Interpolation for extensions of S5-squared

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Bachelor Thesis

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Abstract

Since its introduction the Craig interpolation property has become a standard point of investigation for any logic. In this thesis we examine the related turnstile interpolation property for certain extensions of the modal logic $S5^2$.

In particular, we use the equivalent semantic criterion of amalgamation in modal algebras to provide a full classification of those extensions of $S5^2$ generated by height two frames, that have the turnstile interpolation property.
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1 Introduction

In this thesis we investigate an interpolation property for certain extensions of the product logic $S5^2$. The notion of an interpolation property was first introduced and proved for first order logic by Craig in 1957 [5]. This property, known as Craig interpolation, holds for a logic $L$ when for any pair of formulas $\varphi$ and $\psi$ with $\vdash_L \varphi \to \psi$, there is an interpolant $\chi$ in the shared language of $\varphi$ and $\psi$ such that $\vdash_L \varphi \to \chi$ and $\vdash_L \chi \to \psi$. We examine a variant of this property known as turnstile interpolation, which requires an interpolant $\chi$ for formulas $\varphi, \psi$ with $\varphi \vdash_L \psi$, such that $\varphi \vdash_L \chi$ and $\chi \vdash_L \psi$.

Since it was first introduced, the Craig interpolation property has been investigated for a variety of logics, e.g. intuitionistic and modal logics. Two big names in this investigation are Gabbay and Maksimova. Gabbay proved that the modal systems $K$, $K4$, $T$ and $S4$ have the Craig interpolation property [10]. Maksimova showed that the Craig interpolation property in modal logics is equivalent to the (super)amalgamation property in varieties corresponding to these logics [14, 15, 16], a result we will use extensively. Two more important results by Maksimova are the complete characterisation of those propositional logics intermediate between the intuitionistic and classical that have the Craig interpolation property, as well as the fact that there are at most 37 normal extensions of the modal logic $S4$ that could have this property.

Interest in the logic $S5^2$ arises from Tarski’s algebraization of first order logic. This project, carried out by Tarski and his students, aimed to provide an algebraic analog of first-order logic (FOL), the so-called cylindric algebras [11].

The logic $S5^2$ is one of the cylindric modal logics. These logics, introduced by Venema in [18], are the modal equivalent of cylindric algebras. They correspond to certain fragments of FOL, in particular $S5^n$ corresponds to the $n$-variable fragment of FOL. The logic $S5^2$ is of special interest because it is, in a sense, the largest ‘nice’ fragment of its kind. This is because it, like $S5$, is decidable, has the finite model property and is finitely axiomatizable, amongst other things. The logic $S5^3$ already does not have some of these ‘nice’ properties anymore, as was shown by Maddux in [13] and Kurucz in [12].

In this thesis we specialise the methods used by Maksimova to a class of extensions of $S5^2$ that are generated by specific types of frames. We manage to provide a full classification of logics in this class that have the turnstile interpolation property.

This thesis is divided up into four chapters. Chapter 2 introduces the basic concepts of modal logic, universal algebra, and duality theory. Chapter 3 introduces the notions of interpolation and amalgamation and their equivalence. Chapter 4 defines product frames, and on that basis, product logics. Finally, chapter 5 gives a partial classification of those normal extensions of $S5^2$ with the turnstile interpolation property. We finish with a survey of proved results and possible future work in chapter 6 and finally, chapter 7 is a popular summary.
2 Preliminaries

This chapter provides the necessary background knowledge for the rest of this thesis. In an attempt to be as self-contained as possible, only some knowledge of basic propositional logic is assumed.

We first introduce the basics of modal logic, followed by universal (and in particular, modal) algebra. The final section of this chapter discusses duality theory for finite modal algebras and Kripke frames.

2.1 Modal Logic

2.1.1 Modal Languages

A (propositional) modal language is an extension of the propositional language with one or more modal operators. These operators can have arbitrary (finite) arity but in this thesis we will only examine languages with unary modal operators. We will mostly be concerned with a bi-modal language with two unary modal operators. This language is defined by the BNF:

$$\varphi ::= p \mid \bot \mid \neg \varphi \mid \varphi \lor \psi \mid \Diamond_1 \varphi \mid \Diamond_2 \varphi;$$

where $p$ ranges over some set of propositional variables $\Phi$. There are modal operators $\Box_1, \Box_2$ dual to the $\Diamond_i$ defined by $\Box_i \varphi := \neg \Diamond_i \neg \varphi$ for $i \in \{1, 2\}$. As a matter of convention, we will use the following abbreviations:

$$\top := \neg \bot \neg,$$

$$\varphi \land \psi := \neg (\neg \varphi \lor \neg \psi),$$

$$\varphi \rightarrow \psi := \neg \varphi \lor \psi,$$

$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \leftrightarrow \varphi).$$

2.1.2 Semantics

Notions of truth and validity in modal logic are defined via so-called Kripke frames and Kripke models, we will often refer to these as just frames and models.

**Definition 2.1.** A Kripke frame $\mathcal{F}$ for bimodal language is a triple $(W, R_1, R_2)$ where $W$ a set and $R_1$ and $R_2$ are binary relations on $W$.

The elements of $W$ have many names that are used interchangeably, for example: worlds, nodes, states, and points.
To define a semantics we need to extend our frames with valuations. Just like in the propositional case, these tell us which propositional variables are true. The difference is that now they do so for every world in \( W \). A frame together with a valuation is called a model.

**Definition 2.2.** A Kripke model \( \mathcal{M} \) for the bimodal language is a quadruple \((W, R_1, R_2, V)\) where \((W, R_1, R_2)\) is a Kripke frame and \( V : \Phi \to \mathcal{P}(W) \) is a function that assigns to every propositional variable a set of worlds where it is ‘true’.

We now have all the tools to define truth and validity in the basic and bimodal languages. The truth of a modal formula is defined with respect to a single world in \( W \). The definition is recursive on the complexity of formulas.

**Definition 2.3.** Let \( \mathcal{M} = (W, R_1, R_2, V) \) be a Kripke model and \( w \in W \) a world. Then we define the notion of a formula \( \varphi \) being true or satisfied in \( \mathcal{M} \) at \( w \) (notation \( \mathcal{M}, w \models \varphi \)) as follows:

\[
\begin{align*}
\mathcal{M}, w &\models p \quad \text{iff} \quad w \in V(p), \quad \text{for } p \in \Phi, \\
\mathcal{M}, w &\not\models \bot \quad \text{never}, \\
\mathcal{M}, w &\models \neg \varphi \quad \text{iff} \quad \text{not } \mathcal{M}, w \models \varphi, \\
\mathcal{M}, w &\models \varphi \lor \psi \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi, \\
\mathcal{M}, w &\models \Box_i \varphi \quad \text{iff} \quad \text{there is a } v \in W \text{ s.t. } w R_i v \text{ and } \mathcal{M}, v \models \varphi, \quad i \in \{1, 2\}.
\end{align*}
\]

Though it can be derived from the clauses for \( \neg \varphi \) and \( \Box_i \varphi \) it may be instructive to define a clause for \( \Diamond_i \varphi \) explicitly, the reader is invited to check that both approaches give the same result:

\[
\mathcal{M}, w \not\models \Diamond_i \varphi \quad \text{iff} \quad \text{for all } v \in W \text{ with } w R_i v \text{ we have } \mathcal{M}, v \models \varphi.
\]

We say that a formula \( \varphi \) is valid on a frame \( \mathfrak{F} = (W, R_1, R_2) \) when for all valuations \( V \) and all \( w \in W \) we have \( (\mathfrak{F}, V), w \models \varphi \), notation: \( \mathfrak{F} \models \varphi \). A formula is said to be valid in a class of frames if it is valid in all frames of that class.

**Uni-modal languages**

These definitions can all be restricted to a language with just one unary modal operator \( \Diamond \) (known as the basic modal language) by simply ‘forgetting’ one of the modal operators. Frames and models for this language only have one relation \( R \). We will briefly use this language in chapter 4.

**2.1.3 Soundness and Completeness**

A question of central importance in the study of modal logic is the relation between sets of formulas and the frames on which they are valid.
Normal modal Logics

First we define what is meant by “a modal logic”.

**Definition 2.4.** A set of modal formulas \( L \) is called a *modal logic* if it contains all propositional tautologies and is closed under the rules:

(i) Modus Ponens: if \( \varphi \rightarrow \psi \in L \) and \( \varphi \in L \) then \( \psi \in L \),

(ii) Uniform Substitution: if \( \varphi \in L \), \( p_1, \ldots, p_n \) are variables in \( \varphi \) and \( \psi_1, \ldots, \psi_n \) are modal formulas then \( \varphi[p_1/\psi_1, \ldots, p_n/\psi_n] \in L \). Here \( \varphi[p/\psi] \) is the formula obtained from \( \varphi \) by replacing every occurrence of \( p \) in \( \varphi \) by \( \psi \).

A (bi-)modal logic \( L \) is called *normal* if in addition to the above it contains the formulas

\[(K) \quad \square_i(p \rightarrow q) \rightarrow (\square_i p \rightarrow \square_i q),
\]

\[(\text{Dual}) \quad \Diamond_i p \leftrightarrow \neg \square_i \neg p,\]

for \( i \in \{1, 2\} \), and is closed under *necessitation* (or *generalisation*):

if \( \varphi \in L \) then \( \square_i \varphi \in L \).

The restriction of these definitions to the uni-modal case can be obtained from the above by removing the subscripts. If we have two modal logics \( L_1 \) and \( L_2 \) such that \( L_1 \subseteq L_2 \) we call \( L_2 \) an *extension* of \( L_1 \), if \( L_2 \) is normal it is called a *normal* extension of \( L_1 \).

In this thesis we are only interested in normal modal logics and normal extensions of modal logics. To reduce clutter we will simply write: ‘modal logic’ and ‘extension’, the reader should prefix these with ‘normal’.

We frequently write \( \vdash_L \varphi \) to say \( \varphi \in L \). In this case \( \varphi \) is known as a *theorem* of the logic \( L \). This notation is standard for derivability. The least normal uni-modal logic is known as \( K \).

Various modal logics can be defined from others by adding certain axioms. If \( L_1 \) is a modal logic then \( L_1 + \varphi \) is the least normal extension of \( L_1 \) containing the axiom \( \varphi \).

