The Kodaira dimension of the moduli space of K3 surfaces

Arie Peterson

January 4, 2011
Abstract

Following [4], we determine the Kodaira dimension of the moduli space of polarised K3 surfaces of degree $2d$, for almost all values of $d$. Using a Torelli type theorem, the Kodaira dimension is related to the existence of certain modular forms. The construction of such forms in turn reduces to a combinatorial problem: the existence of points in the lattice $E_8$ that are orthogonal to few roots. This problem is resolved (partially by exhaustive computer search): we conclude that the moduli space is of general type for almost all values of $d$, and we obtain partial information on the Kodaira dimension for some other values of $d$. 
## Contents

0.1 Introduction .................................................. 2

1 Geometry .......................................................... 3
  1.1 K3 surfaces .................................................... 3
      1.1.1 Polarisation .............................................. 3
      1.1.2 The period map and the Torelli theorem .............. 3
  1.2 Kodaira dimension ........................................... 4
  1.3 The Kodaira dimension of $F_2$ ................................ 5
  1.4 The low-weight cusp form .................................... 5

2 Lattices .......................................................... 8
  2.1 Preliminaries ................................................ 8
      2.1.1 Discriminant ............................................. 9
      2.1.2 The lattices $E_8$ and $E_7$ .......................... 9
  2.2 Symmetry ..................................................... 10
      2.2.1 Symmetries of type $D_8$ .............................. 10
      2.2.2 The full symmetry group ............................... 12
  2.3 Lattice computations ....................................... 14
      2.3.1 Lattice points with few orthogonal roots ............ 14
0.1 Introduction

The goal of this master thesis is to determine the Kodaira dimension of the moduli space of K3 surfaces. A K3 surface is one possible generalisation of an elliptic curve; see section 1.1; the Kodaira dimension is a coarse invariant of algebraic varieties, measuring something like the number of dimensions in which the variety is negatively curved (see section 1.2 for more).

Unfortunately, there is no well-behaved moduli space of K3 surfaces. Therefore, we replace it by the moduli space of K3 surfaces with an added polarisation (see section 1.1.1). This moduli space decomposes as a union of an infinite number of components: one for every possible degree of the polarisation (this parameter is typically denoted by $d$). Therefore many questions about the moduli space of K3 surfaces reduce to a set of questions, one for every $d \in \mathbb{N}_+$.

The classical Torelli theorem [9], which describes the moduli space of non-singular curves using the period mapping, has an analogue for K3 surfaces. It is called the global Torelli theorem for projective K3 surfaces [8], and it gives a description of the moduli space of polarised K3 surfaces as an quotient of a complex space by an arithmetic group. This makes questions about the abstract moduli space more concrete, and it brings number theory into play, because of the arithmetic nature of the group in this description.

Now, one possible way to assert that the moduli space of polarised K3 surfaces of degree $2d$ is of general type (i.e., has the highest possible Kodaira dimension), is by proving the existence of certain modular forms. Following [4], we show how the existence of these modular forms can be established, by modifying the so-called Borcherds function $\Phi_{12}$ (first constructed by Borcherds in 1995, see [2]). This procedure is subject to a certain combinatorial condition, on the existence of points in the lattice $E_8$ that are orthogonal to few roots. A main objective of this thesis will be to completely settle this last existence question, for all relevant values of $d$.

For almost all values of $d$, this combinatorial condition can be affirmed by general number-theoretic results. For the remaining cases, we will employ an exhaustive computer search, using the large number of symmetries of the lattice $E_8$ to reduce the necessary amount of computation to manageable proportions.
Chapter 1

Geometry

1.1 K3 surfaces

We will consider only algebraic varieties over \( \mathbb{C} \). In particular, in the following, a surface is an algebraic variety over \( \mathbb{C} \) of dimension 2. One may also consider K3 surfaces over other fields, or complex K3 surfaces that are not algebraic (i.e., can only be defined in the complex-analytic category), but we will not do so.

**Definition 1.1.1.** A K3 surface \( S \) is a surface that is smooth, complete, has trivial canonical bundle, and is regular (i.e., \( H^1(S, \mathcal{O}_S) = 0 \)).

The condition that the canonical bundle be trivial, is the reason we call K3 surfaces a generalisation of elliptic curves. The condition that the surface be regular, excludes precisely the abelian surfaces (i.e., group varieties of dimension 2) (over the complex numbers, these are just the tori).

Examples of K3 surfaces are: smooth quartic surfaces in \( \mathbb{P}^3 \), and smooth intersections of a cubic and a quadric in \( \mathbb{P}^4 \).

1.1.1 Polarisation

There exists no good moduli space of K3 surfaces: one could define a coarse moduli space on a topological level, but it will not be an algebraic variety. The standard way to overcome this, is to look instead at K3 surfaces together with an extra piece of data: a polarisation, a special kind of divisor.

**Definition 1.1.2.** A polarisation of a surface \( S \) is a primitive, pseudo-ample (= nef) divisor on \( S \). The degree of a polarisation \( H \) is its self-intersection number \( H \cdot H \).

For a K3 surface, this self-intersection of a polarisation is always even. We will typically write it as \( 2d \). Alternatively, writing it as \( 2g - 2 \), we call \( g \) the genus of the K3 surface with this polarisation. Sometimes this is confusingly called the genus of the surface itself, but a single K3 surface can have different genera, depending on the choice of divisor.)

1.1.2 The period map and the Torelli theorem

The second integral cohomology group of a K3 surface \( S \), \( H^2(S, \mathbb{Z}) \), with the intersection product, is always isomorphic as a lattice to \( 3U \oplus 2E_8(-1) \) [1, 3.3.(ii)]. We call this the K3 lattice, \( L_{K3} \).

A given polarisation, i.e., a primitive divisor, gives an element of this second cohomology group, corresponding to a point in the lattice \( L_{K3} \) which we call \( h \); if the polarisation has degree \( 2d \), then the orthogonal complement of \( h \) in the lattice is \( L_{2d} := h^\perp \cong 2U \oplus 2E_8(-1) \oplus (-2d) \).

