BRST quantization and string theory spectra

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“Wer keiner Tatsache gewiss ist, der kann auch des Sinnes seiner Worte nicht gewiss sein.”

Ludwig Wittgenstein, Über Gewissheit, §114.iii

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iii “If you are not certain of any fact, you cannot be certain of the meaning of your words either.” - Ludwig Wittgenstein, On Certainty, §114.
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Preface

This thesis deals with the BRST quantization of both bosonic and supersymmetric string theory. String theory is introduced as a world-sheet gauge theory and the general BRST formalism is discussed. As an example, the BRST quantization of the point-particle is given. By applying the BRST machinery to the Polyakov action or its supersymmetric equivalent the spectra of the different theories are derived and investigated.

This thesis can be read as an alternative introduction to string theory. It is an introduction because it doesn’t assume any knowledge about string theory beforehand and it deals with its basics in great detail. It is alternative as opposed to the standard textbook approach, which is usually a combination of canonical quantization and light-cone quantization. With knowledge of these conventional introductions to string theory this thesis can be read as a different perspective towards it, emphasizing different aspects of the theory. Introductory knowledge about conformal field theory is assumed, including the bc and βγ conformal field theories.4

The goal of this thesis is twofold. In the first place it aims to show that string theory can be considered as an ordinary gauge theory on the world-sheet and what the general method of dealing with gauge is. Where the conventional quantization method are slightly ad hoc, in this thesis we will emphasize the generality of the BRST procedure in its application to string theory. This should give more insight about the fundamental aspects of string theory on the world-sheet. In the second place it tries to accustom the reader with the properties of the different string theory spectra. This should provide the reader with a firm background, enabling him to do more advanced research in string theory.

I would like to thank my supervisor, prof. dr. Erik Verlinde, for the many helpful and inspiring conversations, his patience, and for having given me the freedom to develop a solid background of basic knowledge before starting the actual research in more advanced topics.

4See e.g. Ginsparg[1].
1 String theory spectra

1.1 World-sheet coordinates

A world-sheet is a two-dimensional pseudo-Riemannian manifold. Here we give four frequently used coordinate patches. For the moment, we neglect boundary and other topological properties of the world-sheet and we focus on the local aspects of the different coordinate patches.

1. $(\tau, \sigma)$

The metric $\gamma_{\alpha\beta}$ in these coordinates, where $\alpha, \beta \in \{0, 1\}$, is of Lorentzian signature $(-, +)$.

2. $(\sigma^1, \sigma^2)$

They are related to the former coordinate patch through the mapping $(\tau, \sigma) \rightarrow (\sigma^1, \sigma^2)$:

\[
\sigma^1 := \sigma \\
\sigma^2 := i\tau.
\]

We recognize this as part of the so-called Wick rotation, which is a continuation of the time coordinate to imaginary values. It is in this way that these coordinates will be used in string theory. The difference is that after the Wick rotation we’re really having a different theory under consideration and not just the same theory in different coordinates. We do this because calculations sometimes become easier in the new Euclidian\(^2\) theory. To draw conclusions about actual physical phenomena, one has to Wick rotate back to Minkowskian (as opposed to Euclidian) theory.

Since the world-sheet metric is by definition a symmetric, non-degenerate $(0, 2)$-tensor, it transforms under a general world-sheet coordinate transformation $\{\sigma^\alpha\} \rightarrow \{\sigma^\alpha\}$ as

\[
\gamma_{\alpha\beta} \rightarrow \frac{\partial \sigma^\alpha}{\partial \sigma^a} \frac{\partial \sigma^\beta}{\partial \sigma^b} \gamma_{\alpha\beta}.
\]

The metric $g_{ab}$, where $a, b \in \{1, 2\}$, with respect to the coordinate patch $(\sigma^1, \sigma^2)$ is therefore

\[
g_{11} = \gamma_{11} \\
g_{12} = g_{21} = -i\gamma_{12} = -i\gamma_{21} \\
g_{22} = -\gamma_{00}.
\]

We see that this metric is of Euclidian signature $(+, +)$, and therefore call these coordinates Euclidian.

By drawing a picture one immediately sees that a vector $v^\alpha$ with respect to $(\tau, \sigma)$ is in Euclidian coordinates given by the vector $u^\alpha$:

\[
u^1 = v^1 \\
u^2 = iv^0.
\]

\(^1\)Also denoted by $(\sigma^0, \sigma^1)$.

\(^2\)See below for why it’s called Euclidian.
The string theory to be defined on the world-sheet will turn out to have certain gauge symmetries. These symmetries can be used to gauge-fix the world-sheet metric. Two frequently used gauges for the metric are conformal gauge and unit gauge, respectively given by

\[ g_{ab} = e^{2\omega} \delta_{ab} \]

and

\[ g_{ab} = \delta_{ab}, \]

where \( \omega = \omega(\sigma^1, \sigma^2) \) is a function on the world-sheet. Because of the symmetry of the metric, for conformal gauge one generally needs to fix two gauge degrees of freedom and for unit gauge three.

3. \((w, \bar{w})\)

These so-called complex cylindrical coordinates are related to Euclidian coordinates through

\[ w := \sigma^1 + i \sigma^2, \quad \bar{w} := \sigma^1 - i \sigma^2. \]

Although \( \bar{w} \) is the complex conjugate of \( w \), as coordinates they have to be considered independent from each other.

With (1) the metric \( g'_{xy} \), where \( x, y \in \{w, \bar{w}\} \), can be calculated. The \( ww \)-component for example becomes

\[ g'_{ww} = \frac{1}{4} (g_{11} - ig_{12} - ig_{21} - g_{22}). \]

Note that in conformal gauge \( g'_{ww} = g'_{\bar{w}\bar{w}} = 0 \) and that in unit gauge

\[ g'_{ww} = g'_{\bar{w}\bar{w}} = 0, \quad g'_{w\bar{w}} = g'_{\bar{w}w} = \frac{1}{2}. \]

Partial derivatives with respect to complex cylindrical coordinates are denoted by \( \partial' \) and \( \bar{\partial}' \):

\[ \partial' := \partial_w = \frac{\partial \sigma_a}{\partial w} \partial_a = \frac{1}{2} (\partial_1 - i \partial_2) \]

\[ \bar{\partial}' := \partial_{\bar{w}} = \frac{1}{2} (\partial_1 + i \partial_2). \]

4. \((z, \bar{z})\) Finally, there are complex planar coordinates defined by

\[ z := e^{-iw}, \quad \bar{z} := e^{iw}. \]

The origin of the names for the different complex coordinate patches will become clear when we start discussing string theory.

For now, the metric \( \tilde{g}_{xy} \), where \( x, y \in \{z, \bar{z}\} \), can be found to be

\[ \tilde{g}_{zz} = \tilde{g}_{\bar{z}\bar{z}} = -e^{2w} g'_{ww}, \]

\[ \tilde{g}_{z\bar{z}} = e^{i(w-\bar{w})} g'_{w\bar{w}}, \]

\[ \tilde{g}_{\bar{z}z} = -e^{-2w} g'_{\bar{w}w}. \]
Partial derivatives with respect to complex planar coordinates are denoted by $\partial$ and $\bar{\partial}$:

$$\partial := \partial_z = ie^{iw} \partial_w$$
$$\bar{\partial} := \partial_{\bar{z}} = -ie^{-iw} \partial_{\bar{w}}.$$  

After having summed up the different coordinate patches in a formal way, we will now relax the notation a bit. If not ambiguous, metrics, functions, etc. will be assumed to be with respect to the coordinate patch used in the equation.
1.2 Polyakov action

The motion of a bosonic string through $D$-dimensional flat spacetime can be described by a two-dimensional theory defined on a world-sheet. As dynamical variables on the world-sheet live

- $D$ scalar fields $X^\mu(\tau,\sigma)$, where $\mu \in \{0,1,\ldots,D-1\}$,
- and a world-sheet metric $\gamma_{\alpha\beta}(\tau,\sigma)$.

Here, $X^\mu$ signifies the position of the string in spacetime with respect to Cartesian spacetime coordinates. Note that the $X^\mu$ are world-sheet scalars and is a spacetime vector. Also note that quantizing this field theory does not entail second quantization or any particle interpretation. It’s simply a first quantization of dynamical variables which depend on certain continuous parameters.

This two-dimensional field theory is described by the Polyakov action

$$S_{\text{pol}} := -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

$$- \frac{\lambda}{4\pi} \int_M d\tau d\sigma \sqrt{-\gamma} R + \frac{\lambda}{2\pi} \int_{\partial M} d\sigma t^\alpha \gamma^{\alpha\beta} \nabla_{\alpha} n_{\beta}. \quad (2)$$

Here $\gamma := \det \gamma_{\alpha\beta}$, $M$ is the world-sheet manifold, $R$ is the world-sheet Ricci scalar and $\alpha', \lambda$ are undetermined constants. $\alpha'$ is called the Regge slope and has dimensions of spacetime-length-squared. If the manifold has a boundary $\partial M$ - this is the case for open strings - then $t^\alpha$ is the vector tangent to $\partial M$ and $n^\alpha$ is the outward pointing vector perpendicular to the boundary, i.e. $t^\alpha n_\alpha = 0$. $\nabla_\alpha$ is the covariant derivative with respect to $\gamma_{\alpha\beta}$.

1.2.1 Symmetries

This theory has three symmetries:

1. $D$-dimensional Poincaré invariance.
   It is straightforward to see that the action is invariant under a general Poincaré transformation $(\Lambda, a)$ which acts on the fields according to
   \begin{align*}
   X^{\mu}(\tau, \sigma) &= \Lambda^\nu_\mu X^\nu(\tau, \sigma) + a^\mu \\
   \gamma'_{\alpha\beta}(\tau, \sigma) &= \gamma_{\alpha\beta}(\tau, \sigma). 
   \end{align*} \quad (3)

2. World-sheet diffeomorphism invariance.
   This is a general world-sheet coordinate transformation $(\tau, \sigma) \rightarrow (\tau', \sigma')$, under which the fields transform as
   \begin{align*}
   X^{\mu}(\tau', \sigma') &= X^\nu(\tau, \sigma) \\
   \partial_{\sigma'}^\nu \partial_{\sigma'}^{\alpha} \gamma'_{\alpha\beta}(\tau', \sigma') &= \gamma_{\alpha\beta}(\tau, \sigma). \quad (4)
   \end{align*}

---

3For our present purposes it is enough to consider flat $D$-dimensional Minkowski spacetime. This constraint can be relieved by inserting a general spacetime metric $G_{\mu\nu}$ at the appropriate place in the Polyakov action.

4Because of gauge symmetries they are not all independent dynamical variables, as we will see shortly.
This is also easy to see. First remember that the integration measure $d\tau d\sigma \sqrt{-\gamma}$ is coordinate invariant. With the above field transformations it follows that the transformations in the $X^\mu$-part of the action cancel. Requiring the variation of the $R$-part of the integral under a coordinate transformation to be zero gives you the vacuum Einstein equation, but this is trivially satisfied in two dimensions.

As for the boundary part of the action, we know the tangent vector is defined by $t^\alpha := \frac{\partial \sigma^\alpha}{\partial s}$, where $s$ is the proper time with respect to the induced boundary metric. Under the diffeomorphism, it becomes
\[ t'^\alpha = \frac{\partial \sigma'^\alpha}{\partial \sigma^\beta} t^\beta. \tag{5} \]
These transformations cancel the transformation of the covariant derivative of the vector $n^\alpha$, because we know $\nabla_\alpha n_\beta$ has tensorial properties. This makes the Polyakov action diffeomorphism invariant.

3. Two-dimensional Weyl invariance.
This is nothing but a coordinate dependent scaling of the world-sheet metric:
\[ X'^\mu(\tau, \sigma) = X^\mu(\tau, \sigma) \]
\[ \gamma'_{\alpha\beta}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{\alpha\beta}(\tau, \sigma). \tag{6} \]
Because $\sqrt{-\gamma} \gamma^{\alpha\beta}$ is invariant under (6), the whole $X^\mu$-part is. To prove Weyl invariance for the total action, one has to combine the cancelling transformations of the $R$-part and the boundary part.

\[ \chi := \frac{\lambda}{4\pi} \int_M d\tau d\sigma \sqrt{-\gamma} R + \frac{\lambda}{2\pi} \int_{\partial M} ds k, \]
where $k := -t^\alpha t^\beta \nabla_\alpha n_\beta$ is the extrinsic curvature of the boundary. It is straightforward though tedious to show that under (6) the Ricci scalar becomes
\[ \sqrt{-\gamma} R' = \sqrt{-\gamma} (R - 2 \nabla^2 \omega). \]
For the transformation of the extrinsic curvature we note that the only metric dependence is in the Christoffel symbol of the covariant derivative. Under (6) we have
\[ \Gamma_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma + \delta^\gamma_\alpha \omega_\beta + \delta^\gamma_\beta \omega_\alpha - \gamma_{\alpha\beta} \omega^\gamma, \]
leading to
\[ k' = k - t^\alpha t^\beta (\delta^\gamma_\alpha \omega_\beta + \delta^\gamma_\beta \omega_\alpha - \gamma_{\alpha\beta} \omega^\gamma) n_\gamma \]
\[ = k - t^2 n^\sigma \omega_\sigma \]
\[ = k + n^\sigma \omega_\sigma. \]
Here we used that $t^\alpha$ and $n^\alpha$ are perpendicular and that we can normalize the timelike vector $t^\alpha$ to $-1$ using a world-sheet coordinate transformation. For a Weyl transformation applied to the Euler characteristic, we now get
\[ \chi' = \chi - \frac{1}{2\pi} \int_M d\tau d\sigma \sqrt{-\gamma} \nabla^2 \omega + \frac{1}{2\pi} \int_{\partial M} ds n^\sigma \omega_\sigma \]
\[ = -\frac{1}{2\pi} \int_{\partial M} ds n^\sigma \omega_\sigma + \frac{1}{2\pi} \int_{\partial M} ds n^\sigma \omega_\sigma \]
\[ = 0. \]
\[ 5 \]These intermezzo’s are meant for technical details and extra elaborations. The main text can be read without taking notice of the intermezzo’s.
where in the second line we used Stokes’ theorem and the fact that a volume line element is given
by $\sqrt{-\gamma} \, ds = ds \, n_s$, since $s$ is the proper time along the boundary. This proves Weyl invariance of
the Euler characteristic and thus of the Polyakov action.\footnote{Cf. exercise 1.3 in Polchinski\cite{3}}

Note that the first symmetry is a physical symmetry, i.e. related to a physical transformation,
while the other two are gauge symmetries.

### 1.2.2 Sum over topologies

As we will use the BRST formalism to quantize our field theory, we need to work with path
integrals. A path integral calculates the amplitude of a transition from an initial to a final state
by summing over all possible configurations the dynamical variables can have when the obey the
constraints of the initial and final state. This sum is weighted by a factor

$$\exp \left( \frac{iS}{\hbar} \right),$$

where $S$ is the action of the theory and a functional of the dynamical variables. If not stated
differently, in the remainder of the text we will use Planck units, i.e. $c = G = \hbar = k_B = 4\pi\epsilon_0 = 1.$

In the path integral associated with the Polyakov action one sums, among others, over all possible
world-sheet configurations specified by the topology of the world-sheet and the metric $\gamma_{\alpha\beta}$. Since we have noted that the Euler characteristic $\chi$ is invariant under coordinate and Weyl
transformations and is independent of the scalar fields $X^\mu$, we infer that $\chi$ only depends on the
topology of the world-sheet. We can therefore split up the sum over world-sheet configurations
in a sum over topologies and a sum over metrics on these topologies. The path integral would
then be of the form

$$\sum_{\text{topologies}} e^{-i\chi} \left( \int [dX \, d\gamma] e^{iS_X} \right), \quad (7)$$

where

$$S_X := -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu.$$  

For our present purpose of finding the string spectrum we can neglect the topological part of the
action and focus on $S_X$, since the latter specifies the dynamics of the string and thus the string
quantum states we’re looking for.

### 1.2.3 Wick rotation of the Polyakov path integral

A Wick rotation is done in two steps. First, one analytically continues the time coordinate of
the theory to values on the imaginary axis. As the theory of complex analysis doesn’t allow
the contour of integration to hit poles, this deformation of the contour to the imaginary axis
can usually be done in only one way. Second, one defines a new, Euclidian time coordinate
and substitutes this coordinate for the old Minkowskian one. In our case, we rotate the $\tau$-axis

\footnote{Cf. exercise 1.3 in Polchinski\cite{3}}
clockwise into the complex plane and the Euclidian time coordinate $\sigma^2$ is related to $\tau$ through $\tau = -i\sigma^2$.

Now, the Polyakov action takes the form
\[
S_{\text{pol}} = \frac{i}{4\pi\alpha'} \int_M \sqrt{g} g^{ab} \partial_a X^\prime \mu \partial_b X^\prime_\mu + \frac{i\lambda}{4\pi} \int_M \sqrt{g} R' + \frac{i\lambda}{2\pi} \int_{\partial M} ds t^a \nabla_b n_a,
\]
(8)
where $d^2 \sigma = d\sigma_1 d\sigma_2$ and we used some notation introduced in section (1.1). Please note that the boundary term has a different relative sign than in the Minkowskian case. This is because not only the boundary volume element - $ds = \sqrt{-\gamma_{00}} d\tau$ in string theory - produces a factor $-i$ in the Wick rotation, but also the tangent vectors $t^a$. From the definition (5) we namely see that in string theory only the zero-component of the tangent vector is non-zero and it is this component which gives a factor $-i$.

Defining the Euclidian representation of the Polyakov action by
\[
S_{\text{pol,E}} := -iS_{\text{pol}},
\]
(9)
we get an Euclidian path integral weighted by a factor
\[
\exp (iS_{\text{pol}}) = \exp (-S_{\text{pol,E}}).
\]
Here we see already why working in the Euclidian theory is typically more convenient, as contributions to the path integral with large actions get naturally suppressed. If not stated otherwise, we will be working in the Euclidian theory.

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7The primes on the scalars are formally correct, but usually omitted.

8The boundaries of an open string world-sheet are specified by $\sigma = 0, \pi$. 
1.3 Conformal field formalism

1.3.1 Mode expansion of $X^\mu$

The conformal field formalism is usually expressed in complex planar coordinates. Using the standard transformation of the integration measure, starting with (9) we easily see that in complex planar coordinates

$$S_X = \frac{1}{4\pi\alpha'} \int_M d^2z \sqrt{g} g^{xy} \partial_x X \cdot \partial_y X$$

$$= \frac{1}{2\pi\alpha'} \int_M d^2z \partial X \cdot \bar{\partial} X,$$

where the last equality holds for unit gauge.\(^9\) Here we used a "\(\cdot\)" to denote the contraction of the spacetime indices on the scalar fields.

Working in unit gauge, the Euler-Lagrange equations for the $X^\mu(z, \bar{z})$ are

$$\partial \bar{\partial} X^\mu(z, \bar{z}) = 0.$$  

The most general solution to these equations is

$$X^\mu(z, \bar{z}) = X^\mu(z) + \tilde{X}^\mu(\bar{z}).$$

Using the equations of motion for the fields, we see that $\partial X^\mu(z)$ is a holomorphic function and $\bar{\partial} \tilde{X}^\mu(\bar{z})$ is an anti-holomorphic function.

Even without quantizing a field theory or implementing a particle interpretation, it proves to be natural to expand free fields in harmonic oscillator modes. This is because in the mode expansion formalism the $S$-matrix automatically obeys a physical requirement called the cluster decomposition principle.\(^10\) We can do the same for $\partial X^\mu(z)$ and $\bar{\partial} \tilde{X}^\mu(\bar{z})$, which because of their (anti-)holomorphic property can be expanded in a Laurent series. It is convenient to define two coefficients $\alpha^\mu_m$ and $\tilde{\alpha}^\mu_m$ such that the Laurent series become

$$\partial X^\mu(z) = -i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}},$$

$$\bar{\partial} \tilde{X}^\mu(\bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}}.$$  \(11\)

Let’s first express the spacetime momentum $p^\mu$ associated with a solution of the equations of motion in terms of the mode expansion. We know that $p^\mu$ is the conserved charge associated with the spacetime translation symmetry $X'^\mu = X^\mu + \epsilon^\nu \delta^\mu_\nu$, where $\epsilon^\nu$ is infinitesimal. Because there is no boundary term when we vary $S_X$ with respect to a constant infinitesimal translation, we know from Noether’s theorem that there is a conserved current $j^\mu_\nu$ with components\(^11\)

$$j^\mu_\nu = i2\pi \frac{\partial L}{\partial (\partial_z X^\nu)} \delta^\mu_\nu = \frac{i}{\alpha'} \partial^\mu X^\nu,$$

\(^9\)For the $(z, \bar{z})$-metric this means $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} e^{i(w - \bar{w})} = \frac{1}{2} e^{-2\sigma^2}$.

\(^10\)See Weinberg[7], chapter 4.

\(^11\)In string theory it is a convention to add a factor of $2\pi$ to the standard conserved current. As long as one corrects for this in physical quantities like spacetime momentum, this is of no consequence.
\[ j^\nu = i \alpha^{\prime} \partial^\nu X_\nu. \]

The extra factor of \( i \) is because we work in the Euclidian theory now, while we still want the associated Euclidian spacetime momentum to be real. In unit gauge it is convenient to define the current components

\[ j_\mu(z) := j^\mu = i \alpha^{\prime} \partial X_\mu(z), \]
\[ \tilde{j}_\mu(\bar{z}) := j^\mu = i \alpha^{\prime} \bar{\partial} \bar{X}_\mu(\bar{z}). \]

Now, to get the associated conserved charge we need to integrate the world-sheet time component of the current over a spatial hypersurface of constant world-sheet time, i.e.

\[ p_\mu := \frac{1}{2\pi} \int_0^{2\pi} d\sigma^{1} j_\mu \sigma^2. \]

Note that in the Euclidian theory we’re working in now, \( \sigma^2 \) has to be considered as the time coordinate on the world-sheet and \( \sigma^1 \) as the spatial coordinate, while \( \tau, \sigma \) don’t have any meaning in this theory. With this in mind, the formula of the spacetime momentum in the Euclidian theory is equivalent to the Minkowskian one

\[ \frac{1}{2\pi} \int_0^{2\pi} d\sigma^{1} j_\mu \sigma^\tau. \]

Also note that for the moment we focus on closed strings, in which case \( \sigma^1 \in [0, 2\pi) \).