Some standard (uni-)modal logics and their axioms are given in table 2.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( K + \square p \rightarrow p )</td>
</tr>
<tr>
<td>( K4 )</td>
<td>( K + \square p \rightarrow \square \square p )</td>
</tr>
<tr>
<td>( S4 )</td>
<td>( K4 + \square p \rightarrow p )</td>
</tr>
<tr>
<td>( S5 )</td>
<td>( S4 + p \rightarrow \square \Diamond p )</td>
</tr>
</tbody>
</table>

Table 2.1: Some normal modal logics
Soundness and Completeness

Given a set of modal formulas \( L \) we define the \textit{frames} of \( L \) as:

\[
\text{Fr}(L) := \{ \mathcal{F} \mid \text{\( \mathcal{F} \) is a frame and } (\forall \varphi \in L)(\mathcal{F} \models \varphi)\}\}.
\]

Similarly, given a class of frames \( C \) we define the \textit{logic} of \( C \) as:

\[
\text{Log}(C) := \{ \varphi \mid (\forall \mathcal{F} \in C)(\mathcal{F} \models \varphi)\}\}.
\]

We will also call \( \text{Log}(C) \) the \textit{logic generated by} \( C \).

The twin notions of soundness and completeness tie together modal logics and classes of frames in a natural way.

\textbf{Definition 2.5.} Let \( C \) be a class of frames. A modal logic \( L \) is \textit{sound} with respect to \( C \) if \( L \subseteq \text{Log}(C) \). That is, if for every \( \varphi \) and \( \mathcal{F} \in C \) we have \( \vdash_L \varphi \) implies \( \mathcal{F} \models \varphi \).

\textbf{Definition 2.6.} Let \( C \) be a class of frames. A modal logic \( L \) is \textit{complete} with respect to \( C \) if \( \text{Log}(C) \subseteq L \). That is, if for every \( \varphi \), whenever for all frames \( \mathcal{F} \in C \) we have \( \mathcal{F} \models \varphi \) we also have \( \vdash_L \varphi \).

Soundness results are generally proved in a straightforward way, namely: given a modal logic \( L \) and class of frames \( C \), we show that the axioms of \( L \) are valid on \( C \) and that the deduction rules preserve validity on \( C \) (cf. [2, p. 195]). Completeness results tend to require significantly more work. For a survey of methods related to completeness proofs, as well as a broader spectrum of logics, the reader is referred to [2, Chapter 4] and [4, Chapter 5]. A few standard soundness and completeness results are given in table 2.2.

<table>
<thead>
<tr>
<th>Logic</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{K} )</td>
<td>the class of all frames</td>
</tr>
<tr>
<td>( \mathbf{T} )</td>
<td>the class of reflexive frames</td>
</tr>
<tr>
<td>( \mathbf{K4} )</td>
<td>the class of transitive frames</td>
</tr>
<tr>
<td>( \mathbf{S4} )</td>
<td>the class of reflexive, transitive frames</td>
</tr>
<tr>
<td>( \mathbf{S5} )</td>
<td>the class of frames where ( R ) is an equivalence relation</td>
</tr>
</tbody>
</table>

\textbf{Table 2.2: Some soundness and completeness results}

A final part of this section is the following definition.

\textbf{Definition 2.7.} We say a normal modal logic \( L \) has the \textit{finite model property} if for every formula \( \varphi \notin L \) there is a finite \( L \)-frame \( \mathcal{F} \) and valuation \( V \) such that \( (\mathcal{F}, V) \not\models \varphi \).

\section*{2.1.4 Operations on Frames}

For the sake of brevity we will define the following notions for frames of the basic modal language. Extending these definitions to a multimodal logic is relatively straightforward, for a discussion the reader is referred to [2, Section 3.3].

Kripke frames admit several operations that preserve validity, in this section we give three of these. Namely: disjoint unions, generated subframes and bounded morphisms.
Definition 2.8. Let $C = \{ \mathcal{F}_i \}_{i \in I}$ be a class of frames with $\mathcal{F}_i = (W_i, R_i)$. The disjoint union of $C$ (notation: $\bigsqcup_{i \in I} \mathcal{F}_i$) is defined as the frame $\mathcal{F} = (W, R)$ where $W = \bigcup_{i \in I} (W_i \times \{i\})$ and $(w, i)R(v, j)$ iff $wR_i v$ and $i = j$.

Definition 2.9. Let $\mathcal{F} = (W, R)$ be a frame. We call a frame $\mathcal{F}' = (W', R')$ a generated subframe of $\mathcal{F}$ (notation: $\mathcal{F}' \hookrightarrow \mathcal{F}$) when $W' \subseteq W$ and $R' = R \cap (W' \times W')$ such that for all $w, v \in W$: if $w \in W'$ and $wRv$ then $v \in W'$.

Definition 2.10. Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be frames, a bounded morphism (also called a $p$-morphism) from $\mathcal{F}$ to $\mathcal{F}'$ is a function $f$ from $W$ to $W'$ satisfying:

(forth) $wRv$ implies $f(w)Rf(v)$.
(back) If $f(w)R'v'$ then there is some $v \in W$ such that $wRv$ and $f(v) = v'$.

If there is a surjective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$ we call $\mathcal{F}'$ a bounded morphic image of $\mathcal{F}$, (notation: $\mathcal{F} \twoheadrightarrow \mathcal{F}'$). Two frames $\mathcal{F}_1, \mathcal{F}_2$ are called isomorphic if there exists a bijective bounded morphism between them (notation: $\mathcal{F}_1 \cong \mathcal{F}_2$).

The following theorem tells us how the above operations preserve validity, see also [2, Theorem 3.14].

Theorem 2.11. Let $\varphi$ be a modal formula, $\{ \mathcal{F}_i \}_{i \in I}$ a class of frames, and $\mathcal{F}$ and $\mathcal{F}'$ a pair of frames. Then:

(i) If $\mathcal{F}_i \vDash \varphi$ for all $i \in I$ then $\bigsqcup_{i \in I} \mathcal{F}_i \vDash \varphi$.
(ii) If $\mathcal{F}' \hookrightarrow \mathcal{F}$ and $\mathcal{F} \vDash \varphi$ then $\mathcal{F}' \vDash \varphi$.
(iii) If $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ and $\mathcal{F} \vDash \varphi$ then $\mathcal{F}' \vDash \varphi$.

Corollary 2.12. Let $C = \{ \mathcal{F}_i \}_{i \in I}$ be a class of frames, then $\text{Fr}(\text{Log}(C))$ is closed under disjoint unions, generated subframes and bounded morphic images.

Proof. We will only give a proof of the case of bounded morphic images, the other cases are proved analogously. Let $\mathcal{F} \in \text{Fr}(\text{Log}(C))$ and $\mathcal{F}'$ be a frame such that $\mathcal{F} \twoheadrightarrow \mathcal{F}'$. Then for all $\varphi \in \text{Log}(C)$, $\mathcal{F} \vDash \varphi$ and so by Theorem 2.11 (iii) we have $\mathcal{F}' \vDash \varphi$. So we may conclude that $\mathcal{F}' \in \text{Fr}(\text{Log}(C))$. \qed

2.2 Universal Algebra

There exists an intimate connection between logic and universal algebra. Most of the material related to this connection is beyond the scope of this thesis, we will only scratch the surface.

The notion of algebra used here is much broader than the usual groups, rings and fields encountered in the typical undergraduate mathematics curriculum. For a full survey the reader is referred to [3] and [7]. In brief, the following definition provides the general idea.
Definition 2.13. An algebra is a tuple \((A, f_1, \ldots, f_n)\) where \(A\) is a set and each \(f_i\) is an operation on \(A\) of finite arity, such that \(A\) is closed under \(f_i\). \(A\) is called the carrier set or just carrier of the algebra.

2.2.1 Boolean Algebras

Classical propositional logic is related to a class of algebras known as Boolean algebras.

Definition 2.14. A Boolean algebra \(\mathfrak{A}\) is a quintuple \((A, \lor, \land, \neg, \bot, \top)\) such that \(A\) is a set, \(\lor\) and \(\land\) are binary operations on \(A\), \(\neg\) is a unary operation on \(A\), and \(\bot\) and \(\top\) are nullary operations on \(A\) (constants). Further, the following equations must hold on \(A\):

\[
\begin{align*}
(i) & \quad a \lor b = b \lor a \quad \quad \quad a \land b = b \land a \\
(ii) & \quad a \lor (b \land c) = (a \lor b) \land c \quad \quad \quad a \land (b \land c) = (a \land b) \land c \\
(iii) & \quad a \lor \bot = a \quad \quad \quad \quad \quad \quad a \land \top = a \\
(iv) & \quad a \lor \neg a = \top \quad \quad \quad \quad \quad \quad a \land \neg a = \bot \\
(v) & \quad a \lor (b \land c) = (a \lor b) \land (a \lor c) \quad \quad \quad a \land (b \lor c) = (a \land b) \lor (a \land c)
\end{align*}
\]

Here, the operations \(\lor\) and \(\land\) are called join and meet respectively, \(\neg a\) is called the complement of \(a\) and \(\bot, \top\) are called bottom and top. The operations join and meet induce a partial order on \(A\) as follows:

\[
a \leq b \text{ iff } a \lor b = b \quad \text{ iff } a \land b = a
\]

We define the class of Boolean algebras by \(\mathsf{BA}\).

Example 2.15. Let \(X\) be any set and let \(\mathcal{P}(X)\) be the power set of \(X\). Then \((\mathcal{P}(X), \cup, \cap, (\cdot)^c, \emptyset, X)\) is a Boolean algebra, known as the power set algebra of \(X\). Here, join is union, meet is intersection and \((\cdot)^c\) is complement relative to \(X\). Checking that this gives a Boolean algebra is a routine exercise well worth doing for the reader who is first being introduced to the subject, as it gives a very concrete handle on the subject.

There is a natural way to define an algebraic semantics for classical propositional logic using Boolean algebras.

Definition 2.16. Let \(\mathfrak{A} = (A, \lor, \land, \neg, \bot, \top)\) be a Boolean algebra and let \(\Phi\) be some set of propositional variables. A valuation on \(\mathfrak{A}\) is a function \(v : \Phi \to A\). We can inductively extend \(v\) to \(\text{Form}(\Phi)\) as follows:

\[
\begin{align*}
 v(\neg \varphi) &= \neg v(\varphi) \\
v(\varphi \lor \psi) &= v(\varphi) \lor v(\psi) \\
v(\varphi \land \psi) &= v(\varphi) \land v(\psi) \\
v(\bot) &= \bot.
\end{align*}
\]

We then say that a formula \(\varphi\) is true in \(\mathfrak{A}\) under a valuation \(v\) if \(v(\varphi) = \top\), \(\varphi\) is valid in \(\mathfrak{A}\) (notation: \(\mathfrak{A} \models \varphi\)) if it is true under any valuation, and \(\varphi\) is valid in a class of Boolean algebras \(\mathcal{A}\) (notation: \(\mathcal{A} \models \varphi\)) if it is valid in every algebra in \(\mathcal{A}\).
The following theorem is a standard result that can be proved using the so-called Lindenbaum-Tarski construction, see for example [2, Theorem 5.11] or [7, Theorem 11.16].

**Theorem 2.17.** Let \( \varphi \) be a formula in classical propositional logic, then \( \varphi \) is a tautology iff \( \text{BA} \models \varphi \).