Because the canonical bundle of a K3 surface \( S \) is trivial, it has a nowhere-zero global section: in other words, we may choose a holomorphic 2-form \( \omega_S \in H^0(S, K_S) = H^0(S, \Omega^2_S) \) without zeroes.
By the Hodge decomposition $H^2(S,\mathbb{C}) \cong H^0(S,\Omega^2_S) \oplus H^1(S,\Omega^1_S) \oplus H^2(S,\Omega^0_S)$, we may view $\omega_S$ as an element of $H^2(S,\mathbb{C}) \cong H^2(S,\mathbb{Z}) \otimes \mathbb{C} = L_{K3} \otimes \mathbb{C}$. Taking the $\mathbb{C}$-span of $\omega_S$, we get a point of $\mathbb{P}(L_{K3} \otimes \mathbb{C})$. This is, by definition, the period of $S$. The map from the moduli space of K3 surfaces to the projective space $\mathbb{P}(L_{K3} \otimes \mathbb{C})$, sending $S$ to its period, is called the period map. (Note that taking the $\mathbb{C}$-span makes this map well-defined, i.e., independent of the choice of 2-form $\omega_S$, because $H^0(S,\mathbb{C})$ is spanned over $\mathbb{C}$ by $\omega_S$.)

Now, the Torelli theorem, the crucial tool that allows us to answer questions about the moduli space, states that the period of a K3 surface determines the isomorphism class of the surface. (Observe that this is similar to the classical period map for elliptic curves.) Moreover, the theorem says how we may modify the period map, so it gives a coarse moduli space of polarised K3 surfaces, of fixed degree, say 2$d$.

To do so, we must take as codomain of the period map the quotient space $\mathcal{F}_{2d} := \Gamma \backslash \mathcal{D}_{L_{2d}}$, where $\mathcal{D}_L$, for a lattice $L$, is the subspace of $\mathbb{P}(L \otimes \mathbb{C})$ defined as one of the two connected components of

$$\{ C \alpha \in \mathbb{P}(L \otimes \mathbb{C}) : (\alpha,\alpha) = 0 \wedge (\alpha,\overline{\alpha}) > 0 \},$$

and $\Gamma$ is the subgroup of the lattice automorphism group of $L_{K3}$ of elements that preserve the subspace $\mathcal{D}_{L_{K3}}$, and fix the special point $h$ of the lattice.

At this point it may be helpful to realise, that the appearance of the lattice $E_8$ in this description has some profound consequences. Geometric statements about the moduli space correspond to combinatorial questions about the lattice $E_8$, which in turn may sometimes be rephrased as statements about number theory (by the number-theoretic nature of $E_8$). This explains how a purely geometric question, about the Kodaira dimension of the moduli space of polarised K3 surfaces, is (partly) answered using modular forms and combinatorial facts about lattices.

1.2 Kodaira dimension

The Kodaira dimension of an algebraic variety $X$, roughly speaking, measures the number of dimensions of the variety in which it is negatively curved.

Before we give the definition, it is useful to introduce the canonical ring of the variety:

$$R_X := \bigoplus_{k=0}^{\infty} H^0(X,K_X^\otimes k)$$

(where $K_X$ is the canonical bundle). This is a graded ring; the dimension $P_k := \dim H^0(X,K_X^\otimes k)$ of the $k$-graded part is called the $k$-th plurigenus (note that the usual geometric genus is just the first plurigenus $P_1$), and elements of the canonical ring are called pluricanonical forms.

Now, the Kodaira dimension measures how fast the plurigenus $P_k$ grows as $k$ increases. In fact, we take this as our definition:

**Definition 1.2.1.** The Kodaira dimension of $X$, denoted $\kappa(X)$, is defined as the least integer $m$ such that the sequence $(\frac{P_k}{k^m})_{k=1}^\infty$ is bounded. When all plurigenera are 0 (except $P_0$, which is always 1), we say that $\kappa(X) = -\infty$ (or $-1$).

So, a large Kodaira dimension means that the tensor powers of the canonical bundle have many sections.

A more geometric interpretation of the Kodaira dimension is as follows: if the line bundle $K_X^\otimes k$ is effective, it sets up a rational map from $X$ to projective ($P_k - 1$)-space. The dimension of the image of this map becomes constant as $k$ increases, and then equals $\kappa(X)$. (When $K_X^\otimes k$ is not effective, the image of the map is empty, so we may include this case if we adhere to the convention that the dimension of the empty set is negative.) This interpretation shows that the Kodaira dimension, when it is not negative, lies between 0 and the usual dimension of the variety (inclusive). If the Kodaira dimension is maximal, i.e., $\kappa(X) = \dim X$, then $X$ is called of general type. The suggestion raised by this name has a reason: at least in well-understood low-dimensional cases, the part of the moduli space of varieties with this maximal Kodaira dimension is largest (has the highest dimension).
1.3 The Kodaira dimension of $\mathcal{F}_{2d}$

We can now formulate the main theorem.

**Theorem 1.3.1.** The moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$, is of general type (that is, its Kodaira dimension is $19 = \dim \mathcal{F}_{2d}$), for $d \geq 62$, and for $d \in \{46, 50, 52, 54, 57, 58, 60\}$.

In this section and the next, we will give a sketch of the proof. A more complete treatment can be found in our source for this material, the article by Gritsenko, Hulek and Sankaran [4].

Note that, compared to [4, Theorem 1], we include the case $d = 52$. The method employed in [4] could have found this case as well; it was probably missed due to an administrative oversight.

**Proof.** We need to prove that the powers of the canonical bundle, $(K_{\mathcal{F}_{2d}})^{\otimes k}$, have many sections: to be precise, that this number of sections grows as $k^{19}$ as $k$ goes to infinity.

