}(a_n)) \\
&\qquad = \dot{\mathbb{F}}_\Lambda f(\lambda_{\mathbb{U}\mathfrak{X}'}(a_1, \dots, a_{i-1}, \textstyle\bigvee^{\uparrow} B, a_{i+1}, \dots, a_n))
\end{aligned}$$

so $\ddot{\mathbb{F}}_\Lambda f$ is indeed invariant under the congruence. In the $\sim$-step, we use that $\{f^{-1}(b) \mid b \in B\}$ is directed in $\Omega\mathfrak{X}$. Functoriality of $\ddot{\mathbb{F}}_\Lambda f$ follows from functoriality of $\mathbb{Q} \circ \mathbb{T} \circ \mathbb{U}$. So $\ddot{\mathbb{F}}_\Lambda : \mathsf{Top} \to \mathsf{Frm}$ is a functor. □

We are now ready to define the sober Kupke-Kurz-Pattinson lift to Sob for a functor on Set.

4.5 Definition. Define the **sober Kupke-Kurz-Pattinson lift** (KKP lift for short) of $\mathbb{T}$ with respect to Λ to be the functor

$$\ddot{\mathbb{T}}_\Lambda = \mathfrak{pt} \circ \ddot{\mathbb{F}}_\Lambda.$$

This is a functor $\mathsf{Top} \to \mathsf{Top}$ and because $\mathfrak{pt}$ lands in Sob it restricts to an endofunctor on Sob. We shall view $\ddot{\mathbb{T}}_\Lambda$ as an endofunctor on Sob. $\lhd$

The next definition and lemma describe how to lift a predicate lifting to an open predicate lifting. Recall that $\mathbb{Z}$ is the forgetful functor which sends a frame to its underlying set.

4.6 Definition. Let Λ be a collection of predicate liftings for a set functor $\mathbb{T}$. A predicate lifting $\lambda : \breve{\mathbb{P}}^n \to \breve{\mathbb{P}} \circ \mathbb{T}$ in λ induces an open predicate lifting $\ddot{\lambda} : \Omega \to \Omega \circ \ddot{\mathbb{T}}$ for $\ddot{\mathbb{T}}$ via

$$\Omega^n \mathcal{X} \xrightarrow{\lambda_{\mathbb{U}\mathcal{X}}} \mathbb{Z}(\dot{\mathbb{F}}_\Lambda \mathcal{X}) \xrightarrow{q_{\mathcal{X}}} \mathbb{Z}(\ddot{\mathbb{F}}_\Lambda \mathcal{X}) \xrightarrow{\mathbb{Z}k_{\ddot{\mathbb{F}}_\Lambda \mathcal{X}}} \Omega(\mathfrak{pt}(\ddot{\mathbb{F}}_\Lambda \mathcal{X})) = \Omega(\ddot{\mathbb{T}}\mathcal{X}).$$

By $\lambda_{\mathbb{U}\mathcal{X}}$ actually means the restriction of $\lambda_{\mathbb{U}\mathcal{X}}$ to $\Omega^n \mathcal{X} \subseteq \breve{\mathbb{P}}(\mathbb{U}\mathcal{X})$. The map $k_{\mathbb{F}\mathcal{X}}$ is the frame homomorphism given by $a \mapsto \{p \in \mathfrak{pt}(\mathbb{F}_\Lambda \mathcal{X}) \mid p(a) = 1\}$. Let $\ddot{\Lambda} = \{\ddot{\lambda} \mid \lambda \in \Lambda\}$ be the collection of lifted predicate liftings. Then $\ddot{\Lambda}$ is a geometric modal signature for $\ddot{\mathbb{T}}_\Lambda$. $\lhd$

4.7 Lemma. *The assignment $\ddot{\lambda}$ is a natural transformation.*

Proof. Let $f : \mathcal{X} \to \mathcal{X}'$ be a continuous function, then the following diagram commutes in Set,

$$\begin{array}{ccccccc}
\Omega^n \mathcal{X}' & \xrightarrow{\lambda_{\mathbb{U}\mathcal{X}'}} & \mathbb{Z}(\dot{\mathbb{F}}_\Lambda \mathcal{X}') & \xrightarrow{\mathbb{Z}q_{\mathcal{X}'}} & \mathbb{Z}(\ddot{\mathbb{F}}_\Lambda \mathcal{X}') & \xrightarrow{\mathbb{Z}k_{\ddot{\mathbb{F}}_\Lambda \mathcal{X}'}} & \Omega(\mathfrak{pt}(\ddot{\mathbb{F}}_\Lambda \mathcal{X}')) \\
{\scriptstyle (f^{-1})^n}\downarrow & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle \Omega(\mathfrak{pt}(f^{-1}))} \\
\Omega^n \mathcal{X} & \xrightarrow[\lambda_{\mathbb{U}\mathcal{X}}]{} & \mathbb{Z}(\dot{\mathbb{F}}_\Lambda \mathcal{X}) & \xrightarrow[\mathbb{Z}q_{\mathcal{X}}]{} & \mathbb{Z}(\ddot{\mathbb{F}}_\Lambda \mathcal{X}) & \xrightarrow[\mathbb{Z}k_{\ddot{\mathbb{F}}_\Lambda \mathcal{X}}]{} & \Omega(\mathfrak{pt}(\mathbb{F}_\Lambda \mathcal{X}))
\end{array}$$

Commutativity of the left square follows from naturality of λ, commutativity of the middle square follows from the proof of lemma 4.4 and commutativity of the right square can be seen as follows: let $a_1', \dots, a_n' \in \Omega\mathcal{X}$, then

$$\begin{aligned}
\Omega(\mathfrak{pt}(f^{-1})) \circ \mathbb{Z}k_{\mathbb{F}_\Lambda \mathcal{X}'}(\lambda_{\mathbb{U}\mathcal{X}'}(a_1', \dots, a_n')) &= \{q \in \mathfrak{pt}(\mathbb{F}_\Lambda \mathcal{X}) \mid q \circ f^{-1}(\lambda_{\mathbb{U}\mathcal{X}'}(a_1', \dots, a_n')) = 1\} \\
&= \mathbb{Z}k_{\mathbb{F}_\Lambda \mathcal{X}}(f^{-1}(\lambda_{\mathbb{U}\mathcal{X}'}(a_1', \dots, a_n'))).
\end{aligned}$$

So $\ddot{\lambda}$ is an open predicate lifting. $\square$

The nature of the definitions of $\ddot{\mathbb{T}}$ and $\ddot{\Lambda}$ yields the following desirable result.

4.8 Proposition. *Let $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ be a functor and Λ a collection of predicate liftings for $\mathbb{T}$. Then $\ddot{\Lambda}$ is characteristic for $\ddot{\mathbb{T}}_\Lambda$.*

Proof. Let $\mathcal{X}$ be a sober space. We need to show that the collection

$$\{\ddot{\lambda}(a_1, \dots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary}, a_i \in \Omega\mathcal{X}\} \tag{4.1}$$

forms a subbase for the topology on $\ddot{\mathbb{T}}_\Lambda \mathfrak{X}$. An arbitrary open set of $\ddot{\mathbb{T}}_\Lambda \mathfrak{X}$ is of the form $\widetilde{x} = \{p \in \mathfrak{pt}(\ddot{\mathbb{F}}_\Lambda \mathfrak{X}) \mid p(x) = 1\}$, for $x \in \ddot{\mathbb{F}}_\Lambda \mathfrak{X}$. An arbitrary element of $\ddot{\mathbb{F}}_\Lambda \mathfrak{X}$ is the equivalence class of an arbitrary union of finite intersections of elements of the form $\{\lambda_{\mathfrak{U}\mathfrak{X}}(a_1, \dots, a_n)$, for $\lambda \in \Lambda$ and $a_1, \dots, a_n \in \Omega \mathfrak{X}$. So we may write

$$x = \bigcup_{i \in I} \Big(\bigcap_{j \in J_i} [\lambda^{i,j}_{\mathbb{U}\mathfrak{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}})]\Big)$$

for some possibly index set I, finite index sets J_i, $\lambda^{i,j} \in \Lambda$ and open sets $a^{i,j}_k \in \Omega \mathfrak{X}$. Unraveling the definitions gives

$$\begin{aligned} \widetilde{x} &= \bigcup_{i \in I} \Big(\widetilde{\bigcap_{j \in J_i} [\lambda^{i,j}_{\mathbb{U}\mathfrak{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}})]} \Big) \\ &= \bigcup_{i \in I} \Big(\bigcap_{j \in J_i} \ddot{\lambda}^{i,j}_{\mathfrak{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}}) \Big) \end{aligned}$$

This shows that the open sets in (4.1) indeed form a subbase for the open sets of $\ddot{\mathbb{T}}_\Lambda \mathfrak{X}$. □

Let us put our theory to action to (re)obtain the monotone functor on KHaus from definition 2.38.

4.9 Example (The monotone functor)**.** Recall the set functor $\mathbb{D}$ from example 2.7: $\mathbb{D} : X \to \{W \in \breve{\mathbb{P}}\breve{\mathbb{P}} X \mid W$ is up-closed under inclusion order$\}$. The box and diamond are given by the predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \breve{\mathbb{P}} \to \breve{\mathbb{P}} \circ \mathbb{D}$ defined by

$$\lambda^{\Box}_X(a) := \{W \in \mathbb{D}X \mid a \in W\}, \quad \lambda^{\Diamond}_X(a) := \{W \in \mathbb{D}X \mid (X \setminus a) \notin W\},$$

where $X \in \mathsf{Set}$. Furthermore recall from definition 3.37 that for a compact Hausdorff space $\mathfrak{X}$ the space $\mathbb{D}_{\mathsf{kh}} \mathfrak{X}$ is the subset of $\mathbb{D}(\mathbb{U}\mathfrak{X})$ of collections of sets W satisfying for all $u \subseteq \mathbb{U}\mathfrak{X}$ that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. So $\mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \subseteq \mathbb{D}(\mathbb{U}\mathfrak{X})$. The set $\mathbb{D}_{\mathsf{kh}}\mathfrak{X}$ is topologised by the subbase

$$\boxplus a := \{W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \mid a \in W\}, \quad \diamondplus a := \{W \in \mathbb{D}_{\mathsf{kh}}\mathfrak{X} \mid (\mathfrak{X} \setminus a) \notin W\}.$$

By theorem 3.41 the functor $\mathbb{M} : \mathsf{Frm} \to \mathsf{Frm}$ from definition 3.39 is such that $\mathbb{M}(\mathfrak{opn}\,\mathfrak{X}) \cong \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ whenever $\mathfrak{X}$ is a compact Hausdorff space.

Let $\mathfrak{X}$ be a compact Hausdorff space. We claim that $\ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X} = \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$. Define a map $\varphi : \mathbb{M}(\mathfrak{opn}\,\mathfrak{X}) \to \ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X}$ on generators by $\Box a \mapsto [\lambda^\Box(a)]$ and $\Diamond a \mapsto [\lambda^\Diamond(a)]$. This is well-defined because the $[\lambda^\Box(a)], [\lambda^\Diamond(a)]$ satisfy relations (M1) – (M6) from definition 3.39 and it is surjective because the image of φ contains the generators of $\ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X}$.

So we only need to show injectivity of φ. Our strategy to prove this is to define a map $\psi : \ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X} \to \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ and show that it is inverse to φ on the level of sets. Since a set-theoretic inverse suffices we do not need to prove that ψ is a homomorphism; we just want it to be well-defined. Instead of defining $\psi : \ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \to \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ directly, we will give a well-defined map $\psi' : \dot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X} \to \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$ whose kernel contains the kernel of the quotient map $q_{\mathfrak{X}} : \dot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \to \ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X}$. This then yields the map ψ we require. In a diagram:

$$\begin{array}{ccc} \dot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X} & \xrightarrow{\quad \psi' \quad} & \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \\ & {\scriptstyle q_{\mathfrak{X}}} \searrow \quad \nearrow {\scriptstyle \psi} & \\ & \ddot{\mathbb{F}}_{\{\lambda^\Box, \lambda^\Diamond\}} \mathfrak{X} & \end{array} \tag{4.2}$$

Define $\psi' : \dot{\mathbb{F}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathfrak{X} \to \mathbb{M}(\mathfrak{opn}\,\mathfrak{X})$ on generators by $\lambda^\Box(a) \mapsto \boxplus a$ and $\lambda^\Diamond(a) \mapsto \otimes a$. In order to show that this assignments yields a well-defined map (hence extends to a frame homomorphism by remark 3.3) we need to show that the presentation of an element in $\dot{\mathbb{F}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathfrak{X}$ does not affect its image under ψ'. That is, if

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^\Box(a_{i,j}) \cap \bigcap_{j'\in J'_i}\lambda^\Diamond(a_{i,j'})\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^\Box(a_{k,\ell}) \cap \bigcap_{\ell'\in L'_k}\lambda^\Diamond(a_{k,\ell'})\Big), \tag{4.3}$$

where J_i, J'_i, L_k and L'_k are finite index sets, then

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J}\psi'(\lambda^\Box(a_{i,j})) \cap \bigcap_{j'\in J'}\psi'(\lambda^\Diamond(a_{i,j'}))\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\psi'(\lambda^\Box(a_{k,\ell})) \cap \bigcap_{\ell'\in L'}\psi'(\lambda^\Diamond(a_{k,\ell'}))\Big).$$

As stated we have $\mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \subseteq \mathbb{D}(\mathbb{U}\mathfrak{X})$. Observe

$$\psi'(\lambda^\Box(a)) = \boxplus a = \{W \in \mathbb{D}(\mathbb{U}\mathfrak{X}) \mid a \in W\} \cap \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) = \lambda^\Box(a) \cap \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}).$$

and similarly $\psi'(\lambda^\Diamond(a)) = \lambda^\Diamond(a) \cap \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})$. Suppose the identity in (4.3) holds, then we have

$$\begin{aligned}
&\bigcup_{i\in I}\Big(\bigcap_{j\in J}\psi'(\lambda^\Box(a_{i,j})) \cap \bigcap_{j'\in J'}\psi'(\lambda^\Diamond(a_{i,j'}))\Big)\\
&\quad= \bigcup_{i\in I}\Big(\bigcap_{j\in J}(\lambda^\Box(a_{i,j}) \cap \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X})) \cap \bigcap_{j'\in J'}(\lambda^\Diamond(a_{i,j'}) \cap \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}))\Big)\\
&\quad= \bigcup_{i\in I}\Big(\mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \cap \bigcap_{j\in J}\lambda^\Box(a_{i,j}) \cap \bigcap_{j'\in J'}\lambda^\Diamond(a_{i,j'})\Big)\\
&\quad= \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \cap \bigcup_{i\in I}\Big(\bigcap_{j\in J}\lambda^\Box(a_{i,j}) \cap \bigcap_{j'\in J'}\lambda^\Diamond(a_{i,j'})\Big)\\
&\quad= \mathbb{U}(\mathbb{D}_{\mathsf{kh}}\mathfrak{X}) \cap \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\lambda^\Box(a_{k,\ell}) \cap \bigcap_{\ell'\in L'}\lambda^\Diamond(a_{k,\ell'})\Big)\\
&\quad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\psi'(\lambda^\Box(a_{k,\ell})) \cap \bigcap_{\ell'\in L'}\psi'(\lambda^\Diamond(a_{k,\ell'}))\Big).
\end{aligned}$$

So ψ' is well defined.

It is easy to see that $\curlyvee_{b\in B}\lambda(b) \sim \lambda(\curlyvee B)$ implies $\big(\curlyvee_{b\in B}\lambda(b), \lambda(\curlyvee B)\big) \in \ker(\psi)$ for $\lambda \in \{\lambda^\Box, \lambda^\Diamond\}$. Since these pairs generate of the congruence from definition 4.3, we have $\sim\, = \ker(q_{\mathfrak{X}}) \subseteq \ker(\psi')$ and hence there exists a map $\psi : \ddot{\mathbb{F}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathfrak{X} \to \mathfrak{opn}(\widehat{\mathbb{T}}\mathfrak{X})$ such that the diagram in (4.2) commutes. Therefore ψ is (well) defined on generators by $[\lambda^\Box(a)] \mapsto \Box a$ and $[\lambda^\Diamond(a)] \mapsto \Diamond a$. One can easily check that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$ by looking at the action on the generators. It follows that φ is injective.

This entails that for compact Hausdorff spaces $\mathfrak{X}$,

$$\ddot{\mathbb{D}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathfrak{X} = \mathbb{D}_{\mathsf{kh}}\mathfrak{X},$$

Furthermore, it can be seen that for continuous maps $f : \mathfrak{X} \to \mathfrak{X}'$ we have $\mathbb{F}_{\{\lambda^\Box,\lambda^\Diamond\}}f = \mathfrak{opn}(\mathbb{D}_{\mathsf{kh}}f)$. As a consequence, when restricted to KHaus we have

$$(\ddot{\mathbb{D}}_{\{\lambda^\Box,\lambda^\Diamond\}})_{\restriction\mathsf{KHaus}} = \mathbb{D}_{\mathsf{kh}},$$

that is, lifting the monotone functor on Set with respect to the box/diamond lifting yields the monotone functor on KHaus from definition 3.37. ◁

4.10 Example. Using similar techniques as in the previous example, one can show that, when restricted to KHaus, the sober Kupke-Kurz-Pattinson lift of $\mathbb{P}$ with respect to the usual box and diamond lifting coincides with the Vietoris functor. (An algebraic description similar to the one in theorem 3.41 is given in definition 3.35. It is shown in proposition III4.6 of [29] that this algebraic description is dual to $\mathbb{V}_{\mathsf{kh}}$.) ◁

A third example of this method of lifting functors can be found in section 5.2: The functor on KHaus corresponding to geometric conditional frames arises as the restriction of the lift of the functor $\mathbb{C}$ from example 2.10 with respect to the predicate liftings $\lambda^{\Rightarrow}, \lambda^{\Downarrow}$ from example 2.22.

4.2 From Set to Stone

We turn our attention from sober spaces to Stone spaces. Let $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ be a functor and Λ a collection of predicate liftings for $\mathbb{T}$. Along the same lines as the previous section, we construct a lifted version $\widehat{\mathbb{T}}_\Lambda : \mathsf{Stone} \to \mathsf{Stone}$ (the Kupke-Kurz-Pattinson lift).

4.11 Definition. Let $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ be a functor and Λ a set of predicate liftings for $\mathbb{T}$. For a Stone space $\mathfrak{X}$ let $\widehat{\mathbb{B}}_\Lambda \mathfrak{X}$ be the sub-Boolean algebra of $\mathbb{Q}(\mathbb{T}\mathfrak{X})$ generated by the set

$$\{\lambda_{\mathbb{U}\mathfrak{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_1, \ldots, a_n \in \mathbb{C}\mathrm{lp}\,\mathfrak{X}\}. \tag{4.4}$$

That is, $\widehat{\mathbb{B}}_\Lambda \mathfrak{X}$ is obtained by closing the set in (4.4) under taking complements, finite intersections and finite unions. For a continuous map $f : \mathfrak{X} \to \mathfrak{X}'$ let $\widehat{\mathbb{B}}_\Lambda f : \widehat{\mathbb{B}}_\Lambda \mathfrak{X}' \to \widehat{\mathbb{B}}_\Lambda \mathfrak{X}$ be the restriction of $\mathbb{Q}(\mathbb{T}f)$ to $\widehat{\mathbb{B}}_\Lambda \mathfrak{X}'$. ◁

4.12 Lemma. *The assignment $\widehat{\mathbb{B}}_\Lambda$ defines a contravariant functor.*

Proof. We need to show that $\widehat{\mathbb{B}}_\Lambda$ is functorial and that $\widehat{\mathbb{B}}_\Lambda f$ is well defined for every continuous function between Stone spaces $f : \mathfrak{X} \to \mathfrak{X}'$. For the latter, it suffices to show that for all generators $\lambda_{\mathfrak{X}'}(a_1', \ldots, a_n')$ of $\widehat{\mathbb{B}}\mathfrak{X}'$ we have $(\widehat{\mathbb{B}}_\Lambda f)(\lambda_{\mathfrak{X}'}(a_1', \ldots, a_n')) \in \widehat{\mathbb{B}}_\Lambda \mathfrak{X}$, because the homomorphism $\mathbb{Q}(\mathbb{T}f)$ (of which $\widehat{\mathbb{B}}_\Lambda f$ is a restriction) preserves taking complements, finite intersections and finite unions. By naturality of λ we have

$$(\widehat{\mathbb{B}}_\Lambda f)(\lambda_{\mathfrak{X}'}(a_1', \ldots, a_n')) = (\mathbb{T}f)^{-1}(\lambda_{\mathfrak{X}'}(a_1', \ldots, a_n')) = \lambda_{\mathfrak{X}}(f^{-1}(a_1'), \ldots, f^{-1}(a_n'))$$

and because the $f^{-1}(a_i')$ are clopen in $\mathfrak{X}$, the result is in $\widehat{\mathbb{B}}_\Lambda \mathfrak{X}$. Functoriality of $\widehat{\mathbb{B}}_\Lambda$ follows from functoriality of $\mathbb{Q} \circ \mathbb{T}$. □

4.13 Definition. Let $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ be a functor and Λ a set of predicate liftings for $\mathbb{T}$. The **Kupke-Kurz-Pattinson lift** of $\mathbb{T}$ to Stone is

$$\widehat{\mathbb{T}}_\Lambda := \mathfrak{uf} \circ \widehat{\mathbb{B}}_\Lambda.$$ ◁

4.14 Definition. Let Λ be a set of predicate liftings for a functor $\mathbb{T}$. Then any $\lambda : \breve{\mathbb{P}}^n \to \breve{\mathbb{P}} \circ \mathbb{T}$ in Λ induces a clopen predicate lifting $\widehat{\lambda} : \mathbb{C}\mathrm{lp} \to \mathbb{C}\mathrm{lp} \circ \widehat{\mathbb{T}}_\Lambda$ for $\widehat{\mathbb{T}}_\Lambda$ via

$$\mathbb{C}\mathrm{lp}^n\,\mathfrak{X} \xrightarrow{\lambda_{\mathbb{U}\mathfrak{X}}} \mathbb{Z}(\widehat{\mathbb{B}}_\Lambda \mathfrak{X}) \xrightarrow{\mathbb{Z}k_{\widehat{\mathbb{B}}_\Lambda \mathfrak{X}}} \mathbb{C}\mathrm{lp}(\mathfrak{uf}(\widehat{\mathbb{B}}_\Lambda \mathfrak{X})) = \mathbb{C}\mathrm{lp}(\widehat{\mathbb{T}}_\Lambda \mathfrak{X}).$$

(To be precise, the domain of $\lambda_{\mathbb{U}\mathfrak{X}}$ is $\mathbb{Q}^n(\mathbb{U}\mathfrak{X})$. We only need the restriction to $\mathbb{C}\mathrm{lp}^n\,\mathfrak{X}$. We still write $\lambda_{\mathbb{U}\mathfrak{X}}$ for this restriction in order to avoid clutter.) The map $k_{\mathbb{F}\mathfrak{X}}$ is the isomorphism given by Stone duality. Naturality of $\widehat{\lambda}$ follows from naturality of λ. ◁

We state some properties of lifted predicate liftings, the proofs of which are left to the reader.