To relate this expression to the mode expansions we can of course do a series of transformations to \((z, \bar{z})\)-coordinates, but it can be done easier. Note that \( |z| = e^{\sigma^2} \) and \( \arg(z) = -\sigma^1 \). We therefore see that string states at a slice of constant world-sheet time \( \sigma^2 \) live in the \((z, \bar{z})\)-plane on circles of radius \( e^{\sigma^2} \) centered at the origin. This approach to the theory is called radial quantization. To calculate \( p_\mu \) using \((z, \bar{z})\)-coordinates, we therefore have to integrate the radial current component over circles of radius \( R > 0 \). With Gauss’ law we know this is an integral of the divergence of the current over the disk of radius \( R \):

\[ p_\mu = \frac{1}{2\pi} \int_{D_R} d^2 z \sqrt{g} \partial_\mu j^{\mu x} \]
\[ = \frac{1}{2\pi} \oint_{\partial D_R} (\tilde{j}^\mu(\bar{z}) d\bar{z} - j^\mu(z) dz) \]
\[ = \frac{1}{2\pi i} \sqrt{2\alpha^\prime} \oint_{\partial D_R} \left( \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}} dz - \sum_{m=-\infty}^{\infty} \tilde{\alpha}^\mu_m \bar{z}^{m+1} d\bar{z} \right) \]
\[ = \frac{1}{\sqrt{2\alpha^\prime}} (\alpha^\mu_0 + \tilde{\alpha}^\mu_0). \]  

To arrive at (12) one can use the standard trick of starting with a contour integral in polar coordinates \((\rho, \theta)\) and use \( z = \rho e^{i\theta}, \bar{z} = \rho e^{-i\theta} \). In the last line we used that the \( \bar{z} \)-integral is taken in clockwise direction.

Having done this, integration of \( \partial X_\mu(z) \) and \( \bar{\partial} \bar{X}_\mu(\bar{z}) \) to obtain \( X_\mu(z, \bar{z}) \) gives you terms with a complex logarithm in it:

\[ -i \sqrt{\frac{\alpha^\prime}{2}} (\alpha^\mu_0 \log(z) + \tilde{\alpha}^\mu_0 \log(\bar{z})) \]
\[ = -i \sqrt{\frac{\alpha^\prime}{2}} ((\alpha^\mu_0 + \tilde{\alpha}^\mu_0) \log(|z|) + i(\alpha^\mu_0 - \tilde{\alpha}^\mu_0) \arg(z)). \]
as \( \arg(z) = -\arg(\bar{z}) \). Physically you want \( X^\mu(z, \bar{z}) \) to be single-valued, that is, \( X^\mu(z, \bar{z}) = X^\mu(ze^{2\pi i n}, \bar{z}e^{-i2\pi n}) \) for \( n \) integer. Considering the logarithmic terms in \( X^\mu(z, \bar{z}) \), we find the requirement
\[
\alpha_0^\mu = \tilde{\alpha}_0^\mu.
\]
The mode expansion of \( X^\mu(z, \bar{z}) \) therefore becomes
\[
X^\mu(z, \bar{z}) = x^\mu - \frac{\alpha^\prime}{2} p^\mu \log(|z|^2) + i \sqrt{\frac{\alpha^\prime}{2}} \sum_{m=-\infty}^{\infty} \frac{1}{m} \left( \frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m} \right),
\]
where \( x^\mu \) a real, arbitrary integration constant and the spacetime momentum is now
\[
p^\mu = \sqrt{\frac{2}{\alpha^\prime}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha^\prime}} \tilde{\alpha}_0^\mu.
\]

1.3.2 Virasoro generators

The stress-energy-momentum tensor of a field theory, in due course abbreviated as the stress-tensor, is usually defined as
\[
T^{ab}(x) := -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{ab}(x)}.
\]
The manifold\(^{12}\) indices \( a, b \) in this section are arbitrary and not necessarily related to the Euclidean world-sheet indices. The action in this definition is not the full action, but only the matter action - terms of topological origin or for example the Einstein-Hilbert term in general relativity are not included.\(^{13}\) This makes sense, because stress, energy and momentum is associated with matter and not with the structure of spacetime. However, from a fundamental viewpoint this distinction seems arbitrary - the Einstein equations are then simply conservation equations for stress, energy and momentum.

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\(^{12}\)This is the manifold the field theory is defined on. Examples of such manifolds are spacetime itself (e.g. in the case of general relativity) and the string world-sheet of course.

\(^{13}\)Note that variation of the full action with respect to the metric now immediately gives you the Einstein equations.

\(^{14}\)See Wald[6], p. 456.
We suppress the prime on the coordinate, because it's not relevant for further discussion anymore. Under the standard assumption that the variation in the coordinates is zero at infinity or at the boundary of the manifold, to all the Lorentz indices of its tensorially transforming objects need to be contracted, the Lagrangian itself
more to a total derivative. This means that our theory is indeed Lorentz invariant. As expected, the variation in the Lagrangian under an infinitesimal Lorentz transformation is zero up to an infinitesimal parameter has been applied. As expected, the first order corrections obtained by Taylor expanding are (minus) the Lie derivatives of the dynamical object with respect to the vector field \( f^a(x) \).

Variation of the general action under this transformation gives

\[
\delta S := S' - S = -\epsilon \int d^D x' d^D y \sqrt{g} \left\{ \frac{\delta \mathcal{L}(x')}{\delta \phi_i(y)} f^a(x') \nabla_a \Phi_i(x') + \frac{\delta \mathcal{L}(x')}{\delta \partial_a \phi_i(y)} \right\} \left( \frac{\delta \mathcal{L}(x')}{\delta \partial_b \phi_i(y)} f^b(x') \right)
\]

where summation over the field content index \( i \) is implied.

Now focus on Lorentz invariant theories with a flat minkowski metric \( \eta_{ab} \). Our infinitesimal coordinate transformation needs to be restricted now to Lorentz transformations, i.e. transformations which preserve the metric \( \eta_{ab} \). This means that all covariant derivatives in (16) can be replaced by ordinary derivatives. Secondly, the metric preserving condition \( 0 = \delta g_{ab}(x') = \partial_a f_b(x') + \partial_b f_a(x') \) simplifies the variation of the action to

\[
\delta S = -\epsilon \int d^D x' d^D y \sqrt{g} \partial_a \left\{ \frac{\delta \mathcal{L}(x')}{\delta \partial_a \phi_i(y)} f^b(x') \right\} \left( \frac{\delta \mathcal{L}(x')}{\delta \partial_b \phi_i(y)} \right)
\]

As expected, the variation in the Lagrangian under an infinitesimal Lorentz transformation is zero up to a total derivative. This means that our theory is indeed Lorentz invariant. This may seem a surprising conclusion from the general Lagrangian we’re working with. However, since in every term in the Lagrangian all the Lorentz indices of its tensorially transforming objects need to be contracted, the Lagrangian itself transforms as a scalar:

\[
\delta S = -\epsilon \int d^D x' \sqrt{g} f^a(x') \partial_a (f^b(x') \mathcal{L}(x'))
\]

Where in the last equality we used the metric preserving condition of the coordinate transformation. This indeed implies Lorentz invariance on very general accounts.

Combining (17) with (18), we obtain

\[
0 = \int d^D x \sqrt{g} \partial_a \left\{ \int d^D y \left( \frac{\delta \mathcal{L}(x)}{\delta \partial_a \phi_i(y)} \partial_b \phi_i(x) - \delta^b_a \mathcal{L}(x) \right) \right\}
\]

where we defined the canonical stress-energy-momentum tensor as

\[
S^a_{\mu}(x) := \int d^D y \frac{\delta \mathcal{L}(x)}{\delta \partial_a \phi_i(y)} \partial_\mu \phi_i(x) - \delta^a_\mu \mathcal{L}(x).
\]

\(^{15}\)The vector field \( f^a(x) \) is now a Killing vector field with respect to \( \eta_{ab} \).

\(^{16}\)Under the standard assumption that the variation in the coordinates is zero at infinity or at the boundary of the manifold.

\(^{17}\)We suppress the prime on the coordinate, because it’s not relevant for further discussion anymore.
Because we did everything for general Killing vector field \( f^a(x) \), we conclude that locally
\[
0 = \partial_a \left( f_b(x) S^{ab}(x) \right) = \partial_a f_b(x) S^{ab}(x) + f_b(x) \partial_a S^{ab}(x).
\]
Setting\(^{18}\) \( f^a(x) := \delta^{ac} \) and noting that \( S^{ab}(x) \) is independent of \( f^a(x) \), we conclude that for a Lorentz invariant theory the canonical stress-tensor is conserved:
\[
\partial_a S^{ab}(x) = 0.
\]
Having seen this, now return to a manifold with a general metric. With the same argument we gave above the Lagrangian is still a scalar with respect to the coordinate transformation. However, since the transformation is generally not metric preserving we cannot include \( f^a(x) \) in the covariant derivative as we did in (18). Doing this will give us an extra term. Moreover, the non-vanishing of the metric variation gives a second extra term in the equivalent of (19):
\[
0 = \int d^D x \sqrt{g} \left[ \nabla_a \left( f^b(x) \left( \int d^D y \frac{\delta \mathcal{L}(x)}{\delta (\nabla_a \Phi_i(y))} \nabla_b \Phi_i(x) - \delta^a_i \mathcal{L}(x) \right) \right) + \nabla_a f^a(x) \mathcal{L}(x) + 2 \int d^D y \frac{\delta \mathcal{L}(x)}{\delta (g_{ab}(y))} \nabla_a f_b(x) \right].
\]
Substituting the stress-tensor
\[
T^{ab}(x) = -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{ab}(x)} = -\left( g^{ab}(x) \mathcal{L}(x) + 2 \int d^D y \frac{\delta \mathcal{L}(x)}{\delta (g_{ab}(y))} \right)
\]
and the generalized canonical stress-tensor - using covariant derivatives instead of ordinary ones - we obtain
\[
0 = \int d^D x \sqrt{g} \left\{ \nabla_a \left( f_b(x) S^{ab}(x) \right) - \nabla_a f_b(x) T^{ab}(x) \right\} = \int d^D x \sqrt{g} \nabla_a \left\{ \left( f_b(x) S^{ab}(x) - \nabla_a f_b(x) T^{ab}(x) \right) \right\},
\]
where in the last equality we used that the stress-tensor is always conserved.

Again, since the coordinate transformation \( f^a(x) \) can be chosen arbitrary we can now extract the local equation
\[
\nabla_a f_b(x) T^{ab}(x) = \nabla_a f_b(x) S^{ab}(x) + f_b(x) \nabla_a S^{ab}(x).
\]
From this result we can draw two conclusions. First, in a general diffeomorphism invariant field theory the canonical stress-theory is not necessarily conserved. We can try a metric preserving transformation such that \( \nabla_a f_b(x) = 0 \), making the left-hand side of (23) vanish. But a \( S^{ab}(x) \) is not generally symmetric, we cannot conclude that \( \nabla_a S^{ab}(x) \). Similarly we can try a transformation \( f^a(x) = \delta^{ac} \), but now \( \nabla_a f_b(x) = -\Gamma_{ab}^c \) is not necessarily zero. Second, with the above discussion it is not hard to see that the following implications hold:
\[
S^{ab}(x) \text{ symmetric} \Leftrightarrow S^{ab}(x) \text{ conserved} \Leftrightarrow \left( S^{ab}(x) = T^{ab}(x) \right).
\]
This makes the conventional choice of \( T^{ab}(x) \) as symmetric, naturally conserved stress tensor plausible.

Finally, we would like to stress that we haven’t investigated more general Langrangians, for example Lagrangians containing higher spin fields. Although also for these theories \( T^{ab}(x) \) is conventionally defined as the stress-tensor, some of our conclusion might not hold anymore - in particular the conclusions in (24).

In string theory it proves to be convenient to multiply the definition of the stress-tensor with a factor of \( 2 \pi \), just as we did with the spacetime translation symmetry current. With respect to complex planar coordinates, the components of the stress-tensor therefore become
\[
T_{xy}(z, \bar{z}) = \frac{1}{\alpha'} \left( \frac{1}{2} g_{xy} \partial_{\bar{z}} X(z, \bar{z}) \cdot \partial^\bar{z} X(z, \bar{z}) - \partial_x X(z, \bar{z}) \cdot \partial_y X(z, \bar{z}) \right).
\]
\(^{18}\)Obviously, the metric is preserved under this specific choice of coordinate transformation. This may seem trivial, but in what follows we will see that for general curved metrics this crucial step cannot be taken.
In unit gauge, this leads to
\[ T_{zz}(z, \bar{z}) = -\frac{1}{\alpha'} \partial X(z) \cdot \partial X(z) =: T(z) \]
\[ T_{\bar{z}\bar{z}}(z, \bar{z}) = -\frac{1}{\alpha'} \partial \bar{X}(\bar{z}) \cdot \partial \bar{X}(\bar{z}) =: \bar{T}(\bar{z}) \]
\[ T_{zz}(z, \bar{z}) = T_{\bar{z}\bar{z}}(z, \bar{z}) = 0. \] (25)

Using the definition (20) of the canonical stress-tensor\(^{19}\) it is not hard to see that \( S_{xy}(z, \bar{z}) \) is symmetric and equal to \( T_{xy}(z, \bar{z}) \), verifying the result of the above intermezzo.

Also note that \( T_{zz}(z, \bar{z}) \) is holomorphic and \( T_{\bar{z}\bar{z}}(z, \bar{z}) \) is anti-holomorphic, as can be easily verified using the equations of motion for the scalar fields. This is by the way a general property for conformal field theories on a two-dimensional flat manifold, since for these theories the conserved stress-tensor is traceless.

Similar to what we did with the scalar fields we can expand the holomorphic and anti-holomorphic component of the stress tensor in a Laurent series:
\[ T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \quad \bar{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\bar{L}_m}{z^{w+2}}. \]
The Laurent coefficients in these expansions are called Virasoro generators.

Note that this Laurent expansion is nothing but the Fourier decomposition of the stress-tensor on the cylinder at time \( \sigma^2 = 0 \). To see this, recall that such a decomposition is of the standard form
\[ T_{ww}(w) = -\sum_{m=-\infty}^{\infty} e^{(im\sigma^1 - m\sigma^2)} T_m, \]
\[ T_{\bar{w}\bar{w}}(\bar{w}) = -\sum_{m=-\infty}^{\infty} e^{(-im\sigma^1 - m\sigma^2)} \bar{T}_m, \]
where \( T_m \) and \( \bar{T}_m \) are the coefficients of the different energy modes \( m \). Since the transformation between complex cylindrical and complex planar coordinates is a conformal transformations, in order to establish a relation between the stress-tensors on the different coordinate patches one has to use the conformal transformation rule for the stress-tensor. In the present case this is
\[ T_{ww}(w) = (\partial_w z)^2 T(z) + \frac{c}{12} S(z, w), \]
where \( c \) is the central charge of the theory \( S(z, w) \) is the Schwarzian derivative for the transformation \( w \to z \), given by
\[ S(z, w) = \frac{2\partial^3_z z \partial_w z - 3(\partial^2_w z)^2}{2(\partial_w z)^2} = \frac{1}{2}. \]
Therefore we have
\[ T_{ww}(w) = -z^2 T(z) + \frac{c}{24}. \] (26)
Substituting the Fourier decomposition in the LHS and the Laurent expansion in the RHS of (26) and equating the terms with the same power of \( z \),\(^{20}\) we now obtain
\[ T_m = L_m - \delta_m, c \frac{c}{24} \quad \forall m \in \mathbb{Z}, \]
\[^{19}\]Since the field theory includes only world-sheet scalars, the generalized definition of \( S_{xy}(z, \bar{z}) \) coincides with the flat world-sheet definition (20).
\[^{20}\]Remember that \( z = e^{-iw} = e^{-i\sigma^1 + \sigma^2} \).
and similarly for the right-moving modes

\[ \tilde{T}_m = \tilde{L}_m - \delta_{m,0} \frac{\tilde{c}}{24} \quad \forall m \in \mathbb{Z}. \]

The Hamiltonian, for closed strings defined by

\[ H = \frac{1}{2} \int_0^{2\pi} d\sigma^1 \, T^2, \]

becomes in unit gauge\(^{21}\)

\[ H = T_0 + \tilde{T}_0 = L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}. \]

\(^{21}\)See the above intermezzo for why this is true.
1.4 BRST quantization

In this section we will start with a brief discussion of the general formalism of BRST quantization. For this, knowledge about how to deal with a gauge theory path integral and the DeWitt-Faddeev-Popov method is assumed. Before taking up the BRST quantization of the bosonic string in the next section, we will then as an example work out the BRST quantization of the point-particle.

As we have pointed out already, when calculating physical quantities with a path integral one sums over all possible configurations of the dynamical variables constrained by the initial and final state of the system. When applying this formalism to a gauge theory however, we have to make an important adjustment to this statement. The defining property of a gauge symmetry is that there is more than one configuration of the dynamical variables representing the same physical state. Therefore, when summing over all possible configurations we would sum multiple times over the same physical state; we would overcount because of the gauge symmetry. To correct for this overcounting, we divide in the path integral by the volume of the gauge group

\[ \frac{1}{V_{\text{gauge}}} \int \prod_n (D\phi_n) e^{-S[\phi]}, \] (27)

This should make sense of course, as gauge-equivalent configurations give the same value for the path integral and this value is multiplied by the gauge group volume; dividing by this volume seems therefore the appropriate correction. The \( \phi_n \) in the above expression are all the matter and gauge fields in the theory at hand.

1.4.1 DeWitt-Faddeev-Popov method

Instead of summing over all possible configurations - gauge-equivalent or not - and then dividing by the gauge group volume it is more convenient to first fix the gauge and then do the path integral, so that one can ignore the trouble of overcounting completely. The DeWitt-Faddeev-Popov method is what makes this transition.

Suppose the gauge is fixed at the chosen field configuration \( \hat{\phi}_n(x) \). Related to this specific gauge, one can choose the gauge-fixing functionals \( F_i[\phi;x] \) such that

\[ F_i[\phi;x] = 0 \; \forall \; i, x \quad \Leftrightarrow \quad \phi_n(x) = \hat{\phi}_n(x) \; \forall \; n, x. \] (28)

Since the number of independent gauge-fixing functionals is equal to the number of gauge degrees of freedom, the functionals have a gauge group index \( i \). Here \( x \) is the coordinate of the manifold the gauge theory is defined on. One now wants to implement the delta functional \( \delta(F_i[\phi;x]) \) into the path integral, thereby restricting the nonzero contributions to the path integral to those which are gauge-inequivalent to the configuration \( \hat{\phi}_n(x) \) and thus getting around the problem of overcounting. To implement all the conditions in (28) at once, the following product of path integrals can be used:

\[ \prod_{i,j,x,y} \left( \int d\lambda_j(y) \delta \left( F_i[\hat{\phi}^{\lambda};x] \right) \right). \]

Here the product is taken over both the gauge group indices \( i, j \) and discrete manifold points \( x, y \), whereafter the continuum limit of the latter is implicitly taken. Moreover, the \( \lambda_j(y) \) are the gauge transformation parameters and \( \phi_n(x) \) is the field \( \phi_n(x) \) gauge transformed by a gauge transformation \( \lambda(x) \).
To implement this correctly, note that
\[
\prod_{i,j,x,y} \left( \int d\lambda_j(y) \delta\left( F_i[\hat{\phi}^\lambda; x] \right) \right) = \prod_{i,j,x,y} \left( \int d\lambda_j(y) \frac{1}{\delta F_i[\hat{\phi}^\lambda; \lambda=0]} \right)
\]
and that
\[
\prod_{i,j,x,y} \left( \frac{\delta F_i[\hat{\phi}^\lambda; x]}{\delta \lambda_j(y)} \bigg|_{\lambda=0} \right) = \prod_{i,j,x,y} \left( \int db_i(x)dc_j(y) \exp\left( -b_i(x)c_j(y) \Delta_{j,y}F_i[\hat{\phi}, x] \right) \right)
\]
\[
= \int \prod_{k,l} (Db_k Dc_l) \exp \left( -\int d^p x d^p y b_i(x)c_j(y)\Delta_{j,y}F_i[\hat{\phi}, x] \right), \quad (29)
\]
where we defined \( \Delta_{j,y}F_i[\hat{\phi}, x] := \delta F_i[\hat{\phi}^\lambda; x] \bigg|_{\lambda=0} \), we introduced the Grassmannian valued fields \( b_i(x), c_j(y) \), we used a standard Grassmann integral identity, \( k, l \) are also gauge group indices and in the last line summation over repeating gauge group indices is implied. The fields \( b_i(x), c_j(y) \) are called ghost fields and the expression in (29) is the Faddeev-Popov determinant.

Substituting
\[
\prod_{i,j,x,y} \left( \int d\lambda_j(y) \delta\left( F_i[\hat{\phi}^\lambda; x] \right) \right) \prod_{k,l} (Db_k Dc_l) \exp \left( -\int d^p x d^p y b_i(x)c_j(y)\Delta_{j,y}F_i[\hat{\phi}, x] \right)
\]
\[
= 1
\]
into (27), using gauge invariance of the original theory and the Faddeev-Popov determinant and subsequently making a change of field integration variables isolates the path integral over the gauge transformation parameters:
\[
\int \prod_i (D\lambda_i) = V_{\text{gauge}}.
\]
This term cancels against the gauge group volume in the denominator, and what remains is the gauge-fixed path integral
\[
\int \prod_{n,k,l} (D\phi_n Db_k Dc_l) \prod_i (\delta ( F_i[\phi] )) \exp \left( -S[\phi] - \int d^p x d^p y b_i'(x)c_j'(y)\Delta_{j,y}F_i'[\phi, x] \right).
\]
It is important to note that this path integral is through the gauge-fixing functionals dependent on the specific gauge chosen. However, there’s a theorem\(^{23}\) which states that the value of the integral is actually independent of the specific gauge chosen.\(^{24,25}\) To conclude, the second term in the exponential is usually called the ghost action:
\[
S_g := \int d^p x d^p y b_i'(x)c_j'(y)\Delta_{j,y}F_i'[\phi, x].
\]  

\(^{22}\)For the remainder of this section, we suppress the factors \( \sqrt{\mathcal{G}} \) in the volume elements.

\(^{23}\)See Weinberg\([8]\), section 15.5.

\(^{24}\)It is assumed that the chosen gauge is fairly reasonable, i.e. without unphysical peculiarities.