### 2.2.2 Modal Algebras

In the same way that we extended propositional logic to modal logic we can extend Boolean algebras to modal algebras.

**Definition 2.18.** A modal algebra \( \mathfrak{B} \) is a pair \((\mathfrak{A}, \Diamond)\) where \( \mathfrak{A} \) is a Boolean algebra and \( \Diamond \) is a unary operation on \( \mathfrak{A} \) that satisfies

(i) \( \Diamond(a \lor b) = \Diamond a \lor \Diamond b \),

(ii) \( \Diamond \bot = \bot \).

We obtain an operation dual to \( \Diamond \) in the expected way: \( \Box a : = \neg \Diamond \neg a \).

As in the case for Boolean algebras we can define valuations on modal algebras. This is done by extending a valuation \( v \) on the Boolean algebra underlying the modal algebra with the clause:

(iii) \( v(\Diamond \varphi) = \Diamond v(\varphi) \)

We define truth and validity as above. That is, if \( \mathfrak{B} \) is a modal algebra, \( v \) a valuation on \( \mathfrak{B} \) and \( \varphi \) some formula then \( \mathfrak{B}, v \models \varphi \) iff \( v(\varphi) = T \). A formula is valid on \( \mathfrak{B} \) if it is satisfied under every valuation. With these notions in hand we can associate to every normal modal logic a class of Boolean algebras.

**Definition 2.19.** Let \( L \) be a normal modal logic. We define the class of \( L \)-algebras, denoted \( V_L \), to be the class of modal algebras \( \mathfrak{B} \) such that \( \mathfrak{B} \models \varphi \) for all \( \varphi \in L \).

Using a modal version of the Lindenbaum-Tarski argument it can be shown that every modal logic is complete with respect to its algebras. For a proof, see e.g. [2, Theorem 5.27].

**Theorem 2.20.** Let \( L \) be a normal modal logic. Then \( L \) is sound and complete with respect to \( V_L \), that is:

\[ \vdash_L \varphi \iff V_L \models \varphi. \]

### 2.2.3 Operations on Modal Algebras

Similar to the operations on frames (and, as we shall see, closely connected) are the following operations on modal algebras.
Definition 2.21. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be modal algebras. A homomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \) is a map \( f \) that preserves all operations, that is, such that \( f(a \lor \mathfrak{A} b) = f(a) \lor \mathfrak{B} f(b) \), \( f(\neg \mathfrak{A} a) = \neg \mathfrak{B} f(a) \), and so forth. We call \( \mathfrak{B} \) a homomorphic image of \( \mathfrak{A} \) if there is a surjective homomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \). A bijective homomorphism is called an isomorphism, two modal algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) are called isomorphic if there is an isomorphism between them (notation \( \mathfrak{A} \cong \mathfrak{B} \)).

Definition 2.22. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be modal algebras. Then \( \mathfrak{B} \) is a subalgebra of \( \mathfrak{A} \) if the carrier of \( \mathfrak{B} \) is a subset of the carrier set of \( \mathfrak{A} \) and the insertion map \( \iota : \mathfrak{B} \to \mathfrak{A} \) is a homomorphism.

Definition 2.23. Let \( \{ \mathfrak{A}_i \}_{i \in I} \) be a family of algebras of the same type with \( \mathfrak{A}_i = (A_i, f_{i1}, f_{i2}, \ldots, f_{in}) \). The product of \( \{ \mathfrak{A}_i \}_{i \in I} \) (notation: \( \prod_{i \in I} \mathfrak{A}_i \)) is defined as the algebra \( \mathfrak{A} = (A, f_1, f_2, \ldots, f_n) \) with carrier set:

\[
A = \prod_{i \in I} A_i.
\]

And for \( f_j \) with arity \( n \) and \( a_1, \ldots, a_n \in A \) and we have

\[
f_j(a_1, \ldots, a_n)(i) = f^j_i(a_1(i), \ldots, a_n(i)),
\]

that is, \( f_j \) is applied coordinate-wise.

Varieties of Algebras

Varieties are an important part of universal algebra, as they form certain ‘nice’ classes of algebras. For any class of algebras \( C \) we denote by \( H(C) \) the class \( C \) closed under homomorphic images. Similarly, \( S(C) \) and \( P(C) \) denote the class \( C \) closed under subalgebras and products respectively.

Definition 2.24. A class of algebras \( C \) of the same type is called a variety iff it is closed under homomorphic images, subalgebras and products. We let \( V(C) \) denote the smallest variety containing \( C \).

The next result by Tarksi tells us exactly what \( V(C) \) is, see [3, Theorem II.9.5] for a proof.

Theorem 2.25. Let \( C \) be a class of algebras of the same type, then \( V(C) = HSP(C) \).

The following result by Birkhoff characterises varieties in a different way. For a proof, see for example [3, Theorem II.11.9].

Theorem 2.26 (Birkhoff). A class of algebras is a variety iff it is equationally definable.

In fact, given a modal logic \( L \), Theorem 2.20 implies that the class of \( L \)-algebras \( V_L \) is a variety, called the variety of \( L \).
2.3 Duality

The notion of two classes of mathematical objects being dual to each other is common in mathematics. In this section we examine the duality between finite frames and finite modal algebras. We restrict ourselves to the finite case because this turns to be sufficient for our purposes in later chapters.

2.3.1 From Frames to Algebras

The procedure of converting a frame to an algebra is relatively straightforward. But first, we need to define the following operation.

**Definition 2.27.** Let $R$ be a binary relation on a set $W$. We have the following operation on $W$:

$$m_R(X) := \{ w \in W \mid \text{there is a } v \in X \text{ such that } wRv \}.$$ 

This operation gives for every subset $X$ of $W$ the set of points that have a successor in $X$.

**Definition 2.28.** Given a frame $\mathfrak{F} = (W, R)$ we define the complex algebra of $\mathfrak{F}$ (notation: $\mathfrak{F}^+$) as the power set algebra of $W$ extended with the operation $m_R$. That is, the structure $(\mathcal{P}(W), \cup, \cap, (\cdot)^c, \emptyset, X, m_R)$.

The algebra $\mathfrak{F}^+$ is also called the dual of $\mathfrak{F}$, if $C$ is some class of frames we denote by $C^+$ the class of algebras dual to a frame of $C$. That $\mathfrak{F}^+$ is a modal algebra is not immediate but easily proven as an exercise. This result together with the following can be found in [4, Theorem 7.46].

**Theorem 2.29.** Let $\mathfrak{F}$ be a frame and $\varphi$ a modal formula. Then

$$\mathfrak{F} \models \varphi \iff \mathfrak{F}^+ \models \varphi.$$ 

2.3.2 From Algebras to frames

For the conversion of finite modal algebras to frames we first need the notions of atoms and atomic modal algebras.

**Definition 2.30.** Let $\mathfrak{A}$ be a Boolean (or modal) algebra, then an element $a \in A$ with $a \neq \perp$ is called an atom if for all $b \in A$ with $b \leq a$ either $b = a$ or $b = \perp$. We denote the set of atoms of $\mathfrak{A}$ by $\text{At}(\mathfrak{A})$.

**Definition 2.31.** Let $\mathfrak{A}$ be a Boolean algebra. We call $\mathfrak{A}$ atomic if for all $b \in A$ there is a set of atoms $\{a_i\}_{i \in I} \subseteq A$ such that $\bigvee_{i \in I} a_i = b$. A modal algebra $(\mathfrak{A}, \diamond)$ is atomic if $\mathfrak{A}$ is.

The following follows from [7, Lemma 5.4].

**Proposition 2.32.** Every finite modal algebra is atomic.
Definition 2.33. Let \( \mathfrak{A} = (A, \lor, \neg, \bot, \Diamond) \) be a finite modal algebra. We define the frame \( \mathfrak{A}_+ = (\text{At}(\mathfrak{A}), R_\Diamond) \) where At(\( \mathfrak{A} \)) is the set of atoms of \( \mathfrak{A} \) and for \( a, b \in \text{At}(\mathfrak{A}) \): \( aR_\Diamond b \) iff \( a \leq \Diamond b \). If \( \mathcal{C} \) is some class of finite modal algebras we denote by \( \mathcal{C}_+ \) the class of frames dual to some algebra of \( \mathcal{C} \).

The next proposition follows from the Jónsson-Tarski Theorem [2, Theorem 5.43]

Proposition 2.34. Let \( \mathfrak{A} \) be a finite modal algebra and \( \mathfrak{F} \) a finite frame. The following hold:

(i) \( \mathfrak{A} \cong (\mathfrak{A}_+)^+ \),

(ii) \( \mathfrak{F} \cong (\mathfrak{F}^+)_+ \).

Proposition 2.35. Let \( \mathfrak{A} \) be a finite modal algebra and \( \varphi \) a modal formula. Then:

\[ \mathfrak{A} \vDash \varphi \iff \mathfrak{A}_+ \vDash \varphi. \]

Proof. A direct consequence of Theorem 2.29 and Proposition 2.35.

It turns out that there is a strong relation between operations on frames and operations on their dual algebras and vice-versa. We say a frame \( \mathfrak{F}' \) is embeddable in another \( \mathfrak{F} \) if it is isomorphic to a generated subframe of \( \mathfrak{F} \) (notation: \( \mathfrak{F}' \hookrightarrow \mathfrak{F} \)), similarly for algebras: \( \mathfrak{A} \hookrightarrow \mathfrak{B} \) means \( \mathfrak{A} \) is isomorphic to a subalgebra of \( \mathfrak{B} \). See [2, Theorem 5.47]

Theorem 2.36. Let \( \mathfrak{F} \) and \( \mathfrak{F}' \) be frames and \( \mathfrak{A} \) and \( \mathfrak{B} \) be modal algebras. Then:

(i) \( \mathfrak{F} \hookrightarrow \mathfrak{F}' \) implies \( \mathfrak{F}'^+ \hookrightarrow \mathfrak{F}^+ \)

(ii) \( \mathfrak{F} \rightarrow \mathfrak{F}' \) implies \( \mathfrak{F}'^+ \rightarrow \mathfrak{F}^+ \)

(iii) \( \mathfrak{A} \hookrightarrow \mathfrak{B} \) implies \( \mathfrak{B}_+ \hookrightarrow \mathfrak{A}_+ \)

(iv) \( \mathfrak{A} \rightarrow \mathfrak{B} \) implies \( \mathfrak{B}_+ \rightarrow \mathfrak{A}_+ \)
3 Interpolation and amalgamation

The primary property of interest in this thesis is the syntactic interpolation property. This chapter introduces this property and proves it equivalent to the semantic amalgamation property.

3.1 Interpolation

There are several properties of logics that are called interpolation properties, the following two are the ones examined in this thesis.