The first step is to “regularise” the moduli space $\mathcal{F}_{2d}$. By [4, Theorem 2], there is a compactification of $\mathcal{F}_{2d}$ that has relatively benign singularities; resolving those singularities, we get a smooth projective space $\overline{\mathcal{F}}_{2d}$, that has the same Kodaira dimension as $\mathcal{F}_{2d}$.

Next, we try to find modular forms on $\mathcal{D}_{L_{2d}}$ (modular with respect to the group $\Gamma$). Given such a modular form $F$, of weight $19k$, and character $\det^k$ (where $\det : \Gamma \to \mathbb{C}^\times$ is the determinant map), we may form $F(dZ)^{\otimes k} \in H^0(\mathcal{D}_{L_{2d}}, K_{\mathcal{D}_{L_{2d}}}^{\otimes k})$, where $dZ$ is a volume form on $\mathcal{D}_{L_{2d}}$. This pluricanonical form is $\Gamma$-invariant, precisely because $F$ is modular of weight $k$, and it thus descends to a pluricanonical form on $\mathcal{F}_{2d}$, under some restrictions, having to do with the singularities and boundary of the quotient space.

We may construct such modular forms $F$ from two ingredients: firstly, a non-zero cusp form $G$ of weight $a < 19$ and character $\chi$ (where $\chi$ may be arbitrary), that additionally vanishes on the ramification divisor $D$ of the projection $\mathcal{D}_{L_{2d}} \to \Gamma \backslash \mathcal{D}_{L_{2d}}$ (i.e., the subset of points where the action of $\Gamma$ is not so nice); and secondly, a modular form $H$ of weight $(19 - a)k$ and character 1 (where $k$ must be a multiple of some number, depending on $\chi$, but is otherwise arbitrary, so $k$ can be as large as we want). Given such $G$ and $H$, [4, Theorem 1.1] shows that the combination $G^k H$ has all the properties necessary for the associated pluricanonical form to descend to $\mathcal{F}_{2d}$ (note in particular that it has weight $ka + (19 - a)k = 19k$, as needed).

Now we are in good shape: if we are only given the special cusp form $G$, we can use any modular form $H$ of the right weight and character 1 to get the pluricanonical forms we seek; moreover, different such $H$ give different pluricanonical forms (for a fixed choice of $G$). By the general theory of modular forms, the space $M_{(19-a)k}(\Gamma)$ of such forms $H$ is “big enough”: its dimension grows as $k^{19}$, as $k$ increases (here, the exponent 19 appears as the dimension of the space $\mathcal{D}_{L_{2d}}$, on which the modular forms live). We conclude that, once we have a cusp form of “low weight” (i.e., smaller than 19) vanishing on $D$, we can construct of the order of $k^{19}$ different elements of $H^0(\mathcal{F}_{2d}, K_{\mathcal{F}_{2d}}^{\otimes k})$, for any $k$ that is a multiple of a fixed integer, so $\mathcal{F}_{2d}$ must have Kodaira dimension 19, and hence the moduli space $\mathcal{F}_{2d}$ has this Kodaira dimension as well.

1.4 The low-weight cusp form

It remains to obtain a low-weight cusp form $G$ that is zero on $D$. This is accomplished in [4] by using the so-called Borcherds function $\Phi_{12}$ (see [2] for its construction and properties). This is a modular form of weight 12 and character $\det$ on the space $\mathcal{D}_{L_B}$ with respect to the group $\Gamma'$ (in fact, it is the unique such form). Here, $L_B$ is the lattice $2\mathbb{Z} \oplus 3E_8(-1)$, and $\Gamma'$ is the group of lattice automorphisms of $L_B$ that preserve $\mathcal{D}_{L_B}$. In [4, Section 4], it is explained that a modular form on the space $\mathcal{D}_{L_B}$ may actually be viewed as a function $\mathcal{D}_{L_B}^\times \to \mathbb{C}$ on the affine cone $\mathcal{D}_{L_B}^\times$ over $\mathcal{D}_L$; we will use this viewpoint in the following.

Well now, this Borcherds function is a modular form on $\mathcal{D}_{L_{2d}}^\times$, but we need a modular form on $\mathcal{D}_{L_{2d}}$. Recall that the underlying moduli space is $L_{2d} = 2\mathbb{Z} \oplus 2E_8(-1) \oplus (-2d)$, so,

\[1\] Recall that the affine cone $X^\times$ of a subset $X \subseteq \mathbb{P}(V)$ of the projective space of a vector space $V$, is the inverse image of $X$ under the natural map $V \setminus \{0\} \to \mathbb{P}(V)$. In other words, $X^\times := \{v \in V \setminus \{0\} : Cv \in X\}$. 

5
given a lattice point \( v \in \mathbb{E}_8(−1) \) of length \( v^2 = −2d \), we may embed \( L_{2d} \) in \( L_B \), by sending the two copies of \( U \) and \( \mathbb{E}_8(−1) \) to themself, and sending the generator of \( \langle −2d \rangle \) to \( v \) in the third copy of \( \mathbb{E}_8(−1) \) in \( L_B \). This embedding naturally induces an embedding of \( D_{L_{2d}} \) in \( D_{L_B} \), and hence of \( D_{L_{2d}}^* \) in \( D_{L_B}^* \).

At this point, we would like to simply restrict the Borcherds function to \( D_{L_{2d}}^* \) along this embedding (in other words, use the pullback of \( \Phi_{12} \) as our low-weight cusp form). However, this restriction may be zero. To remedy this, we divide \( \Phi_{12} \) by the defining functions of its zero divisor, before pulling it back. We will need a definition.

**Definition 1.4.1.** Given \( r \in L \), a root in a lattice, we define the Heegner divisor \( \mathcal{H}_r \) on the space \( D_L^* \) as the divisor given by the equation \( \langle Z, r \rangle = 0 \) (where \( Z \in D_L^* \); note that the affine cone \( D_L^* \) is a subset of the inner product space \( L \otimes \mathbb{C} \)). We denote the defining function \( Z \mapsto \langle Z, r \rangle \) by \( f_r \).