\(^{25}\)There’s an even stronger statement that the delta functional can be replaced by any reasonable functional \( B(F_i[\phi]) \) without changing the fully connected amplitudes appearing in the \( S \)-matrices of the theory. This can be very convenient. For example, in the DeWitt-Faddeev-Popov procedure applied to Yang-Mills theory it is convenient to replace the delta functional by a term which is Gaussian in the gauge-fixing functionals, as for Gaussian path integrals the exact solution is known.
1.4.2 BRST symmetry

To summarize the above, the result of the DeWitt-Faddeev-Popov method is that by choosing a specific gauge through the gauge-fixing functionals

\[ F_i(\theta; x) = 0, \]

the path integral of the theory takes the form

\[ \int \mathcal{D}\phi_i \mathcal{D}b_j \mathcal{D}c_k e^{(-S - S_{gf} - S_0)}, \]

where we left the products over the field indices implicit and where we used an auxiliary field \( B_i(x) \) - sometimes called the Nakanishi-Lautrup field - to introduce an integral representation of the delta functional, with gauge-fixing action\(^{26}\)

\[ S_{gf} := -i \int d^Dx \sqrt{g} B_i(x) F_i(\phi; x). \]

The ghost action \( S_g \) is as in (30).

The resulting total action \( S_{tot} := S + S_{gf} + S_g \), although not gauge invariant due to the gauge-fixing functional, remarkably still obeys a symmetry: BRST symmetry. The infinitesimal BRST transformation is

\[ \begin{align*}
\delta_B \phi_n(x) &= -i\theta \int d^Dy \sqrt{g} c_i(y) \Delta_{i,y} \phi_n(x) \quad (31a) \\
\delta_B b_i(x) &= 0 \quad (31b) \\
\delta_B c_i(x) &= -\frac{i}{2} \theta \int d^Dy_1 \sqrt{g} \int d^Dy_2 \sqrt{g} f^{i,x}_{j,y_1,k,y_2} c_j(y_1)c_k(y_2), \quad (31c) \\
\delta_B F_i &= 0 \quad (31d) 
\end{align*} \]

where \( \theta \) is an infinitesimal Grassmann parameter and \( f^{i,x}_{j,y_1,k,y_2} \) is the structure function of the gauge transformation:

\[ [\Delta_{j,y_1}, \Delta_{k,y_2}] = \int d^Dy_1 \sqrt{g} f^{i,x}_{j,y_1,k,y_2} \Delta_{i,x}. \]

The proof of BRST invariance of \( S_{tot} \) goes as follows. \( \delta_B S = 0 \), since the BRST transformation of the matter and gauge fields (31a) is just a gauge transformation with gauge parameters \(-i\theta c_i(y)\) and \( S \) is assumed to be the action of a gauge theory. To see that \( \delta_B(S_{gf} + S_g) = 0 \), first note that\(^{27}\)

\[ \begin{align*}
\delta_B \left( \int d^Dy \sqrt{g} \phi_n F_i(\phi; x) \right) &= -\frac{i}{2} \theta \int d^Dy \int d^Dz_1 \int d^Dz_2 f^{i,y}_{k,z_1,l,z_2} c_k(z_1)c_l(z_2) \Delta_{j,y} F_i(\phi; x) \\
&\quad + \theta \int d^Dz_1 \int d^Dz_2 c_k(z_1)c_l(z_2) \Delta_{j,x} \Delta_{l,x} F_i(\phi; x) \\
&= -\frac{i}{2} \theta \int d^Dy \int d^Dz_1 \int d^Dz_2 f^{i,y}_{k,z_1,l,z_2} c_k(z_1)c_l(z_2) \Delta_{j,y} F_i(\phi; x) \\
&\quad + \frac{i}{2} \theta \int d^Dy \int d^Dz_1 \int d^Dz_2 c_k(z_1)c_l(z_2) f^{i,y}_{k,z_1,l,z_2} \Delta_{j,y} F_i(\phi; x) \\
&= 0. 
\end{align*} \]

\(^{26}\)For other functionals \( B(F_i(\phi)) \) one can also construct a gauge-fixing action using an auxiliary field, but in a different form of course.

\(^{27}\)Again, we suppress factors of \( \sqrt{g} \) in the volume elements.

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Because of this and $\delta_B B_i(x) = 0$,

$$\delta_B (S_{gf} + S_g) = -i \int d^D x B_i(x) \delta_B F_i[\phi; x]$$

$$+ \theta \int d^D y \delta_B b_i(x) c_j(y) \Delta_{ij,y} F_i[\phi, x]$$

$$= -\theta \int d^D x B_i(x) c_j(y) \Delta_{ij,y} F_i[\phi, x]$$

$$+ \theta \int d^D y \delta_B B_i(x) c_j(y) \Delta_{ij,y} F_i[\phi, x]$$

$$= 0.$$

This proves the BRST invariance of $S_{tot}$.

### 1.4.3 BRST quantization, general analysis

In the procedure of *BRST quantization* this BRST symmetry is used to derive the physical spectrum of quantum states of the gauge theory. After the DeWitt-Faddeev-Popov method a quantum mechanical amplitude is of the form

$$\langle f | i \rangle = \int D\phi D B_i D b_k D c_l e^{(-S_{gf} - S_g)},$$

where the initial and final state boundary conditions are left implicit. Physically, this amplitude should be independent from the gauge in which one chooses to calculate the path integral. Therefore, it is a physical requirement that under an infinitesimal shift in the gauge-fixing functionals $F_i'[\phi, x] = F_i[\phi, x] + \delta F_i[\phi, x]$ the amplitude between two physical states is invariant:

$$0 = \theta \delta \langle f | i \rangle$$

$$= i \langle f | \delta_B \left( \int d^D x b_i(x) \delta F_i[\phi, x] \right) | i \rangle.$$

A derivation of the second equality goes as

$$\theta \delta \langle f | i \rangle = i \theta \int D\phi D B_i D b_k D c_l e^{(-S_{gf} - S_g)} \left[ \int d^D x i B_i(x) \delta F_i[\phi; x] \right]$$

$$- \langle f \int d^D x \delta B_i(x) c_j(y) \Delta_{ij,y} \delta F_i[\phi; x] \right]$$

$$= i \langle f \int d^D x \left[ i \delta B_i(x) \delta F_i[\phi; x] \right]$$

$$+ i \delta b_i(x) \left( -\theta \int d^D y c_j(y) \Delta_{ij,x} \delta F_i[\phi; x] \right) | i \rangle$$

$$= i \langle f \int d^D x \left( \delta B_i(x) \delta F_i[\phi; x] + b_i(x) \delta F_i[\phi; x] \right) | i \rangle$$

$$= i \langle f \left( \delta_B \left( \int d^D x b_i(x) \delta F_i[\phi; x] \right) \right) | i \rangle.$$
Noether’s theorem tells us that for a general field dependent function $\Phi$,

$$\delta_B \Phi = i\theta [Q_B, \Phi]_{\mp},$$

where $Q_B$ is the conserved Noether charge associated with the BRST symmetry, $[,]_{-} = [\cdot, \cdot]$ and $[,]_{+} = \{\cdot, \cdot\}$, and the sign being minus/plus according as $\Phi$ is bosonic/fermionic. Physical states therefore obey

$$\delta(f|i) = -\langle f| \left[ Q_B, \int d^Dx b_i(x) \delta F_i[\phi, x] \right] \pm |i\rangle,$$

where the sign is now plus/minus if the gauge-fixing functionals are bosonic/fermionic. Since this equality has to hold for arbitrary change $\delta F_i[\phi, x]$ in the gauge choice, it is required that all physical states $|\psi\rangle$ are closed with respect to the BRST charge:

$$Q_B |\psi\rangle = 0.$$

We will call this the (BRST) physicality condition, or the requirement of BRST-invariance for physical states. Note by the way that we assumed here that $Q_B^\dagger = Q$. A simple argument for this is that if they were different there would be another symmetry associated with $Q_B^\dagger$ present in the theory, but there is none. Besides, this can be checked explicitly for the specific theory under investigation.

In the former section we demonstrated BRST invariance of the full gauge-fixed action. In a similar - but more tedious - way one can show that a BRST transformation is nilpotent, i.e. a repeated BRST transformation applied to a general field dependent function $\Phi$ gives zero: $28$

$$\delta_B^2 \Phi = 0.$$

The BRST charge therefore obeys

$$0 = \delta_B^2 \Phi = i\theta [Q_B, i\theta [Q_B, \Phi]_{\mp}]_{\mp} = \theta [Q_B, Q_B, \Phi]_{\mp} = \theta [\theta (Q_B)^2, \Phi].$$

Since this holds for general $\Phi$, $(Q_B)^2$ is either proportional to the unit operator or zero. However, through the Noether procedure it follows from (31a) that $Q_B$ has ghost number 1 and thus has $(Q_B)^2$ ghost number$^{29}$ 2. It therefore cannot be proportional to the unit operator and is thus zero. It is important to note that this nilpotency of the BRST charge is automatically valid in the classical regime, but not in the quantum regime, i.e. in the regularized path integral. For string theory we will see that this has to be imposed explicitly and that this extra condition is highly non-trivial.$^{30}$

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28See Weinberg[8], section 15.7.

29This is a conserved quantity associated to the so-called ghost number symmetry of the ghost action. It counts $c$-field contributions as $+1$ and $b$-field contributions as $-1$. See Polchinksi[3], section 2.7, for more information.

30This might sound related to anomalies in the transition from a classical theory to a quantum theory. In section (1.5.2) we will see that for string theory it is in fact related.
This nilpotency of the BRST charge has important implications for the spectrum of the theory. Suppose a physical state $|\psi'\rangle$ can be written as the BRST charge applied to some other state $|\chi\rangle$: $Q_B|\chi\rangle$. States with this property are called exact with respect to the operator $Q_B$. Using the hermiticity of the BRST charge and the physicality condition, we then find that for any other physical state $|\psi\rangle$ - exact or not - the overlap between the two states is zero:

$$\langle\psi|\psi'\rangle = (Q_B|\psi\rangle)^\dagger|\chi\rangle = 0.$$ 

Moreover, the physical state $|\psi'\rangle$ has zero norm and is therefore also called a null state. The meaning of this is that BRST-exact states cannot occur in any quantum physical experiment, as they have zero overlap with any physical state and thus have zero probability to be observed. Furthermore, two non-exact physical states $|\psi'\rangle$ and $|\psi''\rangle$ which differ from each other by an exact state, $|\psi'\rangle = |\psi''\rangle + Q_B|\chi\rangle$, have the same quantum amplitude with any physical states $|\psi\rangle$:

$$\langle\psi|\psi'\rangle = \langle\psi|(|\psi''\rangle + Q_B|\chi\rangle) = \langle\psi|\psi''\rangle.$$ 

That is, states differing by a BRST-exact state are indistinguishable in any quantum physical experiment. The proper spectrum of our gauge theory is therefore the set of physically distinguishable, non-exact, physical states and from here on we will call the states in this spectrum proper physical states.\footnote{This other state cannot be physical of course.}

Another way to see the proper physical spectrum is as the set of non-exact BRST-closed states, modulo BRST-exact states. This set is usually called the cohomology of the BRST charge $Q_B$: the proper physical spectrum is the cohomology of $Q_B$. These final conclusions will be our guide in determining the spectrum of the point-particle and of open and closed string theory. Besides this, it is important to note that the treatment so far is perfectly general and that the BRST quantization as described above is in principle applicable to any gauge theory.\footnote{Note that in this definition a physical states can only be proper as part of the whole proper spectrum, it cannot be proper in itself. In practice we will neglect this small modification.} This is also a reason why we wanted to derive the string spectrum through BRST quantization, as the more common quantization procedures\footnote{Amongst others, it has been applied to Yang-Mills theory and general relativity.} seem somewhat ad hoc and may therefore mystify the simplicity of string theory as a gauge theory.

As a final remark, note that matrix elements between proper physical states are invariant under a change of the gauge-fixing functionals because of the physicality condition. This implies that if we can show that certain states decouple from the proper physical spectrum in one gauge, they will decouple from the spectrum in any gauge. We are therefore free to choose a gauge and derive the proper physical spectrum in that gauge. This will be particularly useful in section (1.5.6), where we will prove that all ghost states decouple from the proper physical spectrum.\footnote{Canonical quantization, light-cone quantization.}
1.4.4 BRST quantization of the point-particle

Before applying the BRST formalism to the world-sheet theory we described in earlier sections, let’s look at the example of the point-particle of mass $m$.

The Euclidianized version of the point-particle theory in $D$ spacetime dimensions can be described by a world-line field theory with $D+1$ fields:

- $D$ scalar fields $X^\mu(\tau)$, with $\mu \in \{0, \ldots, D-1\}$.
- One tetrad field $e(\tau) := \sqrt{g_{\tau\tau}}(\tau)$, where $g_{\tau\tau}$ is the world-line metric, now of Euclidian signature.

The action of this theory is
\[ S = \frac{1}{2} \int d\tau \left( \frac{1}{e} \dot{X}^\mu \dot{X}_\mu + em^2 \right), \]
where $\dot{f}(\tau) = \frac{\partial}{\partial \tau} f(\tau)$. Using the transformation property of the tetrad field under a world-line reparametrization $\tau \rightarrow \tau' = \tau + \epsilon f(\tau), \epsilon \ll 1$,
\[ e'(\tau') = \frac{\partial \tau}{\partial \tau'} e(\tau'), \]
which is implied by the definition of $e(\tau)$, it is not hard to verify reparametrization invariance of the point-particle action. This is a gauge symmetry.\(^{36}\)

As a basis for the infinitesimal reparametrizations can be chosen $\delta_{\tau_1} \tau(\tau) := \delta(\tau - \tau_1)$. Under such a reparametrization the fields transform as follows:
\[ \delta_{\tau_1} X^\mu(\tau) = -\delta(\tau - \tau_1) \partial_{\tau} X^\mu(\tau) \]
\[ \delta_{\tau_1} e(\tau) = -\partial_{\tau} au(\delta(\tau - \tau_1)e(\tau)). \]

\(^{35}\)See Van Holten\(^{[5]}\) for a different analysis of point-particle BRST quantization.
\(^{36}\)As in the case of string theory, spacetime Poincaré invariance is also present.
We fix the gauge with the analogue of unit gauge for the world-sheet metric: \( e(\tau) = 1 \). Using the gauge-fixing functional

\[
F[x^\mu, e; \tau] = 1 - e(\tau) =: F(\tau),
\]

the gauge fixed action becomes

\[
S = \int d\tau \left( \frac{1}{2e} \dot{X}^\mu \dot{X}_\mu + \frac{1}{2} em^2 + iB(e - 1) - e\right),
\]

where \( B(\tau) \) is the auxiliary bosonic field giving the delta functional and \( b(\tau), c(\tau) \) are the fermionic ghost fields. Note there is only one pair of ghosts because the theory had one gauge degree of freedom.

The BRST transformations (31a)-(31d) applied to the point-particle case become:

\[
\begin{align*}
\delta_B X^\mu(\tau) &= -i\theta \int d\tau' c(\tau') \delta(\tau - \tau') X^\mu(\tau) \\
&= i\theta c(\tau) \partial_\tau X^\mu(\tau) \\
\delta_B e(\tau) &= -i\theta \int d\tau' c(\tau') \delta(\tau - \tau') e(\tau) \\
&= i\theta \partial_\tau e(\tau) c(\tau) \\
\delta_B B(\tau) &= 0 \\
\delta_B b(\tau) &= \theta B(\tau) \\
\delta_B c(\tau) &= -\frac{i}{2} \theta \int d\tau_1 d\tau_2 f_{\tau_1 \tau_2} c(\tau_1) c(\tau_2) \\
&= i\theta c(\tau) \dot{c}(\tau). 
\end{align*}
\]

For the variations of \( e(\tau) \) and \( c(\tau) \) the defining property of the derivative of the Dirac delta function was used:

\[
\int d\tau \partial_\tau \delta(\tau - \tau') f(\tau) = -\partial_{\tau'} f(\tau').
\]
The structure function of the gauge transformation was found to be
\[ f_{\tau_1 \tau_2} = \delta(\tau - \tau_2) \partial_\tau \delta(\tau - \tau_1) - \delta(\tau - \tau_1) \partial_\tau \delta(\tau - \tau_2). \]

Using (32a) twice,
\[ [\delta_{\tau_1}, \delta_{\tau_2}] x^\mu(\tau) = \delta(\tau - \tau_1) \partial_\tau \delta(\tau - \tau_2) - \delta(\tau - \tau_2) \partial_\tau \delta(\tau - \tau_1) \partial_\tau X^\mu(\tau) \]
\[ = \int d\tau_3 f_{\tau_1 \tau_2}^{\tau_3} \partial_\tau X^\mu(\tau) \]
\[ = -f_{\tau_1 \tau_2} \partial_\tau X^\mu(\tau). \]

It turns out to be convenient to integrate out the non-dynamical fields \( B(\tau) \) and \( e(\tau) \), effectively fixing \( e(\tau) \) at 1. The resulting path integral action is then
\[ S = \int d\tau \left( \frac{1}{2} \dot{X}_\mu(\tau) \dot{X}^\mu(\tau) + \frac{1}{2} m^2 - bc \right). \] (35)

The BRST transformations become
\[ \delta_B X^\mu(\tau) = i \theta c \dot{X}^\mu \]
\[ \delta_B b(\tau) = i \theta \left( \frac{1}{2} \dot{X}^\mu \dot{X}_\mu + \frac{1}{2} m^2 - \dot{b}(\tau) c(\tau) \right) \]
\[ \delta_B c(\tau) = i \theta c, \]
where in \( \delta_B b(\tau) \) we eliminated \( B(\tau) \) by using the equation of motion for \( e(\tau) \):
\[ B(\tau) = i \left( \frac{1}{2e(\tau)} \dot{X}^\mu(\tau) \dot{X}_\mu(\tau) + \frac{1}{2} m^2 - \dot{b}(\tau) c(\tau) \right). \]

These transformations are still both a symmetry of the action and nilpotent, but now only if the equations of motion are satisfied. That means that \( Q^2_B \) is only zero when acting on physically acceptable states, i.e. \( Q^2_B \) is an operator equation.

Now, the remaining step is to calculate the BRST charge and to implement the physicality conditions on the complete set of quantum eigenstates of the fields or its conjugated momenta. Because we want to arrive at conditions related to the physical world, we Wick rotate the obtained action back to Minkowski signature and real, physical time \( t = -i\tau \). Using \( f'(t) := \partial_t f(t) = i \partial_\tau f(\tau) = i \dot{f}(\tau) \) and \( S = i S_E \), where \( S_E \) is the Euclidian action (35), we get
\[ S = \int dt \left( \frac{1}{2} X'^\mu X^\mu - \frac{1}{2} m^2 + ibc' \right), \] (36)
where we also performed a partial integration on the last term. Applying the standard formulas...
for the Hamiltonian and conserved charges, we obtain

\[ H = p^2 + ibc' - L = \frac{1}{2}(p^2 + m^2), \]  \hspace{1cm} (37)

\[ Q_B = \frac{1}{2}c(p^2 + m^2) = cH, \]  \hspace{1cm} (38)

where we defined the canonical conjugated spacetime momentum \( p^\mu(t) := X'_\mu(t) \).

Using the Wick rotated variations

\[ \delta_B X'^\mu(t) = \theta c' X'^\mu + \theta c X'^\mu \]
\[ \delta_B b(t) = i\theta \left( \frac{1}{2} X'^\mu X'_\mu + \frac{1}{2} m^2 + ib'c \right) \]
\[ \delta_B c'(t) = \theta c' c' + \theta cc'' , \]

it immediately follows that

\[ \delta_B S = \int dt \frac{d}{dt} \left( \frac{1}{2}c(p^2 - m^2) \right). \]

Thus,

\[ Q_B = p^\mu \Delta_B X_\mu + \pi_4 \Delta_B c - \frac{1}{2}c(p^2 - m^2) \]
\[ = cp^2 + ibc' - \frac{1}{2}c(p^2 - m^2) \]
\[ = \frac{1}{2}c(p^2 + m^2), \]  \hspace{1cm} (39)

where in the last line we used the equation of motion for \( b(t) : c'(t) = 0 \). Note that if we wouldn’t have done a partial integration in the last term of the action (36), we would have got \( Q_B = -cH \). This sign difference is irrelevant however, because one can always multiply a conserved charge with a constant factor without changing its essential properties.

Because of the equal-time canonical commutation relations\(^{38}\)

\[ [X^\mu, p^\nu] = i\eta^{\mu\nu}, \quad \{b, c\} = 1, \]

the ghosts generate a two-state system.\(^{39}\) Therefore a complete set of states is

\[ \{|k, \uparrow\rangle, |k, \downarrow\rangle\}, \]

where

\[ p^\mu|k, \uparrow\rangle = k^\mu|k, \uparrow\rangle \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
This leads to
\[ Q_B |k, \downarrow \rangle = \frac{1}{2} (k^2 + m^2) |k, \uparrow \rangle \]
\[ Q_B |k, \uparrow \rangle = 0. \]

The physicality condition therefore implies that the only physical states are
\[ |k, \downarrow \rangle \quad \text{if } k^2 + m^2 = 0 \]
\[ |k, \uparrow \rangle \quad \text{for all } k^\mu. \]

However, we also see that if \( k^2 + m^2 \neq 0 \), then
\[ |k, \uparrow \rangle = Q_B \left( \frac{2}{k^2 + m^2} |k, \downarrow \rangle \right), \]
\[ \text{i.e. } |k, \uparrow \rangle \text{ for } k^2 + m^2 \neq 0 \text{ is a null state. The distinguishable physical states are therefore:} \]
\[ |k, \downarrow \rangle \quad \text{for } k^2 + m^2 = 0 \]
\[ |k, \uparrow \rangle \quad \text{for } k^2 + m^2 = 0. \]

As expected, proper physical states must satisfy the energy-momentum relation. However, it seems that a physical point-particle with energy-momentum \( k^\mu \) can appear in two different forms, contrary to what we would expect.

There’s however an additional, kinematical condition which prohibits the states \( |k, \uparrow \rangle \) from the physical spectrum: \( b |\psi \rangle = 0 \). This is because for a general proper physical state \( |\psi' \rangle \), we would have
\[ \langle \psi' | k, \uparrow \rangle \propto \delta(k^2 + m^2), \]
\[ \text{since when } k^2 + m^2 \neq 0, |k, \uparrow \rangle \text{ is a null state and has zero overlap with any physical state. But} \]
\[ \text{amplitudes in field theory and string theory never have delta functions.}^{40} \] Therefore we conclude that the only physically acceptable and distinguishable point-particle states are
\[ |k, \downarrow \rangle \quad \text{for } k^2 + m^2 = 0. \]

This is of course exactly what we would expect for a point-particle of mass \( m \). We conclude that the physicality conditions arrived at through the BRST formalism restricts the set of states correctly to the proper physical ones, i.e. it eliminates the unphysical gauge degrees of freedom correctly.

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\(^{40}\)Except when the spacetime dimension \( D = 2 \). This won’t be the case in the string theories we will investigate.

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25
1.5 BRST quantization of the bosonic string

BRST quantization of the string is very similar to that of the point-particle. The first step is to derive the BRST invariant action $S_{BRST} = S_X + S_{gf} + S_g$, which is obtained by the usual gauge-fixing procedure.

