Reasoning in geometry

How first learning to appreciate the generality of arguments helps students come to grips with the notion of proof

MSc Thesis

written by

Olga Grigoriadou

under the supervision of Dr. Wolter Kaper, and submitted in partial fulfillment of the requirements for the degree of

MSc in Mathematics and Science Education

at the University of Amsterdam.

February 2012
Abstract

One of the aims of teaching geometry at school level is to help students understand what counts as an acceptable argument in mathematics (a “proof”) and to move from using inductive arguments to using deductive ones for supporting mathematical statements. The research study described in this thesis focuses on students’ reasoning when they start abandoning inductive arguments for deductive ones. More specifically, we propose a teaching module (partly based on existing literature) which assists students in this learning process, and we present the results of implementing this module on a sample of 20 students in a Greek highschool. We examine the change in the kinds of reasoning used by students at each point of the intervention and the effect of this change on the beliefs of students about mathematics and proof. Our results suggest that laying emphasis on making clear to the students the distinction between deductive and inductive arguments at the beginning of the lessons cycle, can help them to understand better the concept of mathematical proof and to produce proofs.
Acknowledgements

First and foremost I would like to thank Wolter, my supervisor. I still have vivid memories of the moment when I first talked to Wolter about my ideas. He was so enthusiastic that we kept exchanging long emails about the topic for a few days after, sharing our ideas and our enthusiasm. This was a great push forward for me, and made me even more excited to put all my ideas to work. I feel very lucky to have worked with such an enthusiastic, helpful and hardworking supervisor. Wolter’s feedback and support always came right when I needed them, and we exchanged productive criticism at all times. Without him, the way to the end of this project would have been much more difficult.

I would also like to thank the students that participated in the teaching intervention. These students offered with enthusiasm much of their free time in return for new experiences in mathematics. I greatly appreciate the effort they put in, all the way through the intervention, and their feedback. It was my first teaching experience, and I feel lucky that this was with such lovely and motivated students. I thank them all wholeheartedly and I wish them good luck in whatever they do in their lives! Of course, I would not have met these students had it not been for the cooperating teacher, Makis Panteliadis, who gave me permission to work with the students from his classes for this project. His comments, help and support were at all times useful, for which I would like to thank him.

Finally, I would like to thank all the people that supported me during this project: my teachers and fellow students at the AMSTEL institute who guided me and shared their experiences with me; my friends all over the world for supporting me and helping me forget about this project during my holidays; Olga, Rodi, Vassilis and Giannis for taking pictures and videos during the intervention; my professor Anastasios Patronis from the University of Patras for introducing me to the world of mathematics education and for the fruitful discussions we had during the first stages of this project; my parents for teaching me to never stop learning; and my partner Tikitu for proofreading and helping me structure this thesis and for being there for me whatever I do.
# Contents

1 Introduction 1

2 Theoretical Framework 3
   2.1 The meaning of proof 3
   2.2 Deductive versus inductive reasoning 4
   2.3 Geometry as a deductive science 5
   2.4 Previous research on students’ geometrical reasoning 6
      2.4.1 The Van Hiele theory 6
      2.4.2 Schoenfeld on students’ beliefs about proof 7
      2.4.3 The Harel & Sowder proof schemes 8
      2.4.4 The project *Justifying and Proving in School Mathematics* (JPSM) 15
   2.5 Van Hiele levels and Harel & Sowder proof schemes: A correspondence 16
   2.6 Previous research on the Van Hiele theory 17
   2.7 Teaching strategies assisting the move through the Van Hiele levels 18
   2.8 Conclusions and how our research builds on the literature 22

3 Aims and Research Questions 25
   3.1 Aims 25
   3.2 Research Questions 26
   3.3 Hypotheses 26

4 Research Setting 29
   4.1 The place 29
   4.2 The Greek educational system 29
   4.3 Geometry in the Greek curriculum 29
   4.4 The geometry books of the Greek Gymnasium 30

5 Methodology, Teaching Design and Analysis Framework 35
   5.1 Identifying the students’ Van Hiele levels 36
   5.2 Identifying the students’ kinds of reasoning 36
## CONTENTS

5.2.1 Adapting the instruments of the JPSM project to our research ........................... 36
5.2.2 Classifying the students’ kinds of reasoning based on the Harel & Sowder proof schemes ................................................................. 40
5.3 Identifying the students’ beliefs about mathematics and proof ............................ 40
  5.3.1 The adapted Schoenfeld questionnaire ......................................................... 40
  5.3.2 Our questionnaire about what the students believe they learned .................. 40
  5.3.3 An adapted version of certain JPSM project tasks ........................................ 41
5.4 Determining the correlation between proof schemes and Van Hiele levels .......... 41
5.5 Designing and evaluating a teaching strategy .................................................... 41
  5.5.1 The move from the descriptive to the theoretical level in three stages .......... 42
  5.5.2 The five Van Hiele phases for eight concepts in our teaching sequence ...... 44
  5.5.3 Method for evaluating our teaching strategy ................................................ 49
5.6 Other data collection methods ............................................................................ 50

6 The Intervention ....................................................................................................... 51
  6.1 Population and sample ....................................................................................... 51
  6.2 Student participation .......................................................................................... 54
  6.3 Diary of the intervention ................................................................................... 55

7 Results ...................................................................................................................... 65
  7.1 The Usiskin Van Hiele test results ..................................................................... 65
  7.2 The Harel & Sowder proof schemes coding process ........................................... 69
  7.3 Correlations between Van Hiele level and Harel & Sowder proof scheme ......... 78
  7.4 The role of our teaching strategy: A qualitative analysis ...................................... 82
    7.4.1 Moving through the three stages towards deductive proof schemes ........... 82
    7.4.2 Moving through the five Van Hiele teaching phases towards the theoretical Van Hiele level .............................................................. 100
  7.5 Students’ beliefs about mathematics and proof ................................................ 120
    7.5.1 Modified Schoenfeld questionnaire results ................................................ 120
    7.5.2 Students’ own ideas on how their beliefs changed ................................. 124
    7.5.3 Students’ understanding of the validity, convincingness and generality of arguments ................................................................. 131

8 Conclusions and Discussion ..................................................................................... 149
  8.1 Conclusions ...................................................................................................... 149
    8.1.1 Answering SQ.1 and H.1 ........................................................................... 149
    8.1.2 Answering SQ.2 and H.2 ......................................................................... 150
    8.1.3 Answering SQ.3 and H.3 ......................................................................... 151
    8.1.4 Answering our main research question ...................................................... 152
  8.2 Discussion and recommendations ...................................................................... 153

References .................................................................................................................. 159

Further Reading ......................................................................................................... 165
## CONTENTS

### A Pre- and post- tests and questionnaires  
- A.1 The original Usiskin Van Hiele test ................................................................. 168  
- A.2 The Van Hiele test as used in our research ..................................................... 181  
- A.3 The modified Schoenfeld questionnaire ......................................................... 190  
- A.4 Beliefs & reflections questionnaire ................................................................. 192  

### B Task related worksheets  
- B.1 TRIANGLES .................................................................................................. 194  
- B.2 ANGLE SUM ................................................................................................. 200  
- B.3 CHOOSING .................................................................................................... 201  
- B.4 CONCEPTIONS ABOUT PROOF ..................................................................... 205  
- B.5 QUAD ............................................................................................................. 206  
- B.6 OPPANG ......................................................................................................... 207  
- B.7 BIS .................................................................................................................. 208  
- B.8 BIS-GROUP .................................................................................................... 209  
- B.9 EXT-GROUP .................................................................................................... 210  
- B.10 TRIMID .......................................................................................................... 211  
- B.11 TRIMID-GROUP ............................................................................................ 212  
- B.12 QUADMID ..................................................................................................... 213  
- B.13 IF THEN ......................................................................................................... 214  

### C Handouts  
- C.1 Invitation ....................................................................................................... 216  
- C.2 Introduction to the lessons content and expectations ....................................... 218  
- C.3 The pros and cons of arguments ...................................................................... 219  
- C.4 How new truths are discovered in geometry .................................................. 222  
- C.5 What we discovered in the geometry lessons .................................................. 223  
- C.6 Where does it all begin? The map of axioms .................................................. 240  
- C.7 The story of Euclidean geometry ..................................................................... 241  

### D Tables and results  
- D.1 Van Hiele pre-test: Analysis and comparison with other studies .................... 244  
- D.2 Distribution of answers to the VR and EP questions in Task CHOOSING .......... 256  
- D.3 VR and EP scores before and after our intervention ........................................ 258  
- D.4 More results of the beliefs & reflections questionnaire .................................... 258
# List of Figures

2.1 The Harel & Sowder proof schemes ........................................ 10  
2.2 The revised Harel & Sowder deductive proof schemes .................. 10  
2.3 Figure accompanying a proof ............................................... 12  
2.4 An example of a non-causal proof ....................................... 13  
2.5 An example of a causal proof .............................................. 13  

4.1 Introducing new concepts in geometry — the Greek book ............... 33  

6.1 Term grades in mathematics for the participating students .......... 52  
6.2 Making conjectures about the properties of opposing angles .......... 58  
6.3 Figure drawn by the teacher to introduce Task BIS .................... 60  
6.4 Students’ conjectures for Task EXT ...................................... 60  
6.5 Explaining Task QUADMID .................................................. 63  

7.1 Example of the empirical/inductive proof scheme ....................... 72  
7.2 Example of the empirical/inductive proof scheme ....................... 72  
7.3 Example of the empirical/perceptual proof scheme ..................... 73  
7.4 Example of a deductive argument for Task QUAD ...................... 73  
7.5 Example of an incomplete deductive argument for Task TRIMID ....... 74  
7.6 Example of a causal proof scheme for Task OPPANG .................... 75  
7.7 A deductive proof offered to the students in Task CHOOSING .......... 86  
7.8 Example of an empirical argument for Task QUAD .................... 89  
7.9 Example of a deductive argument for Task QUAD .................... 89  
7.10 Example of an argument of an externally convinced student for Task QUAD .... 90  
7.11 A student who correctly identified the non-generality of her argument .... 91  
7.12 A student that mistakenly believed her argument to be general ....... 91  
7.13 A student who offered a correct deductive argument and identified it as convincing .... 91  
7.14 A student who offered a correct deductive argument but still expressed doubts .... 91  
7.15 A deductive and an empirical argument for Task BIS .................. 92  
7.16 A circular and an empirical argument for Task BIS .................... 92  
7.17 The figure used in Task BIS ............................................... 103  
7.18 The figure used in Task TRIMID ......................................... 105  
7.19 Sketch used by the teacher for the discussion of axioms ............... 110  
7.20 Students’ choices for own approach and best grade before and after our intervention .... 133  
7.21 Validity Ratings of four arguments before our intervention .......... 135
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.22</td>
<td>Validity Ratings of four arguments after our intervention</td>
<td>136</td>
</tr>
<tr>
<td>7.23</td>
<td>Explanatory Power of four arguments before our intervention</td>
<td>136</td>
</tr>
<tr>
<td>7.24</td>
<td>Explanatory Power of four arguments after our intervention</td>
<td>137</td>
</tr>
</tbody>
</table>
# List of Tables

2.1 The Van Hiele levels of thinking ................................. 7  
2.2 The Van Hiele levels on which our study focuses ................ 8  
2.3 Correspondence between Van Hiele levels and Harel & Sowder proof schemes 17  
2.4 The stages of the learning sequence we propose for introducing students to proof 22  
2.5 Strategies for introducing students to proof ........................ 23  
4.1 Correspondence of the Van Hiele levels numbering between Fuys, Geddes, and Tischler (1985) and van Hiele (1986) ................................. 31  
5.1 Codes for the Harel & Sowder proof schemes .......................... 40  
5.2 Tasks used for the ‘inquiry’ phase .................................. 45  
5.3 Tasks used for the ‘directed orientation’ phase ...................... 46  
5.4 Methods used for the ‘explication’ phase .............................. 47  
5.5 Tasks used during the ‘free orientation’ phase ....................... 48  
5.6 Hand-out used during the ‘integration’ phase ........................ 49  
5.7 Tasks used for each Van Hiele teaching phase by concept ............. 50  
6.1 Term grades in Mathematics, and Van Hiele levels of the participating students 53  
6.2 Timeline of teaching intervention .................................... 54  
6.3 Students’ absences during the intervention ........................... 55  
7.1 Van Hiele levels of the participants before and after our intervention 68  
7.2 Coding of students’ main proof schemes for all tasks .................. 76  
7.3 Coding of students’ proof scheme subcategories for all tasks .................. 77  
7.4 Variables used for the statistical analysis ................................ 79  
7.5 Correlation (1-tailed) between Van Hiele level and frequency of deductive Harel & Sowder proof schemes at the beginning of the intervention ................................. 80  
7.6 Correlation (1-tailed) between Van Hiele level and frequency of deductive Harel & Sowder proof schemes at the end of the intervention ...................... 80  
7.7 Correlation (1-tailed) between progress in Van Hiele level and progress in Harel & Sowder proof schemes ............................. 81  
7.8 Correlation (1-tailed) between progress in Van Hiele level and Harel & Sowder proof schemes .................................................. 81  
7.9 Kinds of arguments offered by the students for Task ANGLESUM .......... 83  
7.10 Kinds of arguments offered by the students for Task QUAD ........... 90  
7.11 Students who used inductive arguments for Task QUAD ................ 91
7.12 Students who used deductive arguments for Task QUAD .......................... 92
7.13 Kinds of arguments offered by the students for Task BIS ......................... 93
7.14 The kinds of students’ main proof schemes by task (in chronological order) .... 99
7.15 Answers to ‘classroom practice’ questions before the intervention ............... 120
7.16 Answers to ‘classroom practice’ questions after the intervention ................. 121
7.17 Answers to questions related to students’ perceptions about mathematics and
gometry before the intervention ......................................................... 122
7.18 Answers to questions related to students’ perceptions about mathematics and
gometry after the intervention ......................................................... 122
7.19 Answers to questions related to the nature of geometric proofs, reasoning and
constructions before the intervention .................................................. 122
7.20 Answers to questions related to the nature of geometric proofs, reasoning and
constructions after the intervention .................................................... 123
7.21 Categories of answers for the question: ‘Do you believe that what we have done
in these lessons is useful? In what way?’ .............................................. 126
7.22 Categories of answers for the question: ‘Do you believe that the lessons have
affected the ideas you have about mathematics? If yes, how?’ ...................... 128
7.23 Categories of answers for the question: ‘Do you believe that the lessons have
affected the ideas you have about proof? If yes, how?’ .......................... 130
7.24 Distribution of arguments chosen in Task CHOOSING before the intervention 131
7.25 Distribution of arguments chosen in Task CHOOSING after the intervention 132
7.26 Types of arguments in Task CHOOSING ........................................... 132
7.27 Validity Ratings (VR) scoring scheme .............................................. 135
7.28 Explanatory Power (EP) scoring scheme ........................................... 135
7.29 Generality rating of the statement ‘The sum of the angles of a triangle is 180°’
before and after our intervention ....................................................... 137
7.30 Assessing the role of proof before and after our intervention ...................... 138
7.31 Variables used ................................................................................. 139
7.32 Change in students’ VR scores as they move from the descriptive to the theo-
retical Van Hiele level ........................................................................... 139
7.33 Change in students’ VR scores for the students that did not make the move from
the descriptive to the theoretical Van Hiele level .................................... 140
7.34 Change in students’ VR scores as they move from the empirical to the deductive
Harel & Sowder proof schemes ............................................................ 141
7.35 Change in VR scores for the students that did not make the move from the em-
pirical to the deductive Harel & Sowder proof schemes .......................... 141
7.36 Change in students’ EP scores as they move from the descriptive to the theoretical
Van Hiele level .................................................................................... 142
7.37 Change in students’ EP scores for the students that did not make the move from
the descriptive to the theoretical Van Hiele level .................................... 143
7.38 Change in students’ EP scores as they move from the empirical to the deductive
Harel & Sowder proof schemes ............................................................ 144
7.39 Change in EP scores for the students that did not make the move from the em-
pirical to the deductive Harel & Sowder proof schemes .......................... 144
7.40 Change in students’ beliefs about convincingness of arguments as they move from the descriptive to the theoretical Van Hiele level ................. 146
7.41 Change in students’ beliefs about convincingness of arguments for the students that did not make the move from the descriptive to the theoretical Van Hiele level

D.1 Distribution of answers regarding the VR and EP of Stamatis’s (Dylan’s) argument before our intervention .................................................. 256
D.2 Distribution of answers regarding the VR and EP of Stamatis’s (Dylan’s) argument after our intervention .................................................. 256
D.3 Distribution of answers regarding the VR and EP of Georgia’s (Cynthia’s) argument before our intervention ........................................ 256
D.4 Distribution of answers regarding the VR and EP of Georgia’s (Cynthia’s) argument after our intervention ........................................ 256
D.5 Distribution of answers regarding the VR and EP of Lefteris’s (Ewan’s) argument before our intervention ........................................ 257
D.6 Distribution of answers regarding the VR and EP of Lefteris’s (Ewan’s) argument after our intervention ........................................ 257
D.7 Distribution of answers regarding the VR and EP of Kalliopi’s argument before our intervention .................................................. 257
D.8 Distribution of answers regarding the VR and EP of Kalliopi’s argument after our intervention .................................................. 257
D.9 Validity Rating (VR) and Explanatory Power (EP) of the various arguments before and after the intervention .................................................. 258
D.10 Categories of answers for the question: ‘Do you believe you learned something new in the lessons we had? If yes, what?’ ................ 260
D.11 Categories of answers for the question: ‘Did you like the lessons? Would you prefer that anything would be done in a different way? Explain’ ................ 261
D.12 Categories of answers for the question: ‘If we had some more lessons together for the coming weeks, which would be the subject that you would prefer to discuss and why?’ ................ 263
D.13 Categories of answers for the question: ‘Do you believe that what we have done in these lessons is useful? In what way?’ ................ 264
Chapter 1

Introduction

My interest in the topic of this thesis dates back to my years as a mathematics student in Greece, when I first started wondering about the logic behind mathematical proofs. What was it that made a proof correct and indisputable, and what was the relation between mathematical proofs and everyday reasoning? More generally, I was interested in investigating the nature of mathematics, which I thought was to a great extent determined by the notions of proof, truth and argumentation. Back then I had already developed a strong interest in education, and I hoped to become able someday to explain the nature of mathematics to young people who were struggling to find some meaning in the often distorted image of mathematics presented to them in school.

This brought me to Amsterdam, to attend a Masters degree in Logic, where I had the opportunity to study several interesting and interrelated fields such as mathematical logic (in a sense much wider than that of classical propositional/predicate logic as usually taught at undergraduate level), philosophy of language, philosophy of logic, decision theory, formal linguistics, cognitive science and psychology of reasoning. Within this context I chose to examine in my MSc in Logic thesis (Grigoriadou, 2009) the issue of rationality and of logical reasoning from the perspective of philosophy, cognitive science, and psychology of reasoning. The natural next step was to pursue a further study in education, in an attempt to use my knowledge from Logic to understand how students reason while proving mathematical statements, what meaning this process of proof has for them, and how a teacher can effectively guide students through the process of learning how to construct valid arguments.

The focus of this research project is therefore the teaching and learning of Euclidean geometry in school, and specifically in the Greek school. The choice of geometry came quite naturally since the axiomatic system of Euclidean geometry is both a good example of an axiomatic system taught in school and is also tied to an interpretation within physical space which makes it easier to understand for students, since it is less abstract than other fields. The choice of Greece as the context of my research also came rather naturally. Euclidean geometry in the Greek school holds a special place in the curriculum and is taught as a separate subject, which does not always happen in other countries. The way geometry is taught in Greece, however, is traditional: axioms, theorems and propositions are imposed on students instead of letting them discover them for themselves. Knowing this, and also having personal experience as a student in the Greek educational system, I wanted to offer Greek students a learning experience which, unlike those of their ordinary school, would help them discover geometry by themselves and ex-
experience the activities that real mathematicians do: conjecturing, formulating propositions and providing valid arguments for their conjectures. At the same time this would help me examine the students’ reasoning while they engaged in proving activities.

The research study we describe in this thesis is therefore focused on students’ reasoning while doing geometrical proofs, and on a teaching strategy that aims at assisting students in thinking about and doing geometry as a mathematician does. Earlier research on the same subject is plentiful and studying it helped in the formulation of our research questions. In Chapter 2 we give an overview of the most important past research studies and we describe in detail how they helped us in formulating our own research questions and hypotheses, which are presented in Chapter 3. The context of this study, that is, the Greek educational system and the teaching of geometry in the Greek Gymnasium, is described in Chapter 4. In Chapter 5 we offer a detailed presentation of the methodology we followed, the design of the research and the analysis framework. Before presenting the detailed results of our study, in Chapter 7, we offer to the reader a diary of the intervention (Chapter 6) and a timeline of the research study in order to give a clearer picture of what happened when, and what the reactions of the students were. Finally, in Chapter 8 we summarise and discuss our results and we give recommendations for future research.
Chapter 2

Theoretical Framework

2.1 The meaning of proof

Proof is thought to be one of the building blocks of mathematics, since it is via proofs that mathematical truths are established. It is commonly accepted that proof occupies a central position in mathematics (Arsac, 2007). But, what exactly is considered as proof in mathematics?

Most of us are familiar with the everyday, informal meaning of the word ‘proof.’ Often, when we face a claim which is not immediately obvious to us, we ask for stronger arguments in order to be convinced. In this informal sense, ‘proving’ means ‘convincing.’ Besides this informal, conceptual sense of proof there exists a more formal, mathematical sense which is also connected with a specific syntactic structure (Rav, 1999, p. 11). In this formal sense, mathematical proof is the systematic derivation of a conclusion based on a set of axioms by following rules of logic. Proof in this sense is spelled out by following a rigorous syntactic structure commonly accepted by the community of mathematicians. Mathematicians tend to consider formal proofs (in the sense of derivations) as the only mathematically acceptable proofs and claim that outside axiomatic systems there is no rigorous mathematics (Freudenthal, 1973, p. 149). However, often formal proofs do not “fit mathematical practice” and are not “capable of explaining the source of mathematical knowledge and the dynamics of its growth” (Rav, 1999, p. 15). Rav (1999, p. 12) gives an illuminative metaphor for illustrating the difference between conceptual proofs (what he calls “proofs”) and their formal counterparts:

Metaphorically speaking, the relation between a proof and its formalised version is about the same as the relationship between a full-view photo of a human being and a radiograph of that person. Surely, if one is interested in the skeletal structure of an individual for diagnostic purposes, then the X-ray picture yields valuable information. But from a radiograph one cannot reconstruct the ordinary full-fledged view of the individual.

When it comes to education, and specifically to the teaching of proof, the teacher should think about which kind of proof would be more meaningful for students. Are rigorous, formal proofs accessible to students who are being introduced to the world of mathematics? According to Freudenthal (1973, p. 148), students can learn mathematical rigour only by reinvention. This means that students should first understand and experience the need for proofs and the informal
(conceptual) meaning of proofs, and only after that will they be able to abstract from the conceptual meaning and move to the rigor of the syntactic structure of formal proofs. According to Freudenthal (1973, p. 150) experimental mathematics (or the mathematics of free discovery as he calls it) “is much more important than that which is confined to axioms imposed by the teacher or textbook author, and there is no reason to claim that it is any less rigorous. There are levels of rigour and for each subject matter there is a level of rigour adapted to it; the learner should pass through the levels and acquire their rigour.”

Whether formal or informal, proof has certain functions. According to Bell (1976, p. 24), there are three main functions of mathematical proof:

The first is verification or justification, concerned with the truth of a proposition; the second is illumination, in that a good proof is expected to convey an insight into why the proposition is true; this does not affect the validity of a proof, but its presence in a proof is aesthetically pleasing. The third sense of proof is the most characteristically mathematical, that of systematisation, i.e. the organisation of results into a deductive system of axioms, major concepts and theorems, and minor results derived from these.

In order to understand and be able to produce proofs in geometry all three functions mentioned by Bell need to be understood and experienced. The third function is concerned with the process of deduction and it will be one of the main focus points of our study. In mathematics deductive reasoning plays a central role and is very often contrasted with inductive reasoning which is considered less mathematically strong. Inductive reasoning, however, employs quite strongly the second function of proof: illumination. Inductive arguments are based on observations, and as such they serve for getting an insight into which mathematical claims may be generally true. It seems that all three functions cited above are equally important for arriving at the understanding of proofs in geometry (Freudenthal, 1973, p. 149). I will discuss further this well-known distinction between deductive and inductive reasoning in the following section.

2.2 Deductive versus inductive reasoning

Deduction is a specific kind of reasoning concerned with the yielding of true conclusions from true given premises, or else with the yielding of valid conclusions (Johnson-Laird, 1999). In other words, deductive reasoning is about drawing specific conclusions from true given premises by following certain rules of logic. Inductive reasoning on the other hand, is a way of coming up with general conclusions from specific observations. These observations may support the conclusion but they do not ensure it. This is because there might exist counterexamples which we haven’t observed yet and which would make our conclusion actually wrong. Thus, although inductive reasoning might serve as a tool for coming up with interesting conjectures and it is closer

---

1 A classical example of deductive reasoning comes from Aristotle, the ‘father’ of logic and deductive reasoning: “All men are mortal. Socrates is a man. Therefore, Socrates is mortal.” One can easily see that in case the premises (the two first sentences) of this argument are true, then the conclusion (the last sentence) of the argument is also true. We can say thus that the above argument scheme is valid—although of course this does not necessarily give us any information about whether the specific premises of this argument are actually true.

2 For example, if we measure all angles of a specific triangle we will find out (given that our measurements are very accurate) that they add up to 180°. If we repeat this measurement and calculation for more triangles, we will find out that this is the case for any other type of triangle that we come up with. From this observation one could
to our intuition, it certainly gives less valid outcomes when it comes to formal mathematics than
deductive reasoning does.

This of course should not be interpreted as implying that inductive reasoning is of less value
for mathematics in general. Deductive reasoning is indeed the only way for mathematicians to
establish truth. However, for finding truth, empirical and intuitive methods, like induction, are
necessary and helpful (Clements & Battista, 1992, p. 437). This observation is very important
for mathematics education. Young students most often use inductive, intuitive methods for
reasoning about mathematics, which led Schoenfeld characterize them as “naïve empiricists”
(Schoenfeld, 1985, p. 41). Given these observations, the purpose of mathematics teaching should
partly be to show students how to use in a productive way their sense of inductive reasoning on
their way to understanding the need for deductive reasoning in mathematics. This process is after
all similar to the way mathematics was historically developed, as Poincaré (1900, pp. 123-124)
also suggests (as cited by Freudenthal (1973, p. 147)):

When mathematical science becomes rigorous it assumes an artificial charac-
ter which cannot be overlooked. It forgets about its historic origin: it shows how
problems can be solved but not how and why they are posed.

This shows that logic does not suffice; the science of proofs is not the whole
science, intuition is assigned a complementary part, I would say, as counterpart or
antidote of logic.

And especially when the deductive system of geometry is taught to students, it is not ap-
propriate to introduce students to the deductive part of the system without first showing them
the importance of the role of induction in the creation of this system. In the words of Kruyt-
bosch (1936), as cited in the translated version of Dina van Hiele’s dissertation (Fuys, Geddes,
& Tischler, 1984, p. 33), “The stiff deductive teaching of mathematics is wrong, not only from
a didactic perspective, but also from an historic perspective. Many important results have been
found through the inductive method. Why then should these be displaced to an a posteriori role
by the deductive method?” Dina van Hiele supports this claim and adds that “a system of teach-
ing that has the tendency to be based exclusively on deduction or on induction is wrong. In the
initial instruction neither purely deductive nor purely inductive reasoning is appropriate. What
is important here is that the pupils establish relations between empirically obtained results, and
that they gain insight into the empiricism” (Fuys et al., 1984, p. 190).

2.3 Geometry as a deductive science

Geometry is a typical example of a field where deductive reasoning occupies a central position.
From the axioms of geometry, and by using certain rules, one proves propositions and theo-

rems. The focus of this research study will be the area of geometry, and we will look closely
at the process of deductive reasoning. One of the aims will be to examine students’ reasoning
at different levels of thinking3 as well as their reasoning during the transition from one level of

make the generalization that for any triangle, the sum of its angles gives 180°. This is an example of inductive
reasoning. However, this kind of reasoning is not considered to provide a valid mathematical proof since a single
counterexample would be enough to make the conclusion false.

3By ‘levels of thinking’ we mean the levels of thinking in geometry as those were defined by the Van Hieles
(van Hiele, 1986). These levels will be discussed in more detail in the following sections.
Chapter 2. Theoretical Framework

thinking to the next. This transition is not immediately obvious. The reason is that “the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills” (Moore, 1994, p. 250). Therefore, to understand this transition one needs to look at students’ cognitive skills, knowledge background, and beliefs about mathematics and about the world. I will now give a brief overview of some of the relevant research within education which will be useful for our study.

2.4 Previous research on students’ geometrical reasoning

Lately there has been an increasing interest in research which looks at students’ reasoning and the levels of their cognitive development (Hanna, 2000). It is generally observed that students have difficulties with understanding the deductive process and with appreciating the meaning of mathematical (geometrical or other) proof (Usiskin, 1982; Senk, 1985; Schoenfeld, 1985; Porteous, 1986; Clements & Battista, 1992; Moore, 1994; Coe & Ruthven, 1994; Harel & Sowder, 1998; Jones, 2000; Weber, 2001; Szendrei-Radnai & Török, 2007; Hoyles & Healy, 2007; Knuth, Choppin, & Bieda, 2009). Of specific interest for our study is that area of research which focuses on students’ reasoning while they engage in geometrical proofs. Piaget (1953) was one of the first to give a thorough account of the ability of people to make inferences (Johnson-Laird, 1999, p. 110), an ability which is rather fundamental for proving. However, although there is plenty of research the last fifty year which identifies the ‘problem’ of students to understand and appreciate deductive proof, there is not enough research on “students’ opportunities to develop deductive reasoning and to learn skills for evaluating the validity of others’ mathematical arguments” (Bieda, 2009, p. 351). Our intention is therefore to use some of the most prominent theories describing students’ thinking while engaging with mathematical proof and, based on these theories, to carry out a research project which would aim at providing to students opportunities to develop deductive reasoning. These theories will be described in what follows.

2.4.1 The Van Hiele theory

Almost parallel to Piaget, in 1957, a pair of Dutch mathematicians, Pierre and Dina van Hiele, noted that there exist distinct levels of thinking during the process of learning geometry (Fuys, Geddes, & Tischler, 1985, p. 1). Since then the Van Hieles have produced many writings explaining the idea of the levels, and the Van Hiele levels have been established in the mathematics education communities. In his writings, Pierre van Hiele offers an explanation of the Van Hiele levels and their relation to the teaching of geometry (with a primary interest in secondary education). Given the focus of our research (students’ geometrical reasoning) and the wide acceptance of the Van Hiele theory, we chose to use it as a basic element of our research. Below we give a more detailed description of the theory.

Table 2.1 shows the Van Hiele levels as described by van Hiele (1986, p. 53). At the visual level students are able to identify, name, compare and operate on geometric figures (Fuys et al., 1985, p. 5). At the descriptive level students are able to “apply operative properties of a well known figure” (van Hiele, 1986, p.41). For example, students can recognize figures by their properties rather than by how they look, and they empirically discover classes of figures which are defined by the properties of these figures. Moreover, at this level the students are able to use informal arguments in order to operate on the properties they discovered. According to
Van Hiele, “the transition from the base level to the second level is one from a level without a network of relations to a level that has such a network” (p. 49). At the theoretical level, the student “deals with the structures of the descriptive level” (p. 50). During the transition from the descriptive to the theoretical level “the ordering of properties of geometric figures is the object of study,” and eventually, when the theoretical level is reached, “figures are understood as an ordered set of properties” (p. 63). Students are able to reason deductively and understand the axiomatic system of Euclidean geometry. The two final levels, formal logic and nature of logical laws, are not encountered often at school level. More specifically, levels like the fifth or higher are difficult to discern (p. 47) and, according to Van Hiele, checking whether students have attained the fifth or higher levels can only be of theoretical value (p. 47).

| First level: Visual | Second level: Descriptive |
| Third level: Theoretical | Fourth level: Formal Logic |
| Fifth level: The nature of logical laws |

Table 2.1: The Van Hiele levels of thinking.

At every Van Hiele level of thinking the student is able to internally organize the previous level (van Dormolen, 1977, p. 32). Moreover, the Van Hiele levels of thinking have two important properties; they are discontinuous and hierarchical:

[T]he most distinctive property of the levels of thinking is their discontinuity, the lack of coherence between their networks and relations [their structures]. (van Hiele, 1986, p. 49)

The ways of thinking of the base level, the second level, and the third level have a hierarchic arrangement. Thinking at the second level is not possible without that of the base level; thinking at the third level is not possible without thinking at the second level. (p. 51)

In addition, the most important characteristics of the transition between consequent levels are that it is not a natural process and that it depends on language: “The transition from one level to the other is not a natural process; it takes place under the influence of a teaching-learning program. […] The transition is not possible without the learning of a new language.” (van Hiele, 1986, p. 50).

From the five Van Hiele levels described in Table 2.1 the first four are the ones encountered at school level (van Hiele, 1986, p. 47). Particularly at the lower secondary school (ages 12 to 15) students do not even reach level 4 (Formal Logic). Therefore, although we are not opposed to the existence of the other two levels of thinking (fourth and fifth), the first three Van Hiele levels (see Table 2.2) are the ones that we will use in our research.

### 2.4.2 Schoenfeld on students’ beliefs about proof

Schoenfeld is another researcher in the area of education a contemporary of Piaget and the Van Hieles who was interested in explaining students’ behaviour while solving mathematical problems. Schoenfeld was particularly interested in students’ beliefs about proof and about the nature
Chapter 2. Theoretical Framework

<table>
<thead>
<tr>
<th>Label</th>
<th>Level description</th>
</tr>
</thead>
<tbody>
<tr>
<td>vH1:</td>
<td>Visual</td>
</tr>
<tr>
<td>vH2:</td>
<td>Descriptive</td>
</tr>
<tr>
<td>vH3:</td>
<td>Theoretical</td>
</tr>
</tbody>
</table>

Table 2.2: The Van Hiele levels on which our study focuses.

of mathematics, and in how these beliefs affect the way students do mathematics. Schoenfeld (1989) conducted a research project involving secondary geometry students, which revealed that, although the students were highly motivated and scored generally high in the assessments, they had the following beliefs and expectations: they expected to be able to solve very fast the mathematical problems they were asked to solve, and if they could not then they believed it was impossible to solve them; “they behave on construction problems as though their proof-related knowledge were nonexistent”; and, “they claim that mathematics is best learned by memorization” (pp. 348–349).

According to Schoenfeld students appear to be “pure empiricists” (Schoenfeld, 1985, p. 41 and p. 160), that is, students base their solutions to mathematical problems on empirical observations rather than on valid logical arguments. Schoenfeld (1985) has made an attempt to construct a model for the kind of ‘axioms’ students use while solving mathematical problems. It turns out that students’ behaviour fits this model, which Schoenfeld calls the model of pure empiricism and which is comprised not of the logical/deductive axioms we know, but by a set of attitudes which Schoenfeld calls “empirical axioms” (for example, one such ‘axiom’ is that “insight and intuition come from drawings. The more accurate the drawing, the more likely one is to discover useful information” Schoenfeld, 1985, p. 160).

The issue of how students reason, especially when they engage in geometrical, deductive proofs, is still to this day at the center of educational research. More recent work in this area includes: Hanna, 1990; Clements and Battista, 1992; Chazan, 1993; Moore, 1994; Harel and Sowder, 1998; Stanovich and West, 2000; Recio and Godino, 2001; Hoyles and Kücheman, 2002; Harel, 2007; Duval, 2006, 2007; A. J. Stylianides, 2007; Tall, 2008; G. J. Stylianides and A. J. Stylianides, 2008; A. J. Stylianides and G. J. Stylianides, 2009. Moreover, several projects have been funded for investigating students’ reasoning related to proof (for instance the Longitudinal Proof Project which is a continuation of the Justifying and Proving in School Mathematics project, the Proof Project, etc.), and a Newsletter exists especially for the teaching and learning of mathematical proof which includes a section with relevant bibliography.

2.4.3 The Harel & Sowder proof schemes

From the research cited above, Harel and Sowder (1998) and Harel (2007) will be especially significant for our study. After examining the literature we chose to use the Harel and Sowder research in our theoretical framework because they offer a detailed categorisation of students’ attitudes while they prove in mathematics. This would enable us to have a tool for identifying the kinds of reasonings of students who learn geometry. The details of the Harel and Sowder

---

4 [http://www.mathsmed.co.uk/ioe-proof/]
5 [http://www.mathsmed.co.uk/ioe-proof/proof.htm]
6 [http://www.tpp.umassd.edu/]
7 [http://www.lettredelapreuve.it/]
research are explained below.

Harel and Sowder (1998) have created a categorization of cognitive schemes called ‘proof schemes’ which have been consistently observed in their experiments. Each category of proof schemes represents “a cognitive stage, an intellectual ability, in students’ mathematical development” (p. 244). Thus, the classification of proof schemes offered by Harel and Sowder is not merely a classification of the content or the method of proofs, and it is not an a priori classification. Rather, it is based entirely on the observations of students’ conceptions and actions while attempting to do mathematical proofs, that is, when attempting to find convincing arguments for supporting the truth of mathematical statements. In the authors’ words, “it is the individual’s scheme of doubts, truths, and convictions, in a given social context” that underlies their characterization of proof schemes (p. 244). The proof schemes of Harel and Sowder were later revised by Harel (2007). The three main (revised) categories of proof schemes are the following:

- External conviction (EXT)
- Empirical (EMP)
- Deductive (DED)

In the external conviction proof schemes the students believe that mathematical justification is all about ritual and form (Harel & Sowder, 1998, p. 245). Proving in this category might depend on (a) the authority of the teacher, (b) the appearance of an argument or (c) mere symbol manipulation (Harel, 2007, pp. 66-67). These three cases constitute the three different proof schemes which belong to the External Conviction category.

In the empirical proof schemes the students’ attitudes depend heavily on physical facts or sensory experiences (Harel & Sowder, 1998, p. 252). We have two main subcategories here: inductive and perceptual proof schemes. In the inductive proof schemes the students evaluate a conjecture based on one or more specific cases which they generalize over all cases. In the perceptual proof schemes students “ignore transformations on objects or are incapable of anticipating results of transformations completely or accurately” (p. 255).

Finally, the category of deductive proof schemes consists of two main subcategories: the transformational proof schemes, which share the characteristics of generality, operational thought and logical inference Harel (2007, p. 67), and the modern axiomatic proof schemes which include the same characteristics just described but include others too which we will discuss in more detail later. In Figure 2.1 the detailed diagram of the Harel and Sowder (1998) proof schemes is provided, whereas in Figure 2.2 we offer a diagram of the revised deductive proof scheme subcategories and is similar to the one given in Figure 2.1. The diagram of Figure 2.2 is not actually offered by Harel, but we created it based on our understanding of the revised proof schemes as they are described by Harel (2007).

Let us now look at the Harel and Sowder proof schemes in more detail. In what follows we will describe each of the subcategories of the three main proof schemes described above.

---

8 Here Harel and Sowder draw on Piaget and Inhelder, who describe such mental images as constituting “an imitation of actions that can be carried out in thought …[but they] cannot be adequately visualized all the way to [their] ultimate conclusion before [they have been performed]” (Piaget & Inhelder, 1967, p. 259) as quoted in (Harel & Sowder, 1998, p. 252).
The External Conviction Proof Schemes

- **Ritual (RIT)**
  A student is considered to have a ‘ritual’ proof scheme when he or she accepts a proof as valid only when it is written in a specific form which looks ‘mathematical’ although it might be a wrong argument. The same category includes students who would reject a correct proof as wrong because it does not include any mathematical notation. As Harel
2.4. Previous research on students’ geometrical reasoning

and Sowder (1998) state, often students wonder whether specific arguments can be considered as proofs because they do not look like proofs. “Typically, such doubts are raised when the argument is not communicated via mathematical notations and does not include symbolic expressions or computations” (p. 246).

• Authoritarian (AUTH)

A student with the authoritarian proof scheme believes a proof to be valid only because the book or the teacher (i.e., an authority) says so. This proof scheme is re-enforced by curricula which stress truth in mathematics but do not stress the reasons for truth (justification) in mathematics (Harel & Sowder, 1998, p. 247). According to Harel and Sowder the authoritarian proof scheme has (at least) five observed manifestations:

1. Memorizing and applying ready-made formulas as the main expectation.
2. Asking for help for a problem without first making an attempt to solve it.
3. When a sentence is called a ‘theorem’ students stop trying to prove it.
4. Proving a certain statement by rephrasing it into a statement that is a fact.
5. Being reluctant to challenge the teacher’s justification of a conjecture by asking questions.

• Symbolic (SYMB)

The symbolic proof scheme covers those attitudes where students blindly use symbols without considering their meaning. As Harel and Sowder put it, “the main characteristic of the symbolic reasoning is the behavior of approaching the solution of a problem without first comprehending its meaning, that is, without building a coherent image of the problem situation” (Harel & Sowder, 1998, p. 251).

The Empirical Proof Schemes

• Inductive (IND)

According to Harel and Sowder students who possess the inductive proof scheme argue for the truth of a conjecture by evaluating the conjecture only in a few specific cases, and from there they conclude that the conjecture holds for the general case. For example, a student that measures the sum of the angles of a few specific triangles finds them 180° and hence concludes that the sum will be 180° for all triangles, has an inductive proof scheme.

• Perceptual (PER)

In the perceptual proof scheme the transformations of an image are not anticipated, thus the proof becomes specific to the figure drawn. An example of this proof scheme is given in Harel and Sowder (1998, p. 256), where a student makes an attempt to prove that the midpoints of any isosceles trapezoid form a rhombus. In figure 2.3 we can see the shape that the teacher had drawn on the board.

Based on her own figure, the student claimed that she proved that $FH$ and $EG$ are congruent by showing that triangles $FEH$ and $HEG$ are congruent. This student came to a general conclusion based on her perceptual observations made for her own figure, that is, based on observations regarding a specific figure.
Chapter 2. Theoretical Framework

Figure 2.3: Figure accompanying a proof.

The Deductive Proof Schemes

The deductive proof schemes “validate conjectures by means of logical deductions” (Harel & Sowder, 1998, p. 258) and are divided into two main categories: The transformational proof schemes and the modern axiomatic proof schemes. We will look at these two categories and their subcategories separately.

• Transformational (TRAN)

The three essential characteristics of transformational proof schemes (Harel, 2007, p. 67) are:

1. Generality (the individual understands that the goal is to prove a ‘for all’ argument and not an argument relevant only to specific cases),
2. Operational thought (the individual has the ability to set goals and sub-goals and to anticipate outcomes while proving),
3. Logical inference (the individual understands that mathematical proof is based on rules of logic).

Moreover, a distinctive characteristic of transformational proofs is that the operations made on the objects of the proof involve transformations of images by means of deduction (Harel & Sowder, 1998, p. 258). In other words, people possessing a transformational proof scheme are able to foresee the changes in an object and how these will affect the validity of an argument. Based on certain restrictions observed in the proof schemes of the students who participated in the research of Harel and Sowder, the transformational proof schemes were divided into three subcategories depending on the kind of the restriction. The restrictions were “on either the conjecture, the generality of the justification, or the mode of justification,” (p. 267) and the subcategories were named respectively contextual, generic, or constructive/causal. In the revised version of the proof schemes (Harel, 2007) the transformational proof schemes were divided into four categories instead of three, again with the same restrictions. These were: Causal, Constructive, Contextual (includes the Greek Axiomatic and the Arithmetic Symbolic proof schemes) and Generic.

– Causal (CAUS)

A causal proof scheme is one that reveals the cause of a conjecture. For example,
2.4. Previous research on students’ geometrical reasoning

consider the conjecture ‘The sum of the three interior angles of a triangle is equal to 180°.’ A proof of this conjecture which does not reveal the cause of the property suggested by the conjecture would be the following (Harel, 2007, p. 68):

\[
\text{Proof: Construct } CE \text{ parallel to } AB \text{ (See figure 2.4). Then the alternate angles } BAC \text{ and } ACE \text{ are congruent and the corresponding angles } ABC \text{ and } ECD \text{ are congruent. Hence, } m(ABC) + m(BAC) + m(ACB) = m(ECD) + m(ACE) + m(ACB) = 180°.}
\]

Harel refers to a philosophical discussion regarding the lack of causality for the above proof:

What is the case of the property that is proved here, asked these philosophers? The proof appeals to two facts about the auxiliary segment \( CE \) and the external angle \( ACD \). But these facts, they argued, cannot be the true cause of the property. For the property holds whether or not the segment \( CE \) is produced and the angle \( ACD \) considered (Harel, 2007, p. 68).

A proof which is characterized by causality as reported in Harel and Sowder (1998, p. 259), would be the following proof of the same conjecture:

\[
\text{Proof: The two sides } AB \text{ and } AC \text{ of a triangle } ABC \text{ are rotated in opposite directions through the vertices } B \text{ and } C, \text{ respectively, until their angles with the segment } BC \text{ are } 90° \text{(figures a, b). This action transforms the triangle } ABC \text{ into the figure } A'B'CA'', \text{ where } A'B \text{ and } A''C \text{ are perpendicular to the segment } BC. \text{ To create the original triangle, the segments}
\]

Figure 2.4: An example of a non-causal proof.

Figure 2.5: An example of a causal proof.
Chapter 2. Theoretical Framework

$A'B$ and $A''C$ are tilted toward each other until the points $A'$ and $A''$ merge back into the point (figure c). In doing so, we ‘lose two pieces’ from the $90^\circ$ angles $B$ and $C$ (i. e. angles $A'B'A$ and $A''C'A$) but at the same time we ’gain these pieces back’ in creating the angle $A$. This can be better seen if we draw $AO$ perpendicular to $BC$: angles $A'B'A$ and $A''C'A$ are congruent to angles $BAO$ and $AOC$, respectively (figure d) (Harel & Sowder, 1998, p. 259).

The above proof, offered by a student who participated in the research of Harel and Sowder, shows that the student can see the dynamic element of a triangle, which is one of the basic characteristics of a transformational proof scheme. In contrast to the previous one, the above proof shows why the sum of the angles of a triangle is $180^\circ$ (although the argument would need to be continued in order to cover also cases of triangles including right or obtuse angles). So, a causal proof is “an enlightening proof that gives not just mere evidence for the truth of the theorem but the cause of the theorem’s assertion” (Harel, 2007, p. 71).

– Constructive (CONS)

In constructive proof schemes the individual’s doubts “are removed by actual construction of objects - as opposed to mere justification of the existence of objects” (Harel, 2007, p. 71). This proof scheme has emerged from the philosophical idea of constructivism, and proof by contradiction is an example of a non-constructive type of argument which often fails to be convincing. In proof by contradiction, proving that the opposite of a statement cannot be true (because it leads to a contradiction) is sufficient for concluding that the statement is true. A constructivist, however, or someone that has the constructive proof scheme, refuses to accept that something is true merely because if it is not true we end up to a contradiction. Why is it necessary that something exists solely because if it did not we would be in trouble? How can we believe that something exists without ever seeing it?

– Contextual (CONT)

In contextual proof schemes “conjectures are interpreted, and therefore proved, in terms of a specific context” (Harel & Sowder, 1998, p. 268). An example of a contextual proof scheme would be one where the person possessing this scheme is unable to think in terms of abstract structures when trying to prove for instance something about geometry, and is restricted only to what he or she has perceived in the past (e. g. the person cannot understand any interpretation of the axioms of geometry different than the one given in Euclidean geometry).

Examples of mathematical theories that are based on proof schemes which are subcategories of the contextual proof scheme are the Greek axiomatic system and Diophantine Algebra (Harel, 2007, p. 73). The Greek axiomatic system, which is represented by the proof scheme called Greek Axiomatic (GRAX) and is also the one we focused on in our intervention, is based on intuitive axioms and although the Greeks conceived the notion of deductive proof, they “had one single type of mental objects in mind, namely, objects that are idealizations of physical reality, such as a line, plane, triangle, etc.” (p. 73). The Arithmetic Symbolic (ARITHS) which also belongs in the Contextual proof schemes will not be used in our study and therefore will not be further discussed.
2.4. Previous research on students’ geometrical reasoning

- **Generic (GEN)** In generic proof schemes “conjectures are interpreted in general terms but their proof is expressed in a particular context” (Harel & Sowder, 1998, p. 271). An example of this proof scheme offered by Harel and Sowder is one where students were asked to prove that “if a whole number is divisible by 9 then the sum of its digits is divisible by 9,” and instead of using a general argument they did the following they started by taking a specific number “say 867, and saying something to the effect: This number can be presented as $8 \times 100 + 6 \times 10 + 7$, which is $8 \times 99 + 6 \times 9 + 8 + 7 + 6$. Since the first addend, $8 \times 99 + 6 \times 9$, definitely is divisible by 9, the second addend, $8 + 6 + 7$, which is the sum of the number’s digits, must be divisible by 9. Some of these students indicated, in addition, that this process can be applied to any whole number. In so doing, the students were utilizing a generic proof scheme” (Harel & Sowder, 1998, p. 271).

- **(Modern) Axiomatic (AXIOM)**

The essential characteristics of axiomatic proof schemes are the same as the three characteristics of transformational proofs (generality, operational thought, and logical inference) but in addition the person possessing a (modern) axiomatic proof scheme is aware that an axiomatic system is not tied to a specific interpretation. For instance, in the axiomatic system of Euclidean geometry, there is a specific interpretation of the axioms in two- and three-dimensional space. Every axiom and proposition in Euclidean geometry, as opposed to modern axiomatic systems, is therefore connected to a specific interpretation. This is why in the latest version of the Harel and Sowder proof schemes, we see that the Greek Axiomatic proof scheme does not belong to this category of proof schemes but to the Transformational proof schemes.

2.4.4 The project *Justifying and Proving in School Mathematics (JPSM)*

Another project which we will use for our research and which is related to students’ geometrical reasoning is the JPSM project, the focus of which was secondary school students’ geometrical reasoning (Hoyle & Healy, 2007; Healy & Hoyles, 1998). We chose to use some of the instruments created in that project because their aims fit with ours and we found that some of their instruments could be useful to our study. Also, partly using instruments which are already used in other research would allow us to compare our results to those of other studies. Hoyle and Healy (2007) acknowledge the persistent problems students face when it comes to understanding and using proof in mathematics (p. 81) and describe the general aim of their project as follows: “to move beyond analyses that focus only on the individual student or classroom and to begin to identify curricula and school influences on geometrical reasoning” (p. 82). More specifically, the focus of JPSM was on investigating the effects which a then new curriculum applied in England and Wales had on students’ proof conceptions in geometry (p. 83). In that new curriculum, which had been in operation for about 10 years at the time of the project, the aim was “to promote the development of mathematical habits of mind where students would tinker, conjecture, test informally or explain, but in numerical or algebraic contexts rather than geometric ones” (p. 83). Given this emphasis of the curriculum on reasoning but its separation from geometry, the JPSM project set out to investigate its consequences for students’ conceptions and habits of mind as regards to proof in geometry. More specifically, the aims of the JPSM project were to investigate (Hoyle & Healy, 2007, p. 84):
Chapter 2. Theoretical Framework

(i) The characteristics of arguments recognized as proofs by high-attaining students aged 14–15 years,

(ii) The reasons behind their judgments, and

(iii) The ways they constructed proofs for themselves.

The findings of this research show that the majority of students, even after the new curriculum was applied, still appeared to “rely on inductive inference rather than any logical argument to determine the truth of mathematical statements” (Hoyles & Healy, 2007, p. 107). However, the students who followed this curriculum “rarely if ever acted out meaningless formal rituals when producing proofs” as opposed to reports of other research projects in the past (p. 107). Hoyles and Healy (2007) point out that their research raises questions for the planning of curricula for teaching geometry: any curricula that include the teaching of proof in their aims should “explicitly be designed to achieve this goal” (p. 108), and they add that a way to do this is to use research that has already documented ways to effectively teach geometry and proof and to build from this research systematic teaching plans. This is what we aim to do in our research and in the following paragraphs we will show how some of the theories discussed in this chapter can be combined in order to create an effective teaching strategy for the learning of proof. We find the instruments used by Hoyles and Healy (2007) to reveal students’ methods for proving and their beliefs useful and we will explain in detail in Chapter 5 how we used these in our research.

2.5 Van Hiele levels and Harel & Sowder proof schemes: A correspondence

Based on the available literature, we can see a rough correspondence between the Van Hiele levels of thinking and the Harel and Sowder proof schemes. We will explain this correspondence here.

On the one hand we have the Van Hiele levels which, as discussed in Section 2.4.1, are hierarchical. This means that a student cannot be at a later level without passing through the previous one. For example, being at the theoretical Van Hiele level implies that one is able to internally organize the empirical level, that is, the previous level. Moreover, it follows that when you are at any one level, then you are not able to reason (yet) in the way suggested by the following level. Therefore, one who is at the empirical level cannot reason at the theoretical level until he or she reaches that level.

On the other hand, the Harel & Sowder proof schemes do not intend to suggest any hierarchy. However, when a person is at the descriptive Van Hiele level, he or she cannot (by definition of the level) consistently possess any of the analytical Harel & Sowder proof schemes. Only when the theoretical Van Hiele level is reached are students able to reason deductively and begin to understand the axiomatic system of Euclidean geometry. This implies that at the descriptive Van Hiele level or below, a student will not display any analytical proof schemes since he or she is not able to do so yet. Similarly, at the theoretical Van Hiele level, students will not be able to use any of the proof schemes that presuppose understanding of various competing axiomatic systems.

Moreover, although it is not impossible, we do not expect students at the theoretical or higher Van Hiele levels to have external or empirical proof schemes. We may assume that the move to
2.6. Previous research on the Van Hiele theory

A higher thinking level is accompanied more often than not by the abandoning of proof schemes used in the previous levels, especially for the first three levels of thinking (visual, descriptive and theoretical).

Table 2.3 shows our view of the correspondence between the Van Hiele levels of thinking and the Harel & Sowder proof schemes.

<table>
<thead>
<tr>
<th>Van Hiele level</th>
<th>Harel &amp; Sowder proof schemes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: Visual</td>
<td>External and Empirical</td>
</tr>
<tr>
<td>2: Descriptive</td>
<td>Empirical (possibly External)</td>
</tr>
<tr>
<td>3: Theoretical</td>
<td>Deductive-Transformational (possibly Empirical and External)</td>
</tr>
<tr>
<td>4: Formal Logic</td>
<td>Deductive-Modern Axiomatic and Transformational</td>
</tr>
<tr>
<td>5: The nature of logical laws</td>
<td>Deductive-Modern Axiomatic and Transformational</td>
</tr>
</tbody>
</table>

Table 2.3: Correspondence between Van Hiele levels and Harel & Sowder proof schemes.

2.6 Previous research on the Van Hiele theory

The Van Hiele theory, which will play an important role for our research, has been tested several times in various research projects. A large-scale research was conducted in the USA by Usiskin—with a sample of 2700 students selected from 13 schools (Usiskin, 1982, p. 2). Usiskin classified students in Van Hiele levels by using a closed multiple choice test. Fuyse et al. (1985) conducted a large-scale research which focused on how students’ levels of thinking can be affected by the teaching procedure, and on how teachers can be trained to teach geometry in accordance with the Van Hiele theory. An interesting difference between Usiskin (1982) and Fuyse et al. (1985) is that Usiskin used a multiple choice test for classifying students at Van Hiele levels whereas Fuyse et al. (1985) used clinical interviews, which allow for a richer interpretation of students’ replies. In another research project Burger and Shaughnessy (1986), acknowledging the importance of interviews, developed an interview script and a packet for administering and analyzing interviews aiming at classifying students to the Van Hiele levels.

It is generally accepted that an interview is more accurate for assessing the Van Hiele levels of thinking because of the possibility it offers for a richer interpretation of students’ reasoning (A. Gutiérrez & Jaime, 1998, p. 27). A. Gutiérrez and Jaime (1998), acknowledging this fact but also the fact that a multiple choice test is more efficient and easy to administer than an interview (p. 27), have tried to design a test which would combine these two kinds of assessment. They describe (p. 32) the requirements that such a test should meet, and present the results of a longitudinal assessment study based on this framework. Mayberry (1983) conducted a research which showed that the Van Hiele levels form a hierarchy, and Senk (1989) examined the relations between Van Hiele levels and geometry proof-writing achievement for high school level students and found that there is a positive relation between them.

In the Greek context, where our intervention took place, little research has been done on the Van Hiele levels of thinking and what there is involves categorizing students into Van Hiele levels by using questionnaires of the kind used by Usiskin (1982) (not including interviews with the students). Zachos (2000) used a sample of 458 students of B’ Lyceum grade (16-17 year olds). This research supports the results of Mayberry (1983), namely that the Van Hiele levels are hierarchical. Ntziachristos and Koleza (1990) conducted a research which examined
how students of grades 5 and 6 of the elementary school (10-12 year olds) can be classified according to the Van Hiele levels, and what kinds of errors these students make in geometry. Ntziachristos and Zaranis (2001) investigated how a teaching strategy based on the Van Hiele model in combination with the use of educational software in the geometry classroom may improve the achievement of students of A’ Gymnasium level in geometry. Tzifas (2005) in his Master’s thesis summarizes the previous research (pp. 54–57) and offers the results of his own research on a sample of 1838 students from the levels Γ’ Gymnasium and Α’, Β’ Lyceum in 45 schools in Greece. Tzifas classified students in a similar way to Usiskin, by using Usiskin’s multiple choice tests, however he acknowledges that a research with interviews combined with the Usiskin tests would probably have better and more accurate results (Tzifas, 2005, p. 101).

2.7 Teaching strategies assisting the move through the Van Hiele levels

The move from one Van Hiele level to the next does not happen naturally (van Hiele, 1986, p.50). Therefore, appropriate teaching strategies need to be designed and implemented for a successful move through the Van Hiele levels. It is rather encouraging that in the last decades there has been a developing interest in including geometry and proof in the mathematics curricula at all levels (Hanna et al., 2009). However, important as the teaching of geometry has become, it has also been observed that ‘middle school students’ classroom experiences with justification and proof are insufficient for developing desired conceptions of mathematical proof” (Bieda, 2009, p. 65). Moreover, many studies show students’ preference towards empirical arguments over deductive ones (Williams, 1979; Usiskin, 1982; Schoenfeld, 1985; Senk, 1985; Porteous, 1986; Balacheff, 1988; Clements & Battista, 1992; Coe & Ruthven, 1994; Moore, 1994; Harel & Sowder, 1998; Jones, 2000; Weber, 2001; Knuth et al., 2009). But why do students show this preference?

It is a fact that understanding how students’ deductive reasoning develops is difficult and there is still much in this area that is unknown (Dreyfus, 1999). A lot of research is therefore being carried out in this area. In an attempt to answer the above question, a classification of students’ beliefs about empirical and deductive arguments as well as of the reasons behind these beliefs has been made by Chazan (1993). For answering the above question one can claim based on the theory of geometrical thinking of the Van Hieles (van Hiele, 1986; Fuys et al., 1984) that students who show this preference may still be in the descriptive (or earlier) Van Hiele level, and need to move on to the theoretical level before they start showing a preference for deductive arguments.

