Economic Scenario Generators

Mark Plomp

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Supervisors:
Yulia Bondarouk (KPMG)
Jeroen Gielen (KPMG)
Peter Spreij (UvA)
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Abstract
In 2009 the European Union agreed upon introducing a new regulatory framework, called Solvency II, for insurance companies operating in the EU. One of the requirements of the Solvency II directive states that the valuation of assets and liabilities needs to be market consistent. For most insurance companies the only practical method to achieve a market consistent valuation is using a so called economic scenario generator (ESG). An ESG generates future scenarios for different risk factors by Monte Carlo simulation of stochastic models corresponding to these risk factors.

In this thesis the construction and use of a market consistent economic scenario generator is investigated. The main question under investigation is what the sensitivities are in calibrating the ESG and simulating the future scenarios and what their impact is on the estimation of the market consistent value. Four risk factors are modelled for the ESG: interest rates, stocks, real estate and inflation rates. The Hull-White one factor model is used for the modelling of interest rates; stocks and real estate are modelled by the Black-Scholes-Hull-White model and the inflation rates are modelled with the Vasicek model. First on the basis of available literature the stochastic models are discussed and option pricing formulas corresponding to these models are derived. Subsequently the calibration of the stochastic models to market data is discussed; matching the market prices of options to the option pricing formulas derived in the first part is one of the main parts of the calibration. Afterwards, for the analysis of the ESG, two types of insurance policies are modelled. The analysis includes: the impact of the characteristics of the policies on the value, the calibration to different market data, estimation errors due to the Monte Carlo simulation and the impact of changes in the way the models are calibrated.
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Introduction

The goal of this thesis is to discuss the construction and use of a market consistent economic scenario generator (ESG) by insurance companies. A market consistent ESG consists of various stochastic models and produces risk-neutral future scenarios. It is a tool for insurers with which they can calculate the present value of their assets and liabilities, also called technical provisions. In this thesis we will build our own ESG, which will be programmed in Matlab.

The construction of an ESG requires one to make a choice of asset classes (or risk factors) to be modelled, which stochastic models we want to use to simulate these classes and how these models are going to be calibrated. Subsequently, economic scenarios can be generated for the various asset classes and used to calculate the technical provisions for various insurance products.

Typically many risk factors play a role in the asset management of an insurer; for example, interest rates, inflation, credit spreads, equity movements, real estate prices and exchange rates. However, to control the complexity of the model not all of these factors will be modelled. The most important assets in an insurance company’s portfolio are generally bonds, stocks and real estate. Also, inflation can be an important factor for the liabilities of an insurer, because often guarantees embedded in insurance contracts are indexed yearly with the inflation rate. Therefore, we restrict ourselves to modelling these four risk factors, where bonds will be modelled by modelling interest rates.

During the last fifty years a large variety of stochastic models has been developed for the three asset classes. Here the choice is made to use the Hull-White one factor model for the stochastic interest rates, the Hull-White-Black-Scholes model for equity and real estate and the Vasiček model for inflation rates. This choice was made to get realistic results while controlling the complexity of the ESG.

The Hull-White model is chosen because it can give an exact fit to the current term structure, we can derive analytical formulas for interest rate derivatives, like swaptions, which makes it easier to calibrate the model accurately to market data and it is fairly easy to use the equity model under Hull-White interest rates. Moreover, for options the implied volatilities are higher for out-of-the-money
strikes, which is consistent with the volatility skew/smiles implied from market prices of floors and swaptions [26]. This is a very desirable feature of the model since interest rate options embedded in an insurance product are usually written on out-of-the-money strikes.

For equity the Hull-White-Black-Scholes model is chosen because it is easy to use simultaneously with the Hull-White model and it is analytically tractable, which makes calibration to options easy.

The choice for the Vasićek model for inflation rates was made, because it is easy to handle and shows mean reverting behaviour, which is a desirable feature for inflation.

Another analytically tractable interest rate model which can be fitted to the term structure is the CIR++ model [5]. However, the equity model loses its analytical tractability under the CIR++ model. Therefore, the Hull-White model is chosen over the CIR++ model.

As will be discussed in more depth in Chapter 1, it is preferred that the calibration of the model is done with liquidly traded financial instruments if possible. Therefore the interest model and the equity model will be calibrated to swaptions resp. (European) call options. For real estate and inflation, no liquidly traded options or other derivatives are available, therefore we have to resort to historical data to calibrate the models. For real estate we will calibrate the model based on the historical values of a so called appraisal index and for inflation the calibration will be done based on historical euro zone inflation rates.

Two types of insurance contracts will be modelled and valued in this thesis, both of which are life insurance contracts in which at least a guaranteed sum is paid at time of maturity, e.g. at retirement age.

The first contract we will model is called a surplus interest sharing policy. For this policy the return on a portfolio of government bonds is compared to a reference yield. If the return on the portfolio during a period of one year is higher than the reference yield, then part of the excess return is shared with the policy holder resulting in a higher sum of money at time of maturity.

The second contract is called an endowment policy. For this contract the policy holder can choose a fund to invest in, e.g. a mixed fund with bonds, stock and real estate. If at maturity the fund is worth more than the guaranteed sum, then part of the profits are shared with the policy holder.

Under Solvency II the insurance companies are obliged to calculate their technical provisions in a market consistent way; this will be discussed in more depth in Chapter 1. Doing these calculations by using an Economic Scenario Generator had become very popular in the insurance industry [44]. Hence, it is very important for insurance companies, regulators and other parties like KPMG involved with insurance companies, to understand how these models work and
what choices can be made when using a scenario generator. Therefore, the main question about the ESG we develop in this thesis is

*What are the sensitivities in calibrating the Economic Scenario Generator and simulating future scenarios and what is the possible impact of these sensitivities on the outcome of the technical provisions?*

It goes without saying that changes in the calibration and simulation methods have to fall within the market consistent framework. However, often markets aren’t completely liquid and some assumptions have to be made when calibrating the models, because of this we can investigate what reasonable changes we can make to these assumptions and what impact they have on the technical provisions.

**Layout of the thesis**

In Part I the background behind the ESG is discussed. Chapter 1 discusses new regulations for insurance companies, called Solvency II, are discussed, explains in more depth what an ESG is and gives reasons for the use of ESG’s by insurers. Chapter 2 introduces some mathematical concepts which we will use throughout this thesis and will provide a mathematical basis upon which we will build our stochastic models.

Part II covers the mathematical theory behind the models and the option pricing formulas we use in the ESG. This will be done based on literature, which will be referred to during the chapters.

Part III is where the programmed ESG will be analysed and discussed. Chapter 8 will discuss the way in which the ESG is calibrated and provides examples of the outcome of the calibration. In Chapter 10 we will have a look at the technical provisions corresponding to the modelled insurance products, which are explained in Chapter 9, and in what way they are influenced by changing underlying conditions. Finally in Chapter 11 a summary of the thesis is given, the results are discussed and we look at possible improvements of the ESG.
Part I

Background
Chapter 1

General background

1.1 Solvency II

In the wake of the subprime mortgage crisis that started in 2007, in 2009 the Economic and Financial Affairs Council, in which the Economic and Finance ministers of the European Union are seated, approved the Solvency II Directive [44]. Solvency II is a new regulatory framework for the European insurance industry. The main aims of which are to enhance policyholders protection, the stability of the financial system and to unify the insurance market in the EU as a whole, by establishing harmonised solvency requirements across all member states.

In addition, the Solvency II Directive is developed to reflect new and improved risk management standards to define capital requirements and to manage risks and it is partially a response to the previous market turmoil during the financial crisis of 2007-2008.

The European Insurance and Occupational Pensions Authority (EIOPA) has defined three pillars as a way of grouping the Solvency II requirements. Pillar 2 contains the requirements for good governance and risk management of insurers and covers the supervisory activities and powers of regulators. The centerpiece of pillar 2 is the Own Risk And Solvency Assessment (ORSA), which is an internal process the insurer has to undertake to assess the adequacy of its risk management and current and future solvency positions under normal and also under severe stress scenarios. Pillar 3 focusses on disclosure and transparency requirements.

Pillar 1 covers all the quantitative requirements that insurers must satisfy to demonstrate that they have sufficient capital resources. It requires that an insurance firm must maintain technical provisions against liabilities and it defines two capital requirements: the Minimal Capital Requirement (MCR) and the Solvency Capital Requirement (SCR). The MCR is set as a minimal level below which
financial resources should never fall. It is a trigger for regulatory intervention. The SCR is the amount of capital an insurance company should hold under Solvency II and it corresponds to the Value-at-Risk (VAR) of the own funds of an insurer set at a confidence level of 99.5% over a one year period. This can be interpreted as the requirement that only once in 200 years a company won’t have enough funds to meet their obligations.

Initially the Solvency II directive was meant to become effective January 1, 2013. However due to delays the implementation of Solvency II is postponed at least until January 1, 2014. A reason for this is that the negotiations about the Omnibus-2 guideline have come to a halt and will resume in the second half of 2013. Omnibus-2 discusses the transition measures that have to be taken for the implementation of Solvency II and the role EIOPA will have as a supervisor. Another reason is that EIOPA has issued an impact study, the Long Term Guarantee Assessment (LTGA), at the beginning of 2013, which investigates the impact of Solvency II on long term insurance products. It can be expected that following the conclusion of the LTGA further adjustments to the Solvency II framework will be made, this could lead to further postponement of the implementation.

1.2 Market Consistent Valuation

One of the requirements of the Solvency II directive is that insurance companies value their assets and liabilities in a market consistent way [44]. In this section we will look at the meaning of a market consistent valuation; we will generally follow the explanation given in [14].

Before we look at a market consistent valuation, we first have to discuss deep and liquid markets. The Bank of International Settlements defines a deep market as a market where large volume transactions are possible without (drastically) affecting the market price and a liquid market as a market where participants can execute large volume transactions rapidly with a small impact on market prices. In such a market buyers and sellers constantly trade and the observed market price emerges as a consensus opinion of the asset’s value.

It is important to note that the observed market price cannot be identified with the objective values of the traded objects. Sometimes the market can over- or undervalue an object; an example of an overvaluation is the ’dot-com’ bubble at the end of the twentieth century in which the stock value of internet companies and related companies rose rapidly, causing a large overvaluation of these companies [42].

However, market prices have some attractive properties that make them very suitable for assessing the value of an asset. Firstly, they react quickly to changes in relevant information. Secondly, the values are additive; the price of two
securities is the sum of the prices. Thirdly, the price does not depend on who or what is the buyer or the seller. And finally, at any given moment the market price is unique.

When a market is no longer deep or liquid the uniqueness property is lost. Since the number of transactions and the total volume of transactions in such a situation is low, there is uncertainty about the market price at any given point in time. In stock markets this is reflected in the form of a large bid-offer spread, which is the difference between the prices for which an asset can be sold and for which an asset can be bought.

As can be expected, not all markets are deep and liquid. In fact, the number of stocks and bonds that are liquidly traded is very small compared to the number of bonds and stocks that you can trade in. Moreover, most assets aren’t even traded in such a manner, e.g. insurance liabilities. So a priori there seems to be no market consistent value for an insurance liability, since there is no deep and liquid market for these liabilities. We could ask ourselves the question what a fair value for these liabilities is at a given point in time.

The definition for a market consistent value we will adopt here is the following from [31].

A market consistent value of an asset or liability is its market value, if it is readily traded in a deep and liquid market at the point in time that the valuation is struck. For any other asset or liability, a market consistent value is a best estimate of what its market value would have been had it been traded in a deep and liquid market.

For an insurance liability the goal of the market consistent valuation is to transfer the valuation problem into a setting where we have reliable and useful market prices.

The present value of the liabilities of an insurance company are often also referred to as the technical provisions. This is the amount of money an insurance company has to maintain to meet its expected obligations.

An insurance liability is defined by its resulting future cash flows. These cash flows can depend on claims, expenses, changes in the environment and other random events. It can even be argued that some cash flows also depend on the particular insurer holding the liability, e.g. the financial strength and profitability of a company can be part of the agreement.

Thus, to value an insurance liability the cash flows associated to it have to be determined and subsequently the cash flows have to be replicated by using cash flows of deeply traded financial instruments as good as possible. For cash flows that cannot be replicated or only partially, a risk margin has to be taken. For example a cash flow that depends on the value of the value of some underlying
portfolio of stocks and bonds can be replicated perfectly in general, while a cash flow in which mortality rates are a factor might not be entirely replicable and therefore a risk margin has to be taken.

In this thesis we concern ourselves with the replicable parts of the future cash flows.

1.3 Economic Scenario Generator

There are several methods insurers can use to calculate the technical provisions in a market consistent way. One of the most intuitive ways to replicate cash flows, using deeply traded financial instruments, might be to construct a replicating portfolio. Another efficient way would be to use a closed-form solution to calculate the market consistent value. However, in practice it turns out that both methods are often hard to implement because of technical issues, the most important of which is that most insurance contracts contain options that are path-dependent and have high dimensions [44]. Here path-dependent means that the cash flows depend on the way in which the underlying asset value moves during the lifetime of the policy, not just on what asset value is achieved at the end of the contract. High dimensionality means that the various cash flows depend on many underlying risk factors. In these cases it is often hard to find a good replicating portfolio or to derive a closed-form solution and every different insurance contract would need a different replicating portfolio resp. closed-form solution. Therefore, it is more convenient to look for other methods that are easier to implement. Currently one of the most used methods to value insurance liabilities is using a so called economic scenario generator (ESG). We will now discuss what an ESG is and why it can be used for market consistent valuations.

Using an ESG is a simulation based approach to valuation. By using Monte Carlo simulation and various stochastic models, future scenarios are generated. The aim is to model the underlying risk factors of the insurance contracts as good as possible. Subsequently, after the scenarios are generated, they can be used to backward calculate the present value of the insurance liabilities.

An essential feature of the stochastic models and subsequently the generated scenario is that they have to be risk-neutral. This is because the first fundamental theorem of asset pricing implies that if we want to price a claim in a way that avoids arbitrage then this should be done by calculating the expectation in a risk-neutral setting. In the case of a Monte Carlo simulation this means we have to take the average value of what the price of the claim would be under each individual scenario, when the scenarios are generated in a risk-neutral way.

Worthy of addressing here is the fact that even though we generate a set of scenarios in a risk-neutral way, each individual scenario can be considered to be
a real world scenario. The difference between risk-neutral scenarios and real world scenario is not the paths themselves. If a path is possible in a real world setting it is also possible in a risk-neutral setting and vice-versa. The difference is in the probability of these scenarios occurring or more correctly the distribution of the scenarios [22].

As we noted above we would like the ESG to simulate the underlying risk factors of the insurance liabilities. There is a large variety of risk factors to which the cash flows can be sensitive. For example interest rates, inflation, credit spreads, equity movements, real estate prices and exchange rates. Therefore, for an ESG to be market consistent it needs to be able to reproduce the current prices of deep and liquidly traded assets related to these risk factors, e.g. it needs to be able to reproduce the prices of options on equity as good as possible.

There a two ways in which the model can be calibrated in a market consistent way. The first way is to match the model prices as good as possible to the market prices; this is usually done in a least squares sense. The second way is to calibrate the model to historical data, e.g. historical volatility of a certain risk factor.

The former method is the preferred one where liquidly traded assets are available, since this method reproduces the current market prices best. However, sometimes prices of claims related to a specific risk factor aren’t available or the market for these claims is not deep and liquid. In these cases the latter calibration method has to be used.

Sometimes the two methods can be combined. Life insurance liabilities often stretch far into the future and claims with such long maturities generally aren’t available on the market. For example the implied volatility surface of options on equity can be extrapolated based on long time historical volatilities before calibrating the model to market prices.

1.3.1 A note on real world ESG’s

In practice not only market consistent risk-neutral economic scenario generators, but also real world scenario generators are used. The aim of a real world ESG is to produce a realistic distribution of economic scenarios that reflect the way the world is expected to evolve according to its user, for example an insurer or a regulation authority.

The underlying models can be very similar to the models generated by a market consistent ESG, but they differ in the way they are calibrated. A real world ESG is not risk-neutral and therefore includes risk premia. Often volatilities are also determined in a different way; for example based on historical volatilities of equity where option prices and the underlying implied volatilities are used in the market consistent case. This is because implied volatilities are usual higher
than the realised volatilities of returns, because banks add extra margins to the implied volatilities to cover hedging costs, the bank’s cost of capital and liquidity risk [44].

Where a market consistent ESG can just be calibrated to market prices, for the calibration of a real world ESG one needs a lot of ‘expert knowledge’. For example for the determination of the risk premia. Furthermore, as it turns out in practice, a real world ESG is not necessarily arbitrage free and therefore care needs to be taken in their use.

Currently one of the most important uses of a real world is to support the calculation of the earlier mentioned SCR, since this should reflect a 99.5% VAR of the capital of an insurer based on real world events.
Chapter 2

Technical background

This chapter provides some of the basic concepts that we will use throughout this thesis. For a more complete introduction into portfolio theory and option pricing see [4].

First, let us denote the stochastic processes, which correspond to value processes of the financial instruments we want to model, by $Y = (Y_1, ..., Y_m)$; later on in this thesis we will change the notation to reflect the type of asset we are modelling, e.g. equity will be denoted by $S$.

Let us assume we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $Y$ is defined and let the filtration $\mathbb{F} = \{\mathcal{F}_t = \sigma(Y(s) : 0 \leq s \leq t) | t \geq 0\}$ satisfy the usual conditions.

As was mentioned in Section 1.3, we want to generate risk-neutral scenarios. Therefore we want the dynamics of our models to operate under a risk-neutral measure in stead of the real world measure $\mathbb{P}$. Hence, we assume the existence of an equivalent martingale measure (EMM), or risk-neutral measure, $Q$, with respect to the numéraire $B(t)$, which is called the money account and will be defined in Definition 2.8. The dynamics of the stochastic models in this thesis will be given under the risk-neutral measure $Q$.

Portfolios and contingent claims

As we mentioned in the introduction of the thesis, we want to calibrate our models to options. To derive analytic formulas for these options we need to introduce some definitions and assumptions about the market. Before we introduce this definitions, we remind ourselves that a stochastic process $X = X(\omega, t)$ is a progressively measurable process if $\Omega \times [0, t] \ni (\omega, s) \mapsto X(\omega, s) \in \mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable for all $t \geq 0$.

**Definition 2.1.** Suppose we have an $n$-dimensional $\mathbb{F}$-adapted price process $Z = (Z_1, ..., Z_n)$. Then
• a portfolio is any $n$-dimensional progressively measurable process $h = \{h(t) : t \geq 0\}$.