Starting with the non-topological part of the original action of the theory

$$
\frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X
$$

and remembering it has three gauge degrees of freedom (two diffeomorphism invariances, one Weyl invariance), we know that we can use these symmetries to gauge-fix the world-sheet metric to unit gauge. Therefore we impose as our gauge-fixing functional

$$
F_{ab}[X^\mu, g_{ab}; \vec{\sigma}] = \frac{1}{4\pi} (g_{ab}(\vec{\sigma}) - \delta_{ab}) =: F_{ab}(\vec{\sigma}),
$$

where $\vec{\sigma} = (\sigma^1, \sigma^2)$. Note that these equations form indeed three independent conditions, as the metric is a symmetric tensor. The factor $\frac{1}{4\pi}$ is added for convenience, but has to be used consistently in the different parts of the action in order to preserve BRST invariance of the full action.

Using the path integral representation of the delta functional, the gauge-fixing action becomes

$$
S_{gf} = \frac{i}{4\pi} \int d^2\sigma \sqrt{g} B^{ab}(\delta_{ab} - g_{ab}),
$$

where we introduced the symmetric tensor field $B^{ab}(\vec{\sigma})$.

The general form of the ghost action applied to a field theory on a two-dimensional world-sheet is

$$
S_g = \int d^2\sigma \sqrt{g} \int d^2\sigma' \sqrt{g} b^{ab}(\vec{\sigma}) c^i(\vec{\sigma}') \Delta_{i,\vec{\sigma}} F_{ab}(\vec{\sigma}),
$$

where $b^{ab}(\vec{\sigma})$ and $c^i(\vec{\sigma})$ are the Grassmannian ghost fields, the index $i$ runs over the different gauge transformations and $\Delta_{i,\vec{\sigma}} F_{ab}(\vec{\sigma})$ is the finite part of the variation of $F_{ab}(\vec{\sigma})$ under a gauge transformation $i$. Splitting up the gauge symmetries in the standard way, i.e. two diffeomorphism transformations and one Weyl transformation, leads to

$$
S_g = -\frac{1}{2\pi} \int d^2\sigma \sqrt{g} b_{ab}(\vec{\sigma}) \nabla^a c^b(\vec{\sigma}),
$$

where $b_{ab}(\vec{\sigma})$ is symmetric and traceless.

Since $\Delta_{i,\vec{\sigma}} F_{ab}(\vec{\sigma}) = \frac{1}{4\pi} \Delta_{i,\vec{\sigma}} g_{ab}(\vec{\sigma})$, it is sufficient to investigate the behaviour of the metric under gauge transformations.

In the point-particle example we have seen that delta functions form a basis for coordinate transformations. Making an infinitesimal coordinate transformation $\sigma^\nu = \sigma^\nu + \epsilon \delta_i(\vec{\sigma} - \vec{\sigma}_1)$ in one specific direction

41 $b^{ab}(\vec{\sigma})$ is symmetric in its indices.
\[ e \in \{1, 2\}, \text{ we find that} \]
\[ g_b^e(\sigma^e) := g_{ab}(\sigma^a) - e \partial_b \left( \delta(\sigma^e - \sigma_1) \right) g_{ac}(\sigma^c) \]
\[ = g_{ab}(\sigma^a) - e \partial_b \left( \delta(\sigma^e - \sigma_1) \right) g_{bc}(\sigma^c), \]
where in the last equality we used twice that \( \nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} = 0. \) Under a Weyl transformation, we have
\[ g_b^e(\sigma^e) = e^{2\omega(\sigma^e)} g_{ab}(\sigma^a) = g_{ab}(\sigma^a) + 2\omega(\sigma^a) g_{ab}(\sigma^a). \]

Therefore we conclude that the three independent gauge transformations acting on the gauge-fixing functionals give
\[ \Delta_{\omega, \sigma^e} F_{ab}(\sigma^e) = \frac{1}{2\pi} g_{ab} \nabla_b \delta(\sigma - \sigma^e) \]
\[ = \frac{1}{2\pi} g_{ab} \nabla_b \delta(\sigma - \sigma^e), \quad e \in \{1, 2\} \]
\[ \Delta_{\omega, \sigma^e} F_{ab}(\sigma^e) = \frac{1}{2\pi} \delta(\sigma - \sigma^e) g_{ab}(\sigma^e). \]

Substituting these expressions into (43), performing a partial integration\(^4\) in the diffeomorphism terms, using that \( b^{ab}(\sigma^e) \) is a symmetric tensor field and incorporating \( \omega(\sigma^e) \) in a re-definition of the ghost field \( c_{\omega^e}(\sigma^e) \), we obtain
\[ S_g = -\frac{1}{2\pi} \int d^2 \sigma \sqrt{g} \left\{ b^{ab}(\sigma^e) \nabla_a c_b(\sigma^e) + c_{\omega^e}(\sigma^e) b^{ab}(\sigma^e) \right\}. \]

Integrating out the Grassmannian \( c_{\omega^e}(\sigma^e) \)-field in the path integral leads to the condition that \( b^{ab}(\sigma^e) \) is traceless and gives the expression (44).

Because the BRST transformations are tensorial and the mode expansions and Virasoro generators are defined in terms of \((z, \bar{z})\)-coordinates, it is both possible and convenient to proceed the analysis in complex planar coordinates. And since the only contributions to the gauge-fixed path integral come from the world-sheet metric in unit gauge, we can restrict \( S_X \) and \( S_g \) to unit gauge.\(^4\) For \( S_X \) this has been done in (10). For \( S_g \), note that the symmetry and tracelessness of \( b^{ab}(\sigma^e) \) leads in unit gauge to \( b_{zz} = b_{\bar{z}\bar{z}} = 0 \). Furthermore, since \( \Gamma^z_{xx} = \Gamma^\bar{z}_{x\bar{z}} = 0 \) for \( x \in \{z, \bar{z}\} \) in unit gauge and \( \sqrt{g} = \frac{1}{2\pi} = g_{zz} \), we obtain
\[ S_g = \frac{1}{2\pi} \int d^2 z \left( b_{zz} \nabla_z c^z + b_{\bar{z}\bar{z}} \nabla_{\bar{z}} c^\bar{z} \right) \]
\[ = \frac{1}{2\pi} \int d^2 z \left( b_{zz} \partial_z c^z + b_{\bar{z}\bar{z}} \partial_{\bar{z}} c^\bar{z} \right). \]

The equations of motion for the ghost fields are \( \partial_z b_{zz} = \partial_{\bar{z}} b_{\bar{z}\bar{z}} = \partial_z c^z = \partial_{\bar{z}} c^{\bar{z}} = 0 \), making \( b_{zz}(z, \bar{z}) \) and \( c^z(z, \bar{z}) \) holomorphic and \( b_{\bar{z}\bar{z}}(z, \bar{z}) \) and \( c^{\bar{z}}(z, \bar{z}) \) anti-holomorphic functions. Defining \( b(z) := b_{zz}(z, \bar{z}), b(\bar{z}) := b_{\bar{z}\bar{z}}(z, \bar{z}), c(z) := c^z(z, \bar{z}), c(\bar{z}) := c^{\bar{z}}(z, \bar{z}) \) leads to
\[ S_g = \frac{1}{2\pi} \int d^2 z \left( b \partial c + b \bar{c} \bar{b} \right) \]
\[ (45) \]
\(^4\)For the moment consider a closed string manifold. That means a manifold without a boundary and thus a vanishing total derivative term. See section (1.5.3) for the open string analysis of the ghost action. \(^4\)\( S_g \) in unit gauge is zero of course. It doesn’t make sense to impose the gauge-fixing to this part of the action too, as it is exactly this term which restricts the contributions to the path integral to those with unit gauge metric.
as the final form of the ghost action in string theory. Because of the tensorial origins of the ghost fields, we recognize this ghost action as the sum of a \( bc \) CFT and a \( \tilde{b} \tilde{c} \) CFT, both with weight parameter \( \lambda = 2 \). This observation will be useful in due course.

Now, the full path integral becomes
\[
\int D\mathcal{X} Dg D\mathcal{B} D\tilde{\mathcal{B}} Dc D\tilde{c} e^{-\mathcal{S}_X - \mathcal{S}_g - \mathcal{S}_b - \mathcal{S}_\tilde{b} - \mathcal{S}_c - \mathcal{S}_\tilde{c}},
\]
with the different actions are respectively given by (41), (42) and (44). As for the point-particle case, we can gauge-fix the action by integrating out the \( B(\vec{\sigma}) \)-field and consequentially the metric field. The resulting path integral is
\[
\int D\mathcal{X} D\mathcal{B} D\tilde{\mathcal{B}} Dc D\tilde{c} e^{-\mathcal{S}_X - \mathcal{S}_g - \mathcal{S}_c - \mathcal{S}_\tilde{c}},
\]
with the actions now in unit gauge form (10) and (45), respectively.

1.5.1 Derivation of the BRST charge

Regarding the residual BRST symmetry, the point-particle case is again the appropriate example. BRST symmetry for the scalar fields and the ghost fields is still present, if the equations of motion are satisfied. The standard form of the BRST transformations applied to the theory in (46) gives
\[
\begin{align*}
\delta_B X^\mu(z) &= i\theta (c\partial X^\mu + \tilde{c}\bar{\partial} X^\mu) \\
\delta_B b(z) &= i\theta (T_X + T_g) \\
\delta_B \tilde{b}(\bar{z}) &= i\theta (\tilde{T}_X + \tilde{T}_g) \\
\delta_B c(z) &= i\theta c\partial c \\
\delta_B \tilde{c}(\bar{z}) &= i\theta \tilde{c}\bar{\partial} \tilde{c}
\end{align*}
\]
Here, \( T_X (\tilde{T}_X) \) and \( T_g (\tilde{T}_g) \) are the (anti-)holomorphic components of respectively the scalar and ghost part of the action.

Transformation (47a) is trivially obtained, when noting that the scalars fields are invariant under Weyl transformations. For (47b) we use the standard form \( \delta_B b(z) = \delta_B b_{zz}(\vec{z}) = \theta B_{zz}(\vec{z}) \) and substitute the equation of motion from \( g_{zz}(\vec{z}) \) for \( B_{zz}(\vec{z}) \). This equation of motion is obtained by setting the metric variation of the full, non-gauge-fixed action to zero:
\[
0 = \delta g,\vec{\sigma} (\mathcal{S}_X + \mathcal{S}_g + \mathcal{S}_b + \mathcal{S}_c)
= -\frac{\sqrt{-g}}{4\pi} \left( (T_X)^{ab} + (T_g)^{ab} + iB^{ab} \right) \delta g_{ab}(\vec{\sigma}),
\]
where we imposed unit gauge after taking the variation and we used the definition (14) of the stress-tensor. Therefore \( B_{ab} = i((T_X)_{ab} + (T_g)_{ab}) \) and using the tensorial properties of both sides of the expression making the substitution, expression (47b) is obtained. The transformation for \( b(\vec{z}) \) can be derived similarly.

To calculate (47d) we start with the standard expression
\[
\delta_B c^\gamma(\vec{z}) = -\frac{i}{2} \theta \int d^2\tilde{z}_1 d^2\tilde{z}_2 f_{\alpha,\tilde{\gamma},\beta,\tilde{\delta}} c^\alpha(\vec{z}_1) c^\beta(\vec{z}_2),
\]
where the structure functions are defined through
\[
[D_{\alpha,\tilde{\gamma},\beta,\tilde{\delta}}] X^\mu(\vec{z}) = \int d^2\tilde{z}_3 f_{\alpha,\tilde{\gamma},\beta,\tilde{\delta},\gamma,\tilde{\delta}} (\Delta_{\gamma,\tilde{\delta}} X^\mu(\vec{z})).
\]

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Since a Weyl transformation act as the identity on the scalar fields, the commutator including one or two Weyl transformations is zero. For the diffeomorphism transformations on the other hand, one obtains

\[ f_{a,\vec{z}_1 b,\vec{z}_2} = -\delta_a^c \delta(\vec{z}_3 - \vec{z}_1) \partial_a \delta(\vec{z}_3 - \vec{z}_2) + \delta_a^c \delta(\vec{z}_3 - \vec{z}_1) \partial_b \delta(\vec{z}_3 - \vec{z}_2). \]

Substituting this expression indeed gives you (47d). \( \delta_B \bar{c} \bar{z} \) is obtained similarly.

Having arrived at a continuous symmetry for the gauge-fixed action, we employ Noether’s theorem to calculate the BRST current associated with this symmetry. The standard procedure gives

\[ j_B(z) := 2\pi i \cdot (j_B)_z \]
\[ = :eT_X : + \frac{1}{2} :eT_g : = :eT_X : + :bc\partial c : , \]
\[ \tilde{j}_B(\bar{z}) := 2\pi i \cdot (j_B)_{\bar{z}} \]
\[ = :\tilde{e}\tilde{T}_X : + \frac{1}{2} :\tilde{e}\tilde{T}_g : = :\tilde{e}\tilde{T}_X : + :\tilde{b}\tilde{c}\partial \tilde{c} : , \]

where \((j_B)_z\) is the standard Noether definition of the conserved current. As pointed out earlier, the factor \(2\pi i\) is conventional for Euclidianized string theory.

Defining \( \Delta J^x \) through

\[ \delta_B(S_X + S_g) = \frac{\theta}{2\pi\alpha'} \int d^2\bar{z} \partial_\alpha (e^x \partial X \cdot \bar{\partial} \bar{X}) =: \theta \int d^2\bar{z} \frac{1}{2} \partial_\alpha (\Delta J^x) \]

gives the Noether current

\[ (j_B)^x (\bar{z}) := \frac{\delta (L_X + L_g)}{\delta (\partial_\alpha \Phi_i (\bar{z}))} \Delta_B \Phi_i (\bar{z}) = \Delta J^x (\bar{z}) , \]

where we should not forget that there is a factor \( \frac{1}{2} \) in the actions which is part of the measure and not of the lagrangian. This leads to

\[ j_B(z) = 2\pi i \cdot \frac{1}{2} \cdot (j_B)^x \]
\[ = 2\pi i \cdot \frac{1}{2} \cdot \left\{ \frac{1}{\pi\alpha'} \partial X \cdot \Delta_B X - \frac{1}{\pi\alpha'} b \Delta_B c - \frac{i}{\pi\alpha'} \tilde{e} \partial X \cdot \bar{\partial} \bar{X} \right\} : \]
\[ = -\frac{1}{\alpha'} :c\partial X \cdot \partial X : + :bc\partial c : \]
\[ = :eT_X : + :bc\partial c : , \]

and similarly to the given expression for \( \tilde{j}_B(\bar{z}) \).

---

\[ ^{44}\text{For details, refer to the point-particle calculation of the structure functions. This is its trivial generalisation to higher dimensions.} \]

\[ ^{45}\text{Everything here is in unit gauge.} \]
With the assumed knowledge of conformal field theory, it is easy to show that the operator product expansions for the currents are given by

\[ j_B(z)b(w) = \frac{3}{(z-w)^3} :b(w)c(w): + \frac{1}{z-w} (T_X(w) + T_g(w)) + \mathcal{NS} \]

\[ j_B(z)c(w) = \frac{1}{z-w} :c(w)\partial c(w): + \mathcal{NS}, \]  

(50)

and similarly for the antiholomorphic current component and ghost fields.

As in (12), the conserved charge \( Q_B \) is given by a contour integral of the BRST currents:

\[ Q_B = \frac{1}{2\pi i} \oint (dz j_B(z) - d\bar{z} \tilde{j}_B(\bar{z})) \]  

(51)

where the contour is implicitly taken around the origin, i.e. around the poles. Using the standard time/radial ordering argument for contour integrals, the OPE of \( j_B(z) \) with \( b(w) \) and the fact that \( \tilde{j}_B(\bar{z}) b(w) = \mathcal{NS} \) imply that for all \( m \in \mathbb{Z} \)

\[ \{Q_B, b_m\} = \left( \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) j_B(z)b(w)w^{m+1} \]

\[ = \oint \frac{dw}{2\pi i} \{\text{Res}_{z=w} j_B(z)b(w)w^{m+1}\} \]

\[ = \oint \frac{dw}{2\pi i} (T_X(w) + T_g(w)) w^{m+1} \]

\[ = (L_X)_m + (L_g)_m, \]  

(52)

where in the first equality we used that the current is Grassmannian and in the last line the Laurent series of the stress-tensors were substituted.

Before the actual analysis of the physical spectrum can start, the final result we have to obtain is the BRST charge on terms of the mode expansions and Virasoro generators:

\[ Q_B = \sum_{m,n=\infty}^{\infty} \left( c_n(L_X)_{-n} + \tilde{c}_n(L_X)_{-n} \right) \]

\[ + \sum_{m,n=\infty}^{\infty} \left( \frac{m-n}{2} \right) \circ c_m b_{-m-n} + \tilde{c}_m \tilde{b}_{-m-n} \circ + a^0 (c_0 + \tilde{c}_0), \]  

(53)

where the \( \circ \) denote creation-annihilation normal ordering.
For convenience, let’s focus on the contour integral of the holomorphic current component \( j_B(z) = :cT_x + b\partial c + \frac{\theta^2}{2} c^2 : \)

\[
\oint \frac{dz}{2\pi i} j_B(z) = \oint \frac{dz}{2\pi i} \left\{ \sum_{m,n=-\infty}^{\infty} \frac{c_n(L_X)_m}{z^{m+n+1}} - \sum_{l,m,n=-\infty}^{\infty} (m-1) \frac{b_l c_m c_n}{z^{l+m+n+1}} \right\}
\]

\[
+ \frac{3}{2} \sum_{n=-\infty}^{\infty} \frac{(n-1)n c_n}{z^{n+1}} \right\} = \sum_{n=-\infty}^{\infty} c_n(L_X)_{-n} + \sum_{m,n=-\infty}^{\infty} (m-1)c_m c_n b_{-m-n} + 0.
\]

In order to put the terms with three ghost modes in normal order, recall that terms in which all operators are (anti)commuting are automatically equal to their normal ordered counterpart.\(^{46}\) Using that \( \{c_m, c_n\} = 0 \) and \( \{b_m, c_n\} = \delta_{m,-n} \), we get

\[
\sum_{m,n=-\infty}^{\infty} (m-1)c_m c_n b_{-m-n} = \sum_{m,n=-\infty}^{\infty} mc_m c_n b_{-m-n}
\]

\[
= \sum_{m,-\infty}^{\infty} mc_m c_0 b_{-m-n} + \sum_{m,n=-\infty}^{\infty} mc_m c_n b_{-m-n}
\]

\[
= \sum_{m,-\infty}^{\infty} m \circ c_m c_0 b_{-m-n} + a^g c_0
\]

\[
+ \sum_{m,n,-\infty}^{\infty} m \circ c_m c_n b_{-m-n} 0
\]

\[
= \sum_{m,-\infty}^{\infty} m \circ c_m c_n b_{-m-n} 0 + a^g c_0
\]

\[
= \sum_{m,n,-\infty}^{\infty} \left( \frac{m-n}{2} \right) \circ c_m c_n b_{-m-n} 0 + a^g c_0,
\]

where in the first and last equality we used that \( \{c_m, c_n\} = 0 \) and in the third equality we used the convention that \( c_0 \) is considered as a raising operator.

Combining this with a similar expression for the antiholomorphic current component, we obtain (53). Note that when \( \lambda = 2 \) and thus \( a^\theta = -1 \), this form of \( Q_B \) is consistent with (52).

### 1.5.2 Spacetime dimension

Hitherto the spacetime dimension \( D \), or equivalently the number of scalar fields in the Polyakov action, was not specified. In principle, the theory seemed to be consistent for any non-zero spacetime dimension. It will now be shown however that for bosonic string to be a consistent theory the spacetime dimension is fixed at \( D = 26 \).

To see this, note that the standard Jacobi identity of \( Q_B, L_m \) and \( b_n \) can be rewritten as

\[
\{[Q_B, L_m], b_n\} - \{[L_m, b_n], Q_B\} - \{[b_n, Q_B], L_m\} = 0,
\]

\(^{46}\)Since in the definition of the normal ordering symbol the (anti)commutation properties of the operators have been absorbed.
where $L_m$ are the Virasoro generators of the full conformal field theory after the gauge-fixing. In our case $L_m = (L_X)_m + (L_g)_m$. The Jacobi identity was written in this form because it enables us to use some well-known identities; implementing (52) into the Jacobi identity gives

$$\{[Q_B, L_m], b_n\} = \{(m - n)b_{m+n}, Q_B\} + [L_n, L_m]$$

$$= -\frac{c}{12} m(m^2 - 1)\delta_{m,-n} \quad \forall m, n \in \mathbb{Z}.$$  

Here $c$ is the total central charge of the conformal field theory. Rewriting the standard Jacobi identity of $Q_B$, $Q_B$ and $b_n$ in a similar way gives

$$\{[Q_B, Q_B], b_n\} = -\{[Q_B, b_n], Q_B\} - \{[b_n, Q_B], Q_B\}$$

$$= 2\{Q_B, L_n\} \quad \forall n \in \mathbb{Z}.$$  

Now recall that the nilpotence of the BRST transformation imposes a consistency condition on the BRST charge: $(Q_B)^2 = 0$. A consistent bosonic string theory therefore requires $[Q_B, L_m] = 0$ and thus $c = 0$. And since basic conformal field theory tells us that $c = c_X + c_g = D - 26$, we conclude that in a consistent bosonic string theory $D = 26$.