Chazan (1993) found that “explicit attention devoted to discussion of the relative merits of these forms of argumentation [deductive and empirical]” (p. 382) actually helped students understand these two forms better. More specifically, Chazan examined students’ beliefs about proof in mathematics in a context where lessons included empirical exploration with the help of dynamic geometry software, “an emphasis on deductive proof, and a willingness [on the part of the teacher] to use a unit for teaching explicitly about mathematical argumentation” (Chazan, 1993, p. 363). As Chazan (1993, p. 363) mentions, in the sites where the research took place “after the teachers had completed their traditional instruction about what a deductive proof is and how to do simple deductive proofs, they began the instructional unit about the methods of measurement of examples and deductive proof. This unit included activities which were created to highlight differences between measurement of examples and deductive proof.” The
2.7. Teaching strategies assisting the move through the Van Hiele levels

results reported by Chazan were generally positive as to how the juxtaposition of empirical and deductive arguments may affect students’ understanding of the meaning of proof.9 Traditionally, the teaching of Euclidean geometry is done by first introducing students to the Euclidean axioms and propositions and then proceeding to prove theorems and new propositions by using what has already been proven. Usually the teachers and textbooks give to the students the theorems and propositions and then the students are asked to provide a proof for these, knowing already that they are true.10 This teaching method imposes the axiomatic system on students (Freudenthal, 1973; Hanna, 1989; Douek, 2007). The teacher presents the axioms to the students and offers propositions to be proven. In this process the act of conjecturing, an act central to the understanding of the meaning of proof, is totally absent. In the words of Freudenthal, “it is a long way to come to this point in this or that system, and the pupil has to learn this fact, but he will never learn it if an axiomatic system is imposed on him. Didactically this could be a wrong procedure. It is the task of rigour to convince, but ready-made mathematics never convinces. To progress in rigour, the first step is to doubt the rigour one believes in at this moment. Without this doubt there is little learned by letting other people prescribe oneself new criteria of rigour” (Freudenthal, 1973, p. 151). According to Freudenthal (1973, p. 150) experimental mathematics (or the mathematics of free discovery as he calls it) “is much more important than that which is confined to axioms imposed by the teacher or textbook author, and there is no reason to claim that it is any less rigorous. There are levels of rigour and for each subject matter there is a level of rigour adapted to it; the learner should pass through the levels and acquire their rigour.”

What are then some appropriate teaching strategies for teaching geometry and proof that can replace the traditional one? Admittedly there have been attempts within mathematics education at all levels to change this traditional way of teaching and learning into one where the learning experiences are “more cooperative, more conceptual and more connected” (Dreyfus, 1999, p. 85), by asking students more frequently to explain their reasoning. According to Douek (2007, p. 165) the process of asking students to form their own conjecture “fixes very firmly” the conjecture in their minds and thus it becomes more meaningful to them. Research studies from all over the world support this claim (Chazan, 1990; Bussi Bartolini, 2009; Boero, Garuti, & Lemut, 2007; Parenti, Barberis, Pastorino, & Viglienzone, 2007).

As Mariotti (2007b) points out, there have been numerous research projects during the last decades which aimed at designing teaching methods for introducing students to proof and proving. One of the common features of these projects, according to Mariotti, is the attempt to engage students to an investigation of meaningful and thoughtful mathematical activities with an aim to establish a “mathematical community in the classroom” (Mariotti refers here to the work of: Bussi Bartolini, 1991; Arsac, 1992; Maher and Martino, 1996; Yackel and Cobb, 1996; Bussi Bartolini, 1998). In such a ‘community’ students are encouraged to discuss and argue for or against mathematical claims, a process that is connected to proving in mathematics. Discussion as a way of learning more effectively is of course not a new idea, and in the area of the

---

9We think that an appropriate teaching strategy which guides students to the discovery of the differences between empirical and deductive arguments can change the beliefs of students about proof and can eventually lead them to prefer deductive arguments over inductive ones in the context of mathematics, and our claim is supported by the research of Chazan (1993). As we will shortly explain, we believe that if this juxtaposition is placed at the beginning of the introduction of students to deductive proof—rather than after completing the traditional instruction about what deductive proof is, as Chazan’s research suggests)—results may improve even further.

10This way of teaching geometry is followed in Greece, where this research project took place.
teaching of geometry we already read about these considerations in Dina van Hiele’s writings from the year 1957 (Fuys et al., 1984, p. 13): “class conversation is an indispensable part of the class hour. [...] The class conversation is precisely directed towards stimulating the children to as lively a thinking as possible.”

Related to the teacher’s role in the didactics of geometry, Freudenthal suggests that the teacher should only proceed to giving out more information and answers to the students when they have reached the level of wondering why certain things in mathematics are the way they are: “The most important pedagogical quality is patience. One day the child will ask “Why?”. It is useless to start with systematic geometry prior to that day. Stronger yet, it can even be harmful. Geometry instruction should be a means of making the children aware of the power of the human mind - of their own mind. Are we allowed then to rob them of the privilege of making their own discoveries? The secret of geometry is the word “why”. He who does not want to be a spoil-sport should be able to keep a secret.” (In Freudenthal (1956), as cited in the translated version of Dina van Hiele’s dissertation by (Fuys et al., 1984, p. 48)). Dina van Hiele in her dissertation (Fuys et al., 1984, pp. 190-191) makes a similar point: “The first goal is not yet the building of a system of theorems, but the development of thinking. The class as a unit analyzes a certain level, and the pupils learn to analyze by participating in discussion. The teacher only guides discussions and provides favorable learning situations. Parts of a system of theorems are found, and only after insight has been gained into the structure of this system is it appropriate to try and build a system from a certain given set of axioms by using a deductive approach.”

Chazan (1990), in line with the above ideas, offers an approach to teaching geometry in which students are put in the position of mathematicians as they explore the properties of geometrical objects and their relations. This approach differs from the typical instruction since it includes the following points: “exploration and conjecturing; presentation of demonstrative reasoning as explanatory; treatment of proving as a social activity; and emphasis on deductive proofs as part of an exploratory process, not its end point” (p. 19). This approach to teaching geometry gives value to the conclusions students arrive at and gives them a chance to work like real mathematicians do. Students have thus the opportunity to gain a real insight into what might be the uncertainties that mathematicians come across in their area and to appreciate the use not only of deductive reasoning but also of the empirical arguments that are used on the way to ‘firm’ conclusions.

One of the first teaching strategies tested and reported for moving students from one level to the next is that of van Hiele-Geldof (1984a), who reports in her dissertation on the lessons she created for moving students from the visual to the descriptive level. In her dissertation, Dina van Hiele suggests that there are five phases of learning for moving from the visual to the descriptive Van Hiele level. These levels were later summarised by her in a different article (van Hiele-Geldof, 1984b, p. 223):

1. **Information** by means of representative material gathered from the existing substratum of empirical experiences in order to bring the pupils to purposeful action and perception;

2. **Directed orientation** which is possible when the child demonstrates a disposition towards exploration and is willing to carry out the assigned operations;

3. **Explication** through which subjective experiences are objectified and geometric symbols are formed;
4. **Free orientation** which is the willful activity to choose one’s own actions as the object of study in order to explore the domain of abstract symbols;

5. **Integration** which can be recognized as being oriented in the domain, as being able to operate with the figures as a totality of properties.

The above phases of learning can be applied for moving from any of the first three Van Hiele levels to the next. A more general description of these phases, not attached to the move between two specific Van Hiele levels, is given by van Hiele (1984b, p. 247) (note that the name of the first phase has been changed):

1. **Inquiry:** “The student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure.”

2. **Directed orientation:** “The student explores the field of investigation by means of the material. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually.”

3. **Explication:** “Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms. It is during this third phase that the system of relations is partially formed.”

4. **Free orientation:** “The field of investigation is for the most part known, but the student must still be able to find his way there rapidly. This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols.”

5. **Integration:** “The student has oriented himself, but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point, the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows. At the close of the fifth phase a new level of thought is attained.”

Based on the work of the Van Hieles, Fuys et al. (1985) developed and tested three modules for teaching properties of quadrilaterals, angle relationships for polygons and area of quadrilaterals. The study included also clinical interviews with some of the students. One of the main aims of the study was to find out whether it is “possible to follow a didactic as a way of representing material so that the thinking of the child is developed from the lowest level to higher levels [of thinking] in a continuous process” (Fuys et al., 1985, p. 9). The results of the study showed that the instructional method suggested had positive results in moving students from one Van Hiele level to the next for the topics dealt with in the module. It is suggested therefore that there should be further research for creating and evaluating teaching strategies for other topics. It will be our intention in this study to create a teaching strategy for introducing students to deductive proof, and assisting them to make the move from the descriptive to the theoretical Van Hiele level.
Chapter 2. Theoretical Framework

There are also a considerable number of research projects studying how students’ learning of proof and proving can be facilitated by Dynamic Geometry Environments (for example, Chazan and Yerushalmy, 1998; Marrades and Á. Gutiérrez, 2000; Jones, 2000; Patsiomitou and Emvalotis, 2010; Patsiomitou, Barkatsas, and Emvalotis, 2010). These dynamic environments are said to link informal argumentation with formal proof (Mariotti, 2007b, p. 193). Mariotti (2007b) offers a literature review on recent research related to Dynamic Geometry Environments. A rather critical point in teaching strategies using Dynamic Geometry Environments is to make sure that students will not jump to general conclusions based on the exploration of several specific cases, something which is likely in such environments (Mariotti, 2007b, p. 194).

2.8 Conclusions and how our research builds on the literature

We believe that students need to become aware of the distinction between general/deductive and specific/empirical arguments at the beginning of their introduction to (formal) deductive proofs and the Euclidean axiomatic system (regardless the tools and teaching strategies employed). When this distinction is made, the need for deductive arguments becomes more obvious and an axiomatic system is introduced to the students only when they have seen the need for it.

The distinction between empirical and deductive arguments should not be imposed on the students, but rather the students need to be exposed to various arguments, to compare them and argue about their advantages and disadvantages when it comes to proving a ‘for all’ case and therefore to discover on their own the need for such a distinction. To summarise the above, we think that a natural (and effective) learning sequence for proof in geometry should at least include the three stages presented in Table 2.4.

<table>
<thead>
<tr>
<th>Learning Sequence for the Introduction to Deductive Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stage One</strong></td>
</tr>
<tr>
<td><strong>Stage Two</strong></td>
</tr>
<tr>
<td><strong>Stage Three</strong></td>
</tr>
</tbody>
</table>

Table 2.4: The stages of the learning sequence we propose for introducing students to proof.

During these stages students should be encouraged to discuss, conjecture and argue about their findings. We think that guided reinvention is therefore a teaching strategy that can be quite useful and effective. None of the literature that we were able to find takes care of introducing students to the distinction made in the first stage of the above learning sequence before they start producing deductive proofs. Most often the studies concentrate on the second stage, and examine strategies that could lead students to the understanding of deductive arguments which constitute mathematical proof. In our research we will attempt to show how a module for teaching proof that starts by guiding students to discover the distinction discussed above at the beginning of their introduction to formal proof in mathematics, may have positive effects in the learning of proof.

The teaching of the second stage can be organised in various ways. We think that guided reinvention accompanied by frequent group work and discussion organised based on the five phases suggested by the Van Hieles is an appropriate way. We have designed a teaching strategy

\[\text{11}\text{See also (King & Schattschneider, 1997) for more information on the use of dynamic software in the teaching of geometry.}\]
incorporating the three learning stages discussed here and we tested it in the classroom in order
to evaluate it. The detailed design and methodology of the study will be described in Chapter
5, and the results will be offered in Chapter 7.

Various studies have been carried out in order to design and test effective strategies for
proof instruction. To mention only some of the strategies that have been used in research for
this purpose we provide a list of the ones we identified (see Table 2.5). This list is by no means
complete and the use of one teaching strategy does not exclude the others. That is, some of the
following strategies can be (and have been) combined in a single teaching experiment.

<table>
<thead>
<tr>
<th>Teaching Strategies and Tools</th>
<th>Explanation</th>
<th>Research Projects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic Assessment</td>
<td>Solving open ended problems in the presence of an interviewer who provides hints and assesses progress.</td>
<td>Koedinger (1998)</td>
</tr>
<tr>
<td>Guided Reinvention</td>
<td>Conjecturing, arguing, and discussing are at the center of the learning experience. Students 'try out' new ideas and check their validity. The teacher’s role is to guide and assist the students rather than to impose an axiomatic system on them.</td>
<td>Our study, Boero et al. (2007), Bussi Bartolini (2009, 1991, 1998), Chazan (1990), Parenti et al. (2007), Mariotti (2007a)</td>
</tr>
</tbody>
</table>

Table 2.5: Strategies and tools that have been used in research to introduce students to proof.

Based on the literature described in this chapter we formed our ideas presented above about
creating a teaching strategy that would effectively introduce students to deductive proof in ge-
ometry. In the following chapters we will describe our specific aims and research questions as
well as our methodology for carrying out our research.
Chapter 3

Aims and Research Questions

3.1 Aims

In Chapter 2 we discussed some of the main theories related to students’ reasoning: the Van Hiele theory, Piaget’s theory, Schoenfeld’s theory, and Harel & Sowder’s proof schemes. We think there are certain aspects to these theories which, if combined, might result in a better understanding regarding two important aspects of mathematics education: students’ levels of thinking while they engage in geometry activities, and successful strategies for helping students’ transition from the descriptive to the theoretical Van Hiele level.

More specifically, the first aim of our study is:

To further investigate the correspondence that we believe exists between the first three Van Hiele levels and the proof schemes of Harel & Sowder (see Section 2.5).

The second aim is:

To examine and describe students’ kinds of reasoning in geometry while they move from the descriptive to the theoretical Van Hiele level.

Our third aim is:

To create and test a teaching strategy that facilitates the move from the descriptive to the theoretical Van Hiele level.

Related to the aims described above is the need to look closely at the beliefs of students about mathematics in general and about proofs in particular. This will help us understand better the way students interpret and solve proof tasks. For this reason, another aim of this study is:

To find out the beliefs and conceptions of the participating students about mathematics and proof both before and after our intervention (building on the work of Schoenfeld (1989) and Healy and Hoyles (1998)).
3.2 Research Questions

Our main research question is formulated as follows:

\[(RQ)\text{ What kind(s) of reasoning do students use for proving while they are at different levels of thinking in geometry as well as during the transition from the level of informal-visual proofs to the level of formal-logical proofs, and how do these kinds of reasoning relate to students’ beliefs about proof?} \]

There are three subquestions related to the above research question:

SQ.1 What are the beliefs of students about the nature and the meaning of mathematical proof, and mathematics in general, and how do these change when they move from the descriptive to the theoretical Van Hiele level?

SQ.2 Is there a correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking?

SQ.3 Can we make the distinction between deductive and inductive arguments understandable to students before they start producing deductive proofs for themselves, and how can this process be planned according the Van Hiele teaching phases?

The first subquestion (SQ1) is answered by asking students’ conceptions of proof and mathematics before and after our intervention, and by examining the discussions between students and teacher as well as among students. To answer the second subquestion (SQ2) we checked the kinds of proof schemes, as defined in the literature (Harel and Sowder, 1998; Harel, 2007), that are employed by students when proving geometrical propositions. We also checked whether there is a consistent correspondence between the proof schemes used and the students’ Van Hiele levels of thinking. Finally, to answer the third subquestion (SQ3) we designed a series of lessons based on the Van Hiele phases of teaching which specifically aimed at making the distinction between deductive and inductive arguments understandable to students before they start producing deductive proofs for themselves, and we assessed this method by looking at students’ progress through the lessons.

3.3 Hypotheses

We have the following hypotheses:

H.1 As students progress from the descriptive to the theoretical level, their beliefs about mathematics and proof change.

\(^1\) All the research questions are specific to the context of this research study. Whenever the word ‘students’ is used, what is meant is ‘students of the Gymnasium I’ level of the Greek public school curriculum, who have already been taught the geometry chapter of the school book (Argyakis, Bourganas, Mentis, Tsikopoulou, & Chrysovergis, 2007).’

\(^2\) We call this level the ‘descriptive Van Hiele level’ or ‘Van Hiele level 2.’

\(^3\) We call this level the ‘theoretical Van Hiele level’ or ‘Van Hiele level 3.’
3.3. Hypotheses

H.2 Given the correspondence\(^4\) between the Harel & Sowder proof schemes and the Van Hiele levels of thinking, there should be a correlation between corresponding Van Hiele levels and Harel & Sowder proof schemes.

H.3 The move of students from the descriptive to the theoretical Van Hiele level can be described in three stages (related to the arguments students use during proving) which were presented in Table 2.4; a teaching strategy based on these three stages and the five Van Hiele teaching phases—like the one used in this research study—can facilitate this move.

\(^4\)See Section 2.5 for a definition of this correspondence.
Chapter 4

Research Setting

4.1 The place

The research took place in a Greek lower secondary school\(^1\) in Katerini. Katerini is situated in the north of Greece. The school is an ordinary Greek public school. This means that there is no special selection method for students; students who live close to the school go to this school. Thus, one finds students coming from all social backgrounds, as is the case in most public Greek schools, and the abilities of the students are mixed. We chose to do our research in a public school in order to ensure that our sample would be heterogeneous.

4.2 The Greek educational system

Education in Greece is public, and compulsory for all children within the age range 6-15. Compulsory education includes Primary Education (from 6 to 12 years old) and Lower Secondary Education, the Greek ‘Gymnasium’ (from 12-15 years old). Higher Secondary Education, the Greek ‘Lyceum’ (from 15 to 18 years old), is not compulsory and consists of two school types: the Unified Upper Secondary Schools (Eniaia Lykeia) and the Technical Vocational Educational Schools (TEE).

Having a public educational system means (in the case of Greece) that students are not obliged to pay anything at all, not even for books (which are the same for all students in all public schools throughout Greece), during their entire education, from the Elementary School to University. Greek students have thus a constitutionally secured right to free education. However, the level of education within public schools is not always high, therefore students often get help outside school hours, especially when they enter Lyceum, where the demands are higher. At the Gymnasium level, to which our study refers, very few students get help outside school hours.

4.3 Geometry in the Greek curriculum

The way geometry is taught in Greece matches the description offered by Mariotti (2007a, pp. 287–288) of the way geometry is taught in Italy. In Greek secondary schools students are

\(^1\)The school’s web page is: http://5gym-kater.pie.sch.gr/
taught geometry for 5 years (ages 12 to 17). In the first two years of this period (grades A’ and B’ of Gymnasium), they are intuitively introduced to definitions, basic geometrical concepts and their properties. For example: lines, angles, points, symmetry, triangles, parallelograms (at age 12–13), circles and solids (at age 13–14). They are for the first time asked to prove geometrical propositions at the age of 14–15 (grade I’ of Gymnasium), but still they are not explicitly introduced to the idea of a formal-theoretical geometrical proof. Most of the tasks students have to do involve calculating and drawing. In other words, in the Greek Gymnasium students are introduced only to practical geometry which is based on intuition and requires empirical reasoning.

It is only when they enter the upper secondary school (Lyceum) that students are introduced to theoretical geometry. According to the A’ Lyceum geometry textbook the difference between practical and theoretical geometry is that in the latter one has to systematically use logic in order to establish knowledge whereas in the former one depends on measuring and other practical (and thus imperfect from a mathematical point of view) procedures. The aim of mathematics education at the A’ Lyceum level, according to the school textbook, is to introduce students to the formal-deductive method of proof.

The focus of our research is the transition from Gymnasium level geometry to the level of geometry first introduced at Lyceum. Our research has thus been carried out with students that have already been taught the geometry of the last Gymnasium level and involves trying to bring these students to the next level and observe their kinds of reasoning before, during and after the transition. Below we offer a more detailed description of the geometry books of the Greek Gymnasium in order to understand better the background and level of the students involved in our study.

4.4 The geometry books of the Greek Gymnasium

We will examine in this section how the Van Hiele levels are encountered in the Greek Gymnasium geometry textbooks. For doing this we need to establish first the descriptors that correspond to each Van Hiele level. This has already be done by Fuys et al. (1985) and below we present the results of their study.

Descriptors of Van Hiele levels in the literature

Fuys et al. (1985) give a description of the characteristics of each Van Hiele level and provide a detailed characterization of the levels by formulating “specific behavioral and level descriptors” (Fuys et al., 1985, p. 56). In their study they use a slightly different level scale than that used by Van Hiele. Their level 0 which is the level of “recognition of shapes as a whole” (p. 1) corresponds to the Van Hiele visual level. Their levels 1-2 are the levels of “progressing to discovery of properties of figures and informal reasoning about these figures and their properties” (p. 1). Level 1 corresponds to the Van Hiele descriptive level, and level 2 is a level of transition from the descriptive to the theoretical level. Thus, level 2 does not correspond directly with any of the Van Hiele levels as described in van Hiele (1986). Their level 3 corresponds to the Van Hiele theoretical level, and their level 4 to the formal logic level. There is no level in Fuys et al. (1985) that corresponds to the nature of logical laws Van Hiele level. Table 4.1 shows the correspondence of the Van Hiele levels as described by Fuys et al. (1985) with the Van Hiele
levels as described in van Hiele (1986).

Numbering of Van Hiele levels according to Fuys et al. (1985)

| Level 0: The student identifies, names, compares, and operates on geometric figures according to their appearance. |
| Level 1: The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g. by folding, measuring, using a grid or diagram). |
| Level 2: The student logically interrelates previously discovered properties/rules by giving or following informal arguments. |
| Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems. |
| Level 4: The student establishes theorems in different postulation systems and analyzes/comparies these systems |

Numbering of Van Hiele levels according to van Hiele (1986)

| Level 1: Visual |
| Level 2: Descriptive |
| Level 3: Theoretical level |
| Level 4: Formal Logic |

Table 4.1: Correspondence of the Van Hiele levels as described in Fuys et al. (1985) with the Van Hiele levels as described in van Hiele (1986).

Van Hiele levels in the Greek textbooks

The Gymnasium A’ book  At this level the book (Argyrakis et al., 2007) includes a chapter on geometry with practical geometry activities. These activities seem to be at the descriptive level. Students are asked to draw, look at, and play around with specific images, and then important properties of the images as well as of classes of images are stressed. The students are brought to situations in which they can see that images are characterized by certain properties, in order to become able, eventually, to recognize images by their properties. Most activities of the Gymnasium A’ book involve empirical constructions which are meant to probe students to acquire an insight of the properties of the images used. All properties stated in this book are verified by measurements or constructions (e. g. paper folding, using ruler or protractor etc.).

For example, students are involved with identifying and testing relationships among components of figures (level descriptor of level 1 in Fuys et al., 1985, p. 60), such as equality of angles belonging to triangles, or they are expected to discover properties of specific figures empirically and to generalize these properties for the class of the figures (level descriptor of level 1 in Fuys et al., 1985, p. 60), for example they are asked to describe the common characteristics of the angles that they can identify around them, without being affected by their material nature (Vandoulakis, Kalligas, & Markakis, 2007, p. 153). In this book, empirical proof is used whereas deductive proof and its importance for ensuring the truth of geometrical claims are not mentioned for the most part of the book. This makes sense since “it is out of the question that deduction be understood by pupils before they have surveyed such a range of properties [relationships between properties of figures and derivations of new properties of figures from some already known properties], and this is only possible after the attainment of the second level [descriptive level] of thinking” (van Hiele, 1986, p. 64). After finishing level A’ of Gymnasium
the students are somewhat familiar with the language that belongs to the descriptive level of thinking, which includes names of figures and of figure properties.

**The Gymnasium B’ book**  At this level the chapter on geometry of the book (Vlamos, Drout-sas, Presvis, & Rekoumis, 2007) offers activities at the descriptive level (for example, students are asked to find the area of a figure by subdividing the figure into other figures which they already know how to calculate the area of) and seems to be aiming also at giving some first ‘hints’ to students about the transition which leads to the theoretical level (for example, students are given some information about a figure, and they are asked to reach a conclusion which they have to justify based on the information given). The book deals with fewer concepts than the ones of the previous year but it goes into more depth on each concept. General propositions are stated, which are supported mainly by inductive arguments based on observations. Relationships among properties are formulated in an informal way (that is, not proved by deduction but based on observations).

An example of such a relationship is: “Every inscribed angle is equal to the half of a central angle with a corresponding arch” (Vlamos et al., 2007, p. 176). In this book, the students are for the first time introduced to a theorem, the Pythagorean Theorem, and to a proof of the theorem that is based on a construction and on measuring of areas of several shapes. This may be considered as an opportunity for students to discuss informally what theorems are and why they are useful for mathematics.

**The Gymnasium Γ’ book**  At this level, which is also the level of the students we worked with in our project, the book (Argyrakis et al., 2007) aims at building on the descriptive level in order to start introducing students to the theoretical level. The book is mainly dealing with the transition level (Level 2 according to Fuys et al., 1985) from the descriptive to the theoretical level. The book provides activities which correspond to level descriptors like the following: the student “gives informal deductive arguments”, “gives more than one explanation to prove something”, “identifies and uses strategies or insightful reasoning to solve problems”, “recognizes the role of deductive argument and approaches problems in a deductive manner” but “does not grasp the meaning of deduction in an axiomatic sense” (Fuys et al., 1985, p. 67–68).

The teacher’s book accompanying the students’ Gymnasium Γ’ textbook (Argyrakis, Bourgnas, Mentis, Tsikopoulou, & Chrysovergis, 2009) includes guidelines for the teacher. It is worth quoting some parts (our translation): “In many sections the proof of the basic proposition of the section is given. Students of Γ’ Gymnasium level are now in the position to understand better than in the previous years the proving process. For this reason in this book there are opportunities offered for them to practice on simple and short proofs. Initiation of students into the process of proof aims at making students realize the power of proof over the control of the truth of a proposition” (Argyrakis et al., 2009, pp. 9–10). The book uses a basic scheme (p. 10) for the introduction of a new concept (see Figure 4.1), and teachers are required to follow this scheme while teaching:

Most of the sections (in the geometry chapter) of the book start with a concrete activity which is supposed to lead the students to the speculation of a law that is introduced in that section. Then the formulation of the law is offered by the book. Whenever this is appropriate, that is, not always (and in fact in less than half of the cases in the book), a proof of the law is also provided. This is always followed by applications of the law. We conclude that the deductive
process is not always present in the book when a new concept or proposition is introduced to the students. However, as we already mentioned, the students are asked to apply the law for solving exercises which requires to reason deductively in an informal way. Therefore, it seems that the right-hand part of the scheme in Figure 4.1 is used only informally in the book (here “informally” means without “grasping the meaning of deductions in an axiomatic sense,” Fuys et al., 1985, p. 68) and mainly through the exercises/applications of each section.
Chapter 5

Methodology, Teaching Design and Analysis Framework

Given the time frame and size of this research, our study is a small-scale study. The character of our study is both exploratory and explanatory, that is, we aim both at generating certain hypotheses that might be tested in larger studies in the future and at testing some already existing theories (Van Hiele, Harel & Sowder). Moreover, our study is mostly qualitative but has also some quantitative characteristics.

In the following sections we describe our methodology and how it affected the design of our study. The chapter is structured around the research questions and hypotheses which were presented in Chapter 3. Thus, each of the following sections explains the method we used to answer our research questions and check our hypotheses. Below we give an overview of the parts of this chapter.

Our main research question is the following:

What kind(s) of reasoning do students use for proving while they are at different levels of thinking in geometry as well as during the transition from the descriptive to the theoretical Van Hiele level, and how do these kinds of reasoning relate to students’ beliefs about proof?

In order to answer this question we first had to choose a method for identifying the Van Hiele level, the kinds of reasoning, and the beliefs about proof of the students before and after our intervention. This would also allow us to answer our first subquestion and to confirm or reject our first hypothesis. We will present the methods we chose for identifying the Van Hiele level of students, their kinds of reasoning and their beliefs about proof in sections 5.1, 5.2 and 5.3 respectively.

Our second subquestion was about the correlations between students’ proof schemes and

---

1SQ.1: What are the beliefs of students about the nature and the meaning of mathematical proof and how do these change when they move from the descriptive to the theoretical Van Hiele level?

2H.1: As students progress from the descriptive to the theoretical level, their beliefs about mathematics and proof change.
their Van Hiele levels. It was our hypothesis that as students progress from the descriptive to the theoretical Van Hiele level their proof schemes will change from empirical to deductive. In section 5.4 we explain how we checked this hypothesis.

Finally, our third subquestion asked for a teaching method which would be planned according to our three stages idea and the five Van Hiele teaching phases which were explained in Chapter 2. Our related hypothesis was that such a teaching method would facilitate the move of students from the descriptive to the theoretical Van Hiele level. The design of our teaching method, as well as the method we used to evaluate it, is presented in detail in Section 5.5.

5.1 Identifying the students’ Van Hiele levels

In order to make a first selection of our study sample we used Usiskin’s Van Hiele test (see Appendix A.1) for the whole population of the third grade of the chosen school (n = 85). The Usiskin Van Hiele test is a multiple choice test according to which one can classify students to Van Hiele levels. The original test includes 25 questions; these questions are divided in 5 subgroups each of which corresponds to a Van Hiele level, from level 1 to level 5. We used only the first 20 questions of this test, which correspond to Van Hiele levels 1 through 4, since the 5th Van Hiele level is not encountered at the highschool level. We translated the test into Greek before distributing it to the students (the version which we used in our research can be seen in Appendix A.2). We used the same test at the end of our intervention, in order to check the progress of students in their Van Hiele level.

5.2 Identifying the students’ kinds of reasoning

For identifying the kinds of reasoning used by the students and for classifying them into categories, we used the studies of Hoyles and Healy (2007) and Harel and Sowder (1998). Both these studies were discussed in Chapter 2. Here we will discuss separately how we used each of these studies in our research.

5.2.1 Adapting the instruments of the JPSM project to our research

The focus of our research is similar to that of the JPSM project, the aims of which were described in section 2.4.4. Our research is small scale and thus inevitably focuses on the individual student and classroom, however it is still comparable to some extent to the JPSM. The specific aims of JPSM were to investigate (Hoyle & Healy, 2007, p. 84):

3SQ.2: Is there a correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking?
4H.2: Given the correspondence between the Harel & Sowder proof schemes and the Van Hiele levels of thinking there should be a correlation between corresponding Van Hiele levels and Harel & Sowder proof schemes.
5SQ.3: Can we make the distinction between deductive and inductive arguments understandable to students before they start producing deductive proofs for themselves, and how can this process be planned according to the Van Hiele teaching phases?
6H.3: The move of students from the descriptive to the theoretical Van Hiele level can be described in three stages (related to the arguments students use during proving) which were presented in Table 2.4; a teaching strategy based on these three stages and the five Van Hiele teaching phases—like the one used in this research study—can facilitate this move.
5.2. Identifying the students’ kinds of reasoning

(i) The characteristics of arguments recognized as proofs by high-attaining students aged 14–15 years,

(ii) The reasons behind their judgments, and

(iii) The ways they constructed proofs for themselves.

A first comparison to our research shows that there are several important similarities. First of all, our sample shares similar characteristics: 14–15 year old, high attaining students on the one hand (JPSM), and 14–15 year old, high-attaining, motivated students on the other (our research). The difference between the two samples is only that in our case the students that participated in the research were volunteers so they were mainly motivated students (at least for the lesson of mathematics) who were willing to spend some of their time on extra lessons outside normal school hours (we will talk in more detail about the characteristics of our sample in a later chapter).

The first and third—see (i) and (iii) above—aims of the JPSM are immediately related to our main research question. More specifically, the first part of our research question is related to aim (iii) of the JPSM, since an investigation of the ways students construct proofs for themselves may lead to an understanding of the kinds of reasoning students use at different levels of thinking. The second part of our research question is related to aim (i) of the JPSM, since an investigation of the characteristics of arguments recognized as proofs by students may lead to an understanding of students’ conceptions about proof.

Given the similarity between the two research projects we used some of the research instruments (sometimes in slightly modified versions) produced by JPSM. These instruments and how we adapted them are explained in the following paragraphs.

**Research Instruments created by and used in the JPSM project**  
One research instrument used in JPSM was a student proof questionnaire which aimed at probing students’ “views of proof from a variety of perspectives” (p. 86). It contained three parts (Hoyles & Healy, 2007, p. 86):

1. “Students were asked to describe in an open format what they thought about proof and its purposes. Their responses were coded according to a simplified version of de Villier’s classification of the functions of proof, using the categories, truth (verification), explanation and discovery together with a fourth category ‘none/other’ if students wrote nothing or if their contributions appeared to be irrelevant.”

2. “Students were presented with mathematical conjectures and a range of arguments in support of them in a multiple-choice format. They were asked to make two selections from these arguments: the argument that would be nearest to their own approach and the argument they believed would receive the best mark from their teacher.”

3. “Students were asked to judge all the arguments according to their validity and how far they were convincing and explained the conjecture. Two conjectures were included in geometry, one familiar and one unfamiliar.”

The researchers based their choice of arguments presented to the students as proofs of the conjectures on Van Dormolen’s and Balacheff’s analyses (p. 87). There were two empirical
arguments, an argument which “relied on common properties or a generic case” (p. 87), two
deductive arguments (one correct and one incorrect), and a narrative, valid argument which is
commonly used in English mathematics textbooks.

The specific questions asked of the students in order to find out their ideas about the validity
of the arguments, their generality and how convincing they are, were the following:

• For validity:
  1. X’s argument shows that the statement is always true.
  2. X’s argument only shows that the statement is true for some triangles.

• For generality:
  Suppose it has now been proved that, when you add the interior angles of any triangle,
your answer is always $180^\circ$. Zoe asks what needs to be done to prove whether, when
you add the interior angles of any right-angled triangle, your answer is always $180^\circ$. Tick
either A or B.

  (A) Zoe doesn’t need to do anything; the first statement has already proved this.
  (B) Zoe needs to construct a new proof.

• For convincingness:
  1. X’s argument shows you why the statement is true.
  2. X’s argument is an easy way to explain to someone in class who is unsure.

How we adapted the JPSM’s research instruments to our context We adapted the Hoyles
and Healy test to the needs of our research and we used parts of it both as pre- and post-tests.
The adapted version also contained three parts:

Part I: Constructing a proof for a familiar conjecture This part was not taken from the
JSPM project. We created and used the first part of the test only at the beginning of our in-
tervention in order to identify the proof schemes of students at that moment (see also Section
5.2.2). The task was to show that the (familiar) conjecture ‘when you add the interior angles of
a triangle the sum is always $180^\circ$’ is true (see Task ANGLE SUM in Appendix B.2). The reason
for giving this question first as an open format question was to reveal the kinds of arguments
that students use for proving familiar conjectures before they are presented with other arguments
for showing the same conjecture (Part II of the questionnaire) and before our intervention. It is
important to mention that students had possibly heard a couple of times a deductive proof for
this conjecture, however, in general students of this level in Greece are not yet introduced to
deductive proofs as such. Most often the conjectures of geometry are introduced to students
as already given facts, and students are presented only with some examples which verify the
conjecture in order to be convinced of the validity of the conjecture.

7It is included at the end of the book of the previous year as a worked example in the book, but also their teacher
had mentioned it to them that year in class, however without insisting on it. (Personal communication with the
teacher.)
Part II: Choosing a proof of a familiar conjecture

After revealing their arguments in favour of the above conjecture in an open format (Part I), students were provided with a range of arguments in support of that conjecture and were asked to make five selections (instead of the two that were included in the JPSM project) from these arguments: the argument that would be nearest to their own approach, the argument they believed would receive the best grade, the argument that they found most convincing, the argument that they found easiest, and the argument that they found most difficult. In addition to the range of arguments used in the JPSM project, we included an argument borrowed from Harel and Sowder (1998) which was a causal and valid argument (see the last argument in Task CHOOSING Appendix B.3). This means that it was an argument which can easily be generalized for all triangles and which shows a reason why the sum of the angles of any triangle is $180^\circ$. The specific choices the students had to make were the following:

- Whose method would you choose in order to check whether the above statement is true or false? Explain why.
- Whose answer you find more convincing and why?
- Whose answer you think would get the highest grade in a test and why?
- Whose answer is easier for you to understand and whose is more difficult?

After making their choices the students had to answer specific questions related to the validity of the arguments, and how convincing they are. We used the same questions suggested by the JPSM project, although only for 4 out of the range of 8 arguments. We made sure that we included all 4 different kinds of arguments in our range. The questions were the following:

- For validity:
  1. X’s argument shows that the statement is always true
  2. X’s argument only shows that the statement is true for some triangles.

- For convincingness:
  1. X’s argument shows you why the statement is true
  2. X’s argument is an easy way to explain to someone in class who is unsure.

The arguments for which these questions were asked were: Stamatis’s (an empirical, inductive argument), Georgia’s (a theoretical, deductive argument), Lefteris’s (a narrative, practical, but valid argument), and Kalliopi’s (a theoretical, causal, valid argument). The questions can be seen in Appendix B.3.

Part III: Sharing views on the role of proof

In the last part of this test the students had to describe in an open format their ideas about the role of proof. Finally, the students had to answer a question (the same one used by the JPSM project) regarding the generality of the conjecture (see Task CONCEPTIONS ABOUT PROOF in Appendix B.4).
Chapter 5. Methodology, Teaching Design and Analysis Framework

5.2.2 Classifying the students’ kinds of reasoning based on the Harel & Sowder proof schemes

For the classification of students’ arguments and proof attitudes, we used the proof schemes of Harel and Sowder (1998), a description of which was offered in Chapter 2. We created the codes presented in Table 5.1 for these proof schemes.

<table>
<thead>
<tr>
<th>Proof Scheme</th>
<th>Code</th>
<th>Subcategory</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>External Conviction</td>
<td>EXT</td>
<td>Ritual</td>
<td>RIT</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Authoritarian</td>
<td>AUTH</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Symbolic</td>
<td>SYMB</td>
</tr>
<tr>
<td>Empirical</td>
<td>EMP</td>
<td>Inductive</td>
<td>IND</td>
</tr>
<tr>
<td>Deductive (Transformational)</td>
<td>DED</td>
<td>Causal</td>
<td>CAUS</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Constructive</td>
<td>CONS</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Contextual (Greek Axiomatic)</td>
<td>GRAX</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Contextual (Arithmetic Symbolic)</td>
<td>ARITH</td>
</tr>
</tbody>
</table>

Table 5.1: Codes for the Harel & Sowder proof schemes.

5.3 Identifying the students’ beliefs about mathematics and proof

For identifying the students’ beliefs we used various questionnaires which are described below. For students’ beliefs about mathematics, we used a questionnaire by Schoenfeld as well as our own questionnaire. Our own questionnaire, together with a task including questions from the Hoyles and Healy (2007) study was used also for determining students’ beliefs about mathematical proof. In the three short sections below we give a description of these three research instruments.

5.3.1 The adapted Schoenfeld questionnaire

After looking at the relevant literature of previous research with similar aims (see Chapter 2, Section 2.4) we chose to use the methods of Schoenfeld (1989) for determining students’ beliefs about mathematics and proof. In our study we used a modified version of Schoenfeld’s questionnaire in order to check the beliefs of the students participating in our research both before and after the intervention. We used a translation of the modified version of the questionnaire which can be seen in Appendix A.3. Given that the aims of our research were different from those of Schoenfeld, we used methods for analysing our results which were different than the methods used by Schoenfeld. More specifically, we examined the change in the beliefs of our students as a group by administering the Schoenfeld questionnaire both before and after the intervention and then comparing the results, using descriptive statistics.

5.3.2 Our questionnaire about what the students believe they learned

We created a questionnaire (see Appendix A.4) to distribute at the end of our lessons in order to find out the reflections of students on the lessons. More specifically, we wanted to know what the students believe they learned during the lessons, and how they think the lessons affected their
way of thinking as well as their ideas about mathematics and proof. We examined the change in the beliefs of our students as a group after the intervention by using descriptive statistics.

5.3.3 An adapted version of certain JPSM project tasks

We used some of the questions in Task CHOOSING (see Appendix B.3) and Task CONCEPTS ABOUT PROOF (see Appendix B.4) in order to find out the ideas of students about the convincingness, validity and explanatory power of certain types of arguments both before and after our intervention. This would enable us to understand whether students consider mathematical proof as something which is valid and which can explain. For analysing the results, used descriptive statistics and to check whether the change of the students’ beliefs correlates to the change in their Van Hiele level or in their kinds of reasoning. We also compared our results to those of Hoyles and Healy (2007).

5.4 Determining the correlation between proof schemes and Van Hiele levels

As discussed in Section 2.5 there is an obvious correspondence between the Harel & Sowder proof schemes and the Van Hiele levels of thinking. More specifically, we saw (Table 2.3 on page 17) that at the descriptive Van Hiele level students are expected to possess mainly empirical proof schemes while at the theoretical level students are expected to possess mainly deductive-transformational proof schemes. Therefore, it is one of our hypotheses (see hypothesis H.2 in Section 3.3) that, if the test for measuring the Van Hiele level of students and our methods for classifying students’ reasoning to Harel & Sowder proof schemes are valid, there will be a strong correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking.

Our methodology for checking this was to use: (1) the Usiskin Van Hieletest to determine the Van Hiele level of the students before and after the intervention, and (2) the coding scheme for Harel & Sowder proof schemes shown in Table 5.1 for determining the proof schemes of students before and after the intervention, and to use appropriate statistical tests to determine the correlation between the two.

5.5 Designing and evaluating a teaching strategy

We designed a cycle of lessons in order to first make students aware of the distinction between inductive and deductive arguments and to then give opportunities to the students to produce deductive proofs for themselves. As was discussed in Chapter 2 we think that the move from the descriptive to the theoretical Van Hiele level, which was the aim of our lessons, can be done in three stages and at the same time follow the five Van Hiele teaching phases. In the two sections below we present the way we designed our lessons in the three stages we propose (Section 5.5.1), and the way the lessons fit with the five Van Hiele teaching phases (Section 5.5.2). Finally, we present the method we used for evaluating our teaching strategy (Section 5.5.3).
5.5.1 The move from the descriptive to the theoretical level in three stages

We propose that the move from the descriptive to the theoretical Van Hiele level for the concept of proof can be organised in three stages (see hypothesis H.3 in Section 3.3). These stages along with the goals in each stage and the tasks we designed for achieving them are presented below.

1. Understanding the distinction between empirical and deductive arguments. One begins to make the distinction between empirical and deductive arguments when one discovers the basic differences between general (thus more convincing) arguments and arguments specific to the figure (empirical and thus not convincing for all cases). This may happen after evaluating the advantages and disadvantages of both. This first ‘discovery’ which makes one aware of the distinction between empirical and deductive arguments can be thought of as the first part of this stage. The second part would then be when students actually understand this difference through producing different kinds of arguments and classifying them as empirical or deductive.

2. Abandoning the use of empirical arguments in favor of deductive ones. This stage follows from the realisation during the previous stage that empirical arguments are not convincing for all cases. After this realisation, one (by definition) starts showing a preference towards deductive arguments. However, coming up with deductive arguments is much more difficult than coming up with empirical ones, especially when you are doing it for the first time. Therefore, one may stay in this stage for a long period. Moreover, depending on the individual, this stage might take longer or less long. After sufficient practice, finding a good deductive argument starts happening more naturally. During this stage there might be cases when, in the absence of an idea for a general, deductive argument, the student may offer an empirical argument which although not convincing enough may be considered a better option than offering no argument.

We believe that making explicit, and practicing on, the idea that certain properties of figures remain unaltered after applying transformations on the figures may assist the students to making conjectures that hold generally and to finding a deductive proof. This process can be assisted by the use of dynamic software where the transformations are easy to see, however we did not use such software in our research because of the lack of computers in the context of our research. Instead, we used the idea of giving ‘movement’ to a figure in order to introduce to the students this character of generality of certain properties of figures in Euclidean geometry, and we expected that the students would use this idea in their individual attempts to prove their conjectures. This idea is also central to the transformational proof schemes, and especially to the subcategory of causal proof schemes, where the reasons that cause a general property are taken into consideration.

3. Starting to see the axiomatic structure of Euclidean geometry. In this stage the student starts to see that for the arguments that he or she comes up with, only previously proven knowledge can be used, and that it all starts from a few basic assumptions, the axioms.

Based on the above learning sequence, we designed a six session lesson for the group of students selected. The above three stages that take place during the move from the descriptive to the theoretical level played an important role in the way we designed our intervention. Below we explain what the goal of our intervention was in terms of each stage and what our strategy for achieving this goal was.
Stage one: Understanding the distinction between empirical and deductive arguments. The first goal of our first lesson was to make the students aware of the advantages and disadvantages of empirical and deductive arguments. The second goal was to offer them opportunities to begin to understand this distinction better.

To achieve the first goal we designed an activity which would lead students to make the following discovery: empirical arguments are less convincing than deductive ones when it comes to proving that something holds for all existing cases rather than only for a few examples. More specifically, we asked students to share and discuss the advantages and disadvantages of their own arguments (as well as some arguments that had been provided to them) for proving that ‘the sum of the angles of a triangle is $180^\circ$’ (see the tasks in Appendices B.2 and B.3). For that first lesson we prepared a hand-out which would include some examples of arguments the students would have used as well as some of the arguments provided in Task CHOOSING which they had to complete the previous day (Task B.3). The handout would comprise of two columns: the first includes the arguments to be discussed and the second is blank. After discussion in the classroom the students would be asked to fill in the blank column with the advantages and the disadvantages that they believe each argument has. The role of the teacher is to assist and guide the discussion, especially for making clear that we are looking for arguments that will convince us that the proposition ‘the sum of the angles of a triangle is $180^\circ$’ is true for all triangles.

The second goal refers to students’ actual understanding of the distinction. For students to achieve real understanding we believe that they need to practice enough on their own the following: first, providing arguments to support conjectures and then commenting on the advantages and disadvantages of their own arguments (whether empirical or deductive); second, providing both empirical and deductive arguments for supporting the same conjecture. To promote understanding we designed a task (see Task QUAD in Appendix B.5) for the students to complete at the end of the first lesson. The task asks for an argument supporting the proposition ‘The sum of the angles of a quadrilateral is $360^\circ$’ as well as for a description of the advantages and the disadvantages of the argument. With this task we aimed at finding out to what extent the students are able to understand whether their argument is general enough to cover all cases, and thus begin to realise the distinction between empirical and deductive arguments. We also designed a task (see Task BIS in Appendix B.7) explicitly asking for two different kinds of arguments: one specific to the figure and one general. This kind of task can be useful both for the students (to understand better the distinction between the two kinds of argumentation) and for the teacher (for assessing the understanding of students regarding this distinction).

Stage two: Abandoning the use of empirical arguments. This stage may take a long time to be completed. The goal here is to bring students to a point where they will always try to find deductive arguments to prove mathematical statements and never stop looking for such arguments even when they have found an empirical argument because they will be able to realise that empirical arguments are not considered as proof in mathematics. For achieving this, one needs to practice a lot the act of proving and to have many opportunities to see that empirical arguments are not really convincing for all cases.

For this stage we created several tasks new to the students, where they could come up with their own conjectures (to avoid the cases where students would know already from what they learned before that a certain conjecture was actually true) and then try to come up with convincing arguments. We also planned discussion time for the end of each session regarding the
arguments and their convincingness.

**Stage three: Beginning to see the axiomatic structure of Euclidean geometry.** The goal of this stage is to lead students to a point where the axiomatic structure of Euclidean geometry is obvious to them. When students reach this stage they will be able to see that in order to prove anything in geometry you have to rely on the previous propositions you have proven, and that it all starts from a few propositions that we accept as true without proof: the axioms. Given that our intervention was short and that our students started at the empirical level, our goal was limited to just starting off our students with stage three and letting them get just a first glimpse of what this involves.

To achieve this goal, we planned the last lesson for a discussion that would lead students to stage three. We drew together with the students a ‘map’ of all propositions we had proven so far and the connection they had with previous propositions that were used for the proof, which would already have been proven at an earlier stage. By discussing this map and by asking students where they think this ‘going backwards’ process of connecting to the previous propositions proven would end, we planned to help students discover that there is a few assumptions called ‘axioms’ from which all propositions in geometry are derived.

5.5.2 The five Van Hiele phases for eight concepts in our teaching sequence

As was discussed in Chapter 2 (Section 2.7), the Van Hieles presented a teaching sequence comprised of five phases (inquiry, directed orientation, explication, free orientation and integration) for moving students from one Van Hiele level to the next. Given that we wanted to move our students from the descriptive to the theoretical Van Hiele level, all tasks of the three stages in our teaching sequence were designed also based on the five Van Hiele phases (as well as the three stages explained in the previous section).

We need to stress that the five Van Hiele phases are related to our three-stage learning sequence. Our three-stage learning sequence is designed especially for moving students from the descriptive to the theoretical Van Hiele level, for the concept of mathematical (deductive) proof. The teaching sequence suggested by Van Hiele is also designed for this move but can be applied to any concept. Moreover, our three-stage sequence outlines the specific tasks that need to be completed by the student during the learning sequence for proof in mathematics with emphasis given to the first stage (which we consider essential for the learning sequence to be successful); on the other hand the five Van Hiele phases are more general and allow variations for the specific tasks used. The two strategies are related in the following way:

- our *first stage* falls into the area of the *first two Van Hiele phases* (inquiry, directed orientation) and suggests a specific strategy for completing these two phases for the concept of proof in mathematics;
- our *second stage* falls into the area of the *Van Hiele phases 3 and 4* (explication, free orientation);
- our *third stage* falls into the area of the *Van Hiele phases 4 and 5* (free orientation, integration).

Below we repeat the description of each phase and we present the tasks we designed for each phase along with a short explanation of each task and its objectives.
### 5.5. Designing and evaluating a teaching strategy

1. **Inquiry:** “The student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure” (van Hiele, 1984b, p. 247).

Before the beginning of the first phase, students were asked to complete a test (see Task TRIANGLES in Appendix B.1) related to the properties of triangles, which was meant to reveal whether students were in the integration phase of the descriptive Van Hiele level for the concept of triangle and the properties related to triangles. The objective was to check how well students know the properties of figures that would be most often used for the proof tasks that would follow. This test was created by Burger and Shaughnessy (1986) and we adapted it and translated it into Greek in order to use it in our study.

The inquiry phase of the concepts introduced during our intervention was designed by creating the tasks described in Table 5.2. Task CHOOSING\(^8\) in combination with Task ANGLESUM and the discussion that follows it (see later phases) were the first tasks to be used in our intervention.

<table>
<thead>
<tr>
<th>Task</th>
<th>Concept</th>
<th>Description / Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANGLESUM (B.2)</td>
<td>Proving</td>
<td>Students are asked to find an argument for showing that the sum of the angles of a triangle is 180°. (This task is combined with the next.) The aim is to let students <em>get involved with the field of Euclidean geometry and the concept of providing an argument supporting or rejecting a mathematical proposition.</em></td>
</tr>
<tr>
<td>CHOOSING (B.3)- Part 1</td>
<td>Empirical vs. Deductive arguments (Proof), Proving</td>
<td>Students choose, among various arguments, one that they think shows why the sum of the angles of a triangle is 180°. Some of the arguments are empirical and some deductive. Students answer a series of closed and open questions on the convincingness of the arguments. Students will thus realise <em>there exist different kinds of arguments in the field under investigation.</em></td>
</tr>
<tr>
<td>OPPANG (B.6) - Part 1</td>
<td>Conjecture, Theorem</td>
<td>Students make their own conjecture (for a familiar example). The term ‘conjecture’ should not be used at this stage; it is going to be introduced in the explication phase, that is, only after the students have had some experience in making conjectures. Until that point conjectures should be referred to as ‘observations’. The aim is to <em>introduce students to the way mathematicians work when discovering new theorems.</em></td>
</tr>
<tr>
<td>AXIOMS-Part 1</td>
<td>Axioms, Undefined terms, Axiomatic System</td>
<td>The teacher initiates a discussion regarding the previous knowledge that students used in order to prove theorems in class. The teacher asks: ‘What knowledge did you use to prove these theorems? How far back do you think one can go in using previous knowledge for proving new theorems?’ A philosophical discussion is initiated and <em>students are thus directed to the question ‘where does it all begin?’</em> in the context of Euclidean geometry.</td>
</tr>
</tbody>
</table>

Table 5.2: Tasks used for the ‘inquiry’ phase of our teaching sequence.

It is important to note that the tasks of the inquiry phase need not all be given to students at the beginning of the teaching sequence. Various concepts are introduced to the students, and not all at the same time. Therefore, some tasks which belong in this phase are to be

---

\(^8\) An adapted version of a task created by Hoyles and Healy (2007).
used at other points of the intervention than the beginning. (See Table 5.7 on page 50 for a chronological overview.) The structure that we want the students to start discovering is highlighted in Table 5.2. Characteristic of the inquiry phase is the exploration of a new question without being ‘pushed’ towards the desired answer yet.

2. Directed orientation: “The student explores the field of investigation by means of the material. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually” (van Hiele, 1984b, p. 247).

Like before, there are various concepts students work towards during this phase. The tasks we designed (Table 5.3) aim at helping students to discover the characteristic structures of Euclidean geometry through the individual concepts introduced (*Proof, Proving, Theorem, Conjecture, Counterexample, Axioms, Axiomatic System*).

<table>
<thead>
<tr>
<th>Task</th>
<th>Concept</th>
<th>Description / Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHOOSING</td>
<td>Proof, Proving, Counterexample</td>
<td>A thorough discussion, guided by the teacher, on the advantages and disadvantages of all arguments that the students came up with in Part 1 of this task. Students come up with the advantages and disadvantages of each argument and the teacher does not use any of the terminology that is still unknown to the students (like ‘proof’, ‘deductive argument’, ‘inductive argument’).</td>
</tr>
<tr>
<td>QUAD</td>
<td>Proof, Proving</td>
<td>Students give an argument for a certain proposition familiar to them and are directed to offer the advantages and disadvantages of their own argument based to a great extent on the earlier discussion on advantages and disadvantages of other arguments.</td>
</tr>
<tr>
<td>BIS</td>
<td>Proof, Proving, Theorem, Conjecture</td>
<td>Students make their own conjectures for a certain figure and provide two kinds of argument: one practical that depends on the figure (empirical/inductive) and one theoretical that does not depend on the figure (deductive). Students are thus directed to understand better the distinction between these two kinds of arguments by experiencing coming up with both for the same conjecture.</td>
</tr>
<tr>
<td>OPPANG</td>
<td>Proof, Proving, Conjecture</td>
<td>Students provide arguments to convince others about the truth or the falsity of their own conjecture (for a familiar example). Students are thus expected to start abandoning the empirical proof schemes in favour of deductive ones.</td>
</tr>
<tr>
<td>IF-THEN</td>
<td>Implication, Inverse implication</td>
<td>An ‘if then’ proposition (for which the inverse does not hold) is to be proven by the students. Students are then asked to think about the inverse proposition and whether that would also hold. (This task belongs also to the free orientation phase for the concepts of Proving, Proof, and Theorem.) Students’ attention is directed towards the form of the proposition (implication).</td>
</tr>
<tr>
<td>TRIMID</td>
<td>Counterexample</td>
<td>All students have to give arguments for two conjectures. Without them knowing in advance, one of the conjectures should be true and one false. Students are this way directed to discover what kind of arguments are used for disproving conjectures (counterexample).</td>
</tr>
<tr>
<td>AXIOMS</td>
<td>Axioms, Undefined terms, Ax-</td>
<td>The teacher puts on display some of the main theorems proven earlier in class, and asks students what they had used to prove these theorems. As students reply, the teacher draws a backwards diagram of what other propositions were used. The objective is to arrive at a ‘dead end’ which is the axioms that form the basis of Euclidean geometry. This task directs students to discover specific axioms, namely the ones that were used during the intervention.</td>
</tr>
</tbody>
</table>

Table 5.3: Tasks used for the ‘directed orientation’ phase of our teaching sequence.
During this phase the tasks provide a lot of direction to the students by explicitly asking them what to do, in order to increase the chance they make the expected discovery. (The expected discoveries are highlighted in Table 5.3.) As soon as this expected discovery is formulated in a generalised way, this phase is over and the explication phase begins.

3. **Explication:** “Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms. It is during this third phase that the system of relations is partially formed” (van Hiele, 1984b, p. 247).

This phase takes place at various points during the lessons when the teacher presents the students with the mathematical terminology for describing the activities they have been engaged in until that point. The terms to be introduced in this phase are: Conjecture, Argument, Proof, Proving, Conjecture, Theorem, Axioms, Undefined Terms and Axiomatic System. In Table 5.4 we provide a description of how we planned this for our intervention.

<table>
<thead>
<tr>
<th>Task</th>
<th>Concept</th>
<th>Description / Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOW WE DISCOVER (C.4)</td>
<td>Conjecture, Proof, Theorem, Proving</td>
<td>A summary of the process by which mathematicians work to discover new truths in geometry. <em>The goal is to show to the students that all the steps they have experienced up to this point can be described in mathematical terminology,</em> and that what they have been doing is actually what real mathematicians do! The contents of this hand-out need to be discussed with the students to make sure they understand the connections of the terms to their own experiences.</td>
</tr>
<tr>
<td>WHAT WE DISCOVERED (C.5)</td>
<td>Theorem, Proof, Conjecture, Proving</td>
<td>A summary of all the discoveries students made during the lessons. The arguments that students characterise as more general and thus more convincing are what mathematicians call ‘proofs’ (deductive). The arguments specific to the figure (empirical) which are not convincing enough ‘for all’ cases do not fall under the definition of the word ‘proof’ as used in mathematics. During this phase students and teacher establish a common ground by giving the same meaning to the words proof, theorem, conjecture, counterexample and proving.</td>
</tr>
<tr>
<td>TRIMID-Ex.</td>
<td>Counterexample</td>
<td>After completing task TRIMID (B.10) students are introduced through a discussion to the term counterexample and its meaning in mathematics.</td>
</tr>
<tr>
<td>IF-THEN (B.13) - Part 2</td>
<td>Implication, Inverse Implication</td>
<td>The distinction between an implication and its inverse is made explicit and all formal terminology and notation is now introduced.</td>
</tr>
<tr>
<td>AXIOMS-Part 3</td>
<td>Axioms, Undefined terms, Axiomatic System</td>
<td>The teacher names the ‘dead-end’ reached in the corresponding task of the previous phase. <em>The terms introduced are axioms and undefined terms.</em> The teacher gives a hand-out to the students with the map that was drawn on the board showing all the backward steps taken to arrive from the theorems proven in class to some of the axioms of Euclidean geometry.</td>
</tr>
</tbody>
</table>

Table 5.4: Methods used for presenting the terms for the ‘explication’ phase of our teaching sequence.

It is important to stress that this phase needs to take place only after the students have ex-
experienced the meaning of the concepts they are being introduced to. Students at various phases of the intervention, and always after a guided reinvention phase has been completed, are presented with the mathematical terms for the concepts experienced (hand-outs prepared by the teacher may assist this presentation).

4. **Free orientation:** “The field of investigation is for the most part known, but the student must still be able to find his way there rapidly. This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols” (van Hiele, 1984b, p. 247).

During this phase students get the opportunity to make their own conjectures and try out different kinds of arguments for supporting or rejecting these. The tasks we used during this phase are presented in Table 5.5.