Remark that \( r \) and \( −r \) give the same Heegner divisor. Interestingly, the zero divisor of \( \Phi_{12} \) is given just by these Heegner divisors:

**Lemma 1.4.2 ([2]).** The zero divisor of the Borcherds function is given by

\[
\text{div}_0(\Phi_{12}) = \sum_{\{r, −r\} \subset L_B \atop r^2 = −2} \mathcal{H}_r . \tag{1.3}
\]

Now, the restriction of \( \Phi_{12} \) to \( D_{L_{2d}}^* \) is zero if and only if the image under the embedding \( D_{L_{2d}}^* \hookrightarrow D_{L_B}^* \) is contained in the Heegner divisors \( \mathcal{H}_r \), for roots \( r \in L_B \). Moreover, this embedded image is contained in \( \mathcal{H}_r \) precisely when \( r \) is orthogonal to the image of the lattice embedding \( L_{2d} \hookrightarrow L_B \). Recalling the nature of this embedding, this in turn happens if and only if \( r \) is actually contained in the third \( \mathbb{E}_8(−1) \)-summand, and \( r \) is orthogonal to \( l \) (recall that \( l \in \mathbb{E}_8(−1) \) is the “parameter” defining the lattice embedding).

**Definition 1.4.3.** Given \( l \in \mathbb{E}_8(−1) \), we define \( R_{\mathbb{E}_8(−1)}(l) \) to be the set of \( r \in \mathbb{E}_8(−1) \) such that \( r^2 = −2 \) and \( \langle r, l \rangle = 0 \).

In other words, \( R_{\mathbb{E}_8(−1)}(l) \) is the set of roots orthogonal to \( l \). (We will see this definition again in the next chapter (2.1.4), for any lattice \( L \).)

We conclude, that the following modified modular form:

\[
\Phi_{12} \prod_{\{r, −r\} \subset R_{\mathbb{E}_8(−1)}(l)} f_r \tag{1.4}
\]

will have a nonzero restriction to \( D_{L_{2d}}^* \). This restriction is called the *quasi-pullback* of the Borcherds function.

The disadvantage (for our purpose) of this quasi-pullback is, that dividing by the functions \( f_r \) has changed its weight, adding 1 for every such factor. The weight of the quasi-pullback is thus \( 12 + \frac{|R_{\mathbb{E}_8(−1)}(l)|}{2} \); so, to get a modular form of weight less than 19, we must have that \( |R_{\mathbb{E}_8(−1)}(l)| \leq 12 \). Moreover, in order for the form to be a cusp form, we must divide by at least one factor \( f_r \) ([4, Theorem 6.2.(ii)]), so we must also demand that \( |R_{\mathbb{E}_8(−1)}(l)| \geq 2 \). Under these conditions, though, we have constructed the desired low-weight cusp form \( G \).

(A construction similar to the above, allows to conclude that the Kodaira dimension of \( F_{2d} \) is nonnegative, under the milder condition that \( 2 \leq |R_{\mathbb{E}_8(−1)}(l)| \leq 14 \).)

We are therefore left with the following question (we convert from the negative-definite \( \mathbb{E}_8(−1) \) to the positive-definite \( \mathbb{E}_8 \), for ease of notation).

**Question 1.4.4.** Given \( d \in \mathbb{N}_+ \), does there exist a vector \( v \in \mathbb{E}_8 \) of norm squared \( 2d \), such that the number of roots of \( \mathbb{E}_8 \), orthogonal to \( v \), is at least 2 and at most 12?
This is the main question that is answered in the next chapter: see section 2.3.1. The answer is: such a vector $v$ exists if and only if $d \geq 62$ or $d \in \{46, 50, 52, 54, 57, 58, 60\}$. For those values of $d$, we may conclude that $\mathcal{F}_{2d}$ has Kodaira dimension 19: it is of general type.
Chapter 2

Lattices

2.1 Preliminaries

In the course of examining the moduli space of K3 surfaces, we will encounter some very important lattices.

Definition 2.1.1. A lattice is a free $\mathbb{Z}$-module $L$ of finite rank (in other words: a finitely generated free abelian group), together with a symmetric bilinear map $\langle \cdot , \cdot \rangle : L \times L \to \mathbb{Z}$, which we call the inner product.

In other words, a lattice is a discrete subgroup of a finite dimensional vector space with inner product, with the special property that all inner products are integral. In the context of lattices, the quantity $\langle x, x \rangle$, for $x$ an element of a lattice, is called the length of $x$. (Note that this is different from the usual notion of “length” in a vector space with inner product; the latter is the square root of the former.)

A morphism of lattices $f : A \to B$ is a $\mathbb{Z}$-linear (i.e. additive) map that is orthogonal, meaning $\langle a_1, a_2 \rangle_A = \langle f(a_1), f(a_2) \rangle_B$. In particular, isomorphisms of lattices are precisely the orthogonal linear isomorphisms.

Example 2.1.2. For any integer $c$, we define the lattice $\langle c \rangle$ to be the free $\mathbb{Z}$-module generated by a single element $x$, with inner product $\langle x, x \rangle = c$.

Example 2.1.3. The hyperbolic plane lattice $U$ is the free $\mathbb{Z}$-module generated by two elements $e, f$, with inner products $\langle e, e \rangle = \langle f, f \rangle = 0$ and $\langle e, f \rangle = 1$.

We can take the (orthogonal) direct sum $A \oplus B$ of two lattices $A$ and $B$: as a $\mathbb{Z}$-module, it is the direct sum of the underlying modules of $A$ and $B$, and its inner product is given by $\langle a_1 \oplus b_1, a_2 \oplus b_2 \rangle = \langle a_1, a_2 \rangle_A + \langle b_1, b_2 \rangle_B$. The sum of $n$ copies of $L$ is written as $nL$.

We may scale the inner product of a lattice: given a lattice $L$, and an integer $c$, we define $L(c)$ to be the same module as $L$, but with inner product $\langle x, y \rangle = c \langle x, y \rangle_L$.