This rather technical derivation of the spacetime dimension has two more conceptual, though equivalent, counterparts. Since $c \neq 0$ implies $(Q_B)^2 \neq 0$ and thus some breakdown of the BRST symmetry - which is a consequence of the gauge symmetry of the original theory - we might expect a breakdown in the gauge symmetry as well. This is indeed what happens in the form of an anomaly: a breakdown of a symmetry of the classical action when this action is treated as a quantum theory. An infinitesimal Weyl transformation $\delta_W g_{ab}(\vec{\sigma}) = 2\delta \omega(\vec{\sigma}) g_{ab}(\vec{\sigma})$ applied to a quantum expectation value with metric independent insertions “...” is of the form

$$\delta_W \langle ... \rangle = i \frac{1}{2\pi} \int d^2 \sigma \sqrt{\gamma} \delta \omega(\vec{\sigma})(T^a_a(\vec{\sigma}) \ldots),$$

implying that in a Weyl invariant quantum theory the stress-tensor is traceless. One can prove however\footnote{See Polchinski\cite{3}, chapter 3.} that for string theory\footnote{I.e. the quantum theory of the Polyakov action.} in a general spacetime with Ricci scalar $R$

$$T^a_a = -\frac{c}{12} R.$$  

This implies the so-called Weyl anomaly in a curved spacetime if $c \neq 0$, i.e. the Weyl gauge symmetry is broken in the quantum theory of the Polyakov action for $D \neq 26$. In other words, the flat world-sheet CFT can be coupled to a curved metric in a Weyl invariant way only if $D = 26$. To conclude, the inconsistency in the BRST formalism if $D \neq 26$ is resembled by the breakdown of Weyl symmetry in a quantum theory with $D \neq 26$.\footnote{A purist would probably turn around this resemblance, as the BRST invariance is derived from the more fundamental gauge invariance.}

The second counterpart of our derivation comes from light-cone quantization.\footnote{See Polchinski\cite{3}, chapter 1.} In this formalism the spacetime dimension is derived by using the fact that massless particles have $D-2$ momentum degrees of freedom, as opposed to massive particles having $D-1$ momentum degrees of freedom. To see why this is equivalent to the Weyl anomaly, recall that an anomaly means that not all gauge choices lead to the same physics. In different gauges this leads to violation of different physical principles, e.g. covariance or unitarity. In light-cone gauge the breakdown of Weyl...
symmetry leads to a Lorentz anomaly, as the choice of your light-cone axes is a choice of gauge and changes the physics. This Lorentz anomaly is resembled by particles having $D-2$ momentum degrees of freedom not being massless anymore when $D \neq 26$. This shows the equivalence of the Weyl anomaly argument and the light-cone quantization argument for $D = 26$.

In the remainder of this chapter $D = 26$ will be assumed. In the next chapter about the supersymmetric string theory spectrum, the spacetime dimension will be different.

1.5.3 Open string intermezzo

It will turn out to be convenient to analyse the open string spectrum first. The investigation of the closed string spectrum is then merely an extension of the former analysis.

In everything we did so far we had assumed to be working with closed strings, i.e. $\sigma \in [0, 2\pi)$ with the limit points identified and the requirement of single-valuedness of the fields. For open strings, $\sigma \in [0, \pi]$ and the endpoints are not identified anymore. Instead, the principle of least action imposes boundary conditions on the scalar fields. At the boundaries $\sigma = 0, \pi$ the scalar fields obey either the Neumann boundary condition

$$\partial_\sigma X^\mu(\tau, \sigma) = 0,$$

or the Dirichlet boundary condition

$$X^\mu(\tau, \sigma) = d^\mu,$$

where $d^\mu$ a constant on the world-sheet. The boundary conditions in different spacetime directions do not have to be the same and mixed boundary conditions, i.e. different conditions at the two endpoints of one scalar field, are also allowed. When there are no mixed boundary conditions and in $p + 1 \leq 26$ spacetime directions (including the timelike direction) there are Neumann conditions imposed, it is a convention to take the first $p + 1$ spacetime directions as the Neumann directions and to denote these directions with a special Lorentz index $i \in \{0, 1, \ldots, p\}$. The other $25 - p$ spacelike spacetime directions, for which Dirichlet conditions are imposed, are denoted with a Lorentz index $I \in \{p + 1, 25\}$.

The open string analysis differs significantly from the above closed string analysis at three points:

1. For Euclidianized string theory in complex planar coordinates the Neumann and Dirichlet boundary conditions become respectively

$$\partial X^i(z) = \bar{\partial} \bar{X}^i(\bar{z}) \quad \text{if} \ z = \bar{z},$$

and

$$X^I(z, \bar{z}) = d^I \quad \text{if} \ z = \bar{z},$$

where we used that $\partial_\sigma = -iz\partial + iz\partial$. The real axis $z = \bar{z}$ now forms the boundary of the world-sheet; an open string has its $(\sigma^1 = 0)$-endpoint at the positive real axis and its $(\sigma^1 = \pi)$-endpoint at the negative real axis. The points in between are below the real axis: $\text{Im} \ z \leq 0$.

In terms of the mode expansions (11) and (13) of the scalar fields, the boundary conditions are

$$a^i_m = \bar{a}^i_m \quad \forall m \in \mathbb{Z}$$
and
\[ \alpha_0' = \tilde{\alpha}_0' = 0, \quad \alpha_m' = -\tilde{\alpha}_m' \quad \forall m \in \mathbb{Z} \setminus \{0\}, \]
respectively.

2. Compared to the closed string formalism, the spacetime momentum is changed too. In directions for which Dirichlet boundary conditions are imposed translation symmetry is broken and there are therefore no conserved momenta in these directions. The conserved spacetime momentum in the Neumann directions is defined analogously to the closed string spacetime momentum, but with a different integration domain:
\[ p^i = \frac{1}{\pi} \sqrt{\frac{2}{\alpha'}} \int_0^\pi d\sigma^1 \sin^2 \sigma^2. \]
Since we’re not closing the contour in the complex plane now, we have to do the calculation explicitly. This gives
\[ p^i = \frac{\alpha_0'}{\sqrt{2\alpha'}} = \frac{\tilde{\alpha}_0'}{\sqrt{2\alpha'}}, \]
which differs by a factor \( \frac{1}{2} \) from the closed string case.

3. As for the scalar fields, there are also boundary conditions for the scalar fields in the open string case. The BRST formalism is not the easiest way to derive these conditions, but it is nonetheless possible to do so. The resulting boundary conditions are
\[ c(z) = \tilde{c}(\bar{z}) \quad \text{and} \quad b(z) = \tilde{b}(\bar{z}) \quad \text{if} \quad z = \bar{z}, \]

For unmixed boundary conditions, the open string mode expansions are
\[ X^i(z, \bar{z}) = x^i - i \alpha' p^i \log(|z|^2) \quad + i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^i}{m} \left( \frac{1}{z^m} + \frac{1}{\bar{z}^m} \right), \]
\[ X^I(z, \bar{z}) = d^I \quad + i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^I}{m} \left( \frac{1}{z^m} - \frac{1}{\bar{z}^m} \right). \]
To describe the motion of an open string it is therefore sufficient to specify the reduced center-of-mass spacetime momentum \( p^i \) and the contributions from the holomorphic oscillator modes; for an open string the antiholomorphic oscillator modes are redundant.
or equivalently
\[ c_m = \tilde{c}_m \quad \text{and} \quad b_m = \tilde{b}_m \quad \forall m \in \mathbb{Z}. \]

As for the scalar fields, in the theory of open strings it is therefore sufficient to specify the holomorphic ghost modes; the antiholomorphic ghost modes are redundant.

In the derivation of the string theory ghost action (44) from the general form of the BRST ghost action (43) a partial integration was performed. For a world-sheet with boundaries however, the total derivative term does not vanish. The string ghost action for open strings therefore becomes
\[
S_g = -\frac{1}{2\pi} \int d^2 \sigma \sqrt{g} \nabla_a c_b(\sigma) \nabla^a \delta(\sigma - \sigma') + \frac{1}{2\pi} \int d^2 \sigma \sqrt{g} \left( b^{ab}(\sigma) \nabla^a c_b(\sigma) \right)
\]
\[
= -\frac{1}{2\pi} \int d^2 \sigma \sqrt{g} b^{ab}(\sigma) \nabla_a c_b(\sigma)
\]
\[
+ \frac{1}{2\pi} \int d\sigma^2 \left( b^{ab}(\sigma) \nabla^a c_b(\sigma) - b^{ab}(0, \sigma) c_a(\sigma) \right),
\]
where in the second equality Gauss' theorem with boundary vector \( d\Sigma^a = (d\sigma^2, 0) \) was used. Regardless of the boundary terms in the open string ghost action, the principle of least action is still valid. Away from the boundary this leads to the usual equations of motion for the ghosts. To investigate the behaviour of the fields on the \( (\sigma^1 = \pi) \)-boundary, vary the action with respect to a specific variation of the symmetric \( b^{ab}(\sigma) \)-field:
\[
\delta b^{ab}(\sigma) = b^{ab}(\sigma) \delta(\sigma - \sigma') \delta_{\epsilon}^a \delta_{\epsilon}^b \quad \text{for} \quad c \in \{1, 2\}.
\]

Setting \( \delta c, S_g = 0 \) leads to
\[
\nabla_1 c_1 (\pi, 0, \tilde{\sigma}^2) = \delta(0) c_1 (\pi, \tilde{\sigma}^2)
\]
\[
\nabla_2 c_2 (\pi, \tilde{\sigma}^2) = 0.
\]
Continuity therefore requires that \( c_1 (\pi, \tilde{\sigma}^2) = 0 \). Similarly, \( c_1 (0, \tilde{\sigma}^2) = 0 \). And since \( c_1 (\sigma) = i (z(c(z) - \tilde{z}(\tilde{c}(\tilde{z}))) \), we conclude that in unit gauge
\[
c(\sigma) = \tilde{c}(\tilde{\sigma}) \quad \text{if} \quad z = \tilde{z}.
\]

Finally, since the Virasoro generators can be expanded in terms of oscillator modes, in the open string case
\[
(L_X)_m = (\tilde{L}_X)_m \quad \text{and} \quad (L_g)_m = (\tilde{L}_g)_m \quad \forall m \in \mathbb{Z}
\]
and thus \( T_X(z) = \tilde{T}_X(\tilde{z}) \) and \( T_g(z) = \tilde{T}_g(\tilde{z}) \) if \( z = \tilde{z} \).
1.5.4 The open string spectrum

As we have seen before in the point-particle case, the physicality condition for open string states is $Q_B|\psi\rangle = 0$. Also, for the same reason as in section (1.4.4) there’s the additional condition that the $b(z)$-field zero modes annihilate physical states: $b_0|\psi\rangle = 0$. Using (52) we therefore see that a physical open string state obeys

$$L_0|\psi\rangle = (Q_B, b_0)|\psi\rangle = 0,$$

where $L_m := (L_X)_m + (L_g)_m$. For reason that will become clear shortly, this condition implied by BRST invariance and the $b_0$-argument is called the mass shell condition for open bosonic strings.

From the sections (1.3.1) and (1.3.2) it is found that

$$L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + a^X + \sum_{n=1}^{\infty} nc_{-n} b_n + \sum_{n=1}^{\infty} nb_{-n} c_n + a^g = \alpha' p^2 + \sum_{n=0}^{25} n \left( \hat{N}^{\mu}_n + \hat{N}^c_n + \hat{N}^b_n \right) - 1,$$

where we defined the counting operators $\hat{N}^{\mu}_n := \frac{1}{n} \alpha_{-n}^\mu \alpha_n^\mu$ (no summation over $\mu$), $\hat{N}^c_n := c_{-n} b_n$ and $\hat{N}^b_n := b_{-n} c_n$. These counting operators count the number of $n$-th level modes of the designated fields in an open string state, as can be verified using the relevant (anti)commutation relations between the modes. Moreover, the level operator is defined by

$$\hat{N} := \sum_{n=1}^{\infty} n \left( \sum_{\mu=0}^{25} \hat{N}^{\mu}_n + \hat{N}^c_n + \hat{N}^b_n \right),$$

and the associated eigenvalue $N$ of an open string state is called the level of that state.

Now, the mass shell condition implies that physical open string states with level $N$ and spacetime momentum $k^\mu$ are always on the mass shell and have a mass

$$m^2 = -k^2 = \frac{1}{\alpha}(N - 1).$$

It is important to note that the mass shell condition does not replace the physicality condition $Q_B|\psi\rangle = 0$; a state $|\psi'\rangle$ on the mass shell satisfying $b_0|\psi'\rangle = 0$ doesn’t necessarily obey $Q_B|\psi'\rangle = 0$, as long as $b_0 Q_B|\psi'\rangle = 0$. To summarize, physical states automatically obey the mass shell condition, but states on the mass shell are not necessarily physical. This needs to be checked explicitly.

With a simple redefinition $\alpha_0^0 = \alpha_{-m}^0 \quad \forall m \in \mathbb{Z}$, the mode (anti)commutation relations imply that the $\alpha_{-m}^\mu, c_{-m}$ and $b_{-m}$, all with $m > 0$, can be considered as creation operators.\footnote{In terms of the standard forms $[a, a^\dagger] = 1$ for commuting and $\{a, a^\dagger\} = 1$ for anticommuting operators.} By convention, $c_0$ is also considered as a creation operator with respect to the ghost vacuum. Except from the $a_0^\mu$, all other mode operators are considered as annihilation operators with respect to the vacuum.\footnote{These have vanishing commutators with all other mode operators.}

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Note that the vacuum mentioned here is not the spacetime vacuum of quantum field theory. Instead, it is the unexcited open string state with a given spacetime momentum \( k^\mu \). This vacuum is nothing but a level-0 open string and usually denoted by \(|0,k\rangle\), where the first entry denotes the level of the string state and the second denotes its spacetime momentum.

The most general quantum state of a level-\( N \) open string with spacetime momentum \( k^\mu \) is now of the form

\[
|N,k\rangle = \sum_{A} C_A \left[ \prod_{\alpha=0}^{25} \prod_{n=1}^{\infty} \frac{(\alpha^\mu_n)^{N^\mu_n}}{\sqrt{\varepsilon^{N_{c_n}} - N_{b_n}^{N_{c_n}}}} \right] \left[ \prod_{n=0}^{\infty} (c_n)^{N^c_n} \right] \left[ \prod_{n=1}^{\infty} (b_n)^{N^b_n} \right] |0, k\rangle,
\]

where \( A \) is the set of all possible combinations of quantum numbers \( \{N^\mu_n, N^c_n, N^b_n\} \) summing to \( N \) as in (56), \( A \in A \) and \( C_A \) are normalized coefficients. Please note that because of Grassmannian properties of the ghost modes, we can restrict the quantum numbers \( N^c_n \) and \( N^b_n \) to take only the values 0 and 1. Also note that \( c_0 \) is included in the expression, but \( b_0 \) is not. Finally, note that this most general quantum state does not automatically obey any of the three conditions mentioned earlier.

The open string spectrum is conventionally described level by level. We will do the same here.

- \( N = 0 \)

At this level there are two states obeying the mass shell condition:

\[
|0, k\rangle, \quad c_0 |0, k\rangle, \quad \text{with} \quad k^2 = \frac{1}{\alpha'}.
\]

Since \( b_0c_0 |0, k\rangle = |0, k\rangle \) the second state is excluded from the physical spectrum. Because up to a minus sign the \( c_0 \) operator can always be put at the front, we conclude that because of the \( b_0 \)-condition string states with a \( c_0 \)-mode are at any level excluded from the spectrum. We will use this implicitly in due course.

Since \( b_0 \) is considered as an annihilation operator with respect to the string vacuum and has non-trivial (anti)commutation relations only with \( c_0 \), the \( b_0 \)-condition is satisfied for all states without a \( c_0 \)-mode. We therefore don’t need to concern us about this condition anymore.

Regarding the BRST invariance,

\[
Q_B |0, k\rangle = 2 \left( c_0 (L_X)_0 - c_0 \right) |0, k\rangle = 2c_0 (\alpha' k^2 - 1) |0, k\rangle = 0,
\]

because of the mass shell condition. The overall factor of 2 is because the holomorphic and antiholomorphic modes are added in the open string case and is of no physical relevance. Moreover, since \( Q_B c_0 |0, k\rangle = 0 \) there are no exact states at this level. Therefore the only

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53 Note that \( k^\mu \) is replaced by \( \vec{k} \) is the quantum state, as \( k^0 \) is now fixed by the mass shell condition.

54 In textbooks the \( c_0 \)-states are usually not even mentioned, as they play no role in the physical spectrum at all.

55 For this it is also required that \( [\hat{N}, Q_B] = 0 \), as can be checked explicitly. It can also be seen by noting that the mode numbers in any of the terms sum to zero.

37
proper physical states at the $\N=0$-level are
\[ |0, \vec{k}\rangle \quad \text{with} \quad k^2 = \frac{1}{\alpha'}, \]
and each of these states corresponds to a cohomology class.
The particle associated with this set of states is called the \textit{tachyon}. The obvious difficulty with the tachyon is its negative mass squared, $m^2 = -\frac{1}{\alpha'}$, or equivalently its spacelike momentum. The existence of these particles in a flat spacetime is rejected by special relativity and spacelike moving particles are not observed in nature. Luckily, we will see that in supersymmetric string theory tachyonic particles naturally get excluded from the physical spectrum.

\textbullet \ N = 1
From (57) we see that the most general $(\N=1)$-state on the mass shell is of the form
\[ |1, \vec{k}\rangle = (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0, \vec{k}\rangle \quad \text{with} \quad k^2 = 0, \]
where $e^\mu$ is a 26-vector and $\beta, \gamma \in \mathbb{C}$ are constants, leading to $26 + 2$ linearly independent states. Using the (anti)commutation relations of the different operators, the norm of this general state is
\[ \langle 1, \vec{k}' | 1, \vec{k} \rangle = (e^* \cdot e + \beta^* \gamma + \gamma^* \beta) |0, \vec{k}'\rangle |0, \vec{k}\rangle. \] (59)
We therefore see that in the standard orthogonal basis there are $26$ positive-norm states
\[ \alpha_{-1}^{i} |0, \vec{k}\rangle, \quad i \in \{1, 2, \ldots, 25\} \]
and two negative-norm states
\[ \alpha_{-1}^{0} |0, \vec{k}\rangle, \]
\[ \frac{1}{\sqrt{2}} (b_{-1} + c_{-1}) |0, \vec{k}\rangle, \]
\[ \frac{1}{\sqrt{2}} (b_{-1} - c_{-1}) |0, \vec{k}\rangle. \]
The physicality condition applied to the $(\N=1)$-state is
\[ 0 = Q_B |1, \vec{k}\rangle = 2(c_1(LX)_{-1} + c_0(LX)_0 + c_{-1}(LX)_1 + c_0 c_{-1} b_1 + c_1 c_0 b_{-1} - c_0 |1, \vec{k}\rangle \]
\[ = 2(\sqrt{2a'} \beta k \cdot \alpha_{-1} + c_0 e \cdot \alpha_{-1} + \sqrt{2a'} k \cdot e c_{-1} + \gamma c_0 c_{-1} + \beta c_0 b_{-1} - c_0 e \cdot \alpha_{-1} - \beta c_0 b_{-1} \gamma c_0 c_{-1}) |0, \vec{k}\rangle \]
\[ = 2\sqrt{2a'} (\beta k \cdot \alpha_{-1} + k \cdot e c_{-1}) |0, \vec{k}\rangle, \] (60)
where after the second equality we only displayed the non-zero contributions of $Q_B$ applied to $|1, \vec{k}\rangle$. Physical $(\N=1)$-states are therefore required to satisfy the conditions $k \cdot e = \beta = 0$ and the general physical state takes the form
\[ |1, \vec{k}\rangle = (e \cdot \alpha_{-1} + \gamma c_{-1}) |0, \vec{k}\rangle, \quad \text{with} \quad k^2 = 0 \quad \text{and} \quad k \cdot e = 0. \]
To analyse these states further, it is convenient to first consider massless open string states with momentum $k^\mu = (\kappa, \kappa, 0, \ldots, 0)$. An orthogonal basis of states obeying the physicality
conditions $k \cdot e = \beta = 0$ is now
\begin{align*}
c_{-1}|0, \vec{k}\rangle,
k \cdot \alpha_{-1}|0, \vec{k}\rangle, & \quad ( = (-\alpha_{-1}^0 + \alpha_{-1}^1)|0, \vec{k}\rangle) \\
\alpha_{-1}^i|0, \vec{k}\rangle, & \quad \text{with } i = 2, \ldots, 25. \tag{61}
\end{align*}

Note that the second state is physical because it obeys the mass shell condition. Also note that by (59) the first two states have norm zero and thus no overlap with any physical state, while the other 24 states have a positive norm. This gives an indication that the first two states might be exact states with respect to the BRST charge, which is indeed the case. To see this, recall that in deriving the physicality condition (60) we started with the most general open string state
\begin{equation}
|\alpha_{-1}^0, 0, \vec{k}\rangle \quad \text{and where}
\end{equation}

where the most general open string state has the form
\begin{equation}
N \geq 2
\end{equation}

In principle the above machinery could be repeated for all levels. For example, for $N = 2$ the most general open string state has the form
\begin{align*}
|2, \vec{k}\rangle = (A_{\mu \nu} \alpha_{-1}^\nu \alpha_{-1}^\mu + B_{\mu} \alpha_{-1}^\nu b_{-1} + C_{\mu} \alpha_{-1}^\nu c_{-1} + D b_{-1} c_{-1} + A_{\mu} \alpha_{-2}^\nu + B b_{-2} + C c_{-2})|0, \vec{k}\rangle, & \quad \text{with } k^2 = \frac{1}{\alpha_{-1}^\mu},
\end{align*}

and where $A_{\mu \nu}$ is symmetric in its two indices. A basis would consist of $\frac{1}{2} \cdot 26(26 + 1) + 26 + 26 + 26 + 1 + 1 = 432$ states, of which $25 + 26 + 1 + 1 = 54$ are negative-norm states. The application of the physicality condition removes 54 states and leaves another 54 null states in the spectrum. The cohomology of $Q_B$ therefore consists of $432 - 2 \cdot 54 = 324$ proper physical states, namely the 300 states $\alpha_{-1}^1 \alpha_{-1}^2|0, \vec{k}\rangle$ plus the 24 states $\alpha_{-2}^1|0, \vec{k}\rangle$, where the spacetime indices can take values 2, 3, $\ldots$, 25.
Repeating this machinery for higher levels is a tedious exercise without any new insights. There are just a few general properties worth mentioning. One generality is that at every level the number of states projected out in the first step ($Q_B|\psi\rangle = 0$) is equal to the number of exact states projected out in the second step. For $N = 0$ there were 0 and 0 states, for $N = 1$ we had 2 and 2 and for $N = 2$ we had twice 54 states. It’s not hard to see why this is true. Each single physicality condition comes from a specific basis state $|\psi_i\rangle$ which is not necessarily zero when acted upon with the BRST charge. But this feature $Q_B|\psi_i\rangle \neq 0$ is precisely what is needed to have an exact state $|\chi\rangle = Q_B|\psi_i\rangle \neq 0$. Therefore we have for each single physicality condition precisely one exact state,$^{56}$ as the first generality stated.

The other generality is about the remaining proper physical states. For the lowest three states a basis for the proper physical states consisted of all possible combinations of oscillator creation modes with spacetime index running from 2 to 25. That this property is valid at all levels goes under the name of the no-ghost theorem. This theorem states that at every level $N$, what remains after projecting out the states with $X^0(z)-, X^1(z)-, b(z)-$ and $c(z)$-modes are the proper physical states. A proof of this theorem is deferred to section (1.5.6).