<table>
<thead>
<tr>
<th>Task</th>
<th>Concept</th>
<th>Description / Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXT (B.9)</td>
<td>Proof, Proving, Theorem, Conjecture</td>
<td>Students make their own conjectures about an unfamiliar situation. Students may work in groups in order to give a convincing argument (an argument that is general and not specific to the figure they use). This way students are expected to <strong>start abandoning the empirical proof schemes in favour of deductive ones.</strong></td>
</tr>
<tr>
<td>TRIMID (B.10)</td>
<td>Proof, Proving, Conjecture, Theorem</td>
<td>Students make more than one conjecture. The conjectures are discussed as a class and put on display. <strong>Students are asked to prove or disprove the conjectures.</strong> All students have to give arguments for two conjectures. Students are thus <strong>practicing all the concepts/processes which were explicated in the previous phase.</strong></td>
</tr>
<tr>
<td>QUADMID (B.12)</td>
<td>Conjecture, Proof, Theorem, Proving</td>
<td>Students are free to come up with conjectures about a specific figure in an unfamiliar situation which in a way extends the previous task (TRIMID) to quadrilaterals. The class agrees on a common conjecture to prove, which is in fact a theorem. The students are not aware of this until they come up with a convincing deductive argument. Students are thus <strong>practicing all the concepts/processes which were explicated in the previous phase.</strong></td>
</tr>
<tr>
<td>PARALLELOGRAM</td>
<td>Proof, Proving</td>
<td>The teacher draws a parallelogram on the board and guides a discussion on the properties of the parallelogram. Students rediscover the properties of parallelograms by making observations and these observations are discussed. Students are then asked to think about why these properties hold, that is, they are asked to come up with an argument to prove each property of the parallelogram. This way they <strong>use the new concepts explicated in the previous phase for justifying certain properties which were in the past introduced to them as givens.</strong></td>
</tr>
<tr>
<td>IF-THEN (B.13)- Part 3</td>
<td>Proving, Proof, Implication, Counterexample, Inverse implication, Theorem</td>
<td>Students are asked to try and prove or disprove (by finding a counterexample) the inverse proposition introduced in Part 1 of this task, and whether we can know in advance if the inverse holds. This way students <strong>use the concepts they learned to prove or disprove propositions which are in a special form (implication and inverse implication).</strong></td>
</tr>
<tr>
<td>AXIOMS - Part 4</td>
<td>Undefined terms, Axiomatic system, Axioms</td>
<td>The teacher may let students start with different theorems than the ones tried at the directed orientation phase and track them back in the same way. Students should be encouraged to <strong>discover more than one different ‘backwards’ paths from the theorems to the axioms.</strong></td>
</tr>
</tbody>
</table>

Table 5.5: Tasks used during the ‘free orientation’ phase of our teaching sequence.
Students at this point have already been introduced—during the explication phase—to the main concepts behind proving and now it is their task to try and practice this for themselves. The aim of this phase is to get students to a point where they can relatively easily come up with their own conjectures and offer deductive arguments for supporting (or counterexamples for rejecting) their conjectures. When students completely abandon empirical arguments in favor of deductive ones, one can say that they have completed the free orientation phase and are ready for the integration phase.

In our intervention time was limited; however, we think that students should practice many more tasks during this phase.

5. Integration: “The student has oriented himself, but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point, the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows. At the close of the fifth phase a new level of thought is attained” (van Hiele, 1984b, p. 247).

The integration phase of our intervention was designed to take place in the last lesson. With the help of hand-outs and an open discussion the teacher guides students to connect all the concepts they have experienced earlier into a whole: the axiomatic system of Euclidean geometry. The teacher knows whether the students have completed the integration successfully by the way they react and the input they give during the discussion that takes place in this phase. The material used for the integration phase is described in Table 5.6.

<table>
<thead>
<tr>
<th>Task</th>
<th>Concept Description / Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUCLIDEAN GEOMETRY (C.7)</td>
<td>Proof, Proving, Conjecture, Theorem, Counterexample, Implication, Inverse implications, Axiomatic System, Euclidean Geometry</td>
</tr>
</tbody>
</table>

This is the last hand-out of the teaching sequence and gives to the students an overview of Euclidean geometry as an axiomatic system. A discussion needs to go along with the hand-out to facilitate students’ understanding of the connections between the concepts.

Table 5.6: Hand-out used during the ‘integration’ phase of our teaching sequence.

In Table 5.7 on the following page we present all tasks we used in our intervention in chronological order, categorised by concept and by the Van Hiele teaching phase they belong to.

5.5.3 Method for evaluating our teaching strategy

For evaluating our teaching strategy, we took the following steps. First, we looked at the progress of the students as a group in their level of thinking in geometry, since our main aim was to assist students to move from the descriptive to the theoretical Van Hiele level. Secondly, we used the written work of students and their proof scheme classification in order to examine whether the students were able to distinguish between deductive and inductive arguments and to abandon inductive arguments in favour of deductive ones, which was another aim of our teaching intervention. Finally, we made a qualitative analysis of transcripts from the lessons...
Chapter 5. Methodology, Teaching Design and Analysis Framework

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Proving</td>
<td>1 1 2 2</td>
<td>2 2 3 3 4 4 4 4 4 4</td>
<td>4 4 4 5</td>
<td>Empirical vs. Deductive (Proof)</td>
<td>1 2 2 2 2 2 3 3 4 4 4 4 4 4 4 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conjecture</td>
<td>1 2 2 2 2 2 3 3 4 4 4 4 4 4 4 4 5 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorem</td>
<td>1 2 2 3 4 4 4 4 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Counterexample</td>
<td>1 2 3 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implication / Inverse Implication</td>
<td>1 2 3 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Axioms / Undefined Terms</td>
<td>1 2 3 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Axiomatic System</td>
<td>1 2 3 4 5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.7: Tasks (in chronological order from left to right) used in our intervention for each Van Hiele teaching phase by concept. (Numbers 1–5 indicate the corresponding phase of the Van Hiele teaching sequence: 1 = Inquiry, 2 = Directed orientation, 3 = Explication, 4 = Free orientation, 5 = Integration)

and the group work of students, in order to locate those instances where students show progress. In all three of these analyses we used the aims of the three stages (explained in Section 5.5.1) and our expected results for the five phases (explained in Section 5.5.2) as our analysis framework.

5.6 Other data collection methods

As described in the previous sections, the main data collection methods included written tests, tasks, and questionnaires. However, another important way of collecting data was videotaping and voice recording. All teaching sessions as well as student group work were recorded and parts of them were later transcribed, translated and analysed. Students’ conceptions about proof, as well as their exact interpretation of a problem, cannot be fully revealed unless one talks with the students. Discussion was another main tool use throughout the intervention for finding out the conceptions of students about proof and how those were affected by the lessons.
Chapter 6

The Intervention

6.1 Population and sample

Population  The main aim of our study was to find out how the students’ reasoning changes when moving from the descriptive to the theoretical Van Hiele level while doing geometrical proofs, and how their beliefs about mathematics and proof are then affected. Because our main aim is directly related to proofs in geometry and to students who start at the descriptive Van Hiele level, we had to choose a population of students who are likely to be at this Van Hiele level of thinking in geometry at the start of our teaching intervention. As was explained in Chapter 4, in the Greek Gymnasium students are introduced only to practical geometry which is based on intuition and requires empirical reasoning (the descriptive Van Hiele level). It is only when they enter the upper secondary school (Lyceum) that students are introduced to theoretical geometry (the theoretical Van Hiele level). Therefore, our population has to be comprised of students at the last Gymnasium level (level Γ’). In order to ensure that our population is heterogeneous as to the background of students, we chose our population to come from Greek public schools.1

Another characteristic of our population is that it should be comprised of students of relatively good ability in Mathematics. This was necessary in order to ensure that the students had achieved a relatively good mastery of the descriptive Van Hiele level before participating in our intervention. It was important for our research to have students who were not only already in the descriptive Van Hiele level, but also ready to make the move to the theoretical level.

Finally, our population is comprised of motivated students. For practical reasons the research had to take place outside of ordinary school hours. Therefore, we had to ask for volunteers, which made it inevitable to attract the more motivated students. We decided that indeed this population is interesting in itself. To summarise the above, the population about which we would like to generalise can be characterised as follows:

Motivated students of the Γ’ Gymnasium level (14–15 years old) in Greek public schools who have a relatively good mastery of the descriptive Van Hiele level.

Sample selection method  For our teaching intervention we needed to select a small sample of students (no more than 30, given that the researcher would be the only teacher) at the descriptive Van Hiele level. The selection was made based firstly on a pre-test for classifying students into

1The characteristics of Greek public schools were described in Chapter 4.
Van Hiele levels, taken by the whole Γ' Gymnasium level ($n = 85$) of an ordinary public school in the researcher’s home town. To increase validity we used a test that has already been used in previous research. Specifically, we used the first 20 questions of the Usiskin Van Hiele test (see Appendices A.1 and A.2). Permission for using this test was obtained from Usiskin.

The results of the pre-test were analyzed (between January 15 and February 15, 2010) and the students were classified into Van Hiele levels of thinking. Given the small number of students that were classified as being at the descriptive Van Hiele level (level 2), and since we wanted students to have a relatively good mastery of their Van Hiele level, we also took into consideration the students’ term grade in Mathematics. (This selection process will be further explained in Chapter 7.)

A side-aim of our research was to analyse the Van Hiele test results of the whole Γ' Gymnasium level ($n = 85$) and compare it to those of two other studies (Usiskin, 1982, and Tzifas, 2005). Therefore, we will describe here also the larger sample which was used for this analysis. The Γ' Gymnasium level of the school represented adequately both genders: 44 female and 41 male students. Moreover, all students had the same Mathematics teacher for the year 2009–2010, thus they were taught in (roughly) the same way. The abilities of the students were mixed; the term grades of the students in Mathematics can be seen in Figure 6.1. The grading range is from 1 to 20. The passing grade is 9 or above and, for the standards of the Greek highschool, a medium grade is 15.

![Figure 6.1: Mathematics term grades distribution of all the students ($n = 85$) in the Γ' Gymnasium level of the school.](image)

The teacher has not awarded a grade less than 12 to any of the students, although some of the students are too weak in mathematics and would probably be graded more strictly by someone else. Accordingly, the highest grade (20) should not be interpreted as representing an excellent student. The level of the school is generally low, and the grades represent the range of ability of students within each classroom. Grade 12 corresponds to the weakest students in math, whereas grade 20 corresponds to the strongest ones. We can see that there is a variety of abilities of students in mathematics, with 37 out of 85 students (or 43.5%) being above average

---

2Personal communication with the cooperating teacher.
(grades 16–20) and 48 out of 85 students (or 56.5%) being below average (with grades 12–15). The fact that all students are assessed by the same teacher makes comparison of their abilities in mathematics across classrooms easy.

On the basis of the results of the pre-test, which was taken on January 12, 2010, we selected a sample of 40 students to participate in our teaching intervention. We asked those 40 students whether they would like to participate in our intervention, which would take place outside their ordinary school lessons twice a week for 90 minutes; 34 students said they wanted to participate. The final sample of students who actually attended at least four lessons of our intervention was 23 students of which only 4 were male and the rest were female. The results of the Van Hiele test of the selected students (based on both criteria) as well as their term grades in Mathematics are presented in Table 6.1.

<table>
<thead>
<tr>
<th>Name</th>
<th>Pre-test</th>
<th>Math Grade</th>
<th>Gender</th>
<th>Van Hiele Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Katerina</td>
<td>3/5</td>
<td>4/5</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>John</td>
<td>3/5</td>
<td></td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>Debbie</td>
<td>2/5</td>
<td>20/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Flora</td>
<td>2/5</td>
<td>20/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Gayle</td>
<td>2/5</td>
<td>20/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Elli</td>
<td>2/5</td>
<td>19/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Katy</td>
<td>2/5</td>
<td>18/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Sunny</td>
<td>2/5</td>
<td>18/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Elinor</td>
<td>2/5</td>
<td>20/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Olivia</td>
<td>2/5</td>
<td>17/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Theano</td>
<td>2/5</td>
<td>15/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Mary</td>
<td>2/5</td>
<td>19/20</td>
<td>F</td>
<td>20</td>
</tr>
<tr>
<td>Gus</td>
<td>2/5</td>
<td>13/20</td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>Sue</td>
<td>2/5</td>
<td>16/20</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Anna</td>
<td>2/5</td>
<td>15/20</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Nia</td>
<td>1/5</td>
<td>19/20</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Gregory</td>
<td>1/5</td>
<td>20/20</td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>Fiona</td>
<td>no</td>
<td>20/2</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Kelly</td>
<td>no</td>
<td>20/2</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Stian</td>
<td>no</td>
<td>19/2</td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>Thalia</td>
<td>0/5</td>
<td>20/2</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Isis</td>
<td>0/5</td>
<td>14/2</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Nadia</td>
<td>0/5</td>
<td>15/2</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Term grades in Mathematics, and Van Hiele levels of the sample (n = 23) before our teaching intervention (‘no’ = the test could not assign a level).

To summarise the above, the sample of our research has the following characteristics:

23 motivated students (4 male, and 19 female) in the Γ' Gymnasium level (14–15 year old) of an ordinary Greek public school with a relatively good ability in Mathematics. The students were either at the second Van Hiele level as measured by the Usiskin Van Hiele test with the 3/5 criterion, or at a lower level or unclassified but with an above average (between 15 and 20 out of 20) term grade in Mathematics.
6.2 Student participation

On February 26, 2010, the orientation meeting of our intervention took place, in which we briefly discussed the form of the lessons and the schedule of the meetings. The actual lessons began on March 4, 2010 and ended on March 27, 2010. The first and the last meetings of these lessons were entirely reserved for the pre- and post-tests respectively. Out of the 28 students that participated in the intervention, only 19 were present at both these meetings. Between March 4 and 27 there were six meetings in which the students had to complete tasks individually or in groups, and to discuss their ideas. Table 6.2 shows an overview of the meetings’ time line and of the number of students present. We also present the detailed list of students’ nicknames and absences in Table 6.3.

<table>
<thead>
<tr>
<th>Date</th>
<th>Activity</th>
<th>Students present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan 12</td>
<td>Usiskin Van Hiele test</td>
<td>85</td>
</tr>
<tr>
<td>Feb 26</td>
<td>Orientation meeting</td>
<td>20</td>
</tr>
<tr>
<td>Mar 4</td>
<td>Pre-tests</td>
<td>24</td>
</tr>
<tr>
<td>Mar 6</td>
<td>Lesson 1</td>
<td>25</td>
</tr>
<tr>
<td>Mar 11</td>
<td>Lesson 2</td>
<td>22</td>
</tr>
<tr>
<td>Mar 13</td>
<td>Lesson 3</td>
<td>23</td>
</tr>
<tr>
<td>Mar 18</td>
<td>Lesson 4</td>
<td>18</td>
</tr>
<tr>
<td>Mar 20</td>
<td>Lesson 5</td>
<td>16</td>
</tr>
<tr>
<td>Mar 24</td>
<td>Lesson 6</td>
<td>19</td>
</tr>
<tr>
<td>Mar 27</td>
<td>Post-tests</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 6.2: Timeline of our teaching intervention and number of students present.

Only 12 students were present in all meetings from the 4th to the 27th of March. For these students we can follow their progress through the lessons without any problems. An extra five students were absent only once, and this one absence was neither on the pre-test nor on the post-test day; two students were absent only two times and again their absences were neither on the pre-test nor on the post-test day, so for these students we can also follow their progress without much problem. In total, there are 19 students whose progress we can follow from the data we have collected. We will not consider at all for our analysis those students who participated in less than two meetings.
6.3 Diary of the intervention

Below we provide a description of what took place in the classroom in each of the meetings with the students. An overview of the tasks completed by the students as well as the main points that came out of the group discussions and the atmosphere in the classroom is presented.

### 1st meeting: orientation

<table>
<thead>
<tr>
<th>Student name</th>
<th>Van Hele pre-test</th>
<th>Orientation Meeting</th>
<th>Other Pre-tests</th>
<th>Lesson 1</th>
<th>Lesson 2</th>
<th>Lesson 3</th>
<th>Lesson 4</th>
<th>Lesson 5</th>
<th>Lesson 6</th>
<th>All Post-tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Gregory</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>2. Kelly</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>3. Elli</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>4. Nia</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>5. Mary</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>6. John</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>7. Flora</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>8. Katy</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>9. Debbie</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>10. Nadia</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>11. Thalia</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>12. Olivia</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>13. Theano</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>14. Gayle</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>15. Sue</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>16. Katerina</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>17. Fiona</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>18. Isis</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>19. Sunny</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>20. Elinor</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>21. Anna</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>22. Gus</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>23. Stian</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>24. Gary</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>25. Minas</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>26. Chrysal</td>
<td>+</td>
<td></td>
<td></td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>27. Alice</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>28. Elisa</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Total Present</td>
<td>28</td>
<td>20</td>
<td>24</td>
<td>25</td>
<td>22</td>
<td>23</td>
<td>18</td>
<td>16</td>
<td>19</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 6.3: Students’ absences during the intervention (* = present).

### 6.3 Diary of the intervention

Below we provide a description of what took place in the classroom in each of the meetings with the students. An overview of the tasks completed by the students as well as the main points that came out of the group discussions and the atmosphere in the classroom is presented.

#### 1st meeting: orientation

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feb 26, 2010</td>
<td>15 min.</td>
<td>20</td>
<td>Hand-out ‘Geometry and proof with different eyes’</td>
</tr>
</tbody>
</table>

The purpose of the meeting was to introduce the expectations of the lessons to the participating students. To motivate the volunteering students to attend all lessons of the intervention,
they were promised to get a present at the end of the intervention along with a certificate of 
attendance. The condition was that they would not miss any of the lessons (with a maximum of 
one absent in case they had a valid excuse). The students were given a handout called ‘Geometry 
and proof with different eyes…’ (see Appendix C.2) which included information about the 
expectations in the lessons, and what students will gain if they participate in all lessons. Many 
students were absent because some of their school lessons that day were canceled and they did 
not want to spend their whole afternoon at school waiting for our meeting. Absent students 
would be informed about what happened during this meeting by their present classmates.

2nd meeting: pre-tests

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 4, 2010</td>
<td>90 min.</td>
<td>24</td>
<td>Task TRIANGLES (Appendix B.1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task ANGLESUM (Appendix B.2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task CHOOSING (Appendix B.3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task CONCEPTIONS ABOUT PROOF (Appendix B.4)</td>
</tr>
</tbody>
</table>

The students were quite chatty and disappointed they had to do so many tests; they were 
complaining and asking whether the following lessons would also be like this (although they 
had already been told in the previous meeting that this was not going to be the case). In Task 
ANGLESUM the students did not understand what they needed to do and did not know where 
to start. This shows that the students were not familiar with proving (as it was expected). They 
also had difficulty understanding some of the questions, especially the proofs, given in Task 
CHOOSING so they kept asking questions throughout the test. The researcher made sure that 
the answers would clarify the questions of students without revealing the correct answer.

3rd meeting: lesson 1

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 6, 2010</td>
<td>90 min.</td>
<td>25</td>
<td>Discussion / Hand-out Pros and Cons (Appendix C.3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task QUAD (Appendix B.5)</td>
</tr>
</tbody>
</table>

The lesson started with a discussion of students’ answers in Task ANGLESUM. A hand out 
given to the students (see Appendix C.3) with various of their own proofs but also of the 
proofs that were included in Task CHOOSING as sample proofs (Eleni’s, Georgia’s and Left-
eris’s proofs). Beside each proof there was space for writing the advantages and the disadvan-
tages of each proof. Proofs in this first lesson were not referred to by the teacher with the word ‘proof’, but rather with the word ‘arguments’ which all have advantages and disadvantages.

The students came up with interesting ideas about possible advantages and disadvantages 
for each argument. Almost all of them could see that the restriction to one specific figure is a 
disadvantage and that it makes an argument less convincing, although they all agreed that it 
might be more practical and easy. At the beginning of the lesson the teacher stressed that the 
proposition ‘The sum of the angles of any triangle is 180°’ is not already true and that when 
asked to show that this proposition is true, this means that it might be either true or false; both 
options are possible. This was not immediately understood by the students, as the discussion
in class revealed. This was evident also in the work of students, since several students in Task ANGLESUM used a circular argument because they assumed the proposition already to be true.

There was an opportunity to talk about the ‘proof by contradiction’ idea since one girl had tried to prove the proposition by assuming that the sum of the angles of a triangle is either less or more than $180^\circ$, and ending up with a non-triangular shape. There was also a discussion about transformational reasoning, because the teacher stressed that some of the proofs include some kind of ‘movement’. The class agreed on calling this: ‘it has movement’, and decided it is to be added on the ‘advantages’ column. It was also noted that moving the figure helps one consider more cases so it helps to move from a more specific to a more general argument. A discussion about practical and theoretical arguments was also included in this meeting.

In general students were participating and contributing a lot and seemed to enjoy the lesson. For the last ten minutes the teacher asked students to complete Task QUAD. The teacher stressed that now students could take as a given that the sum of the angles of a triangle is $180^\circ$. It is interesting to note that although many students chose to measure the angles of one specific quadrilateral in order to show that the proposition is true (possibly because it is the easiest to do), they commented on it as having the disadvantage of not being very convincing because it is specific to the figure they drew.

**4th meeting: lesson 2**

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 11, 2010</td>
<td>90 min.</td>
<td>22</td>
<td>Task OPPANG (Appendix B.6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task BIS (Appendix B.7)</td>
</tr>
</tbody>
</table>

Task OPPANG started with the teacher showing on the board two intersecting lines (see also Figure 6.2). The teacher said to the students that from now on she would not offer ready mathematical statements to be proven, but students would observe a figure and formulate their own general mathematical statements. Some of the propositions may turn out to be correct for the general case, and some wrong. The students were then asked to guess what relations hold between some of the angles that are formed by the intersecting lines. Here are some of the students’ observations:

1. Adjacent angles add up to $180^\circ$.
2. Opposing angles are equal.
3. All four angles together add up to $360^\circ$.

A fourth observation was brought up by the teacher and it was the following:

4. The sum of the pair of the smaller angles is equal to $60^\circ$.

The teacher had drawn the figure in such a way so that when she measured them they really added up to $60^\circ$. This was done in order to see whether the students would realize that this is wrong because it does not hold in general but depends on the specific figure.

The students very quickly came up with arguments for the first, third and fourth observations (they already knew and took as a given that the straight line has $180^\circ$ and the circle has $360^\circ$).
However, they had difficulty finding an argument for the second guess because they again took it as a given (students had learned this before as a fact in their mathematics lesson). The students were then asked to work on finding a proof individually, and while the sheets were being handed out students were complaining loudly: ‘We know this is true!’ Students were given some time to work on it, and only two or three of them managed to come up with a deductive argument. The teacher talked to them about the ‘movement’ idea in order to give them a hint: she took two pens in her hands as if they were two intersecting lines and turned them around their intersecting point in order to show that something remains always the same, namely the sum of two neighboring angles. So the teacher told them they can use this fact in order to give a general argument about the second proposition.

In total seven students gave a correct deductive argument, three of which had given an empirical argument in the previous lesson; three students gave a wrong deductive argument, probably guided by the teacher but not having enough hints. One of those three had given an empirical argument last time, while the rest had given a deductive argument. When the students finished the individual work on this proof, the teacher presented the problem on the board. The teacher drew two new intersecting lines and said:

“We want to show that these angles which you call opposing are equal. We want to give an argument; a good one. Some of you say ‘let’s measure them.’ I saw this. We go and measure them. This one is…55°. This one is 48°. What did we say? Does this convince us?”

The students said that it was not convincing because the teacher made a mistake. The teacher continued:

“We have said that measurements are not precise. We cannot trust measurements. We want something more convincing, so that we don’t have a problem for any figure. That is, in whatever way we draw the figure our argument should hold.”
and went on giving a hint:

“What did you say earlier yourselves, that these two (pointing at two neighboring angles) make…180°, because they form a straight line. And the same for this pair, and this pair and this pair. [The teacher labeled the angles as 1, 2, 3, 4.] We want to see why these two (2, 4) are equal. Not because you know it or because the book says that! What if what the book says is mistaken?”

The teacher then asked for a more general argument and a few students raised their hands. One student started explaining her argument which was very close to the right one but at some point she was being circular. When the teacher asked her again what she meant, she corrected herself and she realized the circularity, so she gave the correct deductive argument instead. In her task sheet, she provided a circular argument. All students seemed to be really convinced by this argument, but most of them could not come up with it on their own.

Task BIS was even more difficult for the students to do. They had only the last 20 minutes of the lesson to complete it. The teacher drew again the figure on the board and asked the students to observe it and make some guesses. They came up with two. One was that the sum of all angles is 180°, and they gave easily a reason for that (because they form a line). The other was that the sum of two of the angles is 90°, and also that there is a right angle formed between the bisectors.

Only about five students could come up with a deductive proof. At some point the teacher asked them to give two kinds of arguments, an empirical one, where they can measure and do such kind of things, and a more general one. The students seemed to have difficulties but this might also be due to the fact that they were too tired (this lesson took place after the end of a full day in school).

5th meeting: lesson 3

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 13, 2010</td>
<td>90 min.</td>
<td>23</td>
<td>Task BIS - group work (Appendix B.8)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hand-out ‘How new truths are discovered in geometry’ (Appendix C.4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task EXT - group work (Appendix B.9)</td>
</tr>
</tbody>
</table>

Students worked in groups to complete Task BIS (see Figure 6.3).

The students were divided in six groups according to their proof schemes, based on the classification by the teacher of their work during the previous lessons. There were two groups of ‘deductivists’, two groups of ‘inductivists’, and two groups of mixed abilities. All students managed to come up with the solution in groups. The teacher focused less on the ‘deductivists’ and more on the other two groups. They all sooner or later had for Task BIS the intuition that since the two angles together add up to 180° their halves should add up to 90°, but some of them had more difficulty than others with writing this down properly as an argument.

The hand-out ‘How new truths are discovered in geometry’ was given to the students after completing Task BIS and was briefly discussed in class.

Task EXT was next completed by the students and turned out to be easy for them. They easily came up with the two following conjectures (see also Figure 6.4):
Chapter 6. The Intervention

1. The external angle is equal to the two opposite internal ones.

2. The sum of the external and the corresponding internal angle is $180^\circ$

   The argument they came up with was that all the internal angles add up to $180^\circ$, and since the external angle together with one of the internal angles also adds up to $180^\circ$, the external must be equal to the two opposite internal ones.

The students seemed to enjoy working in groups and they asked the teacher to do it again. They also had to fill in evaluation forms about the lesson and the teacher, and 100% of the students said they understood very well the way the teacher talks, that they liked the lessons and enjoyed group work, and that the pace of the lesson was normal (not too slow, not too fast). Most of them also said that the lessons were different from what they are used to because they included more discussion, students had more time to think about the ideas being discussed, and they did not need to worry about exams and homework.

6th meeting: lesson 4

The students were given the hand-out ‘What we discovered in the geometry lessons’ at the beginning of the lesson with a summary of everything that was done in class to that point. All
the propositions along with the arguments supporting them were provided in this document prepared by the teacher. The students were referred to pages 6 and 8 where they could find the arguments they gave in the previous lesson for two of the tasks. The students were then told that they would work on a new task (Task TRIMID) on their own at first, and later they would be put again in groups. Before starting on the task the teacher talked to them about the properties of the parallelogram because they would need to use it in one of the activities.

The teacher drew a parallelogram on the board and asked the students what this figure was called. The properties of the parallelogram came out of observations and a long discussion. The students had to come up with an argument to prove each property. The properties were listed on the board and were kept throughout the lesson.

Then, the students were asked to draw a triangle on their sheets and connect the midpoints of two sides of the triangle (Task TRIMID). The teacher also drew a triangle on the board. Then the students were asked to make observations about the relation that might hold between the line that connects the midpoints of two sides of the triangle and any one of the sides of the triangle. All guesses were written on the board. Each student was then asked to prove or disprove two of these guesses. They first worked individually for a few minutes, and then they worked in groups (Task TRIMID-GROUP).

After students worked in groups there was a discussion as a class. Students were asked to give their arguments to the rest of the class for the guesses which they tried to disprove. The discussion led to the idea of a COUNTEREXAMPLE to which the students were introduced for the first time. Then the teacher presented a deductive argument to prove that the line connecting the midpoints of the two sides is parallel to the third side and equal to its half, since there was no time to discuss the arguments given by the students. Their arguments would be discussed in the following lesson.

7th meeting: lesson 5

The lesson started by summarizing the arguments given for Task TRIMID. The teacher drew a figure on the board of a triangle for which the midpoints of two sides are connected, and reminded students about the idea of COUNTEREXAMPLE. The observations made in the previous lesson were written on the board too (i.e. that the line connecting the midpoint of the two sides on a triangle is parallel to the third side and equal to its half). Also the teacher mentioned that many students gave nice and convincing arguments different than the one given by the teacher. A deductive argument found by the students was presented, as well as some inductive-empirical ones which were not correct. A discussion followed.

Then the teacher drew a random quadrilateral and asked the students whether they think that
Chapter 6. The Intervention

a similar proposition to the one they had just discovered for the triangles holds also for quadrilaterals (see Figure 6.5). The teacher connected the midpoints of the sides of the quadrilateral. The students started saying that the sides will be parallel (based on what they saw). Some claimed that the figure was not random, so the teacher made another one. Now there were two figures on the board.

The students were asked to make observations, and they came up with the one that the teacher had in mind, namely that if you connect the midpoints of neighboring sides of any quadrilateral you get a parallelogram. The hint given by the teacher then was the following: “Do you see any relation to the previous theorem we proved?” Then the students were asked to individually show whether their guess holds generally (Task QUADMID). A student asked: “We have already proven that previous theorem now, right?” and the teacher said: “Yes! So now you can use it. Didn’t we show this last time? So now if you want you can use it. […] You can also use assisting lines if you want, whatever you want.” Another student then said: “Oh, so then we can use now assisting lines”, and the teacher: “Yes, yes in a proof you can use whatever you like, as long as it is convincing.” After they finished working the teacher showed a general deductive argument on the board.

The students then worked on Task IF-THEN. First the teacher drew on the board a line segment AB, found its midpoint, and drew another line ε that passes from the middle of the line segment. The idea was to find the relation that holds between the distances of the edges of AB from line ε. There was a discussion about what the distance is and how we can construct the line that represents that distance. After all lines were drawn, the students said that the distances are the same. The teacher asked why they are so sure, and suggested that they give a convincing argument about their guess. The students were asked to work individually and to use the teacher’s figure from the board.

After the students finished working, the teacher asked them to think about the other direction of the proposition, that is, whether if the edges of a line segment are equidistant from a line that line would pass through the middle of the line segment. The students thought that the inverse would also hold because we know that the one direction holds. Then the teacher gave them a counterexample. So the students were convinced that the inverse of an ‘if, then’ proposition does not actually necessarily hold when the proposition itself does.

Before the end of the meeting the teacher said that whatever they have been proving in class was based on something that we already knew from before and that in the following lesson the discussion would be about how far ‘backwards’ we can go. The question posed to the students to think about was: ‘where did mathematicians start from in order to prove things in geometry?’

8th meeting: lesson 6

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 20, 2010</td>
<td>90 min.</td>
<td>16</td>
<td>Discussion on axioms / hand-out ‘The map of axioms’ (Appendix C.6) Hand-out ‘The story of Euclidean geometry’ (Appendix C.7)</td>
</tr>
</tbody>
</table>

This lesson was dedicated to a discussion about axioms. The students were asked to think about where we start from in order to prove our conjectures. A very interesting discussion followed and students were rather surprised to find out that geometry starts from a few intuitive
assumptions which are taken for granted without any proof (the axioms). A couple of handouts with definitions about Euclidean geometry and axioms were given to the students.

**9th meeting: post-tests**

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration</th>
<th>No. students</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 27, 2010</td>
<td>90 min.</td>
<td>20</td>
<td>Usiskin Van Hiele test (Appendix A.2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task CHOOSING (Appendix B.3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Task CONCEPTIONS ABOUT PROOF (Appendix B.4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Modified Schoenfeld Questionnaire (Appendix A.3)</td>
</tr>
</tbody>
</table>

During this last meeting the students had to complete all post-tests. Students found this a bit boring, as was expected, but they did their work and at the end of the lesson they were given their present and their certificates. Everyone was enthusiastic and several mentioned they were sad that the lessons finished.
Chapter 7

Results

In what follows we describe the results of our research in five sections, each of which corresponds to one of the sections of the Methodology chapter. This chapter is also structured around our research questions and hypotheses, which were presented in Chapter 3. More specifically, we begin by presenting the students’ identified Van Hiele levels and kinds of reasoning (Sections 7.1 and 7.2). Then, we present the statistical results of correlation tests between the Van Hiele levels and kinds of reasoning of the students (Section 7.3). Because of the small number of participants in our research, we also provide a qualitative analysis of how the students’ kinds of reasoning changed while they moved from the descriptive to the theoretical Van Hiele level (Section 7.4.1). Both the above sections are related to our second subquestion and our second hypothesis. An analysis of whether and how our teaching strategy helped the students make this move is also provided (Section 7.4) and is related to our third subquestion and third hypothesis. Finally, we present the results regarding the students’ beliefs about proof (Section 7.5) which is relevant to our first subquestion and hypothesis.

7.1 The Usiskin Van Hiele test results

The context of the Usiskin Van Hiele pre-test The students had been informed by their teacher that a teacher/researcher from the Netherlands would visit the school in order to get some data for her research, so the students were to take a test which was not going to affect their Mathematics grade. On the day of the test both the teacher of the students and the researcher were in class. The researcher introduced herself to the students and handed out the tests and answer sheets. The students were not allowed to open the tests until they were given permission. They were told that they were not expected to know all correct answers and that this test includes also some difficult questions so they should not be disappointed if they could not answer them correctly. It was stressed at least two times before they started that they should not ‘cheat’ for two reasons: first, they would not be graded so there is no problem if they have mistakes, and second, if they copied then the researcher would not be able to find out what their own opinions were about the right answer.

1You can see the original Van Hiele test in Appendix A.1, and the translated version that the students were given in Appendix A.2. The translated version differs from the original only in that it includes just the first 20 questions of the original test.
Chapter 7. Results

After this small introduction, the instructions, which were also included on the first page of the test, were read out by the researcher and the students were asked to fill in their names and the other data on the answer sheet, including the number of the test. Knowing that it is possible that some students might copy from each other, all sheets were marked by the number of the desk each student was sitting in. This helped to later check, during the analysis of the test results, to what extent students sitting next to each other had similar answers.

There was a small typo in one of the questions in the test, which was read out and the students were asked to correct it before they began the test. The students were allowed to ask questions during the test. Both the teacher of the students and the researcher were in the classroom all the time, and the students were closely observed during the test.

Observations   All students present filled in the test. A few of the students (about five in total) filled in the test very quickly, and obviously at random, since they handed it in only about five minutes after the beginning of the test. The researcher sometimes had to talk to some students in person to remind them that they shouldn’t discuss with each other but most students were generally quiet and seemed concentrated on what they were doing.

Difficulties encountered   In three out of four classrooms, the students asked what question 7 meant. The form of question 7 (which is also the form of several of the following questions) was difficult for them to understand. In all these three classrooms the question was explained to all students. Another issue, again common to the same three classrooms, was related to question 11 of the test. In that question, a proposition describes a figure $K$ without giving this figure. This caused problems to some students, who asked the researcher where the figure was, and were wondering how they were supposed to understand this without seeing the figure. This indicates that some students were not yet at a level to understand abstract questions well. The researcher explained to the whole classroom what the meaning of the question was. In the fourth classroom the students did not have these questions, or at least they did not ask anything about them, therefore no explanation was given to them.

The Van Hiele test results   For classifying the students into Van Hiele levels, we used the two criteria defined by Usiskin (1982). The first, ‘strict’ criterion says that a student has reached a certain Van Hiele level if the student has answered correctly four out of the five (4/5) questions for that and all previous levels. The second, ‘loose’ criterion says that a student has reached a certain Van Hiele level if the student has answered correctly three out of the five (3/5) questions for that and all previous levels. For these two criteria, Usiskin (1982, pp. 23–34) states:

The choice of criterion, given the nature of this test, is based upon whether one wishes to reduce Type I or Type II error. Recall that Type I error refers to a decision made (in this case a student meeting a criterion) when it should not have been made.

\[
P(3 \text{ of } 5 \text{ correct by random guessing}) = 0.05792
\]

\[
P(4 \text{ of } 5 \text{ correct by random guessing}) = 0.00672
\]

So the 4 of 5 criterion avoids about 5% of cases in which Type I error may be expected to manifest itself. However, consider the probability of Type II error, the probability that a student who is operating at a given level at, let’s say, 90% mastery, a rather strong criterion, will be found by the test not to meet the criterion.
7.1. The Usiskin Van Hiele test results

\[ P \text{ (less than 3 of 5 correct given 90\% chance on each item)} = .00856 \]
\[ P \text{ (less than 4 of 5 correct given 90\% chance on each item)} = .08146 \]

These are of the same orders of magnitude, in the other direction, as the probabilities associated with Type I error. The 3 of 5 criterion avoids about 7\% of cases in which Type II error may be expected to appear. If weaker mastery, say 80\%, is expected of a student operating at a given level, then it is absolutely necessary to use the 3 out of 5 criterion, for Type II errors with the stricter criterion are much too frequent.

\[ P \text{ (less than 3 of 5 correct given 80\% chance on each item)} = .05792 \]
\[ P \text{ (less than 4 of 5 correct given 80\% chance on each item)} = .26272 \]

Our research population is comprised of students that are of a generally poor level in mathematics, so we chose to use mainly the 3 out of 5 criterion in order to avoid Type II errors, that is, in order to avoid missing classifying to a level students with 80\% mastery of that level. However, we also kept the 4/5 criterion in our analysis since for our teaching intervention we chose a sample of the population which comprised of the best out of all students in terms of abilities in mathematics, based on their term grades and on the information we received from their teacher.

More specifically, we selected a sample of 40 students who were classified at Van Hiele level 2 or above based on the 3/5 criterion, or at a lower (or an unidentified) Van Hiele level but had a good term grade (15 or above) in Mathematics. The final sample of students who took part in our teaching intervention was 24 students, out of which one was not selected by us (Isis) but asked to participate out of her own interest. In table 7.1 we provide the Van Hiele levels, both in the pre- and the post-test, of all the students who participated in the teaching intervention \((n = 24)\) based on both criteria (3/5 and 4/5), the change in Van Hiele level before and after our intervention, and the term grades of the students.

In the fist two columns of the table one can see the pre-test results for each student based on both criteria. In the next two columns the results of the post-test are presented in the same manner; the blank spaces indicate that there is no data because the student was absent. In fact, only 20 students were there in both the pre- and the post-test. Label ‘no’ indicates that the test could not assign a level to the student. This happens when a student has answered to the required amount of questions for a level without having done the same for the questions of the previous level. In the next two columns of the table we calculate the change in Van Hiele level per student before and after our intervention. The blank spaces here indicate that there is no data, either because the student was absent, or because in one of the two tests (pre- and post-) the test could not assign a level to the student. Finally, the last column shows the term grades of the students in Mathematics.

As we can see there are no negative changes to the results which suggests that there was either no change in the level of the students or there was progress which resulted from our intervention. Moreover, in 10 out of the 13 cases for which we can calculate the change in Van Hiele level based on the 3/5 criterion the change was positive. Similarly, in 10 out of the 14 cases for which we can calculate the change based on the 4/5 criterion the change was positive. These results indicated that our teaching intervention helped the students improve their Van Hiele levels. However, one of our main aims in this research was to create a teaching strategy for bringing the students specifically from the descriptive (level 2) to the theoretical (level 3)
Chapter 7. Results

<table>
<thead>
<tr>
<th>Name</th>
<th>Pre-test</th>
<th>Post-test</th>
<th>Change</th>
<th>Math Grade (out of 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3/5 2</td>
<td>4/5 3</td>
<td>1 1</td>
<td>18</td>
</tr>
<tr>
<td>Katerina</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>John</td>
<td>3 2</td>
<td>3 3 0</td>
<td>1 20</td>
<td></td>
</tr>
<tr>
<td>Debbie</td>
<td>2 2</td>
<td>3 3 1</td>
<td>1 20</td>
<td></td>
</tr>
<tr>
<td>Flora</td>
<td>2 2</td>
<td>3 2 1</td>
<td>0 20</td>
<td></td>
</tr>
<tr>
<td>Gayle</td>
<td>2 2</td>
<td>no no</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Elli</td>
<td>2 2</td>
<td>no no</td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>Katy</td>
<td>2 2</td>
<td>no no</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td>Sunny</td>
<td>2 1</td>
<td>3 1 1</td>
<td>0 20</td>
<td></td>
</tr>
<tr>
<td>Elinor</td>
<td>2 1</td>
<td>3 1 1</td>
<td>0 20</td>
<td></td>
</tr>
<tr>
<td>Olivia</td>
<td>2 1</td>
<td>2 2 0</td>
<td>1 17</td>
<td></td>
</tr>
<tr>
<td>Theano</td>
<td>2 1</td>
<td>2 no 0</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Mary</td>
<td>2 1</td>
<td>no 2 1</td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>Gus</td>
<td>2 1</td>
<td></td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>Sue</td>
<td>2 0</td>
<td>no no</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>Fred</td>
<td>2 no</td>
<td></td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>Anna</td>
<td>2 0</td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Nia</td>
<td>1 1</td>
<td>2 2 1</td>
<td>1 19</td>
<td></td>
</tr>
<tr>
<td>Gregory</td>
<td>1 1</td>
<td>3 3 2</td>
<td>2 20</td>
<td></td>
</tr>
<tr>
<td>Fiona</td>
<td>no 2</td>
<td>4 4 2</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Kelly</td>
<td>no 0</td>
<td>3 3 3</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Stian</td>
<td>no 0</td>
<td></td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>Thalia</td>
<td>0 0</td>
<td>3 1 3</td>
<td>1 20</td>
<td></td>
</tr>
<tr>
<td>Isis</td>
<td>0 0</td>
<td>2 0 2</td>
<td>0 14</td>
<td></td>
</tr>
<tr>
<td>Nadia</td>
<td>0 0</td>
<td>1 0 1</td>
<td>0 15</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.1: Van Hiele levels of the participants (n = 24) before and after our intervention based on the Usiskin Van Hiele test ('no' = the test could not assign a level, blank = no data.)

Van Hiele level. Again, we see that for those cases of students for which we can give an answer, 4 out of 6 who started at level 2 based on criterion 3/5 moved to level 3. Similarly, 4 out of 5 of those who started at level 2 based on criterion 4/5 moved to level 3. These results are still good but the number of students is too small to be able to make safe conclusions.

Our results of the Van Hiele pre-test for the whole population (n = 85) were analyzed and compared to those of (Usiskin, 1982) who designed and first used this test, as well as to those of (Tzifas, 2005) since this was a research about the Van Hiele levels of students in Greek highschools and thus shares the same background with our research. The detailed analysis and comparison is presented in Appendix D.1 since these results are not immediately related to our main research question. To give a short summary of the most important results of this comparison, we can say that most of our results agree with those of Tzifas (2005) and Usiskin (1982) apart from one question of the test, question 13, in which Tzifas’ results differ from both ours and Usiskin’s because of an error in the translated version of this question used in Tzifas’ study.²

In general, we think that certain questions of the Usiskin Van Hiele test and the different interpretations that these may be given by different students makes interviews with the students after they have taken the test necessary. Because of time restrictions we could not include interviews in our research, however, we strongly recommend in future research at a larger scale

²A more detailed analysis of this comparison is offered in Appendix D.1.
that interviews accompany the Usiskin Van Hiele test.

Conclusions

When applied to the whole population of our research \((n = 85)\), the Usiskin Van Hiele test revealed results very similar to those of other studies: Usiskin (1982), the first who created and used this test, and Tzifas (2005) who is the only research study that we know using the test on a large sample in Greece.

We chose our sample for the teaching intervention based on the Van Hiele test results as well as on the term grade of the students in Mathematics. The sample \((n = 24)\) included 14 students in Van Hiele level 2 based on the loose criterion \((3/5)\).

Out of the students who also took the Van Hiele test after our intervention \((n = 20)\) a pre/post comparison shows that most of them made progress in their Van Hiele level. More specifically, in 10 out of the 13 cases for which we can calculate the change in Van Hiele level based on the 3/5 criterion the change was positive. Similarly, in 10 out of the 14 cases for which we can calculate the change based on the 4/5 criterion the change was positive.

When we look specifically at the move from the descriptive (level 2) to the theoretical (level 3) Van Hiele level, 4 out of 6 students who started at level 2 based on criterion 3/5 moved to level 3, and 4 out of 5 of those who started at Van Hiele level 2 based on criterion 4/5 moved to level 3. Although the percentages are large, the sample of students is too small to allow us to make safe conclusions without any further analysis of the lessons.

7.2 The Harel & Sowder proof schemes coding process

For coding students’ proof attitudes we used the proof schemes of Harel and Sowder (1998) which were described in detail in Section 2.4.3. The codes we defined and used for the proof schemes were introduced in Section 5.2.2. Here we will give a few examples of proof attitudes of our students during the intervention and how we classified them into the Harel & Sowder proof schemes.

Examples of the external/ritual proof scheme In Task CHOOSING (see Appendix B.3) the students were asked to choose which among several proofs for the conjecture ‘When you add the interior angles of a triangle the sum is always 180°’ would get the best mark. Some students chose the following proof:
Chapter 7. Results

Petros’s argument is wrong because it is circular. However, several students chose it as the one that would get the best mark providing justifications for their choice like the following:

“[…] it represents the rules of mathematicians”

“[…] it uses mathematical operations which we have learned without making many figures or using anything else”

“[…] it shows that we have more knowledge about triangles”

“[…] it is short, correct and easy to understand.”

We see that several students chose Petros’s argument because it ‘looks’ mathematical (uses mathematical notation and is short). We characterised these students’ proof schemes as ‘ritual’ proof schemes.

Example of the external/authoritarian proof scheme At the beginning of our intervention often the students simply believed that a conjecture or a proof was true without seeking for further justification. Since they had never been explicitly and systematically introduced to proof and its significance for mathematics before our intervention, the students came to our lessons already ‘believing’ certain statements because they were written in their book or because their teacher had told them they were true. For example, the conjecture ‘the sum of the internal angles of any triangle is 180°’ was already taken for granted by most of the students without justification. An inductive argument, such as checking the conjecture for a few cases of specific triangles and concluding that it would hold for any triangle, was enough for them to be convinced that the conjecture is true. In fact, most of the students used such an inductive argument when they were asked to argue for the truth of the above conjecture.

The following transcript taken from the first lesson is an example indicative of the external/authoritarian proof scheme. The lesson started with a discussion of the student’s work on the pre-test where they had been asked to first give an argument for the conjecture ‘the sum of the angles of any triangle is 180°,’ and to then choose from a range of given proofs one they would prefer as their own approach. The plan was to discuss the advantages and disadvantages
The Harel & Sowder proof schemes coding process

of several of these arguments. The teacher wrote the conjecture on the board, and presented the
inductive argument that was used by most of the students in the pre-test for supporting the truth
of the above conjecture: we draw one or more triangles, measure their angles and add them up;
because we get a sum of 180° for these triangles, we conclude that the conjecture is true for all
triangles. After the teacher described this argument, the following dialogue took place.

Teacher: [...] the argument that you gave me, many of you, was this one [showing
the inductive argument on the board]. I want you to think about what advantages
and what disadvantages it had, that is, to think for instance was it convincing? Was
it easy or difficult? Was it practical, was it theoretical? Was it.... Whatever you
are thinking which you believe is an advantage. Do you understand it easily, with
difficulty? Whatever. Who wants to say?

Student 1: It was practical and easy to understand.

Teacher: Good. Advantage. You will also write it down [...]. What did you say? It
was practical and easy. [Writes them down on the board] These are basically two.
Practical and easy. It could also be theoretical and easy, right? Anyone else? Can
you think of something else?

Student 2: It is theoretical?

Teacher: Theoretical...Now, the practical, if it is...hmmm...it could also be the-
oretical...Let's see...What do you mean when you say it is theoretical? Can you
explain to me a bit better?

Student 2: I mean, we knew it, we have learned it.

Teacher: Oh, this is what you mean. So, it was...What did you learn? That...?

Student 2: That the sum is 180.

Teacher: This [showing the conjecture]. You had learned it. So you consider it,
let's say...

Students: Given.

Teacher: Given. Hmmm...This is a kind of a trap, right? It is not a given.

In lines 16–21 in the above transcript we see that the student clearly takes the conjecture for
granted because they had ‘learned it’ in class. This is a typical example of taking a conjecture
for granted because of an authority (the teacher, or the book etc.).

Example of the empirical/inductive proof schemes  Before our intervention the students
have been mainly using inductive reasoning in mathematics. This kind of reasoning is very
often used in the books of Gymnasium for introducing new mathematical ideas and concepts.
Therefore, as expected, the students mostly used inductive reasoning in order to argue for the
truth of the conjecture ‘the sum of the angles of any triangle is 180°.’ Figures 7.1 and 7.2 show
two examples of the inductive proof scheme as we encountered it in our students’ reasoning.
Chapter 7. Results

Figure 7.1: Example of the empirical/inductive proof scheme: “For example a triangle has its angles equal and each angle is sixty. Whatever triangle will have always sum 180°.”

Figure 7.2: Example of the empirical/inductive proof scheme: “[Figure 1] All triangles have 3 angles which have sum 180°. - In a right triangle 1 angle is 90 and the other two have sum 90°. 90 + 90 = 180 [Figure 2] In a triangle with 3 unequal angles again the sum is 180° 110 + 55 + 15 = 180 [Figure 3] In an equilateral triangle again the sum is 180° 60 + 60 + 60 = 180 [Figure 4] In an isosceles again the sum is 180° 70 + 70 + 40.”

Example of the empirical/perceptual proof scheme The perceptual proof scheme is one where the transformations of a shape are not taken into consideration thus the proof becomes specific to the figure drawn. This proof scheme is very similar to the inductive proof scheme discussed earlier. The only difference lies on the fact that in the perceptual proof scheme, there is a figure accompanying the proof and the student does not anticipate any transformations of the figure but considers the figure as static. We found several of our students having this proof scheme especially in the beginning of our intervention. In Figure 7.3 we see an example.

In her attempt to show that the sum of the angles of any triangle is 180° this student used a specific instance of a quadrilateral (a square) and used the specific angle measures in order to argue for the general case. Had the student used a random quadrilateral rather than a square, she would have probably realized that her argument does not work.

Examples of the deductive proof schemes The deductive proof schemes are those where students use general, valid arguments to prove a conjecture. We assigned the deductive proof scheme not only to the students that offered correct deductive arguments, but also to the students that gave partially correct or partially complete deductive arguments in which it was obvious
7.2. The Harel & Sowder proof schemes coding process

Figure 7.3: Example of the empirical/perceptual proof scheme: “We know that a rectangle [sic] has all its angles right. If we bring the diagonal of the square there are 2 right triangles formed with one angle = 90° and the other 2 each one 45 since the diagonal (of the square) divides the angle into 2 equal angles. Therefore we have 90° + (45° × 2) = 180°.”

that their attitude corresponds to that of the deductive proof schemes rather than to empirical ones. We did this because as it was explained in earlier sections, the proof schemes characterize the proof attitudes of students and not the proofs themselves. In figures 7.4 and 7.5 we give one example for each of the above cases.

Figure 7.4: Example of a deductive argument for Task QUAD. “We take a random quadrilateral [figure]. We bring the line segment AD. Two triangles are formed. We know that the sum of the angles of any triangle is 180°, therefore 180° + 180° = 360°.”

As explained in Section 5.2.2, there are two main subcategories of the deductive proof schemes: the transformational and the modern axiomatic proof schemes. The latter proof schemes are not encountered at the level of students of our research, therefore we did not use them in the classification of the students’ proof schemes. The transformational proof schemes were divided by Harel (2007) in four subcategories: causal (CAUS), constructive (CONS), contextual (CONT) (within which belongs also the Greek axiomatic—GRAX) and generic (GEN). The example of the proof given in Figure 7.4 belongs to the category of the Greek axiomatic proof schemes.

In our research we encountered very often the Greek axiomatic proof scheme, and some
times the causal proof scheme. As explained in Section 5.5.1, during the second stage of the move towards the theoretical Van Hiele level students need to use the idea of giving ‘movement’ to a figure in order to find out the general properties of figures. We expected to see the students using this idea in their individual attempts to prove their conjectures. This idea is also central to the transformational proof schemes, and especially to the subcategory of causal proof schemes, where the reasons that cause a general property are taken into consideration. Contrary to what we hoped for, the students rarely used causal proof schemes. This may be explained by the short time of the intervention and the fact that the students had not worked with this idea before, or with deductive proofs in general for that matter. An example of the causal proof scheme is given in Figure 7.6.

In order to give an overview of all proof schemes that were used by our students during our teaching intervention we created two tables. In Table 7.2 on page 76 we offer the coding of the proof schemes used by each student (in chronological order from left to right) based on the main proof scheme categories. In Table 7.3 on page 77 we offer the coding of the proof schemes used by each student based on the proof scheme subcategories. One can see that at the beginning of the intervention the students used more often empirical proof schemes, whereas by the end of the intervention the students offered deductive arguments, if at all. We will offer a detailed discussion of the patterns that can be found in these tables in a later section.
7.2. The Harel & Sowder proof schemes coding process

Figure 7.6: Example of a causal proof scheme for Task OPPANG. “If we form 4 angles of 90°, all angles are equal to each other as well as each of them with its opposite. If we move one angle 10° to the right again each angle will be equal to its opposite. They are similar. Therefore opposing angles are equal.

Conclusions

The students before our teaching intervention had mostly empirical proof schemes, whereas at the end of our intervention mostly deductive ones.

Contrary to our expectations, the transformational/causal proof scheme was not encountered much during our teaching intervention. This may be due to the short period of our intervention and to the fact that the idea of giving ‘movement’ to a figure was something entirely new to the students. We assume that use of dynamic software may help students understand this idea in a shorter time.
Table 7.2: Coding of students’ main proof schemes for all tasks. For the classification of students’ proof schemes we used the coding described in Section 5.2.2. (*' = Student Absent, **UNDEC** = Proof scheme could not be identified, ‘no data’ = we did not record data for these students, blank = The student did not offer any argument, * = a failed attempt to offer a deductive argument).

<table>
<thead>
<tr>
<th>Name</th>
<th>ANGLESUM</th>
<th>QUAD</th>
<th>OPPANG</th>
<th>BIS</th>
<th>BIS-group</th>
<th>EXT-group</th>
<th>TRIMID</th>
<th>TRIMID-group</th>
<th>QUADMID</th>
<th>IF-THEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Olivia</td>
<td>EMPEMP</td>
<td>UNDEC</td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gregory</td>
<td>EMPEXT</td>
<td>EMPEMP</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Katerina</td>
<td>DED</td>
<td>EMP</td>
<td>EXT</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>nodata</td>
<td></td>
</tr>
<tr>
<td>Gayle</td>
<td>UNDEC</td>
<td>DED</td>
<td>EMP</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>nodata</td>
<td></td>
</tr>
<tr>
<td>Nadia</td>
<td>EMPEMP</td>
<td>EMPEMP</td>
<td>DEDDED</td>
<td>EMP</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kelly</td>
<td>EXT/EMP</td>
<td>DEDD</td>
<td>EMP</td>
<td>DEDDED</td>
<td>EMP</td>
<td></td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elli</td>
<td>UNDEC</td>
<td>DEDD</td>
<td>EMP</td>
<td>DEDDED</td>
<td>blank</td>
<td></td>
<td>DEDDED</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Mary</td>
<td>EMPEMP</td>
<td>DEDD</td>
<td>EMP</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>nodata</td>
<td></td>
</tr>
<tr>
<td>John</td>
<td>EMP</td>
<td>DEDD</td>
<td>EMP</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elinor</td>
<td>0EMP</td>
<td>DED</td>
<td>EMP</td>
<td>DEDDED</td>
<td>blank</td>
<td></td>
<td>DEDDED</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Thalia</td>
<td>0</td>
<td>EMP</td>
<td>EXTEMP</td>
<td>UNDEC</td>
<td>EXTEMP</td>
<td>UNDEC</td>
<td>UNDEC</td>
<td>EXTEMP</td>
<td>UNDEC</td>
<td>EXTEMP</td>
</tr>
<tr>
<td>Sunny</td>
<td>Anma</td>
<td>Stue</td>
<td>She</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Angie</td>
<td>DEDDE</td>
<td>UNDEC</td>
<td>EMP</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Nia</td>
<td>EXTEMP</td>
<td>EMP</td>
<td>blank</td>
<td>EMP</td>
<td></td>
<td>blank</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gus</td>
<td>EMPEMP</td>
<td>EXTEMP</td>
<td>0</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Minas</td>
<td>0EMP</td>
<td>EXTEMP</td>
<td>0</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Isis</td>
<td>EXTEMP</td>
<td>0</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
</tr>
<tr>
<td>Stian</td>
<td>EMPEMP</td>
<td>EXTEMP</td>
<td>blank</td>
<td>EMPEMP</td>
<td>0</td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Fiona</td>
<td>UNDEC</td>
<td>DEDD</td>
<td>EMP</td>
<td>DEDDED</td>
<td>0</td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Flora</td>
<td>EMP</td>
<td>DEDDED</td>
<td>blank</td>
<td>DEDDED</td>
<td>UNDEC</td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Katy</td>
<td>UNDEC</td>
<td>DEDDED</td>
<td>blank</td>
<td>DEDDED</td>
<td>UNDEC</td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Debbie</td>
<td>UNDEC</td>
<td>EME</td>
<td>EMP</td>
<td>DEDDED</td>
<td>blank</td>
<td></td>
<td>DEDDED</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Anna</td>
<td>EXTEMP</td>
<td>0</td>
<td>EXTEMP</td>
<td>0</td>
<td></td>
<td>blank</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sunny</td>
<td>EXTEMP</td>
<td>EXTEMP</td>
<td>0</td>
<td>DEDDED</td>
<td></td>
<td></td>
<td>blank</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
<tr>
<td>Elinor</td>
<td>0EMP</td>
<td>EMP</td>
<td>blank</td>
<td>EMP</td>
<td></td>
<td>blank</td>
<td>DEDDED</td>
<td></td>
<td>blank</td>
<td></td>
</tr>
</tbody>
</table>
### Table 7.3: Coding of students’ proof scheme subcategories for all tasks. For the classification of students’ proof schemes we used the coding described in Section 5.2.2. (‘0’ = Student Absent, ‘UNDEC’ = Proof scheme could not be identified, ‘no data’ = we did not record data for these students, ‘blank’ = The student did not offer any argument, * = a failed attempt to offer a deductive argument)
Chapter 7. Results

7.3 Correlations between Van Hiele level and Harel & Sowder proof scheme

One of our hypotheses is the following:

H.2 Given the correspondence between the Harel & Sowder proof schemes and the Van Hiele levels of thinking, there should be a correlation between corresponding Van Hiele levels and Harel & Sowder proof schemes.

Moreover, in order to answer our main research question, we had to look at how students’ reasoning changes when students move from the descriptive to the theoretical Van Hiele level. Therefore, we looked both at how the Van Hiele levels and proof schemes of the students correlate before and after our intervention and at how the progress in the Van Hiele level of a student (as identified by the Van Hiele pre-test) correlates with the progress in the student’s Harel & Sowder proof scheme. More specifically, we have the hypothesis that when students are at a lower Van Hiele level than 3 they will use more often empirical rather than deductive proof schemes, whereas when they move to higher Van Hiele levels (3 or 4) they will use more often deductive rather than empirical proof schemes. We have four specific questions related to the above questions and hypotheses:

1. What is the correlation between Van Hiele level and the use of the deductive Harel & Sowder proof schemes before the intervention?