Definition 2.1.4. We define the roots of a lattice $L$ to be the elements of minimal\footnote{For a lattice that is not positive-definite, “minimal” must be read as “of minimal absolute value”.} nonzero length. We denote the set of roots of $L$ by $R(L)$.

Also, we will denote by $R_L(v)$ the set of roots in $L$ that are orthogonal to $v$.

Note that, unless $R_L(v)$ is an empty set, it is the same as the set of roots in the sublattice $v^\perp$ of points in $L$ orthogonal to $v$. 

2.1.1 Discriminant

One fundamental invariant of a lattice is the discriminant: it measures the size of the cells of the lattice. (A cell is a fundamental domain for the translation action on the lattice.)

**Definition 2.1.5.** A basis of a lattice is a basis of the underlying free \( \mathbb{Z} \)-module. The **discriminant** of a lattice is the absolute value of the determinant of a basis. (Note that this is equal to the volume of the parallelepiped spanned by a basis.) A lattice is called **unimodular** if the discriminant equals 1.

This definition of the discriminant is sound, in the sense that it does not depend on the particular basis chosen: the set of ordered bases (written as matrices) is acted on transitively by the general linear group over the integers \( GL(r, \mathbb{Z}) \) (where \( r \) is the rank of the lattice) by right multiplication, and elements of this group all have determinant \( \pm 1 \) (since this determinant must be an invertible element of \( \mathbb{Z} \)).

2.1.2 The lattices \( E_8 \) and \( E_7 \)

The lattice \( E_8 \) is the unique positive-definite, even\(^2 \), unimodular lattice of rank 8. It is highly symmetric; we will examine its symmetry group in the next section.

The condition on the rank is not as arbitrary as it may sound: any even unimodular lattice has rank divisible by 8 (see for instance [3]). Another famous member of this family is the Leech lattice, of rank 24.

We will employ a more basic definition, suitable for calculations.

**Definition 2.1.6.** The lattice \( E_8 \) is the subgroup of \( \mathbb{R}^8 \) of elements such that

- all coordinates are integer or all coordinates are properly half-integer; and
- the sum of all 8 coordinates is an even integer,

with inner product the restriction of the Euclidean inner product of \( \mathbb{R}^8 \).

It arises in Lie theory as the root lattice of the exceptional Lie algebra \( E_8 \). The set \( R(E_8) \), of roots of \( E_8 \), is given by all permutations of

\[
\left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right), \\
\left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right), \\
\left( \frac{1}{2} \frac{1}{2} \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right), \\
\left( \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right), \\
\left( -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right), \\
\left( 1 1 0 0 0 0 0 0 \right), \\
\left( 1 -1 0 0 0 0 0 0 \right), \text{ and} \\
\left( -1 -1 0 0 0 0 0 0 \right).
\]

We see that there are 240 roots. (This is of course a root system of type \( E_8 \) [5, section 2.10].)

We will at some points make use of the following basis for \( E_8 \):

\[
\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\
\alpha_2 = e_1 + e_2, \\
\alpha_i = -e_{i-2} + e_{i-1}, \text{ for } 3 \leq i \leq 8.
\]

However, coordinate descriptions will be with respect to the standard basis \((e_i)_{i=1}^8\), unless stated otherwise.

\(^2\)This means that the length of any element is even.
There is a related lattice of rank one lower than $E_8$, appropriately called $E_7$. We define it as the sublattice of $E_8$ of vectors orthogonal to $e_7 + e_8$ (equivalently, the sublattice generated by $\alpha_i$ for $1 \leq i \leq 7$). This embedding of $E_7$ in $E_8$ will allow us to reuse many results and methods developed for $E_8$.

### 2.2 Symmetry

From its origin as the root system of the corresponding Lie algebra, it can be derived that the lattice $E_8$ has automorphism group $W(E_8)$, the Weyl group of type $E_8$ (a Coxeter group of order 696729600) [5, section 2.11]. In the following, we will denote this group by $G$. This symmetry group plays two roles:

- We will use this high degree of symmetry of $E_8$ to cut down on our calculations. Although not of prime mathematical interest, this changed the running time of the main search from the order of months to the order of seconds.
- We will also be interested in the number of lattice points up to symmetry, having a certain property (in other words, we want to count the $G$-orbits intersecting certain subsets of the lattice).

Both roles require us to develop some practical computational methods for working with $E_8$ up to automorphisms. In particular, we would like to have the following:

A. a fast algorithm to enumerate the lattice points of a given length, where we require only a single representative from each $G$-orbit (although more are allowed); and

B. a fast algorithm to count the number of $G$-orbits intersecting a given set.

Many of our lattice computations will have the following structure:

1. Generate lattice points of given length (algorithm A)
2. Filter lattice points according to some criterium (typically, some property of the number of orthogonal roots)
3. Accumulate results (either a simple exists/does not exist, or a count of orbits (algorithm B))

The amount of time we should invest in eliminating redundancy in the output of algorithm A, depends on the complexity of the middle filtering step.

#### 2.2.1 Symmetries of type $D_8$

Some of the lattice automorphisms of $E_8$ are particularly obvious: the permutations of the 8 coordinates (these form a subgroup $S_8$ of order $8!$), and products of an even number of sign changes of the coordinates (these form a subgroup $C^+$ of order $2^7$).

**Definition 2.2.1.** We define $H$ to be the subgroup of $G$, generated by the two subgroups $S_8$ and $C^+$. Note that $H$ is a semidirect product of $S_8$ and $C^+$: their intersection is trivial, and the conjugation of a sign change by a permutation is again a sign change. Therefore $H$ has order $2^7 \cdot 8! = 5160960$. We note now that it has index $[G : H] = 696729600 / 5160960 = 135$; we will use this later on. We will use $H$ as a first approximation to the automorphism group of $E_8$: it gives a large part of the savings in computations, with little effort (because these symmetries are easy to express in our coordinate description of $E_8$).