**Definition 3.1.** Let $L$ be a modal logic and let $\varphi$ be a modal formula. Define $\text{Var}(\varphi)$ to be the set of propositional variables in $\varphi$. Then we say that:

(CIP). $L$ has the Craig, or Arrow, Interpolation Property if, whenever $\vdash_L \varphi \rightarrow \psi$, there is a formula $\chi$ with $\text{Var}(\chi) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$ such that $\vdash_L \varphi \rightarrow \chi$ and $\vdash_L \chi \rightarrow \psi$.

(TIP). $L$ has the Turnstile Interpolation Property or Interpolation for Derivability if, whenever $\varphi \vdash_L \psi$, there is a formula $\chi$ with $\text{Var}(\chi) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$ such that $\varphi \vdash_L \chi$ and $\chi \vdash_L \psi$.

We will restrict ourselves to Turnstile Interpolation. It is worthwhile to note that, in the case of modal logics, Craig interpolation is stronger than turnstile interpolation in the sense that, if a modal logic has CIP then it also has TIP, see [17, Proposition 3.1]. In the next chapter we will drop ‘turnstile’ completely and just refer to ‘the’ interpolation property.

3.1.1 Some Interpolation Results

Both Craig and Turnstile interpolation have been examined for a wide variety of logics. From classical propositional logic to various modal logics and fragments of first order logic.

**Theorem 3.2.** The following logics all have the (Craig) interpolation property:

(i) Classical Propositional Logic,

(ii) Intuitionistic Propositional Logic,

(iii) First-order logic,
The modal logics $K$, $K4$, $T$ and, $S4$.

The first, second and fourth results are due to Gabbay [10], the third is the original result by Craig [5].

3.2 Amalgamation

Amalgamation is a natural property of classes of (modal) algebras. In this section we will define it, and show it is equivalent to the turnstile interpolation property in modal logics.

3.2.1 Amalgamation and Superamalgamation

**Definition 3.3.** Let $C$ be a class of modal algebras and let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in C$ be algebras with maps $f, g$ forming the diagram:

\[
\begin{array}{c}
\mathfrak{A}_0 \\
\mathfrak{A}_1 \\
\mathfrak{A}_2
\end{array}
\begin{array}{c}
f \\
g
\end{array}
\]

We say that such a diagram has an *amalgam* $\mathfrak{A} \in C$ if there are maps $f', g'$ making the following diagram commute:

\[
\begin{array}{c}
\mathfrak{A}_0 \\
\mathfrak{A}_1 \\
\mathfrak{A}_2
\end{array}
\begin{array}{c}
f \\
g
\end{array}
\begin{array}{c}
f' \\
g'
\end{array}
\mathfrak{A}
\]

A class of algebras $C$ has the *amalgamation* property if every diagram of the form (*) in $C$ has an amalgam.

We can strengthen this definition to the concept of superamalgamation.

**Definition 3.4.** A class $C$ of modal algebras has the *superamalgamation* property if every diagram of the form (*) has an amalgam with the property:

\[a_1 \in \mathfrak{A}_1, a_2 \in \mathfrak{A}_2 \text{ and } f'(a_1) \leq g'(a_2) \Rightarrow \exists a_0 \in \mathfrak{A}_0 (a_1 \leq f(a_0) \text{ and } g(a_0) \leq a_2)\]

\[\text{This terminology deviates somewhat from the established. Eva Hoogland for example, uses the word 'amalgam' to denote the quintuple } \langle \mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, f, g \rangle \text{ in her PhD-thesis. We have chosen to use it for the product of the process of amalgamation instead, since this seems the most natural word for it.}\]
as well as
\[ a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2 \text{ and } g'(a_2) \leq f'(a_1) \Rightarrow \exists a_0 \in \mathcal{A}_0 (a_1 \leq f(a_0) \text{ and } g(a_0) \leq a_2). \]

### 3.2.2 Amalgamation and Duality

In section 2.3 we described a duality between classes of finite algebras and classes of finite frames. In particular, this duality allows us to define a notion of co-amalgamation for classes of frames. The general idea relies on the dual maps obtained in 2.36. This motivates the following definition.

**Definition 3.5.** We say a class of finite frames \( C \) has the **co-amalgamation** property if \( C^+ \) has the amalgamation property.

The above definition requires some explanation. Recall from chapter 2 that every finite modal algebra is dual to a finite frame. In particular, this means that if we have finite modal algebras \( \mathcal{A}_0, \mathcal{A}_1 \) and \( \mathcal{A}_2 \) as in diagram (*) from definition 3.3, these will be dual to finite frames \( \mathcal{F}_i = (\mathcal{A}_i)^+ \). From theorem 2.36 we see that the embeddings \( f, g \) from this diagram turn into surjective bounded morphisms ‘going the other way’. That is, a diagram like (*) corresponds to a diagram of the shape

\[
\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{f^+} & \mathcal{F}_1 \\
\mathcal{F}_1 & \xleftarrow{g^+} & \mathcal{F}_2
\end{array}
\]

It is now straightforward to see that (*) has an amalgam iff there exists a frame \( \mathcal{F} \) with surjective bounded morphisms \( f'_+ \) and \( g'_+ \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{f^+} & \mathcal{F}_1 \\
\mathcal{F}_1 & \xleftarrow{g^+} & \mathcal{F}_2 \\
\mathcal{F} & \xleftarrow{f'_+} & \xrightarrow{g'_+}
\end{array}
\]

We will call this frame \( \mathcal{F} \) a **co-amalgam**.

### 3.3 Interpolation and Amalgamation

The crucial result of this chapter is the following, this equivalence was initially proved for extensions of \( S4 \) by Maksimova over the course of [14, 15, 16], and generalized by Czelakowski in [6]. We refer the reader to [4, Section 14.2] for a concise discussion:
Theorem 3.6. Let $L$ be a normal modal logic, then the following are equivalent:

(i) $L$ has turnstile interpolation,

(ii) $V_L$ has the amalgamation property.

In fact, this result can be strengthened in our case, to do this we first need some definitions.

Definition 3.7. Let $A$ be a modal algebra. We say $A$ is finitely generated if there is some finite subset $B$ of the carrier of $A$ such that no proper subalgebra of $A$ contains $B$.

Definition 3.8. Let $A$ be an algebra and $\{A_i\}_{i \in I}$ a family of algebras, let $\pi_i : \prod_{j \in I} A_j \rightarrow A_i$ be the projection mapping. We say that an embedding $e : A \hookrightarrow \prod_{i \in I} A_i$ is subdirect if for all $i \in I$ we have that $\pi_i \circ e$ is surjective.

We say an algebra $A$ is subdirectly irreducible if for all subdirect embeddings $e : A \hookrightarrow \prod_{i \in I} A_i$, there is an $i \in I$ such that $\pi_i \circ e$ is an isomorphism.

An algebra $A$ is finitely indecomposable\footnote{This terminology is due to Maksimova, in other sources this property is called finitely subdirectly irreducible.} when for any subdirect embedding $e : A \hookrightarrow \prod_{i \in I} A_i$ with $I$ finite, we have an $i \in I$ such that $\pi_i \circ e$ is an isomorphism.

Clearly, every subdirectly irreducible algebra is finitely indecomposable, in the case of finite $S5^2$-algebras these notions are even equivalent. With these definitions we get the following theorem from [9, Theorem 7.9]:

Theorem 3.9. Let $L$ be a normal modal logic, then the following are equivalent:

(i) $L$ has turnstile interpolation,

(ii) If $A_0, A_1, A_2 \in V_L$ are finitely generated and finitely indecomposable, with embeddings $f : A_0 \rightarrow A_1$ and $g : A_0 \rightarrow A_2$, then there is an algebra $A \in V_L$ with embeddings $f' : A_1 \rightarrow A$ and $g' : A_2 \rightarrow A$ such that $f' \circ f = g' \circ g$.\footnote{This terminology is due to Maksimova, in other sources this property is called finitely subdirectly irreducible.}
4 Products of Logics and S5-squared

The main logic of interest in this thesis is $S5^2$. In this chapter we define this logic and give some results concerning its extensions.

4.1 Products of Logics

Product Frames

Underlying the notion of products of logics is the notion of product frames.

**Definition 4.1.** Let $L_1$ and $L_2$ be basic modal logics with operators $\Box_1$ and $\Box_2$ respectively. Further, let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$ be $L_1$ and $L_2$ frames respectively. We define the *product* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ as the frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = (W_1 \times W_2, R_1', R_2')$$

where

$$(w_1, w_2) R'_i (v_1, v_2) \text{ iff } w_j = v_j \text{ and } w_i R_i v_i, \text{ for } i, j \in \{1, 2\}, i \neq j.$$  

The relations $R'_1$ and $R'_2$ on $\mathfrak{F}_1 \times \mathfrak{F}_2$ are in a sense ‘orthogonal’ to each other, because of this they are sometimes referred to as the *horizontal* and *vertical* relations on $\mathfrak{F}_1 \times \mathfrak{F}_2$. Figure 4.1 gives an example of two frames (a) and (b) and their product (c).

![Figure 4.1: two frames and their product](image)

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Product Logics

Products of logics are defined based on products of their frames as follows.

**Definition 4.2.** Let $L_1$ and $L_2$ be modal logics. The *product* $L_1 \times L_2$ is defined by

$$L_1 \times L_2 := \text{Log}(\{F_1 \times F_2 \mid F_1 \in \text{Fr}(L_1) \text{ and } F_2 \in \text{Fr}(L_2)\})$$

Our main logic of interest, $S5^2$, is now defined as:

$$S5^2 := S5 \times S5.$$

Since both relations on $S5^2$-frames are equivalence relations we will refer to them as $E_1$ and $E_2$. There are other, more syntactic, ways to define $S5^2$. One of these is to define $S5^2$ as the least normal modal logic containing the *fusion* of $S5$ with itself (the fusion of two normal modal logics $L_1, L_2$ is the smallest normal modal logic in their shared language that contains $L_1 \cup L_2$) and the *Church-Rosser formula*. See for example [1, Corollary 5.3.2] and [8, Section 5.1]:

$$\text{chr} := \Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p.$$  

It follows that $\text{chr} \in S5^2$. This formula enforces a particular shape to the frames that validate it [1, Theorem 5.2.6].

**Theorem 4.3.** For every frame $\mathfrak{F} = (W, R_1, R_2)$ we have

$$\mathfrak{F} \models \text{chr} \text{ iff } (\forall w, v, u \in W)((wR_1v \land wR_2u) \rightarrow \exists z(vR_2z \land uR_1z)).$$

4.2 Properties of $S5$-squared

**Definition 4.4.** Let $\mathfrak{F} = (W, E_1, E_2)$ be an $S5^2$-frame, we define the following notions:

(i) we say $\mathfrak{F}$ is *rooted* if there is a point $w \in W$ such that for all $v \in W$ we have $(w, v) \in (E_1 \cup E_2)^*$. Where $R^*$ is the reflexive transitive closure of a relation $R$.