2. What is the correlation between Van Hiele level and the use of the deductive Harel & Sowder proof schemes after the intervention?

3. What is the correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in vH level from any vH level to any higher level?

4. What is the correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in vH level from one of the levels 0, 1 or 2 to one of the levels 3 or 4?

Before we look at how we answered the above questions, some definitions are necessary.

• Progress in Van Hiele level

We define progress of a student in their Van Hiele level as the move from a lower Van Hiele level to a higher one. We measured the Van Hiele level of the students with both criteria as defined by Usiskin (1982)—3/5 and 4/5—so we will look at both these criteria. We also use a stricter definition of progress which is the move from Van Hiele levels 0, 1 or 2 to Van Hiele levels 3, or 4, since this is the specific transition we aimed at achieving in this research project and in which we expected the transition from using empirical to using deductive proof schemes to take place. We will use the name PrvH0 for the first kind of progress, and PrvH1 for the second kind of progress. Thus, we have the following two definitions:

– We call ‘PrvH0’ the progress of the students from one Van Hiele level to a next.
7.3. Correlations between Van Hiele level and Harel & Sowder proof scheme

- We call ‘PrvH1’ the progress of the students from Van Hiele levels 0, 1 or 2 to Van Hiele levels 3 or 4.

**Progress in Harel & Sowder proof scheme**

The proof scheme of a student is determined at any stage by the following fraction which represents the frequency of the use of a certain proof scheme by the student:

\[ F = \frac{\text{# of instances of proof scheme } X \text{ in a certain number of tasks}}{\text{# of tasks for which we have data}} \]

If \( F = 1 \), then the student has proof scheme \( X \).

We define progress of a student in their Harel & Sowder proof scheme as the positive change in the value of \( F \) based on the frequency of deductive proof schemes when comparing the first half (\( F_{\text{start}} \)) to the last half (\( F_{\text{end}} \)) of the meetings of the intervention.

\[ F = F_{\text{end}} - F_{\text{start}} = \frac{\text{# of DED in first 5 tasks done}}{a} - \frac{\text{# of DED in last 5 tasks done}}{a}, \]

where \( 0 < a \leq 5 \) is the number of tasks for which the students were present and we have data of their work.

In table 7.4 we include all the variables we used for the statistical analysis of the results.

<table>
<thead>
<tr>
<th>Name of Variable</th>
<th>Scale</th>
<th>Scale Type</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre vH level (3/5)</td>
<td>1 to 4</td>
<td>ordinal</td>
<td>The score of a student in the Van Hiele pre-test as calculated by the ‘loose’ criterion (3 out of 5 questions correct at each level)</td>
</tr>
<tr>
<td>Pre vH level (4/5)</td>
<td>1 to 4</td>
<td>ordinal</td>
<td>The score of a student in the Van Hiele pre-test as calculated by the ‘strict’ criterion (4 out of 5 questions correct at each level)</td>
</tr>
<tr>
<td>Post vH level (3/5)</td>
<td>1 to 4</td>
<td>ordinal</td>
<td>The score of a student in the Van Hiele post-test as calculated by the ‘loose’ criterion (3 out of 5 questions correct at each level)</td>
</tr>
<tr>
<td>Post vH level (4/5)</td>
<td>1 to 4</td>
<td>ordinal</td>
<td>The score of a student in the Van Hiele post-test as calculated by the ‘strict’ criterion (4 out of 5 questions correct at each level)</td>
</tr>
<tr>
<td>PrvH0</td>
<td>1 or 0</td>
<td>binary</td>
<td>1 when there is progress from any Van Hiele level to any higher level, and 0 otherwise.</td>
</tr>
<tr>
<td>PrvH1</td>
<td>1 or 0</td>
<td>binary</td>
<td>1 when there is progress from any of the Van Hiele levels 0, 1, or 2 to any of the Van Hiele levels 3 or 4, and 0 otherwise.</td>
</tr>
<tr>
<td>( F )</td>
<td>0 &lt; ( F ) &lt; 1</td>
<td>ratio</td>
<td>( F = # \text{ of instances of proof scheme } X \text{ in a certain number of tasks / # of tasks} )</td>
</tr>
<tr>
<td>( F_{\text{end}} - F_{\text{start}} )</td>
<td>(-1 &lt; F &lt; 1)</td>
<td>ratio</td>
<td>( F_{\text{end}} - F_{\text{start}} = (# \text{ of DED in first 5 tasks done / } a) - (# \text{ of DED in last 5 tasks / } a), ) where ( 0 &lt; a \leq 5 ) is the number of tasks for which the students were present and we have data of their work.</td>
</tr>
</tbody>
</table>
Chapter 7. Results

In order to answer question 1 (What is the correlation between Van Hiele level and the use of the deductive Harel & Sowder proof schemes before the intervention?), we conducted a Spearman’s rho test (since the variables pre vH level (3/5) and pre vH level (4/5) are ordinal variables). We chose the test to be 1-tailed because our hypothesis was directional. Table 7.5 shows the results of the 1-tailed test.

<table>
<thead>
<tr>
<th>Spearman’s rho</th>
<th>Pre vH level (3/5)</th>
<th>Pre vH level (4/5)</th>
<th>$F_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre vH level (3/5)</td>
<td>Correlation Coefficient</td>
<td>1.000</td>
<td>.637**</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.637**</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>21</td>
<td>20</td>
</tr>
<tr>
<td>Pre vH level (4/5)</td>
<td>Correlation Coefficient</td>
<td>.001</td>
<td>.045</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.001</td>
<td>.045</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>$F_1$</td>
<td>Correlation Coefficient</td>
<td>.363</td>
<td>.370*</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.363</td>
<td>.370*</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>19</td>
<td>22</td>
</tr>
</tbody>
</table>

** Correlation is significant at the 0.01 level (1-tailed).
* Correlation is significant at the 0.05 level (1-tailed).

Table 7.5: Correlation (1-tailed) between Pre Van Hiele level (3/5 and 4/5) and frequency of deductive Harel & Sowder proof schemes at the beginning of the intervention. $F_1$ is Task ANGLESUM (see Appendix B.2).

There was a significant relationship between the vH level of the students (as measured with the stricter criterion, that of 4/5) and the use of deductive proof schemes at the beginning of our intervention (that is, during Task ANGLESUM), $r = .370$, $p$ (one-tailed) < .05. This confirms our hypothesis that the higher the Van Hiele level of a student, the more he or she will use deductive proof schemes.

Similarly, for answering question 2 (What is the correlation between Van Hiele level and the use of the deductive Harel & Sowder proof schemes after the intervention?) we conducted a 1-tailed Spearman’s rho test. Table 7.6 shows the correlations between the Van Hiele level of the students as measured at the end of the intervention, and the frequency of using deductive proof schemes in the final two tasks given to the students (that is, tasks QUADMND and IF-THEN).

<table>
<thead>
<tr>
<th>Spearman’s rho</th>
<th>Post vH level (3/5)</th>
<th>Post vH level (4/5)</th>
<th>$F_{10,11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Post vH level (3/5)</td>
<td>Correlation Coefficient</td>
<td>1.000</td>
<td>.793**</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.793**</td>
<td>.001</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>Post vH level (4/5)</td>
<td>Correlation Coefficient</td>
<td>.001</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.001</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>$F_{10,11}$</td>
<td>Correlation Coefficient</td>
<td>.422</td>
<td>.666*</td>
</tr>
<tr>
<td></td>
<td>Sig. (1-tailed)</td>
<td>.422</td>
<td>.666*</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

** Correlation is significant at the 0.01 level (1-tailed).
* Correlation is significant at the 0.05 level (1-tailed).

Table 7.6: Correlation (1-tailed) between Post Van Hiele level (3/5 and 4/5) and frequency of deductive Harel & Sowder proof schemes at the end of the intervention. $F_{10}$ is Task QUADMID (B.12) and $F_{11}$ is Task IFTTHEN (see Appendix B.13).
The results of the test show that there was a significant relationship between the Van Hiele level as measured at the end of the intervention with the 4/5 criterion and the frequency of using deductive proof schemes, $r = .666$, $p$ (one-tailed) $< .05$. However, we need to be careful in the way we interpret these results, since we see that there is a large drop in the number of the students (from $N = 22$ to $N = 11$) from pre to post test. When the number of participants is lower, a larger correlation would be needed to reach the same level of significance.

For answering question 3 (What is the correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in vH level from any level to any higher level?) we conducted a 1-tailed Pearson test (since we now had to do with binary and ratio scale type variables). Table 7.7 shows that the results of this test were not significant although they are close to significant ($r = .491$).

<table>
<thead>
<tr>
<th></th>
<th>PrvH0 (3/5)</th>
<th>PrvH0 (4/5)</th>
<th>$F_{end} - F_{start}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PrvH0 (3/5) Pearson Correlation</td>
<td>1</td>
<td>-.327</td>
<td>.407</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.178</td>
<td>.095</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>13</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>PrvH0 (4/5) Pearson Correlation</td>
<td>-.327</td>
<td>1</td>
<td>.491</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.178</td>
<td>.053</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>10</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>$F_{end} - F_{start}$ Pearson Correlation</td>
<td>.407</td>
<td>.491</td>
<td>1</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.095</td>
<td>.053</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.7: Correlation (1-tailed) between PrvH0 (3/5 and 4/5) and progress in Harel & Sowder proof schemes during the intervention ($F_{end} - F_{start}$).

It is worth noting that there is a negative correlation between the two Van Hiele test criteria (3/5 and 4/5) which may indicate that the measurement instruments (Van Hiele test) are not precise enough.

We conducted a similar test in order to answer question 4 (What is the correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in vH level from one of the levels 0, 1 or 2 to one of the levels 3 or 4?).

<table>
<thead>
<tr>
<th></th>
<th>PrvH1 (3/5)</th>
<th>PrvH1 (4/5)</th>
<th>$F_{end} - F_{start}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PrvH1 (3/5) Pearson Correlation</td>
<td>1</td>
<td>.408</td>
<td>251</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.121</td>
<td>.204</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>14</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>PrvH1 (4/5) Pearson Correlation</td>
<td>.408</td>
<td>1</td>
<td>-.060</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.121</td>
<td>.426</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>10</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>$F_{end} - F_{start}$ Pearson Correlation</td>
<td>.251</td>
<td>-.060</td>
<td>1</td>
</tr>
<tr>
<td>Sig. (1-tailed)</td>
<td>.204</td>
<td>.426</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>13</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.8: Correlation (1-tailed) between PrvH1 (3/5 and 4/5) and progress in Harel & Sowder proof schemes during the intervention ($F_{end} - F_{start}$).

In Table 7.8 we can see that although the progress in Van Hiele level that we are checking changes to PrvH1 (for a definition see 7.4), the results are still not significant. Moreover, it is interesting to note that with the 4/5 criterion the Van Hiele level of the students shows a slight
change in the opposite direction than the frequency of use of the deductive proof schemes, which was unexpected. One possible explanation for these results might be the small number of the students \((N = 12)\) that we could consider for this statistical test.

### Conclusions

There is a correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking before and after our intervention.

We were unable to give a certain answer to the question of whether there is a correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in vH level based on our statistical analysis, due to the small number of the students for which we had data.

In order to give a final answer to our second hypothesis (H.2), we need to look closely at those moments during our intervention when the students displayed progress in their Van Hiele level and/or moved from having an empirical to having a deductive proof scheme by conducting a qualitative analysis of our data. This will be the subject of the following section.

#### 7.4 The role of our teaching strategy: A qualitative analysis

In what follows we will discuss whether and how we achieved the move of the students from the descriptive to the theoretical Van Hiele level. We will discuss this with respect to two different frameworks.

First (Section 7.4.1), we will present the results of our intervention in terms of the three stages of the learning sequence as these were defined in Table 2.4. We will look at whether the tasks we designed helped us achieve our aim and whether the students managed to move successfully through the three stages in order to understand the concept of proof in mathematics.

Second (Section 7.4.2), we will present the results in terms of the five Van Hiele phases of the teaching sequence (for a description of the phases see Section 2.7 on page 18) and we will look at whether the tasks we designed for each phase assisted the move of the students from the descriptive to the theoretical Van Hiele level.

#### 7.4.1 Moving through the three stages towards deductive proof schemes

**Stage 1: Understanding the distinction between empirical and deductive arguments**

As discussed in Section 5.5.1, we believe that the first stage that must be completed for moving towards the theoretical Van Hiele level (level 3) for the concept of proof has to do with making the distinction between empirical and deductive arguments. We claim that this first stage is completed in two parts: First, one discovers the basic differences between general (and thus more convincing) arguments and arguments specific to the figure (empirical and thus not convincing for all cases); Second, once the distinction has been noticed one needs to practice sufficiently in order to achieve a good understanding of it.
The first lesson of the intervention was designed for the first part described above. We planned a discussion of the various arguments offered by the students themselves during a previous task (see Task ANGLESUM in Appendix B.2) for convincing someone that ‘the sum of the angles of any triangle is 180°.’ Most of the students (16 out of 22, or 73%) had used either empirical or circular arguments. More specifically, 11 students (50%) offered an empirical argument, and 5 (22%) of them a circular argument, whereas only 2 students (9%) offered a deductive argument. The above results are summarised in Table 7.9.

<table>
<thead>
<tr>
<th>Type of argument</th>
<th>Frequency (n = 22)</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deductive</td>
<td>2</td>
<td>9%</td>
</tr>
<tr>
<td>Inductive</td>
<td>11</td>
<td>50%</td>
</tr>
<tr>
<td>Circular</td>
<td>5</td>
<td>23%</td>
</tr>
<tr>
<td>Undecided</td>
<td>4</td>
<td>18%</td>
</tr>
</tbody>
</table>

Table 7.9: Kinds of arguments offered by the students for Task ANGLESUM.

During this first lesson, we discussed the advantages and disadvantages of all the kinds of arguments offered by the students, as well as of the arguments that were provided in Task CHOOSING (see Appendix B.3) which the students had to complete right after they had offered their own arguments. The discussion was long (approximately one hour) and the students came up with many advantages and disadvantages of all the arguments. The advantages and disadvantages the students came up with can be seen in Appendix C.3. Let us look at some parts of the discussion during our first lesson, which let the students discover these advantages and disadvantages.

1 [Discussing the advantages and disadvantages for the following argument: A square can be cut into two parts by bringing the diagonal. The two parts are triangles and each of them is an isosceles and right triangle. The sum of the angles of these triangles is 180°. Therefore the sum of the angles of a triangle is 180°.]
2 Teacher: I am listening. Advantages? Tell me.
3 Gregory: I want to say a disadvantage.
4 Teacher: OK, disadvantage.
5 Gregory: It is not convincing for a different kind of triangle.
6 Teacher: Nice. This [argument] also depends on the figure, right? ‘It depends on the figure’ [writing on the board and the students are copying in their handouts].
7 As far as I understood in all these [showing the arguments discussed earlier on the board], we said that it [the argument] depends on the figure and your fellow student just said that it is not convincing for this reason. Therefore, when we say that it depends on the figure; it is as if we are saying that it is not convincing. Why? Because we want to give an argument for [showing] that every triangle is 180°, but these arguments depend on the figure. Nice. So, we ended up saying that this is a disadvantage. Now. Anything else?
8 Katerina: It is more easy for us to understand.
9 Teacher: Yes, nice. This is an advantage right?
Chapter 7. Results

[The discussion continues and the students offer other advantages and disadvantages]

A lively discussion began when reviewing Eleni’s argument (see Appendix B.3): “I tore the angles up and put them together. It came to a straight line which is $180^\circ$. I tried for an equilateral and an isosceles as well and the same thing happened.” Many students were convinced by this answer at first sight.

*Theano:* It is ‘practical’.

*Teacher:* OK. I will write it as…I imagine…What is this, good?

*Students:* Advantage.

*Teacher:* Advantage. It is easy when we do practical things, right? [Writes it down on the board.] Someone said it is ‘artistic’. Who said this?

*Nia:* Me.

*Teacher:* You? That it is very artistic, I liked this one, I believe this as well and I guess it goes with the…advantages. [Writes it down on the board.]

*Gayle:* It is easy to understand.

*Teacher:* Easy to understand [writes it down with the advantages].

*Anna:* I am convinced.

*Teacher:* You are convinced. Let’s see what the rest of the students think. Let’s vote again. Who was convinced by this? [The majority of the class raises their hands.] Everyone is convinced …Who wasn’t convinced? [About five students raise their hands and others are shouting “for all triangles!” as a counter argument to those that were convinced.] OK, so you [points at a girl] said it is convincing. Why?

*Anna:* Because the way I look at it, that forms the straight line…it convinces me.

*Teacher:* So, you are convinced by this [pointing at the figure of the argument on the board]. So can you now say that it will be the same for any triangle? Are you sure?

*Anna:* Yes.

[At this point many students have their hands raised and want to contribute to the discussion.]

*Teacher:* Nice. You? Are you convinced or not? [Pointing to a girl that is raising her hand.]

*Thalia:* I am not sure, but it is logical.

*Teacher:* It is logical. OK, let’s write this down in the advantages. [Writes it down.]

*Teacher:* Who is not convinced?

*Katerina:* Me, because there is a possibility that we make a mistake when cutting, or whatever…

*Teacher:* When cutting. OK. How did we call this? There is a possibility of error in the measurement. True. If you cut it wrongly…This sounds similar to measuring, right?
Mary: We don’t know if it holds for all triangles…

Student: It holds for all triangles

Mary: …so we need to draw a few more.

Kelly: But she said [Eleni in the proof] that she drew a random triangle.

Teacher: Does she say it is random?

Fiona: She says she tried it also on an equilateral and an isosceles.

Teacher: Nice. OK, so we tried it in a few different kinds of triangles.

Kelly: And one of those is random. It is not something that we chose…

Fiona: We might have been lucky and ended up with the right sum of angles.

Teacher: Yes, this may have happened, isn’t that so?

Students: Yes.

Teacher: Isn’t this as if she measured them and added them up? Only that she did it in a practical way.

[Discussing about the similarity with the argument that uses measurement for a couple of minutes. Students agree.]

Teacher: So are we convinced for all triangles? This is what we want to know. How many are convinced that by this argument we can be sure that for all, infinitely many triangles, it will hold?

[Many students still raise their hands but they are obviously less than before the discussion.]

Teacher: They are less than those that did not raise their hand.

Olivia: But, even if it [the triangle] becomes right, again it will [be 180°] …

Teacher: How do you know?

[Students start arguing with each other, their are divided roughly in two groups and are trying to convince each other. The teacher tries to bring the discussion back to order because they are talking over each other and no argument can actually be heard.]

Teacher: OK, you [to Olivia], tell me why you think it is OK?

Olivia: I believe that even if we take a right triangle we will find the same.

Teacher: But, how do you know? Did you sign a contract with anyone? [Laughs]

Olivia: Because…one angle is 90, and…the other two angles that are left are again 90.

Teacher: How do you know?

Student: How do you know?

Gregory: She tried it! [laughs]

Olivia: I tried it!

Teacher: On all, infinitely many triangles?
Olivia: No, on one!

Fiona: When you want to prove something you cannot use it as a given. You have to doubt it as if it’s not true.

Teacher: So you [to Olivia] assumed that in a right triangle, since one of the angles is 90, the rest will also be 90. But how can we know this? It is this that we are trying to support!

After this discussion a few students were still convinced by Eleni’s empirical argument, whereas the majority of the class agreed to write as a disadvantage of the argument that it depends on the figure. We believe that the fact that the proposition ‘the sum of the angles of a triangle is 180°’ was already known and presented as something true to the students in their ordinary Mathematics lesson, led many of them to circular reasoning (like Olivia in the above example, lines 56 to 66). For this reason, we think that discussing also other, unfamiliar examples with the students at this stage may help the students understand better the distinction between deductive and inductive arguments.

After the above, we discussed a deductive argument, the one offered in Georgia’s argument in Task CHOOSING (see Figure 7.7).

Teacher: So, I am listening. Let’s start from whether this one is more practical or more theoretical.

Students: Theoretical. [Many students say this without hesitation.]

Teacher: What is this, good or bad?

[Most of the students say ‘good’. A couple say it is ‘bad’ because it makes it more difficult.]

Teacher: OK we will put it in as good. […] We did say that practical arguments…
A student: May have errors.

Teacher: They may be good but they also have a risk of having errors. Let’s put it in the advantages.

Kelly: Yes but in the theoretical we need to prove what we say again.

Teacher: So it is more difficult right?

John: The theoretical is more convincing!

Teacher: It is more convincing, eh?

Katy: We learned something on this and we know certainly that it holds. [Referring to the rule about alternate angles.]

Teacher: Something, you mean…from before.

Katy: Yes.

Teacher: So, it is something that depends on previous knowledge.

Katy: Yes.

Teacher: Basically yes, this depends on previous knowledge. And this is a good thing because we are sure, right? [Writes it on the board.] Anything else?

Fiona: I believe that this is the proof.

Teacher: You are convinced. Who is convinced? I want to see hands.

[Almost all students raise their hands.]

Teacher: Convincing! Let’s write this down. Why is it convincing? What difference does this one have? Will the ones that are convinced let me know?

Katy: Because it is based on certain givens.

Teacher: …that we have shown. This we have said, yes, that ‘it depends on previous knowledge’. But there were other arguments that also depend on previous knowledge and you didn’t tell me that they were convincing. [Fiona is raising her hand.] Tell me.

Fiona: It does not depend on the figure because whatever angle there is, it will be, since they are…eh…alternate angles, they will be equal!

Teacher: Do you agree? [To the rest of the class.]

Students: [All with one voice] Yes.

Teacher: All of you? Bravo. OK? Because it does not depend on the figure. This is a good thing, right?

A student: Very good!

Teacher: We will write it down here with the ‘convincing’. Because the other ones [the arguments] that depend on the figure, you said they were not convincing.

[The teacher spends a few minutes on showing to the students how although one can move the sides and angles of the figure to create other possible triangles, the argument will always hold. This is what the class had earlier called ‘the argument
has movement’ (which makes it applicable for infinitely many cases) and it was
considered as an advantage.]

Teacher: So, won’t we find a disadvantage guys?

Students: [Many together] It doesn’t have any!

Fiona: Miss we didn’t put that it is simple, and logical.

Teacher: OK, put it in, ‘simple’ and ‘logical’. Is it difficult?

[Most of them say it is easy but a couple say it is difficult.]

Teacher: Why is it difficult?

Flora: In the theoretical [part]. Because, the alternate, someone might not know it.

Teacher: Yes because it requires [to have certain] knowledge.

John: It depends on previous knowledge.

[Chat about the fact that we had mentioned this earlier.]

Teacher: What other difficult part does it have? Let’s say you start with drawing a
triangle [draws one on the board] and you look at it…we say it is a random one. And
you say ‘what should I do now to show it’…For instance you can start measuring.
That’s the simplest [you can do]. But you were not convinced by it, so we decided
that it was not very convincing because it is for specific triangles, it isn’t for all
[triangles]. What are you going to do?

A student: We will bring a parallel.

Teacher: So? Is this an easy thing to think of?

Fiona: The way of thinking. It is original but…

Teacher: So it requires you to think of drawing an extra line on the figure, you
should think it on your own, and also make the line parallel.

Fiona: It is very original yes, and difficult.

Teacher: We will write ‘original’ in the advantages and ‘difficult to think’ [in the
disadvantages], none of you came up with it. Or, actually, ‘it needs to bring assisting
lines’ that’s how we will write it down. All right?

Students: Miss, they all needed assisting lines!

Teacher: It wasn’t in all. In the one you measured, you measured all angles, did
you bring any [lines]? You did not bring anything. [Checking the previous proofs
with the students to make sure they understand what this means.]

From the above we see that many students (although not yet all) managed to take during
this introductory lesson the first and crucial step of realising the difference between convincing
and non-convincing arguments in relation with the generality of the arguments (lines 13, 15,
33–36). In other words, they became aware of the distinction between empirical and deductive
arguments.

As we already discussed in previous sections, realising the existence of such a distinction
does not imply acquiring a deep understanding of the distinction or immediately stopping being
convinced by inductive arguments. Task QUAD (see Appendix B.5) was designed for promoting
further understanding of this distinction. The task explicitly asked the students to come up with the advantages and disadvantages of their own arguments supporting the proposition ‘the sum of the angles of any quadrilateral is 360 degrees.’ The kinds of arguments that the students gave can be divided into three main categories:

**Category 1: Offering an empirical (inductive) argument**  This category includes 15 (out of 24) students. Some of them used either a rectangle or a square (or both), and calculated 4 times 90° to get 360° for the quadrilateral. Some of them also drew the diagonal of the rectangle or square in order to use also the fact that the sum of the angles of a triangle is 180°. These students were probably confused about how they can use this fact, and they were not sure yet about how they could construct a general argument. The rest of the students did not bring a diagonal, so they did not use the earlier fact. They just measured. Another kind of inductive argument used by these students was to draw random quadrilaterals in addition to squares and rectangles, and measure their angles. None of the students could actually add them up to exactly 360°. Others used a parallelogram and the fact that each pair of angles adds up to 180° in order to support the proposition about any quadrilateral. (See Figure 7.8 for a sample answer.)

![Figure 7.8: Example of an empirical argument for Task QUAD.](image)

**Category 2: Offering a deductive argument**  This category includes 8 (out of 24) students who provided a correct deductive argument that did not depend on specific figures. (See Figure 7.9 for a sample answer.)

![Figure 7.9: Example of a deductive argument for Task QUAD. “We take a random quadrilateral [figure]. We bring the line segment AD. Two triangles are formed. We know that the sum of the angles of any triangle is 180°, therefore 180° + 180° = 360°.”](image)
Chapter 7. Results

Category 3: Being externally convinced  This category includes just one student who used a proof like Eleni’s (see Appendix B.3). He drew three random rectangles and then claimed—but he did not do it!—that he cut the angles and put them together one next to the other and they formed a circle (see Figure 7.10).

Figure 7.10: Example of an argument of an externally convinced student for Task QUAD: “I cut all 4 angles from all quadrilaterals and while I was putting them in one line (all three) I observed that the angles formed a circle (not even one mistake, without shaking).”

To summarise the above, 8 out of 24 students gave a deductive argument and 15 out of 24 students (63%) gave an empirical/inductive argument to show that the sum of the angles of any quadrilateral is 360°. Moreover, it is worth noting that the percentage of students that used an external conviction proof scheme was reduced considerably (from 22% to 4%). This may be explained by the fact that students already had been offered as a given in school that the sum of the angles of a triangle is 180°, which means they possibly took for granted that the statement is true while completing Task ANGLESUM (see Table 7.10).

<table>
<thead>
<tr>
<th>Type of argument</th>
<th>Frequency (n = 24)</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deductive</td>
<td>8</td>
<td>33%</td>
</tr>
<tr>
<td>Inductive</td>
<td>15</td>
<td>63%</td>
</tr>
<tr>
<td>Externally convinced</td>
<td>1</td>
<td>4%</td>
</tr>
</tbody>
</table>

Table 7.10: Kinds of arguments offered by the students for Task QUAD.

Since so many students still used an empirical argument, how can we say whether the first lesson helped them move closer to the deductive proof scheme or not? From the 15 students who used an empirical argument, 8 stated as a disadvantage of their argument either that it depends on the figure so it is not for all quadrilaterals (Figure 7.11), or that it is not convincing (Figure 7.12), or both. Only 6 of them thought their empirical argument was convincing, and one did not say anything at all. The above results are summarised in Table 7.11.
7.4. The role of our teaching strategy: A qualitative analysis

<table>
<thead>
<tr>
<th>Type of answer</th>
<th>Frequency $(n = 15)$</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aware that it is not convincing and/or</td>
<td>8</td>
<td>53%</td>
</tr>
<tr>
<td>depends on the figure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not aware that it is not convincing or</td>
<td>6</td>
<td>40%</td>
</tr>
<tr>
<td>that it depends on the figure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blank</td>
<td>1</td>
<td>7%</td>
</tr>
</tbody>
</table>

Table 7.11: Students who used inductive arguments for Task QUAD.

Figure 7.11: A student who correctly identified the non-generality of her argument (about a rectangle). “Advantages: 1. simple, 2. short, 3. easy, 4. is based on previous knowledge. Disadvantages: It is proven only in this figure, it is therefore a special proof. 2. There is a possibility of mistake when you do it.”

Figure 7.12: A student that mistakenly believed her argument to be general. (The argument this student offered was about a parallelogram.) “Advantages: 1. It is easy; 2. It has theory; 3. It holds for all quadrilaterals. Disadvantages: 1. It is difficult to understand.”

We see in Table 7.11 that after completing Task QUAD, only a quarter of the students still thought that their inductive arguments were convincing and could not see that they depended on the figure. About half of the students who used an empirical argument, already from the first lesson realised that their argument was not convincing enough, although they still used such arguments because they found them easier. We think this is an outcome of the discussion of the advantages and disadvantages of arguments. Moreover, 6 of the 8 (75%) students who used a deductive argument were aware that it does not depend on the figure, while only two thought that it was not convincing enough. (Figures 7.13 and 7.14 show samples of both kinds of answers.)

Figure 7.13: A student who offered a correct deductive argument and identified it as convincing. “Advantages: 1. It has movement, 2. It is theoretical, 3. It is convincing. Disadvantages: 1. It needs to bring assisting lines [diagonal].”

Figure 7.14: A student who offered a correct deductive argument but still expressed doubts. “Advantages: 1. Logical, 2. Theoretical, 3. Simple/Easy, 4. Short. Disadvantages: 1. We can never be sure, there are always doubts.”
Chapter 7. Results

The above results are summarised in Table 7.12.

<table>
<thead>
<tr>
<th>Type of answer</th>
<th>Frequency (n = 8)</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aware that it is convincing</td>
<td>6</td>
<td>75%</td>
</tr>
<tr>
<td>Not aware that it is convincing</td>
<td>2</td>
<td>25%</td>
</tr>
</tbody>
</table>

Table 7.12: Students who used deductive arguments for Task QUAD.

Another task we used in the first stage of our intervention was BIS (see Appendix B.7) where we asked the students to offer two different kinds of arguments: one empirical that depends on the figure and one general that does not depend on the figure. With this task we aimed at assisting the students to understand better the distinction between the two kinds of arguments. Some students gave only empirical arguments similar to the ones offered for previous tasks, some gave both an empirical argument and a correct deductive one (see Figure 7.15), and others gave one circular and one empirical argument (see Figure 7.16).

Figure 7.15: A deductive and an empirical argument for Task BIS. “1. Since all of them [the angles] together are 180°, then the sum of their halves will be 90°. 2. $K_{1,2} = 54°$ therefore $K_1 = K_2 = 27°$. $K_{3,4} = 126$ therefore $K_3 = K_4 = 63$, $K_4 + K_1 = 63° + 27° = 90°”

Figure 7.16: A circular and an empirical argument for Task BIS. “1. Since $KD_1$ perpendicular to $KD_2$, therefore $D_2KD$ = right. 2. Measuring with the protractor we find that angle $D_2KD$ is right.”
7.4. The role of our teaching strategy: A qualitative analysis

Out of the 20 students that completed this task, 11 (55%) gave (at least) an empirical argument, and 6 gave a correct deductive argument. Finally, 3 out of 20 students attempted to give a deductive argument but failed whereas only 2 did not attempt to give any argument. One of the arguments offered by a student could not be categorised in any of the above categories. These results are summarised in Table 7.13.

<table>
<thead>
<tr>
<th>Type of argument</th>
<th>Frequency (n = 20)</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductive</td>
<td>11</td>
<td>55%</td>
</tr>
<tr>
<td>Correct deductive</td>
<td>6</td>
<td>30%</td>
</tr>
<tr>
<td>Undecided</td>
<td>1</td>
<td>5%</td>
</tr>
<tr>
<td>No argument</td>
<td>2</td>
<td>10%</td>
</tr>
</tbody>
</table>

Table 7.13: Kinds of arguments offered by the students for Task BIS. The task asked the students to offer one inductive and one deductive argument

We can conclude from all the above examples discussed for the first stage of our intervention that the students managed to make a move closer to adopting the deductive proof scheme, since they took the first step of realising the non-generality of the empirical proof scheme and therefore its failure to convince. However we can see that still only a bit more than a half of the class offered (or attempted to offer) deductive arguments. We think this is not unexpected since the students in this first stage are only expected to realise the distinction between inductive and deductive arguments and it is only with enough practice, which comes in Stage 2, that they are expected to start completely abandoning empirical arguments.

In what follows we will look at some discussions between the students during Stage 2 in order to locate the further steps the students took towards the theoretical Van Hiele level. We will keep a focus on locating the instances where the students that used mostly empirical arguments were in fact changing their attitude and were starting to look for deductive arguments in order to be more convincing.

Stage 2: Abandoning the use of empirical arguments in favor of deductive ones

There are several examples of cases during our intervention where the students move from an empirical to a deductive proof scheme, a few of which will be discussed here in chronological order. In general, the students made the transition from the empirical to the deductive proof scheme at various moments during the intervention, and some students did not manage to make this transition at all.3 The four examples that will be discussed here are taken from class discussions and group work during which the aim was to provide convincing arguments for showing that certain (general) mathematical statements are true. In the examples we will see various kinds of attitudes (empirical, deductive, and external conviction) towards proof employed by the students, but we will focus on those attitudes which we think show evidence of students’ moving from empirical or external conviction to deductive proof schemes.

The first example of a student making a move closer to the deductive proof scheme takes place in the following context. The students in the previous lesson had to come up with an argument to support the truth of the proposition ‘the sum of the angles of any quadrilateral is $360^\circ$.’ The following lesson started by discussing the arguments in terms of their advantages and disadvantages. The teacher talked about some of the empirical arguments that the students

---

3A complete overview of the progress of each student was offered in Table 7.2 on page 76.
had used, among which those which used a square and said that the sum of its angles is $360^\circ$, and the conclusion of the discussion was that the argument was not convincing. Thus the teacher suggested that someone who had used this argument should try to figure out a more general argument for the general quadrilateral, and also said that they could now use the fact that the sum of the angles of a triangle is $180^\circ$. The discussion below is on this subject.

**Example 1: A look at whole class discussion after task QUAD (see Appendix B.5)**

**Teacher:** Can someone of those that did this previous one [using an empirical argument involving a square] tell me how they would do it for the random quadrilateral? What would they think? What holds always in a quadrilateral? And to also use the fact that the sum of the angles of a triangle is $180^\circ$.

 [...] 

**Olivia:** Again we would bring a diagonal.

**Teacher:** Again. OK let’s do it [draws a diagonal].

**Olivia:** And then we measure the angles…

**Teacher:** Do we need to measure them? Let’s think…

**John:** No! Miss shall I say?

**Olivia:** No it is $180^\circ$. 

[John keeps talking and wants to say the answer.]

**Teacher:** Wait a minute, don’t shout please, I want Olivia to tell me because those of you who already did it there’s no point to tell me, I know that you did it. The point is for everyone to be able to do it. OK? So…tell me.

[In the meantime teacher gives names to all angles formed after bringing the diagonal of the quadrilateral.]

**Olivia:** We know that every triangle is $180^\circ$. If I add the two triangles, that is, $180^\circ$ times 2, it will be $360^\circ$.

Here the teacher noticed that Olivia had the quite simplistic idea that we can multiply $180^\circ$ by two and find the result we want, but this argument is not entirely correct since it holds only in case we divide the quadrilateral into two triangles by bringing a diagonal. If we had divided the quadrilateral in more than two triangles then we would still multiply the number of triangles by 180 (which is of course not correct). Therefore the teacher gave some hints to Olivia to help her state the argument in a more precise way.

**Teacher:** Nice. Are the angles of the triangle also angles of the quadrilateral?

**Olivia:** Yes.

**Teacher:** Why? This angle of the triangle, $B_1$, is also an angle of the quadrilateral?

**Olivia:** It is half.

**Teacher:** Not half...
Olivia: More or less.
Teacher: …one part of \( B \).
Olivia: Yes.
Teacher: So, what do we know for this triangle?
Olivia: That \( A \) plus \( B_1 \) plus \( D_1 \) is equal to 180.

Olivia seemed to grasp the idea of the addition, and she came up with it quite easily without the help of the teacher. She also seemed to see why this would lead to the desired argument.

Teacher: For the other one what do I know?
Olivia: That… \( B_2 \) plus \( D_2 \) plus \( G \) is equal to 180.
Teacher: Now, if we add all this, we get […] 360. Do you see how simple? This one now [the argument], can someone come and tell us that it does not hold for all quadrilaterals?
Olivia: No.
Teacher: What would have to be the case for it not to hold for all quadrilaterals?
There would have to be some quadrilateral that we would not be able to do what in that quadrilateral?
A student: Bring the diagonal.
Teacher: Bring the diagonal, very nice, correct. Is there such a quadrilateral?
Students: No.
Teacher: We have 4 angles. We can always connect the two opposite angles.
[…]
Teacher: So, this is the general argument, which, those who used it, most of them, said that it was convincing. Does anyone of those that did not use it think that this is not convincing?
[No one answers positively.]

In the above passage we see how Olivia, a student who used an empirical/inductive argument to support the statement that ‘the sum of the angles of any quadrilateral is 360°,’ assisted by a discussion with the teacher and her classmates discovers a more general argument for supporting the above statement.

The second example comes from a group task on proving that the angle formed between the bisectors of two adjacent and complementary angles is right. The group was formed by four students of mixed abilities. Each of the students had worked on the same task individually at the end of the previous day and now they had to all together come up with a convincing argument and write it down clearly.
Example 2: Group work for Task BIS (see Appendix B.8).

Sue: We have to prove that angles $\epsilon$ and $\delta$ have sum $90^\circ$. So, we will do the following. We will measure this angle here.

Elli: Without measuring. We have to find a solution theoretically.

Sue: Where is this written? Is there anywhere written ‘theoretically’?

Elli: [Reading] “Write an argument.” That is, we are making an assumption.

Elinor: Argument…that is, anybody who will take this to see it will not start measuring…We have to prove it.

[They discuss again what they need to prove because Anna, another member of the group, has not understood yet. At some point Anna realises that it is not given that the sum of the angles is a right angle.]

Anna: Oh, we don’t know that it is $90^\circ$…

Elinor: [Laughing] No! This is what we are looking for!

Anna: Very easy, we measure $\zeta$ and $\gamma$, we subtract that from $180$, [so] $\epsilon$ and $\delta$ equals $90^\circ$ [she laughs expressing awareness that her idea is not a convincing argument].

Elli: It would be nice but again it needs measurement. We want to find something without measuring.

[Sue insists that she knows how to show it so she is trying to explain her argument to the others.]

Sue: We know that the angle, this one here is right. Therefore, we know that a right angle is $90^\circ$. Eh, isn’t that all guys? If…

Anna: We suppose that we don’t know this! [Anna here is clearly turning from an Empirical proof scheme that she had earlier to a Deductive one.]

Elli: Exactly. We have to find this. So it is wrong this way you are using [to Sue].

Anna: Yes, it starts from the given.

Sue: Wait a minute. [She starts measuring with the protractor again. The whole group reacts and they try to take the protractor from Sue.]

Anna: We want to prove it.

In the above passage we see three different kinds of behaviours. Elli and Elinor are the students who since the beginning of the group work had a deductive proof scheme (lines 3, 6). Sue, on the other hand, is the student that is using an empirical proof scheme and is not yet ready to make the change to a deductive proof scheme (lines 1–2, 18–19, 27–28). In fact until the end of the group work Sue insists on an empirical proof scheme, and one can see that she is confused about what is a general and what is a specific argument. The case that interests us most is that of Anna, who from an empirical proof scheme (lines 11, 13–15) made progress to a deductive one (lines 22, 26, 29). Anna was one of the students that started attending the lessons clearly at the empirical level (based on the results of all the pre-tests and questionnaires). In this transcript we see that Anna suddenly realises the importance of a general argument and the idea of avoiding
circular arguments (i.e. never start by assuming what you want to prove). This happened after a few lessons and we may assume that it happened through discussing the work with her group.

Another example of a move from the empirical to the deductive proof schemes is the work on the same task (B.8) from a different group of students working on the same task, this time four students of lower ability. Most of the students of this group were confused as to what they needed to prove and how they should interpret the figure given.

Example 3: Group work for Task BIS (B.8).

1 Nadia: Neither this one is 90.
2 Stian: Yes, since the figure is like this.
3 Nadia: Yes but she [the teacher] just said that. It is not what it looks like, we have to prove it.
4 [The teacher happens to be passing at that moment and hears the discussion.]
5 Nadia: Isn’t this what you said?
6 Teacher: Yes, yes!
7 Nadia: [To Stian] See?
8 Stian: And how are we going to do it? [Showing the figure to the teacher.] First of all this one and this one are opposing, therefore they are equal.
9 Nadia: Let’s apply what she has been telling us…
10 Sunny: Uff [in a tone of disappointment]! That γ is equal to δ and this kind of things?
11 Nadia: Yes, right!
12 Stian: [To the teacher] Aren’t these, which are opposing, the same?
13 Teacher: How can this be useful for you?
14 George: Shall we do something like what you did on the board?
15 Teacher: What we did on the board, you have to remember, this shows you why it will always be 90. It shows this to you, it…You have to write it somehow yourselves, with a general argument.
16 Someone from the group: Uff…[expressing difficulty] What shall we do guys?
17 Teacher: Can you think why will it always be 90? Think about it. All together [the angles] how much are they?
18 Group: All together 180.
19 Teacher: And that one, why is it 90?
20 Nadia: Because we split in the middle this one and this one.
21 Teacher: Yes!
22 Nadia: Oh, is that it?
23 Teacher: That simple!
24 Nadia: Oh! [She is laughing and she sounds very happy.]
Chapter 7. Results

We see in the above transcript that one of the students (Nadia), realised that there is a distinction between giving an argument specific to the figure and giving a more general, deductive argument (lines 3, 4). The same student, the previous day, when they had to give an argument for the same task individually, used two arguments that were not deductive. The first one was circular (“Because they form a right angle and right angle = 90°”) and the second one was empirical (“If we measure with the protractor we will find out that it is 90°”). So we see that from having an empirical proof scheme the previous day, after the lesson of the following day Nadia tried to find a deductive argument (line 26) although it was much harder than the empirical ones she had come up with the previous day. This change may be due to the frequent comments of the teacher in class that we do not consider arguments depending on the figure as convincing, as well as due to the discussions with the rest of the group.

The last example comes from a group formed by three students who previously displayed mostly empirical proof schemes. Each of the students had worked individually before on the same task. In this group task they had to collaboratively come up with a convincing argument for the TRIMID task, and write it down clearly.

Example 4: Group work for Task TRIMID (see Appendix B.11)

1  [...] 
2  *Isis*: So, we need to show that [...] \( \Delta E \) is parallel to \( \Delta B \).
3  *Gregory*: But you can see that they are guys!

Gregory here has an empirical proof scheme; the figure affects his opinion. Later however, Gregory realises that his argument is not strong and suggests that the two triangles formed are similar and have a similarity ratio, therefore the two lines have to be parallel.

4  *Gregory*: They [the triangles] are similar and have a similarity ratio. And since they 
5  are similar and have a similarity ratio they fall on the same figure [here Gregory 
6  does not remember the correct name of the property he refers to, but he has gotten 
7  the right idea]. [...] because they fall on the same figure then won’t this one be 
8  parallel to that one?

This latter attitude can be described as a deductive proof scheme. Moreover, while the other members of the group have difficulties understanding what to do when it comes time to prove or reject that \( \Delta E \) is half of \( \Delta A \), Gregory seems to understand that since they found a counterexample, the conjecture is false:

8  *Gregory*: It doesn’t hold! Look! This one has to be half of that one.
9  *Nadia*: It is.
10  *Gregory*: In my figure though it isn’t! Look!
11  [...] 
12  *Isis*: Oh! We can say also that we reject it, right?
13  *Gregory*: Yes.
14  *Isis*: So we reject it.
Gregory was a student who, like Anna in the previous example, displayed a clearly empirical proof scheme at the beginning of the intervention. However, through the group work and the individual activities he came to a level at the end of the intervention where he could give general arguments, ones that fit more to the deductive proof schemes.

**An overview of the progress in students’ proof schemes made during Stage 2**

There were several moments like those shown in the above transcripts that occurred during our intervention. In general, different students made the transition from empirical and/or external conviction proof schemes to deductive ones at different moments, and some had more difficulties than others. In Table 7.14 we present and compare the percentages of students giving empirical, external conviction and deductive proof schemes for all the tasks of our intervention in chronological order.

<table>
<thead>
<tr>
<th>Task</th>
<th>Empirical</th>
<th>Deductive</th>
<th>External</th>
<th>blank</th>
<th>undecided</th>
<th>no data</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANGLESUM</td>
<td>50%</td>
<td>9%</td>
<td>23%</td>
<td>0%</td>
<td>18%</td>
<td>0%</td>
<td>22</td>
</tr>
<tr>
<td>QUAD</td>
<td>63%</td>
<td>33%</td>
<td>4%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>24</td>
</tr>
<tr>
<td>OPPANG</td>
<td>23%</td>
<td>36%</td>
<td>23%</td>
<td>0%</td>
<td>18%</td>
<td>0%</td>
<td>22</td>
</tr>
<tr>
<td>BIS</td>
<td>55%</td>
<td>30%</td>
<td>0%</td>
<td>10%</td>
<td>5%</td>
<td>0%</td>
<td>18</td>
</tr>
<tr>
<td>BIS-group</td>
<td>27%</td>
<td>43%</td>
<td>9%</td>
<td>0%</td>
<td>4%</td>
<td>0%</td>
<td>22</td>
</tr>
<tr>
<td>EXT-group</td>
<td>9%</td>
<td>50%</td>
<td>14%</td>
<td>0%</td>
<td>14%</td>
<td>14%</td>
<td>22</td>
</tr>
<tr>
<td>TRIMID</td>
<td>6%</td>
<td>41%</td>
<td>18%</td>
<td>30%</td>
<td>6%</td>
<td>0%</td>
<td>17</td>
</tr>
<tr>
<td>TRIMID-group</td>
<td>28%</td>
<td>55%</td>
<td>17%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>18</td>
</tr>
<tr>
<td>QUAD-MID</td>
<td>0%</td>
<td>69%</td>
<td>6%</td>
<td>0%</td>
<td>25%</td>
<td>0%</td>
<td>16</td>
</tr>
<tr>
<td>IF-THEN</td>
<td>0%</td>
<td>88%</td>
<td>6%</td>
<td>0%</td>
<td>6%</td>
<td>0%</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 7.14: The kinds of students’ main proof schemes by task (in chronological order).

It is clear from Table 7.14 that the percentage of students using either empirical or external proof schemes at the end of Stage 2 shows a considerable decrease. Of course abandoning the empirical or the external conviction proof schemes does not imply that the students will start using deductive proof schemes. However, we can see that the frequency of using a deductive proof scheme increased considerably as well (from 9% to 88%). This may be interpreted as evidence that many students successfully completed the second stage of our intervention. However, we need to be careful in this interpretation since less students completed the last tasks of our lessons and the ‘undecided’ and ‘blank’ cases occur a lot for certain tasks.

Moreover, we need to stress that for Task TRIMID, we have 30% of ‘blank’ answers. This is because coming up with a proof for this task is rather difficult. This is reflected in the lower percentages for the EMP and DED cases for this task. In addition, we should mention that although in the last half of the lessons there are many undecided cases, those cases were mostly students who made an attempt to give a deductive argument but failed because they did not know exactly how to do it (which was expected given that the students were just making their first attempts to give deductive arguments).

Given all the above, we have to say that it seems the lessons helped the students start abandoning their empirical and external proof schemes in favour of deductive ones. However, extra lessons are needed in order to claim with certainty that the students will keep using deductive arguments rather than non-deductive ones consistently, since they completely abandoned empirical proof schemes only during the last two tasks of our intervention and this could be due to the specific choice of tasks.
Chapter 7. Results

Stage 3: Starting to see the axiomatic structure of Euclidean geometry

Stage 3 of our intervention took place in the very last lesson of our intervention. A complete transcript of the discussion is offered in Section 7.4.2, where this stage is discussed in terms of the five Van Hiele phases for the introduction of the concepts ‘axiom’, ‘undefined term’ and ‘axiomatic system’.

Conclusions

The first stage of our teaching strategy was successful, as most of the students (about three quarters) after the completion of this stage were able to understand the distinction between deductive and inductive arguments. However, and as was expected, the students at this point were not always able to offer deductive arguments, something which they are expected to practice during stage two.

The second stage of our teaching strategy was also successful, since the percentage of students using either empirical or external proof schemes at the end of stage two shows a considerable decrease while at the same time the frequency of using a deductive proof scheme shows a considerable increase (from 9% to 88%). However, the number of lessons and of students was small; therefore, we would need to try this teaching strategy also on a larger sample and over a longer period of time in order to be able to draw safer conclusions regarding its effectiveness in general.

7.4.2 Moving through the five Van Hiele teaching phases towards the theoretical Van Hiele level

Here we will describe how the five phases of the teaching sequence suggested by Van Hiele (see Section 2.7 on page 18 for a description of the phases) were realised in our intervention and discuss whether the students completed the phases successfully during the move from the descriptive to the theoretical Van Hiele level for the main concepts introduced. We will discuss the results ordered by groups of concepts, examining whether the aims set for the tasks designed for each phase (see Section 5.5.2 on page 44) were achieved. The integration phase will be discussed only at the end, as it happened at the same time for all concepts in the final lesson of our intervention. A chronological overview of all the concepts introduced and the tasks used for each concept was offered in Table 5.7 on page 50.

Concepts ‘proof’, ‘proving’ and ‘theorem’

Phase 1 - Inquiry: For this phase the Tasks ANGLESUM (see Appendix B.2) and CHOOSING - Part 1 (see Appendix B.3) were used. By letting the students complete Task ANGLESUM we aimed at getting them involved with the field of Euclidean geometry and the concept of providing an argument for supporting or rejecting a mathematical proposition. With task CHOOSING - Part 1 we wanted to allow the students to realise that there exist different kinds of arguments in the field under investigation. The results of this phase were presented in Section 7.4.1 (Stage 1).
Phase 2 - Directed Orientation: During the directed orientation phase the students had to work on various tasks (ANGLESUM, QUAD, OPPANG, BIS, TRIMID). The aims of these tasks were explained in Table 5.3 on page 46. Especially for the concepts of ‘proof’ and ‘proving’ it was our aim to give the students enough opportunities to practice in order to start abandoning empirical proof schemes in favor of deductive ones. In Section 7.4.1 (Stage 2) we presented examples of students who started making this transition. This phase is terminated with the Explication phase in which students and teacher establish a common ground for the meaning of the concepts introduced and the terminology that corresponds to them. Below we will present how the explication phase was realised in our intervention and whether or not it suggests a successful completion of Phase 2.

Phase 3 - Explication: We knew that the students were ready for the explication phase for these concepts from their discussions and group work during the lessons. We have offered some examples of such moments where the students start understanding the idea of giving deductive arguments in Section 7.4.1 (Stage 2) (see Examples 1, 2, and 3). To assist the explication phase for the concepts of ‘proof’, ‘proving’ and ‘theorem’ the teacher prepared two handouts: ‘How we discover new truths in Geometry’ (see Appendix C.4) and ‘What we discovered in the Geometry lessons’ (see Appendix C.5). In the following transcripts we see how the teacher introduced the terminology for these concepts which the students had been practicing during phase 2.

1 Teacher: So, I am saying here [in handout C.4] in short what we have been doing and what I have been telling you in my own words. The first step is ‘observation’; that which usually does not exist in school [meaning the ordinary school lessons], right? To observe and discover on your own some relations. The second step is to ‘conjecture’. Have you ever heard this word? ‘Conjecture’?

2 Students: No.

3 Teacher: That’s OK it doesn’t matter. This word is the one they use in mathematics. What we did, that is, to try and guess what holds generally, to find a relation…you said [for example] this one is probably a right angle [referring to the BIS task (B.7)] this was a ‘conjecture’ that you made. What does ‘conjecture’ mean? A ‘hypothesis’. I guessed something, OK? That is, I am not sure about it yet. When something is a conjecture it means it has not been shown yet, I haven’t given a convincing argument yet. OK? […] Here [referring to the handout] I also gave you an example about the quadrilateral. ‘I make the conjecture that the sum of the angles of every quadrilateral is 360°’, do you remember when each of you measured a quadrilateral and we saw that all of them more or less are close to 360 but not exactly, and we say that probably, I think that, for every quadrilateral the sum of its angles will be 360° but I am not sure, it is a ‘conjecture’, a ‘hypothesis’, right?

4 Then [step 3] we ‘gave movement’ to the figure and we tried to figure out what [lines] we can bring, what we can do in order to show it, and we brought the diagonal. And we saw that whatever way we move it [the figure] in whatever quadrilateral in the world we can bring a diagonal. What does this mean? We divide it into two…into two triangles. This is nice.
And then [step 4] we offered the argument. This is the ‘proof’. And proof in mathematics is what we saw together, the ‘convincing’ arguments. They have to be general, right? We saw many kinds of arguments last time. We saw some that depended on the figure, remember? You measured them…we saw some others who cut them [the angles] and put them together, we saw plenty of arguments. Those that are considered as proof in mathematics are those which are convincing and general, they don’t depend on one figure, OK? [They hold] for all figures of that kind [showing on the board the argument of the task that had been solved earlier].

We will do more of this in the following lessons, that is, we will observe, we will make conjectures, we will give movement to figures and in the end we will prove. I just want you to have this [handout] that gives an overview and if you want take a look at it at home […]

The students were this way introduced to the terminology related to the basic steps behind ‘proving’, and they heard for the first time that ‘proof’ in mathematics is considered to be only the arguments which are convincing for the general case. In a different handout called ‘What we discovered in the Geometry lessons’ (see Appendix C.5) all the conjectures made, and the theorems proven in the lesson to that point were presented in order to assist the students to understand how the explicated concepts related with what they had been doing in the lessons. This was an opportunity to explicate the concept of ‘theorem’ which is the result of the process of proving. Below we offer a transcript showing how this explication took place.

Teacher: I have put here…you remember that handout I gave you with the steps [referring to handout ‘How we discover new truths in Geometry’ (C.4)], right?

Students: Yes.

Teacher: I have put here [in the handout] what we have discovered here together step by step…what we have done in every step. [Explains some of the statements written on the handout] And in the end, what we have proven becomes a theorem. OK? It becomes something that is generally true and that no one can disprove it because you have given a very strong and convincing argument. Whatever [theorems] we have discovered I put them here in blue boxes, at the end of each discovery [I have written] which theorem it is. Some of these, most of them, will be in the book of the first Lyceum level, so when you will see them next year you will recognise them.

Phase 4 - Free orientation: During the free orientation phase the students had to work on various tasks (TRIMID, QUADMID, IF-THEN) in which they had to make their own conjectures, and prove or reject them in order to establish a new theorem. The specific aims of each of the tasks was explained in Table 5.3. Especially for the concepts of ‘proof’ and ‘proving’ it was our aim to give to the students enough opportunities to practice with tasks in order to completely abandon empirical proof schemes in favor of deductive ones and to practice the concepts/processes which were explicated in the previous phase. The results of this phase, presented in Table 7.14, show that the phase was successful.
7.4. The role of our teaching strategy: A qualitative analysis

Concept ‘Conjecture’

Phase 1 - Inquiry: The students were introduced through the first part of Task OPPANG (see Appendix B.6) to the concept of a conjecture. This was achieved by asking students to make their own observations about a figure drawn by the teacher in order to formulate their own general mathematical statements, for which they would later have to offer convincing arguments.

Phase 2 - Directed Orientation: Tasks CHOOSING - Part2, BIS, and OPPANG were used for this phase. Students in these tasks were always asked to make their own conjectures. Students managed to do this rather successfully and they always managed to come up with conjectures that were actually theorems of Euclidean geometry. Here are some examples from the lessons.

Example 1: Transcript from lesson 2, after we had finished the discussion about the arguments the students gave the previous day for task QUAD.

Teacher: Let’s move on. From now on, I will not give you statements like this from the start. We will be drawing a figure, and will be making observations on it together, and this way we will be making our own statements. Some of them will be correct, some of them will be wrong, we don’t know we will see. I will start easy, to see what you can do. We have two lines that are intersecting at a point [starting to talk about task OPPANG (B.6); the teacher had already drawn two intersecting lines on the board]. There are some relations that hold between the angles formed here.

Students came up with the following conjectures:

1. The adjacent angles form 180°.
2. The opposing angles are equal.
3. The sum of all angles is 360°.

Example 2: Transcript from lesson 2, at the beginning of task BIS. Teacher has drawn Figure 7.17 on the board. $K\Delta_1$ is a bisector of angle $AKB$, and $K\Delta_2$ is a bisector of angle $BKT$.

![Figure 7.17: The figure used in Task BIS.](image)
Chapter 7. Results

Teacher: Let’s see now. Like we did earlier, with the opposing angles that we observed some relations, we have to do with this one too. So the first thing that we do, is to observe the figure. Let’s see…let’s think about it for a moment. Do you see anything, a relation?

Mary: All angles together are $180^\circ$.

Teacher: Mary says that all angles together are $180^\circ$. Do you agree?

Students: Yes.

Teacher: This is easy, quick, right? Why? Because it is…

Students: A straight line.

Teacher: A straight line. We know this. Anything else?

Katerina: We can say that, if we let’s say take one angle, $\Delta_1 K$T plus $\Delta_2 K$A they will make 90 [degrees].

Teacher: Hmmm…[Expressing that it sounds interesting.]

Nia: I also thought the same.

Teacher: Did you [to all students] think the same?

John: I did not understand what you said. [He was actually not paying attention, because he was busy drawing the figure in his sheet.]

Sunny: Didn’t you understand that there is a right angle formed?

[The teacher explains again. There is some discussion about the conjecture.]

Teacher: So let’s see, the statement that we want to show, our hypothesis is that the bisectors are perpendicular to each other.

John: How do we know that they are perpendicular?

Teacher: We said, we made a hypothesis. Katerina said that these, and I say these ones also, will be $90^\circ$.

We see from the above transcript (lines 5, 11, 14, 18) that the students were quite comfortable with coming up with conjectures. And this was the case for all of the tasks that they had to complete. This was the evidence that the students were ready for the explication phase.

Phase 3 - Explication: The explication phase for this concept took place together with the explication phase for ‘proof’ and ‘proving’ when the teacher discussed in class handout C.4 (see earlier discussion).

Phase 4 - Free orientation: During this phase the students made more conjectures and they seemed to be quite comfortable with them. This shows that the concept was grasped by them well.
Concept ‘Counterexample’

Phase 1 - Inquiry: The inquiry phase of the concept of ‘counterexample’ coincides with that of the concepts ‘proof’ and ‘proving’ which was presented earlier. During the discussion that took place in our first lesson on the advantages and disadvantages of arguments, we talked about what would have happened if we found a triangle with sum of angles different than 180°. The answer was that this would result in the rejection of the theorem, which is the idea behind a counterexample.