4.15 Lemma. *Let $\mathbb{T}$ be a set functor, Λ a collection of predicate liftings for $\mathbb{T}$, $\widehat{\mathbb{T}}_\Lambda$ the KKP lift of $\mathbb{T}$ with respect to Λ and $\widehat{\Lambda} = \{\widehat{\lambda} \mid \lambda \in \Lambda\}$.*

(i) If $\lambda \in \Lambda$ is monotone, then so is $\widehat{\lambda}$.

(ii) If Λ is monotone, then so is $\widehat{\Lambda}$.

(iii) If Λ is closed under duals, then so is $\widehat{\Lambda}$.

4.16 Remark ([50], proposition 43)**.** As an instance of the Yoneda lemma, the n-ary predicate liftings for a set functor $\mathbb{T}$ are in one-to-one correspondence with subsets of $\mathbb{T}(2^n)$. (The set 2^n is the n-fold product of the two-element space $2 = \{\top, \bot\}$.) To an n-ary predicate lifting λ assign the subset $\lambda_{2^n}(\pi_1^{-1}(\{\top\}), \dots, \pi_n^{-1}(\{\top\})) \subseteq \mathbb{T}(2^n)$, where $\pi_i : 2^n \to 2$ is the i-th projection. Conversely, for $C \subseteq \mathbb{T}(2^n)$, the assignment $\lambda_X^C(a_1, \dots, a_n) := (\mathbb{T}\langle \chi_{a_1}, \dots, \chi_{a_n}\rangle)^{-1}[C] \subseteq \mathbb{T}X$ yields a predicate lifting for $\mathbb{T}$. The angled brackets denote the tupling of the indicator functions χ_{a_i}.

The correspondence gives a canonical collection of predicate liftings for each set functor $\mathbb{T}$, namely the collection of all predicate liftings for $\mathbb{T}$.

Example 10 in [36] states that $\mathbb{P}$ with the collection Λ of all predicate liftings gives $\widehat{\mathbb{P}}_\Lambda = \mathbb{V}_{\mathsf{st}}$. A different way to obtain $\mathbb{V}_{\mathsf{st}}$ as a lifted functor is by lifting $\mathbb{P}$ with respect to the usual predicate lifting for box and diamond. Before we can show this in example 4.19, we need the following definition and lemma. For a proof we refer to [29, Section III.4].

4.17 Definition. The map $\mathbb{N}_{\mathsf{ba}} : \mathsf{BA} \to \mathsf{BA}$ is defined on an object B as the free Boolean algebra generated by $\Box a, \Diamond a$, $(a \in B)$ subject to the relations

(V1) $\Box\top = \top$ (V2) $\Diamond\bot = \bot$

(V3) $\Box(a \wedge b) = \Box a \wedge \Box b$ (V4) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

(V5) $\Box(a \vee b) \leq \Box a \vee \Diamond b$ (V6) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$

For a homomorphism $f : B \to B'$ define $\mathbb{N}_{\mathsf{ba}} f : \mathbb{N}_{\mathsf{ba}} B \to \mathbb{N}_{\mathsf{ba}} B'$ on generators by $\mathbb{N}_{\mathsf{ba}}(\Box a) \mapsto \Box(f(a))$ and $\mathbb{N}_{\mathsf{ba}}(\Diamond a) \mapsto \Diamond(f(a))$. $\lhd$

4.18 Lemma. *If $\mathfrak{X}$ is a Stone space and $\mathbb{V}_{\mathsf{st}}$ is the Vietoris functor on Stone, then*

$$\chi : \mathfrak{clp}(\mathbb{V}_{\mathsf{st}}\mathfrak{X}) \to \mathbb{N}_{\mathsf{ba}}(\mathfrak{clp}\,\mathfrak{X})$$

defined on generators by $\boxplus a \mapsto \Box a$ and $\diamondplus a \mapsto \Diamond a$ is an isomorphism.

We will now show that the Kupke-Kurz-Pattinson lift of the powerset functor $\mathbb{P}$ with respect to the box- and diamond lifting (given below) yields the Vietoris functor on Stone.

4.19 Example. Recall that for the powerset functor $\mathbb{P}$, the box- and diamond-lifting $\lambda^\Box, \lambda^\Diamond : \breve{\mathbb{P}} \to \breve{\mathbb{P}}\mathbb{P}$ are given by

$$\lambda_X^\Box : a \mapsto \{b \in \mathbb{P}X \mid b \subseteq a\}, \quad \lambda_X^\Diamond : a \mapsto \{b \in \mathbb{P}X \mid b \cap a \neq \varnothing\}.$$

Let $\mathcal{X}$ be a Stone space. We claim that $\widehat{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}} = \mathbb{V}_{\mathsf{st}}$.

Let $\mathcal{X}$ be a Stone space. We will show that $\widehat{\mathbb{B}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathcal{X} \cong \mathfrak{clp}(\mathbb{V}\mathcal{X})$. Define $\varphi : \mathfrak{clp}(\mathbb{V}_{\mathsf{st}}\mathcal{X}) \to \widehat{\mathbb{B}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathcal{X}$ on generators by $\boxplus a \mapsto \lambda^\Box(a)$ and $\diamondplus a \mapsto \lambda^\Diamond a$. By lemma 4.18 we may view φ as a morphism $\mathbb{N}_{\mathsf{ba}}(\mathfrak{clp}\,\mathcal{X}) \to \widehat{\mathbb{B}}_{\{\lambda^\Box,\lambda^\Diamond\}}\mathcal{X}$ so in order to show that φ is a well-defined Boolean algebra homomorphism, it suffices to show that the $\lambda^\Box(a)$ and $\lambda^\Diamond(a)$ (where a ranges over $\mathrm{Clp}\,\mathcal{X}$) satisfy relation (V1) – (V6) from definition 4.17. We leave this verification to the reader.

The map φ is surjective because the generators of $\widehat{\mathbb{B}}_{\{\lambda^\Box,\lambda^\Diamond\}}$ are all in its image. So we only need to show injectivity. Observe that, when seen as subsets of $\mathbb{P}(\mathbb{U}\mathcal{X})$, we have

$$\boxplus a = \lambda^\Box(a) \cap \mathbb{V}_{\mathsf{st}}\mathcal{X} \quad \text{and} \quad \diamondplus a = \lambda^\Diamond(a) \cap \mathbb{V}_{\mathsf{st}}\mathcal{X}.$$

Since $\lambda^\Diamond$ is the dual of $\lambda^\Box$, every element of $\mathfrak{clp}(\mathbb{V}_{\mathsf{st}}\mathcal{X})$ is a finite union of finite intersections of generators (i.e. elements of the form $\boxplus a, \diamondplus a$, where $a \in \mathfrak{clp}\,\mathcal{X}$). (Indeed, we do not need to take complements because the generators are closed under complementation in $\mathcal{X}$.) So we may write an arbitrary element $x \in \mathfrak{clp}(\mathbb{V}_{\mathsf{st}}\mathcal{X})$ as $x = \bigcup_{i=1}^n (\bigcap_{j=1}^{m_i} \boxplus a_{i,j} \cap \bigcap_{j'=1}^{m_i'} \diamondplus a_{i,j'})$. It is easy to see that $x = \varphi(x) \cap \mathbb{V}_{\mathsf{st}}\mathcal{X}$. Therefore, if x, x' are elements of $\mathfrak{clp}(\mathbb{V}_{\mathsf{st}}\mathcal{X})$ and $\varphi(x) = \varphi(x')$, then $x = x'$. This shows that φ is injective.

It follows from unraveling the definitions that $\widehat{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}} f = \mathbb{V}_{\mathsf{st}} f$ when f is a continuous map between Stone spaces. Therefore

$$\widehat{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}} = \mathbb{V}_{\mathsf{st}},$$

that is, the Kupke-Kurz-Pattinson lift of $\mathbb{P}$ with respect $\{\lambda^\Box, \lambda^\Diamond\}$ to Stone gives the Vietoris functor on Stone. ◁

We have seen in example 4.10 that, when restricted to KHaus, the lift of $\mathbb{P}$ with respect to $\{\lambda^\Box, \lambda^\Diamond\}$ to a functor on Sob coincides with the Vietoris functor, i.e. $(\ddot{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}})_{\restriction \mathsf{KHaus}} = \mathbb{V}_{\mathsf{kh}}$. This implies

$$(\ddot{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}})_{\restriction \mathsf{Stone}} = \widehat{\mathbb{P}}_{\{\lambda^\Box,\lambda^\Diamond\}}.$$

A natural question that arises is how $\ddot{\mathbb{T}}_\Lambda$ compares to $\widehat{\mathbb{T}}_\Lambda$ in general. We leave this as an interesting open question.

We give another example.

4.20 Example. Let $\mathbb{D}$ be the monotone functor on Set and $\lambda^\Box, \lambda^\Diamond$ the usual predicate liftings for $\mathbb{D}$. Using similar reasoning as in example 4.19, it can be seen that $\widehat{\mathbb{D}}_{\{\lambda^\Box,\lambda^\Diamond\}} = \mathbb{D}_{\mathsf{st}}$. ◁

In section 5.1 the objects of study are descriptive conditional frames. The functor that corresponds to descriptive conditional frames can be obtained by lifting $\mathbb{C}$ from example 2.10 with respect to the predicate liftings $\lambda^{\Rightarrow}, \lambda^{\Downarrow}$ from example 2.22 to a Stone functor. We will see that $\widehat{\mathbb{C}}_{\{\lambda^{\Rightarrow},\lambda^{\Downarrow}\}}$ is *not* the restriction of $\ddot{\mathbb{C}}_{\{\lambda^{\Rightarrow},\lambda^{\Downarrow}\}}$ to Stone.

4.3 The pro-completion lift from Set to Stone

In the final section of this chapter we define a different way to lift a certain class of set functors to Stone functors and we compare this lift with the Kupke-Kurz-Pattinson lift from definition 4.13. We assume that the set functor $\mathbb{T}$ that we work with sends finite sets to finite sets and preserves cofiltered diagrams (that is, $\mathbb{T}$ sends cofiltered diagrams to cofiltered diagrams). Let FinSet and FinStone be the full subcategories of Set and Stone whose objects are finite sets and finite Stone spaces, respectively. We make use of the following important observations.

Observation 1. The category Stone is equivalent to the pro-completion of FinStone [29, Theorem VI2.3]. Essentially, this means that every Stone space is an inverse limit of finite Stone spaces. For details, see chapter VI of [29].

Observation 2. A topology on a finite set is a Stone topology if and only if it is the discrete topology. Therefore every function between finite Stone spaces is continuous. It follows that the category FinSet of finite sets and functions is isomorphic to the category FinStone of finite Stone spaces and continuous functions.

Observation 3. Since $\mathbb{T}$ restricts to FinSet and FinSet is isomorphic to FinStone, the restriction of $\mathbb{T}$ to FinSet defines a functor on FinStone.

Observation 4. A finite quotient $q_{\mathfrak{Y}} : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ of a Stone space $\mathfrak{X}$ is simply a partition of $\mathfrak{X}$ into clopen subsets. For $x \in \mathfrak{X}$, let $[x]$ be the equivalence class of x in $\mathfrak{Y}$. For each $[x] \in \mathfrak{Y}$ the set $q_{\mathfrak{Y}}^{-1}([x])$ is an element of this partition. Moreover, since $\mathfrak{Y}$ is finite $\{[x]\}$ is clopen for each $[x] \in \mathfrak{Y}$, so by continuity of $q_{\mathfrak{Y}}$ the set $q_{\mathfrak{Y}}^{-1}([x]) \subseteq \mathfrak{X}$ is clopen in $\mathfrak{X}$.

Observation 5. The category BA has all limits and colimits because it is the category of models of an algebraic theory (for details see [18]). Since Stone is dually equivalent to BA it has all limits and colimits as well.

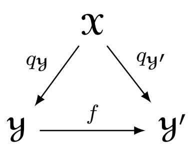
The first observation suggests that we could maybe define a functor $\overline{\mathbb{T}}$ on Stone by defining its action on finite Stone spaces and then taking the pro-completion. Although there is a neat way of lifting $\mathbb{T}$ to a map on the collection of Stone spaces, we have not yet found a suitable way of defining $\overline{\mathbb{T}}f$, for morphisms f in Stone.

In the following definition we define the action of $\overline{\mathbb{T}}$ on objects. When we say that $\mathfrak{Y}$ is a finite quotient of $\mathfrak{X}$, we take this literally, i.e. $\mathfrak{Y} = \mathfrak{X}/{\sim}$ for some equivalence relation $\sim$ on $\mathfrak{X}$. Therefore, each finite quotient $\mathfrak{Y}$ of $\mathfrak{X}$ comes equipped with a unique quotient map $q_{\mathfrak{Y}} : \mathfrak{X} \to \mathfrak{Y}$.

4.21 Definition. Let $\mathbb{T}$ be a set functor which sends finite sets to finite sets. Let $\mathfrak{X}$ be a Stone space. Let $\mathcal{C}$ be the collection of finite quotients of $\mathfrak{X}$,

$$\mathcal{C} = \{\mathfrak{Y} \mid \mathfrak{Y} \text{ is a finite quotient of } \mathfrak{X}\}.$$

For each $\mathfrak{Y} \in \mathcal{C}$ let $q_{\mathfrak{Y}} : \mathfrak{X} \to \mathfrak{Y}$ be the unique corresponding quotient map. For all $\mathfrak{Y}, \mathfrak{Y}' \in \mathcal{C}$, add to $\mathcal{C}$ those continuous maps $f : \mathfrak{Y} \to \mathfrak{Y}'$ that satisfy $f \circ q_{\mathfrak{Y}} = q_{\mathfrak{Y}'}$, i.e. such that the following commutes:



Then $\mathcal{C}$ is a diagram in Stone and in FinStone.

Let $\mathbb{T}\mathcal{C}$ be the diagram obtained from applying $\mathbb{T}$ to the diagram $\mathcal{C}$. Define $\overline{\mathbb{T}}\mathfrak{X}$ to be the categorical limit

$$\varprojlim \mathbb{T}\mathcal{C}$$

of the diagram $\mathbb{T}\mathcal{C}$ in Stone (see figure 4.1). This exists by observation 5 above. We call this the **lift via pro-completion**. ◁

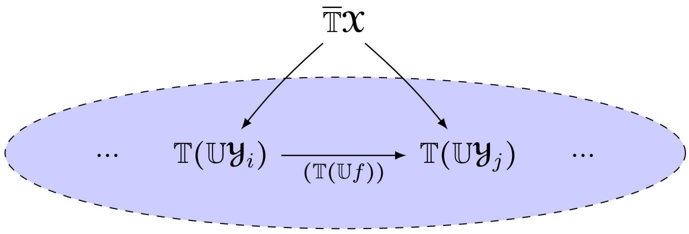


Figure 4.1: The coloured area represents the diagram $\mathbb{T}\mathcal{C}$.

Although, formally, a diagram in C is a functor $F : \mathsf{J} \to \mathsf{C}$, where J is some index category, we shall think of a diagram as a collection of objects and morphisms between them in C.

4.22 Definition. A diagram $\mathcal{C}$ in some category C is **filtered** if:

(i) For any two objects X, X' in $\mathcal{C}$ there exists an object Y and morphisms $f : X \to Y, f' : X' \to Y$ in $\mathcal{C}$; and

(ii) For any two parallel morphisms $f, f' : X \to X'$ in $\mathcal{C}$ there is a morphism $g : X' \to Y$ in $\mathcal{C}$ such that $gf = gf'$.

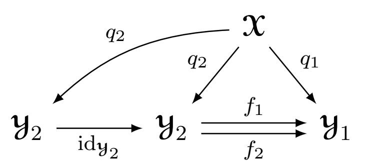
The definition of a **cofiltered diagram** is the same, but with the direction of the morphisms reversed. ◁

The diagram $\mathcal{C}$ from definition 4.21 is cofiltered.

4.23 Lemma. *Let $\mathscr{X}$ be a Stone space and $\mathcal{C}$ the diagram of finite quotients of $\mathscr{X}$ as defined in definition 4.21. Then $\mathcal{C}$ is a cofiltered diagram.*

Proof. For the first item of definition 4.22, suppose $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are two finite quotients of $\mathscr{X}$. We may view them as finite partitions of $\mathscr{X}$ into clopen sets, say, $\mathscr{Y}_1 = \{a_1, \ldots, a_n\}$ and $\mathscr{Y}_2 = \{b_1, \ldots, b_m\}$. Let $\mathscr{Y}_3 := \{a_i \cap b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly the $a_i \cap b_j$ are clopen in $\mathscr{X}$, so $\mathscr{Y}$ is a finite partition of $\mathscr{X}$, hence a finite quotient. Define $f_i : \mathscr{Y}_3 \to \mathscr{Y}_i$ by $f_i([x]_3) \mapsto [x]_i$ for $i = 1, 2$, where $[x]_i$ denotes the equivalence class of x in $\mathscr{Y}_i$. It is easy to check that the f_i are well defined and in $\mathcal{C}$.

For the second item, let $f_1, f_2 : \mathscr{Y}_2 \to \mathscr{Y}_1$ be parallel morphisms in $\mathcal{C}$. Then the identity morphism on $\mathscr{Y}_2$ has the desired property:



Indeed, for all $[x]_2 \in \mathscr{Y}$ we have $f_1(\mathrm{id}_{\mathscr{Y}_2}([x]_2)) = f_1(q_2(x)) = q_1(x) = f_2(q_2(x)) = f_2(\mathrm{id}_{\mathscr{Y}_2}([x]_2))$. □

The question whether this lift via pro-completions and the KKP lift coincide was posed in the conclusion of [36]. We conclude this section by answering this question affirmatively for the objects.

4.24 Theorem. *Let $\mathbb{T}$ be a set functor that restricts to* FinSet *and which preserves cofiltered diagrams. Then on objects the canonical lift given in definition 4.21 and the Kupke-Kurz-Pattinson lift of $\mathbb{T}$ to* Stone *with respect to the collection Λ of all predicate liftings coincide.*

Proof. Let $\mathfrak{X}$ be a Stone space and $\mathcal{C}$ the collection of finite quotients as defined in definition 4.21. We will compare the clopen set Boolean algebras of $\overline{\mathbb{T}}\mathfrak{X}$ and $\widehat{\mathbb{T}}\mathfrak{X}$. The latter, $\widehat{\mathbb{B}}_\Lambda\mathfrak{X}$, is generated as a sub-Boolean algebra of $\mathbb{Q}\mathbb{T}\mathbb{U}\mathfrak{X}$ by the elements

$$\{\lambda_X^C(a_1,\dots,a_n) \mid n \geq 0, C \subseteq \mathbb{T}(2^n), a_i \in \operatorname{Clop}\mathfrak{X}\},$$

where λ^C denotes the n-ary predicate lifting corresponding to the subset $C \subseteq \mathbb{T}(2^n)$, cf. proposition 4.16. Abbreviate $\widehat{\mathcal{B}} \coloneqq \widehat{\mathbb{B}}_\Lambda\mathfrak{X}$. (Note that we implicitly use Stone duality to say that $\mathfrak{clp}(\widehat{\mathbb{T}}\mathfrak{X}) = \mathfrak{clp}(\mathfrak{uf}\widehat{\mathcal{B}}) = \widehat{\mathcal{B}}$.)

Let $\overline{\mathcal{B}}$ be the clopen set Boolean algebra of $\overline{\mathbb{T}}\mathfrak{X}$. Then by the duality of Stone and BA the Boolean algebra $\overline{\mathcal{B}}$ is a colimit of the diagram $\mathfrak{opn}(\mathbb{T}\mathcal{C})$ in BA obtained by applying $\mathfrak{clp} \circ \mathbb{T} \circ \mathbb{U}$ to the diagram $\mathcal{C}$. (After applying $\mathbb{T} \circ \mathbb{U}$ we view the objects as finite Stone spaces rather than finite sets.)

Observe that, if we view $\mathbb{T}(\mathbb{U}(\mathfrak{Y}_i))$ as a Stone space, we have $\mathfrak{clp}(\mathbb{T}(\mathbb{U}(\mathfrak{Y}_i))) = \mathbb{Q}(\mathbb{T}(\mathbb{U}(\mathfrak{Y}_i)))$. Moreover, for all morphisms f in $\mathcal{C}$ we have $(\mathbb{T}(\mathbb{U}f))^{-1} = \mathfrak{clp}(\mathbb{T}(\mathbb{U}f)) = \mathbb{Q}(\mathbb{T}(\mathbb{U}f))$. Accordingly, the map $(\mathbb{T}(\mathbb{U}q_i))^{-1} : \mathfrak{clp}(\mathbb{T}\mathfrak{Y}_i) \to \mathbb{Q}\mathbb{T}\mathbb{U}\mathfrak{X}$ is a Boolean algebra morphism for each of the Boolean algebras $\mathfrak{clp}(\mathbb{T}(\mathbb{U}(\mathfrak{Y}_i)))$. Applying $\mathbb{Q} \circ \mathbb{T} \circ \mathbb{U}$ to the diagram $\mathcal{C}$ shows that these morphisms commute with the morphisms in $\mathfrak{clp}(\mathbb{T}\mathcal{C})$. Therefore, by the definition of a colimit, there exists a homomorphism $f : \overline{\mathcal{B}} \to \mathbb{Q}(\mathbb{T}(\mathbb{U}\mathfrak{X}))$. (See figure 4.2.)

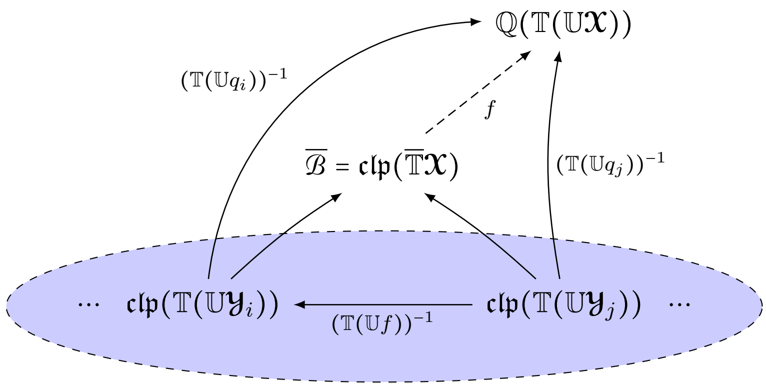


Figure 4.2: The coloured area represents the diagram $\mathfrak{clp}(\mathbb{T}\mathcal{C})$.

Let $\mathcal{B}$ be the sub-Boolean algebra of $\mathbb{Q}\mathbb{T}\mathbb{U}\mathfrak{X}$ generated by the clopen sets $(\mathbb{T}q_i)^{-1}(a)$ where i ranges over I and a ranges over the clopens of $\mathfrak{Y}_i$. Then $\operatorname{im} f = \mathcal{B}$. We will now show that f is in fact injective, hence an isomorphism.

4.24.A Claim. *We have an isomorphism of Boolean algebras $\overline{\mathcal{B}} \cong \mathcal{B}$.*

Proof of claim. We already have an onto homomorphism $f : \overline{\mathcal{B}} \to \mathcal{B}$, so we only need to show that f is injective. By lemma A.8 it suffices to show that $f(z) = \bot$ implies $z = \bot$ for all $z \in \overline{\mathcal{B}}$.