(ii) We call $E_i$ equivalence classes $E_i$-clusters.

(iii) If $\mathfrak{F}$ is a rooted product frame we call it a rectangle.

(iv) A rectangle where the number of $E_1$-clusters equals the number of $E_2$-clusters is called a square.

We will frequently need to refer to the finite rooted frames of logics, if $L$ is an extension of $S5^2$ we denote this class by $\text{Fr}_{\text{root}}(L)$.

For a historical discussion of the following result see [1, page 150].

**Theorem 4.5.** The logic $S5^2$ has the finite model property.

Clearly, $S5^2$ is sound with respect to the class of finite rectangles and the class of finite squares, the following result also gives us completeness, [1, Theorem 6.1.10.]

**Theorem 4.6.** The logic $S5^2$ is complete with respect to the class of finite rectangles and the class of finite squares.
4.3 Characterising Normal Extensions of S5-squared

In the next chapter we will be examining the interpolation property for normal extensions of S5^2. In this section we will set out to define these extensions.

A note on Diagrams

Throughout this section, and particularly the next chapter, we will frequently employ diagrams to clarify definitions and arguments. To keep these diagrams as simple as possible we employ a number of conventions. First, since all of our relations are equivalence relations we will omit arrowheads and arrows denoting reflexivity. We will always use horizontal lines to denote the E_1 relation and vertical lines to denote the E_2 relation. Ellipses with points in them denote E_1 ∩ E_2-clusters, where the number of points indicates the size of the cluster. The figure below illustrates this principle. On the left we have a frame consisting of a three-point E_1 ∩ E_2-cluster, E_1 connected to a single point. On the right we have the square on four points.

![Diagram](image)

4.3.1 Restricting height

As we shall see in the next chapter, the most important type of extensions of S5^2 for our purposes are the logics of specific finite rectangles. To talk about these we need the following notions:

**Definition 4.7.** Let F = (W, E_1, E_2) be a finite rectangle, we define the height of F, also called its E_1-depth (notation d_1(F)), to be the number of E_1-clusters in F. The width, E_2-depth (notation d_2(F)), of F is the number of E_2-clusters. We denote by F_{n,m} the rectangle of width n and height m. The height and width of a class of frames C is defined by

\[ d_i(C) = \sup_{\mathfrak{F} \in C} d_i(\mathfrak{F}). \]

If this supremum does not exist we say that the class has infinite height (or width).

We will say that a logic L has height (or width) n if the height (or width) of its frames is bounded by n and define d_i(L) = d_i(Fr(L)). We define a logic L to be of strict height (or width) n if it is the logic of a class of frames C such that every frame in C is of height (or width) n.
There is nothing special about our choice in associating height with \( E_2 \) and width with \( E_1 \), the converse definition would give the exact same theory. Figure 4.2 shows the frame \( \mathfrak{F}_{4,2} \) and nicely demonstrates why we use this terminology.

![Frame \( \mathfrak{F}_{4,2} \)](image)

Figure 4.2: the rectangle \( \mathfrak{F}_{4,2} \)

We will mostly be concerned with extensions of \( S5^2 \) that take the form Log(\( \mathfrak{F}_{n,m} \)) for a rectangle \( \mathfrak{F}_{n,m} \).

### 4.3.2 A note on extensions of \( S5^2 \)

In the next chapter we will be examining logics generated by finite rooted \( S5^2 \)-frames, that is, logics \( L \) that occur as \( L = \text{Log}(\mathcal{C}) \) for \( \mathcal{C} \) a class of finite rooted \( S5^2 \)-frames. The following lemma will allow us to characterise the class Fr\(_{\text{Fin Root}} \) for such logics.

**Lemma 4.8.** Let \( \mathcal{C} \) be a finite class of finite rooted \( S5^2 \)-frames. Then for all frames \( \mathfrak{F} \in \text{Fr}_{\text{Fin Root}}(\text{Log}(\mathcal{C})) \) there is an \( \mathfrak{F}' \in \mathcal{C} \) such that \( \mathfrak{F}' \rightarrow \mathfrak{F} \).

**Proof.** Let \( \mathcal{C} \) be a class of finite rooted \( S5^2 \)-frames and let \( L = \text{Log}(\mathcal{C}) \), we will examine \( \mathcal{C}^+ \), the class of modal algebras dual to \( \mathcal{C} \).

First note that with \[1\, \text{Theorem 5.4.6} \] and \[1\, \text{Theorem 5.4.14} \] we have that every modal algebra in \( \mathcal{C}^+ \) is subdirectly irreducible. It then follows from \[3\, \text{Corollary 6.10} \] that the finite subdirectly irreducible modal algebras in \( V_L \) are elements of \( \text{HS}(\mathcal{C}^+) \). That is, for every finite subdirectly irreducible algebra \( A \in V_L \) there are finite modal algebras \( B, C \) with \( C \in \mathcal{C}^+ \) such that

\[ A \leftrightarrow B \hookrightarrow C. \]

Via our duality, with Theorem 2.36, we then get that for every frame \( \mathfrak{F}_A \in \text{Fr}_{\text{Fin Root}}(\text{Log} \mathcal{C}) \) there are frames \( \mathfrak{F}_B, \mathfrak{F}_C \) with \( \mathfrak{F}_C \in \mathcal{C} \) such that

\[ \mathfrak{F}_A \leftrightarrow \mathfrak{F}_B \hookrightarrow \mathfrak{F}_C. \]

Now since our frames are rooted, and every point is a root, we must have that \( \mathfrak{F}_A \cong \mathfrak{F}_B \), so that \( \mathfrak{F}_A \leftrightarrow \mathfrak{F}_C \). But this is exactly the result we were after.

\[ ^1 \text{Here we also make use of the fact that } S5^2 \text{-algebras are congruence distributive, that is, their congruence lattices are distributive. Since this is the only place we use this fact, it felt superfluous to introduce all of the material necessary to properly define this.} \]

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Now we can generalise this lemma to arbitrary classes of finite rooted $S5^2$-frames. We do this by using the standard splitting technique (see form example [4, Section 10.5]) to deduce that if for some finite subdirectly irreducible algebra $\mathfrak{A}$ and family of algebras $\{\mathfrak{B}_i\}_{i \in I}$ we have $\mathfrak{A} \in V(\{\mathfrak{B}_i\}_{i \in I})$, then there is an $i \in I$ such that $\mathfrak{A} \in V(\mathfrak{B}_i)$. That is, by Theorem 2.25 we have that $\mathfrak{A} \in HSP(\mathfrak{B}_i)$. The result then follows from Lemma 4.8. This gives the following.

**Lemma 4.9.** Let $\mathcal{C}$ be a class of finite rooted $S5^2$-frames. Then for all frames $\mathfrak{F} \in Fr_{\text{Fin}}^{\text{Root}}(\text{Log}(\mathcal{C}))$ there is an $\mathfrak{F}' \in \mathcal{C}$ such that $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$.

With the tools of chapter 3 in hand, and the extensions of $S5^2$ from this chapter, we are finally able to address the core question of this thesis in the next chapter.
5 Interpolation for extensions of S5-squared

In this chapter we will finally turn towards the purpose of this thesis: to classify those extensions of $S5^2$ that have the interpolation property. With the help of Theorem 5.5 this will involve showing that the class of finite rooted frames of a particular extension has the co-amalgamation property (or not).

5.1 More on interpolation and amalgamation

In chapter 3 we saw a close connection between turnstile interpolation in modal logics and amalgamation of modal algebras. In the case of $S5^2$ we can obtain an even stronger version of Theorem 3.9. Before we can do this we need the following definitions and propositions.

Definition 5.1. Let $L$ be a logic and $\varphi, \psi \in L$. We say $\varphi$ and $\psi$ are equivalent if $\varphi \iff \psi \in L$. We say $L$ is locally tabular if, for any finite set of propositional variables $\Phi$, there are only finitely many pairwise non-equivalent formulas over $\Phi$ in $L$.

Definition 5.2. Let $V$ be a variety of algebras. We say $V$ is locally finite if every finitely generated $A \in V$ is finite.

The following results are (essentially) [1, Theorem 2.3.27] and [1, Corollary 6.2.12] respectively. It is worth noting that $S5^2$ itself is not locally tabular.

Proposition 5.3. A logic $L$ is locally tabular iff $V_L$ is locally finite.

Proposition 5.4. Every proper normal extension $L$ of $S5^2$ is locally tabular.

Finally, we state the theorem that we will use throughout this chapter and sketch a proof.

Theorem 5.5. Let $L$ be a proper normal extension of $S5^2$. Then the following are equivalent.

(i) $L$ has the turnstile interpolation property,

(ii) The class of finite, finitely indecomposable algebras in $V_L$ has the amalgamation property,

(iii) The class $Fr_{\text{Fin Root}}(L)$ has the co-amalgamation property.
Proof. (Sketch). Let $L$ be a proper normal extension of $S5^2$. First, note that (ii) $\Leftrightarrow$ (iii) follows from duality and the combination of [1, Theorem 5.4.6] and [1, Theorem 5.4.14].

Next, (ii) $\Rightarrow$ (i) follows from the fact that (ii) implies Theorem 3.9.(ii) which in turn implies (i). It remains to show (i) $\Rightarrow$ (ii).

Suppose $L$ has the turnstile interpolation property and examine the proof of Theorem 14.11 from [4]. This proof shows how to construct an amalgam $\mathfrak{A}$ for algebras $\mathfrak{A}_i \in V_L, i \in \{0, 1, 2\}$. Now since each $\mathfrak{A}_i$ is finite, this method of construction ensures that $\mathfrak{A}$ is finitely generated. Since $L$ is locally tabular by Proposition 5.4 we know that $V_L$ is locally finite by Proposition 5.3 and so $\mathfrak{A}$ is finite. Furthermore, we see that the amalgam $\mathfrak{A}$ is simple, from which it follows by [1, Theorem 5.4.6] that it is subdirectly irreducible, and so finitely indecomposable.

\section{5.2 Failures}

We will give a survey of extensions of $S5^2$ that do not have interpolation. The first of these results is the most sweeping, a proof can be found in [17, Theorem 4.4].

**Theorem 5.6.** If $L$ is an extension of $S5^2$ such that $\mathfrak{F}_3,3 \in \text{Fr}(L)$, then $L$ does not have the interpolation property.

To see how this excludes a great many logics from having the interpolation property, note that if $k \geq n$ and $l \geq m$ then $\mathfrak{F}_{k,l} \twoheadrightarrow \mathfrak{F}_{n,m}$. This means that any extension of $S5^2$ with a frame of height and width strictly greater than 2 immediately fails to have interpolation, which is why in the rest of this thesis we only examine extensions of $S5^2$ of height at most two.

The proof of the following closely resembles that of [17, Theorem 4.4].