It goes without saying that the no-ghost theorem is very useful for writing down the open string spectrum: at each level one can give all proper physical states without any calculation. Besides this, it is convenient to notice that ghost modes are excluded from the proper physical spectrum, i.e. they cannot be observed in nature and don’t contribute to physical amplitudes. Finally, more than in the specific form of the states we will be interested in the number of proper physical states per level. The no-ghost theorem is a great tool in finding this number, the calculation of it is deferred to section (1.5.7).

1.5.5 The closed string spectrum

As pointed out earlier, the analysis of the closed string spectrum is merely an extension of the open string spectrum analysis. The discussion of the closed string spectrum will therefore be brief at some points. For every annihilation/creation operator in the open string spectrum we now also have its antiholomorphic counterpart. The relation between the two will soon become clear.

All physical closed string states obey the conditions$^{57}$

\[
Q_B|\psi\rangle = 0 \\
b_0|\psi\rangle = \tilde{b}_0|\psi\rangle = 0.
\]

This leads to the mass shell conditions

\[
L_0|\psi\rangle = \tilde{L}_0|\psi\rangle = 0,
\]

$^{56}$This mapping is one-to-one, because otherwise the physicality conditions are not linearly independent and there would be a redundancy in them.

$^{57}$The point-particle argument for $b_0|\psi\rangle = 0$ becomes in the closed string case $(b_0 + \tilde{b}_0)|\psi\rangle = 0$. A detailed analysis of string amplitudes however leads to the separated conditions stated here.
where now

\[ L_0 = \frac{\alpha'}{4} p^2 + \hat{N} - 1 \]

\[ \hat{L}_0 = \frac{\alpha'}{4} p^2 + \hat{\tilde{N}} - 1. \]

A “∼” on an expression means that every holomorphic element of that expression is replaced by its antiholomorphic counterpart. The factor of \( \frac{1}{4} \) is because of the different relation between the spacetime momentum and oscillator zero-modes, as compared to the open string case.

The physicality conditions imply that the holomorphic and antiholomorphic levels of a physical closed string state must be equal, \( N = \tilde{N} \). This is called level matching. To see this, suppose you have a level-(\( N, \tilde{N} \)) closed string state. The mass shell conditions then give

\[ 0 = (L_0 - \hat{L}_0)(N, \tilde{N}, k) = (N - \tilde{N})(N, \tilde{N}, k), \]

implying \( N = \tilde{N} \). We can therefore still speak of the level of a closed string state, meaning both the holomorphic and antiholomorphic level.

- \( N = 0 \)
  
  As in the open string case, we can neglect states with \( c_0 \)- and \( \tilde{c}_0 \)-modes in it. Therefore the only proper physical state at this level is the closed string tachyon:

\[ |0, 0, \vec{k}\rangle, \quad \text{with } k^2 = \frac{4}{\alpha'}, \]

and thus \( m^2 = -\frac{4}{\alpha'} \).

- \( N = 1 \)
  
  Although the trivial extension of the no-ghost theorem - treat the antiholomorphic modes in the same way as the holomorphic modes were treated for open strings - immediately gives you the closed string spectrum at all levels, it is useful to see the derivation of the spectrum at \( N = 1 \) explicitly. The reason for this is that one has to combine holomorphic and antiholomorphic modes and this will also be the case for supersymmetric string theory.

The most general \( N = 1 \) closed string state at the mass shell is

\[ |1, 1, \vec{k}\rangle = \left\{ A_{\mu\nu} \alpha^{\mu'}_{-1} \alpha^{\nu'}_{-1} \right. \]

\[ + e \cdot \alpha_{-1} \tilde{b}_{-1} + \tilde{e} \cdot \tilde{\alpha}_{-1} b_{-1} \]

\[ + d \cdot \alpha_{-1} \tilde{c}_{-1} + \tilde{d} \cdot \tilde{\alpha}_{-1} c_{-1} \]

\[ + B b_{-1} \tilde{b}_{-1} + C c_{-1} \tilde{c}_{-1} \]

\[ + D b_{-1} \tilde{c}_{-1} + E \tilde{b}_{-1} c_{-1} \right\} |0, 0, \vec{k}\rangle, \quad \text{with } k^2 = 0. \tag{63} \]

Note that \( A_{\mu\nu} \) is not necessarily symmetric in its indices. A basis would consist out of
$D^2 + 4D + 4 = 784$ states. A lengthy calculation gives

$$Q_B[1, 1, \tilde{k}] = \sqrt{\alpha'} 2 \left( (e_\mu k_\nu + \bar{e}_\nu k_\mu) \alpha_{-1}^{\mu} \alpha_{-1}^{\nu} \\
+ B((k \cdot \alpha_{-1}) \bar{b}_{-1} - (k \cdot \tilde{\alpha}_{-1}) b_{-1}) \\
+ (Dk_\mu + A_\mu k^\nu) \alpha_{-1}^{\mu} \tilde{c}_{-1} \\
+ (Ek_\mu + A_{\mu} k^\nu) \tilde{\alpha}_{-1}^{\mu} c_{-1} \\
+ (e \cdot k) b_{-1} \tilde{c}_{-1} + (\bar{e} \cdot k) \bar{b}_{-1} c_{-1} \\
+ (d_\mu - \bar{d}_\mu) k^{\mu} \tilde{c}_{-1} c_{-1} \right) \{0, 0, \tilde{k}\}.$$  \hfill (64)

The requirement of BRST invariance results in 104 independent conditions:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Number of conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_\mu k_\nu + \bar{e}<em>\nu k</em>\mu = 0$</td>
<td>51</td>
</tr>
<tr>
<td>$B = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$(d_\mu - \bar{d}_\mu) k^\mu = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$Dk_\mu + A_\mu k^\nu = 0$</td>
<td>51</td>
</tr>
<tr>
<td>$Ek_\mu + A_{\mu} k^\nu = 0$</td>
<td>51</td>
</tr>
</tbody>
</table>

The conditions $e \cdot k = \bar{e} \cdot k = 0$ are included in the first set of conditions. Note that the number of negative-norm states in (63) is also 104 and that they are perfectly corresponding to the BRST conditions.\footnote{There are two negative-norm states related to the choice of the constants $B, C, D$ and $E$, there are 52 states related to $e_\mu, \bar{e}_\mu, d_\mu$, and $\bar{d}_\mu$ and there are 50 states related to $A_{\mu \nu}$.}

Setting $k^\mu = (\kappa, \bar{\kappa}, 0, \ldots, 0)$ with a Lorentz transformation, the BRST conditions lead to the following basis of the remaining 680 physical non-negative-norm states:

<table>
<thead>
<tr>
<th>Basis states</th>
<th># of states</th>
<th># of deleted states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{-1}^a b_{-1} - \alpha_{-1}^a \bar{b}<em>{-1} + \alpha</em>{-1}^a b_{-1} - \alpha_{-1}^a \bar{b}_{-1}$</td>
<td>1</td>
<td>51</td>
</tr>
<tr>
<td>$c_{-1} \tilde{c}_{-1}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{\alpha}<em>{-1}^a c</em>{-1}$</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}<em>{-1}^a \tilde{c}</em>{-1}$</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}<em>{-1}^a c</em>{-1} + \alpha_{-1}^a \tilde{c}_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}<em>{-1}^a c</em>{-1} + \alpha_{-1}^a \tilde{c}_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}<em>{-1}^a c</em>{-1} + \tilde{\alpha}<em>{-1}^a c</em>{-1} + \alpha_{-1}^a \tilde{c}_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a c_{-1} - \alpha_{-1}^a \tilde{c}_{-1}$</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a \tilde{c}<em>{-1} - \alpha</em>{-1}^a \tilde{c}_{-1}$</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a \tilde{c}<em>{-1} + \alpha</em>{-1}^a \tilde{c}<em>{-1} + 2\bar{b}</em>{-1} c_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a \tilde{c}<em>{-1} + \alpha</em>{-1}^a \tilde{c}<em>{-1} + 2\bar{b}</em>{-1} c_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a \tilde{c}<em>{-1} + \alpha</em>{-1}^a \tilde{c}<em>{-1} + 2\bar{b}</em>{-1} c_{-1}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^a \tilde{c}_{-1}$</td>
<td>576</td>
<td></td>
</tr>
</tbody>
</table>

Here, the spacetime indices $i, j$ take values from 2 to 25. The last column displays the number of states projected out by the different conditions, respectively. For $k^\mu = (\kappa, \bar{\kappa}, 0, \ldots, 0)$ the most general exact state is of the form (64), if the $\alpha_{-1}^a \alpha_{-1}^a$-terms are removed. Therefore a basis of proper physical $N = 1$ closed string states is

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j k_{0, 0, \tilde{k}}, \quad \text{with } k^2 = 0 \text{ and } i, j \in \{2, \ldots, 25\}.$$
Now it is important to note that these states transform as a 2-tensor under the 24-dimensional rotation group $SO(24)$. The given representation is reducible: it decomposes into a symmetric tensor, an anti-symmetric tensor and a scalar. The physical significance of this decomposition is that the three irreducible representations do not mix when going from one inertial observer to another. That is, the most general proper physical quantum state $|\psi\rangle$ of a massless closed string can be decomposed as

$$|\psi\rangle = \sum_{i,j=2}^{25} C_{ij} \alpha_{i-1}^{+} \alpha_{j-1}^{+} |0, 0, \vec{k}\rangle,$$

where $C = \text{Tr}(C_{ij})$, and in the case of a measurement every Lorentzian observer will agree in which of the three irreducible representations the state is found to be. The symmetric quantum state is called the graviton, the anti-symmetric state is called the axion and the scalar state is called the dilaton.

- $N \geq 2$

Like in the open string theory, the no-ghost theorem generalized to closed strings can be shown to be valid: at every level $N = \tilde{N}$, what remains after projecting out the states with $X^0(z)$-, $\tilde{X}^0(z)$-, $X^1(z)$-, $\tilde{X}^1(z)$-, $b(z)$-, $\tilde{b}(z)$-, $c(z)$ and $\tilde{c}(z)$-modes are the proper physical states. With this theorem a derivation of the massive closed string spectrum becomes trivial. To conclude, note that the closed string spectrum at a given level is nothing but two copies of the open string spectrum at that level.

1.5.6 No-ghost theorem

In this section we restrict ourselves to a proof of the no-ghost theorem for open string theory. The extension to closed strings is trivial and its use and significance has become clear in the foregoing sections. Moreover, its application will prove to be particularly useful in section (1.5.7).

Define the transverse Hilbert space $\mathcal{H}^{\perp}$ as the set of open string states obeying the mass shell condition and without excitations from the fields $X^0(z)$, $X^1(z)$, $b(z)$ and $c(z)$. Effectively $\mathcal{H}^{\perp}$ consists of all states on the mass shell with excitations only from the oscillator modes with spacetime index running from 2 to 25.

Now, the no-ghost theorem states that the BRST cohomology of $Q_B$, i.e. the set of proper physical states, is isomorphic to $\mathcal{H}^{\perp}$. The proof of this theorem goes in two steps: first it is shown that $\mathcal{H}^{\perp}$ is isomorphic to the cohomology of a different nilpotent operator to be defined, $Q_1$. Then it is shown that the cohomology of $Q_1$ is isomorphic to the BRST cohomology.

As a preliminary step define the light-cone oscillators $\alpha_m^\pm := \frac{1}{\sqrt{2}}(\alpha_m^0 \pm \alpha_m^1)$ and the light-cone oscillator counting operator

$$\hat{N}_{lc} := \sum_{m=-\infty}^{\infty} \frac{1}{m} \circ \alpha_m^+ \alpha_m^- \circ.$$

\[\text{(59)}\]

It is called the transverse Hilbert space because the longitudinal modes in the light-cone formalism are projected out.
Using \([\alpha^+_m, \alpha^-_n] = -m\delta_{m,-n}\) one can derive that \(\hat{N}_{lc} \alpha^+_n \equiv \alpha^+_n\) for \(n \neq 0\), implying that \(\hat{N}_{lc}\) counts the number of \(-\) excitations minus the number of \(+\) excitations, since \(\hat{N}_{lc}\) is normal ordered. Note that \(\hat{N}_{lc}\) does not include zero-modes in its counting.

Now it is possible to uniquely decompose the BRST charge as a sum of operators \(Q_j\), where \(j\) denotes the light-cone oscillator quantum number. Implementing the light-cone operators into \(Q_B\), it is not hard to see that the decomposition takes the form

\[
Q_B = Q_1 + Q_0 + Q_{-1},
\]

as there are no terms with multiple light-cone oscillators of the same sign. On can check that

\[
Q_1 = -2\sqrt{2}\alpha' p^+ \sum_{m=-\infty}^{\infty} c_m \alpha^-_{-m},
\]

\[
Q_{-1} = -2\sqrt{2}\alpha' p^- \sum_{m=-\infty}^{\infty} c_m \alpha^+_{-m},
\]

and that the rest of \(Q_B\) is in \(Q_0\). Using the operator equation \(0 = (Q_B)^2 = (Q_1)^2 + \ldots\) and noting that \((Q_1)^2\) is the only operator with \(N_{lc} = 2\) in the expression, it follows that also \((Q_1)^2 = 0\) as an operator equation. Thus, \(Q_1\) is nilpotent and has a cohomology.

With a Lorentz transformation it is always possible to set \(k^+ \neq 0\) for an open string state \(|\psi\rangle\). Assuming \(k^+ \neq 0\), we can define

\[
R := \frac{1}{2\sqrt{2}\alpha' k^+} \sum_{m=-\infty}^{\infty} b_m \alpha^+_{-m},
\]

\[
S := \{Q_1, R\} = \sum_{n=1}^{\infty} n(\hat{N}_n^b + \hat{N}_n^c + \hat{N}_n^+ + \hat{N}_n^-),
\]

(65)

where \(\hat{N}_n^\pm\) counts the number of \(\pm\) modes separately. It is important to note that because of its nilpotency \(Q_1\) and \(S\) commute. Suppose \(|\psi, s\rangle\) is \(Q_1\)-invariant and is an eigenstate of the \(S\)-operator with eigenvalue \(s\). Then, for nonzero \(s\),

\[
|\psi, s\rangle = \frac{1}{s} \{Q_1, R\} |\psi, s\rangle = Q_1 \left( \frac{1}{s} R |\psi, s\rangle \right),
\]

i.e. \(|\psi, s\rangle\) is an \(Q_1\)-exact state if \(s \neq 0\). Note that because of its nilpotency \(Q_1\) and \(S\) commute. This implies that the cohomology of \(Q_1\) is completely determined by the states with \(s = 0\).\footnote{States with \(s \neq 0\) are exact and \(Q_1\)-exact states with \(s = 0\) can be written as \(Q_1\) applied to another \((s = 0)\)-state.} Suppose now that \(S|\psi\rangle = 0\), and thus that

\[
0 = Q_1 S|\psi\rangle = SQ_1 |\psi\rangle.
\]

Using (65) we see that \(|\psi\rangle\) has ghost number zero. Therefore \(Q_1|\psi\rangle\) has ghost number one and \(S\) in the above expression is invertible,\footnote{\(Q_1|\psi\rangle\) is not in the kernel of \(S\).} implying \(Q_1|\psi\rangle = 0\). Thus, states with \(s = 0\) are automatically \(Q_1\)-invariant and never exact. The cohomology of \(Q_1\) coincides therefore with the kernel of \(S\). But since \(S\) is constructed in such a way that it counts all excitations of the
fields $X^0(z), X^1(z), b(z)$ and $c(z)$, the kernel of $S$ is isomorphic to the transverse Hilbert space $\mathcal{H}^\perp$. This finishes the first part of the proof of the no-ghost theorem: the cohomology of $Q_1$ is isomorphic to $\mathcal{H}^\perp$.

For the second part of the proof, define the operator $U := \{Q_0 + Q_{-1}, R\}$, such that

$$S + U = \{Q_B, R\}.$$  

With similar arguments as for $Q_1$, the cohomology of $Q_B$ is isomorphic to the kernel of $S + U$. The remaining bit is to prove that the kernels of $S$ and $S + U$ are isomorphic. To a state $|\psi\rangle$ in the kernel of $S$ one can associate a state

$$|\psi'\rangle = (1 - S^{-1}U + S^{-1}US^{-1}U - \ldots)|\psi\rangle$$

which is annihilated by $S + U$.\footnote{Note that the inverses $S^{-1}$ are well-defined because $U$ lowers the light-cone oscillator quantum number, making the $S^{-1}$ acting on states with $N_{lc} < 0$ and thus invertible.} This immediately proves that the kernels of $S$ and $S + U$ are isomorphic and thereby finishes the proof of the no-ghost theorem.

1.5.7 Counting states

When dealing with string amplitudes, the number of states per level can be relevant. These numbers are usually encoded in generating function $f(x)$, which is a polynomial in $x$. The coefficient of the term of degree $n$ is the number of proper physical states at level $N = n$.

Let’s derive the generating functional for the open string sector. First, suppose you have a theory with the oscillator $\alpha_{-n}$ as the only creation operator. In that case the generating functional is

$$1 + x^n + x^{2n} + x^{3n} + \cdots = \frac{1}{1 - x^n},$$

since there is one state at the levels $N = pn$, $p \in \mathbb{N}$, namely $(\alpha_{-n})^p |0, \vec{0}\rangle$, and there are zero states at the other levels. For a theory with oscillators $\{\alpha_{-n}, n \in \mathbb{N}\}$ the generating functions of the separate oscillators can be multiplied, since when polynomials are multiplied one adds the degrees of the different terms. This gives a generating function

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^n}.$$  

The open string theory we developed in this chapter consists of 24 copies of the above theory. And again, because of the summation of degrees when multiplying polynomials the generating functional for open string theory is

$$f_{\text{open}}(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{24}}.$$  

A Taylor expansion of this expression gives

$$f_{\text{open}}(x) = 1 + 24x + 324x^2 + 3200x^3 + \ldots,$$

which is consistent with the results of section (1.5.4).
We’ve seen that the closed string spectrum is nothing but two copies of the open string spectrum. However, because of the level matching condition it is not possible to simply square the open string generating function: there would be a huge overcounting from unphysical contributions which do not obey the level matching condition. A possible solution - related to the more advanced treatments of string amplitudes\textsuperscript{63} - is the generating function

\[ f_{\text{closed}}(x, y) = f_{\text{open}}(x)f_{\text{open}}(y), \]

where level matching is assumed. That is, only the coefficients of those terms with the same power in both variables are physically relevant. A Taylor expansion gives

\[ f_{\text{closed}}(x, y) = 1 + 24(x + y) + 24^2xy + 324(x^2 + y^2) + 7776(x^2y^2 + x^2y) + 324^2x^2y^2 + \ldots. \]

As expected, the number of closed string states at level \( N = \tilde{N} \) is the number of open string states at level \( N \) squared.

\textsuperscript{63}See Polchinski\textsuperscript{[3]}, section 7.2.
2 Supersymmetric string theory spectra

In the spectra derived in the former chapter spacetime bosons were found, i.e. quantum states in a tensorial representation with respect to the Lorentz group or a subgroup of the Lorentz group. However, to bring string theory in agreement with nature spacetime fermions, i.e. quantum states in a spinorial representation with respect to the Lorentz group or a subgroup of the Lorentz group, are needed as well. To make this possible, we will have to develop new string theories by adding extra terms to the Polyakov action. These new string theories are called supersymmetric string theories\(^{64,65}\), or superstring theories and the superstring action takes the form

\[
S := \frac{1}{4\pi} \int d^2 z \left( \frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \bar{\psi}^\mu \partial \bar{\psi}_\mu \right). \tag{66}
\]

Here the \(\psi^m(z)\) are Grassmannian, holomorphic fields of weight \((\frac{1}{2}, 0)\) and the action

\[
\frac{1}{4\pi} \int d^2 z \psi \bar{\partial} \psi
\]

is the action of a CFT with stress-tensor\(^{66}\) \(T(z) = -\frac{1}{2} \psi(z) \bar{\partial} \psi(z)\) and central charge \(\frac{1}{2}\). The third term in (66) are the antiholomorphic counterparts of such conformal field theories.

Knowing that superstring theory is a conformal field theory on the world-sheet, the stress-tensor of the full theory is

\[
T_B(z) = -\frac{1}{\alpha'} \partial X^\mu \bar{\partial} X_\mu - \frac{1}{2} \psi(z) \bar{\partial} \psi(z),
\]

with the antiholomorphic component \(\tilde{T}_B\) defined similarly. The subscript \(B\) stands for bosonic and its meaning will become clear shortly.

---

\(^{64}\)In this perspective the string theories in the former chapter are denoted as bosonic string theories. So formally we should have added the word “bosonic” every time we used the words “string theory” in the former chapter.

\(^{65}\)The reason why these theories are called supersymmetric will become clear towards the end of this chapter.

\(^{66}\)In this chapter normal ordering will be implicitly assumed.
2.1 Superconformal symmetry

Interestingly, besides conformal symmetry the superstring action has another remaining gauge symmetry after having fixed the gauge freedom from diffeomorphism and Weyl invariance: superconformal symmetry. It can be checked that the action is invariant under the superconformal transformations

$$\delta_{sc} X^\mu(z, \bar{z}) = \epsilon \sqrt{\alpha'/2} \left( \eta(z) \psi^\mu(z) + \eta(z)^* \tilde{\psi}^\mu(z) \right),$$

$$\delta_{sc} \psi^\mu(z) = -\epsilon \sqrt{\alpha'/2} \eta(z) \partial X^\mu(z),$$

$$\delta_{sc} \tilde{\psi}^\mu(z) = -\epsilon \sqrt{\alpha'/2} \eta(z)^* \bar{\partial} X^\mu(z),$$

and that the associated conserved supercurrents are

$$j^\eta(z) = \eta(z) T_F(z), \quad \tilde{j}^\eta(z) = \eta(z)^* \tilde{T}_F(z),$$

with

$$T_F(z) = i \frac{2}{\alpha'} \psi^\mu \partial X^\mu, \quad \tilde{T}_F(z) = i \frac{2}{\alpha'} \tilde{\psi}^\mu \bar{\partial} X^\mu.$$

Note that consistency requires that $\eta(z)$ and $\eta(z)^*$ are respectively arbitrary holomorphic and antiholomorphic Grassmannian functions on the world-sheet. Also note that $T_F(z)$ and $\tilde{T}_F(z)$ are fermionic and therefore have a subscript $F$.