Given $q_i : \mathfrak{X} \to \mathfrak{Y}_i$, define $r_i : \mathfrak{Y}_i \to \mathfrak{X}$ by choosing for each equivalence class in $\mathfrak{Y}_i$ a representative. Then r_i is a section of q_i, that is, $q_i \circ r_i = \mathrm{id}_{\mathfrak{Y}_i}$.

Let $z \in \mathfrak{clp}(\overline{\mathbb{T}}\mathcal{X}) = \overline{\mathscr{B}}$ such that $f(z) = \bot$. By lemma 4.23 the diagram $\mathcal{C}$ is cofiltered, so since $\mathbb{T}$ preserves cofiltered diagrams the diagram $\overline{\mathbb{T}}\mathcal{C}$ with limit $\overline{\mathbb{T}}\mathcal{X}$ is cofiltered as well, hence by duality the diagram $\mathfrak{clp}(\mathbb{T}\mathcal{C})$ with limit $\overline{\mathscr{B}}$ is filtered. It follows from a routine argument that z comes from some $y \in \mathfrak{clp}(\mathbb{T}\mathcal{Y}_i)$ for some $i \in I$, so $f(z) = (\mathbb{T}q_i)^{-1}(y)$. Then $(\mathbb{T}q_i)^{-1}(y) = \bot$. As $(\mathbb{T}r_i)^{-1} \circ (\mathbb{T}q_i)^{-1} = (\mathbb{T}\,\mathrm{id}_{\mathcal{Y}_i})^{-1} = \mathbb{T}\,\mathrm{id}_{\mathcal{Y}_i} = \mathrm{id}_{\mathbb{T}\mathcal{Y}_i}$, we see that $(\mathbb{T}q_i)^{-1}$ has a left inverse, hence is injective. Therefore $(\mathbb{T}q_i)^{-1}(y) = \bot$ implies $y = \bot$ which in turn implies $z = \bot$. So f is injective. ◇

So it suffices to compare $\widehat{\mathscr{B}}$ and $\mathscr{B}$, which are both sub-Boolean algebras of $\mathbb{Q}\mathbb{T}\mathbb{U}\mathcal{X}$.

4.24.B Claim. *Every generator of $\widehat{\mathscr{B}}$ is in $\mathscr{B}$, hence $\widehat{\mathscr{B}}$ is a sub-Boolean algebra of $\mathscr{B}$.*

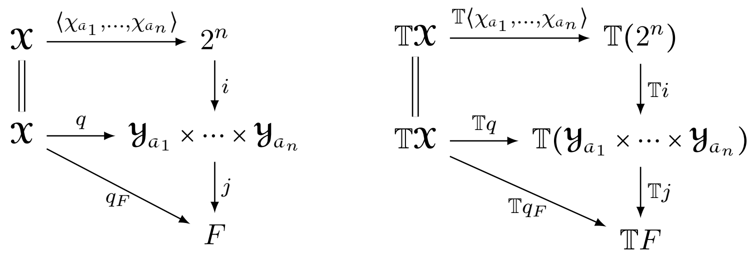
Proof of claim. Pick $n \geq 0$, $C \subseteq \mathbb{T}(2^n)$ and $a_1, \ldots, a_n \in \mathrm{Clp}\,\mathcal{X}$ and consider

$$\lambda^C(a_1, \ldots, a_n) = (\mathbb{T}\langle \chi_{a_1}, \ldots, \chi_{a_n}\rangle)^{-1}[C].$$

This is an arbitrary generator of $\widehat{\mathscr{B}}$. We will show that $\lambda^C(a_1, \ldots, a_n) \in \mathscr{B}$. Recall that χ_{a_1} is the indicator function $\mathcal{X} \to 2$ and $\langle \chi_{a_1}, \ldots, \chi_{a_n}\rangle$ is the tupling of indicator functions.

For every $a \in \mathrm{Clp}\,\mathcal{X}$, let $\mathcal{Y}_a$ be the finite quotient of $\mathcal{X}$ given by $\{a, \mathcal{X} \setminus a\}$ and let q_a be the corresponding quotient map. (This is well-defined because obviously q_a is continuous.) Let $i_k : 2 \to \mathcal{Y}_{a_k}$ be the isomorphism which sends $\top$ to the equivalence class a_k in $\mathcal{Y}_k$ and $\bot$ to $\mathcal{X} \setminus a_k$. Then the tupling $i = \langle i_1, \ldots, i_n\rangle$ is an isomorphism $i : 2^n \to \mathcal{Y}_{a_1} \times \cdots \times \mathcal{Y}_{a_n}$ is an isomorphism. Set $q = \langle q_{a_1}, \ldots, q_{a_n}\rangle$. Then the upper square in the left diagram below commutes.

Let $F := \{\bigcap_{i=1}^n b_i \mid b_i \in \mathcal{Y}_{a_i}\}$ and $j : \mathcal{Y}_{a_1} \times \cdots \times \mathcal{Y}_{a_n} \to F : (b_1, \ldots, b_n) \mapsto \bigcap b_i$. Then F is a finite quotient of $\mathcal{X}$. Call the corresponding quotient map q_F. The lower triangle in the left diagram commutes by construction. Applying $\mathbb{T}$ to the left diagram yields the right diagram.



Since i is an isomorphism, so is $\mathbb{T}i$. Now we get

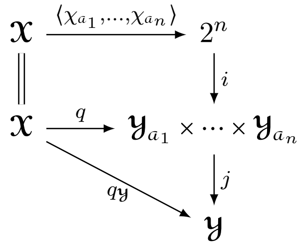
$$(\mathbb{T}\langle \chi_{a_1}, \ldots, \chi_{a_n}\rangle)^{-1}[C] = (\mathbb{T}q)^{-1}[(\mathbb{T}i)[C]] = (\mathbb{T}q_F)^{-1}\underbrace{[\mathbb{T}j \circ \mathbb{T}i[C]]}_{\subseteq \mathbb{T}F}.$$

This shows that $\lambda^C(a_1, \ldots, a_n) \in \mathscr{B}$. Since this holds for any generator $\lambda^C(a_1, \ldots, a_n)$ of $\widehat{\mathscr{B}}$, we have $\widehat{\mathscr{B}} \subseteq \mathscr{B}$. ◇

Our next claim is the converse of claim 4.24.B.

4.24.C Claim. *Every generator of $\mathscr{B}$ is in $\widehat{\mathscr{B}}$, hence $\mathscr{B}$ is a sub-Boolean algebra of $\widehat{\mathscr{B}}$.*

Proof of claim. Let $\mathcal{Y} = \{a_1, \ldots, a_n\}$ and let $q_{\mathcal{Y}} : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ be a finite quotient of $\mathcal{X}$ and let $B \subseteq \mathbb{T}\mathcal{Y}$ be a subset. (Since $\mathbb{T}\mathcal{Y}$ is finite B is automatically clopen.) Then the following diagram commutes



where q, i and j are as before. It is obvious that the diagram commutes. Apply $\mathbb{T}$ to the diagram to see that

$$(\mathbb{T}q)^{-1}[B] = (\mathbb{T}q)^{-1}[(\mathbb{T}j)^{-1}[B]] = (\mathbb{T}\langle \chi_{a_1}, \ldots, \chi_{a_n} \rangle)^{-1}[(\mathbb{T}i)^{-1} \circ (\mathbb{T}j)^{-1}[B]].$$

Note that $(\mathbb{T}i)^{-1} \circ (\mathbb{T}j)^{-1}[B] \subseteq \mathbb{T}(2^n)$. This proves that every generator of $\mathscr{B}$ is in $\widehat{\mathscr{B}}$, hence $\mathscr{B} \subseteq \widehat{\mathscr{B}}$. $\diamond$

We conclude that the clopen set Boolean algebras of $\overline{\mathbb{T}}\mathfrak{X}$ and $\widehat{\mathbb{T}}\mathfrak{X}$ coincide. Therefore $\overline{\mathbb{T}}\mathfrak{X} \cong \widehat{\mathbb{T}}\mathfrak{X}$. This proves the theorem. $\square$

This concludes the general theory that we treat in this thesis. The next chapter will consist of a case study where we will see parts of the developed theory in action.

5

Coalgebraic conditional logic

Recall from section 2.2 that the modality $\Rightarrow$ (the conditional arrow) expresses a notion of conditionality which generally differs from the implication $\to$, and a formula $\varphi \Rightarrow \psi$ may be thought of as "If φ holds, then usually ψ holds as well." Conditional logics have a long history and tradition in philosophical logic [53, 13, 54]. They are used in various applications such as non-monotonic interference, belief change and the analysis of intentions. For more information on conditional logic we refer to [13, 4, 42].

This chapter consists of two parts. The first part contains the generalisation of conditional frames (from example 2.10) to *descriptive* conditional frames. We show that these can be seen as coalgebras for a certain functor $\mathbb{C}_{\mathsf{st}}$ on Stone and give an endofunctor on BA dual to $\mathbb{C}_{\mathsf{st}}$. Also we generalise conditional bisimulations from [7] to descriptive conditional bisimulations and show how they relate to Λ-bisimulations (where Λ is the set of clopen predicate liftings that corresponds to conditional logic), modal equivalence and behavioural equivalence.

In the second part we develop *geometric* conditional logic by adding two modalities, $\Rightarrow$ and $\Downarrow$, to the language of geometric logic via predicate liftings. We generalise conditional frames to *geometric* conditional frames, show that these are coalgebras for a certain functor and give an Isbell dual of this functor.

5.1 Conditional logic on Stone-coalgebras

In this section we develop conditional logic on so-called descriptive conditional models. We define descriptive conditional models and show how they can be viewed as a coalgebras. In subsection 5.1.2 we investigate duality with Boolean algebras. In subsection 5.1.3 we define descriptive conditional bisimulations, which are an adaptation of ideas in [7], and we show that descriptive conditional bisimilarity coincides with modal equivalence, behavioural equivalence and Λ-equivalence (for a suitable Λ).

5.1.1 Descriptive conditional models

Recall from definition 2.8 that a conditional frame is a pair (X, ν) where X is a set and $\nu : X \times \mathbb{P}X \to \mathbb{P}X$ a function satisfying certain properties. We extend this by adding to such a pair a collection $A \subseteq \mathbb{P}X$ of admissible subsets of X. We require A to be a Boolean algebra that is closed under certain properties, such that the truth set of every formula (see definition 5.12) is in A. Besides, A will serve as the clopen subbase for a topology on X. But this is jumping ahead; let us start with the definition of a general conditional frame.

5.1 Definition. A **general conditional frame** is a triple (X, ν, A) where

- X is a set;
- $A \subseteq \mathbb{P}X$ contains $\varnothing$ and X and is closed under Boolean operations and the map $m_\nu : A \times \mathbb{P}X \to \mathbb{P}X$ defined by

$$m_\nu(a, b) := \{x \in X \mid \nu(x, a) \subseteq b\}$$

- $\nu : X \times A \to \mathbb{P}X$ is a map that satisfies for all $x \in X$ and $a, b \in A$,
  (i) if $a \cap b = \varnothing$, then $\nu(x, a) \cap b = \varnothing$,
  (ii) if $a \subseteq b$ and $\nu(x, b) \subseteq a$, then $\nu(x, a) = \nu(x, b)$.

A **general conditional frame morphism** from (X, ν, A) to (X', ν', A') is a map $f : X \to X'$ such that $f^{-1}(a') \in A$ for all $a' \in A'$ and $f[\nu(x, f^{-1}(a'))] = \nu'(f(x), a')$ for all $x \in X, a' \in A'$.

A **general conditional model** is a tuple (X, ν, A, V) where (X, ν, A) is a general conditional frame and $V : \Phi \to A$ is a valuation of the proposition letters. A **general conditional model morphism** $f : (X, \nu, A, V) \to (X', \nu', A', V')$ is a descriptive conditional frame morphism $f : (X, \nu, A) \to (X', \nu', A')$ such that $f^{-1} \circ V' = V$. ◁

As in the set case (cf. remark 2.9) condition (i) is equivalent to $\nu(x, a) \subseteq a$ for all $x \in X, a \in A$.

5.2 Definition. Let $\mathfrak{X} = (X, \nu, A)$ be a general conditional frame. Write $\mathcal{X}$ for the topological space with underlying set X and a topology generated by the subbase A, and $K\mathcal{X}$ for the collection of closed subsets of $\mathcal{X}$. Call a general conditional frame **differentiated** if for all distinct $x, x' \in X$ there is a witness $a \in A$ such that $x \in a$ and $x' \notin a$, **closed** if $\nu(x, a) \in K\mathcal{X}$ for all $x \in X$ and $a \in A$, and **compact** if $\bigcap A_0 \neq \varnothing$ whenever $A_0 \subseteq A$ has the finite intersection property.

A **descriptive conditional frame** is a differentiated, closed, compact general conditional frame. The category of descriptive conditional frames and general descriptive frame morphisms is denoted by DCF. A **descriptive conditional model** is a general conditional model whose underlying general conditional frame is descriptive. Let DCM be the category whose objects are descriptive conditional models and whose morphisms are general conditional model morphisms. ◁

It turns out that descriptive conditional frames can be viewed as coalgebras.

5.3 Definition. Let $\mathbb{C}_{\mathsf{st}}\mathcal{X}$ be the set of maps $h : \mathrm{Clp}\,\mathcal{X} \to K\mathcal{X}$ such that for all $a, b \in \mathrm{Clp}\,\mathcal{X}$,

(C1) $h(a) \subseteq a$; and

(C2) if $a \subseteq b$ and $h(b) \subseteq a$, then $h(a) = h(b)$.

Endow $\mathbb{C}_{\mathsf{st}}\mathcal{X}$ with a topology generated by the subbase

$$\boxplus(a, b) := \{h \in \mathbb{C}_{\mathsf{st}}\mathcal{X} \mid h(a) \subseteq b\}, \quad \otimes(a, b) := \{h \in \mathbb{C}_{\mathsf{st}}\mathcal{X} \mid h(a) \cap b \neq \varnothing\}$$

for $a, b \in \mathrm{Clop}\,\mathcal{X}$. Observe that $\mathbb{C}_{\mathsf{st}}\mathcal{X} \setminus \boxplus(a, b) = \otimes(a, \mathcal{X} \setminus b)$, so this is in fact a *clopen* subbase. For a continuous map $f : \mathcal{X} \to \mathcal{X}'$ between Stone spaces, define $\mathbb{C}_{\mathsf{st}}f : \mathbb{C}_{\mathsf{st}}\mathcal{X} \to \mathbb{C}_{\mathsf{st}}\mathcal{X}'$ by

$$\mathbb{C}_{\mathsf{st}}f(h)(a) = f[h(f^{-1}(a))]$$

for $h \in \mathbb{C}_{\mathsf{st}}\mathcal{X}$ and $a \in \mathrm{Clp}\,\mathcal{X}$. ◁

5.4 Lemma. *Let $f : \mathfrak{X} \to \mathfrak{X}'$ be a continuous map between Stone spaces. The map $\mathbb{C}_{\mathsf{st}}f : \mathbb{C}_{\mathsf{st}}\mathfrak{X} \to \mathbb{C}_{\mathsf{st}}\mathfrak{X}'$ is well defined and continuous.*

Proof. In order to prove that $\mathbb{C}_{\mathsf{st}}f$ is well defined we need to show that for each $h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X}$ and $a \in \mathrm{Clp}\,\mathfrak{X}$ the set $f[h(f^{-1}(a))] \subseteq \mathfrak{X}'$ is closed and that $\mathbb{C}_{\mathsf{st}}f(h)$ satisfies (C1) and (C2). Since a is clopen in $\mathfrak{X}'$, $f^{-1}(a)$ is clopen in $\mathfrak{X}$. By construction $h(f^{-1}(a))$ is closed in $\mathfrak{X}$, and since $\mathfrak{X}$ is compact and $\mathfrak{X}'$ is Hausdorff, f is a closed map so $f[h(f^{-1}(a))]$ is closed in $\mathfrak{X}'$. The map $\mathbb{C}_{\mathsf{st}}f(h)$ satisfies (C1) and (C2) for similar reasoning as in paragraph 2.10, hence it is indeed an element of $\mathbb{C}_{\mathsf{st}}\mathfrak{X}'$.

For continuity, it suffices to show that $(\mathbb{C}_{\mathsf{st}}f)^{-1}(\boxplus(a,b))$ is clopen in $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$, whenever $a, b \in \mathrm{Clp}\,\mathfrak{X}'$. Unravelling the definitions yields

$$
\begin{aligned}
(\mathbb{C}_{\mathsf{st}}f)^{-1}(\boxplus(a,b)) &= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid \mathbb{C}_{\mathsf{st}}f(h) \in \boxplus(a,b)\} \\
&= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid \mathbb{C}_{\mathsf{st}}f(h)(a) \subseteq b\} \\
&= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid f[h(f^{-1}(a))] \subseteq b\} \\
&= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid h(f^{-1}(a)) \subseteq f^{-1}(b)\} \\
&= \boxplus(f^{-1}(a), f^{-1}(b)).
\end{aligned}
$$

Clearly the latter is clopen in $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$. This proves the lemma. □

We have not yet proved that $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is indeed a Stone space whenever $\mathfrak{X}$ is. The next two lemmas will show this.

5.5 Lemma. *Let $\mathfrak{X} = (X, \tau)$ be a topological space. If $\mathfrak{X}$ is zero-dimensional, then so is $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$. If $\mathfrak{X}$ is a Hausdorff space, then so is $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$.*

Proof. *$\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is zero-dimensional.* Follows from the fact that the topology on $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is generated by a clopen subbase.

$\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is Hausdorff. Suppose $\mathfrak{X}$ is a Hausdorff space and $h \neq h'$ in $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$. Then $h(a) \neq h'(a)$ for some $a \in \mathrm{Clp}\,\mathfrak{X}$. Without loss of generality assume that there is $x \in h(a)$ such that $x \notin h'(a)$. Since $\mathfrak{X}$ is a Stone space and $h'(a)$ is closed in $\mathfrak{X}$, there is a clopen $b \in \mathrm{Clp}\,\mathfrak{X}$ such that $x \in b \subseteq X \setminus h'(a)$. Let $d := X \setminus b$, then $h'(a) \subseteq d$ but $h(a) \nsubseteq d$, because $x \notin d$. Therefore $h' \in \boxplus(a, d)$ and $h \notin \boxplus(a, d)$. Furthermore $h' \notin \diamondplus(a, b)$ and $h \in \diamondplus(a, b)$, so $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is a Hausdorff space. □

Showing that $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is a Stone space whenever $\mathfrak{X}$ is a Stone space is more involved.

5.6 Lemma. *Let $\mathfrak{X} = (X, \tau)$ be a Stone space. Then $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is compact.*

Proof. Let $\mathbb{C}'$ be the collection of all functions $h : \mathrm{Clp}\,\mathfrak{X} \to K\mathfrak{X}$ topologized by the subbase $\boxplus(a, b) := \{h \in \mathbb{C}'\mathfrak{X} \mid h(a) \subseteq b\}$ and $\diamondplus(a, b) := \{h \in \mathbb{C}'\mathfrak{X} \mid h(a) \cap b \neq \varnothing\}$, where a, b range over $\mathrm{Clp}\,\mathfrak{X}$. That is, we do not require the elements of $\mathbb{C}'\mathfrak{X}$ to satisfy (C1) and (C2) from definition 5.3. We will first show that $\mathbb{C}'\mathfrak{X}$ is compact, and then that $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is a closed subset of $\mathbb{C}'\mathfrak{X}$.

5.6.A Claim. *Let $\mathfrak{X} = (X, \tau)$ be a compact topological space. Then $\mathbb{C}'\mathfrak{X}$ is compact as well.*

Proof of claim. Suppose

$$
\mathbb{C}'\mathfrak{X} = \bigcup_{i \in I} \boxplus(a_i, b_i) \cup \bigcup_{j \in J} \diamondplus(c_j, d_j) \tag{5.1}
$$

is a cover. Let $A = \{\mathscr{a}_i \mid i \in I\}$ and define B, C, D similarly. Let $D_c = \{\mathscr{d} \in D \mid \diamondtimes(c, \mathscr{d})$ is in the cover$\}$. Define

$$h : \mathbb{C}\mathrm{lp}\,\mathfrak{X} \to K\mathfrak{X} : \mathscr{a} \mapsto \begin{cases} \varnothing & \text{if } \mathscr{a} \notin A \\ X & \text{if } \mathscr{a} \in A \text{ and } \mathscr{a} \notin C \\ X \setminus \bigcup D_{\mathscr{a}} & \text{if } \mathscr{a} \in A \cap C \end{cases}$$

This element is in $\mathbb{C}'\mathfrak{X}$, so it must be in the cover (5.1).

If $h \in \diamondtimes(c_j, \mathscr{d}_j)$ for some $j \in J$, then we must have $h(c_j) \cap \mathscr{d}_j \neq \varnothing$. If $c_j \notin A$ then $h(c_j) = \varnothing$ hence $h(c_j) \cap \mathscr{d}_j = \varnothing$. So this cannot be the case. Therefore we must have $c_j \in A$ and $h(c_j) = (X \setminus \bigcup D_{c_j}) \cap \mathscr{d}_j \neq \varnothing$. Since $\mathscr{d}_j \in D_{c_j}$, this is a contradiction; see figure 5.1. So we must have $h \in \boxplus(\mathscr{a}_i, \mathscr{b}_i)$ for some $i \in I$.

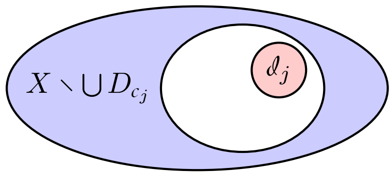


Figure 5.1: $(X \setminus \bigcup D_{c_j}) \cap \mathscr{d}_j = \varnothing$

If $h \in \boxplus(\mathscr{a}_i, \mathscr{b}_i)$ for some $i \in I$, then we must have $h(\mathscr{a}_i) \subseteq \mathscr{b}_i$. If $\mathscr{a}_i \notin C$ then $h(\mathscr{a}_i) = X$ hence $\mathscr{b}_i = X$ (for otherwise $h \notin \boxplus(\mathscr{a}_i, \mathscr{b}_i)$) and we are done, because $\boxplus(\mathscr{a}_i, X) = \mathbb{C}'\mathfrak{X}$. So suppose this is not the case, then we must have $h(\mathscr{a}_i) = X \setminus \bigcup D_{\mathscr{a}_i} \subseteq \mathscr{b}_i$. This implies $\mathscr{b} \cup \bigcup D_{\mathscr{a}} = \mathfrak{X}$, so by compactness of $\mathfrak{X}$ we find a finite number $\mathscr{d}_1, \ldots, \mathscr{d}_n \in D_{\mathscr{a}}$ such that

$$\mathscr{b} \cup \mathscr{d}_1 \cup \cdots \cup \mathscr{d}_n = \mathfrak{X}.$$

As a result

$$\boxplus(\mathscr{a}, \mathscr{b}) \cup \diamondtimes(\mathscr{a}, \mathscr{d}_1) \cup \cdots \cup \diamondtimes(\mathscr{a}, \mathscr{d}_n) \tag{5.2}$$

is a finite subcover of (5.1). (By construction all basic clopens in (5.2) are indeed basic clopens in the cover (5.1).) To see this, we need to show that the finite cover in (5.2) indeed covers all of $\mathbb{C}'\mathfrak{X}$. Let $g \in \mathbb{C}'\mathfrak{X}$ be any element, then either $g(\mathscr{a})$ touches one of the $\mathscr{d}_i$ or $g(\mathscr{a}) \subseteq \mathscr{b}$, so (5.2) is indeed a cover.