**Theorem 5.7.** Let $L$ be an extension of $S5^2$ of height two such that $\text{Fr}(L)$ contains the frame:

![Diagram](image)

Then $L$ does not have the interpolation property.

*Proof.* Let $L$ be an extension of $S5^2$ as in the statement and note that, since $\mathfrak{F}_{2,2}$ can be obtained from this frame by a surjective bounded morphism, corollary 2.12 gives us $\mathfrak{F}_{2,2} \in \text{Fr}^\text{Fin}_{\text{Root}}(L)$. Suppose $L$ has interpolation and with this, that the following diagram has an amalgam.
In this diagram, a point in $\mathcal{F}_1$ or $\mathcal{F}_2$ marked with a diamond is mapped to $a$. Likewise, points marked by a square are mapped to $b$. We will reason about the shape a co-amalgam for this diagram must have and encounter a contradiction, from which it follows that no such amalgam exists.

Let $\mathfrak{F} \in \text{Fr}_{\text{Root}}^\text{Sm}(L)$ be our supposed co-amalgam. Then there are surjective bounded morphisms $f' : \mathfrak{F} \to \mathcal{F}_1$ and $g' : \mathfrak{F} \to \mathcal{F}_2$ such that for all $z \in \mathfrak{F}$ we have $f(f'(z)) = g(g'(z))$. Since $f'$ is surjective there must be some $z_1 \in \mathfrak{F}$ such that $f'(z_1) = x_1$. To guarantee the commutativity of our diagram we must then have w.l.o.g. $g'(z_1) = y_1$ (we could also have chosen $g'(z_1) = y_1$ but this changes nothing about the argument). By the back condition of bounded morphisms and the fact that $x_1 E_1 x_3$ there must be a $z_2 \in \mathfrak{F}$ such that $f'(z_2) = x_3$. Now $g'$ is a surjective bounded morphism as well so there must be some $y_i$ with $g'(z_2) = y_i$ as well as $y_1 E_1 y_i$ and $g(y_i) = b$. These conditions force $g'(z_2) = y_2$. By analogous reasoning we must have a $z_3$ with $f'(z_3) = x_5$ and $g'(z_3) = y_4$.

Since $\mathfrak{F}$ is a frame of $\text{S5}^2$ it satisfies the Church-Rosser formula $\text{chr}$ so we get a point $z_4 \in \mathfrak{F}$ such that $z_1 E_2 z_4$ and $z_3 E_1 z_4$. The facts that $f'$ and $g'$ are bounded morphisms and that our diagram commutes then give us $f'(z_4) = x_4$ and $g'(z_4) = y_3$.

This is the point where our contradiction occurs. Since $x_4 E_2 x_2$ we must have a $z_5 \in \mathfrak{F}$ with $z_4 E_2 z_5$ and $f'(z_5) = x_2$. Now $z_5$ can not be in an $E_1$ cluster different from those of $z_1$ and $z_4$ since $L$ has height 2. It also has to be different from $z_1$ and $z_4$ since $x_1 \neq x_2 \neq x_4$. This leads us to the conclusion that $z_5$ must be in a $E_1 \cap E_2$-cluster with either $z_1$ or $z_4$. The former is impossible because this two-element cluster would have to be mapped to a single point in $\mathcal{F}_2$, but $z_1$ needs to end up at $a$ and $z_5$ at $b$. The latter does not work either. Since, if $z_4 E_1 z_5$ and $f'(z_4) = x_4$ and $f'(z_5) = x_2$, we would have to have $x_4 E_1 x_2$ as well, which is not the case.

There are no more possibilities and we can only conclude that the co-amalgam $\mathfrak{F}$ cannot exist. So $L$ does not have the interpolation property.

The previous result further restricts the possible class of extensions $L$ of $\text{S5}^2$ that have interpolation, by excluding all those that have a frame of height two with an $E_1 \cap E_2$-cluster of more than one point.

**Proposition 5.8.** $\text{Log}((\mathfrak{F}_{2,1}))$ does not have the interpolation property.

**Proof.** Observe the following diagram where diamonds are mapped to the diamond in $\mathcal{F}_{2,1}$ and squares are mapped to the square:
It is easy to see that these maps are surjective bounded morphisms and that $\mathcal{F}_{2,1} \in \text{Fr}(\text{Log}(\mathcal{F}_{3,1}))$. We will show that this diagram does not have a co-amalgam so that by Theorem 5.5, $\text{Log}(\mathcal{F}_{3,1})$ does not have interpolation.

To map a frame $\mathcal{F}$ with a surjective bounded morphism to $\mathcal{F}_{3,1}$ it needs to have at least three worlds. Now with Lemma 4.8, the only finite rooted frame of $\text{Log}(\mathcal{F}_{3,1})$ with three worlds is $\mathcal{F}_{3,1}$ itself, so it is the only candidate for a co-amalgam. However, any surjective bounded morphism would need to map a world to $x_0$ and one to $x_1$ and so by composition, two worlds to $z_0$ and only one to $z_1$. By similar reasoning we would need to map a world to $y_1$ and one to $y_2$ so by composition, two worlds to $z_1$ and only one to $z_0$. This is impossible, so $\mathcal{F}_{3,1}$ cannot be a co-amalgam for our diagram and so the diagram does not have a co-amalgam. It follows that $\text{Log}(\mathcal{F}_{3,1})$ does not have interpolation. \[\square\]

In fact, the above argument can be extended to $\text{Log}(\mathcal{F}_{n,1})$ for any $n \in \mathbb{N}$, as long as $n > 2$. This gives the following theorem.

**Theorem 5.9.** For any $n \in \mathbb{N}$ with $n > 2$, $\text{Log}(\mathcal{F}_{n,1})$ does not have the interpolation property.

If we then note that for any $\mathcal{F}_{n,2}$ we have $\mathcal{F}_{n,2} \rightarrow \mathcal{F}_{n,1}$, another similar argument gives us:

**Theorem 5.10.** For any $n \in \mathbb{N}$ with $n > 2$, $\text{Log}(\mathcal{F}_{n,2})$ does not have the interpolation property.

### 5.3 Height 2

We have already seen that for most $n$, the logic $\text{Log}(\mathcal{F}_{n,2})$ does not have the interpolation property. We only have two logics in this class that might still have the property ($\text{Log}(\mathcal{F}_{1,2})$ and $\text{Log}(\mathcal{F}_{2,2})$) and both of these turn out to have interpolation. There are two more $\text{S5}^2$-frames of height two that generate logics with interpolation. In addition, if we take the logic of the class of all frames $\mathcal{F}_{n,2}$ this has interpolation as well.

We now prove the interpolation property for some strict height 2 extensions of $\text{S5}^2$ via a brute-force method.

**Proposition 5.11.** $\text{Log}(\mathcal{F}_{1,2})$ has the interpolation property.
Proof. Note with Lemma 4.8 that \( \text{Fr}_\text{Fin}^{\text{Root}}(\text{Log}(\mathcal{F}_{1,2})) = \{ \mathcal{F}_{1,1}, \mathcal{F}_{1,2} \} \). Consider the following diagram with \( \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \{ \mathcal{F}_{1,1}, \mathcal{F}_{1,2} \} \)

We will argue that \( \mathcal{F}_{1,2} \) is always a co-amalgam for this diagram. If \( \mathcal{F}_0 = \mathcal{F}_{1,2} \) we must have \( \mathcal{F}_{1,1} = \mathcal{F}_1 = \mathcal{F}_{1,2} \) as well. Mapping \( \mathcal{F}_{1,2} \) to each of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) with the identity mapping then clearly provides a co-amalgam. If \( \mathcal{F}_0 = \mathcal{F}_{1,1} \), any pair mappings from \( \mathcal{F}_{1,2} \) to \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) will, by composition, always end up at the only element of \( \mathcal{F}_0 \) so commutativity is guaranteed. We again see that \( \mathcal{F}_{1,2} \) gives a co-amalgam.

Similar, far more tedious, arguments give the following result

**Proposition 5.12.** The logic \( \text{Log}(\mathcal{F}_{2,2}) \) and the logics of the frames:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\bullet \\
\end{array}
\]

have the interpolation property.

In the above result we only have frames with 2-world \( E_1 \cap E_2 \)-clusters. If we were to take similar frames with more worlds in their clusters, an argument analogous to proposition 5.8 shows that the logic of those frames does not have interpolation.

So far we have only looked at logics generated by single frames of height 2. Next we look at logics generated by classes of more than one frame. First we have a lemma.

**Lemma 5.13.** Let \( L \) be an extension of \( \text{S5}^2 \) generated by a finite class \( C \) of finite, rooted \( \text{S5}^2 \) frames. Then if \( L \) has the interpolation property, there is a frame \( \mathcal{F} \in C \) such that \( L = \text{Log}(\mathcal{F}) \).

**Proof.** Let \( L \) be an extension of \( \text{S5}^2 \) generated by a finite class \( C \) of finite rooted \( \text{S5}^2 \) frames, and suppose there is no \( \mathcal{F} \in C \) such that \( L = \text{Log}(\mathcal{F}) \). Then there are two frames \( \mathcal{F}_1, \mathcal{F}_2 \in C \) such that for all \( \mathcal{F} \in C \) we have \( \mathcal{F} \not\sim \mathcal{F}_1 \) and \( \mathcal{F} \not\sim \mathcal{F}_2 \). Then define \( \mathcal{F}_0 = \mathcal{F}_{1,1} \), we claim that the following diagram, with \( f, g \) the bounded morphisms that identify all points in a given frame, does not have a co-amalgam.

\[
\begin{array}{c}
\mathcal{F}_0 \\
\end{array}
\quad f \leftrightarrow g \\
\begin{array}{c}
\mathcal{F}_1 \\
\mathcal{F}_2 \\
\end{array}
\]
Suppose this diagram does have a co-amalgam $\mathcal{F} \in \text{Fr}_{\text{Fin}}^{\text{Root}}(L)$. Then $\mathcal{F} \rightarrow \mathcal{F}_1$ and $\mathcal{F} \rightarrow \mathcal{F}_2$, with Lemma 4.8 we know that $\mathcal{F}$ would have to be a bounded morphic image of some frame in $C$. Since composition of surjective bounded morphisms again gives a surjective bounded morphism this would imply that there is a frame in $C$ that maps to both $\mathcal{F}_1$ and $\mathcal{F}_2$ with surjective bounded morphisms which gives a contradiction. The lemma follows.

Careful consideration of Lemma 5.13 and the results of Section 5.1 shows that the logics that we have treated in this section so far are the only extensions of $S5^2$ of strict height two, generated by a finite class of frames, that have the interpolation property.