To explain the title of this subsection: $H$ is itself a Weyl group, of type $D_8$ ([5, section 2.10]).
We will use the term “complete system of representatives” for an action of a group \(G\) on a set \(X\); this is simply a complete system of representatives for the equivalence relation induced by the group action, partitioning \(X\) along its \(G\)-orbits.

**Definition 2.2.2.** Let an equivalence relation on a set \(X\) be given. A complete system of representatives gives rise to a projection, a map \(\pi : X \to X\), that sends an element to the elected representative of its equivalence class.

Inversely, we may define a complete system of representatives by giving the projection \(\pi : X \to X\) it induces, provided that \(\pi\) has the following properties:

- it is a projection (i.e. \(\pi^2 = \pi\));
- its fibers are precisely the equivalence classes of \(X\).

Now, to exploit the symmetries expressed by \(H\), we need an effective way to decide if two lattice points are in the same \(H\)-orbit. We proceed to show that the accompanying projection can be computed by a simple sorting procedure.

**Proposition 2.2.4.** The projection function \(\pi : E_8 \to E_8\), induced by the lex representatives, may be described as follows: given \(x\), sort the coordinates of \(x\) by decreasing absolute value (for pairs of coordinates with opposite signs, arbitrarily put negatives before positives); call the resulting vector \(y\). Then

\[
\pi(x) = \sum_{i=1}^{7} |y_i| e_i + \left( \prod_{i=1}^{7} \text{sign } y_i \right) y_8 e_8 .
\]

(In words: flip the sign of all coordinates (except the last) of \(y\) that are negative; flip the sign of the last coordinate if you would otherwise have flipped an odd number of signs.)

**Proof.** Denote by \(\pi'\) the function described by the proposition; we want to show that \(\pi' = \pi\). Now, \(\pi'\) acts by first permuting coordinates, then changing an even number of signs of coordinates, so \(\pi'(x) \in Hx\). We still need to prove that for any \(h \in H\), \(hx\) is not lexicographically greater than \(\pi'(x)\) – it then follows that \(\pi'(x) = \pi(x)\).

Note that \(hx\) and \(\pi'(x)\), being in the same \(H\)-orbit, have the same set of coordinates, up to signs. We may assume that \(hx\) is unequal to \(\pi'(x)\); let \(j\) be the index of the first position where they differ. Call the (common) coordinates of \(hx\) and \(\pi'(x)\) up to position \(j - 1\) the “high” coordinates, and all others the “low” ones. We distinguish two cases:

**Case: \(j < 8\)** We remarked that the set of coordinates of \(hx\) and \(\pi'(x)\) is the same up to individual sign changes; since the high coordinates are pairwise equal, the two sets of low coordinates are also the same, up to individual signs. Then:
– \( \pi'(x)_j \) is not abs-less than any of the low coordinates (by the sorting part of \( \pi' \)); and
– \( \pi'(x)_j \) is positive (by the sign change part of \( \pi' \), since \( j < 8 \)) (it cannot be zero, since then, because of the sorting, all low coordinates would be zero, implying \( \pi'(x) = hx \)), so \( \pi'(x)_j \) is in fact greater than or equal to any other low coordinate. Since \((hx)_j \) is different from \(\pi'(x)_j \), it must be lower, so \(hx \) is smaller in the lexicographical order.

Case: \( j = 8 \) All high coordinates are pairwise equal, so the low coordinates – in this case, only the last – of \(hx\) and \(\pi'(x)\) must be equal up to sign. Now observe that no element of \(H\) can change only a single sign, unless one of the coordinates is zero, but then the last coordinate of \(\pi'(x)\) is certainly zero, by definition of \(\pi'\), implying \(\pi'(x) = hx\). Therefore this case cannot occur.

\[\square\]

2.2.2 The full symmetry group

For some purposes (e.g., counting G-orbits), we need an effective way to decide whether two lattice points belong to the same G-orbit. This will be somewhat harder and messier, compared with the H-orbits, because those symmetries are not as easily reflected in our simple coordinate description of the lattice.

To get a handle on the full symmetry group, we compute a complete system of representatives for the left action of \(H\) on \(G\) (by multiplication) – in other words, a set of (left) coset representatives of \(H\) in \(G\). We will do so using a naive Monte Carlo method:

Algorithm 2.2.5 (Computation of coset representatives of \(H\) in \(G\)). Maintain a set of representatives chosen so far; this set is initially empty. While the number of chosen representatives is less than \([G : H] = 135\):

- pick a random element, say \(g\), of \(G\) (see 2.2.2.1 below);
- compute, for every representative \(r\) already chosen, whether \(g \sim_H r\) (using the efficient method described in 2.2.2.2). If \(g\) is in the H-orbit of any such \(r\), disregard this choice of \(g\) and continue; otherwise, add \(g\) to the list of chosen representatives.

This procedure progresses increasingly slowly, as more representatives are found; the chance of a random element belonging to a new coset decreases, down to \(\frac{1}{135}\) for the last one. Efficiency is not important here, though: we only have to perform this computation once, storing the resulting set of representatives for later use.

2.2.2.1 Generating random elements of \(G\)

Algorithm 2.2.5 depends on the generation of random elements of the group \(G\). We know how to generate random permutations (see for instance [6, pp. 145–146]) and sign changes; this is enough to generate elements of \(H\). To make the step to the full group \(G\), we focus for a moment on a particular element of \(G \setminus H\).

Definition 2.2.6. The Hadamard matrix \(m\) is the following \(4 \times 4\)-matrix:

\[
m = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

Note that \(m^2 = 1\) (so the Hadamard matrix is invertible, of order 2), and that \(m^\top = m\) (it is symmetric).
Lemma 2.2.7. We have \( m \oplus m \in G \setminus H \). That is, viewed as an \( 8 \times 8 \)-matrix, \( m \oplus m \) is a lattice automorphism of \( E_8 \), but it is not a product of permutations and even sign changes.