It follows from the Alexander subbase theorem that $\mathbb{C}'\mathfrak{X}$ is compact. ◇

5.6.B Claim. *Let $\mathfrak{X}$ be a Stone space. Then $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is a closed subspace of $\mathbb{C}'\mathfrak{X}$.*

Proof of claim. Let $h \in \mathbb{C}'\mathfrak{X}$ such that $h \notin \mathbb{C}_{\mathsf{st}}\mathfrak{X}$. Then either there exists $\mathscr{a} \in \mathbb{C}\mathrm{lp}\,\mathfrak{X}$ such that $h(\mathscr{a}) \nsubseteq \mathscr{a}$ or there are $\mathscr{a} \subseteq \mathscr{b}$ in $\mathbb{C}\mathrm{lp}\,\mathfrak{X}$ such that $h(\mathscr{b}) \subseteq \mathscr{a}$ and $h(\mathscr{a}) \neq h(\mathscr{b})$.

In the first case, $\diamondtimes(\mathscr{a}, X \setminus \mathscr{a})$ is an open neighbourhood of h in $\mathbb{C}'\mathfrak{X}$ disjoint from $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ and we are done. In the second case we consider two subcases: If $h(\mathscr{a}) \nsubseteq h(\mathscr{b})$ there exists $x \in h(\mathscr{a})$ such that $x \notin h(\mathscr{b})$. Since $\mathfrak{X}$ is (compact Hausdorff hence) regular and has a clopen subbase we can find disjoint clopens c and $\mathscr{d}$ such that $x \in c$ and $h(\mathscr{b}) \subseteq \mathscr{d}$ (see figure 5.2). But then

$$h \in \boxplus(\mathscr{b}, \mathscr{a}) \cap \diamondtimes(\mathscr{a}, c) \cap \boxplus(\mathscr{b}, \mathscr{d}),$$

which is open in $\mathbb{C}'\mathfrak{X}$. Moreover, it is disjoint from $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ because $\mathscr{a} \subseteq \mathscr{b}$ and for all $h' \in \boxplus(\mathscr{b}, \mathscr{a}) \cap \diamondtimes(\mathscr{a}, c) \cap \boxplus(\mathscr{b}, \mathscr{d})$ we have $h(\mathscr{b}) \subseteq \mathscr{a}$ and $h(\mathscr{a}) \neq h(\mathscr{b})$.

If $h(\mathscr{b}) \nsubseteq h(\mathscr{a})$ we can use similar reasoning to find disjoint clopens c and $\mathscr{d}$ such that $h(\mathscr{a}) \subseteq c$ and $x \in \mathscr{d}$. Then the clopen set $\boxplus(\mathscr{b}, \mathscr{a}) \cap \boxplus(\mathscr{a}, c) \cap \diamondtimes(\mathscr{b}, \mathscr{d})$ contains h and is disjoint from $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$. We conclude that $\mathbb{C}'\mathfrak{X} \setminus \mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is open, so $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is closed in $\mathbb{C}'\mathfrak{X}$. ◇

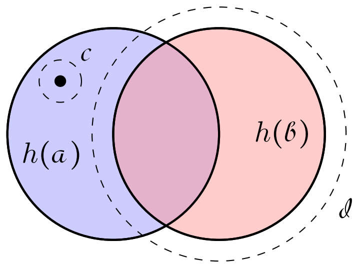


Figure 5.2: The sets $h(a), h(b), c$ and d in $\mathfrak{X}$.

Since closed subspaces of compact spaces are compact, this proves that $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$ is compact. □

We now work our way towards proving that the category of descriptive conditional frames is isomorphic to the category $\mathsf{Coalg}(\mathbb{C}_{\mathsf{st}})$ of $\mathbb{C}_{\mathsf{st}}$-coalgebras.

5.7 Lemma. *Let $\mathfrak{X} = (X, \nu, A)$ be a descriptive conditional frame. Let τ_A be the topology on X generated by A and set $\mathfrak{X} = (X, \tau_A)$. Define $\gamma_\nu : X \to (\mathrm{Clp}\,\mathfrak{X} \to K\mathfrak{X})$ by $\gamma_\nu(x)(a) := \nu(x, a)$. Then $(\mathfrak{X}, \gamma)$ is a $\mathbb{C}_{\mathsf{st}}$-coalgebra.*

Proof. The topological space $\mathfrak{X}$ is zero-dimensional because it has a clopen basis, compact because the frame $\mathfrak{X}$ is compact and Hausdorff because $\mathfrak{X}$ is differentiated. So $\mathfrak{X}$ is a Stone space.

We have $\gamma_\nu(x)(a) \in K\mathfrak{X}$ for all $a \in A$ because $\mathfrak{X}$ is closed (that is, $\nu(x, a) \in K\mathfrak{X}$ for all $\in A$). Also $\gamma_\nu(x)$ satisfies (C1) and (C2) from definition 5.3 because ν satisfies conditions (i) and (ii) from definition 5.1.

We need to show that γ_ν is continuous. Since γ_ν^{-1} preserves taking complements, unions and intersections, it suffices to show that $\gamma_\nu^{-1}(\boxplus(a, b)) \in A$ for all $a, b \in A$. Let $\boxplus(a, b)$ be a clopen set in $\mathbb{C}_{\mathsf{st}}\mathfrak{X}$, then

$$\begin{aligned}
\gamma_\nu^{-1}(\boxplus(a, b)) &= \{x \in X \mid \gamma_\nu(x) \in \boxplus(a, b)\} \\
&= \{x \in X \mid \gamma_\nu(x)(a) \subseteq b\} \\
&= \{x \in X \mid \nu(x, a) \subseteq b\} \\
&= m_\nu(a, b) \in A.
\end{aligned}$$

This shows that γ_ν is continuous. □

5.8 Lemma. *Let $\mathfrak{X} = (X, \tau)$ and let $(\mathfrak{X}, \gamma)$ be a $\mathbb{C}_{\mathsf{st}}$-coalgebra. Define $\nu_\gamma : X \times \mathrm{Clp}\,\mathfrak{X} \to \mathbb{P}X$ by $\nu_\gamma(x, a) := \gamma(x)(a)$. Then $(X, \nu_\gamma, \mathrm{Clp}\,\mathfrak{X})$ is a descriptive conditional frame.*

Proof. Evidently, ν_γ is a map from $X \times \mathrm{Clp}\,\mathfrak{X}$ to $\mathbb{P}X$ that satisfies (i) and (ii) from definition 5.1 (because γ satisfies (C1) and (C2) from definition 5.3). It is obvious that $\mathrm{Clp}\,\mathfrak{X}$ contains $\varnothing$ and X and that it is closed under taking finite unions, finite intersections and complements. Let $a, b \in \mathrm{Clop}\,\mathfrak{X}$, then

$$\begin{aligned}
m_{\nu_\gamma}(a, b) &= \{x \in X \mid \nu_\gamma(x, a) \subseteq b\} \\
&= \{x \in X \mid \gamma(x)(a) \subseteq b\} \\
&= \{x \in X \mid \gamma(x) \in \boxplus(a, b)\} \\
&= \gamma^{-1}(x)(\boxplus(a, b)).
\end{aligned}$$

This is clopen by continuity of γ. So $(X, \nu_\gamma, \mathrm{Clp}\,\mathfrak{X})$ is a general conditional frame.

The frame $(X, \nu_\gamma, \mathrm{Clp}\,\mathfrak{X})$ is closed because $\nu_\gamma(x, a) = \gamma(x)(a) \in K\mathfrak{X}$ for all $x \in X, a \in \mathrm{Clop}\,\mathfrak{X}$. It is compact and differentiated because $\mathfrak{X}$ is compact and Hausdorff. Thus $(X, \nu_\gamma, \mathrm{Clp}\,\mathfrak{X})$ is descriptive. □

It is obvious that the previous two lemmas establish a 1-1 correspondence between objects of DCF and objects of $\mathsf{Coalg}(\mathbb{C}_{\mathsf{st}})$. From now on we will use the terms descriptive conditional frame and $\mathbb{C}_{\mathsf{st}}$-coalgebra interchangeably.

5.9 Proposition. *Let $(\mathfrak{X}, \gamma)$ and $(\mathfrak{X}', \gamma')$ be two $\mathbb{C}_{\mathsf{st}}$-coalgebras and let (X, ν, A) and (X', ν', A') be their corresponding descriptive conditional frames. Let $f : X \to X'$ be a map between sets. Then f is a general conditional frame morphism if and only if it is a $\mathbb{C}_{\mathsf{st}}$-coalgebra morphism.*

Proof. For all $x \in X, a \in A'$ we have

$$(\mathbb{C}_{\mathsf{st}} f)(\gamma(x))(a) = f[\gamma(x)(f^{-1}(a))] = f[\nu(x, f^{-1}(a))]. \tag{5.3}$$

Suppose f is a general conditional frame morphism, then f is continuous because $f^{-1}(a) \in A$ for all $a \in A'$ and it is a $\mathbb{C}_{\mathsf{st}}$-coalgebra morphism because for all $x \in X$ and $a \in A'$ we have

$$\mathbb{C}_{\mathsf{st}} f \circ \gamma(x)(a) = f[\nu(x, f^{-1}(a))] = \nu'(f(x), a) = \gamma'(f(x))(a).$$

The first equality holds by (5.3), the second one because f is a general conditional frame morphism, the third one by the correspondence between objects of DCF and $\mathsf{Coalg}(\mathbb{C}_{\mathsf{st}})$ given in lemma 5.7 and 5.8.

Conversely, suppose f is a $\mathbb{C}_{\mathsf{st}}$-coalgebra morphism. Then $f^{-1}(a) \in A$ for all $a \in A'$ by continuity of f. Furthermore f is a general conditional frame morphism because for all $x \in X$ and $a \in A'$ we have

$$f[\nu(x, f^{-1}(a))] = \mathbb{C}_{\mathsf{st}} f \circ \gamma(x)(a) = \gamma' \circ f(x)(a) = \nu'(f(x), a).$$

The first equality holds by (5.3), the second because f is a $\mathbb{C}_{\mathsf{st}}$-coalgebra morphism and the third one because of the correspondence between objects given in lemma 5.7 and 5.8. This proves the proposition. □

5.10 Theorem. *We have an isomorphism of categories*

$$\mathsf{DCF} \cong \mathsf{Coalg}(\mathbb{C}_{\mathsf{st}}).$$

Proof. Follows from lemma 5.7 and 5.8 and proposition 5.9. □

It follows directly from the definitions that descriptive conditional models are precisely $\mathbb{C}_{\mathsf{st}}$-models. It is an easy exercise to show that general conditional frame morphisms are $\mathbb{C}_{\mathsf{st}}$-model morphisms. This yields the following result:

5.11 Theorem. *There is an isomorphism of categories*

$$\mathsf{DCM} \cong \mathsf{Mod}(\mathbb{C}_{\mathsf{st}}).$$

From now on we will identify descriptive conditional models and $\mathbb{C}_{\mathsf{st}}$-models, and use the terminology interchangeably. We finish this subsection with a definition and remark about the conditional language on descriptive conditional models.

5.12 Definition. Define the clopen predicate liftings $\lambda^{\Rightarrow}, \lambda^{\Downarrow} : \mathrm{Clp}^2 \to \mathrm{Clp} \circ \mathbb{C}_{\mathsf{st}}$ by

$$\lambda^{\Rightarrow}_{\mathcal{X}}(a, b) = \{h \in \mathbb{C}_{\mathsf{st}}\mathcal{X} \mid h(a) \subseteq b\} \quad \text{and} \quad \lambda^{\Downarrow}_{\mathcal{X}}(a, b) = \{h \in \mathbb{C}_{\mathsf{st}}\mathcal{X} \mid h(a) \cap b \neq \varnothing\}.$$

Then $\mathcal{L}(\lambda^{\Rightarrow}, \lambda^{\Downarrow})$ is given by

$$\varphi ::= \bot \mid p \mid \varphi \wedge \psi \mid \varphi \Rightarrow \psi \mid \varphi \Downarrow \psi,$$

where $p \in \Phi$. ◁

5.13 Remark. Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ be a $\mathbb{C}_{\mathsf{st}}$-model, then

$$\mathfrak{X}, x \Vdash \varphi \Rightarrow \psi \quad \text{iff} \quad \gamma(x) \in \lambda^{\Rightarrow}([\![\varphi]\!]^{\mathfrak{X}}, [\![\psi]\!]^{\mathfrak{X}})$$

and

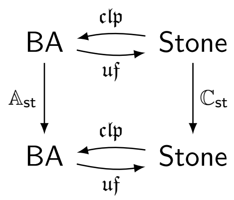
$$\mathfrak{X}, x \Vdash \varphi \Downarrow \psi \quad \text{iff} \quad \gamma(x) \in \lambda^{\Downarrow}([\![\varphi]\!]^{\mathfrak{X}}, [\![\psi]\!]^{\mathfrak{X}}).$$

Indeed, $\mathfrak{X}, x \Vdash \varphi \Rightarrow \psi$ iff $\mathfrak{X}, x \Vdash \neg(\varphi \Downarrow \neg\psi)$.

Moreover, the topology on $\mathbb{C}_{\mathsf{st}}\mathcal{X}$ is generated by $\lambda^{\Rightarrow}(a, b), \lambda^{\Downarrow}(a, b)$ where $a, b \in \mathrm{Clp}\,\mathcal{X}$, so $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$ is a characteristic set of clopen predicate liftings for $\mathbb{C}_{\mathsf{st}}$. This allows us to use various results from chapter 2. In particular, it follows from proposition 2.29 that modal equivalence coincides with behavioural equivalence.

5.1.2 Duality

The goal of this section is to give a functor $\mathbb{A}_{\mathsf{st}}$ on BA, the category of Boolean algebras, such that the following diagram commutes,



We present a functor on BA and then show that it is the dual of $\mathbb{C}_{\mathsf{st}}$.

5.14 Definition. Let B be a Boolean algebra. Let $\mathbb{A}_{\mathsf{st}}B$ be the Boolean algebra generated by the elements $\Box(a, b)$, $\Diamond(a, b)$ where $a, b \in B$ subject to the relations

(A1) $\Box(a, a) = \top_{\mathbb{A}B}$

(A2) $\Diamond(a, \bot_B) = \bot_{\mathbb{A}B}$

(A3) $\Box(a, b) \wedge \Box(a, c) = \Box(a, b \wedge c)$

(A4) $\Diamond(a, b) \vee \Diamond(a, c) = \Diamond(a, b \vee c)$

(A5) $\Box(a, b \vee c) \leq \Box(a, b) \vee \Diamond(a, c)$

(A6) $\Box(a, b) \wedge \Diamond(a, c) \leq \Diamond(a, b \wedge c)$

(A7) $\Box(a, b) \wedge \Box(a, c) = \Box(a, b) \wedge \Box(b, c)$ if $b \leq a$

(A8) $\Box(a, b) \wedge \Diamond(a, c) = \Box(a, b) \wedge \Diamond(b, c)$ if $b \leq a$.

If $f : A \to B$ is a Boolean algebra morphism, define $\mathbb{A}_{\mathsf{st}}f : \mathbb{A}_{\mathsf{st}}A \to \mathbb{A}_{\mathsf{st}}B$ on generators by $\mathbb{A}_{\mathsf{st}}f(\Box(a, b)) = \Box(f(a), f(b))$ and $\mathbb{A}_{\mathsf{st}}f(\Diamond(a, b)) = \Diamond(f(a), f(b))$. (One can easily see that the images of the generators of A under $\mathbb{A}_{\mathsf{st}}f$ satisfy relations (A1) through (A8), so by remark 3.3 $\mathbb{A}_{\mathsf{st}}f$ indeed defines a frame homomorphism.) Then $\mathbb{A}_{\mathsf{st}}$ defines an endofunctor on BA. ◁

5.15 Remark. Relations (A5) and (A6) together imply $\Box(a,b) = \neg \Diamond (a, \neg b)$.

Although it would have been possible to define $\mathbb{A}_{\mathsf{st}}B$ using only the boxes as generators, we have chosen for this presentation, as it provides a smoother transition to the geometric case in section 5.2. Furthermore, we identify ultrafilters of a Boolean algebra B with homomorphisms from B to $2 = \{\top, \bot\}$.

5.16 Lemma. *There is a one-one correspondence between ultrafilters on a Boolean algebra B and (Boolean algebra) homomorphisms $p : B \to 2$.*

The proof of the next theorem is very similar to the proof of theorem 5.38, so we only give a proof sketch here. We decided to spell out the proof of theorem 5.38 because it is slightly more involved than the proof of the proposition below.

5.17 Proposition. *Let $\mathfrak{X}$ be a Stone space. Then $\mathfrak{uf}(\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X})) \cong \mathbb{C}_{\mathsf{st}}\mathfrak{X}$.*

Proof. We view ultrafilters of $\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X})$ as Boolean algebra homomorphisms $p : \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}) \to 2$. Define a map

$$\varphi : \mathbb{C}_{\mathsf{st}}\mathfrak{X} \to \mathfrak{uf}(\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X})) : h \mapsto p_h,$$

where p_h is defined on generators by

$$p_h : \mathbb{A}_{\mathsf{St}}(\mathfrak{clp}\,\mathfrak{X}) \to 2 : \begin{cases} \Box(a,b) \mapsto \top & \text{iff } h(a) \subseteq b \\ \Diamond(a,b) \mapsto \top & \text{iff } h(a) \cap b \neq \varnothing \end{cases}.$$

Conversely, define

$$\psi : \mathfrak{uf}(\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X})) \to \mathbb{C}_{\mathsf{st}}\mathfrak{X}$$

as follows: For an ultrafilter $p : \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}) \to 2$ let $h_p : \mathfrak{clp}\,\mathfrak{X} \to K\mathfrak{X}$ be the map given by

$$h_p(a) = \mathfrak{X} \setminus \bigcup\{b \in \mathfrak{clp}\,\mathfrak{X} \mid p(\Diamond(a,b)) = \bot\}.$$

In order to show that these maps give rise to an isomorphism, we need to show that both φ and ψ are well defined, that they are mutually inverse and that φ is continuous. This is completely similar to the four claims in theorem 5.38. □

The map φ from the previous theorem yields a map $\eta_{\mathfrak{X}} : \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}) \to \mathfrak{clp}(\mathbb{C}_{\mathsf{st}}\mathfrak{X})$.

5.18 Definition. For a Stone space $\mathfrak{X}$ let $\eta_{\mathfrak{X}}$ be the concatenation of $\mathfrak{clp}$, $\mathfrak{uf}$ and $\mathfrak{clp}\,\varphi$, where φ is defined as in theorem 5.17, that is,

$$\eta_{\mathfrak{X}} : \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}) \xrightarrow{\mathfrak{clp}\circ\mathfrak{uf}} \mathfrak{clp}(\mathfrak{uf}(\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}))) \xrightarrow{\mathfrak{clp}\,\varphi} \mathfrak{clp}(\mathbb{C}_{\mathsf{st}}\mathfrak{X}).$$

A routine calculation reveals that this map is given by $\Box(a,b) \mapsto \boxplus(a,b)$ and $\Diamond(a,b) \mapsto \diamondplus(a,b)$. ◁

The collection of isomorphisms $\eta_{\mathfrak{X}}$ constitute a natural isomorphism.

5.19 Proposition. *The collection $\eta = (\eta_{\mathfrak{X}})_{\mathfrak{X}\in\mathsf{Stone}}$ give a natural isomorphism*

$$\eta : \mathbb{A}_{\mathsf{st}} \circ \mathfrak{clp} \to \mathfrak{clp} \circ \mathbb{C}_{\mathsf{st}}.$$

Proof. It follows from proposition 5.17 that $\eta_{\mathfrak{X}}$ is an isomorphism for each Stone space $\mathfrak{X}$. So we need to show naturality of η, that is, for all continuous maps $f : \mathfrak{X} \to \mathfrak{X}'$ in Stone, the following diagram commutes

$$\begin{array}{ccc} \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}_{\mathsf{st}}) & \xleftarrow{\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,f)} & \mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}') \\ \downarrow{\scriptstyle \eta_{\mathfrak{X}}} & & \downarrow{\scriptstyle \eta_{\mathfrak{X}'}} \\ \mathfrak{clp}(\mathbb{C}_{\mathsf{st}}\mathfrak{X}) & \xleftarrow[\mathfrak{clp}(\mathbb{C}_{\mathsf{st}}f)]{} & \mathfrak{clp}(\mathbb{C}_{\mathsf{st}}\mathfrak{X}') \end{array}$$

Let $\Box(\mathfrak{a},\mathfrak{b})$ be a generator in $\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,\mathfrak{X}')$. Then

$$\begin{aligned} \mathfrak{clp}(\mathbb{C}_{\mathsf{st}}f)(\eta_{\mathfrak{X}'}(\Box(\mathfrak{a},\mathfrak{b}))) &= (\mathbb{C}_{\mathsf{st}}f)^{-1}(\boxplus(\mathfrak{a},\mathfrak{b})) && \text{Def of } \mathfrak{clp} \text{ and } \eta \\ &= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid (\mathbb{C}_{\mathsf{st}}f)(h) \in \boxplus(\mathfrak{a},\mathfrak{b})\} \\ &= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid (\mathbb{C}_{\mathsf{st}}f)(h)(\mathfrak{a}) \subseteq \mathfrak{b}\} && \text{Definition of } \boxplus \\ &= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid f[h(f^{-1}(\mathfrak{a}))] \subseteq \mathfrak{b}\} && \text{Definition of } \mathbb{C}f \\ &= \{h \in \mathbb{C}_{\mathsf{st}}\mathfrak{X} \mid h(f^{-1}(\mathfrak{a})) \subseteq f^{-1}(\mathfrak{b})\} \\ &= \boxplus(f^{-1}(\mathfrak{a}), f^{-1}(\mathfrak{b})) && \text{Definition of } \boxplus \\ &= \eta_{\mathfrak{X}}(\Box(f^{-1}(\mathfrak{a}), f^{-1}(\mathfrak{b}))) && \text{Definition of } \eta \\ &= \eta_{\mathfrak{X}}(\mathbb{A}_{\mathsf{st}}(\mathfrak{clp}\,f)(\Box(\mathfrak{a},\mathfrak{b}))) && \text{Def of } \mathfrak{clp} \text{ and } \mathbb{A}_{\mathsf{st}} \end{aligned}$$

The $\Diamond$ case follows from the fact that $\Diamond(\mathfrak{a},\mathfrak{b}) = \neg\Box(\mathfrak{a},\neg\mathfrak{b})$. Therefore η is a natural isomorphism. □

5.20 Corollary. *There is a dual equivalence*

$$\mathsf{Alg}(\mathbb{A}_{\mathsf{st}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathbb{C}_{\mathsf{st}}).$$

Proof. Follows from proposition 5.19 and lemma A.6. □

5.1.3 Bisimulations between descriptive conditional models

In [7] the authors introduce a notion of conditional bisimilarity between conditional models. We modify their notion slightly to work well with $\mathbb{C}_{\mathsf{st}}$-models (descriptive conditional models).