**Proposition 5.14.** Let $L$ be an extension of $S5^2$ generated by a finite class $C$ of finite, rooted $S5^2$ frames, all of height 2, that has the interpolation property. Then $L$ is one of $\text{Log}(\mathcal{F}_1, 2)$, $\text{Log}(\mathcal{F}_2, 2)$ or the logic of one of the frames

![Diagram](attachment:image.png)

Proof. Section 5.1 reduces the possible strict height two extensions of $S5^2$ that have interpolation, and are generated by a single frame, to exactly these four. We have seen that these logics have the interpolation property in Propositions 5.11 and 5.12. The result now follows from Lemma 5.13.

We next examine extensions of $S5^2$ generated by infinite classes of frames. We need the following definition and lemma.

**Definition 5.15.** We define the class of frames $C_{\omega, 2}$ as follows:

$$C_{\omega, 2} = \{ \mathcal{F}_{n, 2} \mid n \in \mathbb{N} \}.$$ 

**Lemma 5.16.** Let $C$ be a class of $S5^2$-frames such that all frames in $C$ are of the form $\mathcal{F}_{n, 2}$, for some $n \in \mathbb{N}$. Then all frames in $\text{Fr}_{\text{Fin}}^{\text{Root}}(\text{Log}(C))$ have one of the following forms.

(a) 
\[ \begin{array}{cc}
\begin{array}{ccc}
\bullet & \cdots & \bullet \\
& \ddots & \\
\bullet & \cdots & \bullet
\end{array}
\end{array} \]

(b) 
\[ \begin{array}{c}
\begin{array}{cccc}
\bullet & \cdots & \bullet & \cdots \\
\end{array}
\end{array} \]

With $k, l, m \geq 0$ and $2k + l \leq n$ as well as $m \leq n$. 

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Proof. Recall from Lemma 4.9 that \( \text{Fr}_{\text{Fin}}^{\text{Root}}(\text{Log}(C)) \) consists entirely of bounded morphic images of frames of the form \( \mathfrak{F}_{n,2} \).

Frames of the form \( \mathfrak{F}_{m,2} \) can be obtained through a bounded morphism from \( \mathfrak{F}_{n,2} \) iff \( m \leq n \). We can do this through repeated application of maps of the form

\[
\begin{array}{c}
\begin{array}{c}
\framebox[0.5in]{\hspace{0.5in}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{m}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{m-1}
\end{array}
\end{array}
\end{array}
\]

where we have ‘identified’ the points that share an ellipse and mapped every other point to itself. In a similar way we obtain \( \mathfrak{F}_{n,1} \) from \( \mathfrak{F}_{n,2} \) by identifying pairs of \( E_2 \)-related points. Putting these two together we get all frames of the form (b) and those of the form (a) with \( k = 0 \). Frames of the form (a) with \( k > 0 \) can be obtained by maps like the following

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\framebox[0.5in]{\hspace{0.5in}}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{m}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hspace{0.5in}\cdots
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{m-1}
\end{array}
\end{array}
\end{array}
\]

where points marked by squares and diamonds respectively in the left frame are mapped to the same point in the two-element cluster of the right frame and points in the ellipse are identified. It is straightforward to check that this type of map (possibly applied to more groups of four points) is necessary to obtain frames of the form (a).

To see that these forms are the only obtainable we note: that we cannot surjectively map frames to other frames of strictly greater height or width and that a map that preserves height cannot create two-element clusters. This gives us the desired result. \( \square \)

### Proposition 5.17

The logic \( \text{Log}(C_{\omega,2}) \) has the interpolation property.

**Proof.** We will show that the class \( C := \text{Fr}^{\text{Fin}}_{\text{Root}}(\text{Log}(C_{\omega,2})) \) has the co-amalgamation property. Note that by Lemma 5.16, the class \( C \) consists of \( C_{\omega,2} \) together with all frames of the forms (a) seen in the lemma.

Now consider the following diagram of frames in \( C \), we will show that a co-amalgam always exists.

\[
\begin{array}{c}
\begin{array}{c}
\mathfrak{F}_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathfrak{F}_1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\leftarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathfrak{F}_2
\end{array}
\end{array}
\end{array}
\]

30
We now define an expanded notion of the width of a frame as follows, it is essentially the number of $E_2$-clusters plus the number of 2-element $E_1 \cap E_2$-clusters:

$$d'_2(\mathcal{F}) = d_2(\mathcal{F}) + \{|x| \text{ is a 2-element } E_1 \cap E_2\text{-cluster in } \mathcal{F}\}|.$$ 

Note that when we examine the proof of Lemma 5.16 we see that for any frame $\mathcal{F}$ in $\text{Fr}(\text{Log}(C_{\omega, 2}))$ we have that $\mathcal{F}_{d'_2(\mathcal{F})}^2$ is a frame of height two such that $\mathcal{F}_{d'_2(\mathcal{F})}^2 \rightarrow \mathcal{F}$.

Set $m = d'_2(\mathcal{F}_1)$ and $k = d'_2(\mathcal{F}_2)$, we now claim that $\mathcal{F} = \mathcal{F}_{m+k,2}$ is a co-amalgam for our diagram. To show this, we first divide $\mathcal{F}$ up into two disjoint parts, $L$ and $R$. Assign to exactly $m$ points the label $l_1$ and to $m$ points the label $l_2$ such that for $w, v \in \mathcal{F}$ with $wE_2v$ we have that $w$ is labeled $l_1$ iff $v$ is labeled $l_2$, and points that are $E_1$ related must have the same label. Then do the same to the remaining $2k$ points with the labels $r_1$ and $r_2$. This gives the following picture:

```
  m      k
 l1    l1    r1    r1
 ...    l1    l2    l2
 ...    r1    r1    r2    r2
  l2    l2    r2    r2
```

Then define the (non-generated) subframes:

$$L := \{w \in \mathcal{F} | w \text{ is labeled } l_i, i \in \{1, 2\}\}$$

and

$$R := \{w \in \mathcal{F} | w \text{ is labeled } r_i, i \in \{1, 2\}\}.$$ 

Now we need to define surjective bounded morphisms $f' : \mathcal{F} \rightarrow \mathcal{F}_1$ and $g' : \mathcal{F} \rightarrow \mathcal{F}_2$. We will do this in three steps, first we define partial bounded morphisms $f'_L : L \rightarrow \mathcal{F}_1$ and $g'_R : R \rightarrow \mathcal{F}_2$, next we use these maps to define partial maps $f'_R : R \rightarrow \mathcal{F}_1$ and $g'_L : L \rightarrow \mathcal{F}_2$. Finally, we combine these partial maps to give our required $f'$ and $g'$.

We already noted that $L$ is the least part of $\mathcal{F}$ that we can map to $\mathcal{F}_1$ with a surjective bounded morphism, likewise for $R$ and $\mathcal{F}_2$. First, map $L$ to $\mathcal{F}_1$ in the natural way, that is, so that $f'_L$ is a bounded morphism of the one of the types described in Lemma 5.16.

Next, do the same thing for $g'_R$, ensuring that for every $a \in \mathcal{F}_0$, there is an $l_1$ point $w$ with $f(f'_L(w)) = a$ iff there is an $r_1$ point $v$ such that $g(g'_R(v)) = a$.

To determine what $f'_R$ (and $g'_L$) should look like, we make use of the already established partial maps. Let $v \in R$ with label $r_1$ (w.l.o.g.) and say $g(g'_R(v)) = a$. Then by our construction, there is a point $w \in L$ with $wE_1v$ and $f(f'_L(w)) = a$. Set $f'_R(v) = f'_L(w)$ and for the unique point $v' \neq v$ with label $r_2$ and $vE_2v'$, set $f'_R(v') = f'_L(v')$, where $w'$ is the unique point with label $l_2$ and $wE_2w'$. Do this with all points labeled $r_1$ and then repeat the procedure for $g'_L$ and points labeled with $l_1$.

We now define the map $f' : \mathcal{F} \rightarrow \mathcal{F}_1$ to be

$$f'(w) = \begin{cases} 
 f'_L & \text{if } w \in L, \\
 f'_R & \text{if } w \in R,
\end{cases}$$

31
with the same for \( g' \). It is easy to see that these maps are well-defined since \( \mathcal{L} \cap \mathcal{R} = \emptyset \) and \( \mathcal{L} \cup \mathcal{R} = \mathcal{F} \).

It is clear that \( f' \) and \( g' \) are surjective. We need to show that (1): for all \( w \in \mathcal{F} \) we have \( f(f'(w)) = g(g'(w)) \), and (2): that \( f' \) and \( g' \) are bounded morphisms.

(1): Note that this statement is clearly true by construction for any point with label \( l_1 \) or \( r_1 \). So suppose w.l.o.g. that \( w \in \mathcal{F} \) with label \( l_2 \), and let \( f(f'(w)) = a \). Then there is a unique \( l_1 \) labeled point \( w' \) such that \( wE_2w' \). We have an \( r_1 \) labeled point \( v' \) such that \( g'(w') = g'(v') \), and this point has a unique \( r_2 \) labeled point \( v \) such that \( vE_2v' \) and so \( g'(w) = g'(v) \). We will show that \( g(g'(v)) = a \). First note that \( f(f'(w')) = g(g'(v')) = c \) by construction. Now since \( f' \) and \( g' \) restricted to \( \mathcal{L} \) and \( \mathcal{R} \) respectively are bounded morphisms and \( f \) and \( g \) are too, we get that \( f(f'(w'))E_2f(f'(w)) \) and \( g(g'(v'))E_2g(g'(v)) \) so that \( cE_2f(f'(w)) \) and \( cE_2g(g'(v)) \). But since \( \mathcal{F}_0 \) is of height at most 2, there can be at most two points \( E_2 \) related to \( c \). If \( a = c \) then \( g(g'(v)) = c \) as well. If \( a \neq c \), then \( g'(v') \) must be \( E_2 \)-related to something that maps to \( a \), but since \( g' \) restricted to \( \mathcal{R} \) is a bounded morphism, the only possibility is \( g'(v) \) so that \( g(g'(v)) = a \) as required. This establishes (1).

(2): We show that \( f' \) satisfies the forth and back conditions, the argument for \( g' \) is identical.

(forth): This is trivially true for points \( w, v \) with \( w = v \) so assume in the following that all points are distinct. First note that if two points \( w, v \in \mathcal{F} \) are both in \( \mathcal{L} \) or \( \mathcal{R} \) and \( wE_2v \) then \( f'(w)E_1f'(v) \) by construction of \( f' \), next note that if \( wE_2v \) and at least one of \( w \) or \( v \) is in \( \mathcal{L} \), then they are in the same part of \( \mathcal{F} \). There are two cases to consider, first: two points \( w, v \), both in \( \mathcal{R} \) such that \( wE_2v \), and second: two points \( w \) and \( v \) such that w.l.o.g. \( w \) has label \( l_1 \) and \( v \) has label \( r_1 \). In the first case we have that one of the points \( w \) and \( v \), say \( w \), has label \( l_1 \), so that \( v \) has label \( r_2 \). Then there are points \( w', v' \in \mathcal{L} \) with \( w' \in \mathcal{L} \) such that \( u \) has label \( l_1 \) and \( v' = f'(w') \), and \( v' \) has label \( l_2 \) and \( f'(v) = f'(v') \). Since \( f' \) restricted to \( \mathcal{L} \) is a bounded morphism we get that \( f'(w')E_2f'(v') \) it immediately follows that \( f'(w)E_2f'(v) \). In the second case we note that there is a point \( v' \) with label \( l_1 \) such that \( f'(v) = f'(v') \). Now since \( f' \) restricted to \( \mathcal{L} \) is a bounded morphism we get that \( f'(w)E_1f'(v) \) and so \( f'(w)E_1f'(v) \) as required.