Proof. By the properties we noted, \( m^T m = mm = 1 \), so \( m \) is an orthonormal matrix, as is \( m \oplus m \). Therefore the linear isomorphism it represents, preserves the standard inner product. To conclude that it is a lattice automorphism, it remains to prove that it maps the lattice \( E_8 \) onto itself. We verify this by computing the effect of \( m \oplus m \) on the lattice basis \( \alpha_i \):

\[
\begin{align*}
(m \oplus m) \alpha_1 &= \left( -\frac{1}{2} \frac{1}{2} \frac{1}{2} -\frac{1}{2} -\frac{1}{2} \right) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \\
(m \oplus m) \alpha_2 &= (1 0 1 0 0 0 0 0) = \alpha_2 + \alpha_4, \\
(m \oplus m) \alpha_3 &= (0 -1 0 -1 0 0 0 0) = -\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \\
(m \oplus m) \alpha_4 &= (0 1 -1 0 0 0 0 0) = -\alpha_4, \\
(m \oplus m) \alpha_5 &= (0 -1 0 1 0 0 0 0) = \alpha_4 + \alpha_5, \\
(m \oplus m) \alpha_6 &= (-\frac{1}{2} \frac{1}{2} \frac{1}{2} -\frac{1}{2} -\frac{1}{2} \frac{1}{2} \frac{1}{2}) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
(m \oplus m) \alpha_7 &= (0 0 0 0 0 -1 0 -1) = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 6\alpha_4 - 5\alpha_5 - 4\alpha_6 - 3\alpha_7 - \alpha_8, \\
(m \oplus m) \alpha_8 &= (0 0 0 0 1 -1 0 0) = -\alpha_8.
\end{align*}
\]

This shows that \( m \oplus m \) maps \( E_8 \) into itself; with respect to the \( \alpha \)-basis, we see that it has determinant

\[
\begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
1 & 1 & -1 & 0 & 0 & 2 & -3 & 0 \\
2 & 0 & -1 & 0 & 0 & 3 & -4 & 0 \\
2 & 1 & -1 & -1 & 1 & 4 & -6 & 0 \\
1 & 0 & -1 & 0 & 1 & 3 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1
\end{vmatrix} = 1 , \quad (2.5)
\]

so, the image of the \( \alpha \)-basis is again a basis of the lattice, and therefore the image of \( m \oplus m \) is the whole lattice \( E_8 \), and \( m \oplus m \in G \). It cannot be generated by permutations and even sign changes, however, because any such matrix has only one nonzero entry on every row and every column, which does not hold for \( m \oplus m \). We conclude that \( m \oplus m \in G \setminus H \).

Now, the utility of \( m \oplus m \) lies in the following: together with the subgroup \( H \), it generates the whole of \( G \). We will not prove this directly, but it follows from the computations we have performed (specifically, from the fact that generating random elements of \( G \) using only \( H \) and \( m \oplus m \), we get elements from all 135 \( H \)-cosets).

We generate random elements of \( G \) using the following scheme: first, generate a random nonnegative integer \( r \). For every \( i \in \{1, \ldots, r\} \), generate a random element \( h_i \in H \). Then the resulting element of \( G \) is the product \( \prod_{i=1}^{r} (m \oplus m)h_i \).

2.2.2.2 Deciding \( H \)-coset equality on \( G \)

The second important part of algorithm 2.2.5, is the procedure that decides whether two given elements of \( G \) belong to the same \( H \)-coset. This is similar to, but different from the work in section 2.2.1; there we had to decide whether two lattice points are in the same \( H \)-orbit.

Note that \( H \) acts on \( G \) by permuting the set of rows, and changing (an even number of) signs of complete rows. Therefore, to fix a complete system of representatives, we must fix an order of the rows, and signs for the rows (but only changing an even number of signs).

Define the following function \( \pi : G \rightarrow G \): first sort the rows of the matrix in nondecreasing lexicographical-absolute value\(^5\). Then change the signs of all rows (except the first) that are lexicographically-absolute value \( \alpha_8 \).

\(^5\)Note that this is only a partial ordering on the set of all rows of matrices in \( G \). However, within a single matrix, there cannot be two rows that are equal up to a total sign change, because then the matrix would be singular, which does not happen for elements of \( G \).

13
negative, and change the sign of the first row if you would otherwise have changed an odd number of rows.

**Proposition 2.2.8.** The function \( \pi \) defines a complete system of representatives for \( H \) in \( G \).

**Proof.** We need to prove that \( \pi \) is a projection, and that its fibres are exactly the \( H \)-cosets.

Let us write \( \pi = c \circ \sigma \), where \( \sigma \) is the function that just sorts the rows, and \( c \) is the part that changes the signs. Now, the crucial observation (and the reason we sorted on lex-abs), is that

\[
\sigma \circ c = c' \circ \sigma, \tag{2.6}
\]

where \( c' \) is some function that only changes signs of rows (albeit in a different way then \( c \)). This is because \( \sigma \) sorts on lex-abs, so first changing some signs and then sorting gives the same order as sorting right away, and the results can differ only in (an even number of) signs of rows.

Using this fact, the projection property is simple to deduce:

\[
\pi^2 = c \circ \sigma \circ c \circ \sigma \\
= c \circ c' \circ \sigma \circ \sigma \\
= c \circ \sigma \circ \sigma \\
= c \circ \sigma \\
= \pi. \tag{2.7}
\]

Now, it is immediate from the definition of \( \pi \) that elements are in the same \( \pi \)-fibre iff they are the same up to permutation of the rows and an even number of sign changes of the rows; the latter happens precisely when these elements are in the same \( H \)-orbit.

\( \square \)

### 2.3 Lattice computations

#### 2.3.1 Lattice points with few orthogonal roots

The main computation we undertook was to answer the following question.

Given \( d \in \mathbb{N}_+ \), does there exist \( v \in E_8 \) of length \( v^2 = 2d \), such that \( 2 \leq |R_{E_8}(v)| \leq 12? \)

Recall that \( R_{E_8}(v) \) is the set of roots of \( E_8 \) that are orthogonal to \( v \) (definition 2.1.4).