5.21 Definition. Let $(\mathfrak{X},\gamma)$ and $(\mathfrak{X}',\gamma')$ be two descriptive conditional frames (viewed as $\mathbb{C}_{\mathsf{st}}$-coalgebras). A closed subset $B \subseteq \mathfrak{X} \times \mathfrak{X}'$ is a **descriptive conditional bisimulation** if, for all B-coherent sets of clopens $(\mathfrak{a},\mathfrak{a}')$ and $(x,x') \in B$ we have

- $\gamma(x)(\mathfrak{a}) \subseteq B^{-1}[\gamma'(x')(\mathfrak{a}')]$ and $\gamma'(x')(\mathfrak{a}') \subseteq B[\gamma(x)(\mathfrak{a})]$.

A conditional bisimulation between descriptive conditional models $\mathfrak{X} = (\mathfrak{X},\gamma,V)$ and $\mathfrak{X}' = (\mathfrak{X}',\gamma',V')$ is a conditional bisimulation between the underlying frames with the extra condition that

- $x \in V(p)$ iff $x' \in V'(p)$ for all $p \in \Phi$.

Two states in two descriptive conditional frames or models are called **descriptive conditional bisimilar** if there is a conditional bisimulation linking them. ◁

5.22 Remark. We can rephrase the first bullet in definition 5.21 as follows: for all B-coherent pairs of clopens $(\mathfrak{a}, \mathfrak{a}')$ and $(x, x') \in B$ we have

- for all $y \in \gamma(x)(\mathfrak{a})$ there exists $y' \in \gamma'(x')(\mathfrak{a}')$ such that $(y, y') \in B$, and vice versa.

This wording resembles more the formulation of definition 5 in [7].

In section 2.3 we defined Λ-bisimulations for Stone coalgebras. A natural question is how these relate to descriptive conditional bisimulations. It turns out that, in terms of bisimilarity, they coincide. The remainder of this subsection is devoted to proving the following theorem.

5.23 Theorem. *Let $(\mathbf{X}, \gamma, V)$ and $(\mathbf{X}', \gamma', V')$ be two descriptive conditional models, $x \in X$ and $x' \in X'$. Then the following are equivalent:*

(i) x and x' are descriptively conditionally bisimilar;

(ii) x and x' are $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$-bisimilar;

(iii) x and x' are modally equivalent;

(iv) x and x' are behaviourally equivalent.

As a corollary of this theorem (combined with proposition 2.33) we find the following relation of descriptive conditional bisimilarity to Aczel-Mendler bisimilarity.

5.24 Corollary. *If two states x and x' in two descriptive conditional frames are Aczel-Mendler bisimilar, then they are descriptive conditional bisimilar.*

We isolate some of the implications of theorem 5.23 as separate propositions.

5.25 Proposition. *Let $\mathfrak{X} = (\mathbf{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbf{X}, \gamma', V')$ be two descriptive conditional models and $B \subseteq \mathbf{X} \times \mathbf{X}$ a descriptive conditional bisimulation. Then B is a $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$-bisimulation.*

Proof. Let $(x, x') \in B$. By definition $x \in V(p)$ iff $x' \in V'(p)$. Suppose $(\mathfrak{a}, \mathfrak{a}')$ and $(\mathfrak{b}, \mathfrak{b}')$ are B-coherent. If $\gamma(x) \in \lambda^{\Rightarrow}_{\mathbf{X}}(\mathfrak{a}, \mathfrak{b})$ then $\gamma(x)(\mathfrak{a}) \subseteq \mathfrak{b}$ and hence

$$\gamma'(x')(\mathfrak{a}') \subseteq B[\gamma(x)(\mathfrak{a})] \subseteq B[\mathfrak{b}] \subseteq \mathfrak{b}', \tag{5.4}$$

so $\gamma'(x') \in \lambda^{\Rightarrow}_{\mathbf{X}'}(\mathfrak{a}', \mathfrak{b}')$. The first inclusion in (5.4) follows from the fact that B is a descriptive conditional bisimulation, the last one from the coherence of $(\mathfrak{b}, \mathfrak{b}')$. The converse direction is proven similarly. For $\lambda^{\Downarrow}$, recall that lemma 2.18 states that $(\mathbf{X} \setminus \mathfrak{b}, X' \setminus \mathfrak{b}')$ is B-coherent whenever $(\mathfrak{b}, \mathfrak{b}')$ is, so

$$\begin{aligned} \gamma(x) \in \lambda^{\Downarrow}_{\mathbf{X}}(\mathfrak{a}, \mathfrak{b}) \quad &\text{iff} \quad \gamma(x) \notin \lambda^{\Rightarrow}_{\mathbf{X}}(\mathfrak{a}, \mathbf{X} \setminus \mathfrak{b}) \\ &\text{iff} \quad \gamma'(x') \notin \lambda^{\Rightarrow}_{\mathbf{X}'}(\mathfrak{a}, \mathbf{X} \setminus \mathfrak{b}) \quad \text{iff} \quad \gamma'(x') \in \lambda^{\Downarrow}_{\mathbf{X}'}(\mathfrak{a}', \mathfrak{b}'). \end{aligned}$$

This proves the proposition. □

5.26 Proposition. *Let $\mathfrak{X} = (\mathbf{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbf{X}', \gamma', V')$ be two descriptive conditional models. Let $x \in X$ and $x' \in X'$ be modally equivalent. Then they are descriptively conditionally bisimilar.*

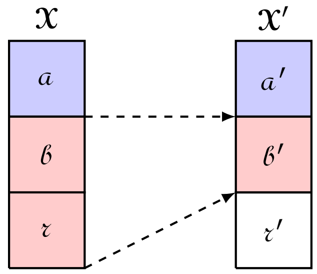


Figure 5.3: The image of $\mathbb{U}\mathfrak{X} \smallsetminus \mathfrak{a}$ under B is $\mathfrak{b}'$.

Proof. Let $B \subseteq \mathfrak{X} \times \mathfrak{X}'$ be the relation of modal equivalence. We will show that B is a descriptive conditional bisimulation. Let $w \in X$ and $w' \in X'$ be modally equivalent and let $(\mathfrak{a}, \mathfrak{a}') \in \mathrm{Clp}(\mathfrak{X}) \times \mathrm{Clp}(\mathfrak{X}')$ be B-coherent. Assume towards a contradiction that B does *not* satisfy the bisimulation property for $\mathfrak{a}$ and $\mathfrak{a}'$, then we claim that this implies that w and w' are not modally equivalent. We will show this in a couple of steps:

Step 1. Show that B is closed in $\mathfrak{X} \times \mathfrak{X}'$.

Step 2. Build a formula α that will play the role of $\mathfrak{a}$ and $\mathfrak{a}'$ in the antecedent. That is $\mathfrak{a} \subseteq [\![\alpha]\!]^{\mathfrak{X}} \subseteq \mathfrak{a} \cup \mathfrak{c}$ with $\mathfrak{c}$ a set of "non-relevant" states, and similarly $\mathfrak{a}' \subseteq [\![\alpha]\!]^{\mathfrak{X}'} \subseteq \mathfrak{a}' \cup \mathfrak{c}'$.

Step 3. Show that $\gamma(w)([\![\alpha]\!]^{\mathfrak{X}}) = \gamma(w)(\mathfrak{a})$ and $\gamma'(w')([\![\alpha]\!]^{\mathfrak{X}'}) = \gamma(w')(\mathfrak{a}')$.

Step 4. Find a contradiction with the assumption that B does not satisfy the bisimulation property for $\mathfrak{a}$ and $\mathfrak{a}'$.

Step 1. Suppose $(x, x') \notin B$, then there exists φ such that $\mathfrak{X}, x \Vdash \varphi$ and $\mathfrak{X}', x' \Vdash \neg\varphi$, so $(x, x') \in [\![\varphi]\!]^{\mathfrak{M}} \times [\![\neg\varphi]\!]^{\mathfrak{M}'}$ which is open in $\mathfrak{X} \times \mathfrak{X}'$ and clearly disjoint from B. Therefore B is closed in $\mathfrak{X} \times \mathfrak{X}'$.

Step 2. Let $\mathfrak{b} = B^{-1}(\mathfrak{X}' \smallsetminus \mathfrak{a}')$ and $\mathfrak{c} = \mathfrak{X} \smallsetminus (\mathfrak{a} \cup \mathfrak{b})$. Observe that $\mathfrak{b}$ is disjoint from $\mathfrak{a}$, for otherwise $B[\mathfrak{a}] \nsubseteq \mathfrak{a}'$. Moreover, elements in $\mathfrak{b}$ are not modally equivalent to elements in $\mathfrak{a}$, because this would imply that $B[\mathfrak{a}] \nsubseteq \mathfrak{a}'$. Observe also that elements in $\mathfrak{c}$ are not related to anything in $\mathfrak{X}'$, that is, they are not modally equivalent to any element in $\mathfrak{X}'$. With this notation, $\mathbb{U}\mathfrak{X}$ is the disjoint union of $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$,

$$\mathbb{U}\mathfrak{X} = \mathfrak{a} \cup \mathfrak{b} \cup \mathfrak{c}.$$

In a similar way we can write $\mathbb{U}\mathfrak{X}' = \mathfrak{a}' \cup \mathfrak{b}' \cup \mathfrak{c}'$. See figure 5.3.

The set $\mathfrak{b}$ is closed in $\mathfrak{X}$. To see this, note that

$$\mathfrak{b} = B^{-1}[\mathfrak{X}' \smallsetminus \mathfrak{a}'] = p[(\mathfrak{X}' \times \mathfrak{a}') \cap B],$$

where $p : \mathfrak{X} \times \mathfrak{X}' \to \mathfrak{X}$ is projection to the first coordinate. Observe that $(\mathfrak{X} \times \mathfrak{a}') \cap B$ is closed in $\mathfrak{X} \times \mathfrak{X}'$. As $\mathfrak{X}'$ is compact, p is a closed map and $\mathfrak{b}$ is closed in $\mathfrak{X}$. For a similar reason $\mathfrak{b}'$ is closed in $\mathfrak{X}'$.

Since elements of $\mathfrak{a}$ are not modally equivalent to elements of $\mathfrak{b} \cup (\mathfrak{X}' \smallsetminus \mathfrak{a}')$, for each $x \in \mathfrak{a}$ and $y \in \mathfrak{b} \cup \mathfrak{b}' \cup \mathfrak{c}'$ there exists $\varphi_{x,y}$ such that $\mathfrak{X}, x \Vdash \varphi_{x,y}$ and $\mathfrak{X}, y \Vdash \neg\varphi_{x,y}$ when $y \in \mathfrak{b}$ and $\mathfrak{X}', y \Vdash \neg\varphi_{x,y}$ when $y \in \mathfrak{b}' \cup \mathfrak{c}'$. Fix $x \in \mathfrak{a}$. Then $\mathfrak{b} \subseteq \bigcup_{y \in \mathfrak{b}} [\![\neg\varphi_{x,y}]\!]^{\mathfrak{X}}$ is an open covering of the closed set $\mathscr{b}$. By compactness there exists a finite subset $I_x \subseteq \mathscr{b}$ such that $\mathscr{b} \subseteq \bigcup_{y\in I_x} [\![\neg\varphi_{x,y}]\!]^{\mathfrak{X}}$. Similarly we have $\mathscr{b}' \cup \mathscr{c}' \subseteq \bigcup_{y\in \mathscr{b}'\cup\mathscr{c}'} [\![\varphi_{x,y}]\!]^{\mathfrak{X}'}$ and by compactness we find a finite $I'_x \subseteq \mathscr{b}' \cup \mathscr{c}'$ such that $\mathscr{b}' \cup \mathscr{c}' \subseteq \bigcup_{y\in I'_x} [\![\varphi_{x,y}]\!]^{\mathfrak{X}'}$ is a finite subcover. Set

$$\chi_x := \bigwedge_{y\in I_x\cup I'_x} \varphi_{x,y}.$$

Then $\mathfrak{X}, x \Vdash \chi_x$ and for all $y \in \mathscr{b} \cup \mathscr{b}' \cup \mathscr{c}'$ we have $y \nVdash \chi_x$. Next, we find an open covering

$$\mathscr{a} \subseteq \bigcup_{x\in\mathscr{a}} [\![\chi_x]\!]^{\mathfrak{X}}$$

and since $\mathscr{a}$ is closed, hence compact, there exists a finite set $J \subseteq \mathscr{a}$ such that $\mathscr{a} \subseteq \bigcup_{x\in J} [\![\chi_x]\!]^{\mathfrak{X}}$. Set

$$\chi := \bigvee_{x\in J} \chi_x.$$

Then χ has the property that $\mathfrak{X}, x \Vdash \chi$ for all $x \in \mathscr{a}$ and $\mathfrak{X}, y \nVdash \chi$ for $y \in \mathscr{b}$ and $\mathfrak{X}', y' \nVdash \chi$ for $y' \in \mathscr{b}' \cup \mathscr{c}'$. In a symmetric way we can define a formula χ' which is true for all $x \in \mathscr{a}'$ and false for all $y \in \mathscr{b}' \cup \mathscr{b} \cup \mathscr{c}$.

Let $\alpha := \chi \vee \chi'$. Then

$$\mathscr{a} \subseteq [\![\alpha]\!]^{\mathfrak{X}} \subseteq \mathscr{a} \cup \mathscr{c} \quad \text{and} \quad \mathscr{a}' \subseteq [\![\alpha]\!]^{\mathfrak{X}'} \subseteq \mathscr{a}' \cup \mathscr{c}'. \tag{5.5}$$

Step 3. Our next goal is to show $\gamma(w)([\![\alpha]\!]^{\mathfrak{X}}) = \gamma(w)(\mathscr{a})$. Let $z \in \gamma(w)([\![\alpha]\!]^{\mathfrak{X}})$. By condition (C1) from definition 5.3 we know $z \in [\![\alpha]\!]^{\mathfrak{X}}$ and by equation (5.5) this means either $z \in \mathscr{a}$ or $z \in \mathscr{c}$.

If $z \in \mathscr{c}$ then z is not modally equivalent to any $x \in \mathfrak{X}'$. Therefore we can make a formula β that is true everywhere in $\mathfrak{X}'$ but that is not true on z: For each $x \in \mathfrak{X}'$, there exists φ_x such that $\mathfrak{X}, z \Vdash \varphi$ and $\mathfrak{X}', x \nVdash \varphi_x$, so the clopen sets $[\![\neg\varphi_x]\!]^{\mathfrak{X}'}$ (where x ranges over $\mathfrak{X}'$) form an open cover of $\mathfrak{X}'$. By compactness there exists a finite subcover, say, $\mathfrak{X}' = [\![\neg\varphi_1]\!]^{\mathfrak{X}'} \cup \cdots \cup [\![\neg\varphi_n]\!]^{\mathfrak{X}'}$. Set

$$\beta := \neg\varphi_1 \vee \cdots \vee \neg\varphi_n,$$

then $\mathfrak{X}, z \nVdash \beta$ while $\mathfrak{X}', x \Vdash \beta$ for all $x \in \mathfrak{X}'$. In particular this means $\gamma(w)([\![\alpha]\!]^{\mathfrak{X}}) \notin [\![\beta]\!]^{\mathfrak{X}}$ and $\gamma'(w')([\![\alpha]\!]^{\mathfrak{X}'}) \subseteq [\![\beta]\!]^{\mathfrak{X}'}$. Therefore $\mathfrak{X}, w \nVdash \alpha \Rightarrow \beta$ while $\mathfrak{X}', w' \Vdash \alpha \Rightarrow \beta$. A contradiction with the assumption that w and w' are modally equivalent. We conclude that $z \in \mathscr{a}$.

As $\mathscr{a} \subseteq [\![\alpha]\!]^{\mathfrak{X}}$ and $\gamma(w)([\![\alpha]\!]^{\mathfrak{X}}) \subseteq \mathscr{a}$, it follows form condition (C2) from definition 5.3 that

$$\gamma(w)([\![\alpha]\!]^{\mathfrak{X}}) = \gamma(w)(\mathscr{a}).$$

In a similar way one may show that $\gamma'(w')([\![\alpha]\!]^{\mathfrak{X}'}) = \gamma'(w')(\mathscr{a}')$.

Step 4. Now suppose for a contradiction that B does not satisfy the bisimulation property for the sets $\mathscr{a}$ and $\mathscr{a}'$. Then without loss of generality, there is $x \in \gamma(w)(\mathscr{a})$ such that for all $x' \in \gamma'(w')(\mathscr{a}')$ we have $(x, x') \notin B$. For each $x' \in \gamma'(w')(\mathscr{a}')$ let $\varphi_{x'}$ be such that $\mathfrak{X}, x \Vdash \neg\varphi_{x'}$ and $\mathfrak{X}', x' \Vdash \varphi_{x'}$. Then the union

$$\bigcup_{x'\in\gamma'(w')(\mathscr{a}')} [\![\varphi_{x'}]\!]^{\mathfrak{X}'}$$

is an open cover of $\gamma'(w')(a')$. Since $\gamma'(w')(a')$ is closed, hence compact, there is a finite set $I \subseteq \gamma'(w')(a')$ such that $\bigcup_{x' \in I} [\![\varphi_{x'}]\!]^{\mathfrak{X}'}$ covers $\gamma'(w')(a')$. Therefore the formula $\psi = \bigvee_{x' \in I} \varphi_{x'}$ that is false at x and true everywhere in $\gamma'(w', [\![\alpha]\!]^{\mathfrak{X}'})$. But then $\mathfrak{X}, w \not\Vdash \alpha \Rightarrow \psi$ and $\mathfrak{X}', w' \Vdash \alpha \Rightarrow \psi$, a contradiction. This shows that B must be a descriptive conditional bisimulation. □

Recall theorem 5.23; the proof is now easy:

5.23 Theorem. *Let $(\mathfrak{X}, \gamma, V)$ and $(\mathfrak{X}', \gamma', V')$ be two descriptive conditional models, $x \in X$ and $x' \in X'$. Then the following are equivalent:*

(i) x and x' are descriptively conditionally bisimilar;

(ii) x and x' are $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$-bisimilar;

(iii) x and x' are modally equivalent;

(iv) x and x' are behaviourally equivalent.

Proof of theorem 5.23. The implication (i) ⇒ (ii) follows from proposition 5.25, (ii) ⇒ (iii) is lemma 2.32, (iii) ⇒ (i) is proposition 5.26 and (iii) ⇔ (iv) is proposition 2.29. □

A notion of bisimulation is called structural if it does not rely on the truth set of any formulas apart from the propositional variables. Structural bisimulations are preferred over non-structural ones. (For a discussion about this in the scope of conditional logic see e.g. [15].) In particular Λ-bisimulations (see definition 2.30) are non-structural. Theorem 5.23 provides a structural charactarization of $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$-bisimilarity (and modal equivalence and behavioural equivalence).

5.2 Geometric conditional logic

In this section we develop (the frames for) geometric conditional logic. We first generalise the conditional frames from example 2.10 to *geometric* conditional frames. We show how geometric conditional frames can be viewed as coalgebras for a functor $\mathbb{C}_{\mathsf{kh}}$. Thereafter, we define two open predicate liftings that correspond to the binary modalities ⇒ and ⇓. In subsection 5.2.2 we give a functor on Frm whose restriction to KRFrm is dual to $\mathbb{C}_{\mathsf{kh}}$.

5.2.1 Geometric conditional frames

5.27 Definition. A **topological conditional frame** is a triple (X, ν, A) where X is a set, $A \subseteq \mathbb{P}X$ is a topology on X and $\nu : X \times A \to \mathbb{P}X$ is a map, such that

- A is closed under the maps $m_\nu, n_\nu : A \times \mathbb{P}X \to \mathbb{P}X$ defined by

$$m_\nu(a, b) := \{x \in X \mid \nu(x, a) \subseteq b\}, \quad n_\nu(a, b) := \{x \in X \mid \nu(x, a) \cap b \neq \varnothing\},$$

- $\nu : X \times A \to K\mathfrak{X}$ satisfies for all $x \in X$, $a, b \in A$;
  (i) if $a \cap b = \varnothing$, then $\nu(x, a) \cap b = \varnothing$,
  (ii) if $a \subseteq b$ and $\nu(x, b) \subseteq a$, then $\nu(x, a) = \nu(x, b)$. ◁

5.28 Definition. Let $\mathfrak{X} = (X, \nu, A)$ be a topological conditional frame and denote by $\mathcal{X}$ the set X topologised by A. We say that $\mathfrak{X}$ is **differentiated** if for all $x, y \in X$ there exist disjoint $a, b \in A$ such that $x \in a$ and $y \in b$, **closed** if $\nu(x, a)$ is closed in $\mathcal{X}$ for all $x \in X$ and $a \in A$, and **compact** if $\mathcal{X}$ is compact.

A **geometric conditional frame** is a compact, closed and differentiated topological conditional frame. A **geometric conditional frame morphism** from (X, ν, A) to (X', ν', A') is a map $f : X \to X'$ such that $f^{-1}(a') \in A$ for all $a' \in A'$ (i.e. f is continuous) and $f[\nu(x, f^{-1}(a))] = \nu'(f(x), a)$ for all $x \in X, a \in A'$. Write GCF for the category of geometric conditional frames and geometric conditional frame morphisms. ◁

Let us take a coalgebraic perspective.

5.29 Definition. For a topological space $\mathcal{X}$, let $\mathbb{C}_{\mathsf{top}}\mathcal{X}$ be the collection of maps $h : \Omega\mathcal{X} \to K\mathcal{X}$ such that for all $x \in \mathcal{X}$ and $a, b \in \Omega\mathcal{X}$ we have

(C1) if $a \cap b = \varnothing$ then $h(a) \cap b = \varnothing$; and

(C2) if $a \subseteq b$ and $h(b) \subseteq a$ then $h(a) = h(b)$.

Endow $\mathbb{C}_{\mathsf{top}}\mathcal{X}$ with the topology generated by the subbase

$$\boxplus(a, b) := \{h \in \mathbb{C}_{\mathsf{top}}\mathcal{X} \mid h(a) \subseteq b\}, \quad \diamond\!\!\!\!\times(a, b) := \{h \in \mathbb{C}_{\mathsf{top}}\mathcal{X} \mid h(a) \cap b \neq \varnothing\}$$

where a and b range over the opens of $\mathcal{X}$. For a continuous map $f : \mathcal{X} \to \mathcal{X}'$ between topological spaces, define $\mathbb{C}_{\mathsf{top}}f : \mathbb{C}_{\mathsf{top}}\mathcal{X} \to \mathbb{C}_{\mathsf{top}}\mathcal{X}'$ by

$$\mathbb{C}_{\mathsf{top}}f(h)(a) = \overline{f[h(f^{-1}(a))]}.$$

(The overline denotes closure.) Then $\mathbb{C}_{\mathsf{top}}$ defines an endofunctor on Top. ◁

5.30 Remark. (i) Condition (C1) can be reformulated as $h(a) \subseteq \overline{a}$. The intuition behind this is as follows: The opens of a topological space will serve as the interpretants of geometric modal formulae and $\nu(x)(\llbracket\varphi\rrbracket)$ indicates the relevant states of $\llbracket\varphi\rrbracket$ for x. If the truth set $\llbracket\varphi\rrbracket$ of some formula φ is disjoint from the truth set $\llbracket\psi\rrbracket$ of ψ, then we require that $\varphi \Rightarrow \psi$ is false, i.e. $h(\llbracket\varphi\rrbracket) \cap \llbracket\psi\rrbracket = \varnothing$. However, we want to allow the situation where every state in $\llbracket\varphi\rrbracket$ is relevant for x, so $a \subseteq h(a)$. The set $h(a)$ must be closed, and the smallest closed set containing a is the closure $\overline{a}$. Therefore, we tolerate $h(a) = \overline{a}$.