• the value process $\Pi_h$ corresponding to the portfolio $h$ is given by

$$\Pi_h(t) = \sum_{i=1}^{n} h_i(t)Z_i(t).$$

• a portfolio $h$ is called self-financing if the value process $\Pi_h$ satisfies the condition

$$d\Pi_h(t) = \sum_{i=1}^{n} h_i(t)dZ_i(t).$$

One of the main assumptions throughout this thesis is that the market is arbitrage free. This is in fact implied by the existence of a risk-neutral measure $\mathbb{Q}$, which we assumed at the beginning of this technical introduction. However, it is still useful to recall the definition of an arbitrage opportunity.

**Definition 2.2.** An arbitrage opportunities in a market is a self-financing portfolio $h$ such that

$$\Pi_h(0) = 0,$$

$$\mathbb{P}(\Pi_h(T) \geq 0) = 1,$$

$$\mathbb{P}(\Pi_h(T) > 0) > 0.$$

A (European) call option $X$ is a contract that is defined on the price $Z_i$ of some underlying asset at a specified maturity time $T$ and with a strike price $K$. At time $T$ the value of the call option is given by

$$\Pi_X(T) = \max(Z_i - K, 0).$$

More general, a contract which is completely defined in terms of one or more underlying assets is called a contingent claim or derivative; the mathematical definition is given below.

**Definition 2.3.** A random variable $X$ is called a contingent claim with maturity $T$ or $T$-claim if it is $\mathcal{F}_T$-measurable. The corresponding price process is denoted by $\Pi_X(t)$ for $0 \leq t \leq T$, where $\Pi_X(T) = X \mathbb{P}$-a.s.

We would like the contingent claims to have a unique price, therefore we need the next definition of attainability and the following proposition.

14
Definition 2.4. We say that a T-claim $X$ is attainable, if there exists a self-financing portfolio $h$ such that

$$\Pi_h(T) = X, \mathbb{P}\text{-a.s.}.$$ 

In this case we call $h$ a replicating portfolio for $X$.

Proposition 2.5. Suppose that the T-claim $X$ is attainable and let $h$ be a replicating portfolio for $X$. Then the only price process $\Pi_X(t)$ which is consistent with the no arbitrage assumption is given by $\Pi_X(t) = \Pi_h(t)$. Furthermore, if $g$ is also a replicating portfolio for $X$, then $\Pi_h(t) = \Pi_g(t)$.

Proof. Suppose at some time $t$ we have $\Pi_X(t) < \Pi_h(t)$ then we can make an arbitrage by selling the portfolio $h$ and buying the claim $X$; vice-versa for $\Pi_X(t) > \Pi_h(t)$. The same argument can also be used to show that $\Pi_h(t) = \Pi_g(t)$ under the no arbitrage assumption. 

Bonds, interest rates and swaptions

The interest rate model plays an important role in the ESG and we will see various types of interest rates and assets based on interest rates during this thesis. Therefore, these are introduced here.

The most basic, to interest rates related, asset is a zero-coupon bond, which is defined as follows.

Definition 2.6. A maturity $T$ zero coupon bond or $T$-bond is a contract ensuring the holder one unit of currency, e.g. one euro, at time $T$. The price of the bond at time $t$ is denoted by $P(t,T)$, for $0 \leq t \leq T$.

Based on the definition of a zero-coupon bond we can define some different types of interest rates.

Definition 2.7. • The simple forward rate for $[T,S]$ at time $t$ is given by

$$F(t,T,S) = \frac{1}{S-T} \left( \frac{P(t,T)}{P(t,S)} - 1 \right).$$

• The simple spot rate for $[t,T]$ is

$$F(t,T) = \frac{1}{T-t} \left( \frac{1}{P(t,T)} - 1 \right).$$

• The continuously compounded forward rate for $[T,S]$ at time $t$ is given by

$$R(t,T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}.$$
• The instantaneous forward rate with maturity \( T \) at time \( t \) is defined as
\[
f(t, T) = \lim_{S \downarrow T} R(t, T, S) = -\frac{\partial \log P(t, T)}{T - t}
\]

• The (instantaneous) short rate at time \( t \) is defined by
\[
r(t) = f(t, t)
\]

With this definition we can define the money account \( B(t) \).

**Definition 2.8.** The money account is defined by the dynamics
\[
 dB(t) = r(t)B(t)dt, \quad B(0) = 1
\]
or equivalently
\[
 B(t) = \exp \left( \int_0^t r(u)du \right).
\]

It can be interpreted as a bank account in which the profits are continuously compounded at the prevailing short rate of interest.

Since we want to calibrate our interest rate model to swaptions, we need to define what an interest rate swap is and what a swaption is, but before we do this we will define a coupon bond.

**Definition 2.9.** A coupon bond is a contract specified by a number of future dates \( T_0 < T_1 < T_2 < \ldots < T_n \), a sequence of coupons \( c_1, \ldots, c_n \) and a nominal value \( N \), such that the owner receives \( c_i \) at \( T_i \) for \( i = 1, \ldots, n \) and receives \( N \) at time \( T_n \). The coupons can either be some fixed amount or floating, in which case the coupons are determined at \( T_i - 1 \) as \( c_i = (T_i - T_{i-1})F(T_i - 1, T_i)N \). Here \( F(T_i - 1, T_i) \) is the simple spot rate defined in Definition 2.7.

It is not hard to determine the \( t \leq T_0 \) price of \( c_i \). In the fixed coupon case we can buy \( c_i P(t, T_i) \) bonds to get a payoff of \( c_i \) at \( T_i \). Therefore the \( t \leq T_0 \) price of \( c_i \) is \( c_i P(t, T_i) \). For the floating case we first have to note that, from the definition of \( F(T_i - 1, T_i) \) we have at \( T_i \)
\[
c_i = \frac{1}{P(T_{i-1}, T_i)} - 1.
\]

At time \( t \) the value of getting \( -1 \) at \( T_i \) is \( -P(t, T_i) \). The time \( t \leq T_0 \) value of \( \frac{1}{P(T_{i-1}, T_i)} \) is \( P(t, T_{i-1}) \), since if we buy at \( t \) a \( T_{i-1} \)-bond for \( P(t, T_{i-1}) \), we get 1 paid
out at $T_{i-1}$ and we can use this to buy $\frac{1}{P(T_{i-1}, T_i)}$ $T_i$-bonds. Therefore the time $t \leq T_0$ value of $c_i$ is

$$P(t, T_{i-1}) - P(t, T_i). \tag{2.1}$$

An interest rate swap is a contract in which floating rate interest payments are exchanged for fixed rate interest payments. We can see this as an agreement between two parties in which one party has a coupon bond which pays a fixed interest rate and the other party has a coupon bond which pays a floating rate. The two parties then decide to exchange the coupons they receive for some period of time, without exchanging the underlying bonds.

**Definition 2.10.** A **payer interest rate swap** settled in arrears is a contract which is defined as follows. There are a number of future dates $T_1, T_2, ..., T_n$ with usually $T_i - T_{i-1} = \delta$, a fixed rate $K$ and a nominal value $N$. At dates $T_1, ..., T_n$ the owner of the contract pays $K\delta N$ and receives floating $F(T_{i-1}, T_i)\delta N$.

At time $T_i$ the net cash flow is equal to

$$N\delta(F(T_{i-1}, T_i) - K),$$

for which, by using the result we derived in (2.1), the $t \leq T_0$ value can be computed as

$$N(P(t, T_{i-1}) - P(t, T_i) - K\delta P(t, T_i)).$$

Now we can compute the $t \leq T_0$ value of the payer swap by summing over $i = 1, ..., n$:

$$\Pi_p(t) = N \left( P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^n P(t, T_i) \right). \tag{2.2}$$

A **receiver interest rate swap** is a swap in which the holder pays a floating rate and receives a fixed rate. Note that in this case the signs of the cash flows above are changed and as a consequence its $t \leq T_0$ value is $\Pi_r(t) = -\Pi_p(t)$.

The remaining question now is how the swap rate $K$ is determined. By definition it is chosen such that at the time the contract is written, the value of the swap equals zero.

**Definition 2.11.** The **forward swap rate** at time $t \leq T_0$ is the fixed rate which gives $\Pi_p(t) = -\Pi_r(t) = 0$ and is denoted by $R_{\text{swap}}(t)$.

We thus get

$$R_{\text{swap}}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^n P(t, T_i)}.$$
We can use this equation to rewrite (2.2) to
\[ \Pi_p(t) = N\delta(R_{swap}(t) - K) \sum_{i=1}^{n} P(t, T_i). \tag{2.3} \]

Now we can define a (payer) swaption, which is an option on a payer swap.

**Definition 2.12.** A payer swaption with strike rate \( K \) is a contract which gives the owner the right at maturity date \( T \) (but not the obligation) to enter into a payer swap with fixed rate \( K \). Usually the maturity date is set to coincide with the first reset date of the underlying swap, i.e. \( T = T_0 \).

An equivalent definition can be given for a receiver swaption. If we use (2.3) for \( t = T_0 \) we can see that the \( T_0 \) payoff of the payer swaption is equal to
\[ N\delta(R_{swap}(T_0) - K)^+ \sum_{i=1}^{n} P(T_0, T_i) \tag{2.4} \]

The market prices of swaptions are given in terms of swaption volatilities that are derived from Black’s formula for swaptions. Therefore we will need Black’s formula to calculate the market prices.

**Theorem 2.13** (Black’s Formula for Swaptions). Black’s formula for the \( t \leq T_0 \) price of a payer swaption is given by
\[ N\delta(R_{swap}(t)\Phi(d_1(t)) - K\Phi(d_2(t))) \sum_{i=1}^{n} P(t, T_i), \]
where
\[ d_1(t) = \frac{\log(R_{swap}(t)/K) + \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}}, \]
\[ d_2(t) = d_1 - \sigma(t)\sqrt{T_0 - t}. \]

Here \( \Phi \) denotes the standard normal distribution function. \( \sigma(t) \) is known as Black’s swaption volatility.

A justification for Black’s formula can be found in Appendix A.
Part II
Theory
Chapter 3

The interest rate model

In this chapter we will discuss the interest rate model we will use in our ESG. We want to calibrate our model to market prices of at-the-money swaptions. Therefore, we will derive an explicit formula for the price of a swaption under the chosen Hull-White model.

The Hull-White model is a short rate model, it defines the dynamics of the short rate. Short rate models were the earliest stochastic short rate models. The relation between the bond prices $P(t, T)$ and the short rate can be derived from the time $t$ value of $P(T, T) = 1$. That is,

$$P(t, T) = B(t) E^Q \left[ \frac{1}{B(T)} \bigg| \mathcal{F}_t \right]$$

The need for an arbitrage free model let Hull and White [24] to an extension of the Vasiček model [45], which can be described by the stochastic differential equation:

$$dr(t) = \lambda (\mu - r(t))dt + \sigma dW(t), \ r(0) = r_0.$$  

The Vasiček model on it’s own is a arbitrage free model, however when used in a real market situation an exact fit of the model to the market term structure is not possible in general. Hence, bond options and other derivatives priced under the Vasiček model would not be consistent with the market prices of bonds and would therefore lead to arbitrage opportunities.

The extension of Hull and White allows for an arbitrage free calibration since it is possible get an exact fit to the currently observed term structure of interest rates. Under the risk-neutral measure $\mathbb{Q}$, with the money market account as numéraire, the dynamics of $r(t)$ under the Hull-White extended Vasiček model
are described by

$$dr(t) = (\theta(t) - a(t)r(t))dt + \sigma(t)dW(t), r(0) = r_0.$$  

Not only can this model be fitted to the term structure with the $\theta(t)$ term, it can also give an exact fit to the spot or forward volatilities with the $\sigma(t)$ term. However, these high degrees of freedom make the model hard to handle analytically in general. Since we want to fit observed market prices of swaptions to model prices in order to calibrate the ESG, it is convenient if we can derive an analytical formula for the swaption prices under the model. Therefore we restrict ourselves to the model known as the Hull-White one factor model \([25]\), where $a(t) = a$ and $\sigma(t) = \sigma$ are constants instead of functions of time. Thus, the dynamics of the Hull-White one factor model are given by:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t), r(0) = r_0. \quad (3.1)$$

An exact fit to the volatility term structure is not possible anymore in this model, but we are still able to exactly fit the term structure and we now can derive an analytic formula for the price of a swaption. Moreover, one of the great strengths of this model is that we can also derive analytical formulae’s for put and call options on equity under Hull-White interest rates.

We will proceed as follows. First we will deduce an explicit formula for the price of a bond $P(t,T)$ under the model and subsequently we will use it to get a formula for the price of a European option on a bond. This formula, together with the so called Jamshidian’s trick \([27]\), will get us the formula for the price of a swaption under the Hull-White model.

### 3.1 The bond price

The first step to deriving a formula for the price of a bond is by analysing the solution of the Hull-White SDE (3.1).

**Proposition 3.1.** The solution to the short rate dynamics in the Hull-White model is given by

$$r(t) = e^{-a(t-s)} \left( r(s) + \int_s^t \theta(u)e^{au}du + \sigma \int_s^t e^{au}dW(u) \right). \quad (3.2)$$

**Proof.** The result can be verified by applying the Itô-formula to (3.2). \qed
From this result it follows that \( r(t) \), conditional on \( F_s \), is Gaussian with conditional expectation and conditional variance

\[
E[r(t)|F_s] = e^{-a(t-s)} \left( r(s) + \int_s^t \theta(u)e^{au}du \right),
\]
\[
\text{Var}[r(t)|F_s] = \mathbb{E} \left[ (\sigma e^{-a(t-s)} \int_s^t e^{au}dW(u))^2 \right | F_s] = \mathbb{E} \left[ \sigma^2 e^{-2a(t-s)} \int_s^t e^{2au}du \right | F_s] = \frac{\sigma^2 e^{2as}}{2a} \left( 1 - e^{-2a(t-s)} \right).
\]

Here we have used the Itô isometry to calculate the variance. Recall that (under our risk-neutral measure \( Q \)) we can represent bond prices as

\[
P(t,T) = \mathbb{E}_Q^Q \left[ e^{-\int_t^T r(s)ds} | F_t \right] = \exp \left( \mathbb{E}[-\int_t^T r(s)ds + \frac{1}{2} \text{Var}[-\int_t^T r(s)ds]] \right).
\]

Since \( r(t) \) is a Gaussian process, one can show that \( \int_t^T r(s)ds \) is normally distributed, which means the bond price is an expectation of a lognormal distribution and of the form

\[
P(t,T) = \exp \left( \mathbb{E}[-\int_t^T r(s)ds + \frac{1}{2} \text{Var}[-\int_t^T r(s)ds]] \right).
\]

Now we remind ourselves that (3.1) is just a short hand notation for

\[
r(t) = r_0 + \int_0^t \theta(s)ds - a \int_0^t r(s)ds + \int_0^t \sigma dW(s).
\]

By rearranging terms and taking the expectation on both sides we get that

\[
\mathbb{E}\left[\int_0^t r(s)ds\right] = \frac{r_0}{a} + \frac{1}{a} \int_0^t \theta(s)ds - \frac{1}{a} \mathbb{E}[r(t)]
\]
\[
= \frac{r_0}{a} + \frac{1}{a} \int_0^t \theta(s)ds - \frac{e^{-at}}{a} \left( r_0 + \int_0^t \theta(s)e^{as}ds \right)
\]
\[
= \frac{r_0}{a} (1 - e^{-at}) + \frac{1}{a} \int_0^t \theta(s)(1 - e^{a(s-t)})ds.
\]

Moreover, from (3.2) we can derive

\[
\text{Cov}[r(t), r(u)] = \sigma^2 e^{-a(t+u)} \mathbb{E}\left[\int_0^t e^{as}dW(s) \int_0^u e^{as}dW(s)\right] = \sigma^2 e^{-a(t+u)} \int_0^{t\wedge u} e^{2as}ds
\]
\[
= \sigma^2 e^{-a(t+u)} \left( e^{2a(t\wedge u)} - 1 \right),
\]

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where we have made use of the Itô isometry and the independence of disjoint Brownian increments. We can use this formula to calculate the variance

\[
\text{Var}\left[ \int_0^t r(s)ds \right] = \mathbb{E}[\left( \int_0^t r(s)ds - \mathbb{E}[\int_0^t r(s)ds] \right)^2]
\]

\[
= \int_0^t \int_0^t \mathbb{E}[(r(s) - \mathbb{E}[r(s)))(r(u) - \mathbb{E}[r(u)])]ds
du
\]

\[
= \int_0^t \int_0^t \text{Cov}[r(s), r(u)]ds
du
\]

\[
= \int_0^t \int_0^t \frac{\sigma^2}{2a} e^{-a(t+u)}(e^{2a(tu)} - 1)ds
du
\]

\[
= \frac{\sigma^2}{2a^3} (2at - 3 + 4e^{-at} - e^{-2at}).
\]

For notational convenience, let us now define

\[
B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})
\]

(3.4)

and insert the results in (3.3). Then

\[
P(t, T) = \exp\left( -\frac{r(t)}{a} (1 - e^{-a(T-t)}) - \frac{1}{a} \int_t^T \theta(s)(1 - e^{a(s-t)})ds
\]

\[
+ \frac{\sigma^2}{2a^3} (a(T - t) - \frac{3}{2} + 2e^{-a(T-t)} - \frac{1}{2}e^{-2a(T-t)}) \right)
\]

\[
= \exp\left( -r(t)B(t, T) - \int_t^T \theta(s)B(T, s)ds
\]

\[
+ \frac{\sigma^2}{2a^2} \int_t^T 1 - 2e^{-a(T-s)} + e^{-2a(T-s)}ds \right)
\]

\[
= \exp\left( -r(t)B(t, T) - \int_t^T \theta(s)B(T, s)ds
\]

\[
+ \frac{\sigma^2}{2} \int_t^T B^2(s, T)ds \right).
\]

(3.5)

The function \( \theta(t) \) gives us the opportunity to fit the theoretical bond prices to the market prices. It follows from the definition of the instantaneous forward rate \( f(t,T) = -\frac{\partial \log(P(t,T))}{\partial T} \) that

\[
P(t, T) = e^{-\int_t^T f(t,s)ds},
\]

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which implies there is a one-to-one correspondence between bond prices \( \{ P(0, T) \mid T > 0 \} \) and the forward rate curve \( \{ f(0, T) \mid T > 0 \} \). Because of convenience we choose to fit the forward rate curve to our model, which results in

\[
f(0, T) = -\frac{\partial}{\partial T} \log(P(0, T))
\]

\[
= -\frac{\partial}{\partial T} B(t, T)r_0 + \frac{\sigma^2}{2} \int_0^T \frac{\partial}{\partial T} B^2(s, T) ds - \int_0^T \theta(s) \frac{\partial}{\partial T} B(s, T) ds
\]

\[
= e^{-aT} r_0 - \frac{\sigma^2}{2a} (1 - e^{-aT})^2 + \int_0^T \theta(s)e^{-a(T-s)} ds.
\]

Here we implicitly used the fact that \( \frac{\partial}{\partial T} B(s, T) = -\frac{\partial}{\partial s} B(s, T) \) to get the last expression. We can solve the last equation for \( \theta(t) \) by writing

\[
f(0, T) = y(T) - g(T),
\]

where

\[
g(T) = \frac{\sigma^2}{2a} (1 - e^{-aT})^2
\]

and

\[
y(T) = e^{-aT} r_0 + \int_0^T \theta(s)e^{-a(T-s)} ds
\]

\[
= \mathbb{E} [r(t)]
\]

We can describe \( y(T) = \mathbb{E} [r(T)] \) by the ordinary differential equation

\[
\frac{d}{dt} y(t) = -ax(t) + \theta(t), \quad y(0) = r(0).
\]

This leads us to the following expression for \( \theta(t) \)

\[
\theta(t) = \frac{d}{dt} y(t) + ay(t)
\]

\[
= \frac{d}{dt} f(0, t) + \frac{d}{dt} g(t) + a (f(0, t) + g(t))
\]

\[
= \frac{d}{dt} f(0, t) + af(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}) \tag{3.6}
\]

Now we can insert \( \theta(t) \) into [3.5] and, skipping some tedious calculations, we arrive at the following result
Proposition 3.2. Under the Hull-White short rate model, with \( \theta(t) \) chosen such that theoretical forward rate curve matches the observed forward rate curve, the theoretical bond price is given by

\[
P(t,T) = \frac{P(0,T)}{P(0,t)} \exp \left( B(t,T) f(0,t) - \frac{\sigma^2}{4a} B^2(t,T) \left( 1 - e^{-2at} \right) - B(t,T)r(t) \right),
\]

where \( B(t,T) \) is given by (3.4).