Phase 2 - Directed Orientation: Students in this phase had to work on Task TRIMID (see Appendix B.10). First, Figure 7.18 was drawn by the teacher on the board and the students had to make their own conjectures about the figure. Students had many ideas and soon the following conjectures were written on the board:

1. $\overline{KL} \parallel \overline{BG}$
2. $\overline{KL} = \overline{AG}/2$
3. $\overline{KL} = \overline{BG}/2$
4. $\overline{KL} = \overline{AB}/2$

Conjectures 1 and 3 are actually true for all triangles whereas conjectures 2 and 4 are only true for specific triangles. Students did not know this in advance, and they were asked to prove or disprove the conjectures they had come up with. Each group had to do this for two conjectures, either 1 and 2, or 3 and 4. Below we see examples from a transcript of students’ group work.

1. John: So, and now that $\overline{KL} = \overline{AG}/2$. We need to prove that this is half of that.
2. Kelly: Or that it is wrong. It could be wrong.
3. John: [To the teacher] Can we show that it is wrong?
4. Kelly: Yes! It says it here!
5. Flora: It doesn’t say…
6. Kelly: But it says it here! […] Here! [Reading out the task instructions] Prove or reject.
Teacher: Yes, if something is not right you will say why.

[…]

Flora: Yes, there, we know it isn’t and then we prove it. This one is 3 [cm] …and this I don’t think it is 3.

[…]

John: We will write that, by measuring them, it is wrong. Theoretically they can’t be right because bla bla bla…Did you get it?

Flora: Or else we could write that we measure them and end of the story. But it is not convincing…

John: Yes, that’s why. By measuring them they turn out to be wrong. And therefore in order to prove it…

Flora: So…

John: So, we measure them and it is wrong.

Kelly: But first we should know whether they are correct!

Flora: Yes, I said that they are not correct.

Kelly: Yes, but why, we did not measure it!

Flora: We [her and John] measured it!

John: We measured it.

Flora: One is 3 and the other is …5?

[…]

Flora: So, we will write that…we will write two arguments. One is that we measure them…

Kelly: And if we don’t know any other [argument] we will not write any other.

John: Come on then, write the first one.

Flora: Write that we measure KΛ and it is […] equal to 7 and AΓ equals 4.

Kelly: But guys, we said for all figures!

John: Eh, yes this is a small proof which if we make a mistake, if we don’t find [another one]…it is our alternative!

Flora: [She is writing] Then…AΛ equals…

Kelly: Therefore it does not hold.

Flora: [She keeps writing] is equal to AΓΔ.

John: We assume. That in order for it to be…in order for them to be equal it has to be either equilateral…ooohhhhh! There is a possibility for them to be equal!

Flora: They are not.

John: But there is a possibility.

Kelly: Eh yes, but we are talking about this [figure]!
John: Yes, that’s why though we depend on the figure […inaudible] if we do this one as well like that, bigger …again this will be the case, it will hold.

Kelly: But…since it does not hold for this [figure] then why would it hold for another one…and even if it holds for a specific figure it is by chance! We cannot say that it holds.

The struggles of these three students to figure out how they can accept or reject a conjecture are obvious in the above passage. They seem to think it is sensible to reject the conjecture just because it does not hold for their own figures, but then they feel that this would be an argument which depends just on one figure. We had discussed during previous lessons that arguments which depend on a specific figure are not convincing and are not considered to be mathematical proofs. However, we see that Kelly eventually discovers (lines 33, 37, 43, 46) that their argument seems to be fine, since the fact that the conjecture might hold in certain figures could be just accidental. Kelly here shows that she is ready for the explication phase. This was the case for other students as well (see for instance Example 4 during the discussion of our Stage 2).

Phase 3 - Explication: The explication phase for the concept of ‘counterexample’ came at the end of task TRIMID where the students had tried to prove or disprove the various conjectures. The teacher initiated the following discussion.

Teacher: Why isn’t this \( K\Lambda = A\Gamma/2 \) correct?
Thalia: Because it depends on the figure.
Teacher: It depends on the figure you think?
Thalia: Because we measure ours, in yours they were equal, but in mine it wasn’t equal…
Katerina: So what does this mean?
Thalia: …so since in some triangle it is not equal it means that it does not hold generally.
Teacher: [To the whole class] Do you agree?
Students: Yes.
Teacher: Have you done the same? You [to a student] I think you told me you had done the same.
Flora: Mine is probably equilateral so I was saying that it is equal. But…
Teacher: But it is, you think, only for yours, right?
Flora: Yes. That it does not hold for all triangles.
Teacher: Let me draw another triangle.
[Starts drawing one on the board and connects the midpoints of the two sides.]
A student: I don’t think that it holds for all.
Teacher: This should be equal to that. It is obvious that it isn’t the case. So, this means that this one here (the conjecture \( K\Lambda = A\Gamma/2 \)) was supposed to be a general...
observation, but we discovered that it isn’t. Why? Because we found an example in which…

**Students:** …It does not hold.

**Teacher:** …it does not hold. OK? For this one? [Points at the conjecture $KA = AB/2$.] If this one here [points at $KA$] we measure it and then we measure that one [points at half of $AB$] do you think they are equal? So this observation that we did here …

**Student:** Depends on the figure.

**Teacher:** …it was only…we were misled by the figure. It [the observation] wasn’t general. What did we do? We found an example, one, which disproves it. That is, **one example is enough in order to reject a conjecture.** Whereas in order to verify and to prove something, in order to give a general argument, one example is not enough.

**Student:** …it is not enough. [Agreeing with what the teacher just said.]

**Teacher:** Do you understand the difference?

**Students:** Yes.

**Teacher:** For example we had said…I will give you another example. We had said that the sum of the angles of a triangle is $180^\circ$. And you told me, look, I will give you an example. [Draws a triangle on the board.] I am measuring them [the angles] I add them, and I get $180^\circ$. Was this…does it convince you?

**Students:** No.

**Teacher:** No. It was an example. But one example cannot prove the general. One example is not enough. If however you had found a triangle that was not $180^\circ$, which would be perfectly constructed, wouldn’t that be enough of an example to reject the general?

**Students:** Yes.

**Teacher:** Like here [shows the TRIMID example on the board]. **One example was enough for rejecting this general observation that we had made. This is called ‘counterexample’ because it is not exactly an example, it is an example that proves the opposite.** [Writes the word on the board.] With a counterexample we can prove that this [the conjecture on the board] is rejected. OK. So you can see now that it is much more difficult to prove that something holds generally than to prove that it does not hold generally. And this is why it is more difficult to construct general theories in mathematics, because it is more difficult to find general arguments.

In one of the last lessons, the students had to find an argument for the IF-THEN task (see Appendix B.13). In this task an implication statement needed to be proven. After the statement was proven, the teacher asked the students whether they thought the inverse would also hold. Most of them said that it would hold. The teacher then offered a counterexample for the inverse of the implication. The students were immediately convinced that now the inverse statement does not hold, and this shows that they had grasped the idea of a ‘counterexample’.
Phase 4 - Free orientation:  We did not plan any task for the free orientation of the concept of ‘counterexample’ because of the limited time we had.

Concepts ‘Axiom’, ‘Undefined term’ and ‘Axiomatic System’

Phase 1 - Inquiry:  The teacher initiated a discussion regarding the previous knowledge that students used in order to prove theorems in class at the end of the 5th lesson. This was the starting point of the discussion on the 6th (and last) lesson.

Teacher:  [Discussion at the end of lesson 5] And next time we will talk about, and it will be the last thing we will say… I have told you many times that what we have been proving is based on something that we took as a given. Have we ever proven anything without taking something else as a given?

Students:  No.

Teacher:  Always we needed something from before. And what we will search for next time is that, we will see together until what point we can go back. So, these mathematicians, where did they begin from to start proving things? For example Euclid who made geometry, where did he start from? Didn’t he need to take something as a given?

Student: [Euclid started] From zero.

Teacher:  How from zero? We will see how we can go backward, and what we can take as a given and what we cannot. OK? Next time.

A lively discussion followed in the final lesson with contributions from many students regarding where they think mathematicians begin from in order to prove their theorems.

Teacher: [Start of lesson 6 - last lesson of the intervention] So, now, all this time what have we been doing guys? We were trying to convince ourselves and others…

Student:  To find arguments.

Teacher:  To find arguments, for some observations that we had made, right? For some statements that we have observed… And what did we do for this? We were looking and looking and looking, and what were we using every time?

Student:  The things from before.

Student:  Our minds.

Teacher:  The things we knew from before, right? That is, some givens, and our minds, who said this [laughs], right? And we were trying to figure out what we know in order to see how we can prove what we want to know. And after we had been proving it, then we could consider it as a given, for the next task, because we now had a proof for it. Is this clear?

Students:  Yes.

Teacher:  Let’s see an example. We will not do anything new today. I mean, you will not be writing and solving, but we need to think what we have done so far.
Chapter 7. Results

Something like...the last lesson, you know...So, [starts writing on the board] we had said that the sum of the angles of any triangle is 180°. Didn’t we say that?

**Student:** The internal angles.

**Teacher:** The internal angles, yes. We had found various arguments, the most convincing was the following.

[Quickly reminds them of the proof by sketching it on the board. See Figure 7.19]

![Figure 7.19: Sketch used by the teacher for the discussion of axioms.](image)

Teacher: What have we used? We will try to make something like a map, a diagram, to try and go backwards and backwards and backwards to see where we can arrive at. Because, if you remember last time I told you...where does this all begin from? That is...we took something as a given, from before, and then we solved the next one. This thing from before, before considering it as a given, we had considered something else as a given in order to prove it. And so on. Can you imagine, do you think this process will stop somewhere? Is there a beginning? Where does it begin from? What do you think? I want to hear your opinions and then we can look at it together. Who wants to say? Do you think there is an end to this? Do you know where it begins? Does it maybe not begin anywhere?

**Nia:** I believe that it has neither a beginning nor an end. It is like the numbers. That is, the numbers do not end, they are infinite.

**Teacher:** In both directions, right?

**Nia:** Yes.

**Fiona:** I believe also that it has neither a beginning nor an end, because you can prove anything, or you can disprove anything. There is nothing for which no proof exists.

We can see in the above passage that the first opinion that emerged from the discussion about ‘where did it all begin’ was the one suggested by Nia and supported by Fiona: “it has neither a beginning nor an end, because you can prove anything, or you can disprove anything. There is nothing for which no proof exists.”

**Teacher:** OK, good. Anyone else? [Students hesitate for a second.] What, you agree, you disagree, you don’t know?
7.4. The role of our teaching strategy: A qualitative analysis

Thalia: I think that we start from a hypothesis. And if this hypothesis helps us prove something, then we can take it as a given later.

Teacher: So you are saying that we start from somewhere, it isn’t as Nia and Fiona said that there is no beginning and...

Thalia: Everything starts somewhere.

Teacher: Everything starts somewhere…Anyone else? Who do you agree with most. The first or the second opinion?

[Some students say that they agree with both. Students start talking to each other.]

Sunny: Isn’t there a middle one?

Nia: [To Thalia] yes, but, what have you done to think of the hypothesis?

Teacher: [To Thalia] What did you say? Repeat what you said before.

Thalia: I say that everything has a beginning.

Teacher: So there are some...

Thalia: Everything starts somewhere.

Teacher: …hypotheses at the beginning. Which we don’t prove, or do we prove them?

Thalia: We don’t know if the hypothesis is true. It’s just a hypothesis. And if this hypothesis helps us prove something that we know, then we can take it as a given. And if we make many experiments with this hypothesis…then the hypothesis becomes a given.

We now see a different opinion emerging from the discussion, that of Thalia which was also supported by many others in the class: “We start from a hypothesis. And if this hypothesis helps us prove something, then we can take it as a given later” (see lines 56, 57). Now the class starts discussing who is right or wrong, and the students who suggested the main two opinions start arguing with each other. More students join the discussion and offer their opinion.

Fiona: The thing is that when we are trying to prove something, we are not trying to show that this is it from the start, but to reject it. And if it is proven, it is proven, otherwise we reject it. Therefore, now, the hypothesis...

Teacher: Yes, but, where does it start from, we say.

Mary: I was also thinking that it all starts from a hypothesis. For example even when we want to talk, we talk for a reason. Everything has some reason. We cannot start from nowhere.

Nia: I believe that, what Thalia said that we start from a hypothesis…Yes but what made her think this hypothesis?

Thalia: This is a different issue.

Nia: First she thought about it and then she made it. So there is no start.

Theano: There is a start because the first thing you think about is to make the hypothesis! This is the start! So everything has a start!
Nia: And how will you think to think to make the hypothesis? [Laughs]

Teacher: [Talking to Theano] Yes but, didn’t we say that in order to make our hypothesis, and in order to take some things as givens in order to do the proof we have to have proven them first. Or, don’t we need to?

Theano: No, we may be asked to prove it at that moment, and we can think…

Teacher: Yeah, OK, I talk in general not in the classroom. Generally, in mathematics, that is, what do mathematicians do?

Theano: Eh, they started somehow!

Kelly: Miss, can I say? When you say the mathematicians, for instance, the first person that thought about these lines and dots, what did he do?

Teacher: Yes, this exactly! What did he do?

John: A hypothesis. [Others agree.]

Kelly: Exactly!

Teacher: Did he make a hypothesis?

Kelly: He didn’t start from some…from something that he had proven.

Student: [He started] out of curiosity!

Teacher: He made a hypothesis, and did he prove this hypothesis?

Kelly: I mean that he did not start from something that is already known, from some givens, he simply started to make hypotheses.

Teacher: And then did he prove them?

Kelly: He proved them?

Teacher: Using what?

Kelly: Nothing! Did the first mathematician know anything from before?

[Laughter.]

Sue: Everything has a beginning, it can’t start from nowhere.

[Students discuss loudly.]

Fiona: Everything starts from an observation. You observe something. After the observation if you observe this more times, in your mind the hypothesis starts to be formed. ‘What if…’ Then you try to prove the hypothesis and this is the first given.

Kelly: The other thing she said, what she [Fiona] said: “Everything starts from an observation.” So, there is a start.

Fiona: They start from the previous hypothesis.

[Students talk all together.]

Olivia: [To Fiona] You disproved your own opinion.

[Fiona says something but it is inaudible.]
At this point the class arrived at a stage where they seemed ready for a directed discussion on how we can start from a specific proof and take a path backwards to see how far back we can go and discover whether there is a beginning or not. This is Stage 2, which will be discussed in what follows.

**Phase 2 - Directed Orientation:** The teacher starts directing a discussion of the proof of the statement ‘the sum of the angles of a triangle is $180^\circ$’ by asking the students to think of all the previous knowledge they had used to prove the statement.

122 *Teacher:* So, let’s look at it together. Let’s try and go backwards, to see…where will we arrive at? What did we use…before ending up with this [pointing at the board] right before, what did we use?

126 *Gregory:* Supplementary angles.

127 *Teacher:* Say this a bit differently, explain it. So, what does this mean? That…

128 *Gregory:* Supplementary angles?

129 *Teacher:* Yes.

130 *Gregory:* That two angles form a sum of $180^\circ$.

131 *Teacher:* Here we have three angles, that’s why I am asking you to explain it differently. They are not supplementary, what are these angles?

133 *Olivia:* Adjacent.

134 *Teacher:* What do they form? [Explains to Gregory that he just needs to say what holds in his own words not by using terminology.]

136 *Gregory:* They have a sum of $180^\circ$.

137 *Teacher:* That’s it. Why? Because this is?

138 *Students:* A straight line.

139 *Teacher:* So we used this. This is the definition of the line actually, let me write it…[Puts it on the board underneath the statement.] […] This we used as a given, right?

141 *Students:* Yes.

143 *Teacher:* What else did we use?

144 *Student:* That alternate angles are equal.

145 [The teacher writes this on the board.]

We can see in the above discussion that the students could remember all the previous knowledge that was used in order to prove the statement ‘the sum of the angles of a triangle is $180^\circ$’ (lines 136, 138, 144). The teacher then asks why this previous knowledge was taken as a given, trying to suggest that everything needs to be proven in mathematics.
Chapter 7. Results

Teacher: Why did we take these as givens guys?

Students: Because we knew them, from before.

Teacher: Did we prove these?

Students: No.

Sunny: What, do we also need to prove these?

Student: The first mathematician proved them.

Teacher: How did he prove them?

[Many opinions are heard, one of which is that he measured them.]

Teacher: But we said that measurements are not convincing. Measurements are only for verifying something, if you like. Let alone that we saw that measurements are not exact. So anyway they are only for us to get an idea, they are not for proving something. So?

Gregory: We assumed they were givens.

Teacher: We assumed they were givens? Did we start from these you think? What do you say?

Gregory: From the observations.

Sunny: Practically.

karina: We took a straight line, and this line […]

[The student is trying to prove why alternate angles are equal but fails. The teacher shows the proof and it is obvious that the proof is based on the definition of parallel lines.]

Teacher: So this, we did not exactly consider it as a given. Actually we did [use it as given], but, it was itself a consequence of something else, the definition of parallel lines. [Puts the definition in the axiom map on the board.] So let’s see what is parallel? What is actually a straight line?

[Students start talking about it.]

After proving the previous knowledge that was used (here the fact that alternate angles are equal), the teacher started directing the students to think about undefined terms such as line and point, and also about properties that we accept without proof (the axioms). The discussion becomes a bit philosophical again as one can see below.

Katerina: It is many points in…

Teacher: What is a line, we need to know what a line is. [Teacher draws in a box the world ‘straight line’ underneath the parallel lines box.]

John: What is a point?

Teacher: Yes! What is a point? Very good question!

Sue: Can I say what a point is? It is the beginning of a line…

Teacher: The beginning of a line? Anyone else?
7.4. The role of our teaching strategy: A qualitative analysis

Theano: The beginning and the end of a line segment.
Fiona: It is any point of a surface.
Teacher: A ‘point’ is any ‘point’?
Fiona: Well, I can’t say it with another word!
Nia: A point is the beginning of a semi-line.4
Teacher: OK.
Kelly: A dot in infinity.
Teacher: [Laughs] That’s a more philosophical opinion.
Teacher: So guys, the ‘point’ has ‘tortured’ a lot of people. Euclid himself…, a
definition of point does not even exist. Do you know how he says what a point is?
It is that which has no dimensions. Can you imagine this?
Debbie: It does have dimensions!
Teacher: None!
Students: It does have!
Teacher: The fact that we are drawing it with the marker is a problem of the prac-
tical process of taking a marker and drawing it. We cannot see it [the point] it is
something that has no dimensions. Can your mind conceive this? So however many
points you put next to each other they will not form a line.
Students: Why?
Teacher: Because the point does not have dimensions. How…?
Debbie: But Miss if we take it this way there exist no lines either!
Teacher: Line, as Euclid said it, is that which has only one dimension. Only in…
John: length.
Teacher: …length. It does not have any width at all. Nothing. Zero.
Kelly: Like a thread.
Teacher: Not even like that, because if a thread had no width we would not be
able to see it. So these are abstract concepts. And another concept is the ‘surface’.
Surface is that which has…
John: Two dimensions!
Teacher: …which has two dimensions. Very nice. So guys, these three are some
hypotheses that Euclid made and he accepted them and this was it. It is the only
three things in the form of definitions that he accepted and said, these I cannot
prove. I take them as they are. And everything else is based on these. But there
are also some other rules that he accepted. Let’s see now from here [points at the
statement on the board ‘A straight angle forms 180°’] can we go backwards? What
will be needed?

4 A line that has a starting point, but no end.
Chapter 7. Results

It is obvious in the last part of the above discussion that a directed orientation of the students towards the concept of undefined terms is attempted. This attempt did not come without reactions from the students. The first reaction came from Fiona, the student who at the beginning of the lesson suggested that all this “has neither a beginning nor an end, because you can prove anything, or you can disprove anything. There is nothing for which no proof exists”:

*Fiona:* Can I ask something? If those three we cannot prove them…we don’t know…we consider them givens, and in any case we don’t know if they are certain] because we can’t prove them, then is there a chance that simply all geometry is wrong? Whatever we have done so far to be wrong?

*Teacher:* [Teacher smiles.] It is not wrong. What does wrong mean?

*Fiona:* Not wrong, but, without a base!

*Teacher:* No! Because we have a base!

[The class becomes very noisy again. They tend to agree with Fiona.]

*Fiona:* But we can’t prove these we just take them as givens. This means that whatever we have proven so far…

*Teacher:* It could be wrong, but it isn’t, because we use these things for two and a half thousand years [smiles].

*Fiona:* And what if for two and a half thousand years we have been wrong?

*Katerina:* And if someone is born and he manages to prove them and they are not this way. Will the whole geometry change?

*Teacher:* If he proves something else, then he can build a new geometry.

*Kelly:* That’s why mathematicians get crazy…

*Teacher:* And they have done this actually…

*Students:* Who?

*Teacher:* Some scientists have made their own hypotheses, and build a different system. Not the system of geometry, a different one.

*Nia:* And is this system continued [today]?

*Teacher:* Yes it is continued and it also has applications.

We see in the above discussion that the students ask what would it mean if the axioms and undefined terms are wrong (since we have no proof for them), which seemed to worry them a lot. Katerina (line 224), asks a quite natural question: what if someone disproves the axioms? This question brings the opportunity to talk about different axiomatic systems that could be built on a selection of different axioms. It was the aim of our lessons to bring the students to the theoretical Van Hiele level, where they are ready to understand that there exist different axiomatic systems (Van Hiele level 4).

After this the teacher tries to continue the directed orientation phase now for going backwards starting from the same statement (‘the sum of the angles of any triangle is 180°’) but taking a different route, by using one of the other proofs we gave for this statement.
7.4. The role of our teaching strategy: A qualitative analysis

Teacher: In any case, for this [points back to the statement on the board] we again need this kind of things. We need to start somewhere. But, the point is that we took a path to go backwards. If you had thought of a different proof, you could have taken a different path. But in the end you will probably end up with the same. For example let’s see what other proof we had found for this. [Draws the proof of Lefteris (see Appendix B.3).] Do you remember, with the external lines etc.? There, we did not use this [the alternate angles] we used something else. What?

Student: The circle.

Teacher: We had said that the angles, the internal with the external together make 180°. But we had said something else too. Do you remember?

Student: About the circle.

Teacher: About the circle.

Student: That it has 360° [inaudible].

Teacher: Yes, we has said that the circle has 360°. [Writes it on the map.] Why?

Who says this? What is a circle?

Mary: Four right angles.

Teacher: Four right angles! So we need also the definition of the circle, right? [Writes it on the map.] […] I can always draw a circle, taking as center a point, whatever point I want, and with radius whatever I want to. This we can always do. Why can we always do it? How are we sure?

Students: It is certain!

Teacher: It is certain, can you prove it? [John suggest something practical.] Yes but this is practical. With an argument like the ones we have been using, can you prove it?

John: No.

Fiona: What should we prove with an argument like the ones we have been using?

Teacher: That I will always be able to bring a circle. Can you?

Fiona: Yes!

Teacher: How?

[Students try to find a proof and make some suggestions.]

Teacher: So, isn’t it very…? Don’t you think ‘it is true!’?

Students: yes!

The discussion led again to undefined terms and axioms. As expected, the students had problems accepting the abstract definitions that the teacher was describing. We discussed all these concepts in a single lesson with the students, but we think that in order for them to understand these concepts and the abstract nature of mathematics better, more time is needed. However, the time we had in our intervention was limited, so at this point the teacher attempted to start the explication of the new concepts and this will be described in the following parts of the transcript.
Phase 3 - Explication: In this part of the discussion, the teacher explicates the concepts ‘axiom’ and ‘undefined term’ by giving a name to the experience the students just had (see lines 267 and 271 below):

Teacher: What just happened to you is what happened to Euclid too. He said there! It seems obvious! This thing and many other statements he called ‘axioms’. Have you ever heard this word? [Writes it on the board.]

Students: No. [Students said they have heard this word only in different contexts because in Greek it also means ‘rank’.]

Teacher: And these ones are called ‘undefined terms’. These three.

Fiona: The point, the line, and?

Teacher: The surface, the point and the line.

Fiona: These are the undefined terms…

Teacher: Undefined terms. We don’t define them, what I told you is the explanation given by Euclid but you don’t need to define them. You begin from here.

Fiona: And the axioms? Which are they, apart from the circle?

Teacher: There are more axioms, I have prepared a list for your [see Appendix C.7] and you will see them. Another axiom for example is that from a point outside a line I can bring only one parallel [to that line]. How can you prove this?

[Students give suggestions.]

Teacher: Anyway, I don’t want you to do it now, you can try to prove it if you like.

[Students want to prove it!]

Teacher: Think about it and let me know next time if you have proven it.

John: What will we win if we prove it?!

The students were quite convinced that the axioms can and should be proven which is a rather natural reaction. It was our aim to make the students discover that there exist some statements in mathematics which we accept without proof, and we expected them to react to this idea, which we think is evidence that they understood what this idea is about.

Phase 4 - Free orientation: The free orientation of these concepts was not realised in our lessons. A plan for this phase could be to let the students follow the same ‘going backwards’ process for other proofs that were discussed in the lessons, until they arrive at something they cannot prove, and also to ask them to find more than one path backwards. The time limitations did not allow us to realise our plans with our students, however we suggest that this phase is necessary for the students to arrive at a complete understanding of the axiomatic system of geometry.

Phase 5 - Integration phase for all concepts

The integration phase for all concepts introduced in our lessons took place during the last part of the discussion, where the concepts of ‘axiom’, ‘undefined term’ and ‘axiomatic system’ were
introduced. During the following discussion also the explication of the concept ‘axiomatic system’ took place, which the teacher named ‘geometry’.

*Teacher:* So, did you understand this process of going backwards? So, there is a beginning after all, which we accept. It is as if we are building something. Geometry is like a big building. That is, you begin from the foundations, three concepts, only three… The good thing…

*Fiona:* So only these three undefined terms exist?

*Teacher:* Yes, and five to ten axioms. OK? From that point on, the point is that mathematicians, and Euclid, have tried to have the least possible axioms and undefined terms. Why? Because they wanted to prove everything, and although at some point geometry was practical, they were measuring fields to see which one is bigger etc., suddenly Euclid came, two and a half thousand years ago, and said we should not be based on experience let’s make geometry based on theory and logic only; only on logical rules. Another such logical rule for instance is [explains the transitivity axiom]. Isn’t this obvious?

*Students:* Yes.

*Teacher:* This is an axiom.

[the recording stops here]

All the concepts learned until that point were connected by the teacher into a whole, that of Euclidean geometry. Unfortunately we don’t have recordings of the final part of the above discussion, however we can say that the students were still at the end of the lessons quite surprised by the idea of the axioms, and the fact that we accept them without proof. They felt that the system of geometry is based on ‘shaky ground’ and expressed some of these opinions in the questionnaires they had to fill in during the last meeting. These will be discussed in the following sections.

### Conclusions

The five Van Hiele teaching phases in our teaching intervention were completed for most concepts successfully. The evidence in the transcripts and in the students’ work shows that the students were ready for each next phase. Specifically, the success of the explication and integration phases was easy to check (by looking at the ways students used the explicated terms). On the other hand, the directed and free orientation phases were short in our intervention, and for some concepts the free orientation phase was not included. Therefore it was not always possible to tell with certainty whether the students as a group were ready for the following phases. To be sure of the success of these phases we would need to apply our teaching strategy over a longer period.

The third stage of our teaching intervention was successful. In this stage the students are supposed to start seeing that, for the arguments they come up with, only previously proven knowledge can be used, and that it all starts from a few basic assumptions: the axioms. The students during our last meeting discovered the existence of axioms and the fact that the axioms are the basic building blocks of Euclidean geometry.
7.5 Students’ beliefs about mathematics and proof

Part of our research question was to examine how the kinds of reasoning used by students when doing proofs in geometry relate to their beliefs about proof. As was described in Section 5.3.1, we planned to use a modified version of Schoenfeld’s questionnaire (1989) (see Appendix A.3) in order to check the beliefs of the students about mathematics and proof both before and after the intervention. In this modified version only the total number of items differs from the original, since we made a selection of those questions that were relevant to our study. The items chosen are worded in the same way as in the original version of the questionnaire, but are translated into Greek.

We also created and used our own questionnaire (see Appendix A.4) which aimed at finding out how the students perceive the change in their beliefs about mathematics and proof. Finally, we used certain questions in Task CHOOSING (see Appendix B.3) and Task CONCEPTIONS ABOUT PROOF (see Appendix B.4) in order to find out the ideas of the students about the convincingness, validity and explanatory power of certain types of arguments both before and after our intervention. The results from these three questionnaires are presented in the three following sections.

7.5.1 Modified Schoenfeld questionnaire results

The items in the modified questionnaire are divided in sections according to the aims of the questions. We followed the categorisation suggested by Schoenfeld (1989). Thus, the items in the first four categories of the modified version of the test were intended to reveal “students’ perceptions of mathematics and school practice” (Schoenfeld, 1989, p. 342), the items in categories 5 and 6 were intended to reveal students’ beliefs about the “nature of geometric proofs, reasoning, and constructions” (Schoenfeld, 1989, p. 342), and the items in the last category were intended to offer to the students “an opportunity to present slightly more extended answers to issues of interest” (Schoenfeld, 1989, p. 342).

We will offer an analysis of the results in parallel with Schoenfeld’s analysis, in order to be able to make a comparison. We will also compare the answers students gave before and after our intervention to the same items of the questionnaire.

Students’ beliefs about mathematics attitude (including geometry)

We will first discuss the answers of the students to items 1 (‘The math that I learn in school is mostly facts and procedures that have to be memorized’), 3 (‘The math that I learn in school is just a way of thinking about space, numbers, and problems’) and 14 (‘The best way to do well in math is to memorize all the formulas’), which reflect classroom practice. Below you see the percentages of answers of the students for each item.

<table>
<thead>
<tr>
<th>Item</th>
<th>Totally agree</th>
<th>Guess I agree</th>
<th>Guess I disagree</th>
<th>Totally disagree</th>
<th>(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 (10%)</td>
<td>6 (30%)</td>
<td>6 (30%)</td>
<td>6 (30%)</td>
<td>2.8</td>
</tr>
<tr>
<td>3</td>
<td>2 (10%)</td>
<td>10 (50%)</td>
<td>6 (30%)</td>
<td>2 (10%)</td>
<td>2.4</td>
</tr>
<tr>
<td>14</td>
<td>3 (15%)</td>
<td>7 (35%)</td>
<td>5 (25%)</td>
<td>5 (50%)</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Table 7.15: Answers to classroom practice items before the intervention \(n = 20\).
In Table 7.15 we see that there is a wide distribution of answers for all three items. The students before our intervention generally disagreed (although not strongly) with the idea that the math they learn in school is mostly facts and procedures that have to be memorized ($M = 2.8$) and with the idea (although again not strongly) that the best way to do well in math is to memorize all the formulas ($M = 2.6$), as opposed to the students in the study of Schoenfeld (1989) where they mostly thought that memorization is really important. Moreover, the students tended slightly towards agreeing that the math they learn in school is just a way of thinking about space, numbers, and problems ($M = 2.4$).

Item 32 of our questionnaire, an open question about whether memorization in mathematics is important or not, gives further evidence for the above. Out of the 15 students who gave an answer, 9 mentioned that memorization is not important (‘In my opinion I think that the best is not memorization but first of all to understand the rule and understand where you could use it. Because if you don’t understand you can’t solve the exercises’, ‘Very little because more important is to understand it.’) and 2 of the 6 (33%) students who thought memorization is very important, also mentioned that it is not enough on its own (‘I think that memorization is important but I believe that a student should first understand the theory and only then memorize it because if he simply knows it and he has not understood it he will not be able to apply it’).

In Table 7.16 we can see the answers of the students for the same items after our intervention.

<table>
<thead>
<tr>
<th>Item</th>
<th>Totally agree</th>
<th>Guess I agree</th>
<th>Guess I disagree</th>
<th>Totally disagree</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6 (30%)</td>
<td>5 (25%)</td>
<td>6 (30%)</td>
<td>3 (15%)</td>
<td>2.3</td>
</tr>
<tr>
<td>3</td>
<td>3 (15%)</td>
<td>9 (45%)</td>
<td>5 (25%)</td>
<td>3 (15%)</td>
<td>2.4</td>
</tr>
<tr>
<td>14</td>
<td>1 (5%)</td>
<td>3 (15%)</td>
<td>7 (35%)</td>
<td>9 (45%)</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Table 7.16: Answers to ‘classroom practice’ questions after the intervention ($n = 20$).

Clearly, the students after the intervention had a much stronger disagreement with the statement (item 14) suggesting memorization as the best way to do well in math ($M = 3.2$). We also see that after experiencing the kind of mathematics we did in our intervention, they were generally in agreement with the statement (item 1) that what they do in ordinary school mathematics is mostly facts and procedures that have to be memorized. As for item 3, the students generally did not change their opinion about what mathematics is ($M = 2.4$).

For item 32, the open question, the picture also changed slightly. The students who thought memorizing was not important were now 12 out of 19 (63% rather than 60% before our intervention), but now 71% (rather than 33%) of the students who said memorizing was important also mentioned that it is not important on its own but you need to understand also what you are doing.

We will now discuss the answers to items 2 (‘The math I learn in school is thought provoking’), 10 (‘In mathematics you can be creative and discover things by yourself’), 20 (‘When I do a geometry proof I get a better understanding of mathematical thinking’), and 22 (‘When I do a geometry proof I can discover things about geometry that I haven’t been taught’). All these items are related to the students’ perceptions of mathematics and geometry. In Tables 7.17 and 7.18 the answers of the students before and after our intervention are presented.

We can see that, although generally the students both before and after the intervention agree with the statement that the math they learn in school is thought provoking (item 2.0), the agreement before the intervention ($M = 1.2$) is much stronger than after the intervention ($M = 2.0$).
Chapter 7. Results

### Table 7.17: Answers to questions related to students’ perceptions about mathematics and geometry before the intervention \((n = 20)\).

<table>
<thead>
<tr>
<th>Item</th>
<th>Totally agree</th>
<th>Guess I agree</th>
<th>Guess I disagree</th>
<th>Totally disagree</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>18 (90%)</td>
<td>2 (10%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>1.2</td>
</tr>
<tr>
<td>10</td>
<td>4 (20%)</td>
<td>9 (45%)</td>
<td>5 (25%)</td>
<td>2 (10%)</td>
<td>2.5</td>
</tr>
<tr>
<td>20</td>
<td>9 (45%)</td>
<td>8 (40%)</td>
<td>3 (15%)</td>
<td>0 (0%)</td>
<td>1.7</td>
</tr>
<tr>
<td>22</td>
<td>4 (20%)</td>
<td>15 (75%)</td>
<td>1 (5%)</td>
<td>0 (0%)</td>
<td>1.9</td>
</tr>
</tbody>
</table>

This may be explained by the fact that our lessons were quite different than the ordinary lessons of the students, and the teacher often mentioned that in ordinary school-books some times the mathematical statements are offered as givens, which the students are expected to accept without questioning. Moreover, we see that the students were not always sure before the intervention \((M = 2.5)\) whether in mathematics you can be creative and discover things on your own (item 10), after the intervention the students tended to agree more \((M = 2.0)\) although many of them still were not totally convinced. The students’ ideas did not change much as to how a geometry proof helps them get a better understanding of mathematical thinking \((M = 1.7)\) before, and \(M = 2.0\) after the intervention) and did not change at all regarding the statement ‘When I do a geometry proof I can discover things about geometry that I haven’t been taught’.

**Students’ beliefs about the nature of geometric proofs, reasoning, and constructions**

In Tables 7.19 and 7.20 we see the distribution of the students’ answers, before and after our intervention respectively, to items 18, 19, 21, 23, 24, 25 and 26 of the questionnaire. We will briefly discuss these answers and the change in the distribution after our intervention.

### Table 7.18: Answers to questions related to students’ perceptions about mathematics and geometry after the intervention \((n = 20)\).

<table>
<thead>
<tr>
<th>Item</th>
<th>Totally agree</th>
<th>Guess I agree</th>
<th>Guess I disagree</th>
<th>Totally disagree</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8 (40%)</td>
<td>8 (40%)</td>
<td>2 (10%)</td>
<td>2 (10%)</td>
<td>2.0</td>
</tr>
<tr>
<td>10</td>
<td>6 (30%)</td>
<td>9 (45%)</td>
<td>4 (20%)</td>
<td>1 (5%)</td>
<td>2.0</td>
</tr>
<tr>
<td>20</td>
<td>7 (35%)</td>
<td>7 (35%)</td>
<td>5 (25%)</td>
<td>1 (5%)</td>
<td>2.0</td>
</tr>
<tr>
<td>22</td>
<td>6 (30%)</td>
<td>11 (55%)</td>
<td>3 (15%)</td>
<td>0 (0%)</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Table 7.19: Answers to questions related to the nature of geometric proofs, reasoning and constructions before the intervention \((n = 20)\). (Note that for question 21, \(n = 19\) since one student left it unanswered.)
7.5. Students’ beliefs about mathematics and proof

Table 7.20: Answers to questions related to the nature of geometric proofs, reasoning and constructions after the intervention \((n = 20)\).

<table>
<thead>
<tr>
<th>Item</th>
<th>Totally agree</th>
<th>Guess I agree</th>
<th>Guess I disagree</th>
<th>Totally disagree</th>
<th>(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>3 (15%)</td>
<td>9 (45%)</td>
<td>6 (30%)</td>
<td>2 (10%)</td>
<td>2.4</td>
</tr>
<tr>
<td>19</td>
<td>11 (55%)</td>
<td>7 (35%)</td>
<td>2 (10%)</td>
<td>0 (0%)</td>
<td>1.6</td>
</tr>
<tr>
<td>21</td>
<td>1 (5%)</td>
<td>4 (20%)</td>
<td>10 (50%)</td>
<td>5 (25%)</td>
<td>3.0</td>
</tr>
<tr>
<td>23</td>
<td>0 (5%)</td>
<td>3 (15%)</td>
<td>7 (35%)</td>
<td>10 (50%)</td>
<td>3.4</td>
</tr>
<tr>
<td>24</td>
<td>11 (55%)</td>
<td>8 (40%)</td>
<td>1 (5%)</td>
<td>0 (0%)</td>
<td>1.5</td>
</tr>
<tr>
<td>25</td>
<td>1 (5%)</td>
<td>7 (35%)</td>
<td>9 (45%)</td>
<td>3 (15%)</td>
<td>2.3</td>
</tr>
<tr>
<td>26</td>
<td>8 (40%)</td>
<td>7 (35%)</td>
<td>3 (15%)</td>
<td>2 (10%)</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Item 18 (‘When I do a geometry proof I can only verify something a mathematician has already shown to be true’) was about the function of proof as establishing truth rather than as simply verification of something given. We see that the average of the answers before our intervention is 2.1 which indicates that the students generally agreed to this statement. After the intervention there was a small change of the average \((M = 2.4)\) towards the direction of disagreement, however the change was small. We think that the item may have been interpreted by the students not as it was intended by Schoenfeld. Obviously the intention of Schoenfeld was to check whether the students understand the difference between verification and proof. However, often the students prove things at school which someone has already proven in the past, so they are simply re-proving them. It is quite probable that the students have not understood the difference between the meanings of the words ‘verification’ and ‘proof’ and this could be the reason why they agreed with the statement of item 18. This is one of the downsides of multiple choice questions where we can never be sure what the student really understood by the question. The results for this question should therefore not be immediately interpreted as suggesting that the students think proof is verification. Because of the nature of this question, an interview with the students would be needed to clarify the meanings of the words used.

The students generally agreed to item 19 (‘When I do a geometry proof the key thing is to get the statements and reasons in proper form’) both before and after our intervention. However, in our research this question has a different significance to that of Schoenfeld (1989) where the teaching of geometrical proof emphasises very much the form of the written proof—proofs have to be written in two columns, one including the statements and the other including the reasons for the statements. Therefore, we believe that the answers to this item indicate that the students feel a need for writing down the arguments in a proof in a proper, mathematical form for them to be understood.

Regarding item 21 (‘When I do a geometry proof I’m finished if I can’t remember the next step’) the students rather clearly disagreed both before \((M = 3.1)\) and after \((M = 3.0)\) our intervention. This indicates that the way these students think in mathematics is not tied to algorithms, rather they are used to thinking of various routes which may lead to the desired answer.

For item 23 (‘When I do a geometry proof I’m doing school math that has nothing to do with the real world’) the students on average (strongly) disagreed both before \((M = 3.6)\) and after \((M = 3.4)\) our intervention. This indicates that the students already thought that geometrical proofs have a connection to the real world and this belief was not affected by our intervention. Item 24 was similar, related to the usefulness of geometrical proofs (‘When I do a geometry proof I feel like I am doing something useful’). The results indicate that the students both before
(M = 1.5) and after (M = 1.5) our intervention clearly agreed with this statement. Items 25 and 26 were related to geometrical constructions, which were not included in our intervention, therefore the results are not relevant to our study and will not be discussed here.

Because the nature of the questions in Schoenfeld’s questionnaire did not cover the scope of our research we had prepared other questionnaires to be filled out by the students. These results will be discussed in the following sections.

### Conclusions

**Beliefs about math in general:** The students after the intervention had a much stronger disagreement with the statement suggesting memorization as the best way to do well in math. Moreover, they were generally in agreement with the statement that what they do in ordinary school mathematics is mostly facts and procedures that have to be memorized. This could be interpreted as an effect of our teaching intervention in which we stressed that geometry is something different that what is usually taught in school, that is, not simply applying a procedure to prove something which was given by the book or the teacher, but rather geometry is an ongoing process of observing, conjecturing, and finding valid arguments to support or reject your conjectures.

**Beliefs about geometry and proof:** The students’ answers to the items of the modified Schoenfeld questionnaire regarding geometrical proofs did not change significantly after our intervention. These items were mainly related to rather general aspects of proof and were not immediately connected with the ideas of validity and generality of arguments. Therefore, the fact that the opinions of the students on these specific aspects of geometry and proof did not change after our intervention is not surprising. In our intervention we focused mainly on the steps of proving and on what exactly it means to prove in mathematics.

### 7.5.2 Students’ own ideas on how their beliefs changed

At the end of our intervention the students had to fill in a questionnaire (see Appendix A.4), which included seven questions related to their opinion about the lessons and what they learned, as well as their beliefs about mathematics and proof. We will discuss here the answers of the students to those questions related to their beliefs about proof and mathematics. The rest of the answers are presented in Appendix D.4.

**Q5: Do you believe that the lessons have affected your way of thinking? If yes, how?**

We asked this question in order to find out how the students think about the ways—if any—their thinking has changed during this lesson. We understand that this meta-question is not easy to answer and that the students’ understanding of how their own thinking has changed might be rather difficult to identify. Therefore we expected that many students would have difficulties answering this question.

We found that only 1 out of 20 students did not answer positively (her answer was ‘Maybe’), and 5 out of 20 students did not give any specific answer as to how the lessons affected their
thinking although 19 of them stated that the lessons did in fact affect it in ‘some’ way. Below are the categories that we identified in the students’ answers.

- **Category 1: Vague**
  Five students did not give a specific explanation on how their thinking changed.

  Yes they affected me positively.

  Yes, in a way. In fact I cannot judge this.

- **Category 2: Don’t accept givens without proof**
  Five students perceived a change in their thinking about mathematical statements. They claimed that, because of the lessons, they had adopted a new, more critical, attitude towards what was offered to them as a ‘given’; they would now think about why a statement is true and would not be immediately convinced without having a proof for it.

  Yes, because now I am thinking of all the reasons why something which I find ready has been proven.

  Yes, because now I don’t accept anything as a given if I don’t prove it first.

  I think yes, because while until now I have been solving the problems as if they were something given, now I look for all the given elements and all the possible solutions.

- **Category 3: Discovering new things**
  Two students claimed that they were now able to make new discoveries, since they had experienced how to do this in the lessons.

  Yes I believe it. They affected my way of thinking positively. That is, now I get more concerned about something that I have to solve (proof) and I don’t restrict myself to the things I already know and as a consequence I can discover more things.

  Yes, they affected my way of thinking. I learned how to discover new things based on what I already know.

- **Category 4: Better (structured) thinking / Strategy / Method**
  Seven students mentioned that now their thinking was better and more structured, and often they connected this with knowing the steps they need to take (the method) in order to arrive at a result in geometry.
I learned to organize and group better my thoughts. To have a beginning and an end. To know where and how to begin to think (in geometry)

Yes it made me think in a specific order the steps I need to take.

Yes, they affected my way of thinking and they made me more observant and also made me think with a strategy in order to solve a problem or a geometry exercise.

- **Category 5: Other**

There were three more kinds of answers, each offered by one student. One was suggesting that after the lessons they are more observant, the second that their understanding has improved, and the last that they could now transfer their knowledge about proof in other instances in their life.

The different kinds of answers given by the students are summarised in Table 7.21.

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency (n = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>19</td>
</tr>
<tr>
<td>Maybe</td>
<td>1</td>
</tr>
<tr>
<td>Vague</td>
<td>5</td>
</tr>
<tr>
<td>Don’t accept givens without proof</td>
<td>5</td>
</tr>
<tr>
<td>Discovering new things</td>
<td>2</td>
</tr>
<tr>
<td>Better (structured) thinking / Strategy / Method</td>
<td>7</td>
</tr>
<tr>
<td>Other</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 7.21: Do you believe that what we have done in these lessons is useful? In what way?

Overall, we can say that most students seemed to be quite aware of how their thinking changed and mentioned the points that we had hoped to change because of our intervention (for example appreciating the nature of proof as explaining why things hold in mathematics and believing that they can discover new theorems on their own).

**Q6: Do you believe that the lessons have affected the ideas you have about mathematics? If yes, how?**

We asked this question in order to find out the beliefs of the students about mathematics, and how they think these have changed during the lessons. It was one of the secondary aims of our study to examine how the students’ beliefs about mathematics and proof change while they move from the descriptive to the theoretical Van Hiele level.

Out of 20 students, 5 said that the lessons did not affect their ideas about mathematics, or that they affected them very marginally:

Not that much but I liked it, now in the [ordinary] classroom where we are being graded I don’t know how I am going to do but here if we did for a little while more I would adore it [math].
7.5. Students’ beliefs about mathematics and proof

Basically not that much…I never liked mathematics…But now it is somehow better…

No, they did not affect it, since mathematics was always an interesting lesson.

Out of 20 students, 15 said that the lessons did affect their idea about mathematics. Below you can see the categories we identified in their answers.

- **Category 1: Better / more interesting / fun**
  Six students thought that after these lessons their ideas about mathematics has improved, but they gave quite vague answers as to how exactly they changed.

    They made it even better. Mathematics is my favourite subject.

    Yes I started liking it more and I consider it to be a great science.

    Yes, now geometry seems to me much more interesting, but also more understandable.

- **Category 2: More difficult / requires more thinking**
  Three students claimed that mathematics seem to them more difficult than they thought before.

    Yes they affected me. It [math] is a bit more difficult to me.

    I found out that mathematics requires more thinking after all.

    Yes…it is more difficult than what I thought.

- **Category 3: It requires more time**
  Two students mentioned that in order to solve a problem you need more time than what they had thought at the beginning.

    They affected the idea I have about mathematics. In the beginning I believed that you don’t need to be involved for too long in order to solve something. I simply knew that this holds and according to this I used to find the rest without looking for the reasons why this holds. I believed that mathematics was just calculations, but after all it is not like that.

- **Category 4: Has a ‘shaky’ beginning; the axioms**
  Three students mentioned that now they knew that mathematics has a beginning. A couple of them mentioned specifically the idea of the axioms and expressed their worries that, since the axioms are not proven, the whole of geometry might be wrong.
Chapter 7. Results

Yes because I didn’t believe that mathematics have a beginning and they start from somewhere. However this attracted my attention.

Yes. Because there are things which we could not prove (axioms) but still we are based on these and we ‘take it as right and if they are wrong maybe the whole geometry is wrong.

Yes. Now I believe that all that which we learn from the elementary school until now might not hold at all (concept of point)\(^5\)!!

- **Category 5: Other**

  In this last category we have the following answers mentioned one time each: mathematics is not just givens without reasons, mathematics can be done cooperatively, mathematics requires imagination, is easy, and is not just calculations.

The different kinds of answers given by the students are summarised in Table 7.22.

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency ((n = 20))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>15</td>
</tr>
<tr>
<td>No</td>
<td>5</td>
</tr>
<tr>
<td>Better / more interesting / fun</td>
<td>6</td>
</tr>
<tr>
<td>More difficult / requires more thinking/time</td>
<td>5</td>
</tr>
<tr>
<td>Has a ‘shaky’ beginning (the axioms)</td>
<td>3</td>
</tr>
<tr>
<td>Is not just givens without reasons</td>
<td>1</td>
</tr>
<tr>
<td>Can be done cooperatively</td>
<td>1</td>
</tr>
<tr>
<td>Requires imagination</td>
<td>1</td>
</tr>
<tr>
<td>Is easy</td>
<td>1</td>
</tr>
<tr>
<td>Is not just calculations</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.22: Do you believe that the lessons have affected the ideas you have about mathematics? If yes, how?

We can see from the above that the students generally thought the lessons affected their beliefs about mathematics, and they offered a plethora of ways in which they believe this has happened.

**Q7: Do you believe that the lessons affected the ideas you have about proof? If yes, how?**

We asked this last question in order to find out the beliefs of the students about proof, and how they think these changed during the lessons. Only three students gave a negative answer to the above question, two of which also provided an explanation.

It hasn’t changed. All mathematics need a proof therefore I don’t believe they have changed it.

No. The idea about proof cannot change from one day to the next.

---

\(^5\)Here the student refers to the discussions we had in class about the fact that the definitions of point and line, which are the undefined terms, were accepted without proof.
The remaining 17 out of a total of 20 students said that the lessons affected the ideas they had about proof in one way or another. In the answers offered by the students on ‘how’ their ideas changed about proof, we could identify the following categories.

- **Category 1: Proof is important**

  Yes. Now I believe that proof is important.

  I learned that proof is very important for the geometrical concepts.

- **Category 2: Proof is useful**

  They are more difficult than I thought and also more useful.

- **Category 3: Proof can explain**

  It affected it. In the past, I did not look for what I was given, however now I am trying to find how the given from an assumption became what it is (given).

  Yes because I didn’t know how to prove and I couldn’t explain many things but now it helps me a lot.

- **Category 4: Proof can be flexible (many different ways of proving the same thing)**.

  Maybe yes. While I did not consider it that important and I considered it something given which I learn by heart, I realized that there are many different ways to prove something.

- **Category 5: Proof is difficult to do**

  They [proofs] are more difficult than I thought and also more useful.

  What I have learned from the lessons about proof is that after all it is not that easy to find, therefore indeed they affected me regarding the idea I had about proof.

  Yes they affected the idea I have about proof. Now I believe that after all it is not that easy to do a proof and that you have to search enough in order to find the right answer.

- **Category 6: Proof is easy and/or fun**

  I don’t know, maybe. I guess yes. At school I found it somehow difficult but here it was somehow easier. And I liked to prove while at school it is somehow…
Chapter 7. Results

Yes, because proof now is not just a make-work but it has become really interesting and it helps me very much in understanding absolutely the knowledge that I already have and to assimilate in an easier way the new elements.

They made me realise that it is not too difficult to prove something as it seemed in the beginning.

• Category 7: Proof is not only for mathematicians

Yes, because in the past I used to believe that proof has been made by very ‘smart’ important people and that it is impossible that we students understand it. But after all it is a fun, easy procedure based on our knowledge.

Some of the students mentioned the fact that now they have learned how to prove and therefore they can understand it better than before as being the only change.

Yes it affected me I can say enough…because I had not understood it very well at school…But now I have understood it perfectly.

Yes because before I did not know how exactly proof is made.

Basically, I never had asked myself what proof is and how we arrive to there, therefore certainly they changed my previous idea!

The different kinds of answers given by the students are summarised in Table 7.23.

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency (n = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>17</td>
</tr>
<tr>
<td>No</td>
<td>3</td>
</tr>
<tr>
<td>Important</td>
<td>2</td>
</tr>
<tr>
<td>Useful</td>
<td>1</td>
</tr>
<tr>
<td>Can explain</td>
<td>2</td>
</tr>
<tr>
<td>Difficult to do</td>
<td>3</td>
</tr>
<tr>
<td>Easy/fun</td>
<td>3</td>
</tr>
<tr>
<td>Not only for mathematicians</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.23: Do you believe that the lessons have affected the ideas you have about proof? If yes, how?

We can see that the students offered plenty of examples on how their beliefs about proof changed. We believe this change is a consequence of our lessons since the students had not been introduced formally to proof in the past. Moreover, the way the lessons were set up offered to the students ample opportunities to experiment with making their own conjectures, coming up with their own arguments and discussing how well their arguments (or their peers’ arguments) explain why the conjectures are true or not. This may have contributed to a deeper understanding of many aspects of proof and led to a change in the students’ earlier ideas about proof.
Conclusions

In the answers of students, we could identify a change in the beliefs of students about proof and mathematics. More specifically:

- Regarding proof: The students after our intervention seem to have appreciated the nature of proof as explaining why things hold in mathematics and they have started believing that they can prove theorems on their own. In 17 out of 20 cases, the students thought that their beliefs about proof had changed after the intervention, in the ways mentioned in Table 7.23.

- Regarding mathematics: The students generally thought the lessons affected their beliefs about mathematics in the ways mentioned in Table 7.22.

### 7.5.3 Students’ understanding of the validity, convincingness and generality of arguments

As explained in Section 5.2.1, following the study of Hoyles and Healy (2007) we asked certain questions to the students before and after our intervention in order to find out their understanding of the validity, convincingness and generality of an argument in mathematics and how this was affected by our intervention. This would contribute to answering the part of our research question referring to the way different kinds of students’ reasoning relate to their ideas about proof. Along these lines we, like Hoyles and Healy, also looked at which argument (out of a given number of arguments) the students would choose as their own approach as well as the extent to which they thought this argument would be the one that receives the best mark by a teacher. Here we present the results of our study regarding these issues and we compare them, whenever possible, to those presented by Hoyles and Healy (2007) and Healy and Hoyles (1998).

In Table 7.24 we present the distribution of the students’ responses to the first four questions in Task CHOOSING before our intervention, and in Table 7.25 those after our intervention. Note that some of the students chose more than one argument for their answer. To make the comparison to the results of Hoyles and Healy (2007) easier, we included the ‘names’ of the arguments as used both in Hoyles and Healy’s research (in the tables they show in brackets) and in our research. The detailed proofs and the questions can be seen in Appendix B.3.

<table>
<thead>
<tr>
<th>Criterion for Choice</th>
<th>Eleni (Amanda)</th>
<th>Georgia (Cynthia)</th>
<th>Petros (Barry)</th>
<th>Stamatis (Dylan)</th>
<th>Lefteris (Ewan)</th>
<th>Tania (Yorath)</th>
<th>Kalliopi</th>
<th>No answer /other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Own approach</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
</tr>
<tr>
<td>Own approach</td>
<td>8 42</td>
<td>3 16</td>
<td>2 10</td>
<td>3 16</td>
<td>2 10</td>
<td>0 0</td>
<td>1 5</td>
<td>0 0</td>
</tr>
<tr>
<td>Best mark</td>
<td>2 10</td>
<td>4 21</td>
<td>4 21</td>
<td>1 5</td>
<td>2 10</td>
<td>1 5</td>
<td>2 10</td>
<td>2 10</td>
</tr>
<tr>
<td>Most convincing</td>
<td>4 21</td>
<td>3 16</td>
<td>2 10</td>
<td>2 10</td>
<td>4 21</td>
<td>0 0</td>
<td>4 21</td>
<td>1 5</td>
</tr>
<tr>
<td>Easiest to understand</td>
<td>9 47</td>
<td>4 21</td>
<td>2 10</td>
<td>4 21</td>
<td>9 47</td>
<td>0 0</td>
<td>0 0</td>
<td>1 5</td>
</tr>
<tr>
<td>Most difficult to und.</td>
<td>0 0</td>
<td>0 0</td>
<td>0 0</td>
<td>0 0</td>
<td>0 0</td>
<td>2 10</td>
<td>12 63</td>
<td>2 10</td>
</tr>
</tbody>
</table>

Table 7.24: Distribution of arguments chosen in Task CHOOSING before the intervention.
Chapter 7. Results

Table 7.25: Distribution of arguments chosen in Task CHOOSING after the intervention.

<table>
<thead>
<tr>
<th>Criterion for Choice</th>
<th>Eleni (Amanda)</th>
<th>Georgia (Cynthia)</th>
<th>Petros (Barry)</th>
<th>Stamatis (Dylan)</th>
<th>Lefteris (Ewan)</th>
<th>Tania (Yorath)</th>
<th>Kalliopi</th>
<th>No answer /other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Own approach</td>
<td>1 5 13 68</td>
<td>0 0 1 5 5 26</td>
<td>1 5 1 5 0 0</td>
<td>1 5 5 26 0 0</td>
<td>1 5 2 10 2 10 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Best mark</td>
<td>1 5 10 53</td>
<td>3 16 0 0 5 26</td>
<td>1 5 1 5 0 0</td>
<td>1 5 5 26 0 0</td>
<td>1 5 2 10 2 10 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Most convincing</td>
<td>0 0 16 84</td>
<td>0 0 1 5 5 26</td>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Easiest to understand</td>
<td>6 32 10 53</td>
<td>0 0 2 10 1 5</td>
<td>0 0 0 0 3 16</td>
<td>2 10 16 84 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Most difficult to und.</td>
<td>0 0 0 0</td>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 3 16</td>
<td>2 10 16 84 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.26: Types of arguments in Task CHOOSING.

<table>
<thead>
<tr>
<th>Name</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eleni’s (Amanda’s)</td>
<td>inductive, practical</td>
</tr>
<tr>
<td>Petros’s (Barry’s)</td>
<td>incorrect, theoretical (formal)</td>
</tr>
<tr>
<td>Georgia’s (Cynthia’s)</td>
<td>deductive, theoretical (formal)</td>
</tr>
<tr>
<td>Stamatis’s (Dylan’s)</td>
<td>inductive, practical</td>
</tr>
<tr>
<td>Lefteris’s (Ewan’s)</td>
<td>deductive, narrative</td>
</tr>
<tr>
<td>Tania’s (Yorath’s)</td>
<td>deductive, practical</td>
</tr>
<tr>
<td>Kalliopi’s</td>
<td>deductive, narrative</td>
</tr>
</tbody>
</table>

Own approach against best mark In Task CHOOSING the students were asked to choose among a pool of four arguments proving the statement ‘the sum of the angles of a triangle is 180°’ the one that they thought matches best their own approach, and the one that they thought would receive the best mark. The names of the four arguments along with a short description of the type of arguments included in this task is offered in Table 7.26.
7.5. Students’ beliefs about mathematics and proof

![Bar chart showing students' choices for own approach and best grade before and after intervention.](image)

**Figure 7.20**: Students’ choices for own approach and best grade before and after our intervention.

different to that of Hoyles and Healy, where the students chose Georgia’s proof to be the one that gets the highest grade both in 1996 and in 2002 (48% and 45% respectively). This could be explained by the fact that we used the test on the same students within a short period of time unlike Hoyles and Healy.

Similarly to the Hoyles and Healy research, the students at the beginning of our intervention showed a slight preference for formal arguments regardless of whether they were correct or not. This indicates that when an argument includes a lot of mathematical symbols, thus being more formally represented, the students think it will get the best mark.