(ii) If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism in KHaus then $f[h(f^{-1}(a))] \subseteq \mathcal{X}'$ is closed, so $\overline{f[h(f^{-1}(a))]} = f[h(f^{-1}(a))]$. Since a is open in $\mathcal{X}'$, $f^{-1}(a)$ is open in $\mathcal{X}$. By construction $h(f^{-1}(a))$ is closed in $\mathcal{X}$, and since $\mathcal{X}$ is compact and $\mathcal{X}'$ is Hausdorff, f is a closed map so $f[h(f^{-1}(a))]$ is closed in $\mathcal{X}'$.

5.31 Lemma. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a continuous map between topological spaces. Then $\mathbb{C}_{\mathsf{top}}f : \mathbb{C}_{\mathsf{top}}\mathcal{X} \to \mathbb{C}_{\mathsf{top}}\mathcal{X}'$ is a well-defined continuous map.*

Proof. We need to show that for each $h \in \mathbb{C}_{\mathsf{top}}\mathcal{X}$, the conditions (C1) and (C2) from definition 5.29 hold, and that $\mathbb{C}_{\mathsf{top}}f$ is continuous. Checking (C1) and (C2) is similar to example 2.21.

For continuity, it suffices to check that for all $a, b \in \Omega\mathcal{X}'$, the sets $(\mathbb{C}_{\mathsf{top}}f)^{-1}(\boxplus(a, b))$ and $(\mathbb{C}_{\mathsf{top}}f)^{-1}(\diamond\!\!\!\!\times(a, b))$ are open in $\mathbb{C}_{\mathsf{top}}\mathcal{X}$. It can be shown by a straightforward computation that

$$(\mathbb{C}_{\mathsf{top}}f)^{-1}(\boxplus(a, b)) = \boxplus(f^{-1}(a), f^{-1}(b)) \quad \text{and} \quad (\mathbb{C}_{\mathsf{top}}f)^{-1}(\diamond\!\!\!\!\times(a, b)) = \diamond\!\!\!\!\times(f^{-1}(a), f^{-1}(b)).$$

Since f is continuous, $f^{-1}(a)$ and $f^{-1}(b)$ are open in $\mathcal{X}$, hence $\boxplus(f^{-1}(a), f^{-1}(b))$ and $\diamond\!\!\!\!\times(f^{-1}(a), f^{-1}(b))$ are open in $\mathbb{C}_{\mathsf{top}}\mathcal{X}$. This proves that $\mathbb{C}_{\mathsf{top}}f$ is continuous. □

5.32 Lemma. *If $\mathfrak{X}$ is a compact Hausdorff space, then so is $\mathbb{C}_{\mathsf{top}}\mathfrak{X}$.*

Proof. *$\mathbb{C}_{\mathsf{top}}$ is Hausdorff.* Suppose $h \neq h'$ in $\mathbb{C}_{\mathsf{top}}\mathfrak{X}$, then $h(a) \neq h'(a)$ for some open set a of $\mathfrak{X}$. Without loss of generality assume there exists $x \in h'(a)$ such that $x \notin h(a)$. Since $h(a)$ is closed and $\mathfrak{X}$ is compact Hausdorff, hence regular, there exist disjoint open neighbourhoods u, v of $h(a)$ and x respectively. Now we have $h \in \boxplus(a, u)$, $h' \notin \boxplus(a, u)$ and $h \notin \diamondplus(a, v)$, $h' \in \diamondplus(a, v)$. For any $j \in \mathbb{C}_{\mathsf{top}}\mathfrak{X}$, whenever $j \in \boxplus(a, u)$ we have $v \cap j(a) = \varnothing$ so $j \notin \diamondplus(a, v)$. Therefore $\boxplus(a, u)$ and $\diamondplus(a, v)$ are disjoint and $\mathbb{C}_{\mathsf{top}}\mathfrak{X}$ is Hausdorff.

$\mathbb{C}_{\mathsf{top}}$ is compact. The proof of this is similar to the proof of lemma 5.6. □

5.33 Definition. Let $\mathbb{C}_{\mathsf{kh}}$ denote the restriction of the functor $\mathbb{C}_{\mathsf{top}}$ to KHaus. Lemma 5.32 entails this is an endofunctor on KHaus. ◁

5.34 Proposition. $\mathsf{GCF} \cong \mathsf{Coalg}(\mathbb{C}_{\mathsf{kh}})$.

Proof. There is an obvious bijection between objects of GCF and objects of $\mathsf{Coalg}(\mathbb{C}_{\mathsf{kh}})$. Let (X, ν, A) and (X', ν', A') be two geometric conditional frames and let $(\mathfrak{X}, \gamma)$ and $(\mathfrak{X}', \gamma')$ be the corresponding $\mathbb{C}_{\mathsf{kh}}$-coalgebras. Let $f : X \to X'$ be a function. We claim that f is a geometric conditional frame morphism if and only if it is a $\mathbb{C}_{\mathsf{kh}}$-coalgebra morphism.

Suppose f is a geometric conditional frame morphism. Then f is continuous as $f^{-1}(a') \in A$ for all $a' \in A'$ and A' is precisely the set of opens in $\mathfrak{X}$. Let $x \in X$ and $a \in \Omega\mathfrak{X}$, then

$$(\mathbb{C}f)(\gamma(x))(a) = f[\gamma(x)(f^{-1}(a))] = f[\nu(x, f^{-1}(a))] = \nu'(f(x), a) = \gamma'(f(x))(a),$$

so f is a $\mathbb{C}_{\mathsf{kh}}$-coalgebra morphism. The converse direction is similar. □

5.35 Remark. The functor $\mathbb{C}_{\mathsf{kh}}$ coincides with the restriction to KHaus of the sober Kupke-Kurz-Pattinson lift (from section 4.1) $\ddot{\mathbb{C}}_{\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}}$ of $\mathbb{C}$ with respect to the predicate liftings $\lambda^{\Rightarrow}, \lambda^{\Downarrow}$ (from example 2.22). This can be proved in a way similar to example 4.9 and using the duality from theorem 5.38 below.

We close this subsection with a brief look at the open predicate liftings that constitute geometric conditional logic.

5.36 Definition. Define the open predicate liftings $\lambda^{\Rightarrow}, \lambda^{\Downarrow} : \Omega^2 \to \Omega \circ \mathbb{C}_{\mathsf{top}}$ by

$$\lambda^{\Rightarrow}_{\mathfrak{X}}(a, b) = \{h \in \mathbb{C}_{\mathsf{top}}\mathfrak{X} \mid h(a) \subseteq b\} \quad \text{and} \quad \lambda^{\Downarrow}_{\mathfrak{X}}(a, b) = \{h \in \mathbb{C}_{\mathsf{top}}\mathfrak{X} \mid h(a) \cap b \neq \varnothing\}.$$

Recall that we write Φ for a set of proposition letters. Then $\mathrm{GML}(\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\})$ is given by

$$\varphi ::= \bot \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee_{i \in I} \varphi_i \mid \varphi_1 \Rightarrow \varphi_2 \mid \varphi_1 \Downarrow \varphi_2,$$

where $p \in \Phi$. ◁

Let $\mathfrak{X} = (\mathfrak{X}, \gamma, V)$ be a $\mathbb{C}_{\mathsf{top}}$-model, then

$$\mathfrak{X}, x \Vdash \varphi \Rightarrow \psi \quad \text{iff} \quad \gamma(x)([\![\varphi]\!]^{\mathfrak{X}}) \subseteq [\![\psi]\!]^{\mathfrak{X}}$$

and

$$\mathfrak{X}, x \Vdash \varphi \Downarrow \psi \quad \text{iff} \quad \gamma(x)([\![\varphi]\!]^{\mathfrak{X}}) \cap [\![\psi]\!]^{\mathfrak{X}} \neq \varnothing.$$

Moreover, the topology on $\mathbb{C}_{\mathsf{top}}\mathfrak{X}$ is generated by $\lambda^{\Rightarrow}(a, b), \lambda^{\Downarrow}(a, b)$ where $a, b \in \Omega\mathfrak{X}$, so $\{\lambda^{\Rightarrow}, \lambda^{\Downarrow}\}$ is a characteristic set of open predicate liftings for $\mathbb{C}_{\mathsf{top}}$.

5.2.2 An Isbell dual for $\mathbb{C}_{\mathsf{kh}}$

We will now work towards a dual functor for $\mathbb{C}_{\mathsf{kh}}$ on the frame side, analogous to theorem 3.41. Recall from definition 3.7 that for an element a in a frame F the negation is defined as $\sim a = \bigvee\{b \in F \mid a \wedge b = \bot\}$.

5.37 Definition. Let B be a frame. Define $\mathbb{A}B$ to be the frame generated by the set $\{\Box(a,b), \Diamond(a,b) \mid a,b \in B\}$ subject to the following relations:

(A1) $\Box(a,b) = \top_{\mathbb{A}B}$ if $b \vee \sim a = \top$

(A2) $\Diamond(a,b) = \bot_{\mathbb{A}B}$ if $a \wedge b = \bot$

(A3) $\Box(a,b) \wedge \Box(a,c) = \Box(a, b \wedge c)$

(A4) $\Diamond(a,b) \vee \Diamond(a,c) = \Diamond(a, b \vee c)$

(A5) $\Box(a, b \vee c) \leq \Box(a,b) \vee \Diamond(a,c)$

(A6) $\Box(a,b) \wedge \Diamond(a,c) \leq \Diamond(a, b \wedge c)$

(A7) $\Box(a,b) \wedge \Box(a,c) = \Box(a,b) \wedge \Box(b,c)$ if $b \leq a$

(A8) $\Box(a,b) \wedge \Diamond(a,c) = \Box(a,b) \wedge \Diamond(b,c)$ if $b \leq a$

(A9) $\bigvee^{\uparrow}_{b \in A} \Box(a,b) = \Box(a, \bigvee^{\uparrow} A)$ if A is directed

(A10) $\bigvee^{\uparrow}_{b \in A} \Diamond(a,b) = \Diamond(a, \bigvee^{\uparrow} A)$ if A is directed

If $f : B \to B'$ is a frame homomorphism, define $\mathbb{A}f : \mathbb{A}B \to \mathbb{A}B'$ on generators by $\mathbb{A}f(\Box(a,b)) = \Box(f(a), f(b))$ and $\mathbb{A}f(\Diamond(a,b)) = \Diamond(f(a), f(b))$. (By remark 3.3 $\mathbb{A}_{\mathsf{st}}f$ is well defined.) ◁

We will now show that there is a duality between $\mathbb{C}_{\mathsf{kh}}$ and a restriction of $\mathbb{A}$. The proof of the next theorem is somewhat similar to the proof of theorem 3.41 and to the proof of proposition III4.6 in [29]. The main difference with the proof in [29] is the fact that we use binary modalities instead of unary ones. Therefore our maps (that constitute the aforementioned duality) are defined differently. The second argument of each of the modalities is treated similarly to the proof in [29].

5.38 Theorem. *If* $\mathfrak{X} = (X, \tau)$ *is a compact Hausdorff space then*

$$\mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X})) \cong \mathbb{C}_{\mathsf{kh}}\mathfrak{X}.$$

Proof. Define a map

$$\varphi : \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \to \mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X})) : h \mapsto p_h,$$

where p_h is defined on generators by

$$p_h : \mathbb{A}(\mathfrak{opn}\,\mathfrak{X}) \to 2 : \begin{cases} \Box(a,b) \mapsto \top & \text{iff } h(a) \subseteq b \\ \Diamond(a,b) \mapsto \top & \text{iff } h(a) \cap b \neq \varnothing \end{cases}.$$

Conversely, for a point $p \in \mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X}))$ define $h_p : \Omega\mathfrak{X} \to K\mathfrak{X}$ by

$$h_p(a) = \mathfrak{X} \setminus \bigcup\{b \in \Omega\mathfrak{X} \mid p(\Diamond(a,b)) = \bot\}.$$

This gives rise to a map $\psi : \mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X})) \to \mathbb{C}_{\mathsf{kh}}\mathfrak{X}$.

Note the absence of the box in the definition of h_p. This is less surprising that it may seem at first sight: Diamonds and boxes interact via relation (A5) and (A6). It follows from claim 5.38.C below that it is also possible to define $h_p(a)$ using boxes, namely via

$$h_p(a) = \bigcap\{b \in \Omega\mathfrak{X} \mid p(\Box(a,b)) = \top\}.$$

We will show that both φ and ψ are well defined, that they are mutually inverse and that φ is continuous. Throughout, we write $X := \mathbb{U}\mathfrak{X}$.

5.38.A Claim. *For each $h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X}$, the map p_h indeed defines a point.*

Proof of claim. Since a point is simply a frame homomorphism $\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) \to 2$ and p_h is defined on generators, by remark 3.3 it suffices to show that the $p(\Box(a,b))$ and $p(\Diamond(a,b))$, where a,b range over $\Omega\mathfrak{X}$, satisfy relations (A1) through (A10).

(A1) Observe that $\sim a = X \setminus \overline{a}$. Let b be such that $b \cup (X \setminus \overline{a}) = X$. Then $\overline{a} \subseteq b$. Since for all h in $\mathbb{C}_{\mathsf{kh}}\mathfrak{X}$ we have $h(a) \subseteq \overline{a}$, we know $p_h(\Box(a,b)) = \top$.

(A2) Let $a,b \in \Omega\mathfrak{X}$ and suppose $a \cap b = \varnothing$. As a and b are open, we have $\overline{a} \cap b = \varnothing$. Since $h(a) \subseteq \overline{a}$ it follows that $h(a) \cap b = \varnothing$, so $p_h(\Diamond(a,b)) = \bot$.

(A3) For all $a,b,c \in \Omega\mathfrak{X}$ we have $p_h(\Box(a,b \cap c)) = \top$ iff $h(a) \subseteq b \cap c$ iff $[h(a) \subseteq b$ and $h(a) \subseteq c]$ iff $[p_h(\Box(a,b)) = \top$ and $p_h(\Box(a,c)) = \top]$ iff $p_h(\Box(a,b) \wedge \Box(a,c)) = \top$.

(A4) For all $a,b,c \in \Omega\mathfrak{X}$ we have $p_h(\Diamond(a,b) \vee \Diamond(a,c)) = \top$ iff $[p_h(\Diamond(a,b)) = \top$ or $p_h(\Diamond(a,c)) = \top]$ iff $[h(a) \cap b \neq \varnothing$ or $h(a) \cap c \neq \varnothing]$ iff $h(a) \cap (b \cup c) \neq \varnothing$ iff $p_h(\Diamond(a,b \cup c)) = \top$.

(A5) Suppose $p_h(\Box(a,b \cup c)) = \top$. If $p_h(\Box(a,b)) = \top$ we are done, so suppose otherwise. Then $p_h(\Box(a,b)) = \bot$, so $h(a) \subseteq b \cup c$ and $h(a) \nsubseteq b$. Then $h(a) \cap c \neq \varnothing$ and hence $p_h(\Diamond(a,c)) = \top$.

(A6) Suppose $p_h(\Box(a,b)) = \top$ and $p_h(\Diamond(a,c)) = \top$. Then $h(a) \cap c \neq \varnothing$, so there is some $x \in h(a)$ such that $x \in c$. Since $h(a) \subseteq b$ this x must be in b, and hence $h(a) \cap (b \cap c) \neq \varnothing$, so that $h_p(\Diamond(a,b \cap c)) = \top$.

(A7) Suppose $b \subseteq a$. If $h(a) \subseteq b$ then by (C2) from definition 5.29 $h(a) = h(b)$, hence $h \in \Box(a,c)$ iff $h \in \Box(b,c)$. So if $p_h(\Box(a,b)) = \top$ we have $p_h(\Box(a,c)) = p_h(\Box(b,c))$, which proves that (A7) holds.

(A8) Suppose $b \subseteq a$. If $h(a) \subseteq b$ then by (C2) from definition 5.29 $h(a) = h(b)$, hence $h \in \Diamond(a,c)$ iff $h \in \Diamond(b,c)$. So if $p_h(\Box(a,b)) = \top$ we have $p_h(\Diamond(a,c)) = p_h(\Diamond(b,c))$ for all c, which proves that (A8) holds.

(A9) We need to show that $\bigvee^{\uparrow}_{b \in A} p_h(\Box(a,b)) = p_h(\Box(a,\bigcup^{\uparrow} A))$. Suppose $\bigvee^{\uparrow}_{b \in A} p_h(\Box(a,b)) = \top$, then $p_h(\Box(a,b)) = \top$ for some $b \in A$. So for this b we have $h(a) \subseteq b$. Since $b \subseteq \bigcup^{\uparrow} A$ this implies $h(a) \subseteq \bigcup^{\uparrow} A$ and hence $p_h(a,\bigcup^{\uparrow} A) = \top$.

Conversely, suppose $p_h(\Box(a,\bigcup^{\uparrow} A)) = \top$, then $h(a) \subseteq \bigcup^{\uparrow} A$. The sets in A form an open cover of $h(a)$ and since $h(a)$ is closed it has a finite subcover. Since A is a directed set, there is some $b \in A$ containing the union of this finite subcover, and therefore $h(a) \subseteq b$. But then $p_h(\Box(a,b)) = \top$ and hence $\bigvee^{\uparrow}_{b \in A} p_h(\Box(a,b)) = \top$.

(A10) We need to show that $\bigvee^{\uparrow}_{b \in A} p_h(\Diamond(a,b)) = p_h(\Diamond(a,\bigcup^{\uparrow} A))$. Suppose $\bigvee^{\uparrow}_{b \in A} p_h(\Diamond(a,b)) = \top$, then there is some $b \in A$ such that $p_h(\Diamond(a,b)) = \top$, so $h(a) \cap b \neq \varnothing$. Since $b \subseteq \bigcup^{\uparrow} A$ this implies $h(a) \cap (\bigcup^{\uparrow} A) \neq \varnothing$, hence $p_h(\Diamond(a,\bigcup^{\uparrow} A)) = \top$.

Suppose $\bigvee^{\uparrow}_{b \in A} p_h(\Diamond(a,b)) = \bot$. Then $h(a) \cap b = \varnothing$ for all $b \in A$. Therefore every $b \in A$ is contained in the open set $X \setminus h(a)$, hence $\bigcup^{\uparrow} A \subseteq X \setminus h(a)$. This implies $h(a) \cap (\bigcup^{\uparrow} A) = \varnothing$, so $p_h(\Diamond(a,\bigcup^{\uparrow} A)) = \bot$.

So for each $h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X}$, the map p_h is indeed a point. ◇

5.38.B Claim. *For each point $p \in \mathsf{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X}))$ the map h_p is an element of $\mathbb{C}_{\mathsf{kh}}\mathfrak{X}$.*

Proof of claim. It is clear that $h_p(a) \in K\mathfrak{X}$, because it is the complement of a union of open sets. We need to show that h_p satisfies conditions (C1) and (C2) from definition 5.29.

First (C1). Let $a \in \Omega\mathfrak{X}$. Using (A2) we find $p(\diamond(a, {\sim}a)) = p(\bot) = \bot$ so $\bigcup\{c \mid p(\diamond(a,c)) = \bot\} \supseteq {\sim}a$ and hence

$$h_p(a) = X \setminus \bigcup\{c \mid p(\diamond(a,c)) = \bot\} \subseteq X \setminus {\sim}a = \overline{a}.$$

Now (C2). Suppose $b \subseteq a$ and $h_p(a) \subseteq b$. We need to show that $h_p(a) = h_p(b)$. Abbreviate $\mathcal{C}_a = \{c \in \Omega\mathfrak{X} \mid p(\diamond(a,c)) = \bot\}$. Then $\bigcup \mathcal{C}_a \in \Omega\mathfrak{X}$. Moreover, $\mathcal{C}_a$ is directed, because $p(\diamond(a,c)) = p(\diamond(a,c')) = \bot$ implies $p(\diamond(a, c \cup c')) = p(\diamond(a,c)) \vee p(\diamond(a,c')) = \bot \vee \bot = \bot$. Now (A10) implies $p(a, \bigcup^{\uparrow} \mathcal{C}_a) = \bot$.

Since $h_p(b) \subseteq a$ we have $X = a \cup (X \setminus h_p(b)) = a \cup \bigcup \mathcal{C}$. Because $p(a, X) = \top$ and $p(a, \bigcup \mathcal{C}_a) = \bot$ it follows from (A5) that $p(\Box(a,b)) = \top$. Therefore, by (A8), $p(\diamond(a,d)) = p(\diamond(b,d))$ for all d so $h_p(a) = h_p(b)$. Clearly this implies $h_p(a) = h_p(b)$, as desired. ◇

5.38.C Claim. *For every point p we have*

(i) $h_p(a) \cap b \neq \varnothing$ *iff* $p(\diamond(a,b)) = \top$*; and*

(ii) $h_p(a) \subseteq b$ *iff* $p(\Box(a,b)) = \top$.

Proof of claim. Abbreviate $\mathcal{C}_a = \{c \in \Omega\mathfrak{X} \mid p(\diamond(a,c)) = \bot\}$. We have seen in the proof of claim 5.38.B that $p(\diamond(a, \bigcup \mathcal{C}_a)) = \bot$. It follows from (A4) that for any $d \subseteq \bigcup \mathcal{C}_a$ we have $p(\diamond(a,d)) = \bot$. By definition $p(\diamond(a,d)) = \bot$ implies $d \subseteq \mathcal{C}_a$. Therefore

$$h_p(a) \cap d \neq \varnothing \quad \text{iff} \quad d \nsubseteq \bigcup \mathcal{C}_a \quad \text{iff} \quad p(\diamond(a,d)) \neq \bot \quad \text{iff} \quad p(\diamond(a,b)) = \top.$$

This proves (i).

Proving (ii) requires some more work. Suppose $h_p(a) \subseteq b$. Then $b \cup \bigcup \mathcal{C}_a = X$ so $p(\Box(a, b \cup \bigcup \mathcal{C}_a)) = \top$. By (A5) we have $p(\Box(a, b \cup \bigcup \mathcal{C}_a)) \leq p(\Box(a,b)) \vee p(\diamond(a, \bigcup \mathcal{C}_a))$ and since $p(\diamond(a, \bigcup \mathcal{C}_a)) = \bot$, we must have $p(\Box(a,b)) = \top$.