Let us refer to the answer to this question as \( G(d) \).

If this is the case, then we know that the moduli space \( \mathcal{F}_{2d} \) of K3 surfaces with a polarisation of degree \( 2d \), is of general type (see sections 1.3 and 1.4).

This question is solved for all but finitely many \( d \) in [4]. For a single value of \( d \), it can be solved by “brute force”: enumerate all vectors of length \( 2d \) (there are only finitely many), and check one by one if they satisfy the above properties (this amounts to taking the inner product with the 240 roots in \( E_8 \)). A much more efficient algorithm is possible, that uses the partial symmetry group \( H \) of the lattice to limit the number of lattice points of which the root set needs to be examined. This algorithm is fully described in [7, section 2.1.B].

The idea is as follows: using proposition 2.2.4, we know that, when enumerating \( G \)-orbits, it suffices to generate all vectors that are lexicographically sorted. It is reasonably straightforward to enumerate all lexicographically sorted vectors of given length \( 2d \), using a recursive procedure (this is detailed in [7]). For each of these generated vectors \( v \), take the inner product with a prepared set of “positive” roots (i.e., one from each set of two opposite roots); as soon as you encounter \( 7 > \frac{12}{2} \) ones that are orthogonal to \( v \), we may dismiss this \( v \); on the other hand, if, after taking all inner products, the number of positive roots orthogonal to \( v \) is at least 1 and at most 6, we have found a \( v \) that asserts \( G(d) \).
We implemented this algorithm, and obtained the following results. For \( d \leq 45 \), \( G(d) \) does not hold. For \( d \geq 62 \), \( G(d) \) does hold (we only needed to verify this for \( d \leq 143 \); for higher values of \( d \), we assume this result from [4]). For the “interesting region” in-between, we get (\( \top \) means true, \( \bot \) means false)\[
\begin{array}{cccccccccccc}
d & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 & 61 \\
G(d) & \top & \bot & \bot & \bot & \top & \bot & \top & \bot & \top & \bot & \bot & \top & \top & \bot & \bot & \bot \\
\end{array}
\]

This confirms the results from [4]: the moduli space of polarised K3 surfaces is of general type when the polarisation of degree 2 satisfies \( d \geq 62 \), or \( d \in \{46, 50, 52, 54, 57, 58, 60\} \). (Note that we do not claim the opposite; there may be some more values for \( d \) such that the moduli space is of general type, but this particular method, of constructing modular forms using a lattice point \( v \in E_8 \) with special combinatorial properties, cannot determine those.) The case \( d = 52 \) is not mentioned in [4], but this is probably due to a bookkeeping error, not to the different method employed there.

### 2.3.1.1 Generalisations

There is a natural generalisation of the question \( G(d) \): we may ask which number of orthogonal roots may occur, given a specified length \( v^2 = 2d \) of the lattice point. In other words, what is the set \( \{ |R_{E_8}(v)| : v \in E_8, v^2 = 2d \} \) for different \( d \in \mathbb{N}_+ \)? (This set is called the root type of \( d \).)

We can use the methods we developed to compute the answer to this question, for any given \( d \in \mathbb{N}_+ \). Simply enumerate the \( G \)-orbits (or actually the \( H \)-orbits, so we have some redundancy), using the same method as above. For every \( v \) so generated, compute the inner products of \( v \) with the positive roots; from this, deduce the number of roots orthogonal to \( v \). Accumulate the set of numbers one gets in this way, for all generated \( v \); after all \( v \) are accounted for, the resulting set is the root type of \( d \).

The resulting list of root types, for increasing values of \( d \), contains a lot of information about the lattice \( E_8 \). Some of this allows a number-theoretic interpretation; for instance, the largest possible set of orthogonal roots (126 = \( |R(E_7)| \)), occurs in the root type of \( d \) if and only if \( d \) is a perfect square.

#### G-orbits

A different generalisation emerges, when we ask not if a lattice point \( v \) exists such that \( 2 \leq |R_{E_8}(v)| \leq 12 \), and \( v^2 = 2d \), but how many such \( v \) exist. Perhaps more naturally, we might ask instead how many exist up to symmetry of the lattice.

We can compute this latter number as follows. First of all, compute a complete system of representatives for \( H \) in \( G \) (see algorithm 2.2.5); let’s call the set of representatives \( R \). Then enumerate the \( G \)-orbits of lattice points of the given length \( 2d \) (redundancy is allowed at this point, so we’ll actually enumerate the \( H \)-orbits, because this is easier). Filter out any points that do not have the wanted number of orthogonal roots. Finally, we need to compute how many different \( G \)-orbits are represented by the resulting list. We do so by using a complete system of representatives for the action of \( G \) on \( E_8 \); for each lattice point in the list, compute its projection along this choice of representatives; then filter the resulting list of lattice points for duplicates; the length of the resulting list is the wanted number of \( G \)-orbits.

It remains to find a complete system of representatives for the action of \( G \) on \( E_8 \) that is as easy to compute as possible. Recall that we already defined a complete system of representatives for the action of the subgroup \( H \) on \( E_8 \), where the projection function can be computed by sorting and changing some signs (see proposition 2.2.4). We may extend this in a naive way to a complete system of representatives for the whole \( G \)-action, by taking as the projection\[
\pi_G(v) := \max_{r \in R} \pi_H(r \cdot v) ,
\]
where \( R \) is the complete system of representatives for \( H \) in \( G \), and \( \pi_H \) is the projection for the lex-representatives. Note that computing \( \pi_G(v) \) requires 135 invocations of the function \( \pi_H \) (and
traversing the resulting list to find the lexicographically-largest element); because $\pi_H$ can be computed very efficiently (by only a little sorting, and a few sign changes), this is still doable for relatively large numbers of lattice points (for very large numbers of points, i.e. high values of $d$, it may be too slow).
Bibliography