Let us now look at the choice of the students regarding the deductive arguments offered in the CHOOSING task. A sizable minority of students (6 out of 19 students, or 32%) chose in the pre-test a deductive, valid argument for their own approach either in a narrative or in a more logical and deductive form (Georgia’s and Lefteris’s proofs). These results are similar to those of Hoyles and Healy (2007, p. 110), where both in 1996 and in 2002, 29% of the students chose for Georgia’s and Lefteris’s arguments as their own approach. The difference between our research and that of Hoyles and Healy is that only 6 out of 19 students (or 32%) in our intervention thought that these two arguments would get the best mark, whereas around 60% of the students in Hoyles and Healy’s research thought that Georgia’s and Lefteris’s argument would receive the best mark (both in 1996 and in 2002).

The picture changes considerably after our intervention. Students do not show a preference any more for inductive, empirical proofs (only 2 out of 19, or 10% of the students would still go for such proofs) and on the contrary, they show a strong preference for deductive proofs (13 students chose Georgia’s answer as their own approach and 5 chose Lefteris’s, both deductive, valid proofs). We interpret this as a result of our intervention where the distinctions between
deductive and inductive arguments were made clear and deductive arguments were now considered ‘mathematical’ as opposed to inductive arguments. Moreover, the fact that we used the same task before and after our intervention does not allow us to be certain as to what caused this change based solely on this questionnaire. It is possible that the students made their choice after the intervention by remembering the discussion of the arguments in class, rather than understanding the subtle distinctions between deductive and inductive arguments. However, the results discussed earlier (see Table 7.14 on page 99) showed that the students displayed a preference towards deductive proofs also in earlier tasks which were unfamiliar to the students.

**Convincingness and difficulty of arguments** In Table 7.24 we see that before our intervention there is no single argument that the majority of the students finds more convincing, although we see that the three most ‘popular’ arguments are Eleni’s, Lefteris’s and Kalliopi’s with 4 students choosing each of these arguments. All these arguments have a practical element to them, therefore this result is not unexpected given that the students were not introduced to theoretical, deductive proofs at that stage. After our intervention these results are considerably different. It is clear that now Georgia’s answer is the most convincing for the majority of the students (16 out of 19) and second most convincing is Lefteris’s answer with 5 out of 19 students choosing it. The two inductive, practical arguments were selected only by 1 student who chose Stamatis’s answer as the most convincing.

Even after the intervention Kalliopi’s argument remained the most difficult to understand. Overall, we can say that whereas the easiest arguments to understand before the intervention were only the inductive, empirical ones (the ones familiar to the students), after the intervention the deductive argument of Georgia is also an easy argument to understand (together with the two inductive ones).

**Validity Rating and Explanatory Power of arguments** In this section we present the results regarding the Validity Rating (VR) and the Explanatory Power (EP) of four of the arguments included in the Task CHOOSING (see Appendix B.3): Stamatis’s, Georgia’s, Lefteris’s and Kalliopi’s. We followed the rating method used by Hoyles and Healy (2007, p. 87) which we will briefly explain here.

The students were given the following two statements relevant to the validity of the four arguments,

- the argument shows that the statement is always true
- the argument only shows that the statement is true for some triangles

and were asked to choose one of three options for each question: ‘I agree’, ‘I disagree’, or ‘I don’t know.’

The Validity Rating (VR) of each of the four arguments was calculated as explained in Table 7.27.

The students were also given the following two statements relevant to the explanatory power of the four arguments:

- the argument shows why the statement is true
- the argument is an easy way to explain to someone in your class who is unsure
7.5. Students’ beliefs about mathematics and proof

<table>
<thead>
<tr>
<th>Student’s choice</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>assessing correctly both statements</td>
<td>2</td>
</tr>
<tr>
<td>assessing correctly one statement</td>
<td>1</td>
</tr>
<tr>
<td>assessing wrongly both statements</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7.27: Validity Ratings (VR) scoring scheme.

and again were asked to choose one of three options for each question: ‘I agree’, ‘I disagree’, or ‘I don’t know.’

The Explanatory Power (EP) of each of the four arguments was calculated as explained in Table 7.28.

<table>
<thead>
<tr>
<th>Student’s choice</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>agreeing with both statements</td>
<td>2</td>
</tr>
<tr>
<td>agreeing with one statement</td>
<td>1</td>
</tr>
<tr>
<td>disagreeing with both statements</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7.28: Explanatory Power (EP) scoring scheme.

The detailed distributions of the answers given by the students to the above questions both before and after our intervention as well as the VR’s and EP’s of each argument are offered in Appendix D. Here we offer an overview of the results (see Figures 7.21, 7.22, 7.23 and 7.24).

As we can see in figures 7.21 and 7.22, the validity ratings for three out of the four arguments were much higher after our intervention (note that when for an argument the right bar in the bar graph is high—there are many 2 scores—it suggest that the students could assess correctly the validity of the argument, whether the argument was deductive or inductive; so more scores of 2 indicates better validity assessment by the students). This indicates that the students after the intervention were better able to correctly assess the validity of arguments. During the lessons the generality of arguments was stressed as a factor in making an argument valid in mathematics. Students evidently could understand better that an argument that is specific to a few cases (an inductive argument) was not considered valid in mathematics, as opposed to deductive arguments which apply to the general case.
Chapter 7. Results

The only argument for which the highest score for the validity rate did not change considerably was Kalliopi’s. This can be explained by the fact that Kalliopi’s argument was the most difficult both before and after our intervention (see also Tables 7.24 and 7.25), and therefore the students had difficulties in assessing its validity. (A look at Table D.9 in Appendix D shows that the most frequent answer the students gave to the questions regarding Kalliopi’s argument was ‘I don’t know.’)

As for the Explanatory Power of the four arguments we see the results before and after our intervention in Tables 7.23 and 7.24.

We can see that after our intervention the scores of Explanatory Power for the deductive arguments that were easy for the students to understand are higher (more instances of score 2), whereas for the inductive argument of Stamatis more students disagree that it can explain well why the statement is true.

Generality of arguments  The question provided to the students in order to assess their understanding of the generality of a valid argument in mathematics, was the following (see also Task CONCEPTIONS ABOUT PROOF in Appendix B.4):

Figure 7.22: Validity Ratings of four arguments after our intervention.

Figure 7.23: Explanatory Power of four arguments before our intervention.
7.5. Students’ beliefs about mathematics and proof

Figure 7.24: Explanatory Power of four arguments after our intervention.

Suppose it has now been proved that, when you add the interior angles of any triangle, your answer is always 180°. Zoe asks what needs to be done to prove whether, when you add the interior angles of any right-angled triangle, your answer is always 180°. Tick either A or B.

(A) Zoe doesn’t need to do anything; the first statement has already proved this.

(B) Zoe needs to construct a new proof.

In Table 7.29 we can see the number and percentage of the students who thought that the argument just proved was general enough for Zoe not to need to prove it again for a right triangle.

<table>
<thead>
<tr>
<th>Assessing the generality of a valid argument</th>
<th>N = 19</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>general</td>
</tr>
<tr>
<td>Before</td>
<td>N %</td>
</tr>
<tr>
<td></td>
<td>15 79</td>
</tr>
<tr>
<td>After</td>
<td>13 68</td>
</tr>
</tbody>
</table>

Table 7.29: Generality rating of the statement ‘The sum of the angles of a triangle is 180°’ before and after our intervention.

Like in Hoyles and Healy’s research, we found that “most students appreciate the generality of valid proof” (1998, p. 3) both before and after our intervention. It is interesting to note that after our intervention fewer students assessed the generality of the same proof correctly. This could be explained by the fact that the students after our intervention often appeared more critical and doubting regarding arguments in general, and they would not accept any argument if they were not absolutely sure about its validity (see also results in Section 7.5.2, p. 124).

Assessing the role of proof The questions asked of the students regarding their ideas on the role of proof were the following (see also Task CONCEPTIONS ABOUT PROOF in Appendix B.4):

• What do you think is “proof” in mathematics? Explain.

• Do we really need to prove things in mathematics? Couldn’t we simply get away without it?
Chapter 7. Results

In Table 7.30 we present the results of the answers of the students to the first of the above questions. As was mentioned in the methodology chapter, for the coding of the answers we used “a simplified version of de Villier’s classification of the functions of proof, using the categories, truth (verification), explanation and discovery together with a fourth category ‘none/other’ if the students wrote nothing or if their contributions appeared to be irrelevant” (Hoyles & Healy, 2007, p. 86).

<table>
<thead>
<tr>
<th></th>
<th>Truth</th>
<th>Explanation</th>
<th>Discovery</th>
<th>No/Other</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
<td>N %</td>
</tr>
<tr>
<td>Before</td>
<td>4 21</td>
<td>6 32</td>
<td>0 0</td>
<td>6 32</td>
</tr>
<tr>
<td>After</td>
<td>7 37</td>
<td>3 16</td>
<td>3 16</td>
<td>5 26</td>
</tr>
</tbody>
</table>

Table 7.30: Assessing the role of proof before and after our intervention.

Healy and Hoyles (1998, p. 17) present results similar to ours for most of the aspects apart from that of proof as Truth. At the beginning of our research 21% of the students considered proof as a means to establish truth, a percentage which by the end of our intervention was changed to 37%. We explain this change by the fact that this aspect of proof was stressed a lot during the lessons. This percentage in the research of Hoyles and Healy was much bigger (50%). However, the other three functions of proof in Hoyles and Healy’s research were as follows: Explanation 35%, Discovery 1% and None/Other 28%. We see that these numbers match our numbers before the intervention. However, there is a significant difference in the Discovery aspect of proof which after our intervention changed from 0 to 16%. Discovery was one of the aspects of proof also stressed in our intervention and the students were frequently asked to make their own conjectures and try to provide arguments for them, which we think helped them understand the aspect of discovery that lies within the concept of proof. Finally, we observe that a sizeable percentage of students do not know what proof is both before (32%) and after (26%) our intervention.

Changes in students’ beliefs about proof  It was part of our research question (see Section 3.2, p. 26) to examine how the Van Hiele level of the students and their kinds of reasoning (proof schemes) relate to their beliefs about proof both before and after our intervention. In line with this research question we formulated the following hypothesis:

As students progress from the descriptive to the theoretical level, their beliefs about mathematics and proof change.

The ‘change’ we expected was that students would believe after the intervention that proof in mathematics is something that explains why a statement is generally true. The small number of students who were assessed to have made the move from the descriptive to the theoretical level by the Usiskin Van Hiele test ($n = 10$ based on the 3/5 criterion and $n = 7$ based on the 4/5 criterion) and the Harel and Sowder proof schemes ($n = 10$) does not allow us to run statistical tests to check the above hypothesis. Therefore, in order to check our hypothesis we will look at the instances of students who moved during our intervention from the descriptive to the theoretical Van Hiele level and examine whether their scores about VR and EP of arguments and their beliefs about the convincingness of an argument changed in a positive direction.
Similarly we will also look at the students whose progress in Van Hiele level was not measured to be positive by any of the instruments used in this research. The variables we use for this comparison are presented in Table 7.31

Table 7.31: Variables used.

<table>
<thead>
<tr>
<th>Name of Variable</th>
<th>Scale</th>
<th>Scale Type</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in vH level</td>
<td>yes or no</td>
<td>binary</td>
<td>‘yes’ corresponds to a move (with either criterion in both pre- or post-test) from Van Hiele level 2 to Van Hiele level 3 or 4 as well as from Van Hiele level 2 to no level (the test did not give results) and from no level to Van Hiele level 3 or 4; ‘no’ corresponds to the rest of the cases.</td>
</tr>
<tr>
<td>Change in VR</td>
<td>-2 to 2</td>
<td>ordinal</td>
<td>The change in the validity rating score before and after our intervention.</td>
</tr>
<tr>
<td>Change in EP</td>
<td>-2 to 2</td>
<td>ordinal</td>
<td>The change in the explanatory power score before and after our intervention.</td>
</tr>
<tr>
<td>Change in PS</td>
<td>yes or no</td>
<td>binary</td>
<td>1 corresponds to moving from Empirical to Deductive proof schemes, and 0 to not making that move.</td>
</tr>
</tbody>
</table>

Changes in the appreciation of validity of arguments  We will first look at the changes in the VR for three different arguments (a familiar deductive one, Georgia’s; an unfamiliar deductive one, Lefteris’s; and an inductive one, Stamatis’s) of those students that were assessed to have moved from the descriptive to the theoretical Van Hiele level with both criteria (3/5 and 4/5). We expected that for the familiar argument the results would not change very much before and after our intervention since the students had seen the argument before therefore their Validity Rating was expected to be correct both before and after our intervention. For the other two arguments however, the unfamiliar deductive and the inductive one, we expected a positive change in the Validity Ratings of students. We can see the results and the scores of these students in Table 7.32.

Table 7.32: Change in students’ VR scores as they move from the descriptive to the theoretical Van Hiele level.
Chapter 7. Results

As was expected, there is not much change in the VR scores of the students for the familiar deductive argument. Students had heard this argument in the past, and in fact the large majority of them could easily assess correctly the validity of this argument both before and after our intervention. As for the unfamiliar deductive argument, we see that there is mostly a positive change in the VR score of the students (7 out of 12); only 4 out of 12 students did not show any change (three of those students had a VR score of 2—correct assessment of validity—both before and after our intervention, and one of those students had a VR score of 0—wrong assessment of validity—both before and after our intervention) and only 1 student had a negative change. Finally, for the inductive argument we see an even more positive change: 9 out of 12 students could now better assess the validity of an inductive argument. This was a result which we aimed at achieving. Only 1 student did not show a positive change but the student had assessed correctly the validity both before and after our intervention, and 1 student had a negative change.

We will now look at the case of students that were measured to not have made a move from the descriptive to the theoretical Van Hiele level by the instruments used in our research (see Table 7.33).

<table>
<thead>
<tr>
<th>Students ($n = 7$)</th>
<th>Change in vH 3/5</th>
<th>Change in vH 4/5</th>
<th>Change in VR ded/fam</th>
<th>Change in VR ded/unfam</th>
<th>Change in VR ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gregory</td>
<td>no</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Theano</td>
<td>no</td>
<td>no</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Thalia</td>
<td>no</td>
<td>no</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Isis</td>
<td>no</td>
<td>no</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Nadia</td>
<td>no</td>
<td>no</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Nia</td>
<td>no</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Olivia</td>
<td>no</td>
<td>no</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td><strong>Expected change</strong></td>
<td></td>
<td></td>
<td>positive</td>
<td>zero</td>
<td>zero</td>
</tr>
<tr>
<td><strong>Average of Outcome</strong></td>
<td></td>
<td></td>
<td>0.6</td>
<td>0.9</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 7.33: Change in students’ VR scores for the students that did not make the move from the descriptive to the theoretical Van Hiele level based on the Usiskin Van Hiele test results.

Out of the 7 students 4 showed no change in the validity rating of Georgia’s familiar argument, of which 3 had assessed the validity of the argument correctly both before and after our intervention. As expected, a bigger ratio (3/7) of students now showed a change in the assessment of this argument’s validity compared to the students whose Van Hiele level was measured to have changed in the desired direction (2/12). As for the unfamiliar deductive argument, the large majority of students (5/7) made a positive change in the assessment of its validity. Similarly, for the inductive argument the students assessed more correctly its validity after our intervention (5/7), a result that was unexpected (although it was an aim of our intervention) if we consider the assessment of the Van Hiele level of the students.

The above results were unexpected and may be interpreted in the three following ways. First, they may be due to the way we assessed the Van Hiele level of the students (the Usiskin test is a multiple choice test which does not always provide a complete overview of the abilities of the students that are being tested). Second, perhaps the questions we used from the study of Healy and Hoyles (1998) regarding the VR and the EP scores were not sufficient and were not touching a wide enough range of arguments in order to be able to reveal the beliefs of the students regarding the validity and the explanatory power of proof in mathematics. Third, it may be possible that the students’ changes in beliefs about mathematics and the students’ changes in
Van Hiele level are really independent, when measured over a short period of time.

Given the above interpretations, we can draw the conclusion that the positive change in the assessment of the validity of arguments that was measured for these students resulted from our intervention and it is not immediately connected to their Van Hiele level as this was measured by the instruments we used.

When we look at the changes in the VRs for the same three arguments but this time of those students that were measured to have moved from the Empirical (EMP) to the Deductive (DED) Harel and Sowder proof schemes the picture is similar although there is a slight change in the groups of students (we remind the reader that the correlation between change in Van Hiele level and change in proof schemes was not significant as explained in Section 7.3). The results are shown in Tables 7.34 and 7.35.

<table>
<thead>
<tr>
<th>Students (n = 10)</th>
<th>Change in Proof Scheme</th>
<th>Change in VR ded/fam</th>
<th>Change in VR ded/unfam</th>
<th>Change in VR ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thalia</td>
<td>yes</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Nia</td>
<td>yes</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Gregory</td>
<td>yes</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Olivia</td>
<td>yes</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Mary</td>
<td>yes</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Debbie</td>
<td>yes</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Gayle</td>
<td>yes</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Elli</td>
<td>yes</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Katy</td>
<td>yes</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Katerina</td>
<td>yes</td>
<td>0</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Expected change: zero or positive  positive  positive
Average of Outcome: 0.2  1  0.9

Table 7.34: Change in students’ VR scores as they move from the empirical to the deductive Harel & Sowder proof schemes.

In Table 7.34 see that our expectations were confirmed in the case of all three arguments. For the unfamiliar deductive argument, only 3 out of 10 students did not show any change but we should mention that those three students had a VR score of 2—correct assessment of validity—both before and after our intervention.

<table>
<thead>
<tr>
<th>Students (n = 6)</th>
<th>Change in Proof Scheme</th>
<th>Change in VR ded/fam</th>
<th>Change in VR ded/unfam</th>
<th>Change in VR ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theano</td>
<td>no</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Isis</td>
<td>no</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Nadia</td>
<td>no</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Kelly</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Flora</td>
<td>no</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>John</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Expected change: zero or positive  zero  zero
Average of Outcome: 0.7  0.8  1

Table 7.35: Change in VR scores for the students that did not make the move from the empirical to the deductive Harel & Sowder proof schemes.

In Table 7.35 we see that our expectations are most of the time not confirmed in the case of the students who did not show a move from inductive to deductive proof schemes, although
from a teaching perspective our aims were achieved since after our intervention more students
could assess the validity of all three kinds of arguments correctly.

Like before, we draw the conclusion that the positive change in the assessment of the validity
of arguments that was measured for the students who were not measured to have moved from
using empirical to using deductive proof schemes, resulted from our intervention and that it is
not immediately connected to their proof schemes as those were identified in our study.

**Changes in the appreciation of the explanatory power of arguments** As in the case of the
Validity Ratings, we examined the change in the beliefs of the students about the Explanatory
Power of three different arguments by dividing the group of students into two main categories:
first, those that were measured to have made a move from the descriptive to the theoretical
Van Hiele level by at least one of the criteria (3/5 and 4/5) or from having empirical proof
schemes to having deductive proof schemes (see Tables 7.36 and 7.38), and second, those who
were not measured to make a move in either of the above cases (see Tables 7.37 and 7.39). We
expected that students of the first category would mostly display a positive change regarding
the explanatory power of the two deductive arguments and a negative change regarding the
inductive argument. For the students of the second category, we expected to see a negative or 0
change for the deductive arguments and a positive or 0 change for the inductive arguments.

<table>
<thead>
<tr>
<th>Students (n = 12)</th>
<th>Change in vH 3/5</th>
<th>Change in vH 4/5</th>
<th>Change in EP ded/fam</th>
<th>Change in EP ded/unfam</th>
<th>Change in EP ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debbie</td>
<td>yes</td>
<td>yes</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Gayle</td>
<td>yes</td>
<td>yes</td>
<td>1</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>Eli</td>
<td>yes</td>
<td>yes</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Katy</td>
<td>yes</td>
<td>yes</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Fiona</td>
<td>yes</td>
<td>yes</td>
<td>1</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Sunny</td>
<td>yes</td>
<td>no</td>
<td>−1</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Mary</td>
<td>yes</td>
<td>no</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Flora</td>
<td>yes</td>
<td>no</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Kelly</td>
<td>yes</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Sue</td>
<td>yes</td>
<td>no</td>
<td>0</td>
<td>−1</td>
<td>−2</td>
</tr>
<tr>
<td>Katerina</td>
<td>no</td>
<td>yes</td>
<td>0</td>
<td>1</td>
<td>−1</td>
</tr>
<tr>
<td>John</td>
<td>no</td>
<td>yes</td>
<td>−1</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td><strong>Expected change</strong></td>
<td></td>
<td></td>
<td>positive</td>
<td>positive</td>
<td>negative</td>
</tr>
<tr>
<td><strong>Average of Outcome</strong></td>
<td></td>
<td></td>
<td>0.5</td>
<td>0.1</td>
<td>−0.6</td>
</tr>
</tbody>
</table>

Table 7.36: Change in students’ EP scores as they move from the descriptive to the theoretical Van Hiele
level.

Table 7.36 shows that our expectations were confirmed to a large extent regarding the fa-
miliar deductive argument: 6/12 positive changes, and all zero changes were answers in which
the explanatory power of this deductive argument was considered high (score of 2) both be-
fore and after our intervention. The same holds regarding the inductive argument: 7/8 negative
changes, and one zero change was an answer in which the explanatory power of the inductive
argument was considered zero both before and after our intervention. However, we see that
for the unfamiliar deductive argument the students generally did not change their ideas about
its explanatory power. In fact, most of the ‘0’ changes concerned ratings of 1 both before and
after our intervention. This indicates that possibly the students found the type of the deductive
argument (a narrative rather than a formal argument) more difficult to understand, and they were
not actually rating the explanatory power of a deductive argument in general.

<table>
<thead>
<tr>
<th>Students (n = 7)</th>
<th>Change in EP ded/fam</th>
<th>Change in EP ded/unfam</th>
<th>Change in EP ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gregory</td>
<td>no</td>
<td>no</td>
<td>1</td>
</tr>
<tr>
<td>Theano</td>
<td>no</td>
<td>no</td>
<td>−1</td>
</tr>
<tr>
<td>Thalia</td>
<td>no</td>
<td>no</td>
<td>−1</td>
</tr>
<tr>
<td>Isis</td>
<td>no</td>
<td>no</td>
<td>−1</td>
</tr>
<tr>
<td>Nadia</td>
<td>no</td>
<td>no</td>
<td>1</td>
</tr>
<tr>
<td>Nia</td>
<td>no</td>
<td>no</td>
<td>1</td>
</tr>
<tr>
<td>Olivia</td>
<td>no</td>
<td>no</td>
<td>2</td>
</tr>
<tr>
<td><strong>Expected change</strong></td>
<td>zero</td>
<td>zero</td>
<td>zero</td>
</tr>
<tr>
<td><strong>Average of Outcome</strong></td>
<td>0.3</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table 7.37: Change in students’ EP scores for the students that did not make the move from the descriptive to the theoretical Van Hiele level based on the Usiskin Van Hiele test results.

Our expectations are met to a smaller degree for the students that were measured, based on the Usiskin Van Hiele test results, to not have moved from the descriptive to the theoretical Van Hiele level (see Table 7.37). Many students (3/7) thought after the intervention that the familiar deductive argument was less explanatory than they thought before the intervention which shows that these students were not convinced by our teaching that deductive arguments have good explanatory power. Moreover, one of the positive changes (Nadia’s) was from a score of 0 to a score of 1 which can be also classified as a student that was still unsure about the explanatory power of this argument.

As for the inductive argument, as expected none of the students in this subgroup thought that an inductive argument is less explanatory than they thought at the beginning of our intervention (no negative numbers shown in the table) and only one student (Olivia) of those who did not change their opinion had considered the explanatory power of the inductive argument to be 0 both before and after our intervention. The case of the unfamiliar deductive argument is different as 4 out of 7 students gave a higher explanatory power score to the argument after our intervention. However, only 2 of those students (Gregory and Nia) gave the maximum score (2) to the explanatory power of this argument, which means that the majority of the students did not think that this argument is particularly explanatory after our intervention.

We conclude that there is a relation between change from the descriptive to the theoretical Van Hiele level (as this was measured by the Usiskin Van Hiele test) and change in beliefs about the explanatory power of certain arguments. More specifically, we found that as the students moved from the descriptive to the theoretical Van Hiele level, they found deductive arguments more explanatory and inductive arguments less explanatory. However, we cannot conclude that the change in Van Hiele level has caused the change in beliefs about the explanatory power, as the change in beliefs could be a result of our teaching intervention independent of the Van Hiele level of students.

For the students that made a move from using empirical to using deductive proof schemes the results are presented in Table 7.38.

For the familiar deductive argument we see that there are mostly positive changes which was expected. Similarly, for the unfamiliar deductive argument there were mainly positive changes (6/10) although not as big as for the familiar deductive argument, possibly because of the type of the unfamiliar argument (narrative and relatively difficult to understand). However, for the
Chapter 7. Results

<table>
<thead>
<tr>
<th>Students (n = 10)</th>
<th>Change in Proof Scheme</th>
<th>Change in EP ded/fam</th>
<th>Change in EP ded/unfam</th>
<th>Change in EP ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thalia</td>
<td>yes</td>
<td>−1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Nia</td>
<td>yes</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Gregory</td>
<td>yes</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Olivia</td>
<td>yes</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mary</td>
<td>yes</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Debbie</td>
<td>yes</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Gayle</td>
<td>yes</td>
<td>1</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>Elli</td>
<td>yes</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Katy</td>
<td>yes</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Katerina</td>
<td>yes</td>
<td>0</td>
<td>1</td>
<td>−1</td>
</tr>
</tbody>
</table>

Expected change positive positive negative
Average of Outcome 0.9 0.5 0.2

Table 7.38: Change in students’ EP scores as they move from the empirical to the deductive Harel & Sowder proof schemes.

inductive argument, we see almost as many positive as negative changes and many zero changes. Of the zero changes only two (Elli and Olivia) were of students who thought both before and after our intervention that the explanatory power of the inductive argument is 0.

The results regarding the students who did not make the move from using empirical to using deductive proof schemes are presented in Table 7.39.

<table>
<thead>
<tr>
<th>Students (n = 6)</th>
<th>Change in Proof Scheme</th>
<th>Change in EP ded/fam</th>
<th>Change in EP ded/unfam</th>
<th>Change in EP ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theano</td>
<td>no</td>
<td>−1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Isis</td>
<td>no</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>Nadia</td>
<td>no</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Kelly</td>
<td>no</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Flora</td>
<td>no</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>John</td>
<td>no</td>
<td>−1</td>
<td>0</td>
<td>−1</td>
</tr>
</tbody>
</table>

Expected change zero zero zero
Average of Outcome −0.2 0 −0.3

Table 7.39: Change in EP scores for the students that did not make the move from the empirical to the deductive Harel & Sowder proof schemes.

Here, as was expected there is not much positive change: students’s beliefs about how explanatory the deductive arguments in question are did not change after our intervention. This was an expected result since these students did not manage to make a move from using empirical to using deductive proof schemes. As for the inductive argument we see two negative changes which was unexpected, however the averages are close to zero, and we should possibly consider them as equal to zero within measurement accuracy.

To summarise, we found that the students who had a deductive proof scheme more often than an empirical one at the end of our intervention, found deductive arguments more explanatory which was expected, but they also found inductive arguments more or equally explanatory before our intervention. On the contrary, the beliefs about the explanatory power of the students who kept having empirical proof schemes both before and after our intervention changed in an unexpected direction: after our intervention the change in the beliefs about the explanatory power of the inductive argument either did not change or changed in the opposite direction (the
students now thought that inductive arguments were less explanatory than before).

We conclude that there is a relation between change in proof schemes and change in the beliefs about the explanatory power of deductive arguments, but not of inductive arguments. Of course, we cannot conclude that the change in proof schemes has caused the change in beliefs about the explanatory power of arguments, as the change in beliefs could be a result of our teaching intervention independent of the students’ proof schemes. Students after our intervention have obviously started considering deductive arguments more explanatory than before, but still have not entirely changed their ideas about the explanatory power of inductive arguments. This can be explained by the fact that the students before our intervention had been using only inductive arguments therefore abandoning the belief that they are explanatory turned out not to be possible in only a few weeks.

**Change in the beliefs about the convincingness of arguments**  
One of the questions in Task CHOOSING (see Appendix B.3) was about choosing the most convincing of the given arguments. A brief description of the type of each argument was given in Table 7.26. To briefly remind the reader about the types of arguments we provide the following list:

- Georgia’s (GEO), Lefteris’s (LEF), Tania’s (TAN) and Kalliopis’s (KAL) arguments were deductive (we label these as DED) arguments of various forms,
- Eleni’s (ELE) and Stamatis’s (STA) were inductive (we label these as IND) arguments, and
- Petros’s (PET) argument was an incorrect, circular and ‘formal’ argument (we label this as EXT).

We expected that as the students move from the descriptive to the theoretical Van Hiele level or from using empirical to using deductive proof schemes, they would find deductive arguments convincing rather than inductive ones. In other words, we expected that these students would either find a deductive argument convincing both before and after our intervention or only after our intervention. On the other hand, we expected that the students who did not make a change in Van Hiele level or in proof scheme would continue finding inductive arguments more convincing than others after our intervention.

In Tables 7.40 and 7.41 we see how the change in the students’ beliefs about the convincingness of arguments relates to the change in the Van Hiele level of students.

In Table 7.40 we see that for those students who moved from the descriptive to the theoretical Van Hiele level (based on at least one of the criteria) their beliefs about the convincingness of arguments either changed positively (in the 6 cases where before the intervention the choice was an inductive argument) or remained unaltered (in the 6 cases where before the intervention the choice was already a deductive argument). This result agrees with our expectations. In only one case (Sue), a student who was measured to have made a move in the Van Hiele levels believed an inductive argument to be convincing both before and after our intervention (it is worth mentioning that this student had been consistently showing that she was at the descriptive Van Hiele level).

In Table 7.41 we see that although some students were not measured by the Usiskin Van Hiele test to have made the move from the descriptive to the theoretical Van Hiele level, several of them (4/7) did think after our intervention that a deductive argument was more convincing.
Chapter 7. Results

<table>
<thead>
<tr>
<th>Change in beliefs about Convincingness</th>
<th>Change in Change in Change in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students ((n=12))</td>
<td>vH 3/5</td>
</tr>
<tr>
<td>Debbie</td>
<td>yes</td>
</tr>
<tr>
<td>Gayle</td>
<td>yes</td>
</tr>
<tr>
<td>Elli</td>
<td>yes</td>
</tr>
<tr>
<td>Katy</td>
<td>yes</td>
</tr>
<tr>
<td>Fiona</td>
<td>yes</td>
</tr>
<tr>
<td>Sunny</td>
<td>yes</td>
</tr>
<tr>
<td>Mary</td>
<td>yes</td>
</tr>
<tr>
<td>Flora</td>
<td>yes</td>
</tr>
<tr>
<td>Kelly</td>
<td>yes</td>
</tr>
<tr>
<td>Sue</td>
<td>yes</td>
</tr>
<tr>
<td>Katerina</td>
<td>yes</td>
</tr>
<tr>
<td>John</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 7.40: Change in students’ beliefs about convincingness of arguments as they move from the descriptive to the theoretical Van Hiele level.

<table>
<thead>
<tr>
<th>Change in beliefs about Convincingness</th>
<th>Change in Change in Change in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students ((n=7))</td>
<td>vH 3/5</td>
</tr>
<tr>
<td>Gregory</td>
<td>no</td>
</tr>
<tr>
<td>Theano</td>
<td>no</td>
</tr>
<tr>
<td>Thalia</td>
<td>no</td>
</tr>
<tr>
<td>Isis</td>
<td>no</td>
</tr>
<tr>
<td>Nadia</td>
<td>no</td>
</tr>
<tr>
<td>Nia</td>
<td>no</td>
</tr>
<tr>
<td>Olivia</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 7.41: Change in students’ beliefs about convincingness of arguments for the students that did not make the move from the descriptive to the theoretical Van Hiele level based on the Usiskin Van Hiele test results.

than the ones that they had chosen before our intervention (two of them had chosen inductive arguments, and two of them had made no distinctions between arguments and were classified as having the external proof scheme). Moreover, those students whose beliefs about which argument was most convincing did not change, had chosen as most convincing a deductive argument both before and after our intervention. We conclude that the change in the beliefs of the students about the convincingness of arguments does not depend on the move from the descriptive to the theoretical Van Hiele level (as the levels were measured by the Usiskin Van Hiele test) but is a result of the teaching intervention in which it was very often stressed that an argument is more convincing when it is general and does not depend on the figure.

The picture is exactly the same when we look at the relation between change in proof schemes and change in the beliefs about the convincingness of arguments, since the group of students remains the same only it is divided differently and we have already seen that the large majority of students after our intervention found a deductive argument convincing.

We conclude that the change in the beliefs of the students about the convincingness of deductive or inductive arguments does not correlate with the change in their Van Hiele level or their proof schemes. The positive result in the change of students’ beliefs can be interpreted as a result of our teaching intervention.
Conclusions

Students before the intervention would mostly choose as their own approach an inductive argument although they would not think this argument gets the best mark. After our intervention the students mostly chose deductive arguments both as their own approach and as the one that would get the best mark.

• Students before our intervention did not show strong preference regarding the convincingness of arguments, whereas after our intervention deductive arguments were chosen to be most convincing.

• Students understood that if a general statement is proven, then it holds for all specific cases. (Similar to Healy and Hoyles (1998, p. 3) who found that “Most students appreciate the generality of a valid proof.”)

• Our hypothesis that ‘as students progress from the descriptive to the theoretical level, their beliefs about mathematics and proof change’ was confirmed for the beliefs of students about the explanatory power of mathematical proof, however it was not confirmed for the beliefs of students regarding the validity or the convincingness of mathematical proof. We conclude that the change in the assessment of the validity and the convincingness of arguments resulted from our intervention and is not immediately connected to the change in their Van Hiele level or in their proof schemes as this was measured by the instruments we used.
Chapter 8

Conclusions and Discussion

In this chapter we will summarize the results of our research (Section 8.1) discuss them and offer our recommendations for future research (Section 8.2).

8.1 Conclusions

Our main research questions, our subquestions and our hypotheses were presented in Chapter 3 on page 25. Here we will summarize the results of our research study by addressing our research questions and hypotheses one by one.

8.1.1 Answering SQ.1 and H.1

Our first subquestion and hypothesis were the following:

SQ.1: What are the beliefs of students about the nature and the meaning of mathematical proof, and mathematics in general, and how do these change when they move from the descriptive to the theoretical Van Hiele level?

H.1: As students progress from the descriptive to the theoretical level, their beliefs about mathematics and proof change.

These were addressed by asking students’ conceptions of proof and mathematics before and after our intervention (with the help of questionnaires), and by examining the discussions between students and teacher as well as among students (see Section 7.5).

The results from using the adapted version of the Schoenfeld questionnaire (explained in detail in Section 7.5.1) showed that, regarding mathematics in general, the students after our intervention thought that ordinary school mathematics is mostly facts and procedures that have to be memorised. In addition, after the intervention the students thought even more than before that memorisation is not the best way to do well in math. This could be interpreted as an effect of the kind of teaching in which we stressed that geometry and mathematics in general is something different that what is usually taught in school; that is, it is not simply an application of procedures but rather an ongoing process of observing, conjecturing, and finding valid arguments to support or reject your conjectures. The students themselves also thought in general that the lessons affected their beliefs about mathematics in various ways (the results of the relevant questionnaire
were analysed in Section 7.5.2). Some of the ways they mentioned included that they now thought mathematics is better/more interesting/fun, more difficult/requires more thinking and time, has a ‘shaky’ beginning (the axioms), is not just givens without reasons, can be done cooperatively, requires imagination, is easy, is not just calculations.

Regarding the beliefs of students about geometry and proof, the results we got from the modified Schoenfeld questionnaire only showed the changes related to rather general aspects of proof and were not immediately connected with the ideas of validity and generality of arguments which we introduced in our intervention. Therefore, although the students’ answers to the relevant items of the Schoenfeld questionnaire did not show any change, we found that students’ beliefs about more specific aspects of geometry and proof did change based on the other two instruments we used.

More specifically, based on our questionnaire (see Section 7.5.2) we found that the students after our intervention seemed to have appreciated the nature of proof as explaining why things hold in mathematics and they had started believing that they can prove theorems on their own. In 17 out of 20 cases, the students thought that their beliefs about proof have changed after the intervention in ways such as that they now thought proof is: important, useful, can explain, easy/fun, difficult, not only for mathematicians.

Based on the instruments we used from Healy and Hoyles (1998), we found that after our intervention the students mostly chose deductive rather than inductive arguments both as their own approach and as the one that would get the best mark, and showed a strong preference for deductive arguments as being the ones that are most convincing (see Section 7.5.3). Moreover, after our intervention students understood better that if a general statement is proven, then it holds for all specific cases. This result is the same as for Healy and Hoyles (1998, p. 3) who found that “Most students appreciate the generality of a valid proof.”

Although in general there was a change in the beliefs of students both about mathematics in general and about proof in particular, this change was not always related to the change of the Van Hiele level of students from the descriptive to the theoretical level. Based on the detailed analysis offered in Section 7.5.3 we can say that our first hypothesis was confirmed for the beliefs of students about the explanatory power of mathematical proof, however it was not confirmed for the beliefs of students regarding the validity or the convincingness of mathematical proof. We conclude that the change in the way students assessed the validity and the convincingness of arguments after our intervention resulted from our teaching and is not immediately connected to the change in their Van Hiele level or in their proof schemes as measured by the instruments we used.

8.1.2 Answering SQ.2 and H.2

Our second subquestion and hypothesis were the following:

**SQ.2:** Is there a correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking?

**H.2:** Given the correspondence1 between the Harel & Sowder proof schemes and the Van Hiele levels of thinking, there should be a correlation between corresponding Van Hiele levels and Harel & Sowder proof schemes.

---

1See Section 2.5 on page 16 for a definition of this correspondence.
8.1. Conclusions

To address these we checked both before and after the intervention the kinds of proof schemes employed by the students when proving geometrical propositions (see Section 7.2) as well as their Van Hiele levels (see Section 7.1). By conducting statistical tests we checked whether there is a consistent correspondence between the proof schemes used and the students’ Van Hiele levels of thinking (see Section 7.3). We found that there is a correlation between the Harel & Sowder proof schemes and the Van Hiele levels of thinking before and after our intervention. However, we were unable to give a certain answer to the question of whether there is a correlation between the progress in Harel & Sowder proof schemes (from EMP to DED) and the progress in the Van Hiele level based on our statistical analysis, due to the small number of the students for which we had data.

Therefore, in order to further investigate our second hypothesis (H.2), we conducted a qualitative analysis of our results by looking at those moments during our intervention when the students displayed progress in their Van Hiele level and/or moved from having an empirical to having a deductive proof scheme. This was presented in Section 7.4. The qualitative analysis of the results showed cases of students whose progress in Van Hiele level happened together with their move from an inductive to a deductive proof scheme. However, as before, a larger number of students would be needed in order to confirm statistically our second hypothesis.

8.1.3 Answering SQ.3 and H.3

Our third subquestion and hypothesis were the following:

SQ.3: Can we make the distinction between deductive and inductive arguments understandable to students before they start producing deductive proofs for themselves, and how can this process be planned according the Van Hiele teaching phases?

H.3: The move of students from the descriptive to the theoretical Van Hiele level can be described in three stages (related to the arguments students use during proving); a teaching strategy based on these three stages and the five Van Hiele teaching phases—like the one used in this research study—can facilitate this move.

To address these we designed a series of lessons based on the five Van Hiele teaching phases (inquiry, directed orientation, explication, free orientation and integration) and on our idea of a three stage learning sequence:

• Stage 1: Understanding the distinction between empirical and deductive arguments.
• Stage 2: Abandoning the use of empirical arguments in favor of deductive ones.
• Stage 3: Starting to see the axiomatic structure of Euclidean geometry.

We found (see Section 7.4.1) that the first stage of our teaching strategy was successful, as most of the students (about three quarters) after the completion of this stage were able to understand the distinction between deductive and inductive arguments. However, and as was
expected, the students at this point were not always able to offer deductive arguments, something which they are expected to practice during stage two. The second stage of our teaching strategy was also successful, since the percentage of students using either empirical or external proof schemes at the end of stage two shows a considerable decrease while at the same time the frequency of using a deductive proof scheme shows a considerable increase (from 9% to 88%). However, the number of lessons and of students was small; therefore, we would need to try this teaching strategy also on a larger sample and over a longer period of time in order to be able to draw safer conclusions regarding its effectiveness in general. The third stage of our teaching intervention was successful. In the third stage the students are supposed to start seeing that, for the arguments they come up with, only previously proven knowledge can be used, and that it all starts from a few basic assumptions: the axioms. The students during our last meeting discovered the existence of axioms and the fact that the axioms are the basic building blocks of Euclidean geometry.

The five Van Hiele teaching phases in our teaching intervention were also completed for most concepts successfully (see Section 7.4.2). However, both the directed and the free orientation phases were short, and for some concepts the free orientation phase was not included. We believe that our teaching strategy can be more successful (all students arriving at Van Hiele level 3 or above, all students consistently using deductive arguments) if it is applied over a longer period of time.

Our teaching strategy helped the students as a group to move to a greater Van Hiele level than what they started with. Of the students who took the Van Hiele test both before and after our intervention ($n = 20$) a pre/post comparison showed that most of them made progress in their Van Hiele level. More specifically, in 10 out of the 13 cases for which we can calculate the change in Van Hiele level based on the 3/5 criterion the change was positive. Similarly, in 10 out of the 14 cases for which we can calculate the change based on the 4/5 criterion the change was positive. Moreover, when we look specifically at the move from the descriptive (level 2) to the theoretical (level 3) Van Hiele level, 4 out of 6 students who started at level 2 based on criterion 3/5 moved to level 3, and 4 out of 5 of those who started at Van Hiele level 2 based on criterion 4/5 moved to level 3. The few students for which there was no positive change in their Van Hiele level in any of the above, remained at the same Van Hiele level as they started with. These results, in combination with the qualitative analysis of the lessons, allow us to conclude that our teaching strategy was successful in terms of moving students to a higher Van Hiele level.

8.1.4 Answering our main research question

Our main research question was the following:

(RQ) What kind(s) of reasoning do students use for proving while they are at different levels of thinking in geometry as well as during the transition from the level of informal-visual proofs to the level of formal-logical proofs, and how do these kinds of reasoning relate to students’ beliefs about proof?

Our intention was to look at the specific kinds of reasoning that students use when they move from the descriptive to the theoretical Van Hiele level, especially when they follow a learning path where the distinction between deductive and inductive arguments is made obvious at the beginning. We expected that students would use a lot the transformational/causal proof scheme
which would help them see the ‘movement’ in the figures involved in their conjectures. We also expected that students would rather quickly abandon the empirical proof schemes, because of the distinction made between inductive and deductive arguments early in the intervention.

Indeed we found that the students before our teaching intervention had mostly empirical proof schemes (see also Tables 7.2 and 7.3 on pages 76 and 77), whereas at the end of our intervention they had mostly deductive ones. Contrary to our expectations, however, the transformational/causal proof scheme was not encountered much during our teaching intervention.

The second part of our research question, related to the correlations between change in the kinds of reasoning and change in the beliefs, was answered by answering our first subquestion. As we mentioned before, although overall there was a change in the beliefs of students both about mathematics in general and about proof in particular, this change was not always related to the change of the proof schemes (kinds of reasoning) from empirical to deductive, or to the change of the Van Hiele level of students from descriptive to theoretical.

8.2 Discussion and recommendations

In what follows we will summarise and evaluate the way we used past research in our study. We will talk about what we learned and what we would do differently.

The Van Hiele theory and the Usiskin Van Hiele test We used the Van Hiele levels of thinking in our research both for selecting a sample of students and for designing our teaching intervention. The first researcher to design a multiple choice test for classifying students into Van Hiele levels was Usiskin (1982) and since then his test has been very frequently used in research all over the world. In our analysis of the results of this test (see Appendix D.1), we found very similar results both to those of Usiskin (1982) and to those of Tzifas (2005), who used the Van Hiele test on a large sample of students in Greece. The only significant differences in the results were regarding questions 13 and 17 of the Van Hiele test.

For question 13, our research gave similar results to that of Usiskin (1982), however, because of an error in the Greek translation of Tzifas’ question, the results in the study of Tzifas for this question are very different. Tzifas in his thesis (Tzifas, 2005, p. 79) mentions that during the test many students asked why a specific answer was missing from the range of answers for question 13, however the explanation for this is not offered by Tzifas.

For question 17 our results are similar to those of Tzifas but different from those of Usiskin. We think this may be due to the different educational background of our students and students in the United States (where Usiskin’s research took place).

In general, we noticed that for questions 13 and after the students seem to start answering more randomly which at some cases shows in the different results between our study and that of Usiskin and Tzifas. One possible interpretation of this is that in our study, the students were all of the same age (14 to 15), whereas in Tzifas’ and Usiskin’s study the students participating were of a larger range of ages (14 to 17). Therefore, in our sample of students there are less students who could potentially be at higher Van Hiele levels which makes them less able to answer correctly the last questions of the questionnaire.

As we have already discussed, a multiple choice test is quite restrictive and does not in itself allow the researcher to have a clear picture of the level of the student. We think that the risk of interpreting differently a question in the multiple choice test makes interviews with
Chapter 8. Conclusions and Discussion

the students after they have taken the test necessary. Because of time restrictions we could not include interviews in our research, however we strongly recommend in future research that interviews accompany the test.

Moreover, in our study some of the students were not at the desired Van Hiele level (level 2) at the beginning of the intervention. Having a larger sample of students at the desired level would allow for making safer conclusions regarding the success of the teaching intervention.

The Harel & Sowder proof schemes We used the Harel & Sowder proof schemes for classifying the kinds of reasoning of our students. During this process we sometimes had difficulty in deciding the category to which certain students’ proof schemes belonged. The examples offered by Harel and Sowder (1998) and Harel (2007) were not always enough for making clear what the characteristics of each of the deductive proof schemes were. For future research we recommend that a more specific coding of proof schemes is made based on clear descriptors for each proof scheme. In addition, we think that, as is the case with the Van Hiele levels, the proof scheme of an individual is not easy to find out only based on written work or on group work discussions. Interviews with selected students are strongly recommended for this reason.

Another point we would like to raise is related to the causal proof schemes. As mentioned in the main body of this thesis, we believe that during the move towards the theoretical Van Hiele level students can be helped if they use the idea of giving ‘movement’ to a figure in order to find out the general properties of figures. We hoped to see the students using this idea in their individual attempts to prove their conjectures. This idea is also central to the transformational proof schemes, and especially to the subcategory of causal proof schemes, where the reasons that cause a general property are taken into consideration. Contrary to our expectations, however, the transformational/causal proof scheme was not encountered much during our teaching intervention. This may be due to the short period of our intervention and to the fact that the idea of giving ‘movement’ to a figure was something entirely new to the students. We suggest the use of dynamic software for assisting students to understand this idea in a shorter time.

Schoenfeld’s questionnaire We used a modified version of a questionnaire designed by Schoenfeld (1989) as part of our instruments for identifying students’ beliefs about mathematics and proof. We found that the questions we used were related to general aspects of mathematics and proof which turned out not to be as useful for our research as we had thought. Our intervention was focused on certain central aspects of proof, such as validity and convincingness, therefore we do not recommend the use of this questionnaire if this research is repeated. Moreover, interviews with students can again be a more effective way to accurately reveal students’ beliefs.

The Hoyles & Healy instruments We found the Hoyles & Healy instruments quite useful. Especially Task CHOOSING (see Appendix B.3) was rather useful for identifying the proof schemes of students at the beginning of our intervention. However, a couple of the proofs in this task, such as Lefteris’, Tania’s and Kalliopi’s, caused confusion among the students since they did not understand them very well. We recommend that in future research the kinds of proofs in this task are adapted to the students’ background and, if necessary, to dedicate some time before the completion of the task to ensure that all students understand the meaning of the proofs.\footnote{This was taken care of by Hoyles and Healy in their own research.} Moreover, in our intervention Task CHOOSING was partly used for assessing the
change in the choices of types of arguments by the students for supporting the proposition “the sum of the angles of a triangle is $180^\circ$.” However, we realised that had we used a different task at the end of the intervention for assessing change we would have been able to get more useful results regarding this change: since we used the same task after the intervention the students were already familiar (to an extent) with this task which may have affected their answers.

Given the difference in the research questions between our research and that of Hoyles and Healy (2007) we could make a comparison only regarding certain results of the Hoyles and Healy study. We used instruments from Hoyles and Healy’s research in order to find out the students’ choices of arguments as their own approach or as the ones that would receive the best mark, their understanding of validity, convincingness and generality of arguments, as well as their appreciation of the explanatory power of arguments and of the role of proof. We will briefly discuss these results and compare it to those of Hoyles and Healy.

Regarding the choice of arguments of students as their own approach, before the intervention more than half of the students chose an inductive, pragmatic argument, and only 3 students (16%) thought that such an argument would get the best mark. This result is similar to the findings of Hoyles and Healy (2007, p. 110) where the two empirical arguments were chosen as own approach by 46% of the students, whereas only 9% of the students thought these arguments would get the best mark. In fact, we see that at the beginning of our intervention there is no strong preference of the students on which proof would get the best mark and their choices are rather evenly spread over all answers (although there is a slight preference for best mark for the two formal arguments, Georgia’s and Petros’s—21% each). This can be explained by the level of the students in geometry at that stage; the students were not familiar with the distinctions between different kinds of arguments. In Hoyles and Healy’s results the students chose Georgia’s proof to be the one that gets the highest grade both in 1996 and in 2002 (48% and 45% respectively).

Similarly to the Hoyles and Healy research, the students at the beginning of our intervention showed a slight preference for formal arguments regardless of whether they were correct or not. This indicates that when an argument includes a lot of mathematical symbols, thus being more formally represented, the students think it will get the best mark. Regarding deductive arguments, a sizable minority of students (32%) chose in the pre-test a deductive, valid argument for their own approach either in a narrative or in a more logical and deductive form (Georgia’s and Lefteris’s proofs). These results are similar to those of Hoyles and Healy (2007, p. 110) where, both in 1996 and in 2002, 29% of the students chose for Goergia’s and Lefteris’s arguments as their own approach. The difference between our research and that of Hoyles and Healy is that only 6 out of 19 students (or 32%) in our intervention thought that these two arguments would get the best mark, whereas around 60% of the students in Hoyles and Healy’s research thought that Georgia’s and Lefteris’s argument would receive the best mark (both in 1996 and in 2002).

The picture changes considerably after our intervention. Students do not show a preference any more for inductive, empirical proofs (only 2 out of 19, or 10% of the students would still choose such proofs) and on the contrary, they show a strong preference for deductive proofs (13 students chose Georgia’s answer as their own approach and 5 chose Lefteris’s, both deductive, valid proofs). We interpret this as a result of our intervention where the distinctions between deductive and inductive arguments were made clear and deductive arguments were now considered ‘mathematical’ as opposed to inductive arguments. However, the fact that we used the same task before and after our intervention does not allow us to be certain as to what caused this change. It is possible that the students made their choice after the intervention by remembering
Chapter 8. Conclusions and Discussion

the discussion of the arguments in class, rather than by understanding the distinctions between deductive and inductive arguments. However, based also on other results of our study (see Table 7.14 on page 99) which show that the students displayed a preference towards deductive proofs also in earlier tasks, we can conclude that the change was caused by our intervention.

Regarding the appreciation of the validity of proof, as in Hoyles and Healy’s research we found that “most students appreciate the generality of valid proof” (1998, p. 3) both before and after our intervention. It is interesting to note that after our intervention fewer students assessed the generality of the same proof correctly. This could be explained by the fact that the students after our intervention often appeared more critical and doubting regarding arguments in general, and they would not accept any argument if they were not absolutely sure about its validity (see also results in Section 7.5.2, p. 124).

Finally, regarding the role of proof, Healy and Hoyles (1998, p. 17) present results similar to ours for some of the aspects (explanation, discovery) but not for others (truth). At the beginning of our research 21% of the students considered proof as a means to establish truth, a percentage which by the end of our intervention was changed to 37%. We explain this change by the fact that this aspect of proof was stressed a lot during the lessons. This percentage in the research of Hoyles and Healy was much bigger (50%). The other three functions of proof in Hoyles and Healy’s research were as follows: Explanation 35%, Discovery 1% and None/Other 28%. These numbers match our numbers before the intervention. However, there is a significant difference in the Discovery aspect of proof which after our intervention changed from 0 to 16%. Discovery was one of the aspects of proof also stressed in our intervention and the students were frequently asked to make their own conjectures and to try to provide arguments for them, which we think helped them understand the aspect of discovery that lies within the concept of proof. Finally, we observed that a sizeable percentage of students did not know what proof is both before (32%) and after (26%) our intervention. This is similar to the result mentioned by Healy and Hoyles (1998, p. 4) where students with little or no sense of proof were “over one quarter of the sample.”

The design of our teaching intervention  Our teaching intervention was largely based on past research, more specifically the Van Hiele theory and the method of discovery learning where students have to make observations about specific figures, formulate conjectures, and try to argue for their truth. However, there was also an innovative element in our design, namely the idea of first making students aware of the distinction between inductive and deductive arguments before offering them opportunities to produce deductive proofs for themselves. Several tasks were designed for this first stage of making the distinction between inductive and deductive arguments. The results of our study showed that this teaching strategy was successful, since quite early in the intervention the students started realising that there is an important distinction between arguments that are general and arguments that are specific to a figure.

Although our intervention was generally successful, we consider it necessary for it to be repeated on a larger number of students and on a more varied sample (as opposed to our sample of students motivated in mathematics) in order to make safer conclusions about its effectiveness. We also recommend that more tasks are designed for each stage. Our intervention was rather short and it would take a longer period of time to ensure that the students have arrived at the desired Van Hiele level.

We do think that practicing the steps of proving and of discovering geometrical truths in the same way that mathematicians do, which is how we did it in our intervention, allows students
to have more confidence in themselves and to understand better the concepts introduced.

**Concluding remarks** To conclude, I would like to stress that the experience gained during this research for me as a teacher was extremely enjoyable and valuable. Regardless of the specific aims of this project, it was a pleasure to see the students enjoying doing mathematics, and to see them get excited when discovering mathematical truths on their own, something they did not think they could do before these lessons. Especially the fact that the students discovered through discussion that Euclidean geometry is based on certain axioms made a huge impression on them and seemed to be an unforgettable experience. I believe that giving to the students the opportunity to discover geometry rather than imposing on them theorems and propositions may improve their abilities in geometry at all levels.
References


REFERENCES


Further Reading


Appendix A

Pre- and post- tests and questionnaires
A.1 The original Usiskin Van Hiele test

The Van Hiele level theory offers an explanation and a remedy for student difficulty with higher-order cognitive processes required for success in secondary school geometry. This document reports results of a study which involved about 2700 students in 13 high schools, selected to provide broad representation of community socio-economics in the United States. The investigation looked at:

1. How are entering geometry students distributed with respect to the levels in the Van Hiele scheme?
2. What changes in Van Hiele levels take place after a year's study of geometry?
3. To what extent are levels related to concurrent geometry achievement?
4. To what extent do levels predict geometry achievement after a year's study?
5. What generalizations can be made concerning the entering Van Hiele level and geometry knowledge of students who are later found to be unsuccessful in their study of geometry?
6. To what extent is geometry being taught to students appropriate to their level?
7. To what extent do geometry classes in different schools and socio-economic settings differ in content appropriateness to student level. (MP)
A.1. The original Usiskin Van Hiele test

VAN HIELE GEOMETRY TEST

1. Which of these are squares?
   (A) K only
   (B) L only
   (C) M only
   (D) L and M only
   (E) All are squares.

2. Which of these are triangles?
   (A) None of these are triangles.
   (B) V only
   (C) W only
   (D) W and X only
   (E) V and W only

3. Which of these are rectangles?
   (A) S only
   (B) T only
   (C) S and T only
   (D) S and U only
   (E) All are rectangles.
4. Which of these are squares?

(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

5. Which of these are parallelograms?

(A) J only
(B) L only
(C) J and M only
(D) None of these are parallelograms.
(E) All are parallelograms.

6. PQRS is a square.

Which relationship is true in all squares?
(A) PR and RS have the same length.
(B) QS and PR are perpendicular.
(C) PS and QR are perpendicular.
(D) PS and QS have the same length.
(E) Angle Q is larger than angle R.
7. In a rectangle $GHJK$, $GJ$ and $HK$ are the diagonals.

Which of (A)-(D) is not true in every rectangle?
(A) There are four right angles.
(B) There are four sides.
(C) The diagonals have the same length.
(D) The opposite sides have the same length.
(E) All of (A)-(D) are true in every rectangle.

8. A rhombus is a 4-sided figure with all sides of the same length.

Here are three examples.

Which of (A)-(D) is not true in every rhombus?
(A) The two diagonals have the same length.
(B) Each diagonal bisects two angles of the rhombus.
(C) The two diagonals are perpendicular.
(D) The opposite angles have the same measure.
(E) All of (A)-(D) are true in every rhombus.
Appendix A. Pre- and post-tests and questionnaires

9. An isosceles triangle is a triangle with two sides of equal length. Here are three examples.

Which of (A)-(D) is true in every isosceles triangle?
(A) The three sides must have the same length.
(B) One side must have twice the length of another side.
(C) There must be at least two angles with the same measure.
(D) The three angles must have the same measure.
(E) None of (A)-(D) is true in every isosceles triangle.

10. Two circles with centers P and Q intersect at R and S to form a 4-sided figure PRQS. Here are two examples.

Which of (A)-(D) is not always true?
(A) PRQS will have two pairs of sides of equal length.
(B) PRQS will have at least two angles of equal measure.
(C) The lines PQ and RS will be perpendicular.
(D) Angles P and Q will have the same measure.
(E) All of (A)-(D) are true.
11. Here are two statements.
   Statement 1: Figure F is a rectangle.
   Statement 2: Figure F is a triangle.
   Which is correct?
   (A) If 1 is true, then 2 is true.
   (B) If 1 is false, then 2 is true.
   (C) 1 and 2 cannot both be true.
   (D) 1 and 2 cannot both be false.
   (E) None of (A)-(D) is correct.

12. Here are two statements.
   Statement S: \( \triangle ABC \) has three sides of the same length.
   Statement T: In \( \triangle ABC \), \( \angle B \) and \( \angle C \) have the same measure.
   Which is correct?
   (A) Statements S and T cannot both be true.
   (B) If S is true, then T is true.
   (C) If T is true, then S is true.
   (D) If S is false, then T is false.
   (E) None of (A)-(D) is correct.
Appendix A. Pre- and post- tests and questionnaires

13. Which of these can be called rectangles?

(A) All can.
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only

14. Which is true?
(A) All properties of rectangles are properties of all squares.
(B) All properties of squares are properties of all rectangles.
(C) All properties of rectangles are properties of all parallelograms.
(D) All properties of squares are properties of all parallelograms.
(E) None of (A)-(D) is true.

15. What do all rectangles have that some parallelograms do not have?
(A) opposite sides equal
(B) diagonals equal
(C) opposite sides parallel
(D) opposite angles equal
(E) none of (A)-(D)
16. Here is a right triangle ABC. Equilateral triangles ACE, ABF, and BCD have been constructed on the sides of ABC.

From this information, one can prove that AB, BE, and CF have a point in common. What would this proof tell you?

(A) Only in this triangle drawn can we be sure that AB, BE and CF have a point in common.

(B) In some but not all right triangles, AB, BE and CF have a point in common.