For the converse, suppose $h_p(a) \nsubseteq b$. Then $X \setminus \bigcup \mathcal{C}_a \nsubseteq b$, so

$$b \cup \bigcup \mathcal{C}_a \neq X. \tag{5.6}$$

If $b' \lessdot b$ then by lemma 3.8 ${\sim}b' \vee b = \top$, so by (5.6) ${\sim}b' \nsubseteq \bigcup \mathcal{C}$ and $h_p(a) \cap {\sim}b \neq \varnothing$; hence by part (i) of this proof $p(\diamond(a, {\sim}b')) = \top$. Relation (A8) gives

$$p(\Box(a,b')) \wedge p(\diamond(a, {\sim}b')) \leq p(\diamond(a, b' \wedge {\sim}b')) = p(\diamond(a, \bot)) = \bot$$

and since $p(\diamond(a, {\sim}b')) = \top$ this implies $p(\Box(a,b')) = \bot$. It then follows from (A10) and regularity of $\mathfrak{X}$ that

$$p(\Box(a,b)) = p(\Box(a, \textstyle\bigcup^{\uparrow}_{b' \lessdot b} b')) = p(\bigcup^{\uparrow}_{b' \lessdot b} \Box(a,b')) = \bigcup^{\uparrow}_{b' \lessdot b} p(\Box(a,b')) = \bot.$$

We may conclude that $p(\Box(a,b)) = \top$ iff $h_p(a) \subseteq b$. ◇

5.38.D Claim. *The maps φ and ψ define a bijection between $\mathbb{C}_{\mathsf{kh}}\mathfrak{X}$ and $\mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X}))$.*

Proof of claim. We will show that for all $p \in \mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X}))$ we have $p = p_{h_p}$ and for all $h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X}$ we have $h = h_{p_h}$. We start with showing $p = p_{h_p}$.

It suffices to show that p and p_{h_p} coincide on the generators of $\mathbb{A}(\mathfrak{opn}\,\mathfrak{X})$. It follows from the definition of p_h (for a point p) and claim 5.38.C that

$$p(\Box(a,b)) = \top \quad \text{iff} \quad h_p(a) \subseteq b \quad \text{iff} \quad p_{h_p}(\Box(a,b)) = \top$$

and

$$p(\Diamond(a,b)) = \top \quad \text{iff} \quad h_p(a) \cap b \neq \varnothing \quad \text{iff} \quad p_{h_p}(\Diamond(a,b)) = \top.$$

Therefore $p = p_{h_p}$.

Now let us prove $h = h_{p_h}$. It suffices to show that for each $a, b \in \Omega\mathfrak{X}$ we have $h(a) \subseteq b$ iff $h_{p_h}(a) \subseteq b$. By definition of p_h we have $h(a) \subseteq b$ iff $p_h(\Box(a,b)) = \top$. It follows from claim 5.38.C that $p_h(\Box(a,b)) = \top$ iff $h_{p_h}(a) \subseteq b$. This proves the claim. $\Diamond$

5.38.E Claim. *The map φ is continuous.*

Proof. The opens of $\mathfrak{pt}(\mathbb{A}(\mathfrak{opn}\,\mathfrak{X}))$ are generated by $\widetilde{\Box(a,b)} = \{p \mid p(\Box(a,b)) = \top\}$ and $\widetilde{\Diamond(a,b)} = \{p \mid p(\Diamond(a,b)) = \top\}$. We have

$$\varphi^{-1}(\widetilde{\Box(a,b)}) = \varphi^{-1}(\{p \mid p(\Box(a,b)) = \top\}) = \{h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \mid h(a) \subseteq b\} = \boxplus(a,b)$$

and similarly $\varphi^{-1}(\widetilde{\Diamond(a,b)}) = \diamondplus(a,b)$. This proves continuity of φ. $\Diamond$

This completes the proof of the theorem. $\Box$

An immediate corollary of this theorem is the fact that the functor $\mathbb{A}$ on Frm preserves compact regularity. Let $\mathbb{A}_{\mathsf{kr}}$ be the restriction of $\mathbb{A}$ to KRFrm. Theorem 5.38 entails that the frames $\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X})$ and $\mathfrak{opn}(\mathbb{C}_{\mathsf{kh}}\mathfrak{X})$ are isomorphic for every $\mathfrak{X} \in \mathsf{KHaus}$.

5.39 Definition. For a compact Hausdorff space $\mathfrak{X}$, let $\eta_{\mathfrak{X}}$ be the map

$$\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) \xrightarrow{\mathfrak{opn}\circ\mathfrak{pt}} \mathfrak{opn}(\mathfrak{pt}(\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}))) \xrightarrow{\mathfrak{opn}\,\varphi} \mathfrak{opn}(\mathbb{C}_{\mathsf{kh}}\mathfrak{X}).$$

It is routine to verify that, on generators, this is given by $\Box(a,b) \mapsto \boxplus(a,b)$ and $\Diamond(a,b) \mapsto \diamondplus(a,b)$. $\lhd$

It turns out that the the collection $(\eta_{\mathfrak{X}})_{\mathfrak{X}\in\mathsf{KHaus}}$ constitutes a natural isomorphism between $\mathbb{A}_{\mathsf{kr}} \circ \mathfrak{opn}$ and $\mathfrak{opn} \circ \mathbb{C}_{\mathsf{kh}}$.

5.40 Proposition. *Let $\eta = (\eta_{\mathfrak{X}})_{\mathfrak{X}\in\mathsf{KHaus}}$. Then $\eta : \mathbb{A}_{\mathsf{kr}} \circ \mathfrak{opn} \to \mathfrak{opn} \circ \mathbb{C}_{\mathsf{kh}}$ is a natural isomorphism.*

Proof. It follows from theorem 5.38 that $\eta_{\mathfrak{X}}$ is an isomorphism for each compact Hausdorff sapce $\mathfrak{X}$, so we only need to show naturality. Let $f : \mathfrak{X} \to \mathfrak{X}'$ be a continuous map between compact Hausdorff spaces. We need to show that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}) & \xleftarrow{\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,f)} & \mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\,\mathfrak{X}') \\
{\scriptstyle \eta_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathfrak{X}'}} \\
\mathfrak{opn}(\mathbb{C}_{\mathsf{kh}}\mathfrak{X}) & \xleftarrow[\mathfrak{opn}(\mathbb{C}_{\mathsf{kh}}f)]{} & \mathfrak{opn}(\mathbb{C}_{\mathsf{kh}}\mathfrak{X}')
\end{array}$$

Let $\boxdot(a, b)$ be a generator in $\mathbb{A}_{\mathsf{kr}}(\mathrm{Clop}\,\mathfrak{X}')$. Then

$$
\begin{aligned}
&\mathfrak{opn}(\mathbb{C}_{\mathsf{kh}} f)(\eta_{\mathfrak{X}'}(\boxdot(a, b))) \\
&\qquad = (\mathbb{C}_{\mathsf{kh}} f)^{-1}(\boxplus(a, b)) && \text{Def of } \mathfrak{opn} \text{ and } \eta_{\mathfrak{X}'} \\
&\qquad = \{h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \mid (\mathbb{C}_{\mathsf{kh}} f)(h) \in \boxplus(a, b)\} \\
&\qquad = \{h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \mid (\mathbb{C}_{\mathsf{kh}} f)(h)(a) \subseteq b\} && \text{Definition of } \boxplus \\
&\qquad = \{h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \mid f[h(f^{-1}(a))] \subseteq b\} && \text{Definition of } \mathbb{C}_{\mathsf{kh}} f \\
&\qquad = \{h \in \mathbb{C}_{\mathsf{kh}}\mathfrak{X} \mid h(f^{-1}(a)) \subseteq f^{-1}(b)\} \\
&\qquad = \boxplus(f^{-1}(a), f^{-1}(b)) && \text{Definition of } \boxplus \\
&\qquad = \eta_{\mathfrak{X}}(\boxdot(f^{-1}(a), f^{-1}(b))) && \text{Definition of } \eta_{\mathfrak{X}} \\
&\qquad = \eta_{\mathfrak{X}}(\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\, f)(\boxdot(a, b))) && \text{Def of } \mathbb{A}_{\mathsf{kr}} \text{ and } \mathfrak{opn}
\end{aligned}
$$

With a similar argument it can be shown that

$$(\mathfrak{opn}(\mathbb{C}_{\mathsf{kh}} f)(\eta_{\mathfrak{X}'}(\diamond(a, b))) = \eta_{\mathfrak{X}}(\mathbb{A}_{\mathsf{kr}}(\mathfrak{opn}\, f)(\diamond(a, b))).$$

This proves that η is a natural transformation. □

Applying lemma A.6 to proposition 5.40 yields the following corollary.

5.41 Corollary. *There is a dual equivalence*

$$\mathsf{Alg}(\mathbb{A}_{\mathsf{kr}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathbb{C}_{\mathsf{kh}}).$$

This section provided an initial attempt at the development of geometric conditional logic. We finish this section on the claim that there are still many interesting questions concerning geometric conditional logic, some of which can be found in chapter 6.

6

Conclusion

We have started building a framework for coalgebraic *geometric* logic and investigated some examples of concrete funtors. There are still many unanswered and interesting questions. We outline possible directions for further research.

Bisimulations In [6] the authors define Λ-bisimulations (which are inspired by [17]) between set coalgebas. In this thesis we define Λ-bisimulations between Stone coalgebras (section 2.3) and between sober coalgebras (section 3.4). This raises the question whether a more uniform treatment of Λ-bisimulations is possible, which encompasses all these cases.

Lifting functors In section 4.1 we give a method to lift a set functor $\mathbb{T}$ together with a set of predicate liftings Λ for $\mathbb{T}$ to an endofunctor $\ddot{\mathbb{T}}_\Lambda$ on Sob. We know three instances where this lifted functor preserves the compact Hausdorff property: when lifting the powerset functor, the monotone functor and the conditional functor (with respect to a suitable Λ). An interesting question is whether we can find general conditions (on $\mathbb{T}$ or Λ) which imply preservation properties of $\ddot{\mathbb{T}}$, like preserving compactness or the Hausdorff property. This search could be inspired by the cases where we know that the compact Hausdorff property is preserved.

Besides, in [57] the authors give a method for lifting a certain class of set functors to endofunctors on Frm, which leads to geometric logic with an added modality ∇. We suspect that there is a connection with our way of lifting set functors to sober functors, in the sense that, for a cleverly chosen set of predicate liftings Λ, the lift of $\mathbb{T}$ to a functor on Frm is dual to $\ddot{\mathbb{T}}_\Lambda$.

Examples in theory As mentioned at the end of chapter 3, it is not completely clear how the concrete examples of functors we have seen fit in the general theory that we developed. An interesting direction for further research is to investigate how examples fit within this general theory.

Proof theory Throughout this thesis we have focussed on the semantics of certain coalgebraic logics. An interesting course of action is to investigate the syntactic side of the logics involved. One could either try to do this in full generality, or (as a first step) develop proof theory for examples such as monotone and conditional geometric logic.

7

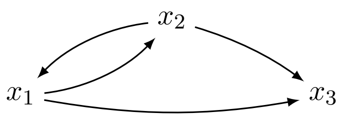
Populaire samenvatting

We schetsen wat een set-coalgebra is en hoe zulke coalgebras verschillende structure beschrijven.

Coalgebras Laat $\mathbb{T}$ een formule zijn die van elke verzameling X een (andere) verzameling $\mathbb{T}X$ maakt (dat noemen we een functor). Een voorbeeld van zo'n formule is de machtsverzameling functor $\mathbb{P}$, die stuurt een verzameling X naar de machtsverzameling $\mathbb{P}X = \{V \mid V \subseteq X\}$. Een $\mathbb{T}$-coalgebra is dan simpelweg een paar (X, γ) van een verzameling X en een afbeelding $\gamma : X \to \mathbb{P}X$. Coalgebras zijn een algemeen raamwerk om structuren te beschrijven. We geven twee voorbeelden van zulke structuren en hoe ze als coalgebra gezien kunnen worden.

Transitie systemen Een transitie systeem is een verzameling X met daarop een relatie R. Een relatie is een deelverzameling van $X \times X$. Als x en y twee punten in X zijn dan zeggen we "x ziet y" als $(x, y) \in R$.

Een voorbeeld van een transitie systeem is



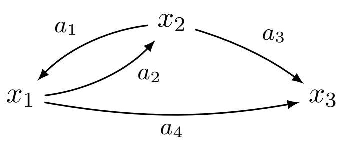
Dit is een weergave van (X, R) met $X = \{x_1, x_2, x_3\}$ en

$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_3)\}.$$

We kunnen dit zien als een $\mathbb{P}$-coalgebra door $\gamma_R : X \to \mathbb{P}X$ te definiëren als $\gamma(x_i) = \{x_j \in X \mid (x_i, x_j) \in R\}$. Dan beschrijft de $\mathbb{P}$-coalgebra (X, γ_R) precies het transitie system (X, R); dat wil zeggen, als we alleen (X, γ_R) zouden weten kunnen we daaruit aflezen wat (X, R) is en andersom.

In het algemeen corresponderen transitie systemen één-op-één met $\mathbb{P}$-coalgebras.

Gelabelde transitie systemen Stel nu dat we aan elke pijl een label toevoegen. Zij A een collectie labels waaruit we kunnen kiezen. Een gelabeld transitie systeem een paar (X, L) met X een verzameling en $L \subseteq X \times A \times X$ een gelabelde relatie. Een voorbeeld van een gelabeld transitie systeem is



Dit is (X_1, L_1) met $X_1 = \{x_1, x_2, x_3\}$ en

$$L_1 = \{(x_1, a_2, x_2), (x_1, a_4, x_3), (x_2, a_1, x_1), (x_2, a_3, x_3)\}.$$

Om dit als een coalgebra te zien kunnen we niet meer $\mathbb{P}$ gebruiken maar moeten we een andere functor, $\mathbb{L}$, maken.

Voor een verzameling X definiëren we

$$\mathbb{L}X = \mathbb{P}(A \times X).$$

We kunnen het gelabelde transitie systeem (X_1, L_1) nu beschouwen als $\mathbb{L}$-coalgebra: Laat $\gamma_{L_1}(x_i) = \{(a, x_j) \mid (x_i, a, x_j) \in L\}$. Dan is

$$\gamma_{L_1} : X \to \mathbb{P}X : \begin{cases} x_1 \mapsto \{(a_2, x_2), (a_4, x_3)\} \\ x_2 \mapsto \{(a_1, x_1), (a_3, x_3)\} \\ x_3 \mapsto \varnothing \end{cases}$$

en dan kunnen we uit het paar (X, γ_{L_1}) het gelabelde transitie systeem (X, L) achterhalen. In het algemeen corresponderen gelabelde transitie systemen precies met $\mathbb{L}$-coalgebras.

Deze scriptie Coalgebras zoals hierboven, maar in een meer algemene vorm, vormen de basis van deze thesis. We definiëren logica op coalgebras en bestuderen eigenschappen van verschillende klassen coalgebras.

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Index

A

Appendix

A.1 Notation

Throughout this thesis, Φ denotes a denumerable set of proposition letters. Also we have number of notational conventions. Sets come in many forms and, when the setting is not yet set, are written as capital letters X, X', Y. Elements of a set are denoted by lower case letters x, x', y, u. Subsets of a given set X are written in lower case calligraphy, a, b. Collections of subsets of a given set are written as W, W', V. Topological spaces are indicated by bold upper case calligraphic letters, $\mathcal{X}, \mathcal{X}'$, $\mathcal{Y}$ and their underlying sets by the upper case normal font of the letter.

Categories are denoted in $\mathsf{this\ font}$. Usually C stands for an arbitrary category. We fix some categories:

- Set is the category of sets and functions;
- Top is the category of topological spaces and continuous functions;
- $\mathsf{Top}_0, \mathsf{Sob}, \mathsf{KTop}, \mathsf{KSob}, \mathsf{KHaus}$ and Stone are the full subcategories of Top with as objects T_0-spaces, sober spaces, compact spaces, compact sober spaces, compact Hausdorff spaces and Stone spaces respectively;
- BA is the category of Boolean algebras and Boolean algebra homomorphisms;
- Frm is category of frames and frame homomorphisms;
- $\mathsf{SFrm}, \mathsf{KFrm}, \mathsf{KSFrm}$ and KRFrm are the full subcategories of Frm with objects spatial frames, compact frames, compact spatial frames and compact regular frames respectively.

More categories are defined as we go.

Functors are usually denoted by a blackboard font letter, $\mathbb{A}, \mathbb{B}, \mathbb{C}$. An arbitrary functor is written as $\mathbb{T}$ and its the domain and range should be clear from the context. The following functors will be used throughout the thesis:

- $\mathbb{U} : \mathsf{Top} \to \mathsf{Set}$ is the forgetful functor sending a topological space to its underlying set;
- $\mathbb{Z} : \mathsf{Frm} \to \mathsf{Set}$ is the forgetful functor sending a frame to its carrier in Set, in particular this restricts to a functor $\mathsf{BA} \to \mathsf{Set}$;
- $\mathbb{P} : \mathsf{Set} \to \mathsf{Set}$ is the covariant powerset functor;

- $\mathbb{Q} : \mathsf{Set} \to \mathsf{BA}$ is the contravariant functor sending a set to its powerset Boolean algebra and a function on Set to its inverse image.

Some functors do not adhere to the rule of being a blackboard letter:

- $\mathfrak{uf} : \mathsf{BA} \to \mathsf{Stone}$ is the contravariant functor which sends a Boolean algebra B to the Stone space of ultrafilters topologised by $\{\widetilde{a} \mid a \in B\}$, where $\widetilde{a} = \{u \in \mathfrak{uf} B \mid a \in u\}$, and a homomorphism $f : B \to B'$ to $\mathfrak{uf} f : \mathfrak{uf} B' \to \mathfrak{uf} B$ defined by $(\mathfrak{uf} f)(u') = \{f^{-1}(a') \mid a' \in u'\}$;
- $\mathfrak{clp} : \mathsf{Stone} \to \mathsf{BA}$ is the contravariant functor which sends a Stone space to its Boolean algebra of clopen sets and a continuous function f to f^{-1};
- $\mathfrak{pt} : \mathsf{Frm} \to \mathsf{Top}$ is the contravariant functor which sends a frame to its space of points, this is defined in definition 3.13;
- $\mathfrak{opn} : \mathsf{Top} \to \mathsf{Frm}$ is the contravariant functor which sends a topological space to its frame of open sets and a continuous map f to f^{-1}, see definition 3.6.

The functors $\mathfrak{uf}$ and $\mathfrak{clp}$ constitute Stone duality; the functors $\mathfrak{pt}$ and $\mathfrak{opn}$ constitute a duality between sober spaces and spatial frames. Furthermore we have the abbreviations

- $\breve{\mathbb{P}} = \mathbb{Z} \circ \mathbb{Q} : \mathsf{Set} \to \mathsf{Set}$, known as the contravariant powerset functor;
- $\mathbb{C}\mathrm{lp} = \mathbb{Z} \circ \mathfrak{clp} : \mathsf{Stone} \to \mathsf{Set}$ sends a Stone space to its set of clopen subsets;
- $\Omega = \mathbb{Z} \circ \mathfrak{opn} : \mathsf{Top} \to \mathsf{Set}$ sends a topological space to its set of open subsets.

Along the way, more functors will be defined when required.

We often use diamonds and boxes. When we use these to as formal symbols who generate a frame we use an "empty" box and diamond, $\square$ and $\diamond$. When using boxes and diamonds to define open or clopen sets in some topological space, we use "crossed" boxes and diamonds, $\boxplus$ and $\diamondplus$.

Sometimes we have split up long proofs into several claims. The claims are then numbered with letters within the proposition or theorem. The proof of such a claim is closed by a diamond, $\diamond$, instead of a box, in order to differentiate between to proof and it subproofs.

A.2 The monotone functor on Frm

Recall that the monotone functor $\mathbb{M} : \mathsf{Frm} \to \mathsf{Frm}$ is defined as follows: For a frame F the frame $\mathbb{M}F$ is the frame generated by the set $\mathbb{M}F = \{\square a, \diamond a \ (a \in F)\}$ subject to the relations

(M1) $\square(a \wedge b) \leq \square a$

(M2) $\diamond a \leq \diamond(a \vee b)$

(M3) $\square a \wedge \diamond b \leq 0$ whenever $a \wedge b \leq 0$

(M4) $\square a \vee \diamond b \geq 1$ whenever $a \vee b \geq 1$

(M5) $\square \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\square a \mid a \in A\}$

(M6) $\diamond \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\diamond a \mid a \in A\}$,

where $a, b \in F$ and A is a directed subset of F. For a homomorphism $f : F \to F'$ define $\mathbb{M}f : \mathbb{M}F \to \mathbb{M}F'$ on generators by

$$\square a \mapsto \square f(a), \quad \diamond a \mapsto \diamond f(a).$$

We have seen that $\mathbb{M}$ preserves regularity in proposition 3.40. Via the duality from theorem 3.41 it follows from lemma 3.38 that $\mathbb{M}$ preserves compactness as well. But we can also prove this fact without using the duality to move the problem to the topological side. Before we prove this, we give an equivalent definition of $\mathbb{M}F$, which is inspired by theorem 4.2 from [57]. We then show that this equivalent frame is compact whenever F is. The proof is similar to the proof of theorem 4.2 from [57].

A.1 Definition. Let D be a frame. Define

$$\begin{aligned}
\mathbb{M}'D := \mathrm{Fr}\langle \wp_\omega D \times \wp_\omega D \ \ &(\text{qua } \vee\text{-semilattice}) \ \mid \\
(i') \quad & (\gamma \cup \{a \wedge b\}, \delta) \leq (\gamma \cup \{a\}, \delta) \\
(ii') \quad & (\gamma, \delta \cup \{a\}) \leq (\gamma, \delta \cup \{a \vee b\}) \\
(iii') \quad & (\gamma \cup \{a\}, \delta) \wedge (\gamma, \delta \cup \{b\}) \leq (\gamma, \delta) \quad \text{if } a \wedge b = 0 \\
(iv') \quad & \top \leq (\gamma \cup \{a\}, \delta \cup \{b\}) \qquad \text{if } a \vee b = 1 \\
(v') \quad & (\gamma \cup \{\textstyle\bigvee^{\uparrow} A\}, \delta) \leq \textstyle\bigvee^{\uparrow}_{a \in A} (\gamma \cup \{a\}, \delta) \\
(vi') \quad & (\gamma, \{\textstyle\bigvee^{\uparrow} A\} \cup \delta) \leq \textstyle\bigvee^{\uparrow}_{a \in A} (\gamma, \{a\} \cup \delta) \\
& \rangle.
\end{aligned}$$

Here we use $\top$ to denote the top element of $\mathbb{M}'D$ and 1 for the top element of D. The join structure is given by $(\gamma, \delta) \vee (\gamma', \delta') = (\gamma \cup \gamma', \delta \cup \delta')$. ◁

A.2 Lemma. *Let D be a frame. Then $\mathbb{M}D \cong \mathbb{M}'D$.*

Proof. Define

$$\mu : \mathbb{M}D \to \widehat{\mathbb{M}}D : \begin{cases} \Box a \mapsto (\{a\}, \varnothing) \\ \Diamond a \mapsto (\varnothing, \{a\}) \end{cases}$$

and

$$\eta : \widehat{\mathbb{M}}D \to \mathbb{M}D : (\gamma, \delta) \mapsto \bigvee_{c \in \gamma} \Box c \vee \bigvee_{d \in \delta} \Diamond d.$$

These maps obviously give a bijection. We will show that both assignments preserve the relations from definition 3.39 and A.1, wherefore they can be lifted to frame homomorphisms. Moreover, both maps are frame homomorphisms.