3.1.1 Alternative representation of the short rate

Equation (3.6) gives us the means to derive an alternative representation for the short rate \( r(t) \). This representation will be used in the next chapter, in which the equity model is discussed, and it will also be used for the simulation of the short rate.

Let us look at (3.2) again and insert (3.6) for \( \theta(t) \). This results in

\[
r(t) = f(0,t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2 + \sigma \int_0^t e^{-a(t-s)} dW(s).
\]

Now we can write the short rate as \( r(t) = x(t) + \phi(t) \) where

\[
\phi(t) = f(0,t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2 \text{ and }
\]

\[
x(t) = \sigma \int_0^t e^{-a(t-s)} dW(s).
\]

Here is \( \phi(t) \) the deterministic part of the short rate and \( x(t) \) is a solution to

\[
dx(t) = -ax(t)dt + \sigma dW(t), \; x(0) = 0.
\]

3.2 Bond option

In this section we will derive an analytic formula for the price of an option on a zero coupon bond. First we take a look at a general option pricing formula which we will derive by using a change of measure technique. Here, for the most part, we will follow the arguments presented by German et al (1995) [21]. Afterwards the formula is applied to the Hull-White setting.

From the first fundamental theorem of asset pricing we know that a market is arbitrage free if and only if there exists a (local) equivalent martingale measure
Moreover, if the market is also complete or a contingent claim is attainable, we know it has a unique arbitrage free price, which we can calculate by taking the expectation of the discounted claim under the risk-neutral measure. What we should realize, however, is that this is a risk-neutral measure or EMM only relative to a chosen numéraire. The classical choice for this numéraire is the money account \( B(t) = e^{\int_0^t r(s)ds} \). In fact, we have assumed that the dynamics of our short-rate model are under the EMM with \( B \) as numéraire. However, we could imagine that, under stochastic interest rates, the money account is not the most convenient choice for a numéraire. Suppose, for example, we have a \( T \)-claim \( X \), then the price of \( X \) at \( t = 0 \) is given by

\[
\Pi_X(0) = E_Q\left[ \frac{X(T)}{B(T)} \right] = E_Q\left[ e^{-\int_0^T r(t)dt} X(T) \right]. \tag{3.9}
\]

Since we’re dealing with stochastic interest rates, we would need to consider the joint distribution of \( X \) and \( B \) and integrate with respect to that distribution. This can turn out to be rather hard work. We might want to consider an alternative numéraire. Because \( P(0, T) = E_Q[e^{-\int_0^T r(t)dt}] \), which is observable in the market, \( P(t, T) \) might be a good choice of numéraire, therefore we introduce a new measure corresponding to this numéraire.

**Definition 3.3.** \( T \)-forward measure For some fixed time \( T > 0 \) the \( T \)-forward measure \( Q^T \) is the equivalent martingale measure for the numéraire \( P(t, T) \).

As we will see in Corollary 3.5 the fair value of \( X \) at \( t = 0 \) can be calculated as

\[
\Pi_X(0) = P(0, T)E^{Q^T}[X(T)],
\]

which looks more manageable than the expectation in (3.9).

The first objective now is to find what the corresponding Girsanov-transformation is when we change from \( Q \) to \( Q^T \).

To prove the following proposition, we need a result on conditional expectations known as the Abstract Bayes’ Formula, which is given in Appendix B.

**Proposition 3.4.** If \( Q \) is the risk-neutral measure relative to the money account \( B \), then the likelihood process

\[
L^T(t) = \frac{dQ^T}{dQ} \bigg|_{\mathcal{F}_t}, \quad 0 \leq t \leq T
\]

---

\(^1\)In fact this is not completely true in a continuous time setting, where we need a stronger concept as ‘no free lunch with vanishing risk’ to replace the arbitrage free condition. For more information on this subject see Björk [4] or the more technical paper of Delbaen and Schachermayer [12].
which makes $Q^T$ a risk-neutral measure for numéraire $P(t, T)$ is given by

$$L^T(t) = \frac{P(t, T)}{P(0, T)B(t)}. \quad (3.10)$$

**Proof.** For $Q^T$ to be a martingale measure, we have to show that every (sufficiently integrable) price process $X(t)$ normalized by $P(t, T)$ is a $Q^T$ martingale. For such a price process, we already know $X(t)/B(t)$ is a $Q$-martingale and in particular so is $L^T(t)$. Applying the Abstract Bayes’ Formula, we can derive the required result:

$$E_{T}^{Q}[X(t)P(t, T)|\mathcal{F}_s] = \frac{E_{T}^{Q}[L^T(t)\frac{X(t)}{P(t, T)}|\mathcal{F}_s]}{L^T(s)} = \frac{1}{P(0, T)}E_{T}^{Q}\left[\frac{X(t)}{B(t)}|\mathcal{F}_s\right] \cdot \frac{1}{P(0, T)}\frac{X(s)}{B(s)} = \frac{X(s)}{P(s, T)}.$$

\[ \square \]

**Corollary 3.5.** Let $X$ be a $T$-claim s.t. $E_{T}^{Q}\left[\frac{|X|}{B(T)}\right] < \infty$, then

$$E_{T}^{Q}[|X|] < \infty$$

and the price of $X$ under $Q^T$ at time $t$ is given by

$$\Pi_X(t) = P(t, T)E_{T}^{Q}[X|\mathcal{F}_t].$$

**Proof.** Both follow from Proposition 3.4 since

$$E_{T}^{Q}[|X|] = E_{T}^{Q}[L^T(T)\cdot |X|] = E_{T}^{Q}\left[\frac{|X|}{P(0, T)B(T)}\right] < \infty$$

and using Bayes’ Formula and the fact that $\frac{B(t)}{P(t, T)}$ is a martingale under $Q^T$

$$\Pi_X(t) = B(t)E_{T}^{Q}\left[\frac{X}{B(T)}|\mathcal{F}_t\right] = B(t) \frac{E_{T}^{Q}\left[\frac{1}{L^T(T)}\cdot \frac{X}{B(T)}|\mathcal{F}_t\right]}{E_{T}^{Q}\left[\frac{1}{L^T(T)}|\mathcal{F}_t\right]}$$

$$= P(t, T)E_{T}^{Q}[X|\mathcal{F}_t].$$

\[ \square \]
Proposition 3.6. Suppose the $\mathbb{Q}$-dynamics of a $T$-bond are of the form
\[ dP(t, T) = r(t)P(t, T)dt + v(t, T)P(t, T)dW(t), \]
where $W$ is a $\mathbb{Q}$-Brownian Motion, then the dynamics of $L^T$ are given by
\[ dL^T(t) = v(t, T)L^T(t)dW(t) \]
and as a consequence the dynamics of $W$ under $\mathbb{Q}^T$ are given by
\[ dW(t) = v(t, T)dt + dW^T(t) \]
where $W^T$ is a $\mathbb{Q}^T$-Brownian Motion.

Proof. Remembering that $dB(t) = r(t)B(t)dt$, the first result follows from applying the Itô formula to $L^T(t)$ (3.10) and the second result follows from applying the Girsanov Theorem, see for example [4].

We can use the change of measure to derive a general formula for pricing European call option $X$ with strike $K$ maturing at time $T$ on an $S$-bond with $S > T$. The time $T$ value of $X$ is thus
\[ X = \max(P(T, S) - K, 0). \]

To ensure the existence of a unique arbitrage-free price, we assume that the option is attainable in the sense of Definition 2.4.

Proposition 3.7. Let $X$ be a European call option as above, then the $t = 0$ price can be written as
\[ \Pi_X(0) = P(0, T)\mathbb{E}^{Q^T}\left[ \left( \frac{P(T, S)}{P(T, T)} - K \right)^+ \right] \]
or
\[ \Pi_X(0) = P(0, S)\mathbb{Q}^S(P(T, S) > K) - KP(0, T)\mathbb{Q}^T(P(T, S) > K). \]

Here $\mathbb{Q}^S$ denotes the $S$-forward measure corresponding to numéraire $P(t, S)$.

Proof. The first expression is a direct result from Corollary 3.5. The second result follows from using an indicator function and changing the measure as follows
\[
\mathbb{E}^{Q} \left[ \frac{(P(T, S) - K)^+}{B(T)} \right] = \mathbb{E}^{Q} \left[ \frac{P(T, S)}{B(T)} 1_{P(T, S) > K} \right] - K \mathbb{E}^{Q} \left[ \frac{1}{B(T)} 1_{P(T, S) > K} \right]
\]
\[ = P(0, S)\mathbb{E}^{Q_S} \left[ \frac{P(0, S)B(T)}{P(T, S)} \cdot \frac{P(T, S)}{B(T)} 1_{P(T, S) > K} \right] - KP(0, T)\mathbb{Q}^T(P(T, S) > K).
\]

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Now the task at hand is to find conditions under which we can evaluate the probabilities in (3.11). As it turns out, deterministic volatilities are sufficient for computability. So let us assume that the process

$$Z(t) = \frac{P(t,S)}{P(t,T)}$$

for $t \leq S \wedge T$

has a stochastic differential under $Q$ of the form

$$dZ(t) = Z(t)\mu(t)dt + Z(t)\sigma(t)dW(t)$$

for $t \leq s \wedge t$, (3.12)

where $\sigma(t)$ is deterministic (and $\mu(t)$ is sufficiently integrable). Note that $Z$ is a $Q^T$-martingale and since the volatility process is unaffected by a Girsanov-transformation the dynamics of $Z(t)$ are

$$dZ(t) = Z(t)\sigma(t)dW^T(t)$$

under $Q^T$. Moreover, by applying the Itô-formula we can see that an explicit solution is given by

$$Z(t) = \frac{P(0,S)}{P(0,T)}\exp\left(-\frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW^T(s)\right).$$

The exponent has a normal distribution with variance

$$\Sigma^2(T) = \int_0^T \sigma^2(t)dt$$

and we see that $Z(T)$ has a lognormal distribution. This leads us to the following calculation for the second probability in (3.11).

$$Q^T(P(T,S) > K) = \Phi\left(\frac{\log\left(\frac{P(0,S)}{KP(0,T)}\right) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}\right) =: \Phi(d_2),$$

where $\Phi$ is the cumulative distribution function of a standard normal random variable. Now the first probability in (3.11) is calculated in a similar fashion. First note that

$$Q^S(P(T,S) > K) = Q^S\left(\frac{P(T,T)}{P(T,S)} < \frac{1}{K}\right) = Q^S\left(Y(T) < \frac{1}{K}\right).$$
So \( Y(t) = \frac{P(t, T)}{P(t, S)} = \frac{1}{Z(t)} \) and it is a martingale under \( Q^S \). Applying the Itô-formula to \( \frac{1}{Z(t)} \) results in

\[
Y(t) = \frac{P(0, T)}{P(0, S)} \exp \left( -\frac{1}{2} \int_0^t \sigma^2(s)ds - \int_0^t \sigma(s)dW^T(s) \right),
\]

and

\[
Q^S (P(T, S) > K) = \Phi \left( d_2 + \sqrt{\Sigma^2(T)} \right) =: \Phi(d_1)
\]

### 3.2.1 Bond option under the Hull-White model

We want to apply the general bond option formula we derived in the previous section to the Hull-White model. Therefore we need to check if the assumptions we made in the derivation of the option formula are compatible with the Hull-White model. That is, we need to check if \( Z(t) = \frac{P(t, S)}{P(t, T)} \) has an SDE of the form \( (3.12) \) where \( \sigma(t) \) is deterministic. From the explicit formula we derived for the bond price \( P(t, T) \), equation (3.7), we can write \( Z(t) \) in the form

\[
Z(t) = \exp \left( A(t, S) - A(t, T) - (B(t, S) - B(t, T))r(t) \right),
\]

where

\[
A(t, T) = B(t, T)f(0, t) - \frac{\sigma^2}{4a} B^2(t, T) \left( 1 - e^{-2at} \right).
\]

Applying the Itô-formula gives us the dynamics of \( Z(t) \), which are of the form

\[
dZ(t) = Z(t)(\ldots)dt - Z(t)\sigma [B(t, S) - B(t, T)] dW(t),
\]

where

\[
-\sigma [B(t, S) - B(t, T)] = \frac{\sigma}{a} e^{at} \left[ e^{-aS} - e^{-aT} \right] =: \sigma(t).
\]

Thus \( \sigma(t) \) is deterministic and we can apply the previous results to get

**Proposition 3.8. Hull-White bond option** In our (restricted) Hull-White model, the price of a put option at time \( t \leq T \), with strike \( K \) and maturity \( T \) on a zero-coupon bond maturing at \( S \), is given by

\[
ZB_P(t, T, S, K) = KP(t, T)\Phi(d_1) - P(t, S)\Phi(d_2),
\]

where

\[
d_1 = \frac{\ln \left( \frac{KP(t, T)}{P(t, S)} \right) + \frac{1}{2} \Sigma^2}{\sqrt{\Sigma^2}},
\]

\[
d_2 = \sqrt{\Sigma^2} - d_1
\]

\[
\Sigma^2 = \frac{\sigma^2}{2a^2} \left[ 1 - e^{2a(T-t)} \right] \left( 1 - e^{-a(S-T)} \right)^2.
\]
The swaption formula

We want to configure our model to swaption prices, so we still need to find an explicit formula for swaptions. As it turns out, because of a clever trick developed by Jamshidian in 1989 [27], it possible for us to write a swaption as a portfolio of put options on bonds.

**Proposition 3.9.** Let the definition of a swaps and a swaption be as in Definition 2.10 and Definition 2.12. Then, under the Hull-white short rate model, the price of a payer swaption strike $K$ at time $t \leq T_0$ is given by

$$\text{Swapt}_p(t, T_0, T_1, ..., T_n, N, K) = N \left( ZB_p(t, T_0, T_n, K_n) + K \delta \sum_{i=1}^{n} ZB_p(t, T_0, T_i, K_i) \right),$$

where

$$K_i = P(T_0, T_i; r^*)$$

and $ZB_p$ is given by Proposition 3.8.

**Proof.** Recall the formula for the price of a swap (2.2), which leads to the following $T_0$ value of a swaption

$$\Pi(T_0) = N \left( 1 - P(T_0, T_n) - K \delta \sum_{i=1}^{n} P(T_0, T_i) \right)^+. \tag{3.13}$$

To express the dependence of the bond prices on the short rate of interest $r$ we write $P(T_0, T_i) = P(T_0, T_i; r)$. The following two steps that finish the proof together are known as Jamshidian’s trick. The most important observation of this trick is that $P(T_0, T_i; \cdot)$ is monotone decreasing in $r$ as we can see from (3.7). We can define $r^*$ as the rate for which

$$P(T_0, T_n; r^*) + K \delta \sum_{i=1}^{n} P(T_0, T_i; r^*) = 1. \tag{3.14}$$

Plugging this into (3.13) and using the monotonicity we get

$$N \left( (P(T_0, T_n; r^*) - P(T_0, T_n; r)) + K \delta \sum_{i=1}^{n} (P(T_0, T_i; r^*) - P(T_0, T_i; r)) \right)^+$$

$$= N(P(T_0, T_n; r^*) - P(T_0, T_n; r))^+ + K \delta N \sum_{i=1}^{n} (P(T_0, T_i; r^*) - P(T_0, T_i; r))^+. \notag$$

$\square$
Chapter 4

The equity model

In this chapter we will discuss the equity model we will use in our ESG. In agreement with what we did with the short rate model we want to calibrate our model to market prices of (European) options on equity. Therefore, we will derive an explicit formula for the price of an option in our model. In this chapter we basically follow the derivation presented in [5].

We would like our equity model to be consistent with the stochastic short rate we modelled in the previous chapter. Moreover, we would like our model to be analytically tractable. We will see that both of these properties are met by the Hull-White-Black-Scholes model; this is the classic Black-Scholes equity model, but with the stochastic Hull-White interest rates in stead of a constant rate. The dynamics under $Q$ for this equity model are thus given by

$$dS(t) = S(t)r(t)dt + \sigma SS(t)dW_S(t), \quad S(0) = S_0,$$

where $r(t)$ is the short rate under the Hull-White model, so

$$dr(t) = (\theta(t) - ar(t))dt + \sigma_r dW_r(t), \quad r(0) = r_0.$$

Here the two Brownian motions are correlated with

$$d\langle W_r, W_S \rangle_t = \rho dt$$

for $-1 \leq \rho \leq 1$.