By acting on the fields twice one can check that the commutator of two superconformal transformations is a conformal transformation, that the commutator of a superconformal and a conformal transformation is a superconformal transformation and that the commutator of two conformal transformations is again a conformal transformation. This shows that the conformal and superconformal transformations together are closed, they form a superconformal algebra. This can also be checked by deriving the OPE’s of the generators of the transformations, $T_B(z)$ and $T_F(z)$. Using the known OPE’s of the different fields and again denoting the spacetime dimension by $D$, we find

$$T_B(z)T_B(w) = \frac{3D}{4(z-w)^4} + \frac{2}{(z-w)^2} T_B(w) + \frac{1}{z-w} \partial T_B(w) + \mathcal{NS},$$

$$T_B(z)T_F(w) = \frac{3}{2(z-w)^2} T_F(w) + \frac{1}{z-w} \partial T_F(w) + \mathcal{NS},$$

$$T_F(z)T_F(w) = \frac{D}{(z-w)^3} + \frac{2}{z-w} T_B(w) + \mathcal{NS},$$

and similarly for their antiholomorphic counterparts. Because on the right-hand side of the equations all operators are again generators of either a conformal or superconformal transformation, the algebra is indeed closed. Furthermore, we see from the second OPE that $T_F(z)$ is a tensor of weight $(\frac{3}{2}, 0)$. More importantly, the first OPE tells us that the total central charge of the theory in (66) is

$$c = \frac{3}{2} D.$$

This makes sense, as we’ve seen that scalar fields have central charge 1 and the new $\psi(z)$-fields have central charge $\frac{1}{2}$: $D + \frac{1}{2} D = \frac{3}{2} D$. 48
2.1.1 Spacetime dimension

Having established this, we are basically done. We can repeat the whole machinery of fixing the gauge appropriately, determining the ghost action, finding the mode expansion of the BRST charge and performing the BRST quantization. However, there are three important differences:

1. The first is that the periodicity conditions for the fermionic fields are not determined by the physical interpretation of strings moving through spacetime, as it was the case for the periodicity conditions for the scalar fields for closed strings.

2. The second difference is that the fermionic fields are Grassmannian and have different conformal weights as compared to the scalar fields. We will see that this has important consequences for the algebra of the mode expansions and the ground state spectra of the theory.

3. Finally, as we have seen above the gauge symmetry of the action (66) is enlarged with a superconformal symmetry. And since BRST symmetry is directly related to the gauge symmetry of the original action, we might expect important changes when going through the procedure.

Discussion of the first two point is deferred to the next section. For the moment, let’s focus on a first aspect of the third difference.

Because of the bigger gauge symmetry the gauge-fixed path integral action has more ghost fields. The full ghost action of the superstring theory is

\[ S_{gh} = \frac{1}{2\pi} \int d^2z \left( \bar{b}\partial c + \beta \partial \gamma + \bar{\beta}\partial \bar{c} + \bar{\beta}\partial \bar{\gamma} \right), \]

where the \( b(z), \bar{b}(\bar{z}), c(z) \)- and \( \bar{c}(\bar{z}) \)-fields are the same Grassmannian fields as in the bosonic string theory ghost action, and the fields \( \beta(z), \gamma(z) \) and \( \bar{\beta}(\bar{z}), \bar{\gamma}(\bar{z}) \) are respectively holomorphic and antiholomorphic commuting fields, with weight \( \frac{3}{2} \) for the \( \beta \)-fields and \( -\frac{1}{2} \) for the \( \gamma \)-fields.

The central charge of the ghost part of the action can be calculated and is \(-15\). As was showed in section (1.5.2), the requirement of nilpotency of the BRST charge in the quantum theory forces the total central charge of the path integral action of a conformal field theory to vanish. In our case this requirement becomes

\[ 0 = c_{\text{tot}} = c_{\text{matter}} + c_{\text{ghost}} = \frac{3}{2}D - 15 \]

and thus we conclude that a consistent superstring theory can only exist in ten spacetime dimensions:

\[ D = 10. \]
2.2 Ramond and Neveu-Schwarz sectors

As said, the fermion fields on the closed string cylinder do not necessarily have to be periodic under the identification of \( w \) and \( w + 2\pi \). But because of this identification the matter fermion action on the cylinder,

\[
\frac{1}{4\pi} \int d^2w \left( \psi^\mu \partial_w \psi_\mu + \bar{\psi}^\mu \partial_w \bar{\psi}_\mu \right),
\]

must be invariant under a transformation \( w \rightarrow w + 2\pi \). This yields two possible conditions: the Ramond periodicity condition (R)

\[
\psi^\mu (w + 2) = \psi^\mu (w) \quad \forall \mu = 0, \ldots, D - 1,
\]

or the Neveu-Schwarz periodicity condition (NS)

\[
\psi^\mu (w + 2) = -\psi^\mu (w) \quad \forall \mu = 0, \ldots, D - 1.
\]

The reason that the condition must be the same in all spacetime direction is that the fermion fields carry a Lorentz index and therefore mix under a Lorentz transformation; and you don’t want to lose Lorentz invariance. For the right-moving fields the same two conditions are possible.

The possible conditions are conventionally summarized as

\[
\psi^\mu (w + 2\pi) = e^{2\pi i \nu} \psi^\mu (w),
\]

\[
\bar{\psi}^\mu (\bar{w} + 2\pi) = e^{-2\pi i \tilde{\nu}} \bar{\psi}^\mu (\bar{w}),
\]

with \( \nu \) and \( \tilde{\nu} \) taking values 0 and \( \frac{1}{2} \). Thus there are four different sets of conditions, denoted by \( (\nu, \tilde{\nu}) \), or by

\[
\text{NS} - \text{NS} \doteq (1/2, 1/2)
\]

\[
\text{NS} - \text{R} \doteq (1/2, 0)
\]

\[
\text{R} - \text{NS} \doteq (0, 1/2)
\]

\[
\text{R} - \text{R} \doteq (0, 0).
\]

These four options for the closed superstring are called sectors and we will see that the different periodicity conditions in each sector produce their own Hilbert space and that these Hilbert spaces are different from each other.

A Fourier expansion of the fermion fields on the cylinder and a transformation to the plane\(^{67}\) gives

\[
\psi^\mu (z) = \sum_{r \in \mathbb{Z} + \nu} \frac{\psi^\mu_r}{z^{r+1/2}}, \quad \bar{\psi}^\mu (\bar{z}) = \sum_{r \in \mathbb{Z} + \nu} \frac{\bar{\psi}^\mu_r}{\bar{z}^{r+1/2}}.
\]

(67)

Of course the periodicity condition is resembled in the mode expansion, but it is crucial to note that when circling around on the plane it is the Ramond condition (\( \nu = 0 \)) that gives the minus sign, while the Neveu-Schwarz condition gives the plus sign. This difference between the plane and the cylinder can be confusing. The OPE of the fermions fields\(^{68}\) now gives the anticommutation relations between the fermion oscillator modes:

\[
\{ \psi^\mu_r, \psi^\nu_s \} = \{ \bar{\psi}^\mu_r, \bar{\psi}^\nu_s \} = \eta^{\mu \nu} \delta_{r,-s}.
\]

\(^{67}\)Use that the fermion fields carry weight \( \frac{1}{2} \).

\(^{68}\)Recall: \( \psi(z)\psi(w) = \frac{1}{z-w} + \mathcal{NS} \), similar for the antiholomorphic fields.
2.2.1 The open superstring sectors

For open strings we can employ a trick. In this case the possible boundary conditions come from the requirement that the boundary terms in the equations of motion vanish, what leads to:

\[
\psi^\mu(0, \sigma^2) = e^{2\pi i\nu'} \tilde{\psi}^\mu(0, \sigma^2), \quad \text{for } \nu' = 0, \frac{1}{2}
\]

\[
\psi^\mu(\pi, \sigma^2) = e^{2\pi i\nu''} \tilde{\psi}^\mu(\pi, \sigma^2), \quad \text{for } \nu'' = 0, \frac{1}{2}.
\]

However, a re-definition of \( \tilde{\psi}^\mu(\vec{\sigma}) = e^{2\pi i\nu} \tilde{\psi}^\mu(\vec{\sigma}) \) eliminates the condition at \( \sigma^1 = \pi \) and leaves over the condition

\[
\psi^\mu(0, \sigma^2) = e^{2\pi i\nu} \tilde{\psi}^\mu(0, \sigma^2), \quad \text{for } \nu = 0, \frac{1}{2}.
\]

In the open superstring there are therefore two sectors. Now comes the trick. Define

\[
\psi^\mu(\sigma^1, \sigma^2) = \tilde{\psi}^\mu(2\pi - \sigma^1, \sigma^2), \quad \text{for } \sigma^1 \in \pi, 2\pi.
\]

The open string fermion fields are now combined in one holomorphic field on the circle and the boundary conditions are transformed into the periodicity conditions as we saw them in the closed string case:

\[
\psi^\mu(0, \sigma^2) = e^{2\pi i\nu} \tilde{\psi}^\mu(0, \sigma^2) = e^{2\pi i\nu} \psi^\mu(0, \sigma^2), \quad \text{for } \nu = 0, \frac{1}{2}.
\]

It is therefore justified to call the two open string sectors after the closed string sectors: a Ramond sector (\( \nu = 0 \)) and a Neveu-Schwarz (\( \nu = \frac{1}{2} \)) sector. Besides this, the mode expansions (67) of the holomorphic field and the associated algebra can be used for the open superstring as well.

2.2.2 Current mode expansions

Focussing on the holomorphic part of the theory and remembering the weight of the bosonic stress tensor and the fermionic supercurrent, the natural mode expansions are

\[
T_B(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \quad T_F(z) = \sum_{r \in \mathbb{Z} + \frac{\nu}{2}} \frac{G_r}{z^{r+3/2}}.
\]

Substituting the field mode expansions into these conserved quantities and keeping track of the creation-annihilation normal ordering gives

\[
L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha^\mu_{m-n} \alpha^\mu_{m-n} \circ + \frac{1}{4} \sum_{r \in \mathbb{Z} + \frac{\nu}{2}} (2r - m) \circ \psi^\mu_{m-r} \psi^\mu_{m-r} \circ + a^m \delta_{m,0},
\]

\[
G_r = \sum_{n \in \mathbb{Z}} \alpha^\mu_{r,n} \psi^\mu_{m-r,n}.
\]

Note that these expansions depend on the sector, and so does the normal ordering constant \( a^m \), where the superscript \( m \) stands for “matter fields”. We know from the bosonic treatment that a scalar contributes \( -\frac{1}{24} \) and one can calculate that a periodic fermion contributes \( \frac{1}{24} \) and an antiperiodic fermion \( -\frac{1}{48} \). Remembering that the transition to the plane adds a term \( \frac{1}{24} \) to \( L_0 \) leads to:

\[
R : a^m = \frac{1}{16} D, \quad NS : a^m = 0.
\]
2.3 BRST quantization

Having elaborated on most of the major differences between supersymmetric and bosonic string theory, we can now apply this knowledge to the BRST quantization of supersymmetric string theory. We will focus on the holomorphic part, as the antiholomorphic part is just a copy of it.

The extra gauge symmetry identified as the superconformal symmetry gives a rather natural extension of the BRST current: the extra set of ghost fields which are associated to the superconformal symmetry extend the former BRST current

\[ j_B(z) = cT_X + \frac{1}{2}cT_g \]

where the new ghost stress-tensors come from our knowledge of the \( bc\beta\gamma \) ghost superconformal CFT:

\[ (T_m)_B = (\partial b)c - 2\partial(bc) + (\partial\beta)\gamma - \frac{3}{2}\partial(\beta\gamma) \]
\[ (T_g)_F = -\frac{1}{2}(\partial\beta)c + \frac{3}{2}\partial(\beta c) - 2b\gamma. \]

Using (51), this leads to a BRST charge

\[ Q_B = \sum_{m \in \mathbb{Z}} c_{-m}(L_m)_m + \sum_{r \in \mathbb{Z} + \nu} \gamma_{-r}(G_m)_r \]
\[ - \sum_{m,n \in \mathbb{Z}} \frac{1}{2}(n - m) \circ b_{-m-n}c_m \circ c_n \circ \]
\[ + \sum_{m \in \mathbb{Z}} \left[ \frac{1}{2}(2r - m) \circ \beta_{-m-r}c_m \gamma_r \right. \]
\[ \left. - \circ b_{-m-r} \gamma_{r-\nu} \circ \right] + a^3 c_0, \tag{68} \]

where the first subscript of \((L_m)_m\) and \((L_m)_m\) means that only the matter contributions are included. Furthermore, the OPE's of the ghost fields imply that

\[ \{Q_B, b_n\} = L_n, \quad [Q_B, \beta_r] = G_r. \]

The first equation is well-known and leads to the mass shell condition\(^{69}\), while the second equation is new. Treating the \( \beta\gamma \)-system analogously to the \( bc \)-system leads to the convention that \( \beta_0 \) is an annihilation operator and \( \gamma_0 \) a creation operator. With the same argument as before physical states \( |\psi\rangle \) therefore obey the additional conditions

\[ b_0|\psi\rangle = \beta_0|\psi\rangle = 0, \]

and using the physicality condition thus also

\[ L_0|\psi\rangle = G_0|\psi\rangle = 0. \]

It is crucial to note that the last condition only holds for the R sector, as there is no \( G_0\)-mode in the NS sector.

\(^{69}\)Note that the \( L_0 \) mode expansion has extra fermionic terms, leading to a different mass spectrum than in bosonic string theory.
To conclude, proper physical superstring states are in the cohomology of the BRST charge (68) and satisfy the additional conditions
\[ b_0|\psi\rangle = \beta_0|\psi\rangle = L_0|\psi\rangle = 0, \]
where in the R sector there is also the condition
\[ G_0|\psi\rangle = 0. \]
Furthermore, the supersymmetric counterpart of the bosonic no-ghost theorem can be proved in a very similar way. The main result is that excitations of the fields \(X^0(z), X^1(z), \psi^0(z), \psi^1(z), b(z), c(z), \beta(z) \) and \(\gamma(z)\) are excluded from the proper physical spectrum. Therefore, obtaining the ground states for the different sectors is the final key to know the full proper physical spectra of the supersymmetric string theories.

Before investigating the ground states of the different spectra, let’s make things slightly more explicit. Because of the no-ghost theorem the \(L_0\)-generators in the open string case effectively are
\[
\text{R : } L_0 = \alpha' p^2 + \hat{N}_\alpha + \hat{N}_\psi,
\]
\[
\text{NS : } L_0 = \alpha' p^2 + \hat{N}_\alpha + \hat{N}_\psi - \frac{1}{2}.
\]
In the closed string case the first term has an extra factor \(\frac{1}{4}\). The \(\hat{N}_\alpha\) is the usual counting operator of modes of the scalar fields, and \(\hat{N}_\psi\) counts in a similar way the fermionic way. Note that the fermionic mode numbers in the NS sector are half-integer. Finally, the extra term \(-\frac{1}{2}\) in the NS sector comes from the ordering constants: \(a^m + a^g = 0 - \frac{1}{2} = -\frac{1}{2}\). In the R sector the ordering constants of the matter and ghost part cancel.

### 2.3.1 Ground states

As already pointed out, we focus on a theory with a single holomorphic set of operators, i.e. open string theory. The ground states in supersymmetric string theory are a direct product of the standard scalar ground state, the ghost ground state and the fermionic ground state. Neglecting the ghost ground state because of the no-ghost theorem, we can denote the ground state by
\[ |0; \vec{k}\rangle_{\text{sector}}, \]
where the subscript is either R or NS, depending on the sector we’re in.

Let’s start with the simple case: the NS ground state. Define the fermionic part of the NS ground state such that it is annihilated by all \(\psi^r\), with \(r > 0\). From (69) we then see that the full NS ground state is tachyonic with mass \(m^2 = -\frac{1}{2\alpha'}\). For the moment, this is all we can say about the NS ground state. Other NS states are created by acting with bosonic and fermionic creation operators on \(|0, \vec{k}\rangle_{\text{NS}}\).

The complication with the R ground states is that they are degenerate because of the \(\psi^0\)-modes. Define the fermion ground states as the states annihilated by all modes with \(r > 0\). The \(\psi^0\)-modes cannot be considered as creation of annihilation operators. Instead, they satisfy the Clifford algebra when multiplied by \(\sqrt{2}\):
\[ \{\psi^r_0, \psi^s_0\} = \eta^{rs}. \]
And because \( \{ \psi^r, \psi^0 \} = 0 \) for \( r \neq 0 \), the \( \psi^0 \)-modes carry ground states into ground states. We therefore conclude that the R ground states form a representation of the gamma matrix algebra: they are spinors. In \( D = 10 \) this is the 32-dimensional Dirac representation.

A basis for this representation is formed by

\[
|\vec{s}\rangle_R := |s_0, s_1, \ldots, s_4\rangle_R = \frac{1}{2^5} (\psi^0_0 + \psi^1_0)^{s_0 + 1/2} (\psi^2_0 + i\psi^3_0)^{s_1 + 1/2} \ldots (\psi^8_0 + i\psi^9_0)^{s_4 + 1/2} \zeta,
\]

with \( s_a = \pm \frac{1}{2} \) for \( a = 0, \ldots, 4 \). The spinor \( \zeta \) is annihilated by \( (-\psi^0_0 + \psi^1_0) \) and \( (\psi^{2a}_0 - i\psi^{2a+1}_0) \) for \( a = 1, \ldots, 4 \). In analogy with the gamma matrices we can define the spacetime Lorentz generators by

\[
\Sigma_{\mu\nu} := -\frac{i}{2} \sum_{r \in \mathbb{Z}^+} [\psi^\mu_r, \psi^\nu_{-r}],
\]

which satisfy the Lorentz algebra. It can be checked that the 32 basis states are eigenstates of the operator \( S_a = i\delta_{a,0} \Sigma^{2a,2a+1} \) with eigenvalue \( s_a \). Because of the half-integer eigenvalues we therefore see that R ground states are spacetime fermions. Similarly, the NS ground state is annihilated by the spacetime Lorentz generators and is thus a spacetime boson. And because excitations change the eigenvalues under \( \Sigma^{2a,2a+1} \) by integer values, all open string states in the R sector are spacetime fermions and all open string states in the NS sector are spacetime bosons.

It is known that in 10 dimensions the Dirac representation is reducible to two irreducible Weyl representations \( 16 + 16' \), which are distinguished by their eigenvalues \( \pm 1 \) under \( \Gamma := \Gamma^0 \Gamma^1 \ldots \Gamma^9 \). Similarly, one can define the operator

\[
e^{i\pi F}, \quad \text{with} \quad F := \sum_{a=0}^{4} S_a,
\]

where \( F \) is called the world-sheet fermion number. It can be shown that

\[
e^{i\pi F} |0\rangle_{\text{NS}} = -|0\rangle_{\text{NS}},
\]

\[
e^{i\pi F} |\vec{s}\rangle_R = |\vec{s}'\rangle_R \Gamma_{\vec{s},\vec{s}'},
\]

where \( \Gamma := 32S_0S_1S_2S_3S_4 \) and \( \Gamma_{\vec{s},\vec{s}'} \) is diagonal with \( +1 \) if the \( s_a \) include an even number of \( -\frac{1}{2} \) and \(-1\) if the \( s_a \) include an odd number of \( -\frac{1}{2} \). In this way the R ground states are split up in two 16-dimensional irreducible representations, distinguished by the world-sheet fermion number.

Having discussed the open string ground states, let’s now focus on the closed string ground states. In principle, these are simply the products of two copies of the open string ground states: one for the holomorphic and one for the antiholomorphic part. There’s slightly more to say about it however. In the first place both the NS-NS and the R-R sector have integer spin, i.e. are spacetime bosons. For the R-R sector this may seem surprising, but remember that combining two half-integer spins gives integer spins. On the other hand, the NS-R and the R-NS sector form spacetime fermions. Note by the way that because of level matching, in the mixed sectors one cannot simply multiply the open string ground states. We’ll get back to this later.

The ground states of the R-R sector transform as the product of two Dirac representations: \( |\vec{s}, \vec{s}'\rangle_R \). We know that the product of two spinor representations is equivalent to the direct sum

\[
|\vec{s}, \vec{s}'\rangle_R.
\]

\[\text{Here the ghost ground states are included, introducing an extra factor } -1 \text{ in the NS sector and } -i \text{ in the R sector.}\]
of anti-symmetric tensors of different rank. In the R-R sector this means:

$$32 \times 32 = [0] + [1] + [2] + \cdots + [10] = [0]^2 + [1]^2 + \cdots + [4]^2 + [5],$$

where $[n]$ is the set of anti-symmetric tensors of rank $n$. In the second equality we used the a tensor of rank $n$ is dual to a tensor of rank $D - n$ because of the $\epsilon$-tensor.

As the holomorphic and antiholomorphic part decompose separately into two Weyl representations, the R-R sector decomposes into four irreducible representations of the 10-dimensional Lorentz group. This can be summarized as follows:

<table>
<thead>
<tr>
<th>$(\exp(i\pi F), \exp(i\pi \tilde{F}))$</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(+1,+1)$</td>
<td>$16 \times 16 \cong [1] + [3] + [5]_+$</td>
</tr>
<tr>
<td>$(+1,-1)$</td>
<td>$16 \times 16' \cong [0] + [2] + [4]$</td>
</tr>
<tr>
<td>$(-1,+1)$</td>
<td>$16' \times 16 \cong [0] + [2] + [4]$</td>
</tr>
<tr>
<td>$(-1,-1)$</td>
<td>$16' \times 16' \cong [1] + [3] + [5]_-$</td>
</tr>
</tbody>
</table>

The subscripts $+/-$ are there because anti-symmetric tensors of rank half the spacetime dimension are self-dual. Their number of independent components is therefore halved. By the way, in general we have

$$\dim([n]) = \frac{D(D-1)(D-2)\ldots(D-n+1)}{n!}$$

for $1 \leq n \leq k + 1$ and $D = 2k + 2$ or $D = 2k + 3$.

### 2.3.2 Physical states

Using the BRST formalism and with help of the above discussion about ground states in the different sectors, we will now derive the tachyonic and massless proper physical superstring states. As said before, massive string states are a trivial extension because of the no-ghost theorem.