(C) In any right triangle, AB, BE and CF have a point in common.

(D) In any triangle, AB, BE and CF have a point in common.

(E) In any equilateral triangle, AB, BE and CF have a point in common.

17. Here are three properties of a figure.

Property D: It has diagonals of equal length.

Property S: It is a square.

Property R: It is a rectangle.

Which is true?

(A) D implies S which implies R.

(B) D implies R which implies S.

(C) S implies R which implies D.

(D) R implies D which implies S.

(E) R implies S which implies D.
18. Here are two statements.
   I. If a figure is a rectangle, then its diagonals bisect each other.
   II. If the diagonals of a figure bisect each other, the figure is a rectangle.

Which is correct?
   (A) To prove I is true, it is enough to prove that II is true.
   (B) To prove II is true, it is enough to prove that I is true.
   (C) To prove II is true, it is enough to find one rectangle whose diagonals bisect each other.
   (D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
   (E) None of (A)-(D) is correct.

19. In geometry:
   (A) Every term can be defined and every true statement can be proved true.
   (B) Every term can be defined but it is necessary to assume that certain statements are true.
   (C) Some terms must be left undefined but every true statement can be proved true.
   (D) Some terms must be left undefined and it is necessary to have some statements which are assumed true.
   (E) None of (A)-(D) is correct.
A.1. The original Usiskin Van Hiele test

20. Examine these three sentences.

(1) Two lines perpendicular to the same line are parallel.
(2) A line that is perpendicular to one of two parallel lines is perpendicular to the other.
(3) If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines m and p are perpendicular and lines n and p are perpendicular. Which of the above sentences could be the reason that line m is parallel to line n?

(A) (1) only
(B) (2) only
(C) (3) only
(D) Either (1) or (2)
(E) Either (2) or (3)

21. In F-geometry, one that is different from the one you are used to, there are exactly four points and six lines. Every line contains exactly two points. If the points are P, Q, R, and S, the lines are \{P,Q\}, \{P,R\}, \{P,S\}, \{Q,R\}, \{Q,S\}, and \{R,S\}.

Here are how the words "intersect" and "parallel" are used in F-geometry. The lines \{P,Q\} and \{P,R\} intersect at P because \{P,Q\} and \{P,R\} have P in common.

The lines \{P,Q\} and \{R,S\} are parallel because they have no points in common.

From this information, which is correct?
(A) \{P,R\} and \{Q,S\} intersect.
(B) \{P,R\} and \{Q,S\} are parallel.
(C) \{Q,R\} and \{R,S\} are parallel.
(D) \{P,S\} and \{Q,R\} intersect.
(E) None of (A)-(D) is correct.
22. To trisect an angle means to divide it into three parts of equal measure. In 1847, P.L. Wantzel proved that, in general, it is impossible to trisect angles using only a compass and an unmarked ruler. From his proof, what can you conclude?

(A) In general, it is impossible to bisect angles using only a compass and an unmarked ruler.

(B) In general, it is impossible to trisect angles using only a compass and a marked ruler.

(C) In general, it is impossible to trisect angles using any drawing instruments.

(D) It is still possible that in the future someone may find a general way to trisect angles using only a compass and an unmarked ruler.

(E) No one will ever be able to find a general method for trisecting angles using only a compass and an unmarked ruler.

23. There is a geometry invented by a mathematician J in which the following is true:

The sum of the measures of the angles of a triangle is less than 180°.

Which is correct?

(A) J made a mistake in measuring the angles of the triangle.

(B) J made a mistake in logical reasoning.

(C) J has a wrong idea of what is meant by “true.”

(D) J started with different assumptions than those in the usual geometry.

(E) None of (A)-(D) is correct.
24. Two geometry books define the word rectangle in different ways. Which is true?
   (A) One of the books has an error.
   (B) One of the definitions is wrong. There cannot be two different definitions for rectangle.
   (C) The rectangles in one of the books must have different properties from those in the other book.
   (D) The rectangles in one of the books must have the same properties as those in the other book.
   (E) The properties of rectangles in the two books might be different.

25. Suppose you have proved statements I and II.
   I. If p, then q.
   II. If s, then not q.
   Which statement follows from statements I and II?
   (A) If p, then s.
   (B) If not p, then not q.
   (C) If p or q, then s.
   (D) If s, then not p.
   (E) If not s, then p.
Appendix A. Pre- and post-tests and questionnaires

Please print

Last First Middle

Date

Grade in School (circle): 8 9 10 11 12 Sex (circle): M F

Birth date

Test date

Cross out the correct answer

1. A B C D E
2. A B C D E
3. A B C D E
4. A B C D E
5. A B C D E
6. A B C D E
7. A B C D E
8. A B C D E
9. A B C D E
10. A B C D E
11. A B C D E
12. A B C D E
13. A B C D E
14. A B C D E
15. A B C D E
16. A B C D E
17. A B C D E
18. A B C D E
19. A B C D E
20. A B C D E
21. A B C D E
22. A B C D E
23. A B C D E
24. A B C D E
25. A B C D E

Test Number

Project Use Only

ID

Space for drawing or figuring (You may also use the other side)
A.2 The Van Hiele test as used in our research

A.2. The Van Hiele test as used in our research

Ari jmìc Test:

TEST GEWMETRIAS
VAN HIELE
ODHGIES

Μη αναθέτετε αυτό το τεστ μέχρι να σας δοθεί η ενσωματωμένη απάντηση.

Αυτό το τεστ περιέχει 20 ερωτήσεις. Δεν απαντάμε άρα για να γνωρίζετε τα πάντα που ζητούνται σε αυτό το τεστ.

Πάνω δεξιά σε αυτή τη σελίδα υπάρχει γραμμής ο αριθμός του τεστ. Αντάπτριαστε αυτό τον αριθμό στο αντίστοιχο σημείο στο Φύλλο Απαντήσεων.

Όταν δοθεί η ενσωματωμένη απάντηση:

1. Διαβάστε κάθε ερώτηση προσεχτικά.
2. Αποκαλύψτε τους απαντήσεις νόμιζατε ότι είναι η σωστή. Μόνο μία απάντηση σε κάθε ερώτηση είναι σωστή. Κάλωστε το γράφμα του ανατομική στην απάντηση στο Φύλλο Απαντήσεων.
3. Χρησιμοποιήστε το κείμενο στο Φύλλο Απαντήσεων για να επιλέξετε τον σωστό τετελέσθηκε την προηγούμενη απάντησή σας.
4. Αν θελήσατε να αλλάξατε μια απάντηση, σβήστε προηγούμενη απάντησή σας.
5. Αν χρειαστείτε άλλη μάζα, σήμερα το κείμενο σου.
6. Θα έχετε 35 λεπτά για να επιλέξετε αυτό το τεστ.

Περιμένετε μέχρι να σας δοθεί η ενσωματωμένη απάντηση από τον καθηγητή σας να ξεκινήσετε.

Copyright ©1980 by the University of Chicago. Reprinted with the permission of the University of Chicago.
Τέστ Γεωμετρίας Van Hiele

1. Ποιά από τα παρακάτω είναι τετράγωνα?
   (A) Μόνο το Κ.
   (B) Μόνο το Λ.
   (C) Μόνο το Μ.
   (D) Το Λ και το Μ.
   (E) Όλα είναι τετράγωνα.

2. Ποιά από τα παρακάτω είναι τρίγωνα?
   (A) Κανένα από αυτά δεν είναι τρίγωνο.
   (B) Μόνο το Ξ.
   (C) Μόνο το Ο.
   (D) Μόνο το Ο και το Π.
   (E) Μόνο το Ξ και το Ο.

3. Ποιά από τα παρακάτω είναι ορθογώνια?
   (A) Μόνο το Ρ.
   (B) Μόνο το Σ.
   (C) Μόνο το Ρ και το Σ.
   (D) Μόνο το Ρ και το Τ.
   (E) Όλα είναι ορθογώνια.

Copyright ©1980 by the University of Chicago. Reprinted with the permission of the University of Chicago.
4. Ποιά από τα παρακάτω είναι τετράγωνα?

(A) Κανένα από αυτά δεν είναι τετράγωνο.
(B) Μόνο το Φ.
(C) Μόνο το Χ και το Ψ.
(D) Μόνο το Ψ και το Φ.
(E) Όλα είναι τετράγωνα.

5. Ποιά από τα παρακάτω είναι παράλληλογράμματα?

(A) Μόνο το Α.
(B) Μόνο το Γ.
(C) Μόνο το Α και το Β.
(D) Κανένα από αυτά δεν είναι παράλληλογράμμα.
(E) Όλα είναι παράλληλογράμμα.

6. Το \(AB\Gamma\Delta\) είναι τετράγωνο.
Ποιά σχέση είναι αδιάκοπη για κάθε τετράγωνο?

(A) \(\Lambda\Gamma\) και \(\Gamma\Delta\) έχουν το ίδιο μήκος.
(B) \(\Delta\Gamma\) και \(\Gamma\Delta\) είναι κάθετες.
(C) \(\Delta\Gamma\) και \(\Gamma\Delta\) είναι κάθετες.
(D) \(\Delta\Gamma\) και \(\Gamma\Delta\) έχουν το ίδιο μήκος.
(E) Η γωνία Β είναι μεγαλύτερη από την γωνία Γ.
7. Σε ένα ορθογώνιο EZHΘ, α ΕΗ και ΖΘ είναι α βαρύνσεις.

```plaintext
E

Z

H
```

Ποια από τις απαντήσεις (A)-(D) δεν είναι αδύνατη για κάθε ορθογώνιο;

- (A) Έχει τέσσερις φύλλες γωνίες.
- (B) Έχει τέσσερις πλευρές.
- (C) Οι διαγώνιοι έχουν το ίδιο μήκος.
- (D) Οι απεναντί πλευρές έχουν το ίδιο μήκος.
- (E) Οι απαντήσεις (A)-(D) είναι όλες αδύνατες για κάθε ορθογώνιο.

8. Ρήμα είναι ένα 4-πλευρο σχήμα με όλες τις πλευρές ίσες. Δίνετε τρεις παραπάνω:

- (A) Οι δύο διαγώνιοι έχουν το ίδιο μήκος.
- (B) Κάθε διαγώνιος διαιρεί τις ρίζες του πλήθους.
- (C) Οι δύο διαγώνιοι είναι συμμετρικοί.
- (D) Οι απεναντί γωνίες είναι ίσες.
- (E) Οι απαντήσεις (A)-(D) είναι όλες αδύνατες για κάθε ρήμα.
9. Ισοσκελής είναι ένα τρίγωνο με δύο ίσες πλευρές. Δείτε τρία παράδειγματα:

Ποιό από τις απαντήσεις (A)–(D) είναι αλήθεια για κάθε ισοσκελής τρίγωνο;

(A) Οι τρεις πλευρές πρέπει να έχουν το ίδιο μήκος.
(B) Μια πλευρά πρέπει να έχει το δεσμάδι μέρος μιας άλλης.
(C) Πρέπει να έχει τουλάχιστον δύο γωνίες ίσες.
(D) Οι τρεις γωνίες πρέπει να έχουν ίσες.
(E) Καμία από τις απαντήσεις (A)–(D) δεν είναι αλήθεια για κάθε τρίγωνο.

10. Δύο κύκλοι με κέντρα A και Γ τονίζονται στα σημεία B και Δ και σχηματίζουν ένα 4-πλευρο σχήμα AΓBD. Δείτε τέσσερα παράδειγματα:

Ποιό από τις απαντήσεις (A)–(D) δεν είναι πάντα αλήθεια;

(A) Το AΓBD θα έχει δύο ζεύγη πλευρών του ίδιου μήκους.
(B) Το AΓBD θα έχει τουλάχιστον δύο γωνίες ίσες.
(C) Το πελάτηριο βέλη AΓ και BD θα είναι αλήθεια.
(D) Οι γωνίες Α και Γ θα είναι ίσες.
(E) Είναι (A)–(D) είναι όλες αλήθειες.
11. Δίνονται δύο προτάσεις.

Πρόταση 1: Το σχήμα Κ είναι τετράγωνο.
Πρόταση 2: Το σχήμα Κ είναι τρίγωνο.

Ποιό είναι σωστό?

(A) Αν η 1 είναι αλήθεια, τότε η 2 είναι αλήθεια.
(B) Αν η 1 είναι ηθική, τότε η 2 είναι αλήθεια.
(C) Οι 1 και 2 δε μπορούν να είναι και οι δύο αληθείες.
(D) Οι 1 και 2 δε μπορούν να είναι και οι δύο ηθικές.
(E) Κανένα από τα (A)–(D) δεν είναι σωστό.

12. Δίνονται δύο προτάσεις.

Πρόταση S: Το τρίγωνο ABG έχει τρεις ίσες πλευρές.
Πρόταση T: Στο τρίγωνο ABG, οι γωνίες B και Γ είναι ίσες.

Ποιό είναι σωστό?

(A) Οι προτάσεις S και T δε μπορούν να είναι και οι δύο αληθείες.
(B) Αν η S είναι αλήθεια, τότε η T είναι αλήθεια.
(C) Αν η T είναι αλήθεια, τότε η S είναι αλήθεια.
(D) Αν η S είναι ηθική, τότε η T είναι ηθική.
(E) Κανένα από τα (A)–(D) δεν είναι σωστό.

13. Ποιό από τα παρακάτω μπορούν να σχηματίσουν εδρανογωνία;

| Π | Ρ | Σ |

(A) Όλα μπορούν.
(B) Μόνο το Ρ.
(C) Μόνο το Σ.
(D) Μόνο το Π και το Ρ.
(E) Μόνο το Ρ και το Σ.

Copyright ©1980 by the University of Chicago. Reprinted with the permission of the University of Chicago.
14. Ποιό είναι αλήθεια;

(A) Όλες οι διάταξης των ορθογώνιων είναι διάταξης όλων των τετραγωνίων.
(B) Όλες οι διάταξης των τετραγωνίων είναι διάταξης όλων των ορθογώνιων.
(C) Όλες οι διάταξης των ορθογώνιων είναι διάταξης όλων των παραλληλόγραμμων.
(D) Όλες οι διάταξης των τετραγωνίων είναι διάταξης όλων των παραλληλόγραμμων.
(E) Κανένα από τα (A)-(D) δεν είναι αλήθεια.

15. Τι έχουν όλα τα ορθογώνια το απέναντι κάθετο παραλληλόγραμμα δεν το έχουν:

(A) Απέναντι πλευρές τους.
(B) Θετικά αριθμοί.
(C) Απέναντι πλευρές τους.
(D) Απέναντι γωνίες τους.
(E) Τέσσερα από τα (A)-(D).

16. Παρακάτω δίνεται ένα ορθογώνιο τρίγωνο ΑΒΓ. Ισόλευκα τρίγωνα ΒΓΔ, ΑΒΕ και ΑΓΖ έχουν συγκεκριμένες πλευρές του ΑΒΓ.

From these data, it appears that the angles of the squares formed by the sides of the triangle BZ, AD and GE are a common measure. To be verified, we need to confirm that the hypothesis:

(A) Μόνο στο συγκεκριμένο σχήμα το δίνεται παραδείγμα μορφόπλασμα να πλέκεται σήμαντη που τα BB, AD και GE έχουν ένα κοινό σήματο. Το χαμηλότερο από αυτή την απόδειξη;
(B) Σε κάθε από τα ορθογώνια τρίγωνα BB, AD και GE έχουν ένα κοινό σήματο.
(C) Σε κάθε ορθογώνιο τρίγωνο τα BB, AD και GE έχουν ένα κοινό σήματο.
(D) Σε κάθε τρίγωνο τα BB, AD και GE έχουν ένα κοινό σήματο.
(E) Σε κάθε ισολευκό τρίγωνο τα BB, AD και GE έχουν ένα κοινό σήματο.

Copyright ©1980 by the University of Chicago. Reprinted with the permission of the University of Chicago.
17. Δότω σας τρεις ιδιότητες ενός σχήματος.

Ιδιότητα A: Έχει ίσες διαγώνιες.
Ιδιότητα B: Έχει τετράγωνο.
Ιδιότητα G: Έχει ορθογώνιο.

Ποιό από τα παρακάτω είναι αλήθεια;

(A) Από το A συνεπεγέται το B από το οποίο συνεπεγέται το G.
(B) Από το A συνεπεγέται το G από το οποίο συνεπεγέται το B.
(C) Από το B συνεπεγέται το G από το οποίο συνεπεγέται το A.
(D) Από το G συνεπεγέται το A από το οποίο συνεπεγέται το B.
(E) Από το G συνεπεγέται το B από το οποίο συνεπεγέται το A.

18. Δότω δύο προτάσεις.

I. Αν ένα σχήμα είναι ορθογώνιο, τότε οι διαγώνιες του διατηρούν την άλλη.
II. Αν οι διαγώνιες ενός σχήματος διατηρούν την άλλη, τότε το σχήμα είναι ορθογώνιο.

Ποιό από τα παρακάτω είναι σωστό?

(A) Για να αποδείξουμε ότι η I είναι αλήθεια, χρειάζεται να αποδείξουμε ότι η II είναι αλήθεια.
(B) Για να αποδείξουμε ότι η II είναι αλήθεια, χρειάζεται να αποδείξουμε ότι η I είναι αλήθεια.
(C) Για να αποδείξουμε ότι η II είναι αλήθεια, χρειάζεται να βρεθεί μια ορθογώνιο ένα οποίο οι διαγώνιες του διατηρούν την άλλη.
(D) Για να αποδείξουμε ότι η II είναι αλήθεια, χρειάζεται να βρεθεί μια μη-ορθογώνιο ένα οποίο οι διαγώνιες του διατηρούν την άλλη.
(E) Κανένα από τα (A)–(D) δεν είναι σωστό.
19. Στη γεωμετρία:

(A) Κάθε έννοια μπορεί να οριστεί και κάθε ακρή μπορεί να αποδειχθεί ότι είναι ακρή.

(B) Κάθε έννοια μπορεί να οριστεί αλλά είναι απαραίτητο να υπολογίσετε ακρές ακρής μπορεί να αποδειχθεί ότι είναι ακρή.

(C) Κάθε έννοια μπορεί να παραμετροστεί χωρίς ακρή αλλά κάθε ακρή μπορεί να αποδειχθεί ότι είναι ακρή.

(D) Κάθε έννοια μπορεί να παραμετροστεί χωρίς ακρή και κάθε ακρή μπορεί να θίγεται κάτω έννοιες προσθέτες τις οποίες υπολογίσετε ότι είναι ακρή.

(E) Κανένα από τα (A)-(D) δεν είναι σωστό.

20. Εξετάστε τις παρακάτω τρεις προσδόκες.

1. Δύο ευθύγραμμα καθότι στην ίδια ευθύγραμμα είναι μεταξύ τους παράλληλες.
2. Μία ευθύγραμμα είναι καθότι σε μία από τις δύο παράλληλες ευθύγραμμα είναι καθότι και στην άλλη.
3. Έτσι δύο ευθύγραμμα αποκλίνουν μία από την άλλη τόπο είναι παράλληλες.

Στην παρακάτω σχάρα, δηλώνεται ότι οι ευθύγραμμα (c) και (η) είναι καθότι και σε αυτό συνθέτες (d) και (η) είναι καθότι. Ποτέ από το το παρακάτω προσδόκης θα μπορούσε να αποδείξει το λόγο για τον οποίο η ευθύγραμμα (c) είναι παράλληλη με την ευθύγραμμα (d):

(A) Μόνο η (1).
(B) Μόνο η (2).
(C) Μόνο η (3).
(D) Είναι η (1) είναι η (2).
(E) Είναι η (2) είναι η (3).
A.3 The modified Schoenfeld questionnaire

| Number: ____ |

Mathematics Questionnaire

Please fill in:
First Name: 
Last Name: 

We are interested in your ideas about mathematics. Your answers to the questions that follow will help us to understand what you think mathematics is all about. This questionnaire is not part of your regular school work, and you will not be graded. There are no ‘correct’ answers to this questionnaire. We would simply like to know your opinions. Please tell us what you really believe. If you are not honest the results of the questionnaire will not be useful. In case you would like, you have the right to not fill in the questionnaire. In that case, you have to sit quietly in your seat and wait until the rest of your classmates are finished.

Thank you very much for your help!

Instructions: For each statement on the left side, circle the number under the answer that best describes what you think or feel.

<table>
<thead>
<tr>
<th>Category 1</th>
<th>I totally agree</th>
<th>I guess I agree</th>
<th>I guess I disagree</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Mostly facts and procedures that have to be memorized.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2. Thought provoking.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3. Just a way of thinking about space, numbers, and problems.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category 2</th>
<th>Always</th>
<th>Usually</th>
<th>Occasionally</th>
<th>Never</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. You have to remember the right answer to answer it correctly.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>5. There are lots of possible right answers you might give.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>6. You have to think really hard to answer it.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>7. The students who understand only need a few seconds to answer correctly.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category 3</th>
<th>I totally agree</th>
<th>I guess I agree</th>
<th>I guess I disagree</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>8. In mathematics something is either right or it’s wrong.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>9. Everything important about mathematics is already known by mathematicians.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>10. In mathematics you can be creative and discover things by yourself.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>11. Math problems can be done correctly in only one way.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>12. Real math problems can be solved by common sense instead of the math rules you learn in school.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>13. To solve math problems you have to be taught the right procedure, or you can’t do anything.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>14. The best way to do well in math is to memorize all the formulas.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
A.3. The modified Schoenfeld questionnaire

<table>
<thead>
<tr>
<th>Category 4</th>
<th>I totally agree</th>
<th>I guess I agree</th>
<th>I guess I disagree</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>When you get the wrong answer to a math problem...</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>15. It is absolutely wrong—there’s no room for argument.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>16. You only find out when it’s different from the book’s answer or when the teacher tells you.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>17. You have to start all over in order to do it right.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category 5</th>
<th>I totally agree</th>
<th>I guess I agree</th>
<th>I guess I disagree</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>When I do a geometry proof...</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>18. I can only verify something as mathematician has already shown to be</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>19. The key thing is to get the statements and reasons in proper form.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>20. I get a better understanding of mathematical thinking.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>21. I’m finished if I can’t remember the next step.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>22. I can discover things about geometry that I haven’t been taught.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>23. I feel like I am doing something useful.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category 6</th>
<th>I totally agree</th>
<th>I guess I agree</th>
<th>I guess I disagree</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometrical constructions have little to do with other things in geometry like proofs and theorems.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>25. A construction is easy to figure out even if I’ve forgotten exactly how to do it.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Category 7

Instructions: Answer each of the following questions in a sentence or two. Write your answer in the space below each question.

27. Why do you think so much time in geometry classes is spent learning to write proofs?

28. What can you do if you get stuck while doing a math problem?

29. In what way, if any, is the math you’ve studied useful?

30. Do you think that students can discover mathematics on their own, or does all mathematics have to be shown to them? Please explain.

31. If you understand the material, how long should it take to solve a typical homework problem? What is a reasonable amount of time to work on a problem before you know it’s impossible.

32. How important is memorizing in learning mathematics? If anything else is important, please explain how.
A.4 Beliefs & reflections questionnaire

Our thoughts about the lessons

Name of student: ________________________________

1. Do you think you learned anything new in the lessons we had? If yes what?

2. Did you like the lessons? Would you prefer that anything would be done in a different way? Explain.

3. If we had some more lessons together for the coming weeks, which would be the subject that you would prefer to discuss and why?

4. Do you think that what we have done in the lessons is useful? In what way?

5. Do you believe that the lessons affected your way of thinking? If yes, how?

6. Do you believe that the lessons affected the idea you have about mathematics? If yes, how?

7. Do you believe that the lessons affected the idea you have about proof? If yes, how?
Appendix B

Task related worksheets

B.1 TRIANGLES

TASK 1 (Triangles)

Name of student: __________________________________________

1. Draw in the following box a triangle.

2. Draw in the following box a triangle that is different from the previous in some way, and describe in what way it is different.

3. Draw in the following box a triangle that is different from the previous two in some way, and describe in what way it is different.
4. How many different triangles you think you could draw? Can you give some more examples and describe how they would be different from the ones you have already drawn in the previous page? Use the space in the following box to do that.
5. Put a T on each triangle on this sheet.
6. Choose at least two of the figures of the previous page on which you put a T, and explain here below why you put a T on them:

I put a T on figure number _____ because:

I put a T on figure number _____ because:


7. Choose at least two of the figures of the previous page on which you did not put a T, and explain here below why:

I did not put a T on figure number _____ because:

I did not put a T on figure number _____ because:
What would you tell someone to look for to pick out all the triangles on a sheet of figures? Write your answer below:

What is the shortest list of things you could tell someone to look for to pick out all the triangles on a sheet of figures? Write your answer below.

Which of the triangles in the figure of the following page are alike in some way? Write down their numbers and in what way they are alike.
11. What other groupings of triangles that are alike in some way can you make? Write down in the space below the groups of triangles and the way in which they are alike.

First Group:
This group’s triangles are alike in the following way:

Second Group:
This group’s triangles are alike in the following way:

Third group:
This group’s triangles are alike in the following way:

Fourth group:
This group’s triangles are alike in the following way:
B.2 ANGLE SUM

Task 3 (part 1)

Name of student: __________________________________________

Show that the sum of the angles of any triangle is 180 degrees. Give explanations for your arguments.
Eleni, Petros, Georgia, Stamatis, Lefteris and Tania, were trying to prove whether the following statement is true or false:

**When you add the interior angles of a triangle the sum is always 180°**

Eleni’s answer:
I tore the angles up and put them together. It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

So Eleni says it’s true.

Petros’s answer:
I drew an isosceles triangle with \( \alpha \) equal to 65°.

\[
\alpha = 180°-2\beta \quad \text{(because base angles in isosceles triangles are equal)}
\]

\[
\beta = 180°, \quad (\alpha+\beta) = 65°
\]

Therefore \( \alpha + \beta + y = 180° \)

So Petros says it’s true.

Georgia’s answer:
I drew a line parallel to the base of the triangle

\[
\alpha = 180°-2\gamma \quad \text{(because alternate angles between two parallel lines are equal)}
\]

\[
\gamma = \beta \quad \text{(because alternate angles between two parallel lines are equal)}
\]

\[
\alpha+\alpha+\gamma=180° \quad \text{(because they are angles on a straight line)}
\]

So, \( \alpha+\beta+\gamma=180° \)

So Georgia says it is true.

Stamatis’s answer:
I measured the angles \( \alpha, \beta, \gamma \) of all sorts of triangles accurately and made a table.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( y )</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>34</td>
<td>36</td>
<td>180</td>
</tr>
<tr>
<td>95</td>
<td>43</td>
<td>42</td>
<td>180</td>
</tr>
<tr>
<td>35</td>
<td>72</td>
<td>73</td>
<td>180</td>
</tr>
<tr>
<td>10</td>
<td>27</td>
<td>143</td>
<td>180</td>
</tr>
</tbody>
</table>

They all added up to 180°.

So Stamatis says it’s true

(continued on the next page)
If we walk around the edge of triangle ABC starting at A (and looking at A) passing through B and then through C ending up back to A making an extra turn to end up looking towards the point we started (that is, F), then we must have turned 360° because we came back in exactly the same position. We have made three turns: One at B (angle αB), then one at C (angle βC), and finally one at A in order to return looking towards the same point (angle γA). These three turns together must be 360°.

We can see that every exterior angle when added to the corresponding interior gives 180° because together they form a straight line. This gives as a sum of:

\[(αB + βC + γA) = 180° + 180° + 180° = 540°.\]

In order to calculate the sum of α, β, and γ we subtract from the above sum the sum of the turns we made, that is we subtract the sum of αB + βC + γA which we know is 360°.

Therefore α + β + γ = 540° - 360° = 180°

So Leften’s says it’s true.

Tania’s answer

I drew a tessellation of triangles and marked all the equal angles. All triangles on the tessellation are equal, and each triangle has a yellow, a red and a blue angle. I know that the angles around a point add up to 360°. I can conclude that the sum of a red, a blue and a yellow angle is 180°.

So Tania says it’s true.

Kalliopi’s answer

The two sides of a triangle ABC are being rotated in opposite directions through the vertices B and C, respectively, until their angles with the segment BC are 90° (Figures a, b). This action transforms the triangle ABC into the figure A’BCA”, where A’ B and A” C are perpendicular to the segment BC. In figure b, the sum of the angles B and C is 180°. To create the original triangle, the segments A’ B and A” C are tilted towards each other until the points A’, A” merge back into a point (Figure c). In doing so we lose two pieces (i.e. angles A’ BC and A” CB) from the 90° angles B and C, but at the same time we gain these pieces back in creating angle A. So A’+B+C=180°.

So, Kalliopi says it’s true.
Answer the following questions:

1) Whose method would you choose in order to check whether the above statement is true or false? Explain why.

2) Whose answer you find more convincing and why?

3) Whose answer you think would get the highest grade in a test and why?

4) Whose answer is easier for you to understand and whose is more difficult?

In the following boxes circle the number that suits your ideas best.

<table>
<thead>
<tr>
<th>Stamatios's answer</th>
<th>Agree</th>
<th>Don't know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shows that the statement is always true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Only shows that the statement is true for some triangles</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Shows you why the statement is true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Is an easy way to explain to someone in class</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Comments: [In this box you can write any comments you might have on the answers you gave above.]

3
### Appendix B. Task related worksheets

<table>
<thead>
<tr>
<th></th>
<th>Agree</th>
<th>Don't know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shows that the statement is always true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Only shows that the statement is true for some triangles</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Shows you why the statement is true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Is an easy way to explain to someone in class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Comments: (In this box you can write any comments you might have on the answers you gave above)
B.4 CONCEPTIONS ABOUT PROOF

Task 3 (part 3)

Name of student: ___________________________________________________

Please answer the following questions:


2. Do we really need to prove things in mathematics? Couldn’t we simply get away without it?

3. Suppose it has now been proved that, when you add the interior angles of any triangle, your answer is always 180°.

   Zoe asks what needs to be done to prove whether, when you add the interior angles of any right-angled triangle, your answer is always 180°.

   Tick either A or B.

   (A) Zoe doesn’t need to do anything; the first statement has already proved this.

   (B) Zoe needs to construct a new proof.

   Please explain your choice.
Appendix B. Task related worksheets

B.5 QUAD

Task 4 (QUAD)

Student name: ________________________________

Give convincing arguments in order to support the following proposition:

The sum of the angles of a quadrilateral is 360°

After you give your arguments, write in the box what you think are the advantages and the disadvantages of your arguments.

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
B.6 OPPANG

Task 5 (OPPANG)

Name of student: ________________________________________________________________

1. Write down your own “theory” for the relations that holds between the angles that are formed when two lines are intersected.

2. Give arguments in order to support your “theory” so that no one can disprove you.
Appendix B. Task related worksheets

B.7 BIS

Task 6 (BIS)

Student Name: _____________________________________________

1. Give arguments in order to support the proposition which is written on the board [the angle formed between the bisectors of two consecutive and complementary angles is right]. Give two kinds of arguments. One practical that depends on the figure and one theoretical that does not depend on the figure.
TASK 7 (BIS) GROUP WORK

Names of students:
__________________________________________________________________________________________________________________________________________________________
__________________________________________________________________________________________________________________________________________________________

**Exercise:** Give convincing arguments for the following proposition:

*The angle formed between the bisectors of two consecutive and complementary angles is right.*

Begin by writing clearly the givens and what you are looking for. Then, decide as a group which is a convincing argument in order to show that the above proposition holds in general. Write down your argument clearly at the back of this page.

Givens:

Looking for:


**TASK 8 (EXT) GROUP WORK**

Names of students:

__________________________________________________________________________________________________________________________________________

__________________________________________________________________________________________________________________________________________

**Exercise:** Give convincing arguments for the following proposition:

*Every external angle of a triangle is equal to the sum of the two opposite internal angles of the triangle.*

Begin by writing clearly the givens and what you are looking for. Then, decide as a group which is a convincing argument in order to show that the above proposition holds in general. Write down your argument clearly at the back of this page.

![Diagram of a triangle with external angles](image)

**Givens:**

__________________________________________________________________________________________________________________

**Looking for:**

__________________________________________________________________________________________________________________
B.10  TRIMID

Task 9 (TRIMID)

Student Name:_______________________________________________

1. Draw a triangle and connect the midpoints of two of its sides (whichever sides you want). Then make as many observations as you can about the relationship that holds between the line that connects the midpoints of the two sides and the sides of the triangle and write them below your figure. You can make measurements if you like in order to come up with observations. Finally, give in the back of the page a convincing argument in order to support that each of your observations holds in general.
B.11 TRIMID-GROUP

Exercise: Write down the assumptions (conjectures) which you will try to prove or disprove:

Give convincing arguments in order to support, as a group, that each of the above conjectures holds or not. Write your arguments down as clearly as you can. (Advice: Always start by writing clearly the givens and what you are looking for.)
Task 10 (QUADMID)

Name of student:__________________________________________________

Show that:

If we connect the midpoints of the adjacent sides of a quadrilateral, a parallelogram will be formed.
B.13 IF THEN

Task 11 (IF THEN)

Name of students:__________________________________________________

Show that:

If a line $\varphi$ passes through the midpoint of a line segment, then the ends of the line segment are equidistant from the line $\varphi$. 
Appendix C

Handouts
C.1 Invitation

**Geometry and proof with different eyes...**

### Introduction

Hello again! We met each other some time ago, when you filled in a multiple choice test in geometry. First of all, I want to thank you wholeheartedly again for your help. The test results were very interesting and helped me a lot for my research. Without you and your help this would not be possible. My research is continued, and here I will explain to you the second part of my research and I will invite you to help me also in its second phase.

**How was the choice of students made for the second phase?**

Based on the results of the test which you took a while ago, I chose 40 students that math with what I want to do in the second phase of my research. This does not mean that I chose the most able or the weakest students. I chose 40 students regardless of how good they are in general in mathematics, based only on whether they have answered some specific items of the test correctly. It would be more interesting to work with all students, but unfortunately I am only one and you are 85, so this is practically impossible! From the chosen 40 students, the maximum number that can take part in the lessons of the second phase is 25. Therefore, there will be a second selection process in case those who state they want to participate are more than 25.

**What does the second phase consist of and what will you gain if you participate?**

In the second phase of the research we will have a series of geometry lessons. The basic concept that will be discussed in the lessons is the concept of proof in geometry. Why do we need proofs? Which proofs are accepted by mathematicians and why? Why do we spend time on proofs at school? How will proofs be useful to us? These are only some of the questions that we will try to answer together in the lessons I have prepared for you.

These lessons **will not be like the lessons you are used to.** What will be different?

1. They will include some tasks that will provoke your thought in an exciting way.
2. You will be given the chance to explore and discover mathematical concepts on your own.
3. You will be given the chance to assimilate and practice the knowledge you already have gained from your normal mathematics class.
4. You will prepare for some of the concepts you will be busy with in Lyceum.
5. They will include a lot of discussion time. Everyone’s opinion matters equally and will be welcome.
6. You will not have any homework! I don’t want, under any circumstances, to put more work load on your shoulders than the one you already have! You will be asked to do some exercises and tests in the classroom, but nothing outside.

7. You will have the rare experience to participate in a research project. The results of the research will be read by dozens of people thus you will help in your own way the research community.

The programme

Let’s move now to the more practical matters. How many lessons will you have to follow, and when will the lessons take place? Will your participation play any role for your semester grades? In what follows you will find to answer to these but also to other questions. Before you decide to participate in the lessons, it is important to read carefully all the following points!

- There will be 8 lessons in total, lasting 90 minutes each.
- The lessons will take place outside normal school ours.
- The lessons will take place between 1-26 March, 2 lessons per week.
- The lessons will take place in a classroom of your school.
- Some of the lessons will be videotaped, and all of them will be voice recorded, so that I can analyze the lessons later. No face will be clearly visible in the videos and in my thesis your real names will not be used.
- I will ask from some students to have a face to face meeting with me, some moment that they will have free during one of the school days, so that I can ask them certain questions related to the lessons (a kind of interview). These meetings will not exceed two per person.
- The lessons will not count for your grading in school. You will gain, however, some knowledge that will help you achieve better in your normal math class.

What do you need to do in order to participate in the lessons?

Participation is voluntary, which means that you are not obliged to participate. If you are interested in participating, you have to do within a week from today the following:

- Inform your teacher that you want to participate.
- Discuss it with your parents so that you can get their permission.
- Fill in the following page with the days of the week that you could stay after school hours for 90 minutes, and specify also if this is possible in weekends.

Epilogue

I hope that you decide to participate in the lessons, and I am sure that you will not regret it! Thank you one more time for your interest, and I am looking forward to receiving the list of students with which I will have the pleasure to work. Don’t forget to fill in next page and give it to your teacher the latest in a week from today. If you are not interested in participating, just say that to your teacher. Have a nice continuation!

Olga Grigoriadou (Mathematician)
Amsterdam, February 8, 2010
C.2 Introduction to the lessons content and expectations

Geometry and proof with different eyes...

Introduction

The purpose of this text is to inform you for the cycle of lessons “Geometry and proof with different eyes...” in more detail. What will we be doing in the lessons? What will you have to bring with you? What will you gain from these lessons?

What form will the lessons have?

- Main theme: the concept of proof in geometry and mathematics.
- Main teaching method: Discussion and problem solving in class.
- You will not be using a specific book.
- You will not be graded.
- You will often be asked to fill in task sheets which the teacher will collect at the end of each lesson.
- You will be thinking and writing at the end of each lesson about what you did or did not like of the lesson, what you think you learned what you think you did not understand.
- The lessons will be videotaped and recorded.

What are you expected to do in the lessons

- Participate in the discussion and ask questions.
- Do not hesitate to tell your opinion. Your opinion matters whichever it is!
- Respect and do not interrupt your teacher and your classmates.
- Attend all lessons of the cycle.
- Be punctual.

What you need to bring with you each time.

- Pencil and eraser
- Pen
- Ruler
- Pair of compasses
- Protractor

What will you gain?

- You will get the chance to understand how mathematicians work and you will discover yourselves, with the guidance of the teacher, new mathematical concepts.
- You will find out how imagination, creativity, and your ability to participate in discussions can be used in the lesson of mathematics.
- You will understand better what proof is in mathematics and why we are using it.
- You will get prepared for a part of the curriculum that you will be taught in next year’s A’ Lyceum level.

If you follow all lessons you will receive a lesson and a certificate of attendance!
C.3 The pros and cons of arguments

Proposition: The sum of the interior angles of any triangle is 180°

<table>
<thead>
<tr>
<th>Arguments</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I measure with the protractor the measures of one or more angles and I find it to be 180°. Therefore, the angles in any triangle have a sum of 180°.</td>
<td>1. Practical 2. Easy 3. Short</td>
<td>1. Not convincing for every triangle (it depends on the figure) 2. Practically impossible to measure all triangles 3. There is a possibility of error in the measurement</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. I assume that the sum of the angles of a triangle is less than 180°. The figure that is formed by three angles that do not have a sum of 180° is not a triangle. The same happens if I assume that the sum of the angles of a triangle is more than 180 degrees. Therefore the angles in any triangle must have a sum of 180°.</td>
<td>1. Nice way of thinking (it starts from the opposite of what it tries to show and ends up in something that does not make sense) 2. Smart/original</td>
<td>1. Depends on the figure (not convincing) 2. Complicated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. If I take a square and bring the diagonal, 2 right isosceles triangles are formed. Each triangle has one right angle and two equal angles of 45° degrees each. Therefore the sum of the angles of the triangle is 90°+45°+45°=180°. So the angles in any triangle have sum of 180°.</td>
<td>1. Easy 2. Uses previous knowledge 3. Short</td>
<td>1. Depends on the figure (it is not convincing) 2. It needs to bring extra lines</td>
</tr>
</tbody>
</table>
4. The circle is 360°. Therefore a triangle that is formed by a triangle that is inscribed on a semicircle will be half a circle, that is, 180°.

I tore the angles up and put them together. It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

6. Georgia’s answer (from the hand-out):
I drew a line parallel to the base of the triangle.

\[
\begin{align*}
\angle \epsilon &= \gamma \text{ (because alternate angles between two parallel lines are equal)} \\
\angle \delta &= \beta \text{ (because alternate angles between two parallel lines are equal)} \\
\angle \epsilon + \alpha + \delta &= 180° \text{ (because they are angles on a straight line)} \\
\therefore \alpha + \beta + \gamma &= 180°
\end{align*}
\]
7. Lefteri’s answer:

If we walk around the edge of triangle ABD starting at A (and looking at Γ) passing through Γ, and then through B ending up back to A making an extra turn to end up looking towards the point we started (that is, Γ), then we must have turned 360° because we came back in exactly the same position.

We have made three turns: One at Γ (angle γ), then one at B (angle β), and finally one at A in order to return looking towards the same point (angle α). These three turns together must be 360°.

We can see that every exterior angle when added to the corresponding interior gives 180° because together they form a straight line. This gives as a sum of (α+γ)+(β+δ)+γ+δ=180+180+180=540°.

In order to calculate the sum α+β+γ we subtract from the above sum of the turns we made, that is we subtract the sum of α+β+γ, which we know is 360°.

Therefore α+β+γ=540-360=180°.
C.4 How new truths are discovered in geometry

The process by which new truths are discovered in geometry is not simply a mystery. The mathematicians who were involved with geometry, among which was the father of Geometry, Euclid, followed a very simple method in order to discover new mathematical truths. In the school textbooks, unfortunately, this method is usually not obvious. That is, usually it is not asked from the students to discover something on their own and they are simply presented with “ready” truths. However, the magic element of mathematics is based on this method, and when one understands it one can realize what exactly geometry, and mathematics in general, is.

STEP 1: Observe

The first and foremost step followed by the mathematicians doing geometry is the observation of various shapes around them. This observation often includes measurements and other practical methods. During observations, mathematicians try to discover relations which hold among the elements of the shape (for instance its angles, or sides).

Example: I observe that if I draw many quadrilaterals, measure their angles and add them together, I get approximately (depending on the accuracy of my measurements) 360°.

STEP 2: Conjecture (that is, I GUESS)

After observing the shapes, mathematicians try to make guesses, or as we call it in mathematics make conjectures, about which general relations may hold among certain elements of the shapes in question. During this step, we cannot yet be sure whether our conjecture is generally true. We have an intuition that our conjecture is true based on our observations, but we cannot be sure yet whether it is generally true because it is practically impossible to observe infinitely many shapes.

Example: I make the conjecture that the sum of the angles of every quadrilateral is 360°.

STEP 3: I give Movement to the shape

After making the conjecture, mathematicians try to give movement to the shape in order to check whether their conjecture is rejected or not when the shape gets different forms, and to observe whether there are any general relations that remain unchanged in their shape.

Example: I move the vertices of the quadrilateral and observe that I can always divide it into two triangles.

STEP 4: Prove (that is, I give CONVINCING ARGUMENTS)

In order to reject or confirm a conjecture, mathematicians try to give convincing arguments, that is, arguments which hold generally and do not depend on the figure. These kinds of arguments are the ones which are called in mathematics PROOF. These arguments often use certain truths which have already been proven in the past. Those conjectures for which convincing arguments are found and are considered important, mathematicians call them THEOREMS.

Example of convincing argument: I know that the sum of the angles of any triangle is 180°. Every quadrilateral can be divided into two triangles so that the angles of the quadrilateral are formed by the angles of the two triangles. The sum of the angles of the two triangles is 360°, therefore the sum of the angles of any quadrilateral is 360°.

YOU CAN REMEMBER THE ABOVE METHOD FROM THE 4 INITIALS: O.C.M.P.
Τί ανακαλύφαμε στα μαθήματα Γεωμετρίας

Στα μαθήματα γεωμετρίας που κάναμε ανακαλύφαμε διάφορες γενικές αλήθειες. Η διαδικασία που ακολουθούσαμε αποτελούνταν από 4 βήματα: Παρατήρηση, Εικασία, Κίνηση σχήματος και Απόδειξη (Π.Ε.Κ.Α.). Δες την αναλυτική εξήγηση των βημάτων αυτής της διαδικασίας στο φυλλάδιο «Πως ανακαλύπτουνταν κανονικές αλήθειες στη γεωμετρία». Παρακάτω είναι συγκεντρωμένα όλα όσα ανακαλύφαμε μαζί στα μαθήματα του κύκλου «Γεωμετρία με άλλα μάτια...».

1ο ΑΝΑΚΑΛΥΨΗ

ΒΗΜΑ 1: Παρατήρηση

Παρατηρήσαμε κάποιες μετρήσεις με μοφογραμμό σε διάφορα τρίγωνα ότι το άθροισμα των γωνιών πελλόν τριγώνων μοιάζει να προσεχίζει τις 180 μοίρες. Επεκτά οι μετρήσεις δεν είναι τέλειες, δεν μπορούμε να είμαστε σίγουροι μόνο από τη μέτρηση ότι κάθε τρίγωνο θα έχει άθροισμα γωνιών 180 μοίρες, αλλά οι μετρήσεις μας βοηθάνε να υποθέσουμε αυτή την υπόθεση και να περάσουμε στο επόμενο βήμα.

ΒΗΜΑ 2: Εικασία

Υποθέσαμε ότι

Εάν μετρήσουμε τις γωνίες οποιοδήποτε τριγώνου και τις προσθέσουμε το άθροισμά τους θα είναι 180°.

Αυτή την υπόθεση (εικασία) δε μπορούμε να την δεχτούμε ως αλήθεια μέχρι να βρούμε ένα πειστικό επεξήγηση. Αλλά, προσπάθησατε ότι αυτή η υπόθεση δεν έρευνε αν είναι αλήθεια ή όχι μέχρι να βρούμε το πειστικό επεξήγησις για να δεχθούμε ή να απορρίψουμε την υπόθεσή μας.

ΒΗΜΑ 3: Κίνηση στο σχήμα

Προσπάθησαμε να δούμε τι παραμένει διότι ότι κοινόμε ένα τρίγωνο προς όλες τις κατευθύνσεις. Κοινόντας το τρίγωνο διαπιστώσαμε ότι το να μετράς τις γωνίες δεν συνέβαλε ένα τρίγωνο και να βρίσκεις ότι κάνουν 180 μοίρες να μας βοηθά να καταλάβουμε γιατί σε υποθέσεις τρίγωνο θα δίνουμε άθροισμα γωνιών 180 μοίρες. Εσείς, ανακαλύψαμε άλλες ιδέες για πιο πειστικά επεξήγηση που δεν εξαρτώνται από το σχήμα.

ΒΗΜΑ 4: Απόδειξη

Ανακαλύφαμε δύο πειστικά επεξήγησις που δεν εξαρτώνται από το σχήμα:
1ο πειστικό επιχείρημα

Ζωγραφίζουμε ένα τυχαίο τρίγωνο. Φέρνουμε μια γραμμή παράλληλη προς μια βάση του τριγώνου.

Είναι:
\[ e=\gamma \quad (\text{Γιατί οι εντός εναλλάξ γωνίες είναι ίσες}) \]
\[ d=\beta \quad (\text{Γιατί οι εντός εναλλάξ γωνίες είναι ίσες}) \]
\[ e+\alpha+\gamma=180^\circ \quad (\text{Γιατί είναι γωνίες που το άθροισμά τους σχηματίζει ευθεία γραμμή}) \]

Αρα, \( \alpha+\beta+\gamma=180^\circ \)

(Παρατήρηση: Το επιχείρημα αυτό αποφασίσαμε ως τάξη ότι είναι πειστικό γιατί δεν εξαρτάται από το συγκεκριμένο σχήμα, δηλαδή ισχύει για κάθε τρίγωνο.)

2ο πειστικό επιχείρημα

Αν περιπατήσουμε γύρω από το περίγραμμα ενός τυχαίου τριγώνου \( \triangle ABF \) ξεκινώντας από το \( A \) και κοιτώντας προς το \( \Gamma \) περνώντας από το \( \Gamma \), μετά από το \( B \) και καταλήγοντας πίσω στο \( A \) κάνοντας μια επιπλέον στροφή για να επιστρέψουμε κοιτώντας προς την κοτεύθυνση που ξεκίνησαμε (δηλαδή προς το \( \Gamma \)), τότε πρέπει να έχουμε στρίβει 360° διότι επιστρέφουμε ακριβώς στην ίδια θέση.

Έχουμε κάνει τρεις στροφές:
Μια στο \( \Gamma \), την \( \gamma_{\Gamma} \), μετά μία στο \( B \), την \( \beta_{\Gamma} \), και τέλος μία στο \( A \) για να επιστρέψουμε κοιτώντας προς το ίδιο σημείο, την \( \alpha_{\Gamma} \). Αυτές οι τρεις στροφές μαζί πρέπει να κάνουν 360°.
Μπορούμε να δούμε ότι κάθε εξωτερική γωνία όταν προστεθεί στην αντίστοιχη εσωτερική πρέπει να δίνει 180° γιατί σχηματίζουν μαζί μια ευθεία γραμμή.

Αυτό μας δίνει σύνολο:

\[(α+α_2)+(β+β_2)+(γ+γ_2) = 180°+180°+180° = 540°\]

Για να υπολογίσουμε το άθροισμα α+β+γ που είναι και το ίζτούμενο, αφαιρούμε από το παραπάνω άθροισμα το άθροισμα των στροφών που κάναμε, δηλαδή αφαιρούμε το άθροισμα α_2+β_2+γ_2 το οποίο ξέρουμε ότι είναι 360°.

Οπότε α+β+γ=540-360=180°

(Παρατήρηση: Το επιχείρημα αυτό αποφασίσαμε ότι είναι πειστικό γιατί δεν εξαρτάται από το συγκεκριμένο σχήμα, δηλαδή ισχύει για κάθε τρίγωνο.)

Αφού βρούμε ένα πειστικό, γενικό επιχείρημα (αυτό που λέμε δηλαδή στα μαθηματικά «απόδειξη») για να υποστηρίξουμε μια ευκαιρία μας, η ευκαιρία μπορεί πλέον να ονομαστεί ΘΕΟΡΗΜΑ.

Ετσι φτάσαμε τελικά στο παρακάτω θεώρημα:

**ΘΕΟΡΗΜΑ 1: (Βιβλίο Α’ Λυκείου, σελ. 83)**

Το άθροισμα των εσωτερικών γωνιών κάθε τριγώνου είναι 180°.

---

**2ο ΑΝΑΚΛΑΨΗ**

**ΒΗΜΑ 1**: Παρατήρηση

Σχεδίασαμε ο καθένας ένα διαφορετικό τετράπλευρο, μετρήσαμε τις γωνίες του, τις προσθέσαμε και βρήκαμε σχεδόν 360°.

**ΒΗΜΑ 2**: Ευκαιρία

Κάναμε την υπόθεση (ευκαιρία) ότι αυτό θα ισχύει για κάθε τετράπλευρο. Δηλαδή διετυπώσαμε την παρακάτω ευκαιρία: Το άθροισμα των εσωτερικών γωνιών κάθε τετράπλευρου είναι 360°.
**ΒΗΜΑ 3:** Κίνηση στο σχήμα

Παρατηρήσαμε ότι όπως και να είναι οι κορυφές του τετραπλέυρου, μπορούμε να φέρουμε πάντα τη διαγώνιο.

**ΒΗΜΑ 4:** Απόδειξη

Βρήκαμε το παρακάτω πειστικό επιχείρημα για να υποστηρίξουμε την εικασία μας.

Ζωγράφιζαμε ένα τυχαίο τετράπλευρο $ABCD$ και φέραμε μία διαγώνιο του, έστω την $\Delta$ (θα μπορούσα να φέρω και την $\Delta$, δε θα είχε καμία διαφορά). Τότε το τετράπλευρο χωρίζεται σε δύο τρίγωνα: $ABD$ και $\Gamma$.

Ξέρω ότι το άθροισμα των εσωτερικών γωνιών κάθε τριγώνου (δες ΘΕΩΡΗΜΑ 1 και τα πειστικά επιχειρήματα για το θεώρημα 1 που δώσαμε στην τάξη) είναι 180°.

Οι γωνίες του τετραγώνου αποτελούνται από το άθροισμα των γωνιών των δύο τρίγωνων. Όπως το άθροισμα των γωνιών του τετραγώνου είναι ίσο με $180°+180°=360°$.

Αφού βρήκαμε ένα πειστικό, γενικό επιχείρημα (αυτό που λέμε δηλαδή στα μαθηματικά «απόδειξη»), για να υποστηρίξουμε μια εικασία μας, η εικασία μπορεί πλέον να ονομαστεί ΘΕΩΡΗΜΑ.

Έτσι φτάσαμε τελικά στο παρακάτω θεώρημα:

**ΘΕΩΡΗΜΑ 2:**

Το άθροισμα των εσωτερικών γωνιών κάθε τετραπλέυρου είναι 360°.
3η ΑΝΑΚΑΛΥΨΗ

ΒΗΜΑ 1: Παρατήρηση

Σχεδιάζουμε στον πυκνά δύο τεμνόμενες ευθείες, όπως φαίνεται στο σχήμα.

Καναμε διάφορες παρατηρήσεις, ανάμεσα στις οποίες οι «επένδυες γωνίες» ή άλλως οι «κατακορυφή γωνίες» είναι ίσες. Δηλαδή, α-γ και β-δ. Κάναμε αυτήν την παρατήρηση γιατί έτσι έμαθαμε στο συγκεκριμένο σχήμα. Κάναμε και άλλα παρόμοια σχήματα όπου οι γωνίες είχαν διαφορετικά μέγεθη, και σύμφωνα με τις μετρήσεις μας φάνηκε ότι πάλι ισχύει το ίδιο. Δηλαδή ότι σε κάθε περίπτωση μιαζί ισχύει ότι οι κατακορυφή γωνίες είναι ίσες.

ΒΗΜΑ 2: Εικασία

Δέχεται των παραπάνω παρατηρήσεων κάναμε την υπόθεση ότι το ίδιο θα ισχύει για κάθε ξενύχτι κατακορυφή γωνίων που σχηματίζονται από δύο τεμνόμενες ευθείες. Δηλαδή διατυπώσαμε την παρακάτω εικασία:

Οι κατακορυφή γωνίες είναι ίσες (γενικά).

Δεν είχαμε άλλα σχέδια στο στοιχείο ακόμη η η παραπάνω πρόταση ήταν πάντα αλήθεια, αλλά αυτό μας έλεγε η διαδικασία. Αυτό μας τονίζει ότι η εικασία είναι ίση.

ΒΗΜΑ 3: Κίνηση στο σχήμα

Παρατηρήσαμε ότι όταν και να περιστρέφομε τις τεμνόμενες γωνίες γύρω από το σημείο τους, το άθροισμα δύο διαδοχικών γωνιών θα είναι πάντα 180 μέτρα γιατί θα σχηματίζουν ευθεία. Δηλαδή, πάντα θα έχει 180° για + β = 180°, γ = 180°, γ + β = 180° και β = a = 180. Αυτό μας έδωσε μια ιδέα για να βρούμε το παραδείγματα επιχείρηση.

ΒΗΜΑ 4: Απόδειξη

Έχουμε το πάντα να ισχύει:

α+β=180°, α + δ = 180°

Θέλω να δείξω ότι β=δ

Είναι α+β=180 => β=180-α
Και α+δ=180 => δ=180-α

Οπότε η β και η δ γωνίες είναι ίσες με το ίδιο πρόγραμμα (το 180-α), αρα είναι ίσες και μεταξύ τους.
Me állo lógya, oi β kai oi δ γωνίες einai parapληρωματικές tis ídias γωνίας, δηλαδή της α. Άρα είναι Íses.

Γενικά, αι κατακρυφή γωνίες είναι parapληρωματικές tis ídias γωνίας άρα είναι Íses.

Αφού βρήκαμε ένα πειστικό, γενικό επιχείρημα (αυτό που λέμε δηλαδή στα μαθηματικά «απόδειξη»), για να υποστηρίζουμε μια ευκαιρία μας, η ευκαιρία μπορεί πλέον να ονομαστεί ΘΕΟΡΗΜΑ.

Έτσι φτάσαμε τελικά στο παρακάτω θεώρημα:

**ΘΕΟΡΗΜΑ 3: (Βιβλίο Α’ Λυκείου, σελ. 19)**

Οι κατακρυφή γωνίες είναι Íses.

### 4η ΑΝΑΚΑΛΥΨΗ

**ΒΗΜΑ 1: Παρατήρηση**

Σχεδίασαμε το παρακάτω σχήμα στον πίνακα. Στο σχήμα αυτό οι γωνίες α και β είναι εφεξής και parapληρωματικές, και οι κόκκινες γωνίες είναι η διχότωμος των γωνίων α και β, από τον χωρίζον τη γωνία α σε δύο ίσα μέρη (δηλ.) και τη γωνία β δύο ίσα μέρη (δηλ.).

Κάναμε διάφορες paratήρησες για το parapληρώμα του σχήματος ανάμεσα στις αποδείξεις ήπειρου και ελεύθερης.

Ανάμεσα στις διχότομες του σχήματος mouzhе η σχηματίζεται μια ορθή γωνία. Δηλαδή με άλλα λόγια, οι γωνίες e και δ mouzhе ζητήσει ζητήσει 90 μορφές. Διακοπτόμασε να ζωγραφίσουμε και άλλα σχήματα, όπως αι γωνίες a και b είχαν άλλη μεγάλη ήταν μικρές εφεξής και parapληρωματικές. Άρα σχηματίζει ότι πολὺ μουζή να σχηματίζεται ορθή γωνία ανάμεσα στις διχότομες των δύο γωνίων.
BHMA 2ο: Ευκασία

Λόγω των παραπάνω παρατηρήσεων κάναμε την υπόθεση ότι το ίδιο θα ισχύει για κάθε δύο ευθείες και παραπληρωμικές γωνίες. Δηλαδή διατυπώσαμε την παρακάτω ευκασία:

Η γωνία που σχηματίζεται μεταξύ των διευθύνσεων δύο ευθείες και παραπληρωμικών γωνιών είναι ορθή.

Δεν είχαμε έχει κάνει καίνες άσκηση, αν η παραπάνω πρόταση ήταν αλήθεια, αλλά αυτό μας έλεγε η διαδικασία της Γ. Γι αυτό την ονομάσαμε ευκασία.

BHMA 3ο: Κίνηση στο σχήμα

Παρατηρήσαμε ότι όπως μεγέθυναν και να έχουν οι γωνίες α και β, οι διαγώνιοι τους θα της χωρίσουν στη μέση και η γωνία που σχηματίζεται από τις δύο διεύθυνσες θα αποτελείται από το μισο της γωνίας α και το μισο της γωνίας β.