**Proposition 4.1.** A solution to the dynamics of the equity model is given by

$$S(t) = S(s) \exp \left( \int_s^t r(u)du - \frac{1}{2} \sigma_S^2(t-s) + \sigma_S (W_S(t) - W_S(s)) \right) \quad (4.1)$$

**Proof.** The result can be verified by applying the Itô-formula to (4.1). □
Because \((W_r(t), W_s(t))\) is a jointly Gaussian process, one can see from (4.1) that \(\log S(t)\) has a normal distribution conditional on \(\mathcal{F}_s\) and we can calculate its conditional expectation and conditional variance. Recall that in Section 3.1.1 we saw that we can write \(r(t) = x(t) + \phi(t)\). From the SDE of \(x(t)\), equation (3.8), we can see that

\[
\int_s^t x(u) du = \frac{x(s) - x(t)}{a} + \frac{\sigma_r}{a} \int_s^t dW_r(u)
\]

and the explicit solution for \(x(t)\) is given by

\[
x(t) = x(s) e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_r(u).
\]

Combining the last two equations results in

\[
\int_s^t x(u) du = \frac{1 - e^{-a(t-s)}}{a} x(s) + \frac{\sigma_r}{a} \int_s^t (1 - e^{-a(t-u)}) dW_r(u).
\]

Using the last equality, we have that

\[
\log \left( \frac{S(t)}{S(s)} \right) = \int_s^t r(u) du - \frac{1}{2} \sigma_S^2 (t-s) + \sigma_S (W_S(t) - W_S(s))
\]

\[
= \frac{1 - e^{-a(t-s)}}{a} x(s) + \sigma \int_s^t e^{-a(t-u)} dW_r(u) + \int_s^t \phi(u) du
\]

\[
- \frac{1}{2} \sigma_S^2 (t-s) + \sigma_S (W_S(t) - W_S(s)),
\]

and we can calculate its conditional expectation by inserting \(r(s) - \phi(s)\) for \(x(s)\) and integrating \(\phi(t)\)

\[
\mathbb{E} \left[ \log \left( \frac{S(t)}{S(s)} \right) \middle| \mathcal{F}_s \right] = \frac{1 - e^{-a(t-s)}}{a} \left( r(s) - f(0,s) - \frac{\sigma_S^2}{2a^2} (1 - e^{-as})^2 \right)
\]

\[
- \frac{1}{2} \sigma_S^2 (t-s) + \log \left( \frac{P(0,s)}{P(0,t)} \right)
\]

\[
+ \frac{\sigma_S^2}{2a^2} \left[ t - s + \frac{2}{a} (e^{-at} - e^{-as}) - \frac{1}{2a} (e^{-2at} - e^{-2as}) \right].
\]

We can also derive the conditional variance from equation (4.2):

\[
\text{Var} \left[ \log \left( \frac{S(t)}{S(s)} \right) \middle| \mathcal{F}_s \right] = \text{Var} \left[ \sigma \int_s^t e^{-a(t-u)} dW_r(u) + \sigma_S \int_s^t dW_S(u) \middle| \mathcal{F}_s \right]
\]

\[
= \sigma_r^2 \left[ t - s - \frac{2}{a} (1 - e^{-a(t-s)}) + \frac{1}{2a} (1 - e^{-2a(t-s)}) \right]
\]

\[
+ \frac{1}{2} \sigma_S^2 (t-s) + 2 \rho \frac{\sigma_r \sigma_S}{a} \left[ t - s - \frac{1}{a} (1 - e^{-a(t-s)}) \right].
\]

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4.1 change of measure

As in the case of the bond options, it is convenient to change the measure to the $T$-forward measure if we want to derive an explicit formula for an equity option. Since we’re dealing with two correlated Brownian motion, we have to take this correlation in to account when applying a Girsanov-transformation for the Brownian motions. However, recall that any multivariate normal distributed random variable $Z \sim N(\mu, \Sigma)$ can be written in the form $Z = \mu + L \cdot \bar{Z}$ where $\bar{Z}$ is a vector of independent standard normal random variables and $\Sigma = L \cdot L^T$. It is therefore easier for us to write our processes in terms of two independent Brownian motions before we do a measure transform. Applying a Cholesky transformation leads us to

$$dW_r(t) = d\bar{W}_r(t),$$
$$dW_S(t) = \rho d\bar{W}_r(t) + \sqrt{1 - \rho^2} d\bar{W}_s(t)$$

where $\bar{W}_r$ and $\bar{W}_s$ are two independent Brownian motions and so

$$dr(t) = (\theta(t) - ar(t))dt + \sigma_r d\bar{W}_r(t),$$
$$dS(t) = S(t)r(t)dt + S(t)\sigma_S\rho d\bar{W}_r(t) + S(t)\sigma_S\sqrt{1 - \rho^2} d\bar{W}_s(t).$$

If we apply the Itô-formula to (3.7) we see that the dynamics of the bond price are of the form

$$dP(t, T) = P(t, T)(\ldots)dt - P(t, T)\frac{\sigma}{\alpha} [1 - e^{-a(T-t)}] dW(t)$$

and we can apply the Girsanov-transformation, which is similar to Proposition 3.6 for multidimensional $W$. This results in two independent Brownian motions $\bar{W}_r^T$ and $\bar{W}_s^T$ under the measure $\mathbb{Q}^T$, defined by

$$d\bar{W}_r(t) = d\bar{W}_r^T(t) - \frac{\sigma_r}{\alpha} [1 - e^{-a(T-t)}] dt$$
$$d\bar{W}_s(t) = d\bar{W}_s^T(t).$$

Hence, under the measure $\mathbb{Q}^T$, the dynamics of $r(t)$ and $S(t)$ are described by

$$dr(t) = \left(\theta(t) - \frac{\sigma_r^2}{\alpha} [1 - e^{-a(T-t)}] - ar(t)\right) dt + \sigma_r d\bar{W}_r^T(t),$$ (4.3)
$$dS(t) = S(t) \left( r(t) - \frac{\sigma_r \sigma_S \rho}{\alpha} [1 - e^{-a(T-t)}] \right) dt + S(t)\sigma_S\rho d\bar{W}_r^T(t) + S(t)\sigma_S\sqrt{1 - \rho^2} d\bar{W}_s^T(t).$$
We can find solutions for these SDE’s which can be checked easily by using \( r(t) = x(t) + \phi(t) \) and the Itô-formula. For all \( t \leq T \) we have

\[
\begin{align*}
r(t) &= r(s)e^{-a(t-s)} + \int_s^t \theta(u)e^{-a(t-u)}du \\
&\quad - \frac{\sigma^2}{a} \int_s^t e^{-a(t-u)} \left[ 1 - e^{-a(T-u)} \right] du + \sigma \int_s^t e^{-a(t-u)}d\bar{W}_r^T(u) \\
&= x(s)e^{-a(t-s)} - \frac{\sigma^2}{a} \int_s^t e^{-a(t-u)} \left[ 1 - e^{-a(T-u)} \right] du \\
&\quad + \sigma \int_s^t e^{-a(t-u)}d\bar{W}_r^T(u) + \phi(t).
\end{align*}
\]

(4.4)

and

\[
S(T) = S(t)\exp\left( \int_t^T r(u)du - \frac{\sigma \sigma S \rho}{a} \int_t^T (1 - e^{-a(T-u)}) du \\
- \frac{1}{2} \frac{\sigma^2}{S}(T - t) + \sigma S \rho \int_t^T d\bar{W}_r^T + \sigma S \sqrt{1 - \rho^2} \int_t^T d\bar{W}_s^T(u) \right).
\]

(4.5)

From (4.3) we see that under \( Q^T \) we have

\[
dx(t) = - \left( \frac{\sigma^2}{a} \left[ 1 - e^{-a(T-t)} \right] + ax(t) \right) dt + \sigma_r d\bar{W}_r^T(t),
\]

which leads to

\[
\int_t^T x(u)du = \frac{x(t) - x(T)}{a} - \frac{\sigma^2}{a} \int_t^T (1 - e^{-a(T-u)}) du + \sigma_r \int_t^T \bar{W}_r^T(u)
\]

and again since \( r(t) = x(t) + \phi(t) \) from (4.4) we get

\[
\int_t^T x(u)du = \frac{1 - e^{-a(T-t)}}{a} x(t) \\
- \frac{\sigma^2}{a^2} \int_t^T (1 - e^{-a(T-u)})^2 du + \frac{\sigma r}{a} \int_t^T (1 - e^{-a(T-u)}) d\bar{W}_r^T(u).
\]
We can insert this into (4.5), which results in
\[
S(T) = S(t) \exp \left( \frac{1 - e^{-a(T-t)}}{a} x(t) - \frac{\sigma^2}{2a^2} \int_t^T (1 - e^{-u})^2 \, du \right) \\
+ \frac{\sigma_r}{a} \int_t^T (1 - e^{-a(T-u)}) \, dW_r(u) + \int_t^T f(0, u) \, du \\
+ \frac{\sigma_r^2}{2a^2} \int_t^T (1 - e^{-a(T-u)}) \, du - \frac{\sigma_r \sigma_S \rho}{\sigma^2} \int_t^T (1 - e^{-u}) \, du \\
- \frac{1}{2} \sigma_s^2 (T - t) + \sigma_S \rho \int_t^T \Phi(u) \, du + \sigma_S \sqrt{1 - \rho^2} \int_t^T dW_s^T(u)
\]

Now the expectation of \( \log S(t) \) conditional on \( \mathcal{F}_t \) under \( \mathbb{Q}^T \) can be calculated
\[
\mathbb{E}^{\mathbb{Q}^T} [\log (S(T)) | \mathcal{F}_t] \\
= \frac{1 - e^{-a(T-t)}}{a} x(t) - \frac{\sigma^2}{2a^2} \left( T - t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right) \\
+ \log \left( \frac{P(0, t)}{P(0, T)} \right) + \frac{\sigma^2}{2a^2} \left( T - t + \frac{2}{a} (e^{-aT} - e^{-at}) - \frac{1}{2a} (e^{-2aT} - e^{-2at}) \right) \\
- \frac{\sigma_r \sigma_S \rho}{\sigma^2} \left( T - t + \frac{1}{a} (1 - e^{-a(T-t)}) \right) - \frac{1}{2} \sigma_s^2 (T - t) + \log(S(t)) \\
= \frac{1 - e^{-a(T-t)}}{a} x(t) - V(t, T) + \log \left( \frac{P(0, t)}{P(0, T)} \right) \left( V(0, T) - V(0, t) \right) \\
- \frac{\sigma_r \sigma_S \rho}{\sigma^2} \left( T - t + \frac{1}{a} (1 - e^{-a(T-t)}) \right) - \frac{1}{2} \sigma_s^2 (T - t) + \log(S(t)) \\
= B(t, T) (r(t) - \phi(t)) - V(t, T) + \log \left( \frac{P(0, t)}{P(0, T)} \right) \left( V(0, T) - V(0, t) \right) \\
- \frac{\sigma_r \sigma_S \rho}{\sigma^2} \left( T - t + \frac{1}{a} (1 - e^{-a(T-t)}) \right) - \frac{1}{2} \sigma_s^2 (T - t) + \log(S(t))
\]

where
\[
V(t, T) := \frac{\sigma^2}{a^2} \left( T - t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right).
\]

By applying the bond price formula (3.7) to the last equation, we arrive at the following result:
\[
\mathbb{E}^{\mathbb{Q}^T} [\log (S(T)) | \mathcal{F}_t] = \log \left( \frac{S(t)}{P(t, T)} \right) - \frac{\sigma_r \sigma_S \rho}{\sigma^2} \left( T - t + \frac{1}{a} (1 - e^{-a(T-t)}) \right) \\
- \frac{1}{2} \sigma_s^2 (T - t) - \frac{1}{2} V(t, T) := m(t, T).
\]
For the variance we get the following

\[
\text{Var}^Q_T [\log(S(T)) | \mathcal{F}_t] = \text{Var}^Q_T \left[ \frac{\sigma_r}{a} \int_t^T (1 - e^{-a(T-u)}) \, d\tilde{W}_r(u) \right.
\]

\[+ \sigma_s \rho \int_t^T d\tilde{W}^T_s (u) + \sigma_s \sqrt{1 - \rho^2} \int_t^T d\tilde{W}_s(u) \right| \mathcal{F}_t \]

\[= V(t, T) + \sigma_s^2(T - t) \]

\[+ 2 \frac{\sigma_r \sigma_s \rho}{a} \left( T - t - \frac{1}{a} \left(1 - e^{-a(T-t)}\right) \right) \]

\[=: v^2(t, T) \quad (4.7) \]

**Proposition 4.2.** The price at time \(t\) for a European call option \(X\) with maturity \(T\) and strike \(K\) under the Black-Scholes-Hull-White model is given by

\[
\Pi_X(t) = S(t) \Phi \left( \frac{\log \left( \frac{S(t)}{K P(t, T)} \right) + \frac{1}{2} v^2(t, T)}{v^2(t, T)} \right) - K P(t, T) \Phi \left( \frac{\log \left( \frac{S(t)}{K P(t, T)} \right) - \frac{1}{2} v^2(t, T)}{v^2(t, T)} \right) \]

**Proof.** From Corollary 3.5 we know that

\[
\Pi_X(t) = P(t, T) \mathbb{E}^Q_T \left[ (S(T) - K)^+ | \mathcal{F}_t \right].
\]

Since \(S(T)\) has a lognormal distribution this expectation can be calculated as follows:

\[
\mathbb{E}^Q_T \left[ (S(T) - K)^+ | \mathcal{F}_t \right] = \int_{-\infty}^{\infty} \frac{(e^x - K)^+}{\sqrt{2\pi} v^2(t, T)} e^{-\frac{1}{2} \left( \frac{e^x - m(t, T)}{v^2(t, T)} \right)^2} \, dx
\]

\[= \int_{\log(K)}^{\infty} \frac{(e^x - K)}{\sqrt{2\pi} v^2(t, T)} e^{-\frac{1}{2} \left( \frac{e^x - m(t, T)}{v^2(t, T)} \right)^2} \, dx
\]

\[= \frac{1}{\sqrt{2\pi}} \int_{\log(K) - m(t, T)}^{\infty} \frac{1}{\sqrt{v^2(t, T)}} e^{-\frac{1}{2} (z - v^2(t, T))^2} \, dz
\]

\[= e^{m(t, T) + \frac{1}{2} v^2(t, T)} \int_{\log(K) - m(t, T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (z - v^2(t, T))^2} \, dz
\]

\[- K \int_{\log(K) - m(t, T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \, dz
\]
\[ e^{m(t,T) + \frac{1}{2} v^2(t,T)} \Phi \left( \frac{-\log(K) - m(t,T) + v^2(t,T)}{\sqrt{v^2(t,T)}} \right) \]

\[ - K \Phi \left( -\frac{\log(K) - m(t,T)}{\sqrt{v^2(t,T)}} \right). \]

The result follows from the formula’s for \( m(t,T) \), equation (4.6) and \( v^2(t,T) \), equation (4.7).
Chapter 5

The real estate model

In this chapter the real estate model is discussed. Unlike the equity and bond markets, there is no central regulated market for real estate transactions. As is argued in [19] transactions in real estate are less frequent than on the bond and equity market and usually happen between two private parties. As a result the quality of information on real estate prices is considered to be poor. In the case of real estate, the lack of information on transactions has led to the use of appraisals as a basis for tracking price movements. Appraisals are often aggregated to construct an appraisal index. Since real estate is a risky asset it assumed that it has similar dynamics under $\mathbb{Q}$ as equity and because of that we choose to model it by

$$dRE(t) = RE(t)r(t)dt + \sigma_{re}RE(t)dW_{re}(t), \quad RE(0) = RE_0.$$

Since, except for the appraisal indices, there is almost no market data available, the model will be calibrated based on the historical volatility of real estate returns. For the remaining part of this chapter we will discuss appraisal indices and how to derive returns on real estate from these indices.

In an optimal setting, the appraisal index would give us an accurate reflection of current market prices. However, there is substantial empirical evidence implying the existence of appraisal smoothing or ‘appraisal lag’ [3][7][9]. It is suggested that market agents use information about previous appraisals of property as well as current market information, when appraising a property. On an individual level this may be the optimal, if not the only, way to appraise a property, but when combined in an index this may not be optimal and its value may differ from the real market value. In addition to this, properties can be appraised at different points in time, but are averaged out in the appraisal indices which are usually quoted quarterly or yearly. This implies an index value can be seen as a moving average of appraisal values.

Moreover, in [10] it is argued that the returns of appraisal indices have an abnormally low volatility compared to other risky assets, while the expected
returns are of the same order of magnitude. We can therefore argue that Economic theory suggests that the appraisal indices don’t reflect the real market prices. This is because market players show risk averse behaviour, this means that for a higher volatility of an asset they demand a higher average rate of return. But this also means that if two traded assets have the same rate of return, they should also be equally volatile, because else no one would buy the most volatile asset.

5.1 Unsmoothing

To derive the real return series of estate and consequently the real historical volatility of the returns, the appraisal index has to be unsmoothed. Over the years, various unsmoothing techniques have been suggested for appraisal indices and assets with similar characteristics as real estate. Two of the most well known and most often used methods are developed by Geltner [18] and by Fisher, Geltner and Webb [17]. In this thesis we choose to use the second unsmoothing method, because in comparison to the other method it allows for more information to be extracted from the data itself and because less assumptions are needed. We will now discuss the model of Fisher et al. For the Geltner model, see Appendix C.

The method proposed by Fisher et al. is called the Full-Information Value index method. This name reflects that the unsmoothing of the appraisal index by this method is done as if the real estate market is an efficient market i.e. the underlying true returns are uncorrelated across time. The smoothing model is represented by

\[ r_t^* = w_0 r_t + w(B) r_{t-1}. \]  

Here \( w(B) \) is shorthand for

\[ w(B) = w_1 + w_2 B + w_3 B^2 + \ldots, \]

where \( B \) is the lag operator i.e. \( Br_t = r_{t-1}, B^2 r_t = r_{t-2}, \) etc. Like before \( r_t^* \) is the smoothed index return and \( r_t \) is the unsmoothed or real return. Substituting and expressing \( r_t^* \) in present and past values of \( r_{t-1}^* \), one can see that Equation 5.2 corresponds to an AR-model given by

\[ r_t^* = \phi(B) r_{t-1}^* + e_t, \]

where

\[ \phi(B) = \phi_1 + \phi_2 B + \ldots \]

and

\[ e_t = w_0 r_t. \]

40
Expression (5.3) can be inverted to obtain an explicit expression for $r_t$ as a function of $r_t^*, r_{t-1}^*, \ldots$. This results in

$$r_t = \frac{r_t^* - \phi(B)r_{t-1}^*}{w_0}. \quad (5.4)$$

We assumed unpredictable returns, which implies that $e_t$ is white noise and gives us the opportunity to estimate the parameters $\phi$ through standard univariate time-series estimation. However, to derive $r_t$ we need to make an additional assumption to evaluate $w_0$. In the paper of Fisher et al. an assumption on the volatility of $r_t$ is made. It is widely perceived by institutional investors that the volatility of true commercial property returns is approximately half the real volatility of stock returns $\sigma_{stock}^2$. So

$$SD[r_t] = \frac{\sigma_{stock}^2}{2}$$

and by taking the standard deviation on both sides in (5.4) this gives

$$w_0 = \frac{2SD[r_t^* - \phi(B)r_{t-1}^*]}{\sigma_{stock}}.$$ 

Where $SD[r_t^* - \phi(B)r_{t-1}^*]$ is calculated as the empirical standard deviation based on the historical smoothed return series $r_t^*$ and the estimated values of $\phi$. Now the real returns $r_t$ can be derived.