(back): This is trivially true for \( w \in \mathcal{L} \) since \( f' \) restricted to \( \mathcal{L} \) is a bounded morphism, so suppose \( w \in \mathcal{R} \) with \( f'(w) = x \) and w.l.o.g. label \( l_1 \). Then there is a point \( v \in \mathcal{L} \) such that \( f'(v) = x \) as well. Now suppose we have a \( y \in \mathcal{F}_1 \) such that \( xE_1y \). Then there is an \( u \in \mathcal{L} \) such that \( uE_1y \) since \( f' \) restricted to \( \mathcal{L} \) is a bounded morphism. So we get \( uE_1w \) as required. The case for \( E_2 \) is similar.

All these results together establish that \( \mathcal{F} \) together with \( f' \) and \( g' \) forms a co-amalgam for our diagram, which in turn establishes that \( \text{Fr}^\text{Fin}_{\text{Root}}(\text{Log}(C_{\omega,2})) \) has the co-amalgamation property. We conclude from Theorem 5.5 that \( \text{Log}(C_{\omega,2}) \) has the interpolation property as required.

There are three more infinite classes of \( S^{5^2} \) frames of height 2 that generate logics with the interpolation property. These classes consist of frames of height two and width one, like the following:
That is, frames consisting of exactly one $E_2$-cluster, and two $E_1$-clusters of $n$ and $m$ points respectively. We will denote these frames by $\mathfrak{G}_{n,m}$. Note that, due to the fact that $E_2$ is an equivalence relation, the frame $\mathfrak{G}_{n,m}$ is the same as $\mathfrak{G}_{m,n}$. We define classes of such frames.

**Definition 5.18.** We define the class of frames $D_{n,\omega}$ for each $n \in \mathbb{N}$ as follows

$$D_{n,\omega} = \{ \mathfrak{G}_{n,m} \mid m \in \mathbb{N} \}.$$  

In addition we define

$$D_{\omega,\omega} = \{ \mathfrak{G}_{n,m} \mid n, m \in \mathbb{N} \}$$

The proof of the following is essentially the same as that of [4, Theorem 14.21].

**Proposition 5.19.** The logics $\text{Log}(D_{1,\omega})$, $\text{Log}(D_{2,\omega})$ and $\text{Log}(D_{\omega,\omega})$ all have the interpolation property.

At the time of writing we do not know whether the logics $\text{Log}(D_{1,\omega})$, $\text{Log}(D_{2,\omega})$ and $\text{Log}(D_{\omega,\omega})$ are different. This does not negatively impact our results, it only might mean that Proposition 5.19 says the same thing three times.

The results we have seen in this chapter so far allow us to provide a full classification of those proper extensions of $S5^2$ of strict height 2 that have the interpolation property.

**Theorem 5.20.** Let $L$ be a proper extension of $S5^2$ of strict height 2, then $L$ has the interpolation property iff it is one of the following: $\text{Log}(\mathfrak{F}_{1,2})$, $\text{Log}(\mathfrak{F}_{2,2})$, $\text{Log}(C_{\omega,2})$, $\text{Log}(D_{1,\omega})$, $\text{Log}(D_{2,\omega})$, $\text{Log}(D_{\omega,\omega})$ or the logics of one of the frames

$$\begin{align*}
\begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \bullet \\
\end{array}
\end{array}
\quad \text{or} \quad 
\begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \bullet \bullet \\
\end{array}
\end{array}
\end{align*}$$

**Proof.** In section 5.1 we saw our possible class of extensions reduced to exactly this set. The various propositions of this section collectively prove that these all have interpolation, establishing the result.

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6 Conclusion

In this thesis we have been able to provide a full classification of those extensions of $\text{S}5^2$ of strict height 2 that have the interpolation property. In fact, we have done something more. It is easy to see that all results pertaining to extensions of $\text{S}5^2$ of strict height 2, apply equally well to those of strict width 2, defined in the natural way.

Something else to note is the absence of height 1 extensions in this thesis. Perhaps somewhat counter-intuitively, extensions of height 1 turn out to be more difficult to classify than those of strict height 2. Due to limitations in time (and space) we regretfully had to leave off examining these. Some results about extensions of height 1 can be inferred from the results of the last chapter however. Proposition 5.8 gives a clear limitative result and the proofs of Propositions 5.11 and 5.12 can be adapted to show that the logics generated by $\mathfrak{S}_{2,1}$ and for example $\mathfrak{S}_{2,1}$ with the relations $E_1$ and $E_2$ ‘swapped out’ have the interpolation property. More work would be required to provide a full classification of extensions of $\text{S}5^2$ that have interpolation, and the mechanisms of chapter 5 could prove useful in this pursuit.
7 Populaire samenvatting

Logica is de studie van kwalitatieve informatie. Wat we hiermee bedoelen is eigenlijk heel simpel. Op de middelbare school leren we dat wiskunde over getallen gaat, we rekenen, lossen vergelijkingen op, en bepalen oppervlakte en inhoud van allerlei vormen. Steeds gaat het over kwantitatieve informatie, dat wil zeggen: informatie over hoeveelheid.

In de hogere wiskunde hebben we het liever over eigenschappen. Getallen kunnen priem zijn, functies continu, en ruimtes ‘nul-dimensionaal, Hausdorff en compact’. Dit soort informatie noemen we kwalitatief, het gaat over de eigenschappen van objecten.

Wiskunde is niet het enige vakgebied dat zich bezighoudt met eigenschappen, ook de informatica, filosofie en linguïstiek zijn velden die zich voornamelijk met kwalitatieve vraagstukken bezighouden. Een informaticus vraagt zich bijvoorbeeld af of een bepaald computerprogramma ooit zal stoppen met draaien, en als een filosoof zich afvraagt: “wat is ‘weten’?”, dan wil hij precies die eigenschappen vinden die het concept ‘weten’ perfect omschrijven.

Logica ligt in de doorsnede van deze vier vakgebieden, het is in feite een manier om dit soort kwalitatieve informatie zo precies mogelijk te beschrijven, en zo beter te begrijpen. Omdat er veel manieren zijn om over eigenschappen te praten zijn er ook veel verschillende logische systemen, het logische systeem dat ik in deze scriptie gebruik heet modale logica.

In de propositielogica kunnen we spreken over simpele uitspraken die gewoonweg waar of onwaar zijn, bijvoorbeeld: ‘het regent’ en ‘als Jan naar het feest komt, dan komt Marietje niet’. De modale logica breidt dit uit door een manier te geven om te praten over uitspraken die zogeheten modaliteiten bevatten. Modaliteiten zijn woorden, of zinsdelen, die de betekenis van een zin aanpassen. Denk bijvoorbeeld aan ‘noodzakelijk’, ‘bewijsbaar’ of ‘in de toekomst’, zo kunnen we het ook over concepten als geloof en kennis hebben. Hiermee krijgen we zinnen zoals: ‘het is noodzakelijk dat het regent’ en ‘als Anna weet dat Jan naar het feest komt, dan gelooft ze dat Marietje niet komt’.

Om op een goede manier over deze verschillende concepten te kunnen praten hebben we verschillende modale systemen nodig, die ieder andere aannames maken. Zo is het bijvoorbeeld redelijk om aan te nemen dat we iets alleen maar kunnen weten als het ook echt waar is, maar kunnen we best onware dingen geloven. Het specifieke systeem dat ik bekijk is interessant omdat het een behoorlijk grote uitdrukkingskracht heeft, maar zich toch nog op een, volgens logici, ‘nette’ manier gedraagt.

De eigenschap die ik voor dit systeem onderzoek heet interpolatie. Dit is een eigenschap die logische systemen hebben als ze op een consistente manier theorieën (stukken informatie) die elkaar niet tegenspreken bij elkaar kunnen voegen. Deze eigenschap is bijvoorbeeld nuttig als twee bedrijven gaan fuseren, en ze hun databanken willen samenvoegen. Ik onderzoek deze eigenschap aan de hand van speciale soorten algebra’s.
Op de middelbare school leer je dat algebra iets is waarbij je een ‘$x$’ moet vinden. In de wiskunde nemen we het idee van algebra veel breder. Met ‘een algebra’ bedoelen we een verzameling objecten, zoals bijvoorbeeld getallen, en een aantal operaties op die verzameling. In het geval van de vergelijkingen waar je een waarde voor ‘$x$’ moet vinden bestaat de algebra waar je mee werkt uit de reële getallen en de operaties van optellen en vermenigvuldigen (aftrekken en delen zijn de ‘omgekeerde’ van deze operaties). Logici hebben ook hun eigen algebra’s, een ander soort voor ieder logisch systeem. Net zoals we met getallen kunnen rekenen, kunnen we dat ook in zekere zin met uitspraken. De uitspraak ‘het regent en de straat wordt nat’ is te zien als het ‘product’ van de uitspraken ‘het regent’ en ‘de straat wordt nat’. Net zoals dat het product van twee getallen niet-nul is alleen maar wanneer ze allebei niet-nul zijn, zo is ook de *conjunctie* van twee uitspraken (dat wil zeggen, er het woordje ‘en’ tussen zetten) alleen maar waar als allebei de uitspraken al waar zijn.

Bij modale logica hoort (verrassend genoeg) *modale algebra*. Bij ieder systeem van modale logica hoort een bepaald type modale algebra, en het blijkt dat een modaal systeem de interpolatie-eigenschap heeft precies wanneer de klasse modale algebra’s die erbij horen de zogenaamde *amalgamatie*-eigenschap heeft. Deze eigenschap geldt voor een klasse algebra’s wanneer we individuele algebra’s uit die klasse op een ‘nette’ manier kunnen samenvoegen. Dit feit gebruik ik om voor een bepaald deel van het logische systeem dat ik gebruik te laten zien dat het de interpolatie-eigenschap heeft, en voor een ander deel dat het de eigenschap niet heeft.

Hiermee is het werk nog niet klaar. Er zijn nog veel stukken van mijn gekozen systeem waarvan we nog niet weten of ze de interpolatie-eigenschap hebben. Daarnaast zijn er nog veel andere systeem van modale (en andere) logica waarvoor we de eigenschap nog moeten onderzoeken.
Bibliography