Let’s start with the open string states. In the NS sector the tachyonic ground state from the former section obeys all the physicality conditions, as there is no $G_0$-condition in this sector. We therefore have the physical state

$$|0; \vec{k}\rangle_{\text{NS}}, \quad \text{with } m^2 = -k^2 = -\frac{1}{2\alpha'}. $$

Note that $e^{i\pi F} = -1$ for this state. This tachyonic open string sector - transforming as a 1-dimensional representation under the Lorentz group - is therefore denoted as NS−. The most general first excited state in the NS sector is

$$|\frac{1}{2}; k\rangle_{\text{NS}} = (e \cdot \psi_{-1/2} + B\gamma_{-1/2} + C\beta_{-1/2}) |0; \vec{k}\rangle_{\text{NS}}.$$  

The $L_0$-condition makes this state massless: $k^2 = 0$. The physicality condition becomes

$$0 = Q_B |\frac{1}{2}; \vec{k}\rangle_{\text{NS}}$$

$$= (\sqrt{2\alpha'}\gamma_{-1/2}(k \cdot e) + \sqrt{2\alpha'}C(k \cdot \psi_{-1/2}) \frac{1}{2}Bc_0\gamma_{-1/2} + \frac{1}{2}Cc_0\beta_{-1/2}) |0; \vec{k}\rangle_{\text{NS}},$$

(70)
leading to the requirements $k \cdot e = B = C = 0$. From (70) it follows immediately that the only physical, BRST-exact state is of the form

$$k \cdot \psi_{-1/2} |0; \vec{k}\rangle_{\text{NS}}.$$ 

Setting $k^\mu = (\kappa, \kappa, 0, \ldots, 0)$, a basis of proper physical states is

$$\psi^i_{-1/2} |0; \vec{k}\rangle_{\text{NS}}, \quad \text{with } i = 2, \ldots, 9.$$ 

As in the bosonic case, we have a massless vector with $D-2$ spacelike polarisations; a photon. This sector is denoted by NS+, as the extra fermionic mode makes the state having eigenvalue +1 under $e^{i\pi F}$.

The most general ground state in the open string R sector is

$$|0; \vec{k}\rangle_{\text{R}} = u_{\vec{s}} |\vec{s}; \vec{k}\rangle_{\text{R}}, \quad \text{with } k^2 = 0,$$

where we immediately imposed the mass shell condition and summation over the 32 different configurations of $\vec{s}$ is implied. We see that in the R sector there is no tachyonic state. With the definition of the R sector ground states in the former section, it is clear that this state is BRST-closed. The remaining $G_0$-condition gives:

$$0 = G_0 |0; \vec{k}\rangle_{\text{R}} = \sqrt{2\alpha'} (k \cdot \psi_0) u_{\vec{s}} |\vec{s}; \vec{k}\rangle_{\text{R}},$$

leading to $0 = (k \cdot \psi_0) u_{\vec{s}}$. Now, because we’ve seen that $\sqrt{2} \psi_0^{\mu} \gamma^\mu$ we recognize the massless Dirac equation in this condition:

$$(k \cdot \Gamma) u_{\vec{s}} = k \cdot \Gamma_{\vec{s}\vec{s}} u_{\vec{s}} = 0.$$ 

This makes sense, as the ground states in the open string R sector are massless spinors with respect to the Lorentz group. As we know, the 32-dimensional Dirac representation is reducible into two Weyl representation with opposite “chirality”, i.e. differen world-sheet fermion number. Characterizing these representations by their $e^{i\pi F}$-number, the open string R sector consists of two 16-dimensional irreducible spinor representations: R±.

A discussion of the lowest states in the closed string spectrum is simple now: the closed string is two copies of the open string, and the physicality conditions are essentially the same. The only important extra feature is level matching: because the closed string has to obey two mass shell conditions - for $L_0$ and for $\tilde{L}_0$ - this gives the extra condition that

$$N_{\text{tot}} - \nu = \tilde{N}_{\text{tot}} - \tilde{\nu},$$

with $\nu, \tilde{\nu}$ taking values $0, \frac{1}{2}$ depending on the sector. The sectors NS+, R+ and R− can therefore be combined in closed string states, but the sector NS− cannot be combined with any of the other three sectors. The physical closed string sectors are thus (NS−, NS−), which is tachyonic with mass $m^2 = -\frac{2}{\alpha'}$, and the massless sectors (NS+, NS+), (R+, R+), (R−, R−), (NS+, R+), (R + NS+), (NS+, R−), (R−, NS+), (R+, R−) and (R−, R+)..

2.3.3 $SO(8)$ representations

In $D$ dimensions massless states are classified as $SO(D-2)$ representations, since these transformations leave the spacetime momentum invariant. In superstring theory there are three options:
a state transforms as a scalar under $SO(8)$, which is the trivial representation. Or a state transforms as an 8-vector under $SO(8)$, which is usually denoted by $8_v$. Or a state transforms as an eight-component spinor under $SO(8)$, usually denoted by $8_s$. From here onwards we will denote the spinor representation with $e^{i\pi F} = +1$ by $8_s$ and the spinor representation with $e^{i\pi F} = -1$ by $8_c$. The $s$ stands for spinor, and the $c$ stands for the complex conjugate. The first is also called the chiral representation, and the latter is called the anti-chiral representation.

For open strings it is obvious now that the NS− sector is a scalar under $SO(8)$, while the NS+ sector is a vector under $SO(8)$. With $k^\mu = (\kappa, \kappa, 0, \ldots, 0)$ the massless Dirac operator becomes $k \cdot \Gamma = -\kappa (\Gamma^0 - \Gamma^1) = -2\kappa \Gamma^0 (S_0 + \frac{1}{2})$ and thus is the physical state condition:

$$(S_0 + \frac{1}{2})u_s |\vec{s}; \vec{k}\rangle_R = 0.$$ 

The states with $s_0 = -\frac{1}{2}$ are singled out. And as we know from our group theory knowledge, under $SO(9,1) \rightarrow SO(1,2) \times SO(8)$ the Weyl representations decompose as

- $16_s \rightarrow 8_s (s_0 = +\frac{1}{2}) + 8_s (s_0 = -\frac{1}{2})$,
- $16_c \rightarrow 8_c (s_0 = +\frac{1}{2}) + 8_s (s_0 = -\frac{1}{2})$.

Therefore the Dirac condition leads us to the conclusion that in the R open string sector there is a $8_s$ chiral representation and a $8_c$ anti-chiral representation. The tachyonic and massless open string sectors as $SO(8)$ representations can be summarized as:

<table>
<thead>
<tr>
<th>Sector</th>
<th>$SO(8)$ representation</th>
<th>$m^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS−</td>
<td>8_c</td>
<td>$-\frac{1}{2\alpha'}$</td>
</tr>
<tr>
<td>NS+</td>
<td>8_s</td>
<td>0</td>
</tr>
<tr>
<td>R+</td>
<td>8_s</td>
<td>0</td>
</tr>
<tr>
<td>R−</td>
<td>8_s</td>
<td>0</td>
</tr>
</tbody>
</table>

For closed strings we can use our knowledge about products of representations. For example, we know that the (NS+, NS+) sector forms an $8_v \times 8_s$ representation, which can be decomposed in a scalar, an anti-symmetric 2-tensor and a traceless symmetric 2-tensor: $8_v \times 8_s \rightarrow [0] + [2] + (2)$, where (2) denotes the traceless symmetric 2-tensor. Furthermore, products of spinor representations have been discussed in section (2.3.1) already. What is new is a product of a vector representation with a spinor representation. For example, a state in $8_v \times 8_s$ can be denoted by $|i, \vec{s}\rangle$, where the first index is the vector index and the second index is the spinor index. From these states we can construct eight $SO(8)$-invariant states:

$$|i, \vec{s}\rangle \Gamma_{i\vec{s}}^i,$$

where summation over repeated indices is implicit. Because the free index is anti-chiral we see that $8_v$ is an irreducible representation in $8_v \times 8_s$. It can be shown that the other 56 states form an irreducible vector-spinor representation $56_s$ : these states $|i, \vec{s}\rangle$ obey the condition $|i, \vec{s}\rangle \Gamma^i_{i\vec{s}} = 0$ and are constructed through $|i, \vec{s}\rangle := |i, \vec{s}\rangle - \frac{1}{m^2-2} \Gamma_{i\vec{s}}^i \Gamma^j_{\ell\vec{s}} |j, \ell\rangle$. [71]

[71]See Johnson[2], section 7.1, for more information.

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All this results in the following summary of the tachyonic and massless states of the closed superstring:

<table>
<thead>
<tr>
<th>Sector</th>
<th>SO(8) representation</th>
<th>$m^2$</th>
<th>Tensor representation</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(NS^-, NS^-)$</td>
<td>1</td>
<td>$-\frac{2}{m'}$</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>$(NS^+, NS^+)$</td>
<td>$8_v \times 8_v$</td>
<td>0</td>
<td>$[0] + [2] + (2)$</td>
<td>$1 + 28 + 35$</td>
</tr>
<tr>
<td>$(R^+, R^+)$</td>
<td>$8_s \times 8_s$</td>
<td>0</td>
<td>$[0] + [2] + [4]_+$</td>
<td>$1 + 28 + 35_+$</td>
</tr>
<tr>
<td>$(R^-, R^-)$</td>
<td>$8_c \times 8_c$</td>
<td>0</td>
<td>$[0] + [2] + [4]_-$</td>
<td>$1 + 28 + 35_-$</td>
</tr>
<tr>
<td>$(NS^+, R^+)$</td>
<td>$8_v \times 8_s$</td>
<td>0</td>
<td>-</td>
<td>$8_s + 56_c$</td>
</tr>
<tr>
<td>$(NS^-, R^-)$</td>
<td>$8_v \times 8_c$</td>
<td>0</td>
<td>-</td>
<td>$8_s + 56_c$</td>
</tr>
<tr>
<td>$(R^+, R^-)$</td>
<td>$8_s \times 8_c$</td>
<td>0</td>
<td>$[1] + [3]$</td>
<td>$8_v + 56_t$</td>
</tr>
</tbody>
</table>

The subscript $t$ is to make clear that this 56-dimensional anti-symmetric tensor representation differs from the 56-dimensional vector-spinor representations of the NS-R sectors.

This finishes the BRST quantization of open and closed superstring theory and we have thereby reached the goal of our thesis: we derived the spectra of bosonic and supersymmetric string theory from first principles through BRST quantization. However, because of the different sectors in superstring theory there’s slightly more to say about it. Consistency conditions related to amplitudes between states of different sectors forbids certain combinations of sectors and forces other combinations to be present together. This will be the topic of the final section.
2.4 Superstring theories

Here we will very briefly discuss the extra conditions imposed on superstring theories and the three consistent superstring theories that remain after imposing the conditions. For a more comprehensive treatment of these theories we refer to Polchinski[4], section 10.6, and Johnson[2], section 7.1.

2.4.1 Closed superstring theories

Let’s start with consistent closed string theories. We’ve seen that because of level matching the $2^4 = 16$ sectors reduced to 10 closed string sectors. This means that there are potentially $2^{10}$ different closed superstring theories. Consistency conditions and comparison with what is observed in nature will luckily reduce this number to 2. Denoting the different sector by $(\alpha, F, \tilde{\alpha}, \tilde{F})$, using a new mod 2 parameter $\alpha := 1 - 2\nu$, the conditions can be summarized as:

1. If both $(\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1)$ and $(\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2)$ are in the spectrum, then they must satisfy
   \[ F_1 \alpha_2 - F_2 \alpha_1 - \tilde{F}_1 \tilde{\alpha}_2 + \tilde{F}_2 \tilde{\alpha}_1 \in 2\mathbb{Z}. \]
   The origin of this condition lays in products of vertex operators which come about in path integral amplitudes. These products are required to be single-valued and thus there may not be a resulting phase when one operator circles the other in a contour integral. This phase can only be present in case of a branch cut, i.e. for R sectors ($\alpha \neq 0$). Furthermore, this phase in proportional to the number of fermion fields in the other operator. This explains the appearance of the products $F\alpha$ in the condition.

2. There’s nothing that forbids the application of a vertex operator of one sector in the theory to a state of another sector in the theory: the resulting state should still be in the theory. Therefore, if the sectors $(\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1)$ and $(\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2)$ are in the theory, then
   \[ (\alpha_1 + \alpha_2, F_1 + F_2, \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{F}_1 + \tilde{F}_2) \]
   should also be in the theory.

3. Finally, one-loop amplitudes between arbitrary sectors are not modular-invariant, which is related to the conformal symmetry which remains after the gauge-fix. The condition resulting from the requirement of modular-invariance is: there must be at least one left-moving R sector and one right-moving R sector.

The applications of these conditions reduces the $2^{10}$ theories to four physically distinguishable consistent closed superstring theories: 0A, 0B, IIA and IIB. The sectors in each of these theories are:

- **0A**: $(\text{NS}+, \text{NS}+) (\text{NS}+, \text{NS}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+$).
- **0B**: $(\text{NS}+, \text{NS}+) (\text{NS}+, \text{NS}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+$).
- **IIA**: $(\text{NS}+, \text{NS}+) (\text{R}+, \text{NS}+) (\text{NS}+, \text{NS}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+$).
- **IIB**: $(\text{NS}+, \text{NS}+) (\text{R}+, \text{NS}+) (\text{NS}+, \text{NS}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+) (\text{R}+, \text{R}+$).

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Since the the 0A and 0B theories there is a tachyonic sector and there are no spacetime fermions,\textsuperscript{72} we conclude that these theories are not interesting. The two potentially interesting theories are therefore IIA and IIB.

A convenient way of summarizing these theories is

\begin{align*}
\text{IIA} & : (8_v + 8_s) \times (8_v + 8_c), \\
\text{IIB} & : (8_v + 8_s) \times (8_v + 8_s).
\end{align*}

Systematically writing out the product and using the results at the end of the former section, we obtain:

\begin{align*}
\text{IIA} & : (8_v + 8_s) \times (8_v + 8_c) \\
& = (8_v \times 8_v) + (8_v \times 8_c) + (8_s \times 8_v) + (8_s \times 8_c) \\
& \equiv (\text{NS}+,\text{NS}+) + (\text{NS}+,\text{R}-) + (\text{R}+,\text{NS}+) + (\text{R}+,\text{R}-) \\
& \equiv ([0] + [2] + [2]) + (8_s + 56_s) + (8_c + 56_c) + ([1] + [3]) \\
& \equiv ([0] + [1] + [2] + [3] + [2]) + (8_s + 8_c + 56_s + 56_c),
\end{align*}

where in the last line we grouped the spacetime bosons ans fermions, respectively. Note that the IIA massless spectrum consists of 256 independent proper physical states. Also note that precisely half of these states are spacetime bosons and the other half are spacetime fermions. This is a sign of \textit{spacetime supersymmetry}; it can be shown that this is indeed the case. In the massless spectrum we recognize the dilaton, axion and graviton states. The [1]- and [3]-states are called \textit{Ramond-Ramond states}, after their origin in the Ramond-Ramond sector. Furthermore, the $8_{s,c}$ are \textit{gauginos} and the $56_{s,c}$ are \textit{gravitinos}. The II in IIA is because there are two gravitino representations in the spectrum. Similarly for IIB, and in 0A and 0B there were none.

A similar analysis of the IIB spectrum yields

\begin{align*}
\text{IIB} & : (8_v + 8_s) \times (8_v + 8_s) \\
& = (8_v \times 8_v) + (8_v \times 8_s) + (8_s \times 8_v) + (8_s \times 8_s) \\
& \equiv (\text{NS}+,\text{NS}+) + (\text{NS}+,\text{R}+) + (\text{R}+,\text{NS}+) + (\text{R}+,\text{R}+) \\
& \equiv ([0] + [2] + [2]) + (8_s + 56_s) + (8_c + 56_c) + ([0] + [2] + [4] + [4]) \\
& \equiv ([0]^2 + [2]^2 + [4]_+ + [2]) + (8_s^2 + 56_s^2 + 56_c^2),
\end{align*}

where a square means that a representation occurs twice. Again note a total of 256 independent massless states: 128 spacetime bosons and 128 spacetime fermions. The states originating from the R-R sector are again called Ramond-Ramond states and need to be distinguished from the dilaton and axion, which are in similar $SO(8)$ representations. It is important to note that the gauginos in IIB have the same chirality, as opposed to IIA (the same holds for the gravitinos). We say that the IIA spectrum is \textit{non-chiral}, i.e. invariant under spacetime parity what switches the chiralities of the representations. On the other hand, the spectrum of IIB is chiral.

These chirality properties lead to another way of defining the different theories: the IIB theory can be defined by keeping all sectors with $e^{i\pi F} = e^{i\pi \tilde{F}} = +1$, and the IIA theory can be defined by keeping all sectors with $e^{i\pi F} = +1$ and $e^{i\pi \tilde{F}} = (-1)^\tilde{a}$. This is called the \textit{GSO-projection}.

\textsuperscript{72}Remember that spacetime fermions arise when and NS and R sector are combined in a closed string.
2.4.2 Open superstring theories

Without going in any detail, the only consistent theory with open strings turns out to be a type I theory of unoriented open superstrings with gauge group $SO(32)$ and unoriented closed superstrings. The fact that in a theory with open string states there always need to be closed string states as well makes sense, because two open string states can interact and thereby form a closed string state. The massless content of this theory is

$$[0] + [2] + (2) + 8_c + 56_s + (8_v + 8_s)_{SO(32)},$$

where the last two representations are open string states. Again, there is a dilaton, an axion and a graviton state. There is also one gravitino (type I) and there are two gaugino’s. Finally, there is also a photon state. There is a total of 128 independent states, of which 64 are spacetime bosons and 64 are spacetime fermions (the photon is a boson, the open string gaugino is a fermion).

This result can also be obtained through the GSO-projection, which regarding the open string states prescribes to keep only the states with $e^{i\pi F} = +1$. Effectively this means that you keep the NS+ and R+ sectors, and that the tachyon is projected out.

2.4.3 Counting states

Like we did for open bosonic strings, we can construct generating functions for the number of states per level of an open superstring theory. For a single fermion mode $\psi_{-r}$, $r \in \mathbb{Z} + \nu$, the generating function is

$$f_r(x) = 1 + x^r,$$

as each mode can occur at most once due to anticommutativity. The generating function for all fermionic modes in one spacetime direction is thus

$$f_{\text{fermionic}}(x) = \prod_{r=\nu}^{\infty} (1 + x^r).$$

Applying this to the physical spectra of our ten-dimensional open superstring theory, we get:

$$f_{\text{NS}}(x) = \prod_{n=1}^{\infty} \left( 1 + \frac{x^{n-1/2}}{1 - x^n} \right)^8,$$
$$f_{\text{R}}(x) = 16 \prod_{n=1}^{\infty} \left( \frac{1 + x^n}{1 - x^n} \right)^8,$$

where the factor of 16 is because the R ground states are degenerate and form a 16-dimensional irreducible Weyl representation.

Now focus on the open string states in the type I theory discussed in the former section. Because of the GSO-projection these are formed by the NS+ and R+ sectors. The R+ sector is simply half of the R sector, since half of the R ground states are in the R+ sector. Therefore,

$$f_{\text{R+}}(x) = 8 \prod_{n=1}^{\infty} \left( \frac{1 + x^n}{1 - x^n} \right)^8.$$

One can check that $f_{\text{R+}}(x) = 8 + 128x + 1152x^2 + \ldots$, which is in agreement with the $8_s$ in the massless content of type I theory. Finding $f_{\text{NS+}}(x)$ is slightly more difficult. Doing the Taylor
expansion one can see that in
\[ \prod_{n=1}^{\infty} \left( \frac{1 - x^{n-1/2}}{1 - x^n} \right)^8 \]
only the terms representing states with an odd number of fermionic modes got an extra minus sign with respect to \( f_{NS}(x) \). Since precisely these states form the NS+ sector, we have
\[ f_{NS+}(x) = \frac{1}{2} \left( \prod_{n=1}^{\infty} \left( \frac{1 + x^{n-1/2}}{1 - x^n} \right)^8 - \prod_{n=1}^{\infty} \left( \frac{1 - x^{n-1/2}}{1 - x^n} \right)^8 \right). \]

Finally, because of spacetime supersymmetry we expect the number of NS sector states (bosons) equal to the number R sector states (fermions) at each mass level. Correcting for the \(-\frac{1}{2}\) term in \( L_0 \) for the NS sector, we therefore expect
\[
\frac{1}{2\sqrt{x}} \left( \prod_{n=1}^{\infty} \left( \frac{1 + x^{n-1/2}}{1 - x^n} \right)^8 - \prod_{n=1}^{\infty} \left( \frac{1 - x^{n-1/2}}{1 - x^n} \right)^8 \right) = 8 \prod_{n=1}^{\infty} \left( \frac{1 + x^n}{1 - x^n} \right)^8.
\]
This identity was proven by Jacobi in 1829 and is called the *abstruse identity*. It plays a crucial role in the supersymmetric properties of string theory and in more advanced topics, like the interaction between D-branes. Note that this identity only holds for \( D = 10 \).

### 2.4.4 A note to strings in background fields

Although everything in this thesis was done for strings in a flat empty background, this is neither the most general nor the most interesting situation. More generally, a string is moving through a spacetime filled with other strings. Instead of treating each interaction separately it is much more convenient to describe the other strings as fields coupled to the single string we’re looking at. These fields will then appear in the Polyakov action for general backgrounds and are related to the states in our superstring theories. For example, a coherent state of gravitons forms a symmetric 2-tensor field \( G_{\mu\nu}(x) \) which couples to the moving string as if it were the spacetime metric:
\[
S = -\frac{1}{4\pi\alpha'} \int_M \mathrm{d}^4x \sqrt{-g} \gamma^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu.
\]
This enables us to generalize string theory to curved spacetimes. Similarly there is an antisymmetric 2-tensor field \( B_{\mu\nu}(X) \), the *Kalb-Ramond field*, related to axion states and coupling to the moving string. And there is a scalar dilaton field \( \Phi(X) \) related to dilaton states.

Unfortunately we do not have enough time to investigate the enormous amount of beautiful theory in this direction. But after having thoroughly investigated the basics of string theory and the bosonic ans supersymmetric string theory spectra in flat spacetime, further steps should be much easier and more fruitful.
Summary

It was shown that string theory can be considered as a gauge theory with scalar fields, a metric field and (possibly) spinors on the world-sheet. The world-sheet gauge symmetries of the original action are diffeomorphism invariance and Weyl invariance. Fixing the gauge leads to a theory with (super)conformal symmetry, including ghost fields on the world-sheet. This action is BRST invariant. The requirement that amplitudes in the theory are independent of the specific gauge-fix leads to physicality conditions on states. Using the (super)conformal properties this gives you a method of projecting out the unphysical states and determining the spectrum. It was shown that this procedure precisely eliminates the negative- and zero-norm states from the proper physical spectrum, as well as the ghost states.

Furthermore, we’ve seen that only certain supersymmetric theories can be considered as physically realistic. First of all because in bosonic and other supersymmetric theories tachyons were present, or spacetimes fermions were absent. And secondly because string interactions impose conditions on the combinations of different sectors. The upshot is that there are three consistent superstring theories, obtained by the GSO-projection: IIA closed superstring theory, IIB closed superstring theory and type I unoriented open and closed superstring theory. None of these theories incorporates tachyonic states. It was also observed that in the massless sectors there is an equal number of spacetime bosons and fermions, which is a sign of the spacetime supersymmetry of the theory.

Finally, formulas were derived for the number of open string states at the different mass levels. This verified spacetime supersymmetry for massive open superstring states as well and is an important starting point for further research, for example the study of D-brane interactions.
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