One of the most important assumptions in this model is that the real estate market is informational efficient. However, as we argued before, the real estate market is not an efficient market. Thus, a way to interpret the resulting values are the values that would have prevailed if the market was efficient.

A simple extension of the model had been proposed by [8]. This is because it had been found that in the model by Fisher et al. the error term $e_t$ not necessarily has a zero expectation and the derived series of return still show positive autocorrelation. The solution that is proposed by Cho et al. is that a constant drift term should be added to the model and two step differences $\Delta r_t = r_t - r_{t-2}$ are used to get independent errors.
Chapter 6
The inflation model

The last risk factor that is modelled in the economic scenario generator is inflation. A lot of life insurance and pension products are 'indexed' yearly on the basis of inflation. This means that the payment at the maturity of the insurance contract is incremented each year with the inflation.

Unlike interest rates and stock indices, inflation rates are not constantly observable. The inflation rates in the eurozone are calculated monthly on the basis of changes in the consumer price index (CPI); for the inflation in the Netherlands this is done by the Dutch Central Bureau for Statistics (CBS) and for the inflation in the whole eurozone this is done by Eurostat, the statistics bureau of the European Union. The CPI measures changes in the price level of a market basket of consumer goods and services purchased by households. The composition of the market basket is standarised by EU guidelines. The most important categories in the CPI are housing, water, electricity and gas (24.5% of the total weight), transport (11.6%) and food and non-alcoholic beverages (11.3%).

There is strong evidence that suggests that inflation rates follow a mean reverting process [2][34]. Because the Vasiček model exhibits mean reverting behaviour [35] and because it is not too hard to implement, given that we already discussed the extended Vasiček model of Hull-White for the modelling of interest rates, we choose to use it to model inflation in our ESG. The stochastic differential equation is given by

$$dI(t) = \lambda(\mu - I(t))dt + \sigma_I dW(t), \quad I(0) = I_0.$$  

Here $\mu$ is the long term mean, $\lambda$ is the mean reverting speed and $\sigma_I$ is the volatility of the inflation.

Since there are no deep and liquid markets from which we can infer a term structure of inflation or on which inflation derivatives are traded, the model will be calibrated based on historical data. To do so we need the solution to the SDE.
Because the Vasiček model is a special case of the Hull-White model we can apply Proposition 3.1 to get the solution to the short rate dynamics. So

\[ I(t) = e^{-\lambda t} \left( I_0 + \int_0^t \lambda \mu e^{\lambda s} ds + \sigma \int_0^t e^{\lambda s} dW(s) \right), \]

which gives us the following expectation and variance,

\[
\mathbb{E}[I(t)] = e^{-\lambda t} I_0 + \mu \left( 1 - e^{-\lambda t} \right), \tag{6.1}
\]

\[
\text{Var}[I(t)] = \sigma^2 \left( \frac{1 - e^{-2\lambda t}}{2\lambda} \right). \tag{6.2}
\]

In chapter (reference) we will discuss how these explicit formula’s can be used to calibrate the inflation model to historical data.
Chapter 7
Simulation

When the parameters for each model and the correlation between the various risk factors have been estimated, we want the economic scenario generator to generate future scenarios; this is done with Monte Carlo simulation. In this section the simulation method is described and we will discuss two types of errors that can arise when doing such a simulation: discretisation errors and distributional errors.

7.1 Euler Scheme

If we want to simulate our stochastic models, we must discretise the stochastic processes, because we cannot generate continuous paths. While there are several discretisation methods, we choose to use the Euler-Maruyama scheme \[35\], because it is the most straightforward to implement.

Let us assume we have a stochastic process of which the dynamics are described by a stochastic differential equation of the form

\[
dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t),
\]

(7.1)

where \(W\) is a Brownian motion. We wish to simulate \(X\) over the interval \([0, T]\) and we have discretised the interval as \(0 = t_1 < t_2 < \ldots < t_m = T\). Integrating the SDE from \(t_i\) to \(t_{i+1}\) gives us

\[
X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(X(s), s)ds + \int_{t_i}^{t_{i+1}} \sigma(X(s), s)dW(s).
\]

(7.2)

The starting point of any discretisation method should be to approximate this expression.

Now let us have a look at (7.2) in more depth. The first integral is a regular Riemann integral, but the second integral is a stochastic integral. Without
discussing the technical details, a convenient way to interpret a stochastic integral is

\[
\int_0^t \sigma(X(s), s) dW(s) = \lim_{h \to 0} \sum \sigma(X(t_i), t_i) (W(t_{i+1}) - W(t_i)),
\]

(7.3)

where \( h = \max_i |t_{i+1} - t_i| \) is the width of the partition. Important to note is that the \( \sigma(X(t_i), t_i) \) are evaluated at the left-hand point of the time interval; this can be interpreted as the inability to see into the future.

The Euler method is derived directly from the interpretation in (7.3). It approximates the integrals using the 'left point rule'. Hence (7.2) is approximated by

\[
X(t_{i+1}) = X(t_i) + \mu(X(t_i), t_i) \Delta t_{i+1} + \sigma(X(t_i), t_i) \Delta W_{i+1},
\]

(7.4)

where \( \Delta t_{i+1} = (t_{i+1} - t_i) \) and \( \Delta W_{i+1} = (W(t_{i+1}) - W(t_i)) \). Here the \( \Delta W_i \) are increments of a Brownian motion and thus the \( \Delta W_i \) are independent random variables with a \( N(0, \Delta t_i) \) distribution.

An alternative to the Euler method is given by the Milstein method [37]. It is based on the expansion of the coefficients \( \mu(X(t), t) \) and \( \sigma(X(t), t) \) by using the Itô formula. The Milstein discretisation method is given by

\[
X(t_{i+1}) = X(t_i) + \mu(X(t_i), t_i) \Delta t_{i+1} + \sigma(X(t_i), t_i) \Delta W_{i+1} + \frac{1}{2} \sigma(X(t_i), t_i) \frac{\partial \sigma}{\partial X}(X(t_i), t_i) (\Delta W_{i+1}^2 - \Delta t_{i+1}).
\]

(7.5)

Generally the Milstein method has a higher order of convergence than the Euler method, but when the \( \sigma(X(t), t) \) term is a deterministic function of time equation (7.5) coincides with equation (7.4). For our ESG this is the case, since the volatility terms of the stochastic models are all deterministic.

As was mentioned in the introduction of this chapter, all the stochastic models will be simulated using the Euler method. However, for the interest rates model, we won’t directly simulate the Hull-White SDE in equation (3.1). We will use the representation \( r(t) = x(t) + \phi(t) \) derived in Section 3.1.1. The advantages of simulating \( r(t) \) by using the SDE for \( x(t) \) and adding \( \phi(t) \) instead of simulating the SDE of \( r(t) \) (3.1) directly is that we don’t need the derivative of the instantaneous forward rate \( f(0, t) \), which is part of \( \theta(t) \).

### 7.2 Discretisation and distributional error

Recall that the purpose of the economic scenario generator is to calculate technical provisions of various insurance products. The technical provisions are defined as
the expected value (under the risk neutral measure) of the discounted future cash flows. Let $X$ be a stochastic process, with a corresponding stochastic differential equation as in (7.1), then calculating the technical provisions comes down to calculating expected values of the form $E^{Q}[f(X(T))]$, for some given function $f$ and discounting it with the current term structure.

Approximating the expectation by using the Monte Carlo simulation can result in an error in the estimated value of the expectation. Generally this approximation error consists of two components: a statistical error and a discretisation error.

The values realised by the Monte Carlo simulation are used to approximate the expectation; that is

$$
E[f(X(T))] \approx \sum_{j=1}^{N} \frac{f(\bar{X}^{(m)}(T;\omega_j))}{N},
$$

where $\bar{X}^{(m)}$ is a discrete approximation of $X$, e.g. the result from the Euler method, corresponding to partition $0 = t_1 < t_2 < \ldots < t_m = T$. We can describe the error of the Monte Carlo method as

$$
\mathbb{E}[f(X(T))] - \sum_{j=1}^{N} \frac{f(\bar{X}^{(m)}(T;\omega_j))}{N}
$$

(7.6)

$$
= \mathbb{E}[f(X(T)) - f(\bar{X}^{(m)}(T))] + \left( \mathbb{E}[f(\bar{X}^{(m)}(T))] - \sum_{j=1}^{N} \frac{f(\bar{X}^{(m)}(T;\omega_j))}{N} \right).
$$

(7.7)

On the right hand side of (7.6) the first part of the error is what we call the time discretisation error and the second part is the statistical error.

Let us first discuss the statistical error. The two theorems fundamental to understanding the statistical error of Monte Carlo methods are the Law of Large Numbers and the Central Limit Theorem. According to the (strong) Law of Large Numbers, under the condition of i.i.d. distribution of the random variables $f(\bar{X}^{(m)}(T;\omega_j))$, the statistical error converges almost surely to zero. However, this doesn’t give us any clue about the rate of convergence. This is done by the Central Limit Theorem. Under the additional condition of finite variance $\nu^2$ of the $f(\bar{X}^{(m)}(T;\omega_j))$, the theorem states that

$$
\sqrt{N} \left( \sum_{j=1}^{N} \frac{f(\bar{X}^{(m)}(T;\omega_j))}{N} - \mathbb{E}[f(\bar{X}^{(m)}(T))] \right) \overset{\text{weak}}{\underset{N \to \infty}{\rightarrow}} \mathcal{N}(0,\nu^2).
$$

Here the $\overset{\text{weak}}{\rightarrow}$ stands for weak convergence or convergence in distribution. The theorem implies roughly that the statistical error tends to zero as $\frac{1}{\sqrt{N}}$. For a full
in depth treatment of the Law of Large numbers and the Central Limit Theorem see for example [30].

While an intuitive explanation was given for the Euler-Maruyama scheme in the last section, we actually didn’t discuss whether or not the scheme actually gives us a feasible approximation, i.e. whether or not the discrete process \( \bar{X} \) converges to the process \( X \). Discussing the discretisation error only makes sense if the Euler-Maruyame scheme converges.

As it turns out the Euler scheme converges both in a strong sense, \( L^1 \) convergence, as in a weak sense. We will only discuss the weak convergence, since the concept is the most appropriate one for our problem. For more on the strong convergence see [32]. The following theorem not only gives us the weak convergence of the discrete process it also gives us the rate of convergence.

**Theorem 7.1.** Assume that \( \mu \), \( \sigma \) and \( f \) are smooth functions and decay sufficiently fast as \( |x| \to \infty \). Then there exists a \( C > 0 \) such that for \( m \) large enough the following holds:

\[
|\mathbb{E}[f(X(T)) - f(\bar{X}^{(m)}(T))]| \leq C \max |\Delta t_i|.
\]

**Proof.** For a proof of the theorem see [32]. \( \square \)
Part III

Analysis and results
Chapter 8

Calibration and simulation

In this chapter the calibration of the various models in the ESG is discussed. We look at the market data used for calibrating the models, the model parameters that are the result from fitting market prices to the model prices are given and the differences between market prices and the model prices are noted.

8.1 Interest rate model

As we noted before, the Hull-White model will be calibrated on the basis of at-the-money (ATM) swaptions. The model could also be calibrated to caplets or floorlets, but there are only up to 20 ATM caplets resp. floorlets quoted at a given time, while there can be up to 200 quotes for ATM swaption; this is because swaptions are quoted both for different option maturities as for different tenors of the underlying swap. Moreover, because more high maturity/high tenor swaptions are quoted, we can calibrate the model better to higher maturities, which can be very important for (life) insurance contracts with a high maturity.

The model is calibrated to the 6 months Libor ATM swaptions traded in EUR on the over the counter market (OTC) on 31-12-2012 quoted in terms of Black volatilities and taken from Bloomberg, see figure 8.1. The interest rate term structure to which the model is fitted is the corresponding Libor swaprate curve on 31-12-2012 provided by Bloomberg, see figure 8.2.
Since swaptions are quoted in terms of implied volatilities, we have to use Black’s formula for swaptions, given in Theorem 2.13, to derive the market prices. Because we have ATM swaption volatilities, the strike price $K$ is equal to the swaprate $R_{\text{swap}}$ given by (2). Bloomberg also provides the swap rates together with the volatilities, hence we can check whether or not we get the same swap rates when we derive the bond prices from the term structure and insert them in Formula (2). The calculated swap rates are not exactly the same as those provided by Bloomberg. The rooted mean square error can be found in Table 8.1. A possible explanation for these relatively small differences is that we just used $\frac{1}{2}$ for the time steps of 6-months in the tenor structure, so we did not account for the underlying
day count convention. Because of the size of the error and because it is not likely to influence the behaviour of the model, we won’t bother ourselves too much with these (small) differences.

<table>
<thead>
<tr>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.86 · 10⁻⁴</td>
</tr>
</tbody>
</table>

**Table 8.1:** The RMSE between the swaprate given by Bloomberg and the swap rate derived from the term structure. The average of the swap rates is 2.25 · 10⁻²

Calibrating the Hull-White model in a market consistent way is equivalent to finding the values of $a$ and $\sigma_r$ for which

\[
\sqrt{\frac{\sum_{i=1}^{N} w_i (\text{MarketValue}_i - \text{ModelValue}_i(\psi))^2}{N}}
\]

(8.1)

is minimal. Here $\psi = (a, \sigma_r)$, $N$ is the number of available market prices and $w_i$ is the weight attached to the $i$-th market value price. The model values of swaptions are given by Proposition 3.9. The values for $a$ and $\sigma_r$ derived by the least mean square estimation of (8.1) with weights $w_i = 1$ for all $i$ are given in Table **8.2**

<table>
<thead>
<tr>
<th>$a$</th>
<th>2.34 · 10⁻²</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_r$</td>
<td>0.82 · 10⁻²</td>
</tr>
</tbody>
</table>

**Table 8.2:** Parameters for the Hull-white model calibrated to swaption prices

### 8.2 Equity model

The equity model will be calibrated to ATM European call options on the Euro Stoxx 50 index. The Euro Stoxx 50 is a stock index based on 50 of the most important stocks in the Eurozone. The index reflects the total (weighted) return of the stocks and is corrected for paid dividends by ‘reinvesting’ them in the index constituents in proportion to their weights. The options traded on the index are among the most liquid options in the world. The prices and implied volatilities of the ATM options traded on 31-12-2012 were taken from Bloomberg.

In contrast to swaptions, the index options are only liquidity traded up to a maturity of two years, but since insurance liabilities often have a much higher
maturity we would like to fit our model to maturities up to 50 years. If we want to be able to do this, we must extrapolate the market data.

The extrapolation method we use here, is a (simplified) version of the method proposed by the CRO Forum [11]. The CRO Forum is a group of professional risk managers from the insurance industry that focuses on developing and promoting industry best practices in risk management. The extrapolation method is based on the market wide accepted assumption and observation that volatility follows a mean reverting process. In the model the forward implied volatility is discussed, which is related to the implied volatility in the same way as forward rates are related to the zero-coupon interest rate structure. So we have

\[ IV^2(T) = \frac{1}{T} \int_0^T IV^2_{FWD}(t) dt \]  

(8.2)

They propose that the forward volatility should converge to a long term forward volatility in the following way for \( t \geq T \):

\[ IV^2_{FWD}(t) = \text{Market} IV^2_{FWD}(T)e^{-K(t-T)} + \text{LongTerm} IV^2_{FWD}(1 - e^{-K(t-T)}). \]

Here is \( T \) the time of the last liquid data point, Market\( IV^2 \) the forward implied volatility derived from market quotes, LongTerm\( IV^2_{FWD} \) the long term implied volatility and \( K \) the mean reversion speed. The long term forward implied volatility can be derived using historical market data. In general, implied volatilities are higher than historical volatilities which is supposed to be a consequence of the cost of hedging/replicating an option. Furthermore there is a lot of uncertainty about what the long term implied volatility really is and this should also be accounted for in the form of a surcharge on the historical long term implied volatility.

Because the determination of the long term implied volatility requires a lot of data and expert knowledge, we will use the following simplification of the model proposed above:

\[ IV^2_{FWD}(t) = IV^2_0 e^{-\alpha t} + IV^2_\infty (1 - e^{-\alpha t}). \]

Implementing this to (8.2) we get

\[ IV^2_{model}(t) = IV^2_\infty + \frac{1}{\alpha T} (1 - e^{-\alpha T})(IV^2_0 - IV^2_\infty) \]

and we can obtain the values by, once again, minimising the rooted mean square error between the quoted market prices and the model prices.

\[ RMSE = \sqrt{\frac{\sum_i (IV_{model}(t_i) - IV_{market}(t_i))^2}{N}} \]
Figure 8.3: The extrapolated option implied volatility curve and the implied volatilities quoted in the market as circles

$N$ is the number of quoted market prices. See Figure [8.3] for a graphical comparison between the extrapolated volatilities and the market volatilities.

The calibration can be done by minimizing the sum of squared errors between market and model prices, see (8.1), where $\psi = (\sigma_S, \rho)$. The market prices are derived from the extrapolated implied volatility curve by using the term structure we also used with the interest rate model together with the famous Black-Scholes option pricing formula. The results can be seen in table (reference)

| $\sigma_S$ | 0.24 |
| $\rho$    | -0.27 |

Table 8.3: Parameters for the Hull-white model calibrated to swaption prices

### 8.3 Real estate model

For the calibration of the real estate model the ROZ/IPD Property Index is used. The ROZ/IPD Property Index was established in 1997 and measures direct property returns in the Netherlands dating back to 1995. It is published annually.

As we discussed before, the index has to be unsmoothed before we can use it for the calibration of our model. The unsmoothing method used, is that of Fisher et al. [17].
Quan and Quigley [39] [40] have shown that in the case of an annual index an AR(1) model will adequately capture the individual appraisers behaviour. In the case of a quarterly index an AR(4) index would be the best choice, because some properties are only appraised every half year or year in stead of every quarter.