BHMA 4ο: Απόδειξη

1ο πειστικό επιχείρημα

Έχουμε ότι:

\[ a + \beta = 180 ^ \circ \] (διάτι στις γωνίες α και β είναι παραπληρωμικές)

και

\[ \gamma = \delta = \frac{a}{2} \] (διάτι φέραμε τη διεύθυνση της γωνίας α η οποία την χωρίζει σε δύο ισομέρη γ και θ θ στο θαυμάτων είναι ίσο με το μισό της α)

\[ \epsilon = \zeta = \frac{\beta}{2} \] (διάτι φέραμε τη διεύθυνση της γωνίας β η οποία την χωρίζει σε δύο ισομέρη ε και ζ που το καθενα είναι ίσο με το μισό της β)

\[ \theta \] θέλω να δείξω ότι:

\[ \epsilon + \delta = 90 ^ \circ \]

Είναι \[ a + \beta = 180 ^ \circ \Rightarrow \]

\[ (a + \beta)/2 = 180 ^ \circ /2 \Rightarrow \]

\[ a/2 + \beta/2 = 90 ^ \circ \Rightarrow \]

\[ \epsilon + \delta = 90 ^ \circ \]

(δηλαδή έχουμε την κάτι που έχουμε και καταλήξαμε σε αυτό που ζήταμε να δείξουμε)

2ο πειστικό επιχείρημα

Το δεύτερο επιχείρημα που βρήκαμε είναι ίδιο με το πρώτο απλώς είναι διατυπωμένο με λόγια αντί για σύμβολα. Έχουμε ότι α και β μας κάνουν 180 μούρες γιατί είναι παραπληρωμικές. Έχουμε επίσης ότι η κάθε γωνία είναι ίση με τη δ και ότι κάθε γωνία είναι ίση με το μισο της α γιατί αν γ και δ δημιουργήσουμε όπως φέραμε τη διεύθυνση της γωνίας α. Με τον ίδιο τρόπο, έχουμε ότι η κάθε γωνία είναι ίση με το μισο της β. Επειδή αρχικά αγωνία α και β είχαν άθροισμα 180 μούρες και επειδή η περά
το μισό κάθε μίας και το πρόσθεσα, αυτό που θα πάρω στο τέλος συνιστάκει θα είναι μισό του 180, δηλαδή 90 μοίρες. Με άλλα λόγια, το άθροισμα των μισών δύο γωνιών που μας κάνουν 180 μοίρες, θα είναι το μισό, δηλαδή 90 μοίρες.

Αφού βρέχει ένα πιεστικό, γενικό επιχείρημα (αυτό που λέμε δηλαδή στα μαθηματικά «απόδειξη»), για να υποστηρίζουμε την εικασία μας, η εικασία μπορεί πλέον να ονομαστεί ΘΕΩΡΗΜΑ.

Ετσι φτάσαμε τελικά στο παρακάτω θεώρημα:

θεώρημα 4: (βιβλίο α’ Λυκείου, σελ. 20)

η γωνία που σχηματίζεται μεταξύ των διχοτόμων δύο εφεξής και παραπληροματικών γωνιών είναι ορθή.

5η ΑΝΑΚΛΑΥΣΗ

βήμα 1: Παρατήρηση

Σχεδίασαμε ένα τρίγωνο στον πίνακα και φέραμε μια τυχαία εξωτερική του γωνία όπως φαίνεται στο παρακάτω σχήμα:

Κάναμε την εξής παρατήρηση (μετά από μετρήσεις) για τη σχέση που μπορεί να έχει η εξωτερική γωνία του τριγώνου με τις εσωτερικές του γωνίες: 

Μιαίνει η εξωτερική γωνία του τριγώνου (B2) να είναι ίση με το άθροισμα των δύο απέναντι εσωτερικών γωνιών (A+Γ).

Σχεδίασαμε και άλλα τρίγωνα, και είδαμε ότι το ίδιο ισχύει και γ’αυτό.

βήμα 2: Εικασία

Από της παραπάνω παρατήρησες καταλήξαμε στην υπόθεση ότι μίλαμε πρέπει να ισχύει γενικά η παρατήρηση μας, δηλαδή μίλαμε γενικά για κάθε τρίγωνο:

κάθε εξωτερική γωνία ενός τριγώνου είναι ίση με το άθροισμα των δύο απέναντι εσωτερικών γωνιών του τριγώνου.

Δεν είμασταν σίγουροι ακόμα αν η παραπάνω πρόταση ήταν πάντα αλήθεια, δηλαδή αν ισχύει όντως για κάθε τρίγωνο (διότι είναι πρακτικά αδύνατον να μετρήσουμε αν ισχύει σε όπερα τρίγωνο) αλλά αυτό μας έλεγε η διαίσθησή μας, γι αυτό την αναφέρουμε «εικασία».
ΒΗΜΑ 3: Κίνηση στο σχήμα
Παρατηρήσαμε ότι όπως και να «κουνήσουμε» το σχήμα οι Β1 και οι Β2 θα είναι πάντα 180 μοίρες, και επίσης το άθροισμα των επιτεκτικών γωνιών θα είναι και αυτό 180 μοίρες (αυτό το έχουμε αποδείξει δήθεν και το δικτυωόμασε ως θεώρημα 1 – δες σελίδα 3)

ΒΗΜΑ 4: Απόδειξη
Δουλεύοντας σε ομάδες καταψήφισες να βρούμε δύο πειστικά επιχειρήματα.
1ο πειστικό επιχείρημα
Ξέρω ότι:
\[A + B + \Gamma = 180^\circ\] (ως άθροισμα γωνιών τριγώνου)
\[B_1 + B_2 = 180^\circ\] (ισοτιμία ευθεία) 
Θέλω να δείξω ότι:
\[B_2 = A + \Gamma\]
Εκκινώντας από αυτά που ξέρω, έχω:
\[A + B + \Gamma = B_1 + B_2 \Rightarrow A + \Gamma = B_2\] (σφάλμα στο Β1 και από τα δύο μέρη της εξώσης)
Οπότε κατέληγε σε αυτό που ήθελα να δείξω.

2ο πειστικό επιχείρημα
Ξέρω ότι:
\[A + B + \Gamma = 180^\circ\] (ως άθροισμα γωνιών τριγώνου)
\[B_1 + B_2 = 180^\circ\] (ισοτιμία ευθεία) 
Θέλω να δείξω ότι:
\[B_2 = A + \Gamma\]
Φέρω μια γραμμή και παράλληλη προς την ΑΓ η οποία να περνάει από το Β. Η ευθεία ε χωρίζει τη γωνία B2 σε δύο (όχι απαραίτητα ίσες) γωνίες x και y.
Παρατηρώ ότι:
Α = x (ώς εντός ευκλάξ γωνίες)
Γ = y (ώς εντός αυτός και επί τα αυτά γωνίες)
B2 = x + y
Όπως: B2 = x + y = A + Γ, το σποί είναι ακριβώς αυτό που ήθελα να δείξω.

Αφού βρήκαμε ένα πειστικό, γενικό επιχείρημα (αυτό που λέμε δηλαδή στα μαθηματικά «απόδειξη»),
για να υποστηρίζουμε την εικασία μας, η εικασία μπορεί πλέον να συναπτεί ΘΕΩΡΗΜΑ.

Στον φάσσαμε τελικά στο παρακάτω θεώρημα:

**ΘΕΩΡΗΜΑ 5: (Βιβλίο Α’ Λυκείου, σελ. 83-84)**
Κάθε εξωτερική γωνία τριγώνου είναι ίση με το άθροισμα των απέναντι εσωτερικών γωνιών του τριγώνου.
6η ΑΝΑΚΑΛΥΨΗ

ΒΗΜΑ 1: Παρατήρηση

Φέρμα δύο ξενάρια παράλληλων γραμμών στον πίνακα όπως φαίνεται παρακάτω και
παρατηρήσετε το τετράγωνο που σχηματίζεται. Είστε ότι το
tετράγωνο το οποίο κατασκευάζεται ήφετεμα δύο ξενάρια
παράλληλων ευθειών λέγεται παραλληλόγραμμο.

ΟΡΙΣΜΟΣ ΠΑΡΑΛΛΗΛΟΓΡΑΜΜΟΥ: (Βιβλίο Α’ Λυκείου,
σελ. 97)

Παραλληλόγραμμο λέγεται το τετράγωνο που έχει τις
απέναντι πλευρές του παράλληλες.

Παρατηρώντας το σχήμα ανακαλύφθηκα μαζί κάποιες ιδιότητες του παραλληλόγραμμού οι οποίες
προκύπτουν από τον ορισμό του. Αρχικά διατυπώσαμε τις σκέψεις μας ως ευκαιρίες, και μετά βρήκαμε
πειστικά επιχειρήματα για να δείξουμε ότι είναι αλήθεια γενικά και ότι δεν εξαρτώνται από το σχήμα
(δε τα παρακάτω βήματα).

ΒΗΜΑ 2: Ευκατα

Με τη βοήθεια μετρήσεων και άλλων πρακτικών διαδικασιών, καταλήξαμε στις παρακάτω
υποθέσεις (ευκαιρίες). Αυτές οι υποθέσεις δε μπορούμε να είμαστε σίγουροι ότι ισχύουν γενικά,
γιατί τις κόνταμε για το συγκεκριμένο σχήμα.

• Ευκατα 1: Όταν πλευρές είναι ίσες,
• Ευκατα 2: Όταν απέναντι γωνίες είναι ίσες,
• Ευκατα 3: Όταν διαγώνιες διχοτομούνται.

ΒΗΜΑ 3: Κίνηση στο σχήμα

Διαπιστώσαμε ότι εκεί διόγγονται κίνηση στις πλευρές του παραλληλόγραμμου προκύπτουν πολλά
άλλα τετράγωνα άλλες φορές γενικά παραλληλόγραμμα, ή άλλες φορές συγκεκριμένου τύπου
παραλληλόγραμμα πους για παράδειγμα τετράγωνα, ορθογώνια, ρήμβη κλπ. Παρατηρήσαμε όμως
ότι ένα πρόγμα παραμένει αποθετερό και αυτό είναι ότι οι απέναντι πλευρές είναι πάντα
παράλληλες (από την ορισμό του παραλληλόγραμμου).

C.5. What we discovered in the geometry lessons
ΒΗΜΑ 4η: Απόδειξη

- Απόδειξη 1ης εικασίας: Οι απέναντι πλευρές ενός παραλληλογράμμου είναι ίσες.

Φέρομε τη διαγώνια ΒΔ του παραλληλογράμμου, και παρατηρήσαμε ότι σχηματίζουν δύο τρίγωνα. Τα ΑΒΔ και ΒΓΔ τα οποία μοιάζουν να είναι ίσα. Για να το διαπιστώσουμε τα συγκρίναμε. Έχουν

ΒΔ κουνή
Γωνία ΔΒΓ = Γωνία ΒΔΑ (ώς εντός εναλλάξ)
Γωνία ΓΔΒ = Γωνία ΑΒΔ (ώς εντός εναλλάξ)

Αρα από το κριτήριο Γ-Γ συμπεραίνουμε ότι τα τρίγωνα είναι ίσα. Επομένως αντίστοιχες πλευρές τους είναι ίσες, που σημαίνει: ΑΒ=ΔΓ και ΑΔ=ΒΓ.

Δείξαμε λοιπόν ότι οι απέναντι πλευρές ενός τυχαίου παραλληλογράμμου είναι ίσες. Αυτό το ονομάζουμε, αφού το αποδείξαμε, γενική ιδιότητα των παραλληλογράμμων.

ΙΔΙΟΤΗΤΑ 1η: (Βιβλίο Α' Αυκείου, σελ. 97)
Οι απέναντι πλευρές ενός παραλληλογράμμου είναι ίσες.

- Απόδειξη 2ης εικασίας: Οι απέναντι γωνίες ενός παραλληλογράμμου είναι ίσες.

Η σύμφωνη των τριγώνων που κάναμε στην προηγούμενη απόδειξη της εικασίας 1, μπορεί να χρησιμοποιηθεί και για την δεύτερη εικασία. Από την εισόδη των τριγώνων προκύπτει ότι οι απέναντι γωνίες είναι ίσες.

Δείξαμε λοιπόν ότι οι απέναντι γωνίες ενός τυχαίου παραλληλογράμμου είναι ίσες. Αυτό το ονομάζομε, αφού το αποδείξαμε, γενική ιδιότητα των παραλληλογράμμων.

ΙΔΙΟΤΗΤΑ 2η: (Βιβλίο Α' Αυκείου, σελ. 97)
Οι απέναντι γωνίες ενός παραλληλογράμμου είναι ίσες.
• Απόδειξη 3ου εικασίας: Οι διαγώνιες ενός παραλληλογράμμου διχοτομούνται.

Συγκρίνοντας τα τρίγωνα AOD και BOG διαπιστώνουμε ότι είναι ίσα διότι έχουν:

\[ \Delta \Delta \Delta \beta \] (λόγω της ιδιότητας 1 που αποδείξαμε προηγουμένως)
Γωνία OAD = Γωνία OGB (ως ενός εναλλάξ)
Γωνία ADO = Γωνία GBO (ως ενός εναλλάξ)

Αρα τα τρίγωνα είναι ίσα, οπότε:
AO=OG και BO=OD, δηλαδή οι διαγώνιες διχοτομούνται.

Δείξαμε λοιπόν ότι οι διαγώνιες ενός τυχαίου παραλληλογράμμου διχοτομούνται. Αυτό το ονομάσαμε, αφού το αποδείξαμε, γενική ιδιότητα των παραλληλογράμμων.

6η ΑΝΑΚΛΑΨΗ

ΒΗΜΑ 1: Παρατήρηση

Ζωγραφίσαμε ένα τρίγωνο στον πίνακα και ενώσαμε τα κέντρα δύο πλευρών του όπως φαίνεται στο παρακάτω σχέma. Έπειτα σύραμε το καθένα μας ζωγράφισε ένα διά το τρίγωνο στο τετραδίο του και ένωσε τα μέσα δύο πλευρών του.

Κάναμε διάφορες παρατηρήσεις για τη σχέση της ευθείας KL με τις υπόλοιπες πλευρές του τριγώνου.

Μία από αυτές ήταν ότι η ευθεία KL είναι παράλληλη προς την AG. Άλλη ήταν ότι η KL είναι ίση με το μισό της AG. Επίσης παρατηρήσαμε ότι μοιάζει σαν να είναι KL=AB/2 και KL=BI/2.

Αυτές τις εικασίες μας τις διαπιστώσαμε ως εικασίες δηλαδή ως υποθέσεις, όπως φαίνεται στο παρακάτω βήμα.

IDIOTHTA 3': (Βιβλίο Α' Λυκείου, σελ. 97)

Οι διαγώνιες ενός παραλληλογράμμου διχοτομούνται.
Appendix C. Handouts

**ΒΗΜΑ 2ο: Εικασία**

Κάνουμε 4 διαφορετικές εικασίες.

- **Εικασία 1ος:** Η ευθεία που ενώνει τα μέσα του δύο πλευρών ενός τριγώνου είναι παράλληλη προς την τρίτη πλευρά. (Στο σχήμα: KL // AG)

- **Εικασία 2ος:** Η ευθεία που ενώνει τα μέσα του δύο πλευρών ενός τριγώνου είναι ίση με το μισό της τρίτης πλευράς. (Στο σχήμα: KL = AG / 2)

- **3ος και 4ος Εικασία:** Η ευθεία που ενώνει τα μέσα του δύο πλευρών ενός τριγώνου είναι ίση με την αριστερή (ή τη δεξιά αντίστοιχη) πλευρά από τις δύο πλευρές που ενώνονται. (Στο σχήμα: KL = AB / 2 ή KL = BI / 2)

Οι παραπάνω εικασίες ήταν απλά κάποιες υποθέσεις που τις βασίσαμε στις παραπρόθεσεις μας. Αργόν δυστυπούσαμε αυτές τις εικασίες, προσπαθήσαμε να βρούμε πειστικά επεξεργαστή ως τα να τις αποδείξουμε ή να τις απορρίψουμε.

**ΒΗΜΑ 3ο: Κίνηση στο σχήμα**

Καινούντας τις καρφιές του τριγώνου, δηλαδή φτιάχνοντας πολλά διαφορετικά τρίγωνα κάθε είδους, είδαμε ότι κάποιες από τις παραπάνω εικασίες μοιάζουν πράγματι να ισχύουν γενικά, και κάποιες άλλες όχι. Συγκεκριμένα, οι εικασίες 1 και 2 φαίνεται να είναι αλήθειες σε κάθε σχήμα, ενώ οι εικασίες 3 και 4 φαίνεται να εξαρτώνται από το σχήμα. Αυτό μας έδωσε κάποιες ιδέες για να βρούμε ένα γενικό και πειστικό επεξεργασία, δηλαδή μία απόδειξη για να υποστηρίξουμε ή να καταρρίψουμε κάθε μία από τις παραπάνω εικασίες.

**ΒΗΜΑ 4ο: Απόδειξη**

- **Απόδειξη 1ος εικασίας.**

  **(1ος τρόπος)**

  Έλεγξα ότι το K είναι το μέσο της AB. Άρα

  BK / BA = 1 / 2

  Έλεγξα ότι το Λ είναι το μέσο της BG. Άρα

  BL / BI = 1 / 2

  Επειδή BK / BA = BL / BI και λόγου του θεωρήματος του Θοδώρη, οι KL, AB θα είναι παράλληλες. Οπότε αποδείχθηκε η πρώτη εικασία.

  **(2ος τρόπος)**

  Επειδή BK / BA = BL / BI και λόγου του θεωρήματος του Θοδώρη, οι KL, AB θα είναι παράλληλες. Οπότε αποδείχθηκε η πρώτη εικασία.

**ΒΗΜΑ 5ο: Απόδειξη**

- **Απόδειξη 2ος εικασίας.**

  **(1ος τρόπος)**

  Αναλύουμε την ευθεία ΚΛ προς το Λ κατά μέρος ΔΜ ίσο με το ΚΛ.
C.5. What we discovered in the geometry lessons

Еάν ενώσουμε τα σημεία Κ, Β, Μ, Λ, σηματίζεται ένα τετράπλευρο. Παρατηρούμε ότι στο τετράπλευρο αυτό ισχεί ότι οι διαγώνιες διχωτούνται, δηλαδή ΒΛ/ΛΓ και ΚΛ/ΛΜ λόγω του τρίγωνου με τον οποίο το κατασκευάζομε. Ετσι, μπορούμε να συμπεράνουμε ότι το τετράπλευρο ΚΒΜΓ είναι παραλληλόγραμμο.

Γνωρίζουμε ήδη τον ορισμό του παραλληλόγραμμου και έχουμε αποδείξει τις ιδιότητές του. Αρα γνωρίζουμε ότι

ΚΒ/ΜΓ και ΚΒ=ΜΓ (ως απέναντι πλευρές παραλληλόγραμμου)

Που σημαίνει ότι και η KA, εφόσον βρίσκεται πάνω στην ίδια ευθεία με την KB, θα είναι παράλληλη με τη ΜΓ. Επιπλέον έχουμε ότι BK=KA και BK=MG, άρα KA=MG.

Οπότε στο τετράπλευρο ΚΜΓΑ έχουμε ότι οι απέναντι πλευρές (ΚΑ και ΜΓ) είναι παράλληλες και ίσες, που σημαίνει ότι είναι παραλληλόγραμμο. Αφού είναι παραλληλόγραμμο, θα έχει τις απέναντι πλευρές του παραλληλές, και άρα KM//AG. Όμως η ΚΑ βρίσκεται πάνω στην ΚΜ, άρα ΚΛ//ΑΓ.

• Απόδειξη 2ου ευκαταστάθμου:

Στο προηγούμενο σχήμα, φέρνοντας την ευθεία ΛΛ' παράλληλη προς την ΚΑ. Εφύσσουν ΚΛ\//ΑΓ' (το δείχνει προηγούμενος) και ΚΑ//ΛΝ (διότι έτσι φέραμε την ΛΝ) συμπεράνουμε ότι το ΚΛΛ'Α' είναι παραλληλόγραμμο. Που σημαίνει ότι οι απέναντι πλευρές του είναι ίσες (λόγω των ιδιότητων του παραλληλόγραμμου). Αρα

ΚΛ=AN

Τα ίδια ισχύουν και για το τετράπλευρο ΛΜΝΓ, άρα

ΛΜ=NG

Αλλά γνωρίζω ότι

ΚΛ=ΛΜ

Από τις τρεις παραπάνω σχέσεις προκύπτει ότι AN=NG δηλαδή AN=AG/2

Οπότε δείχνουμε ότι ΚΑ=ΛΓ/2

15
Οι παραπάνω δύο ευκασίες που αποδείχθηκαν αποτελούν μαζί ένα θεώρημα.

**ΘΕΩΡΗΜΑ 6: (Βιβλίο Α’ Λυκείου, σελ. 104)**

Το ευθύγραμμο τμήμα που ενώνει τα μέσα δύο πλευρών τριγώνου είναι παράλληλο προς την τρίτη πλευρά και ίσο με το μισό της.

- **Απόρριψη 3\textsuperscript{η} και 4\textsuperscript{η} ευκασίες:**

  Παρατηρήσαμε ότι η 3\textsuperscript{η} και η 4\textsuperscript{η} ευκασία εξαρτώνται από το σχήμα. Δηλαδή, μπορούμε να βρούμε τρίγωνα στα οποία να μην ισχύουν. Για παράδειγμα στο διπλάνο σχήμα η ΚΛ είναι πολύ μικρότερη και από τη δύο πλευρές τις οποίες ενώνει.

  Αυτό το παράδειγμα που βρήκαμε για να απορρίψουμε την ευκασία μας λέγεται ΑΝΤΙΠΑΡΑΔΕΙΓΜΑ.

  Αντιπαράδειγμα δηλαδή είναι ένα παράδειγμα που αντικρούει μια γενική υπόθεση και είναι αυτό που μπορούμε να απορρίψουμε το επιχείρημά αυτό. Αντίθετα, είδαμε ότι για να υποστηρίζουμε μια γενική υπόθεση, δηλαδή για να δώσουμε ένα γενικό επιχείρημα που να δεχθεί η η υπόθεση είναι αλήθεια, δεν φαίνεται να δώσουμε μόνο ένα παράδειγμα.

7\textsuperscript{η} ΑΝΑΚΑΛΥΨΗ

**ΒΗΜΑ 1: Παρατήρηση**

Σωστογράφησας στον πίνακα ένα ευθύγραμμο τμήμα ΑΒ, βρήκαμε το μέσο του Μ, και φέρουμε μια τυχαία ευθεία ε η οποία να περνάει από το Μ (όπως φαίνεται στο σχήμα της επόμενης σελίδας). Έπειτα φέρουμε τις αποστάσεις ΑΛ και ΒΚ των σημείων Α και Β, αντίστοιχα, από την ευθεία ε.

Παρατηρήσαμε ότι οι αποστάσεις αυτές μοιάζουν να είναι ίσες.
C.5. What we discovered in the geometry lessons

**BHMA 2°: Eikasia**

Καταλήξαμε στην παρακάτω εικασία:

Οι αποστάσεις ΑΔ και ΚΒ θα έχουν ίσες όπως και να φέρουμε την ευθεία ε, αρκεί να περνάει από το μέσον του ΑΒ.

**BHMA 3°: Κίνηση στο σχήμα**

Φέρνοντας διάφορες ευθείες που να σχηματίζουν διαφορετικές γωνίες με το ΑΒ, παρατηρήσαμε ότι η εικασία μας μοιάζει να σχετίζει όντως γενικά αλλά δεν το γνωρίζουμε ακόμη μέχρι να δώσουμε ένα γενικό και πειστικό επιχείρημα. Αυτό που παρατηρούμε όμως ψηλότερα διάφορες ευθείες ε, είναι ότι στο σχήμα μας πάντα δημιουργούνται δύο τρίγωνα τα οποία μοιάζουν να είναι ίσα. Έτσι σκεφτόμαστε να χρησιμοποιήσουμε σύγκριση τριγώνων για να αποδείξουμε την εικασία μας.

**BHMA 4°: Απόδειξη**

Συγκρίνομε τις τρίγωνα ΑΜΔ και ΚΜΒ. Τα τρίγωνα είναι ορθογώνια και έχουν:

ΛΜ=ΜΒ (γιατί το Μ είναι το μέσον του ΑΒ)  
ΛΜΔ = ΚΜΒ (ως κατακόρυφη γωνία)

Αρα τα τρίγωνα είναι ίσα. Επειδή ως θα είναι ΑΔ=ΚΒ.  
Καταλήγουμε στο εξής συμπέρασμα:

Εάν μια ευθεία ε περνάει από το μέσο ενός ευθύγραμμου τρίματος, τότε τα άκρα του ευθύγραμμου τρίματος ισαπέχουν από την ευθεία ε.

**Παρατήρηση:**

Διαπιστώσαμε ότι το αντίστροφο αντίς της πρότασης δεν σχετίζει. Δηλαδή η πρόταση «Εάν τα άκra ενός ευθύγραμμου τρίματος ισαπέχουν από μια ευθεία ε, τότε η ευθεία ε περνάει από το κέντρο του ευθύγραμμου τρίματος» δεν είναι πάντα αλήθεια.

**Αντιπαράδεξια:** Στην περίπτωση που η είναι παράλληλη στην ΑΒ, τα άκρα του ΑΒ ισαπέχουν από την ε, όμως δεν η ε δεν περνάει από το μέσον του ΑΒ.

Γενικά, δεν είναι απαραίτητο στον μια πρόταση είναι αλήθεια και η αντίστροφη της να είναι αλήθεια.
C.6 Where does it all begin? The map of axioms

The colors in the following diagram represent the following:

- blue: definitions
- pink: axioms (propositions accepted as being true without proof, since we need to begin somewhere)
- white: theorems and propositions
- purple: undefined terms (certain concepts which remain without definition)

**Discovery 1:** The sum of the angles of any triangle is 180°.

**Discovery 2:** The sum of angles of any quadrilateral is 360°.

**Discovery 3:** Opposing angles are equal.

**Discovery 4:** The angle formed by the bisectors of two adjacent supplementary angles is right.

**Discovery 5:** An external angle of a triangle is equal to the sum of the two corresponding internal angles.

**Discovery 6:** The line which connects the midpoints of two consecutive sides of a triangle, is parallel to the third side and equal to its half.

**Discovery 7:** If we connect the midpoints of two consecutive sides of a quadrilateral, a parallelogram is formed.

**Alternate angles are equal**

**From a point outside a line we can bring only one parallel to that line.**

**If equals are subtracted from equals, then their remainders are also equal.**

\[ a + x = b + x \]

\[ a = x \]

**Properties of parallelogram**

**Parallel lines**

**Angle**

**Straight line**

**Point**

**Parallellogram**
C.7 The story of Euclidean geometry

Euclidean Geometry

What is geometry and when was it discovered?

Geometry is the science which studies the space around us and the shapes on a surface. Geometry is one of the most ancient sections of knowledge. Before it was established as a science, geometry was empirical, that is, it had to do with measurements and practical methods. This “empirical” geometry we call Practical Geometry, and it is the one you are taught in Gymnasium. However, geometry at some point has started to be founded as a science. This happened when it stopped being depended on experience (measurements etc.) and started to depend only on logical rules. That is, when we stopped being interested simply on the fact that something can be verified by measurement and we started being interested on why it holds generally. Then geometry became what we now call Theoretical Geometry. This transition from practical to theoretical geometry was made first by Euclid and this is why it is known in history as Euclidean Geometry. In Euclidean Geometry we do not need protractors and numbered rulers simply because measurements are not considered absolutely precise and thus we should not depend on them. On the contrary, theoretical geometry is based on observation and on the concept of proof. You will study theoretical geometry in Lyceum.

Where does the foundation of geometry begin?

The foundation of geometry is based on the simple idea that we have to accept a few initial propositions as true, the minimum possible, on which every other proposition that we discover will be based. Those initial propositions which Euclid thought of were called undefined terms and axioms.

The undefined terms

We call undefined terms the first basic concepts upon which Euclidean geometry was founded. It is very difficult to understand the undefined terms which Euclid has established are:

1. Point (point is that which has no dimensions)
2. Line (line is that which has only one dimension)
3. Plane (plane is that which has only two dimensions)

Axioms

Axioms are certain propositions which we accept without proof. The truth of these propositions is obvious. All other propositions in geometry need to be proven and are based on these axioms. That is, the axioms are the basic building blocks of Euclidean Geometry. We begin by the axioms and we prove whatever else exists in geometry. Euclid established the following axioms:

1. A straight line can be drawn between any two points.
2. Any line segment can be infinitely extended to both the directions of its edges in order to form a line.
3. It is possible to draw a circle, with any given center and radius.

4. All right angles are equal to each other.

5. If a line which intersects with two other lines forms internal angles with these on the same side, with a sum of less than 2 right angles, then if the two lines are extended to infinity, they intersect on that side on which the sum of the above angles was less than 2 right angles.

6. Those things which are equal to the same thing, they are also equal to one another.

7. If equals are added to equals, then their sums are also equal.

8. If equals are subtracted from equals, then their remainders are also equal.

9. Those which coincide are equal.

10. The whole is bigger than each of its parts.

11. Equal wholes have equal parts: equal quarters, equal halves etc.
Appendix D

Tables and results
D.1 Van Hiele pre-test: Analysis and comparison with other studies

The following diagram shows the percentage and range of students’ (n=85) answers to all 20 test questions.

We can see that many questions were either not answered by all of the students or their answer was invalid (this is indicated by "---" in the diagram above). As the difficulty of the questions increases - each question is more difficult than the previous - the number of no or invalid answers increases. However, we see that the percentage of students that do not give an answer remains relatively low even for the last question (5%).

A closer look at students’ answers for each question of the test reveals some interesting aspects both of students’ level of thinking in geometry and of the van Hiele test. We will now discuss the answers on each van Hiele test item and comment both on what we can conclude about the students’ level of thinking and on the reliability of the van Hiele test.
The first question turned out to be rather easy for the level of the students. It is the very first question corresponding to van Hiele level 1. As is shown in the pie chart to the right, most of the students (72 students, or 85% of the total sample) had no problem finding the correct answer. It was expected that students at this age would be able to distinguish between a square, a triangle and a rectangle. A considerable number of the rest of the answers given (11 out of 12) were answer d, namely the one that suggests that both the rectangle and the square in the pictures are squares. Two of the students that gave d as the answer are students who have 20 (out of 20) as a term grade in Mathematics. This is rather peculiar, and we suspect that these students have given this answer accidentally. This is one of the downsides of multiple choice tests.

The results for this first question agree with the results by Tzifas (p. 75, 2006) in which roughly 86% of the students answered correctly, whereas roughly 11% of them wrongly chose answer d. The results also agree with Usiskin’s results (1982, p. 170) in which 85% of the students answered correctly, whereas 8% chose answer d.

For the second question a slightly smaller percentage of students (56 students, or 66%) answered correctly. The wrong answer that appears mostly (24 students, or 28%) is c. We might interpret this result as an inability to recognize a triangle by its properties, since when they are in forms that have not been seen before by the student (for example tall, thin, and stretched triangles like the one designed by Usiskin for this question) the students think they are not triangles. These students are apparently still in the first van Hiele level. Again here, some of the students that gave c as an answer are above average students. This might suggest that the “stretched” triangle is not very obviously a triangle for some students, in the sense that they cannot tell clearly whether it has three straight sides. An interview would be needed again here to ask students why they gave c as an answer to this question. If the “stretched” triangle works as a kind of optical illusion, then this item of the test should be reconsidered.
Here the results are a bit different than those found by Tzifas (2006). For our research, the test question which our students found the easiest was the first question, whereas for Tzifas’s research this question, with 96% success rate, was the easiest. Actually for the students of our study, the second question turned out to be the second most difficult of the level 1 questions. However, when comparing our results to Usiskin’s, we see that they are very similar, since only 63% of the students of their study gave the correct answer, whereas 30% gave answer c.

Moving to question 3, we see again that most students (61 students, or 72%) got the correct answer. However, we see that there is more variation for the alternative choices. Among the wrong choices, the most popular are a and e. Eleven students (13%) answered that all given figures are rectangles (answer e), which may be interpreted as thinking that when a figure has at least two right angles it can be called a rectangle, and 6 students (7%) answered that only the first figure is a rectangle (answer a), which may be interpreted as failing in recognizing that a figure is a rectangle when it is tilted. Again, several above average students gave wrong answers; this fact raises questions which would need an interview in order to be answered and indicates that a multiple choice test on its own is insufficient to reveal with certainty the van Hiele levels of students.

Our results on question 3 are very similar to those by Tzifas (2006), where 86% of the students gave answer c, 6.4% gave answer e, and 4.9% gave answer a. Usiskin’s results, however, gave a much higher percentage (93%) of correct answers to this question.
Question 4, along the same lines, has a relatively high percentage of successful answers. More than half of the students (57 students, or 67%) gave the correct answer. A considerable number of students gave a different answer than the correct one, the two most popular being e (12%) and c (11%). One way this may be interpreted is that some students may confuse the word “rectangle” with the word “four-sided”. In Greek, these two words look and sound quite similar. For the rectangle we use the word “τετράγωνο” which literally means “four-angled” and for any figure with four sides we use the word “τετράπλευρο” which literally means “four-sided.” These two words and their definitions may be confused by the students. In Usiskin’s study, 80% of the students answered correctly. The difference of his results to ours in this case may be explained because of the difference in language.

Another explanation, however, may be the fact that the test in Usiskin’s case was given to a wider range of ages (14-17), which means that more students in his sample were more experienced in geometry than the students of our sample, and probably they were in higher van Hiele levels.

Question 5, as expected, turned out to be the most difficult of the first level. Students need to recognize parallelograms. One of the figures given was a rhombus. According to the van Hiele theory, if a student is not yet in the second van Hiele level he or she will not be able to characterize a rhombus as a parallelogram. Here we see that, although the most popular answer to question 5 is e (the correct one) with 34 students (or 40%) selecting it, the next most popular answer is c with almost the same popularity rate (33 students, or 39%). Answer c suggest that the rhombus given in the question is not a parallelogram. One may conclude from this that students selecting answer c are still at the first van Hiele level, the visual level, where one characterizes a shape only by how it looks and not by its properties. Tzifas does not offer any results for this question, while Usiskin’s percentage results for this question (Usiskin, 1982, p. 170) do not add up to 100%. Thus, we cannot compare our results to either of theirs for this question.
Question 6 deserves more discussion. Before the van Hiele test was distributed to the students, it was tried on a single student by the researcher in order to see how long it would take for a good student to complete it. The student on which it was tried is an excellent student in Mathematics, and she is one year older than the students of our sample. We expected that she would know without difficulty how to answer the first 10 questions of the test which correspond to the first 2 van Hiele levels, and that she wouldn’t need more than 30 minutes to finish the test. Indeed, the students finished the test in 25 minutes and she was taking her time while doing it. She could answer correctly all questions from 1 to 10, but one: question 6. This was rather surprising, and it made us think her mistake was accidental. She must know for sure, we thought, that two opposite sides of a square cannot be perpendicular. What made her give answer c to question 6? A few hours after the test, and after the researcher had already checked the answers of the student, we went back to her and asked what she would answer for question 6. She looked at it carefully, without knowing what she had written on the answer sheet, and she said she would choose b. The answer she had given earlier was c. When she was told that her previous answer was c she laughed and said: “of course these lines are not perpendicular to each other!”

This incident worried us a bit, although of course the criteria for classifying a student to a level judging by the answers they give to this test allow for one (the strict, 4 / 5, criterion) or two (the loose, 3 / 5, criterion) errors that could be excused as being accidental. So, we thought, it would be OK in the end to use a multiple choice test if we allow “accidents”. However, looking at the results of the test on the whole population, we can see that a big percentage of students (52 students, or 61%) made exactly the same mistake! Is this accidental? Is it possible that there is something wrong with the test question? One possible explanation of this unexpected result is that students interpret answer c as saying that the two opposite sides of the square are perpendicular to the third side of the square and not to each other. For finding out what exactly the interpretation of the students was, and why they chose c as the right answer one would have to interview the students for which we did not have time in our research.

What made us even more suspicious about this question is the results of the other two studies. In Tzifas (2006), 33% of the students chose answer c to question 6. The way Tzifas interpreted this result is that there is a surprising percentage of students that believe the opposite sides of a square to be perpendicular (Tzifas, 2006, p. 76). We think that this is not necessarily the case since as explained earlier there might be other reasons (e.g. different interpretations of the answers by the students) that might explain this phenomenon. In Usiskin’s study the answer choice c was selected by 61% of the students, which is quite a high percentage. However, it is important to note that in Usiskin’s study the question was formulated differently, asking about the sides of a rectangle instead of a square.
there are surprisingly similar results: only 33% of the students chose the correct answer to question 6, while 40% of the students chose answer c. We believe that the case of this question should be further investigated.

For question 7, things ran more smoothly. Most students (65 students, or 76%) figured out the correct answer. As mentioned earlier, in three out of four classrooms students had difficulties understanding the structure of the question and what exactly it was asking for. For question 8, things were quite different. Less than half the students managed to answer correctly (38 students, or 45%); all four wrong answers to question 8 are more or less equally popular, with answer e slightly standing out. This might be interpreted as unfamiliarity of students with the properties of the rhombus.

The percentages of correct answers to questions 7 and 8 in Tzifa’s study were 75.5% and 58.5% respectively, whereas the corresponding percentages in Usiskin’s study were 66% and 38% respectively.
For questions 9 and 10 the results are similar. Only just more than half of the students (45 students, or 53%) managed to answer question 9 correctly, and only 36 students (42%) answered question 10 correctly. There is again variety in the alternative (wrong) choices. For question 9, the most popular wrong answer is d. This may mean that some students confuse the isosceles with the equilateral triangle, since answer d suggests that all isosceles triangles have all their angles equal. Another source of confusion, however, may be the structure of the question. Questions 7 and 8 asked which of the answers are not true in every figure of the kind indicated by the questions, whereas question 9 asks which of the answers are true in every figure of the kind indicated. If a student does not read this question carefully enough, he or she might think that it is still a question of the same kind as 7 and 8. The students while taking the test asked a couple of times whether more than one answer is right in some cases, although at the beginning they were told that only one answer can be correct. Question 9 might be one of the questions that made them wonder if more than one answer is correct, if they were confused about what exactly the question asks for. Once more, it turns out that an interview with some students would be very useful in order to understand why so many wrong answers were given.

The case of question 10 is slightly different. The risk of confusion of the kind involved in question 9 is not the case here, but the question involves many more concepts and it is generally more difficult than all the rest of the vH level 2 questions. It is therefore not surprising that most of the students gave the
wrong answer. The percentages of correct answers to questions 9 and 10 in Tzifa’s study were 65% and 37% respectively, whereas the corresponding percentages in Usiskin’s study were 69% and 38% respectively. We see that in both these studies, like in ours, considerably more students were able to answer correctly question 9 than question 10.

Questions 11 to 15 correspond to van Hiele level 3. As was expected (given the age and school level), most students failed to give the correct answer to these questions. The pie charts below show the exact answer counts to all 5 questions. Question 14 turned out to be the most difficult with a success rate of 8%, while question 13 was the easiest to answer with a success rate of 42%.

Our results for these five questions are not always similar to the results of Tzifas and Usiskin. We can say that in general, and as the difficulty of the questions and the van Hiele level of the questions increases, the students seem to start answering more randomly. In our study, the students were all of the same age (14-15), whereas in Tzifas’ and Usiskin’s study the
students participating were of a larger range of ages (14-17). Maybe it is because of this that our results for questions 11 to 20 differ more than the ones for questions 1-10.

It is worth mentioning that Tzifas’ results for question 13 are very different from both ours and Usiskin’s. This is because of the different interpretation of the original version of question 13 in the translated version used in Tzifas’ study. In the translated version of question 13 as it is presented by Tzifas (2005, p.129) the order of the figures is different to the one in the original test made by Usiskin, which gives to the translated version a very different meaning. The question, as Usiskin provides it, has on the left a square and in the middle a very tall and thin rectangle. This question aims at revealing whether the students can understand that also squares can be called rectangles, and that even a rectangle that looks a bit strange compared to the ones we are used to because it is very tall and very thin can be called ‘rectangle’. Most of the students in Usiskin’s research as well as the 42% of the students in our study gave answer e to this question, which corresponds to the two rectangles of the picture. Obviously, the correct answer is a, however most of the students were not at a level appropriate to recognise this.

In the case of Tzifas’ study, answer e does not correspond to the two rectangles of the figure, but to the square and to one of the rectangles, fact which explains first, why the large majority of the students in Tzifas’ research answered a and second, why “during the test many students asked why the answer ‘Only K and M’ is not among the options” (Tzifas, 2005, p. 79, our translation). The option the students were looking for in fact corresponds to the two rectangles of the picture.

Therefore, if question 13 in Tzifas’ research would have been equivalent to Usiskin’s question 13, then the results for this questions would also have been different; possibly the results for the classification of students into van Hiele levels in Tzifas’s study would have been different as well. For instance, let us assume that a student has answered correctly at least three of the questions which correspond to van Hiele levels 1 and 2. Let us also assume that the same student has answered correctly three out of four questions corresponding to van Hiele level 3, one of which is question 13 which they would have probably answered wrongly if its meaning was as in the original test. Then, the student has been classified by Tzifas at least at van Hiele level 3, whereas normally he or she would have been assigned to van Hiele level 2.
Questions 16 to 20 correspond to van Hiele level 4, a level which students of this age in a Greek public school are usually expected not to have acquired. As the pie charts show, in the first two questions of the level, 16 and 17, the most popular answer is the correct one. However, only about one third of the students answered these two questions correctly (24 students, or 28% for question 16, and 28 students, or 33% for question 17). In general, the failure to answer correctly the last five questions of the test shows that the students are not yet able to understand the logical relation between propositions in geometry, something which was anticipated. Also, their answers seem to be more random than in the previous questions of the test.

Answer counts to vH question 16
(Correct answer is c)

Answer counts to vH question 17
(Correct answer is c)

Answer counts to vH question 18
(Correct answer is d)
The overall percentages of students classified in each van Hiele level based on both criteria are shown in the following pie charts.

The overall percentages of students classified in each van Hiele level based on both criteria are shown in the following pie charts.
For classifying students to van Hiele levels we need to make sure that "the student at level \( n \) satisfies the criterion not only at that level but at all preceding levels" (Usiskin, 1982, p. 25). By following this property and the loose criterion (3 out of 5), we could not classify 20 students (24%) to any level. This is indicated by "no" in the pie chart of the previous page. Similarly, for the strict criterion we could not classify 6 students (7%). We could classify 26 students (31%) to the second van Hiele level based on the loose criterion and only 12 students (14%) based on the strict criterion. The following diagram shows the percentages of students that we were able to classify to a van Hiele level considering both criteria, and compares it to the corresponding results of Usiskin and Tzifas.

There were several cases of very good students (with term grades between 18 and 20 – out of 20) which we were unable to classify to any level. This was often because in the Usiskin van Hiele test questions corresponding to level 2 they had one correct answer less than the correct answers they gave for the van Hiele level 3 questions.
### D.2 Distribution of answers to the VR and EP questions in Task CHOOSING

<table>
<thead>
<tr>
<th>Statements</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>shows that the statement is always true</td>
<td>13</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>only shows that the statement is true for some triangles</td>
<td>5</td>
<td>3</td>
<td>11</td>
<td>2</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>shows why the statement is true</td>
<td>12</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>is an easy way to explain to someone in your class who is unsure</td>
<td>13</td>
<td>4</td>
<td>2</td>
<td>13</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Table D.1: Distribution of answers regarding the VR and EP of Stamatis’s (Dylan’s) argument before our intervention ($N = 19$).

<table>
<thead>
<tr>
<th>Statements</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>shows that the statement is always true</td>
<td>3</td>
<td>2</td>
<td>14</td>
<td>3</td>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>only shows that the statement is true for some triangles</td>
<td>13</td>
<td>3</td>
<td>3</td>
<td>13</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>shows why the statement is true</td>
<td>8</td>
<td>3</td>
<td>8</td>
<td>8</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>is an easy way to explain to someone in your class who is unsure</td>
<td>11</td>
<td>5</td>
<td>3</td>
<td>11</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

Table D.2: Distribution of answers regarding the VR and EP of Stamatis’s (Dylan’s) argument after our intervention ($N = 19$).

<table>
<thead>
<tr>
<th>Statements</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
<th>Agree</th>
<th>Don’t know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>shows that the statement is always true</td>
<td>15</td>
<td>3</td>
<td>1</td>
<td>15</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>only shows that the statement is true for some triangles</td>
<td>2</td>
<td>4</td>
<td>13</td>
<td>2</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>shows why the statement is true</td>
<td>13</td>
<td>1</td>
<td>4</td>
<td>17</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>is an easy way to explain to someone in your class who is unsure</td>
<td>17</td>
<td>1</td>
<td>1</td>
<td>17</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table D.3: Distribution of answers regarding the VR and EP of Georgia’s (Cynthia’s) argument before our intervention ($N = 19$).
D.2. Distribution of answers to the VR and EP questions in Task CHOOSING

<table>
<thead>
<tr>
<th></th>
<th>shows that the statement is always true</th>
<th>only shows that the statement is true for some triangles</th>
<th>shows why the statement is true</th>
<th>is an easy way to explain to someone in your class who is unsure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Agree</strong></td>
<td>17 89</td>
<td>15 89</td>
<td>17 89</td>
<td>13 68</td>
</tr>
<tr>
<td><strong>Don’t know</strong></td>
<td>2 10</td>
<td>1 5</td>
<td>1 5</td>
<td>5 26</td>
</tr>
<tr>
<td><strong>Disagree</strong></td>
<td>0 0</td>
<td>17 89</td>
<td>1 5</td>
<td>1 5</td>
</tr>
</tbody>
</table>

Table D.4: Distribution of answers regarding the VR and EP of Georgia’s (Cynthia’s) argument after our intervention ($N = 19$).

<table>
<thead>
<tr>
<th></th>
<th>shows that the statement is always true</th>
<th>only shows that the statement is true for some triangles</th>
<th>shows why the statement is true</th>
<th>is an easy way to explain to someone in your class who is unsure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Agree</strong></td>
<td>9 47</td>
<td>4 21</td>
<td>12 63</td>
<td>4 21</td>
</tr>
<tr>
<td><strong>Don’t know</strong></td>
<td>8 42</td>
<td>7 37</td>
<td>7 37</td>
<td>4 21</td>
</tr>
<tr>
<td><strong>Disagree</strong></td>
<td>2 10</td>
<td>8 42</td>
<td>0 0</td>
<td>11 58</td>
</tr>
</tbody>
</table>

Table D.5: Distribution of answers regarding the VR and EP of Lefteris’s (Ewan’s) argument before our intervention ($N = 19$).

<table>
<thead>
<tr>
<th></th>
<th>shows that the statement is always true</th>
<th>only shows that the statement is true for some triangles</th>
<th>shows why the statement is true</th>
<th>is an easy way to explain to someone in your class who is unsure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Agree</strong></td>
<td>15 79</td>
<td>1 5</td>
<td>16 84</td>
<td>5 26</td>
</tr>
<tr>
<td><strong>Don’t know</strong></td>
<td>3 16</td>
<td>2 10</td>
<td>3 16</td>
<td>6 32</td>
</tr>
<tr>
<td><strong>Disagree</strong></td>
<td>1 5</td>
<td>15 84</td>
<td>0 0</td>
<td>8 42</td>
</tr>
</tbody>
</table>

Table D.6: Distribution of answers regarding the VR and EP of Lefteris’s (Ewan’s) argument after our intervention ($N = 19$).

<table>
<thead>
<tr>
<th></th>
<th>shows that the statement is always true</th>
<th>only shows that the statement is true for some triangles</th>
<th>shows why the statement is true</th>
<th>is an easy way to explain to someone in your class who is unsure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Agree</strong></td>
<td>3 16</td>
<td>4 21</td>
<td>8 42</td>
<td>2 10</td>
</tr>
<tr>
<td><strong>Don’t know</strong></td>
<td>12 63</td>
<td>10 53</td>
<td>8 42</td>
<td>7 37</td>
</tr>
<tr>
<td><strong>Disagree</strong></td>
<td>4 21</td>
<td>4 21</td>
<td>2 10</td>
<td>10 53</td>
</tr>
</tbody>
</table>

Table D.7: Distribution of answers regarding the VR and EP of Kalliopi’s argument before our intervention ($N = 19$).
Appendix D. Tables and results

Table D.8: Distribution of answers regarding the VR and EP of Kalliopi’s argument after our intervention ($N = 19$).

<table>
<thead>
<tr>
<th></th>
<th>agrees</th>
<th>N</th>
<th>%</th>
<th>only shows that the statement is true for some triangles</th>
<th>shows why the statement is true</th>
<th>is an easy way to explain to someone in your class who is unsure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agree</td>
<td>5</td>
<td>26</td>
<td>4</td>
<td>21</td>
<td>11</td>
<td>58</td>
</tr>
<tr>
<td>Don’t know</td>
<td>10</td>
<td>53</td>
<td>9</td>
<td>47</td>
<td>6</td>
<td>32</td>
</tr>
<tr>
<td>Disagree</td>
<td>4</td>
<td>21</td>
<td>6</td>
<td>32</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Table D.9: Validity Rating (VR) and Explanatory Power (EP) of the various arguments before and after the intervention ($N = 19$).

D.3 VR and EP scores before and after our intervention

<table>
<thead>
<tr>
<th>Argument</th>
<th>VR</th>
<th>EP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Stamatis</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>before</td>
<td>10</td>
<td>53</td>
</tr>
<tr>
<td>after</td>
<td>13</td>
<td>68</td>
</tr>
<tr>
<td>Georgia</td>
<td>16</td>
<td>84</td>
</tr>
<tr>
<td>before</td>
<td>7</td>
<td>37</td>
</tr>
<tr>
<td>after</td>
<td>14</td>
<td>74</td>
</tr>
<tr>
<td>Lefteris</td>
<td>3</td>
<td>16</td>
</tr>
<tr>
<td>before</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>after</td>
<td>13</td>
<td>68</td>
</tr>
</tbody>
</table>

D.4 More results of the beliefs & reflections questionnaire

Q1: Do you believe you learned something new in the lessons we had? If yes, what?

We asked the students this question in order to find out what they got out of the lessons, and whether what they learned was considered by them something different than what they had already learned at school.

Out of the 20 students who filled in this question, 19 answered positively. The only negative answer was a ‘no’ with no further explanation. The positive answers can be grouped under the following categories. (The categories are non-exclusive.)

- **Category 1: Vague answers**

  Some of the students (4 out of 20) gave a vague answer as to what new things they learned during the lessons. For example:

  Yes, I believe that I learned a lot which will help me from this year to Lyceum.
Yes. Basically whatever we did was something new which hasn’t happened to us before.

- **Category 2: Proving**
  Many of the students (9 out of 20) explicitly mentioned proving as a new thing they learned in these lessons.

  Yes, I believe that we learned. We learned how to prove a theorem. We learned to observe, to make conjectures, and finally how to prove it.

  In the mathematics that we did I learned to prove some of the theories of mathematics.

- **Category 3: Explaining why the givens are true**
  Some of the students (4 out of 20) said that the new thing they learned was how to explain why what they were taught before as a given actually holds. Here are some of the students’ own words:

  Yes. I learned to observe, to prove and to give movement to things that are known but I would never have been able to say whether they are true and why. They made me think and look at geometry with different eyes.

  Yes, I learned new things like e.g. some rules which we simply had them as given and we started from them in order to solve an exercise here we proved and we explained these rules.

  This shows that the lessons helped the students appreciate the explanatory function of proof, and its function as a way of establishing truth in mathematics. Generally, most of the students mentioned that learning and understanding what proof is was something new to them.

- **Category 4: Other**
  Other things mentioned by the students, which do not fall in any of the above categories, are the following. The new thing learned was undefined terms (“Yes, the undefined terms and how to prove something.”); they learned how all knowledge is connected into a system when it comes to proving things in mathematics (“Yes I learned. I learned that in order to prove something I have to take into consideration everything and most of all my previous knowledge!”); the lessons gave direction to their thinking (“Yes I learned in what direction more or less I have to direct myself in order to find a solution.”); and they learned about the meaning of proof in geometry (“Yes, I learned many things until today. I learned the meaning of proof in geometry, the need to doubt the givens and many things about geometry of course.”).

  The different kinds of answers given by the students are summarised in Table D.10. The categories of students’ answers are non-exclusive since sometimes more than one suggestion was offered by a single student.
Appendix D. Tables and results

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency (n = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>19</td>
</tr>
<tr>
<td>No</td>
<td>1</td>
</tr>
<tr>
<td>Vague</td>
<td>4</td>
</tr>
<tr>
<td>Proving</td>
<td>9</td>
</tr>
<tr>
<td>Explain why givens are true</td>
<td>4</td>
</tr>
<tr>
<td>Other</td>
<td>4</td>
</tr>
</tbody>
</table>

Table D.10: Do you believe you learned something new in the lessons we had? If yes, what?

We can conclude that almost all students found that the lessons helped them learn many new things about geometry and especially proof and its explanatory power.

Q2: Did you like the lessons? Would you prefer that anything would be done in a different way? Explain.

This question aimed at finding out how the students liked the structure of the lessons and the teaching strategies that we employed. We expected the students to give generally positive answers since the lessons we designed were quite different to the traditional way mathematics is taught in their school. Our expectations were correct since all of the students answered this question positively and many of them were rather enthusiastic about the lessons. Here are some answers.

The lessons were super. I simply believe that we could have given less time in some lessons so that we could have done more things.

I liked the lessons a lot. The teacher was giving us time to think and I can say that we were playing a game which helped us learn many things.

Yes, they were fun because each one of us was saying his opinion and sometimes it was funny …

No the lessons were exceptional. They were making us interested whatever their subject was. Moreover, the responsible person for the program has taken care so that the environment is friendly and very approachable for us.

The students enjoyed very much having opportunities for discussion in the lessons and being free to say their own opinion (all answers, whether right or wrong, were in fact constantly encouraged by the teacher). Students also seem to have liked the fact that there was group work so they had the opportunity to discuss with their peers and cooperate in order to come up with their own ideas. Many students contrasted this with their ordinary school lessons:

I liked the lessons and the way they were done was nice. All students had an opportunity to speak and there was cooperation.
Yes I liked them very much because contrary to the school lessons we prove on our own problems and we ‘function’ in teams. I wouldn’t want something to happen in a different way.

I liked them and I wouldn’t change anything. I prefer them to the normal classes of school.

One of them commented that it was unexpected that they would have a good time during a mathematics lesson.

Yes I liked them very much. The time was passing very pleasantly learning mathematics, which I considered very unlikely.

There were only 2 students that suggested that something could be done differently in the lessons (the second part of the question). One of them suggested that we could have more work in groups, and the other suggested the opposite (this was a girl that worked in a group with someone she didn’t like). Out of the rest of the students, 10 suggested that nothing would be done differently (for various reasons), and 8 did not mention at all their opinion on this. The different kinds of answers given by the students are summarised in Table D.11.

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>20</td>
</tr>
<tr>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>Included suggestions</td>
<td>2</td>
</tr>
</tbody>
</table>

Table D.11: Did you like the lessons? Would you prefer that anything would be done in a different way? Explain.

Overall we can say that the students prefer the guided reinvention teaching strategy to the traditional one, and that this method of teaching seems to make the lessons more ‘fun’ for them.

**Q3: If we had some more lessons together for the coming weeks, which would be the subject that you would prefer to discuss and why?**

We asked this question in order to find out what subjects they would be motivated to continue learning after our cycle of lessons. Only two students said that they would not want any other subject.

Nothing because it would be chosen as usual by the others.

I don’t know. Already with what we have done I can’t complain!

Out of the answers that gave some suggestions we identified the following categories.

- **Category 1: Geometry & Proofs**
  Seven students said that they want to do more geometry (some of them suggested also specific areas such as constructions, polygons, and the circle) and/or proof.
I would like to prove more hypotheses or to reject them. Specifically, I would like to be involved with the circle.

To continue the various proofs. Like in the previous lessons to try and give our own arguments to explain our observations.

• **Category 2: Axioms & Undefined terms**

Because the last lesson of our intervention related to the axiomatic system, the axioms, and the undefined terms which surprised the students and made them really interested, five of the answers given suggested that we could continue discussing about these concepts.

I would like to discuss more about the undefined terms and the axioms because I got interested in them and I believe that it would be very useful knowledge since they are the ‘beginning’ of geometry.

I would want to discuss the subject with the point [undefined terms] which had concerned all of us in the previous lesson.

I would like to discuss about the undefined terms and the axioms.

• **Category 3: History of Geometry and Mathematics**

Three students suggested that the following lessons should have a more ‘philosophical’ character, related to the history of geometry and mathematics in general.

Maybe I would prefer to be involved with how did the great mathematicians begin in order to arrive to the mathematics of today.

I would prefer to discuss for Geometry generally. Where it began from, how and why. I believe that it would be nice because we would get a wider view of geometry.

I would like to discuss things like, the history of geometry, and what we can construct by using geometry.

• **Category 4: Other**

There were also four students suggesting different subjects like algebra, trigonometry, arithmetic etc.

The subject I would choose would be trigonometry because I want to understand where we get the numbers of the table from.

The different kinds of answers given by the students are summarised in Table D.12.

Overall we can say that the students seemed to be motivated to continue learning about geometry and especially about the concepts of geometry that are related to the axiomatic system and the philosophy of mathematics.
Table D.12: If we had some more lessons together for the coming weeks, which would be the subject that you would prefer to discuss and why?

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency (n = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nothing</td>
<td>2</td>
</tr>
<tr>
<td>Geometry and proofs</td>
<td>7</td>
</tr>
<tr>
<td>Axioms and undefined terms</td>
<td>5</td>
</tr>
<tr>
<td>History of Geometry and Mathematics</td>
<td>3</td>
</tr>
<tr>
<td>Other</td>
<td>4</td>
</tr>
</tbody>
</table>

Q4: Do you believe that what we have done in these lessons is useful? In what way?

We asked this question in order to find out the opinion of the students on which part of the lessons was most useful to them (if at all). All 20 students said that the lessons were useful for one reason or another. The answers given can be divided into five categories based on the justification they gave.

- **Category 1: For the future (Lyceum)**
  Students were told by the teacher during the lessons that all the theorems we proved in class would be in the book of their next school level (A’ level of Lyceum), so 9 out of 20 students thought that this was rather useful.

  Yes because it will help us in A’ Lyceum and of course also for this year when we will do what you have told us.

  It was useful because we will know them already in Lyceum where they will be asked to.

- **Category 2: When you learn something new it is always good**
  Six of the answers were a bit vague, mainly mentioning that whatever one learns is good, therefore what we did was good as well.

  Yes it is useful because everything we learned taught us something new and when you learn something new it is never useless.

  I believe yes. Whatever one learns is good.

  Yes. It is a nice way in understanding better some things.

- **Category 3: Improved/changed our way of thinking**
  Three students mentioned that the lessons helped them improve or change their way of thinking.

  It is useful, because it has taught us a new way of thinking.

  I believe that yes. I don’t know if I will use it in the future but it helped in improving the way I am thinking.
• **Category 4: We now understand why certain things are true**

Finally, there were 5 students that mentioned as useful that they now understand why certain things hold in geometry (and mathematics in general).

I believe that what we did will help me in understanding better for what reason some things hold in geometry.

It is very useful because in school we have the proofs ready and we don’t “put our mind to work” in order to understand the reason why something is this or the other way, having as a result difficulties in the future.

The different kinds of answers given by the students are summarised in Table D.13.

<table>
<thead>
<tr>
<th>Answer</th>
<th>Frequency (n = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>20</td>
</tr>
<tr>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>For the future (Lyceum)</td>
<td>9</td>
</tr>
<tr>
<td>When you learn something new it is always good</td>
<td>6</td>
</tr>
<tr>
<td>Improved/changed our way of thinking</td>
<td>3</td>
</tr>
<tr>
<td>We now understand why certain things are true</td>
<td>5</td>
</tr>
</tbody>
</table>

Table D.13: Do you believe that what we have done in these lessons is useful? In what way?

We can conclude from the above that the students were quite satisfied regarding the usefulness of the lessons. We had hoped to make students see that proof can function as an explanation in mathematics, and 5 students offered this as something useful that they learned.