The map μ is a homomorphism because it is defined on the generators and is well-defined for it preserves the relations from definition 3.39.

We first show that the assignment μ preserves (i) – (vi).

(i) First (i),

$$\mu(\Box(a \wedge b)) = (\{a \wedge b\}, \varnothing) \overset{(i')}{\leq} (\{a\}, \varnothing) = \mu(\Box a).$$

(ii) Second,

$$\mu(\Diamond a) = (\varnothing, \{a\}) \overset{(ii')}{\leq} (\varnothing, \{a \vee b\}) = \mu(\Diamond(a \vee b)).$$

(iii) For (iii), suppose $a \wedge b = 0$, then

$$\mu(\Box a \wedge \Diamond b) = (\{a\}, \varnothing) \wedge (\varnothing, \{b\}) \overset{(iii')}{\leq} (\varnothing, \varnothing) = \mu(0).$$

(iv) Now suppose $a \vee b = 1$, then

$$\mu(1) = \top \overset{(iv')}{\leq} = (\{a\}, \varnothing) \vee (\varnothing, \{b\}) = \mu(\Box a) \vee \mu(\Diamond b) = \mu(\Box a \vee \Diamond b).$$

(v) For (v), first observe

$$\mu(\Box {\textstyle\bigvee^{\uparrow}} A) = (\{{\textstyle\bigvee^{\uparrow}} A\}, \varnothing) \overset{(v')}{\leq} \bigvee_{a \in A} (\{a\}, \varnothing) = {\textstyle\bigvee^{\uparrow}_{a \in A}} \mu(\Box a) = \mu({\textstyle\bigvee^{\uparrow}_{a \in A}} \Box a).$$

From (i') it follows that

$$(\{a\}, \varnothing) = (\{({\textstyle\bigvee^{\uparrow}} A) \wedge a\}, \varnothing) \leq (\{{\textstyle\bigvee^{\uparrow}} A\}, \varnothing)$$

so that $\mu(\Box a) \leq \mu(\Box \bigvee^{\uparrow} A)$ and hence $\mu(\bigvee^{\uparrow}_{a \in A} \Box a) = \bigvee^{\uparrow}_{a \in A} \mu(\Box a) \leq \mu(\Box \bigvee^{\uparrow} A)$.

(vi) From (vi') we get $\mu(\Diamond \bigvee^{\uparrow} A) \leq \mu(\bigvee^{\uparrow}_{a \in A} \Diamond a)$. From (ii') it follows that $(\varnothing, \{a\}) \leq (\varnothing, \bigvee^{\uparrow} A)$ so that $\mu(\Diamond a) \leq \mu(\Diamond \bigvee^{\uparrow} A)$ for all $a \in A$ and hence $\mu(\bigvee^{\uparrow}_{a \in A} \Diamond a) = \bigvee^{\uparrow}_{a \in A} \mu(\Diamond a) \leq \mu(\Diamond \bigvee^{\uparrow} A)$. This proves equality.

In order to show that η defines a frame homomorphism we need to prove that it preserves joins and relation (o') – (vi'). The preservation of joins is obvious. For (i'),

$$\begin{aligned}
\eta(\gamma \cup \{a \wedge b\}, \delta) &= \bigvee_{c \in \gamma} \Box c \vee \Box(a \wedge b) \vee \bigvee_{d \in \delta} \Diamond d \\
&\leq \bigvee_{c \in \gamma} \Box c \vee \Box a \vee \bigvee_{d \in \delta} \Diamond d \\
&= \eta(\gamma \cup \{a\}, \delta).
\end{aligned}$$

In a similar way (ii') can be treated.

For (iii'), suppose $a \wedge b = 0$, then

$$\begin{aligned}
\eta(\gamma \cup \{a\}, \delta) \wedge \eta(\gamma, \delta \cup \{b\}) &= \big(\bigvee \Box c \vee \Box a \bigvee \Diamond d\big) \wedge \big(\bigvee \Box c \vee \bigvee \Diamond d \vee \Diamond b\big) \\
&= \bigvee \Box c \vee \bigvee \Diamond d \vee (\Box a \wedge \Diamond b) \\
&\leq \bigvee \Box c \vee \bigvee \Diamond d \vee 0 \\
&= \eta(\gamma, \delta).
\end{aligned}$$

The dual notion (iv'), can be treated similarly.

For (v'),

$$\begin{aligned}
\eta(\gamma \cup \{{\textstyle\bigvee^{\uparrow}} A\}, \delta) &= \bigvee_{c \in \gamma} \Box c \vee \Box {\textstyle\bigvee^{\uparrow}} A \vee \bigvee_{d \in \delta} \Diamond d \\
&= \bigvee_{c \in \gamma} \Box c \vee {\textstyle\bigvee^{\uparrow}_{a \in A}} \Box a \vee \bigvee_{d \in \delta} \Diamond d \\
&= {\textstyle\bigvee^{\uparrow}_{a \in A}} \big(\bigvee \Box c \vee \Box a \vee \bigvee \Diamond d\big) \\
&= {\textstyle\bigvee^{\uparrow}_{a \in A}} (\gamma \cup \{a\}, \delta).
\end{aligned}$$

Lastly (vi') is similar. □

A.3 Theorem. *Suppose D is compact. Then $\mathbb{M}D$ is compact.*

Proof. The frame $\mathbb{M}D$ is compact iff there is a preframe homomorphism $\varphi : \mathbb{M}D \to 2$ that is right adjoint to the unique frame homomorphism $! : 2 \to \mathbb{M}D$ (cf. [57, Theorem 4.2]).

By proposition A.2 $\mathbb{M}D \cong \mathbb{M}'D$, so we may work with the latter. Because all the relations in definition A.1 are join stable, we can use the preframe coverage theorem (theorem 5.1 in [32]) to find

$$\mathbb{M}'D \cong \mathrm{PreFr}\langle\, \wp_\omega D \times \wp_\omega D \ \text{ (qua poset) } \mid \text{ same relations as definition A.1 } \rangle.$$

Define $\varphi : \mathbb{M}'D \to 2$ by

$$\varphi(\gamma, \delta) = \begin{cases} 1 & \text{iff there are } c \in \gamma \text{ such that } c \vee (\bigvee \delta) = 1 \\ 0 & \text{otherwise} \end{cases} .$$

We need to check that φ is indeed a pre-frame homomorphism. Since φ is defined on generators, it suffices to show that it preservers the relations (i') – (vi'), because if it does it can be lifted in a unique way to a frame homomorphism $\mathbb{M}'D \to 2$. We check that φ preserves the relations one by one.

(i') Suppose $\varphi(\gamma \cup \{a\}, \delta) = 0$, then $c \vee \bigvee \delta = 0$ for all $c \in \gamma$ and $(a \wedge b) \vee \bigvee \delta \leq a \vee \bigvee \delta = 0$.

(ii') If $\varphi(\gamma, \delta \cup \{a\}) = 1$, then $c \vee \bigvee(\delta \cup \{a\}) = 1$ for some $c \in \gamma$, so $c \vee \bigvee(\delta \cup \{a \vee b\}) \geq c \vee \bigvee(\delta \cup \{a\}) = 1$.

(iii') Suppose $\varphi(\gamma \cup \{a\}, \delta) = 1$ and $\varphi(\gamma, \delta \cup \{b\}) = 1$. Then either there is some $c \in \gamma$ such that $c \vee \bigvee \delta = 1$, which implies $\varphi(\gamma, \delta) = 1$, or $a \vee \bigvee \delta = 1$. In the latter case, note that we also have some $c' \in \gamma$ such that $c' \vee \bigvee \delta \vee b = 1$, so that

$$c' \vee \bigvee \delta = c' \vee \bigvee \delta \vee (a \wedge b) = (a \vee \bigvee \delta \vee c') \wedge (c' \vee \bigvee \delta \vee b) = 1 \wedge 1 = 1.$$

The first equality holds because $a \wedge b = 0$. Again we find $\varphi(\gamma, \delta) = 1$.

(iv') If $a \vee b = 1$, then $a \vee \bigvee(\delta \cup \{b\}) = 1$ so $\varphi(\gamma \cup \{a\}, \delta \cup \{b\}) = 1$.

(v') Suppose $\varphi(\gamma \cup \{\bigvee^{\uparrow} A\}, \delta) = 1$, then either $c \vee (\bigvee \delta) = 1$ for some $c \in \gamma$, or $1 = (\bigvee^{\uparrow} A) \vee (\bigvee \delta) = \bigvee^{\uparrow}_{a \in A}(a \vee (\bigvee \delta))$ (note that the latter is indeed a directed set, because A is). By compactness of D this gives $a \vee (\bigvee \delta) = 1$ for some $a \in A$. So both cases yield $\varphi\big(\bigvee^{\uparrow}_{a \in A}(\gamma \cup \{a\}, \delta)\big) = 1$.

(vi') Suppose $\varphi(\gamma, \{\bigvee^{\uparrow} A\} \cup \delta) = 1$, then, for some $c \in \gamma$, we have

$$1 = c \vee \bigvee(\{\bigvee^{\uparrow} A\} \cup \delta) = \bigvee^{\uparrow}(c \vee a \vee \bigvee \delta)$$

and by compactness we must have $c \vee \bigvee^{\uparrow}(\{a\} \cup \delta) = 1$ for one of the a. (The set $\{c \vee a \vee \bigvee \delta \mid a \in A\}$ is directed and by (ii').)

Lastly, we need to check that φ is right-adjoint to $! : 2 \to \mathbb{M}'L$ (defined by $1 \mapsto \top = (1, 1)$ (note that $(1, 1)$ is in the equivalence class of $\top$), and $0 \mapsto (\varnothing, \varnothing)$). It suffices to show that $\varphi(!(p)) \geq p$ and $!(\varphi(\gamma, \delta)) \leq (\gamma, \delta)$. For the first, suppose $p = 1$, then $!(p)$ is the equivalence class of $(1, 1)$ and $\varphi(!(p)) = 1$. For the second, if $\varphi(\gamma, \delta) = 1$, then there are $c \in \gamma$ such that $c \vee (\bigvee \delta) = 1$ (in particular $\delta \neq \varnothing$) and hence

$$1 = (\{1\}, \delta) = (\{c \vee (\bigvee \delta)\}, \delta) \leq (\{c\}, \delta) \leq (\gamma, \delta).$$

The first inequality follows from recalling that δ is a finite set and applying (vi') repeatedly. ☐

Proposition 3.40 and theorem A.3 combined yield the following result.

A.4 Corollary. *The functor* $\mathbb{M}$ *on* Frm *preserves compactness.*

A.3 For reference

A.5 Lemma. *Let* C *and* D *be categories and* $\mathbb{I} : \mathsf{C} \to \mathsf{D}$*,* $\mathbb{J} : \mathsf{D} \to \mathsf{C}$ *two functors that constitute a dual equivalence between* C *and* D*. Let* $\mathbb{S} : \mathsf{C} \to \mathsf{C}$ *and* $\mathbb{T} : \mathsf{D} \to \mathsf{D}$ *be endofunctors. The following are equivalent*

(i) there is a natural isomorphism $\eta : \mathbb{S} \circ \mathbb{J} \to \mathbb{J} \circ \mathbb{T}$*;*

(ii) there is a natural isomorphism $\mu : \mathbb{T} \circ \mathbb{I} \to \mathbb{I} \circ \mathbb{S}$*.*

Proof. (i) to (ii): define μ by

$$\mathbb{T}\mathbb{I} \xrightarrow{\imath_{\mathbb{T}\mathbb{I}}} \mathbb{I}\mathbb{J}\mathbb{T}\mathbb{I} \xrightarrow{\mathbb{I}\eta_{\mathbb{I}}} \mathbb{I}\mathbb{S}\mathbb{J}\mathbb{I} \xrightarrow{\mathbb{I}\mathbb{S}\epsilon} \mathbb{I}\mathbb{S}.$$

These are all natural isomorphisms. (ii) to (i) is similar. □

A.6 Lemma. *Let* C *and* D *be categories and* $\mathbb{I} : \mathsf{C} \to \mathsf{D}$, $\mathbb{J} : \mathsf{D} \to \mathsf{C}$ *two contravariant functors that constitute a dual equivalence between* C *and* D*. Let* $\epsilon : \mathrm{Id} \to \mathbb{J} \circ \mathbb{I}$ *and* $\imath : \mathrm{Id} \to \mathbb{I} \circ \mathbb{J}$ *be the corresponding natural isomorphisms. Let* $\mathbb{S} : \mathsf{C} \to \mathsf{C}$ *and* $\mathbb{T} : \mathsf{D} \to \mathsf{D}$ *be endofunctors. Suppose there is is a natural isomorphism* $\eta : \mathbb{S} \circ \mathbb{J} \to \mathbb{J} \circ \mathbb{T}$*. Then there is a dual equivalence*

$$\mathsf{Alg}(\mathbb{S}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathbb{T}).$$

Proof. We define two functors $\mathbb{X} : \mathsf{Alg}(\mathbb{S}) \to \mathsf{Coalg}(\mathbb{T})$ and $\mathbb{Y} : \mathsf{Coalg}(\mathbb{T}) \to \mathsf{Alg}(\mathbb{S})$ and we will give two natural isomorhisms $\mathrm{Id}_{\mathsf{Alg}(\mathbb{S})} \cong \mathbb{Y} \circ \mathbb{X}$ and $\mathrm{Id}_{\mathsf{Coalg}(\mathbb{T})} \cong \mathbb{X} \circ \mathbb{Y}$.

Definition of $\mathbb{X}$*.* By lemma A.5 we get a natural isomorphism $\mu : \mathbb{T} \circ \mathbb{I} \to \mathbb{I} \circ \mathbb{S}$.

Let $\delta : \mathbb{S}X \to X$ be an $\mathbb{S}$-algebra. Applying $\mathbb{I}$ gives $\mathbb{I}\delta : \mathbb{I}X \to \mathbb{I}\mathbb{S}X$ and composition with μ_X^{-1} gives

$$\mathbb{I}X \xrightarrow{\mathbb{I}\delta} \mathbb{I}\mathbb{S}X \xrightarrow{\mu_X^{-1}} \mathbb{T}\mathbb{I}X. \tag{A.1}$$

So if we put $\gamma_\delta := \mu_X^{-1} \circ \mathbb{I}\delta$ the pair $(\mathbb{I}X, \gamma_\delta)$ is a $\mathbb{T}$-coalgebra. Define $\mathbb{X}(X, \delta) := (\mathbb{I}X, \gamma_\delta)$. Let $f : X \to X'$ be a $\mathbb{S}$-algebra morphism from (X, δ) to (X', δ'). The the following diagram commutes,

$$\begin{array}{ccc} \mathbb{S}X & \xrightarrow{\mathbb{S}f} & \mathbb{S}X' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ X & \xrightarrow[f]{} & X' \end{array} \tag{A.2}$$

Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathbb{I}X & \xleftarrow{\mathbb{I}f} & \mathbb{I}X' \\ {\scriptstyle\mathbb{I}\delta}\downarrow & & \downarrow{\scriptstyle\mathbb{I}\delta'} \\ \mathbb{I}\mathbb{S}X & \xleftarrow{\mathbb{I}\mathbb{S}f} & \mathbb{I}\mathbb{S}X' \\ {\scriptstyle\mu_X^{-1}}\downarrow & & \downarrow{\scriptstyle\mu_{X'}^{-1}} \\ \mathbb{T}\mathbb{I}X & \xleftarrow[\mathbb{T}\mathbb{I}f]{} & \mathbb{T}\mathbb{I}X \end{array}$$

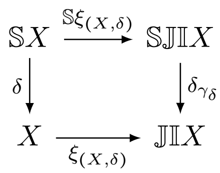
Commutativity of the upper square follows from applying $\mathbb{I}$ to the diagram in (A.2) and commutativity of the lower square follows from the fact that μ is a natural isomorphism.

This shows that $\mathbb{I}f$ is a $\mathbb{T}$-coalgebra morphism from $(\mathbb{I}X, \gamma_\delta)$ to $(\mathbb{I}X', \gamma_{\delta'})$. Define $\mathbb{X}f := \mathbb{I}f$.

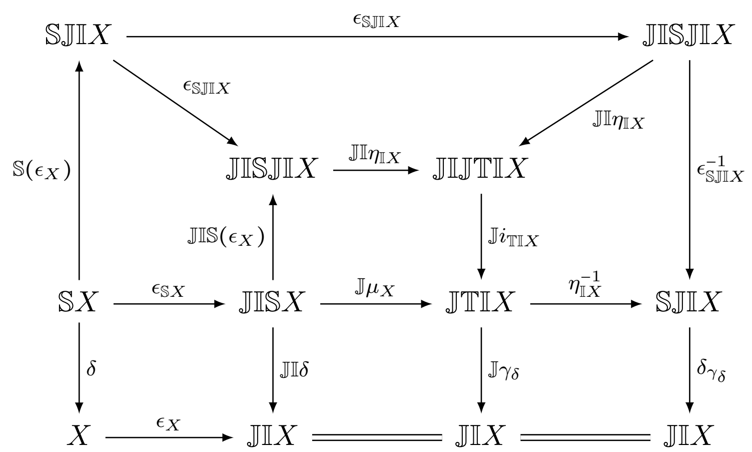
Definition of $\mathbb{Y}$. Define $\mathbb{Y} : \mathsf{Coalg}(\mathbb{T}) \to \mathsf{Alg}(\mathbb{S})$ in a similar way, where $\mathbb{Y}(X, \gamma) = (\mathbb{J}X, \delta_\gamma)$ and $\mathbb{Y}f = \mathbb{J}f$. Here δ_γ is the composition

$$\mathbb{S}\mathbb{J}X \xrightarrow{\eta_X} \mathbb{J}\mathbb{T}X \xrightarrow{\mathbb{J}\gamma} \mathbb{J}X. \tag{A.3}$$

The natural isomorphism $\xi : \mathrm{Id}_{\mathsf{Alg}(\mathbb{S})} \to \mathbb{Y} \circ \mathbb{X}$. We need an algebra isomorphism $\xi_{(X,\delta)} : (X, \delta) \to \mathbb{Y} \circ \mathbb{X}(X, \delta) = (\mathbb{J}\mathbb{I}X, \delta_{\gamma_\delta})$. That is, the following diagram must commute



Let $\xi_{(X,\delta)} := \epsilon_X$. We claim that with this definition the diagram above commutes. Obviously this yields an isomorphism. For commutativity:



Since $\mathbb{I}$ and $\mathbb{J}$ form an adjunction, we have $\mathrm{Id}_{\mathbb{J}} = \epsilon_{\mathbb{J}}^{-1} \circ \mathbb{J}\imath^{-1}$ (see e.g. [40] section IV.1 theorem 1), so $\epsilon_{\mathbb{J}}^{-1} = \mathbb{J}\imath$.

The upper square commutes trivially. The middle square commutes by definition of μ. The squares left and right of the middle square commute because ϵ is a natural transformation, as does the left lower square. For the middle right square, recall $\epsilon_J^{-1} = \mathbb{J}\imath$. The middle and right lower squares commutes by definition of γ_δ and δ_γ. It follows that

$$\epsilon_X \circ \delta = \delta_{\gamma_\delta} \circ \mathbb{S}(\epsilon_X),$$

as desired. This gives the natural isomorphism $\mathrm{Id}_{\mathsf{Alg}(\mathbb{S})} \to \mathbb{Y} \circ \mathbb{X}$.

The natural isomorphism $\zeta : \mathrm{Id}_{\mathsf{Coalg}(\mathbb{T})} \to \mathbb{X} \circ \mathbb{Y}$. This can be treated similar to ξ.

Conclusion. This proves that there is a dual equivalence $\mathsf{Alg}(\mathbb{S}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathbb{T})$. □

As we often work with Stone duality and Isbell duality, the previous lemma provides a powerful tool.

A.7 Lemma. *(i) Pullbacks in* Sob *are computed as in* Top.

(ii) Pullbacks in KHaus *are computed as in* Top.

Proof. (i) Let

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xleftarrow{f'} \mathcal{X}' \tag{A.4}$$

be a cospan in Sob. The pullback of this diagram in Top is $\mathcal{P} = \{(u, u') \in \mathcal{X} \times \mathcal{X}' \mid f(u) = f'(u')\}$ viewed as a subspace of $\mathcal{X} \times \mathcal{X}'$. Let $\pi : \mathcal{P} \to \mathcal{X}$ and $\pi' : \mathcal{P} \to \mathcal{X}'$ be projections. Since dual equivalences send limits to colimits and vice versa the diagram

$$\begin{array}{ccc} \mathfrak{pt}(\mathfrak{opn}\,\mathcal{X}) & \xrightarrow{\mathfrak{pt}(\mathfrak{opn}\,\pi')} & \mathfrak{pt}(\mathfrak{opn}\,\mathcal{X}') \\ {\scriptstyle \mathfrak{pt}(\mathfrak{opn}\,\pi)}\downarrow & & \downarrow{\scriptstyle \mathfrak{pt}(\mathfrak{opn}\,f')} \\ \mathfrak{pt}(\mathfrak{opn}\,\mathcal{X}) & \xrightarrow[\mathfrak{pt}(\mathfrak{opn}\,f)]{} & \mathfrak{pt}(\mathfrak{opn}\,\mathcal{Y}) \end{array}$$

is a pullback diagram in Top. By proposition 3.14 the cospan

$$\mathfrak{pt}(\mathfrak{opn}\,\mathcal{X}) \xrightarrow{\mathfrak{pt}(\mathfrak{opn}\,f)} \mathfrak{pt}(\mathfrak{opn}\,\mathcal{Y}) \xleftarrow{\mathfrak{pt}(\mathfrak{opn}\,f')} \mathfrak{pt}(\mathfrak{opn}\,\mathcal{X}')$$

is isomorphic to the cospan in (A.4), hence $\mathfrak{pt}(\mathfrak{opn}\,\mathcal{P})$ is also a pullback of the cospan in (A.4). Since pullbacks are unique up to isomorphism $\mathcal{P} \cong \mathfrak{pt}(\mathfrak{opn}\,\mathcal{P})$. By definition 3.13 the latter is sober, therefore $\mathcal{P}$ is sober.

(ii) Suppose the diagram of (A.4) is in KHaus and we take the pullback $\mathcal{P}$ in Top. The product of two Hausdorff spaces is a Hausdorff space and the subspace of a Hausdorff space is Hausdorff, so $\mathcal{P}$ is Hausdorff. Suppose $(u, u') \in \mathcal{X} \times \mathcal{X}'$ is not in $\mathcal{P}$. Then $f(u) \neq f'(u')$ in $\mathcal{Y}$ so there exist disjoint open sets a, a' in $\mathcal{Y}$ containing u and u' respectively. Then $a \times a'$ is open in $\mathcal{X} \times \mathcal{X}'$ and disjoint from $\mathcal{P}$. Therefore $\mathcal{P}$ is closed. As the product of two compact sets is again compact and closed subsets of compact sets are compact, $\mathcal{P}$ is compact. □

A.8 Lemma. *Let* $f : A \to B$ *be a Boolean algebra homomorphism. Then* f *is injective if and only if* $f^{-1}(\top_B) = \{\top_A\}$.