Remember that we needed to assume that the volatility of the real returns was half the volatility of stock returns. Because we chose to model our equity on the Euro Stoxx 50 index, the volatility of stock returns has been derived from historical data of the Euro Stoxx 50 index over the time period corresponding to the time period over which we unsmooth the real estate index (1995-2012). The volatility over this period was 25.99% annualized.

The result of the unsmoothing can be seen in Figure 8.4 where we plotted both the original data as the unsmoothed data.

![Figure 8.4: Smoothed and unsmoothed real estate return (left) and real estate index (right).](image)

As we can see the resulting volatility of the unsmoothed returns is significantly higher than the unsmoothed returns and the resulting unsmoothed index values are considerably lower at this point in time. A very desirable feature of the unsmoothed index is that it reflects the 2007-2008 financial crisis, the resulting crash in stock markets and the following decline of real estate prices. Moreover, it also gives a good reflection of the current price level compared to the level before the crisis. The Dutch Central Bureau for Statistics (CBS) reports a decline of 21.3% in housing prices since the top of the housing bubble end 2007 [6]. This is fairly close to the 24% decline which can be derived from the unsmoothed index.

To calibrate our model we need the volatility of real estate $\sigma_{RE}$ as well as the correlations of real estate with interest rates $\rho_{r,RE}$ and equity $\rho_{S,RE}$. In equation (5.1) we defined the drift term for our real estate returns as the, at that time, prevailing short rate. Therefore, to derive the volatility of real estate from
the unsmoothed index and to get correct correlations, we first have to correct the
index for the volatility of interest rates.

The correction is done as follows. First the index values are discounted
by the historical interest rates; that is

\[ ROZ_{\text{disc}}(t) = ROZ(t) \exp \left( - \int_0^t r(s) ds \right), \]

for \( t = 0 \) at 31 December 1994 and \( t = 1, 2, \ldots, 18 \). The historical interest rates
used for the discounting are those published by the Dutch Central Bank (DNB) at
their website. Secondly the (yearly) log-returns of the discounted index values are
taken. Finally, the volatility of real estate is taken to be the empirical volatility over
these log returns and the correlations are calculated by using the historical interest
rates and the historical yearly log returns of the Eurostoxx50 index. The results
are given in table (reference). (Eventueel plot Stoxx50 en smoothed/unsmoothed
vergelijken nog)

<table>
<thead>
<tr>
<th>( \sigma_{RE} )</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_{r,RE} )</td>
<td>-0.09</td>
</tr>
<tr>
<td>( \rho_{s,RE} )</td>
<td>0.38</td>
</tr>
</tbody>
</table>

**Table 8.4**: Parameters for the real estate model derived from historical data.

### 8.4 Inflation model

The calibration of the inflation model is based on the eurozone inflation rates from
1991 to 2012 published by Eurostat.

As was already mentioned in the previous chapter, the mean and variance
of the Vasicek model are used to calibrate the model. If the sample time of data
in the data set is \( \delta \) then from (6.1) and (6.2) we can write the dynamics of the
inflation as

\[ I_{i+1} = I_i e^{-\lambda \delta} + \mu \left( 1 - e^{-\lambda \delta} \right) + \sigma \sqrt{\frac{1 - e^{-2\lambda \delta}}{2\lambda}} Z, \]

where \( Z \sim N(0, 1) \). Thus the relation between consecutive observations \( I_i, I_{i+1} \) is
linear with an iid normal random term \( \epsilon \):

\[ I_{i+1} = aI_i + b + \epsilon. \]
Now the data can be fitted in a least square sense to a linear function and subsequently we can calculate the model parameters from the following equations:

\[
\begin{align*}
\lambda &= -\frac{\log a}{\delta}, \\
\mu &= \frac{b}{1 - a}, \\
\sigma_I &= s \delta \sqrt{\frac{-2 \log a}{\delta (1 - a^2)}}.
\end{align*}
\]

For the calibration yearly data is used, so \(\delta = 1\). In figure (referentie) the result of the least squares regression on the data is visualised. The model parameters and the correlations, which are calculated based on the historical data of the various risk factors, can be found in Table 8.5.

![Figure 8.5: The percentage of inflation plotted against inflation in the previous year (red) and the result of least squares fitting a line to the data (blue) ](image)

**8.5 simulation**

When all the model parameters and the correlations between the various models have been estimated, the future paths of the interest rates, stock index, real estate index and the inflation can be simulated.

The simulation of each of the models is done by following the Euler scheme described in Chapter (referentie). The four stochastic models in our ESG are correlated. Therefore a 4-dimensional correlated Brownian motion or, under the discrete
Table 8.5: Parameters for the real estate model derived from historical data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.9812</td>
</tr>
<tr>
<td>$\mu$</td>
<td>2.1342</td>
</tr>
<tr>
<td>$\sigma_I$</td>
<td>1.08</td>
</tr>
<tr>
<td>$\rho_{r,I}$</td>
<td>0.30</td>
</tr>
<tr>
<td>$\rho_{s,I}$</td>
<td>-0.76</td>
</tr>
<tr>
<td>$\rho_{RE,I}$</td>
<td>-0.34</td>
</tr>
</tbody>
</table>

Suppose now $\Sigma$ is the correlation matrix that is the outcome of the calibration, then we want to generate values from a $N(0, \Sigma)$ distribution. The way to do this has already been mentioned in Chapter 4, but will also be repeated here. First a 4-dimensional vector $Z = (Z_1, ..., Z_4)$ of independent standard normal random variables is simulated. Secondly, we apply the Choleski decomposition to $\Sigma$ to get a lower triangle matrix $L$ for which it holds that $\Sigma = LL^\top$. The last and final step is then to multiply $L$ and $Z$, since then $LZ \sim N(0, \Sigma)$.

While the Choleski decomposition of a correlation matrix into a lower triangle matrix is unique, there are multiple ways to decompose the correlation matrix into $\Sigma = LL^\top$; any of these decompositions would also lead to $LZ \sim N(0, \Sigma)$.

When the timestep $t_{i+1} - t_i$ for the Euler method has been set, e.g. a month, a year etc., and the number of scenarios to be generated has been determined, we can let the ESG produce scenarios. In Figure 8.6, 8.7, 8.8 and 8.9 an example of generated scenarios together with their average is shown.
Figure 8.6: Generated interest rate scenarios (left) and the mean of the scenarios (right) based on market data at 31-12-2012

Figure 8.7: Generated scenarios of the Eurostoxx50 index (left) and the mean of the scenarios (right) based on market data at 31-12-2012

Figure 8.8: Generated scenarios of the ROZ/IPD real estate index (left) and the mean of the scenarios (right) based on market data at 31-12-2012
Figure 8.9: Generated scenarios of the inflation rate in the eurozone (left) and the mean of the scenarios (right) based on market data at 31-12-2012
Chapter 9

Insurance Products

The whole purpose of the construction of an ESG, calibrating it to market data and the generation of future scenarios is to calculate the technical provisions for insurance contracts. Therefore, two types of life insurance products are modelled in this thesis and will be used to study some of the properties of our ESG. In this chapter we will discuss these two products and the way they are modelled.

The two life insurance products that will be modelled are a *surplus interest sharing* policy and an *endowment* policy. For both policies the maturity of the contract is defined as the time to retirement of the policyholder, which for convenience is set at 67 for all policy holders. The distribution of the number of policies as a function of the age at the inception of the contract is given in Figure 9.1. This is a (simplified version of a) distribution as it could be found in the real world for an insurance policy.

![Figure 9.1: Number of policies as a function of the age of the policy holder at inception.](image)

At inception, the policyholders of both products are guaranteed a minimal amount of money at maturity. We model the products as single-premium assurance policies; this means that no premiums are paid during the term of the policy. Also,
it is simply assumed that all costs the policy holder has to pay to the insurer for their services are deducted from the single-premium. Because someone who is older usually has more savings and therefore can invest more, the guaranteed amount of cash also depends on the age of the policy holder at the start of the contract. The guaranteed amounts are plotted in Figure 9.2.

![Figure 9.2: Guaranteed amount of money at inception as a function of age.](image)

Mortality rates are also modelled, since these also have an impact on when certain cash flows take place. Often with these products a life insurance policy is sold. This ensures that in the event of the demise of the policy holder before the maturity date the guaranteed sum is paid to the heirs. In the model it is assumed that every policy is sold with a life insurance policy. In practice, the chance of passing is modelled as a function of age, the year of birth, gender and other factors like yearly income. For convenience, mortality is modelled only as a function of age. The yearly mortality rates in the model can be seen in Figure 9.3; these mortality rates are in line with empirical results [1].

![Figure 9.3: Yearly mortality rates based on empirical results.](image)

To calculate the technical provisions the future cash flows have to be discounted. Echoing the opinion of the European Commission and in anticipation
of the introduction of Solvency II, the Dutch Central Bank (DNB) imposed the use of the so called Ultimate Forward Rate curve (UFR curve) from 30 June 2012 for the discounting of cash flows of insurers [13]. For terms below 20 years, the UFR curve is derived from the European swap rates and therefore for these terms equal to the curve we used when calibrating the interest rate model. For terms longer than twenty years, the forward rate is extrapolated from the twenty years forward rate to the Ultimate Forward Rate, which is set at 4.2%. The 4.2% is based on historical data for inflation (2%) and real interest rates (2.2%). The convergence time is 40 years, so the 60 years forward rate is 4.2%.

There are two main reason for the introduction of the UFR. The first one is that bond markets are not liquid for terms higher than 20 years and therefore market prices are not reliable. The second reason for introducing the UFR is to make insurers with long term obligations, e.g. the obligations corresponding to the insurance products that we use here, be less influenced by short term changes in interest rates. One of the arguments why this is a better method compared to using the market rates is that beyond twenty years the liquidity of swaps drops and therefore aren’t trustworthy. Of course the risk is that the current low interest rates don’t rise in the long run, which would mean that the current technical provisions are too low with respect to the future liabilities [41]. The UFR curve is published every month by DNB. The curve at 31-12-2012 is given in Figure 9.4.

![Figure 9.4: The Ultimate Forward Rate curve published by DNB for 31-12-2012.](image)

**Surplus interest sharing policy**

For the surplus interest sharing policy the amount of money paid at maturity depends on the yearly return on a portfolio of government bonds with respect to a fixed discount rate.
In the Netherlands the portfolio of Dutch government bonds is standardized by the Association of Insurers. The portfolio consists of bonds with a term between 2 and 15 years. The return on the portfolio is called ‘U-rendement’ and is reported every month by the Centre for Insurance Statistics (CVS).

In the model the yearly returns are calculated in the same way as is done in practice. However, it should be noted that we make the assumption here that the risk-free return in our model is the same as the return on Dutch government bonds.

The returns on the bond portfolio are compared every year with a fixed discount rate, which in our model is three percent. If the return is higher than the fixed rate the guaranteed sum will be increased with half of the surplus return. This means that if the return on the portfolio at the end of a year is 4% then the guaranteed sum will be increased with 0.5%. Only a part of the surplus return is given to the policy holder to protect the insurer against returns below the fixed rate in future years.

An important feature of this policy is that if the returns fall below the fixed discount rate, the guaranteed sum won’t be increased, but more importantly it won’t be lowered either.

Also, because the return is based on a standardized portfolio which is monitored by the CVS, the return is not necessarily connected to the actual returns the insurance company is making.

**Endowment policy**

The endowment policy gives the policy holders a choice between different asset mixes to invest in. At the end of the term the insurer pays the policy holder the value of the investment at that time or the guaranteed sum if the value of the investments has fallen below the guarantee.

There are three types of portfolios the policy holder can invest in: a ‘fixed income’ portfolio, with only bonds with a remaining term between 2 and 15 years, a mixed portfolio, with 60% bonds, 25% stock and 15% invested in real estate and a ‘risky’ portfolio, which for 70% consists of stocks and 30% of real estate investments. At the end of every year the different portfolios are rebalanced to match their initial weights. For example, the risky portfolio might consist of 80% stocks and 20% real estate at the end of a year due to price movements in the last year, then stocks have to be sold and real estate has to be bought in order to get 70% stocks and 30% real estate in the portfolio at the start of next year.

Also, at the end of every year the guaranteed amount will be increased with the inflation percentage over that year. This way the purchasing power of the policy holder at maturity remains in tact.
Chapter 10

Analysis of the technical provisions

In this chapter we will have a look at the technical provisions (TP) corresponding to the modelled insurance products and in what way they are influenced by changing underlying conditions. First we will look at in what way the conditions of the policies influence the TP, then we will look at changes in the market conditions, thirdly estimation errors will be discussed and last but not least we will look at changes that can be made to the calibration procedure.

10.1 Policy conditions

Now some insights will be given into what kind of impact some aspects of the insurance policies have on the technical provisions. The calculations are done with scenarios generated based on the market data at 31-12-2012. The technical provisions for the situation described above, which will be referred to as the 'standard situation' from now on, can be found in Table 10.1:

<table>
<thead>
<tr>
<th>Product Type</th>
<th>Standard situation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>199.3</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>15.3</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>36.4</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>90.5</td>
</tr>
</tbody>
</table>

Table 10.1: Technical provisions in millions of euros for the standard situation.

What might be notable is that the TP of the surplus interest sharing policy are much higher than that of the endowment policies. This is because for
the endowment policy the underlying portfolio is modelled and therefore the cash flows are only due to differences of the guaranteed with the guarantee, while no underlying portfolio is modelled for the interest sharing policy.

distribution of the cash flows

When we look at the TP of the endowment policies there is also a big difference between the TP of the fixed income portfolio and the riskier portfolios. The policy holder gets all the extra funds when the value of the portfolio is above the guarantee, but the insurance company has to make up the whole difference if the value is below the guarantee. Because equity and real estate are more volatile than bonds, the probability of a low value of the portfolio is bigger for the 'mixed' and 'risky' portfolio than for the 'fixed income' portfolio. Therefore, the technical provisions corresponding to the mixed and risky portfolio are higher. In Figure 10.1, this is illustrated with a histogram, for the risky and the fixed income portfolio, of the discounted value of the cash flows resulting from the different scenarios.

![Histogram of discounted cash flows](image)

**Figure 10.1:** Current value of the cash flows resulting from the different scenarios; for the fixed income portfolio (blue) and the risky portfolio (red)
mortality rates

An aspect which gives us a good opportunity to look at some of the properties of the model and the insurance policies is the mortality rate. The technical provisions are calculated for two different situations. In the first situation the mortality rates are set to zero, so no one dies before the age of 67. This is not a realistic assumption, but it gives us some insight in what impact mortality has on the technical provisions of a life insurance company. In the second situation the products are modelled without the life insurance policy. In the case of the surplus interest sharing policy the absence of a life insurance policy means that no money will be paid upon death, while in the case of the endowment policy it means that there is no guaranteed sum; the value of the investment at the time of death will be paid, no matter whether it is below or above the current guaranteed sum. The technical provisions are given in Table 10.2.

<table>
<thead>
<tr>
<th>Policy Type</th>
<th>No mort.</th>
<th>diff.</th>
<th>No life ins.</th>
<th>diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>194.9</td>
<td>−2.19%</td>
<td>170.6</td>
<td>−14.37%</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>15.0</td>
<td>−2.09%</td>
<td>13.1</td>
<td>−14.29%</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>36.8</td>
<td>1.08%</td>
<td>31.9</td>
<td>−12.43%</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>92.6</td>
<td>2.29%</td>
<td>79.9</td>
<td>−11.75%</td>
</tr>
</tbody>
</table>

Table 10.2: Technical provisions in millions and the percentage difference w.r.t. the standard situation for when the mortality rates are set to zero and when policy holders have no extra life insurance policy.

The difference between the standard situation and the situation without a life insurance policy is the most easily explained. For the no life insurance case, when someone dies, the insurance company has to pay nothing to the heirs of a surplus interest rate policy and only has to pay the current value of the investments in the case of the endowment policy, which results in a net cash flow of zero. In the standard case the insurer has to pay (at least) the guaranteed amount to the heirs and therefore it is obvious that the technical provisions in that case are higher.

The impact of zero mortality rates on the technical provisions is more complex and we need to discuss the policies separately. The lower technical provisions for the surplus interest rate policy are due to the discounting of cash flows. If a policy holder lives at least until the age of 67, the guaranteed sum has to be paid later than when someone dies before this age. Because the cash flows have to be discounted, the same cash flow at a later point in time gives a lower current value of that cash flow. Of course, because of surplus interest sharing, the guaranteed amount could have gone up between the moment of death and when the policy holder would have reached the age of 67, but the surplus interest is most of the
time lower than the yearly discount rates and therefore doesn’t raise the current value. A visual representation of the difference between the yearly discount rates and the surplus interest can be found in Figure 10.2.

For the ‘fixed income’ portfolio sort of the same thing holds as for the surplus interest sharing policy. Remember that the guarantee is increased with the inflation rate every year. Since on average the return made on the portfolio of bonds minus the inflation is lower than the yearly discount rate, the current value of a cash flow resulting from someone passing away before the age of 67 is higher than the current cash flow resulting from someone reaching the age of 67. Figure 10.2 also gives a visual representation of the difference between the yearly discount rates and the average return minus the inflation.

![Figure 10.2: Yearly discount rates derived from the UFR curve (blue) compared with the average surplus interest (green) and the average return on a bond reduced with the inflation rate (red).](image)

Now the question is why the technical provisions go up for the riskier portfolios, since one could argue that because of the risk neutral way in which we modelled equity and real estate the value of those asset classes, on average, grows at the rate of the modelled interest rates and therefore the same argument as before should hold. However, the mean growth rate is not the prevailing factor in this case. The coverage ratio is the value of the portfolio divided by the total value of the remaining guarantees. As we saw in Figure 10.1 the variance of the coverage ratio is much larger for the risky portfolios than for the bond portfolio. In Figure 10.3 we can see that, while the mean of the coverage ratio goes up, the variance grows larger in time and the median drops. This implies that the probability of a low coverage ratio is larger further into the future and in turn
this leads to a higher expectation of the value of a future cash flow. Therefore the technical provisions are higher when the mortality rates are set to zero.

Figure 10.3: The mean of the coverage ratio for the risky portfolio (red), the median (blue) and the 16th and 84th percentile given by the error bars.

Coverage ratio

What might also be interesting is to investigate how sensitive the technical provisions are to changes in the coverage ratio of the endowment policies by changing the coverage ratio at the start of the simulation. In the standard case we implicitly assumed that the coverage ratio was 100%.

It would be even more interesting if we knew the probabilities with which the coverage ratio could go up or down in the coming year, because then we would be able to give a confidence interval for the TP. However, we need to remind ourselves that we are dealing with risk-neutral scenarios and not real world scenarios. Therefore, the construction of a confidence interval is not possible.

In Table 10.3 the TP are given for different coverage ratios at the start of the simulation and in Table 10.4 the percentage difference compared to the standard situation are given.

As could be expected, the relative changes of technical provisions in Table 10.4 are the largest for the fixed income portfolio. This is because bonds are less volatile than stocks and real estate and hence the chances of the coverage ratio getting back to 100% are smaller.

It is also worthwhile to look at the shape of the curve of the technical provisions plotted against the coverage ratio. For the fixed income portfolio the curve is given in Figure 10.4. The shape of the curve is similar to that of the price
### Table 10.3: Technical provisions in millions for different coverage ratios at the start of the simulation.

<table>
<thead>
<tr>
<th>Coverage ratio</th>
<th>80%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
<th>101%</th>
<th>105%</th>
<th>110%</th>
<th>120%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment: fixed</td>
<td>49.7</td>
<td>30.3</td>
<td>22.1</td>
<td>16.6</td>
<td>14.2</td>
<td>10.2</td>
<td>6.5</td>
<td>2.6</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>63.0</td>
<td>48.2</td>
<td>41.2</td>
<td>37.5</td>
<td>35.4</td>
<td>31.6</td>
<td>27.4</td>
<td>20.7</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>108.7</td>
<td>99.0</td>
<td>94.7</td>
<td>91.4</td>
<td>89.9</td>
<td>86.9</td>
<td>83.4</td>
<td>77.0</td>
</tr>
</tbody>
</table>

### Table 10.4: Percentage change of the technical provisions compared to when the coverage ratio is 100%.

<table>
<thead>
<tr>
<th>Coverage ratio</th>
<th>80%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
<th>101%</th>
<th>105%</th>
<th>110%</th>
<th>120%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment: fixed</td>
<td>223.9%</td>
<td>97.0%</td>
<td>44.0%</td>
<td>8.0%</td>
<td>−7.6%</td>
<td>−33.8%</td>
<td>−57.7%</td>
<td>−83.1%</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>73.1%</td>
<td>32.5%</td>
<td>15.3%</td>
<td>2.9%</td>
<td>−2.8%</td>
<td>−13.3%</td>
<td>−24.9%</td>
<td>−43.3%</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>19.9%</td>
<td>9.2%</td>
<td>4.5%</td>
<td>0.9%</td>
<td>−0.9%</td>
<td>−4.2%</td>
<td>−8.0%</td>
<td>−15.1%</td>
</tr>
</tbody>
</table>

of a put option plotted against the strike price. This is not too strange, since from the insurers point of view the insurance product can actually be seen as a portfolio of short put options on the chosen portfolio. If the value of the portfolio is below the strike price (the guaranteed amount) at maturity, the insurer has to pay the difference and if it is above the strike price it doesn’t have to pay anything.

### 10.2 Market conditions

The market prices are used as an input for the ESG and cannot be influenced. However, it can still be important to monitor the way a model behaves under changing market conditions to check whether or not the changes in the estimated technical provisions are consistent with the changes in the market conditions.

Because gathering a lot of market data and calibrating the model to all the different market data is very time consuming, extensive research about how the model behaves with respect to changes in market data is not feasible. However, to get an idea of what the course of the technical provisions during a year can be and how much the technical provisions change during a year, the ESG is calibrated to end of month data starting at 31-12-2011 and ending at 31-12-2012. The development of the technical provisions during the year can be found in Figure 10.5 and Figure 10.6. The standard deviation of the TP over the data can be found in Table 10.5.

One of the most pronounced changes in the technical provisions is the one corresponding to the end of May 2012. This gives an interesting insight in how the
Figure 10.4: The technical provisions for the fixed income portfolio plotted against the coverage ratio.

<table>
<thead>
<tr>
<th></th>
<th>Absolute</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>2.12</td>
<td>1.05%</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>3.29</td>
<td>26.79%</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>3.75</td>
<td>11.57%</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>4.74</td>
<td>5.34%</td>
</tr>
</tbody>
</table>

Table 10.5: Standard deviation (absolute value in millions and as a percentage of the mean) of the end of month data of the technical provisions ranging from 31-12-2011 to 31-12-2012.

TP can be influenced by turmoil in the financial market. In May 2012 elections of the parliament of Greece were held, but Greece failed to form a government and new elections were scheduled for June. Because of the bad financial situation of Greece, this caused turmoil on the financial markets during May.

From the standard deviations we can see that the technical provisions can be quite volatile. Therefore it is important that an insurance company tries to hedge their risks and where this is not entirely possible holds an additional capital to ensure it is able to meet its obligations.

10.3 Estimation error

As we discussed in Chapter 7 the Monte Carlo simulation of the stochastic models produces a statistical or distributional error and a discretisation error. In this
section we will look at the influence of the number of scenarios that is generated, which corresponds to the statistical error, and the influence of size of the time steps in the generated scenarios, which corresponds to the discretisation error.

Number of scenarios

To get good results it is important that the distributional error is minimized. Because of the central limit theorem we know that the more simulations are done, the smaller this error will become. However, doing more simulation comes at the price of computation time. Therefore, we need to find a balance between the number of scenarios and the distributional error.

The number of scenarios that were generated for the calculation of the TP ranged from 10 to 15000. For each number of scenarios the ESG was run 100 times, which means that 100 estimates of the TP were generated for each preset number of scenarios. In Table 10.6 the standard deviations of the estimated technical provisions are given for each category and in Figure 10.7 the technical provisions for the mixed portfolio in the endowment policy are given as a function of the generated scenarios for a set of 10000 scenarios.

A criterium was set for choosing an optimal number of generated scenarios, i.e. the number of scenarios for which the distributional error is small enough, while keeping the computation time as low as possible. The criterium used was that the standard deviation of the estimated technical provisions has to be less
than 1% of the mean of the estimated technical provisions. As it turns out this criterium is only met for all products when 10000 or 15000 scenarios are generated and therefore to limit the computation time the standard number of scenarios to generate was set at 10000. The percentages for 5000, 10000 and 15000 are given in Table 10.7.

Let us take a brief look at what this really means to set this criterium at 1%. In Table 10.1 the given estimate of the technical provisions for the endowment policy with the risky portfolio was 90.5 million euro. This means that following our criterium when doing simulations one standard deviation still corresponds to almost one million euro. Therefore, it is very important for insurance companies to limit the distributional error that is the result of Monte Carlo simulation, since errors that are relatively small can lead to big absolute errors in terms of money.

<table>
<thead>
<tr>
<th>Nr. of scenarios</th>
<th>10</th>
<th>100</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
<th>15000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>5.99</td>
<td>1.65</td>
<td>0.82</td>
<td>0.50</td>
<td>0.41</td>
<td>0.19</td>
<td>0.15</td>
<td>0.13</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>3.26</td>
<td>8.87</td>
<td>0.49</td>
<td>0.29</td>
<td>0.20</td>
<td>0.17</td>
<td>0.10</td>
<td>0.07</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>12.52</td>
<td>3.93</td>
<td>1.73</td>
<td>1.10</td>
<td>0.76</td>
<td>0.60</td>
<td>0.37</td>
<td>0.23</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>23.27</td>
<td>7.02</td>
<td>3.26</td>
<td>2.19</td>
<td>1.48</td>
<td>1.06</td>
<td>0.66</td>
<td>0.49</td>
</tr>
</tbody>
</table>

Table 10.6: Standard deviation (in millions) of the technical provisions for different numbers of generated scenarios.
Figure 10.7: Technical provisions for the mixed portfolio of the endowment policy calculated by each time adding one scenario.

<table>
<thead>
<tr>
<th>Nr. of scenarios</th>
<th>5000</th>
<th>10000</th>
<th>15000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>0.09%</td>
<td>0.07%</td>
<td>0.07%</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>1.10%</td>
<td>0.68%</td>
<td>0.45%</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>1.64%</td>
<td>0.99%</td>
<td>0.65%</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>1.17%</td>
<td>0.73%</td>
<td>0.54%</td>
</tr>
</tbody>
</table>

Table 10.7: Standard deviation of the estimated TP as a percentage of the mean of the estimated TP.

The number of steps

As we saw in Chapter 7 in addition to the distributional error there is also a discretisation error which is the result of approximating our continuous stochastic process with a discrete stochastic process by means of the Euler-Maruyama method. If we choose to do our simulation with smaller time steps, this comes at the cost of computation time.

Simulations were done with time steps of 1 year, 6 months, 3 months, 2 months and 1 month; the resulting technical provisions are given in Table 10.8. Afterwards the differences of the consecutive refinements were taken.

All the differences fall within two times the standard deviations given in Table 10.6 for 10000 scenarios, the majority within one standard deviation and no clear trend is visible in the refinement steps. Therefore, it is very hard to distinguish the discretisation error from the distributional error. The choice has been made to simulate the scenarios with a time interval of 6 months.
<table>
<thead>
<tr>
<th>Size of the steps</th>
<th>1 year</th>
<th>6 months</th>
<th>3 months</th>
<th>2 months</th>
<th>1 month</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>199.4</td>
<td>199.4</td>
<td>199.5</td>
<td>199.3</td>
<td>199.1</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>15.2</td>
<td>15.3</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>35.5</td>
<td>35.6</td>
<td>36.3</td>
<td>36.0</td>
<td>36.4</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>90.8</td>
<td>89.9</td>
<td>90.3</td>
<td>89.9</td>
<td>90.0</td>
</tr>
</tbody>
</table>

Table 10.8: Technical provisions corresponding to different time steps taken during the simulation

10.4 Changes in the calibration

Until now we discussed some properties of the calculation of the technical provisions that were either due to the Monte Carlo simulation method we use or due to the characteristics of the insurance products. However, another very important way to look at the model is by looking at it from a more actuarial point of view. This means that we look at the freedom someone has in using this model and in which ways the choices resulting from this freedom can influence the outcomes of the model, e.g. an insurance company could make choices that lower the technical provisions.

What cannot be influenced are the cash flows that are related to the insurance products, because those are contractual. Nor can the market data be changed. One could choose to calibrate your model to different instruments, e.g. calibrate the interest rates to caps, but regulating authorities like DNB require insurance companies to calibrate their models in a consistent way over time for the calculation of the TP, so changing instruments whenever it is convenient isn’t possible.

As a result, the only freedom in the use of the ESG can be found in the calibration of the model. Therefore, we have to look for (sometimes implicit) assumptions we made when calibrating the models. We will have a closer look at choices made in the calibration of the interest rate model and the equity model.

Equity model

The most obvious assumption in the calibration of the ESG was made in the equity model. Because options are only deep liquidity traded for relatively short maturities, we had to extrapolate the implied volatility curve. We chose to fit a curve to the market data and one of the output variables was the long term forward implied volatility. However, in the extrapolation method which is proposed by the CRO Forum[11] the long term forward implied volatility is an input variable, which is mainly based on the historical volatility of the stock index. In the paper of the
CRO Forum the long term volatility of the Eurosoxx50 index, based on data up to 2008, is given as 23.06%. From our own data, which we got from Bloomberg and which goes up to 2012, a volatility of 25.99% is derived. These two numbers are used as a reference in the investigation of the impact of calibrating the model to different long term volatilities. It might be interesting to note that the long term volatility that is a result of our 'standard' calibration is 25.52%, which is not a bad result if we look at the two other percentages. The model has been calibrated for long term volatilities ranging from 20% to 28% with steps of 2%. The TP and the $\sigma_S$ parameter of the equity model can be found in Table 10.9; the percentage difference of the TP compared to the standard situation can be found in Table 10.10. We only calculated the TP for the mixed and risky portfolios since equity has no influence on the other products.

<table>
<thead>
<tr>
<th>Long term vol.</th>
<th>20%</th>
<th>22%</th>
<th>24%</th>
<th>26%</th>
<th>28%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment: mixed</td>
<td>31.6</td>
<td>33.2</td>
<td>35.1</td>
<td>36.7</td>
<td>38.1</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>77.6</td>
<td>81.9</td>
<td>87.7</td>
<td>92.3</td>
<td>96.7</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.228</td>
<td>0.237</td>
<td>0.246</td>
<td>0.255</td>
<td>0.264</td>
</tr>
</tbody>
</table>

Table 10.9: Technical provisions as a result of different long term implied volatilities and the $\sigma_S$ parameter for equity.

<table>
<thead>
<tr>
<th>Long term vol.</th>
<th>20%</th>
<th>22%</th>
<th>24%</th>
<th>26%</th>
<th>28%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment: mixed</td>
<td>-13.20%</td>
<td>-8.93%</td>
<td>-3.60%</td>
<td>0.89%</td>
<td>4.73%</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>-14.33%</td>
<td>-9.51%</td>
<td>-3.12%</td>
<td>1.96%</td>
<td>6.75%</td>
</tr>
</tbody>
</table>

Table 10.10: Percentage change of the TP compared to the standard situation, where te long term volatility is 25.52%.

As we can see the choice of a long term forward implied volatility can have a significant impact on the outcome of the technical provisions. The change of 25.52% to 26% already leads to a change of almost two percent in the technical provisions, which is more than 1.5 million in absolute value.

**Interest rate model**

In the calibration of the interest rate model to the swaption volatility surface, or equivalent, to the prices of swaption we made an assumption regarding the least square fitting. In equation (8.1) we can choose weights $w_i$ for the each of the market prices and we chose to set the weights equal to one and didn’t wonder if this was the right or best decision.
However, for the different maturities and different tenors we could argue that setting all the weights to one is not the only feasible choice. There are a lot quoted prices with a short maturity and tenor and only few prices with longer maturities and tenors as can be seen in Figure[10.8] Therefore, the standard calibration, which gives weight one to each quoted price, assigns more weight to short terms (0-5 years) and in comparison less weight to medium terms (5-15 years) and to long terms (15+ years). The modelled insurance policies have the majority of their obligations at medium and long terms, so assigning different weights seems justifiable.

![Swaption volatility matrix for 31-12-2012 from Bloomberg.](image)

A different method to deal with this problem would be to interpolate the volatility surface and calculate the prices at equidistant points. However, to avoid further complications with choosing the right extrapolation method, we will adjust the weights.

Before we propose how we can assess the weights for all the quoted prices, let us look at an example, where we compare two quotes: a 2 year tenor and 5 year maturity swaption and a 25 year tenor and 20 year maturity swaption. The distance to the next tenor and the last tenor for the first price is 1 year, when keeping the maturity at 5 years and the same holds for the distance to the next and last maturity when keeping the tenor at 2 years. For the second quoted price the distance to the next tenor and the last tenor is 5 years, when you keep the maturity at 20 years and the same holds for the distance to the next and last maturity when keeping the tenor at 25 years. To keep the weight \( w_i \) proportional to the distance to the neighbouring points we can set a weight of \( 1 \times 1 = 1 \) for the first point and \( 5 \times 5 = 25 \) for the second point

Let \( i_1, ..., i_n \) be the different maturities and let \( j_1, ..., j_m \) be the different

<table>
<thead>
<tr>
<th>Exp yr</th>
<th>1YR</th>
<th>2YR</th>
<th>3YR</th>
<th>4YR</th>
<th>5YR</th>
<th>6YR</th>
<th>7YR</th>
<th>8YR</th>
<th>9YR</th>
<th>10YR</th>
<th>12YR</th>
<th>15YR</th>
<th>20YR</th>
<th>25YR</th>
<th>30YR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MO</td>
<td>67.01</td>
<td>80.35</td>
<td>77.6</td>
<td>68.57</td>
<td>60.1</td>
<td>51.05</td>
<td>45.36</td>
<td>40.9</td>
<td>39.06</td>
<td>36.66</td>
<td>33.96</td>
<td>29.9</td>
<td>28.08</td>
<td>29.3</td>
<td></td>
</tr>
<tr>
<td>2 MO</td>
<td>69.03</td>
<td>86.13</td>
<td>78.72</td>
<td>67.38</td>
<td>55.18</td>
<td>45.78</td>
<td>40.46</td>
<td>36.64</td>
<td>33.94</td>
<td>31.19</td>
<td>29.17</td>
<td>29.95</td>
<td>29.65</td>
<td>29.3</td>
<td></td>
</tr>
<tr>
<td>3 MO</td>
<td>68.68</td>
<td>80.53</td>
<td>73.25</td>
<td>64.86</td>
<td>56.99</td>
<td>50.01</td>
<td>45.36</td>
<td>41.97</td>
<td>39.53</td>
<td>37.32</td>
<td>35.06</td>
<td>30.65</td>
<td>29.92</td>
<td>29.71</td>
<td>29.67</td>
</tr>
<tr>
<td>4YR</td>
<td>68.19</td>
<td>74.76</td>
<td>67.72</td>
<td>61.57</td>
<td>54.86</td>
<td>48.45</td>
<td>44.36</td>
<td>41.22</td>
<td>38.65</td>
<td>36.72</td>
<td>34.12</td>
<td>30.93</td>
<td>29.67</td>
<td>29.59</td>
<td>29.65</td>
</tr>
</tbody>
</table>

...
tenors and let us change the notation of the weights to $w_{ik, jl}$. Then, in agreement with the example, we can give a formula for the weights as follows for inner points:

$$w_{ik, jl} = \frac{i_k+1 - i_k}{2} \times \frac{j_l+1 - j_l}{2} \text{ for } 1 < k < n, 1 < l < m. \quad (10.1)$$

For the different border and corner cases the formula is adjustment in similar ways, therefore we only give one case:

$$w_{ik, jl} = \frac{i_k+1 - i_k}{2} \times \frac{j_l+1 - j_l}{2} \text{ for } k = 1, 1 < l < m. \quad (10.2)$$

To give some insight into the way the technical provisions change by changing the weights according to (10.1) (10.2) it was decided to do the optimisation with $w^k_{i, j}$ for $k = 0, 0.25, 0.5, 0.75, 1$. Note that $k = 0$ corresponds to the original situation. The results are given in Table 10.11.

<table>
<thead>
<tr>
<th>Size of the steps</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surplus interest</td>
<td>199.3</td>
<td>200.5</td>
<td>200.4</td>
<td>200.4</td>
<td>200.5</td>
</tr>
<tr>
<td>Endowment: fixed</td>
<td>15.3</td>
<td>15.7</td>
<td>15.7</td>
<td>15.8</td>
<td>15.5</td>
</tr>
<tr>
<td>Endowment: mixed</td>
<td>36.4</td>
<td>36.0</td>
<td>36.1</td>
<td>36.5</td>
<td>35.5</td>
</tr>
<tr>
<td>Endowment: risky</td>
<td>90.5</td>
<td>89.9</td>
<td>90.4</td>
<td>90.9</td>
<td>89.4</td>
</tr>
<tr>
<td>Parameter: $a$</td>
<td>0.0420</td>
<td>0.0422</td>
<td>0.0424</td>
<td>0.0424</td>
<td>0.0425</td>
</tr>
<tr>
<td>Parameter: $\sigma_r$</td>
<td>0.0103</td>
<td>0.0106</td>
<td>0.0107</td>
<td>0.0107</td>
<td>0.0107</td>
</tr>
</tbody>
</table>

Table 10.11: Technical provisions and the parameters $a$ and $\sigma_r$ corresponding to different weights in the calibration of the interest rate model.

The most striking is that the parameters of the interest rate model do not change much under the change of the weights. This is also reflected in the technical provisions, since the changes in TP fall within the two times the standard deviation of the distributional error we saw in Table 10.6. Moreover, there is no clear trend visible in the technical provisions.
Chapter 11

Summary, Conclusions and Discussion

In this thesis we have constructed an Economic Scenario Generator (ESG) with the purpose to calculate technical provisions (TP) of insurance contracts in a market consistent way. The reason for constructing an ESG was that other methods for the calculation of TP, e.g. using a replicating portfolio or a closed form solution, were too complex to implement because of the path dependency of the contacts and the multiple underlying risk factors.

The choice was made to model interest rates, equity, real estate and inflation rates. The interest rates were modelled with the Hull-White one factor model, equity and real estate were modelled with the Hull-White-Black-Scholes model and the inflation rates were modelled by the Vasiček model.

To get a market consistent ESG the models had to be calibrated to market prices, i.e. the market prices have to match the model prices as good as possible. Therefore, in (referentie chapters/secties interest rate and equity) we derived explicit formulas for swaptions and European options for the interest rate resp. equity model. Because no options or other liquidly traded derivatives were available for real estate and inflation rates in (referentie chapters/secties real estate en inflatie) we looked at methods to calibrate these models to historical data.

Subsequently in Chapter 8 we gave an example calibration of the models to market data at 31-12-2012. This was done by least square fitting the model prices to the market prices of at-the-money euro swaptions and European options on the Eurostoxx50 index. Here we first had to extrapolate the implied volatilities of the European options because only quotes of liquidly traded options with a short maturity were available. The real estate model was calibrated to the ROZ-IPD index and the inflation to the historical eurozone inflation. Afterwards scenarios were generated based on these calibrations.

Two types of insurance products were modelled: a surplus interest shar-
ing policy and an endowment policy for which one could choose between three different portfolios to invest in. These products were described in Chapter 9 and in Chapter 10 they were used to study some properties of the ESG in relation to the calculation of technical provisions for the policies. We looked at the impact of changes in the conditions of the policies, changes in market conditions, errors in the estimation of the TP and ways in which the calibration procedure can be changed and how that influences the technical provisions.

The most striking sensitivities in the model were the distributional error that is the result of Monte Carlo simulation and the choice of a long term implied volatility for the calibration of the equity model.

For the distributional error we saw that we needed at least 10000 scenarios in order to get the standard deviation of the estimated TP below 1% of the mean of the estimates. In practice this can still result in a large absolute error, since 1% of the TP can already correspond to millions of euros. The ESG of Barrie & Hibbert, which is one of the best and most used scenario generators in the world [15], can only generate up to 5000 scenarios. Their ESG uses different models than ours, e.g. the Displaced Diffusion Stochastic Volatility Libor Market Model and the Stochastic Volatility Jump Diffusion model, so we can’t really derive the error from the number of scenarios. Still, when using an ESG in practice, the distributional error is a very important factor to keep track of.

In the case of the extrapolation of the equity implied volatility, we saw from the paper of the CRO Forum [11] that the determination of the long term forward implied volatility relies a lot on expert knowledge. As a consequence, the determination of the long term volatility heavily depends on what data and argumentation one uses to determine it. Moreover, we saw that a change of only 0.5% of the long term volatilities can already lead to a 2% change in the technical provisions.

11.1 Improvement of the ESG

The ESG could be improved in several ways. It is possible to include models for other risk factors like credit spreads and exchange rates. Also, for each of the stochastic models we used, there are more advanced models that capture the market behaviour better than the models used here. We did not use these models, because the chosen models already capture market behaviour adequately and because those models are too complex to discuss thoroughly in this thesis, hard to program correctly and/or would take too much time to calibrate and simulate due to extra parameters and stochastic factors in these models.

One must always keep in mind what the purpose of the model is and think about the level of complexity that is needed. If an insurance company
has various interest rate derivatives in its portfolio, but no equity derivatives, it might be necessary to improve the interest rate model, while improving the equity model won’t be very useful. Implementing a more complex model also increases the computation time of both the calibration, because there are more parameters, and the simulation, due to more risk factors in the model. Moreover, an important requirement of Solvency II is that the models and methods should be well understood by senior management. Therefore, simplicity, transparency and robustness of the ESG are desirable features and one must always ask oneself if the benefit of implementing a more advanced model outweighs the increased complexity resulting from those models.

**Interest rate model**

In our ESG we chose to use the Hull-White one factor model for the interest rates. The advantages for this model were that we could fit the model exactly to the current term structure, we could derive analytical formulas for interest rate derivatives, like swaptions, which makes it easier to calibrate the model accurately to market data and it is fairly easy to use the equity model under Hull-White interest rates.

The model has some limitations. The model can produce negative interest rates. In practice however this doesn’t lead to any issues in the valuation, among others because the probability of this happening is small because of the mean reversion; see Pelsser [38] and Brigo and Mercurio [5]. Also, the model is a one factor model and therefore, the interest rates are perfectly correlated, i.e. the model can only produce parallel changes in the interest rates. The last important limitation is the constant volatility, because of which we cannot get an exact fit to the complete volatility surface.

To deal with the perfect correlation of the interest rates, we could add extra stochastic factors, driven by an extra Brownian motion, to the model. While one factor explains 68% to 75% of the changes, two factors, which allows for shifts in the level and slope of the yield curve, can already account for 85% to 90% of the changes according to Jamshidian and Zhu [29]. Therefore, an improvement would be to extend the model to the Hull-White two factor model, which is also known as the G2++ model. The advantages of this model are that it is still analytically tractable and it is still easy to use the equity model simultaneously with this model.

We could also add time-dependent or stochastic volatilities to the model. However, implementing stochastic volatilities can be very complex and the model would lose its easy tractability. Implementing time-dependent volatilities can give additional problems with getting an exact fit with the term structure of interest rates and also the use of the equity model [5].
The most advanced model for interest rates currently available is the Displaced Diffusion Stochastic Volatility Libor Market Model (DDSV-LMM). The model allows for an exact fit to the swaption implied volatility surface and is capable of capturing the risk of changing implied volatilities, due to the stochastic volatilities. One of the disadvantages is that it is not analytically tractable.

Equity

We used the Black-Scholes model with stochastic interest rates for our equity, which can also be referred to as the Hull-White-Black-Scholes model in our case. The advantages were that it is easy to use simultaneously with the Hull-White model and it is analytically tractable, which makes calibration to options easy.

There are some limitations with the model. The lognormal distribution of the stock doesn’t behave appropriate in extreme market conditions, since the heavy-tailed returns that can be seen in practice can’t be incorporated in this model. The model is also incapable to give an exact fit to the at-the-money implied volatilities, because it has a constant volatility factor. Finally, we can’t calibrate the model to different strikes because it is a one factor model.

The model could be improved by allowing for time dependent volatility. If the volatility parameter is defined as a piecewise constant function, then the adjusted model can be made to exactly fit the observed at-the-money implied volatilities. Again, as it turns out, it is not too hard to adapt the option pricing formulas to piecewise constant volatilities, thus retaining the analytical tractability. This can for example be seen in the model documentation of ING insurance, where they use the Hull-White two factor model and the equity model with piecewise constant volatility in their ESG.

The most advanced model currently Stochastic Volatility Jump Diffusion model, which is basically a combination of Heston’s Stochastic Volatility model and the Merton’s Jump Diffusion model. Due to the stochastic volatility, the model is capable of capturing the overall shape of the implied volatility surface. The merton model allows for rare events, jumps, in the equity value and is therefore able to capture the skewness of stock returns better than other models.

Other models

We modelled real estate the same way as our equity model and because of that it suffers from the same drawbacks as the equity model. Therefore, the model could be improved by a more complex model, in the same way as we suggested for the equity model. However, as we noted, not much accurate market data is available and no options are liquidly traded for real estate, so calibrating a complexer model to real estate data might not be feasible. In practice real estate can be modelled
with a simple model, like the one we used, that is linked to the more complex equity model and thus also demonstrates the desirable features of those models. This is for example done in the economic scenario generator of Barrie & Hibbert.

For the eurozone inflation we used the Vasiček model. Since the economies in the eurozone are economically linked to each other, it can be assumed that long term inflation in each eurozone country is (almost) the same, but short term inflation may differ a lot. Therefore the model could be improved by adding a model for the spread between the eurozone inflation and the country specific inflation, where the spread has a mean reversion to zero.
Appendix A

Derivation of Black’s formula

In this appendix we give a justification for Black’s formula which was given in Theorem 2.13 and which we state below for completeness.

**Theorem A.1** (Black’s Formula for Swaptions). Black’s formula for the \( t \leq T_0 \) price of a payer swaption is given by

\[
N \delta(R_{\text{swap}}(t) \Phi(d_1(t)) - K \Phi(d_2(t))) \sum_{i=1}^{n} P(t, T_i),
\]

where

\[
d_1(t) = \frac{\log(R_{\text{swap}}(t)K) + \frac{1}{2} \sigma(t)^2(T_0 - t)}{\sigma(t) \sqrt{T_0 - t}},
\]

\[
d_2(t) = d_1 - \sigma(t) \sqrt{T_0 - t}.
\]

Here \( \Phi \) denotes the standard normal distribution function. \( \sigma(t) \) is known as Black’s swaption volatility.

Suppose the risk-neutral dynamics of \( R_{\text{swap}}(t) \) are given by

\[
dR_{\text{swap}}(t) = R_{\text{swap}}(t)\sigma dW(t),
\]

where \( \sigma > 0 \) is a constant and \( W \) a Brownian motion. Then we can derive the solution for \( R_{\text{swap}}(t) \). That is,

\[
R_{\text{swap}}(T) = R_{\text{swap}}(t) \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma(W(T) - W(t)) \right).
\]
Note that \( R_{\text{swap}} \) has a lognormal distribution. Now we can calculate the time \( t \) value of \((R_{\text{swap}}(T) - K)^+\) for some strike \( K > 0 \).

\[
\mathbb{E} \left[ (R_{\text{swap}}(T) - K)^+ \mid \mathcal{F}_t \right] = \mathbb{E} \left[ (R_{\text{swap}}(T) - K)1_{\{R_{\text{swap}}(T) > K\}} \mid \mathcal{F}_t \right] \\
= \int_{-\infty}^{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)} (e^z - K) f(z) dz \\
= \int_{-\infty}^{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)} e^z f(z) dz - K \int_{-\infty}^{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)} f(z) dz,
\]

(A.1)

where \( f \) is the density of a normal random variable \( Z \) with distribution \( N(-\frac{1}{2}\sigma^2(T - t), \sigma^2(T - t)) \) and we implicitly used the fact that \( W \) has independent increments to deal with the conditional expectation. Let us first look at the second term in (A.1).

\[
K \int_{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)}^{\infty} f(z) dz = K \left( 1 - \int_{-\infty}^{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)} f(z) dz \right) \\
= K \left( 1 - \Phi \left( \log \left( \frac{K}{R_{\text{swap}}(T)} \right) + \frac{1}{2} \sigma(t)^2(T_0 - t) \over \sigma(t)\sqrt{T_0 - t} \right) \right) \\
= K \Phi \left( \log \left( \frac{R_{\text{swap}}(T)}{K} \right) - \frac{1}{2} \sigma(t)^2(T_0 - t) \over \sigma(t)\sqrt{T_0 - t} \right) 
\]

(A.2)

For the first term in (A.1) we have

\[
\int_{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)}^{\infty} e^z f(z) dz = \int_{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)}^{\infty} e^z \over \sigma\sqrt{(T - t)2\pi} e^{-\frac{1}{2} \left( \frac{(z + \frac{1}{2}\sigma^2(T - t))}{\sigma^2(T - t)} \right)} dz \\
= \int_{\log \left( \frac{K}{R_{\text{swap}}(T)} \right)}^{\infty} \over \sigma\sqrt{(T - t)2\pi} \over e^{-\frac{1}{2} \left( \frac{(z + \frac{1}{2}\sigma^2(T - t))}{\sigma^2(T - t)} \right)} dz \\
= 1 - \Phi \left( \log \left( \frac{K}{R_{\text{swap}}(T)} \right) - \frac{1}{2} \sigma(t)^2(T_0 - t) \over \sigma(t)\sqrt{T_0 - t} \right) \\
= \Phi \left( \log \left( \frac{R_{\text{swap}}(T)}{K} \right) + \frac{1}{2} \sigma(t)^2(T_0 - t) \over \sigma(t)\sqrt{T_0 - t} \right).
\]

(A.3)

Inserting (A.2) and (A.3) into (A.1) and combining it with (2.4) gives us Black’s formula for swaptions.
Appendix B

Abstract Bayes’ Formula

In this appendix the Abstract Bayes’ Formula is given together with a proof. The formula was used in the proof of Proposition 3.4.

Lemma B.1 (Abstract Bayes’ Formula). Let $X$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{Q} \ll \mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$ with Radon-Nikodym derivative

$$L = \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ on } \mathcal{F}.$$  

Furthermore, let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathcal{F}$ a sigma-algebra. Then

$$E_{\mathbb{Q}}[X|\mathcal{G}] = \frac{E_{\mathbb{P}}[LX|\mathcal{G}]}{E_{\mathbb{P}}[L|\mathcal{G}]} \text{, } \mathbb{Q}\text{-a.s..}$$

Proof. Basically, the main trick in this proof is repeatedly applying the definition of a conditional expectation. Take some $G \in \mathcal{G}$, then

$$\int_G E_{\mathbb{Q}}[X|\mathcal{G}] E_{\mathbb{P}}[L|\mathcal{G}] d\mathbb{P} = \int_G E_{\mathbb{P}}[L E_{\mathbb{Q}}[X|\mathcal{G}]|G] d\mathbb{P}$$

$$= \int_G L E_{\mathbb{Q}}[X|\mathcal{G}] d\mathbb{P}$$

$$= \int_G E_{\mathbb{Q}}[X|\mathcal{G}] d\mathbb{Q}$$

$$= \int_G X d\mathbb{Q}$$

$$= \int_G LXd\mathbb{P}$$

$$= \int_G E_{\mathbb{P}}[LX|\mathcal{G}] d\mathbb{P}.$$
Which proves that
\[ \mathbb{E}^Q[X|\mathcal{G}] \mathbb{E}^P[L|\mathcal{G}] = \mathbb{E}^P[L \cdot X|\mathcal{G}], \text{ } \mathbb{P}\text{-a.s.} \]
and hence \( \mathbb{Q}\text{-a.s.} \). We only need to show that \( \mathbb{E}^P[L|\mathcal{G}] \neq 0 \) \( \mathbb{Q}\text{-a.s.} \) to finish the proof. This follows from
\[
\mathbb{Q} \left( \mathbb{E}^P[L|\mathcal{G}] = 0 \right) = \int_{\{ \mathbb{E}^P[L|\mathcal{G}] = 0 \}} d\mathbb{Q} \\
= \int_{\{ \mathbb{E}^P[L|\mathcal{G}] = 0 \}} L d\mathbb{P} \\
= \int_{\{ \mathbb{E}^P[L|\mathcal{G}] = 0 \}} \mathbb{E}^P[L|\mathcal{G}] d\mathbb{P} = 0.
\]
\[\square\]
Appendix C

Geltner’s real estate unsmoothing method

In this appendix we look at the unsmoothing method which was suggested by Geltner [18]. This model, which is sometimes referred to as the simplest reverse-engineering model [20], assumes the appraiser uses a simple 'Bayesian' updating rule to estimate the property value at each point in time. Formally, this must be understood as follows. Suppose we denote by $V_{t-1}^*$ the previous appraised value and suppose the appraiser also has some (imperfect) empirical data $V_t^E$ about the current real market value $V_t$. Thus

$$V_t^E = V_t + e_t,$$  \hspace{1cm} (C.1)

where $e_t$ is some random error. Then, as we will see shortly, the optimal appraisal value of the property should be given by

$$V_t^* = \alpha V_t^E + (1 - \alpha)V_{t-1}^*,$$  \hspace{1cm} (C.2)

where $0 \leq \alpha \leq 1$, which is given by

$$\alpha = \frac{Var[r_t]}{Var[r_t] + \sigma_t^2}.$$  \hspace{1cm} (C.3)

Here $\sigma_t^2$ is the variance of the error term $e_t$, which is proportional to $V_t$, and $r_t$ is the real market return $r_t = \frac{V_t - V_{t-1}}{V_{t-1}}$. If we now combine (C.1) with (C.2) we get

$$V_t^* = \alpha V_t + \alpha e_t + (1 - \alpha)V_{t-1}^*.$$  

From this equation it can be made intuitively clear why this is the optimal appraisal method for a real estate agent. If we take $\alpha \approx 1$, a lot of weight will be attached to the error term $e_t$, which can result in a large error in the appraised value of
the property. If \( \alpha \approx 0 \), (almost) no weight is assigned to the real market value \( V_t \) or the change in market value since \( t - 1 \). Since in an appraisal index appraisals of different properties are aggregated, the error term \( e_t \) will diversify away. As a result, the index will be updated in the following way

\[
V_t^* = \alpha V_t + (1 - \alpha)V_{t-1}^*,
\]

(C.4)

which is obviously not optimal for \( \alpha < 1 \). We can also see that we can derive the absolute real returns \( r_t = V_t - V_{t-1} \) from the absolute index returns \( r_t^* \) by

\[
r_t = \frac{r_t^* - (1 - \alpha)r_{t-1}^*}{\alpha}.
\]

One of the strengths of this model is that we don’t need to assume that the real estate market is efficient. In an efficient market we would have to assume that true market returns are uncorrelated, since all information is already included in the current market price. In [20] it is implicated that the real estate market is not as informational efficient as the listed exchange market for, for example, stocks. This information inefficiency implies that the real estate market reacts slowly to new information and as a consequence it has correlated returns.

Another reason why the model is widely used, is that after determining \( \alpha \) the model is really easy to use. However, one of the problems in this model lies in the determination of the smoothing factor \( \alpha \) or equivalently in determining the variance of \( r_t \) and the variance of the error term \( e_t \) in (C.3). It is conventional among institutional investors that the variance of \( r_t \) is half that of the stock market [18] [17]. It is however much harder to determine \( \sigma_t^2 \) since it depends on the variance of individual real estate transactions and only limited data on real estate transactions is available.
Bibliography


