Instructional Strategies to Support Students’ First Steps into Proving and Reasoning

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ABSTRACT

Proof and reasoning are the core aspects of the nature of mathematics. This study emphasizes the importance of these aspects in the mathematics classroom and the engagement of students in meaningful geometrical proving tasks. The main aim of my intervention was to investigate how young students (13-14 years old) with no or minimum familiarity with proving statements, conceive the notions of proof and convincing in the mathematics classroom. For this purpose, I designed and taught a four lesson sequence to 21 students in the Amsterdam International Community School. I examined the quality of students’ proofs and the types of reasoning used. In addition, I evaluated the effectiveness of my instructional scaffolding practices during these initial attempts of students to prove geometrical statements. The results suggest that the students could in general engage in proving activities and were able to move from simple to more complex empirical reasoning or to deductive reasoning when working on the designed tasks. Carefully applied instructional scaffolding guidance was found to support students’ autonomy and independence to carry out the tasks.
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# Table of Contents

List of Figures ................................................................................................................. iii
List of Tables .................................................................................................................. vii

1 Introduction ..................................................................................................................... 1

2 Theoretical Framework .................................................................................................. 3

2.1 What is Proof? ............................................................................................................ 3

2.2 Proof and Reasoning in the Mathematics Classrooms ............................................... 3

2.3 Proof Taxonomies .................................................................................................... 6

2.4 Proof and the Dynamic Learning Environment .......................................................... 9

2.5 A Learning Environment Appropriate for Proving and Reasoning .......................... 11

2.6 The Instructor’s Role in Establishing Good Communication: Instructional Scaffolding .................................................................................................................. 12

3 Research Questions and Research Setting .................................................................... 17

3.1 Research Questions and General Expectations ....................................................... 17

3.2 Research Setting ...................................................................................................... 19

4 Research Design ........................................................................................................... 21

4.1 Design Strategies ...................................................................................................... 21

4.2 Lesson Design ......................................................................................................... 24

4.2.1 Lesson 1 ............................................................................................................ 25

4.2.2 Lesson 2 ............................................................................................................ 29

4.2.3 Lesson 3 ............................................................................................................ 36

4.2.4 Lesson 4 ............................................................................................................ 38

4.3 Design of Interviews ............................................................................................... 41

4.4 Summary of Data Sources and Implementation Timeline ....................................... 42

5 Analytical Framework .................................................................................................. 45

5.1 Quality of Students’ Proofs ..................................................................................... 46

5.2 Types of Students’ Reasoning .................................................................................. 46

5.3 The Instructional Scaffolding Strategies ................................................................ 48

6 Analysis and Results .................................................................................................... 51

6.1 Lesson 1 .................................................................................................................. 52

6.2 Lesson 2 .................................................................................................................. 66

6.3 Lesson 3 .................................................................................................................. 96
List of Figures

Figure 2.1 Harel & Sowder's Proof Schemes (1998; 2001) .......................................................... 8
Figure 4.1 Angles around us ..................................................................................................... 26
Figure 4.2 Kinds of angles ....................................................................................................... 26
Figure 4.3 Worksheet ‘Angles on the Grid’ ........................................................................... 27
Figure 4.4 Applet for Vertically Opposite Angles ................................................................. 28
Figure 4.5 Worksheet ‘Vertically Opposite Angles’ ............................................................... 29
Figure 4.6 Worksheet ‘Corresponding Angles’ ..................................................................... 31
Figure 4.7 Worksheet ‘Co-interior Angles’ ............................................................................. 32
Figure 4.8 Tangram ................................................................................................................ 33
Figure 4.9 Tangram Triangles ............................................................................................... 34
Figure 4.10 Quadrilaterals .................................................................................................... 34
Figure 4.11 Worksheet ‘Compare Triangles’ ........................................................................ 35
Figure 4.12 Applet ‘Angle Sum of Triangle’ ....................................................................... 36
Figure 4.13 Worksheet ‘Angle Sum of Triangle’ .................................................................. 37
Figure 4.14 Applet ‘Thales’ Theorem’ ................................................................................ 38
Figure 4.15 Worksheet ‘Thales’ Theorem’ ........................................................................ 39
Figure 4.16 Worksheet ‘Trapezium’ .................................................................................... 40
Figure 5.1 Phoebe & Cassie's work ..................................................................................... 50
Figure 6.1 Angles on the Grid ............................................................................................. 52
Figure 6.2 Applet for Vertically Opposite Angles ................................................................. 55
Figure 6.3 Worksheet ‘Vertically Opposite Angles’ ............................................................. 56
Figure 6.4 Haruki & Sean's work ......................................................................................... 57
Figure 6.5 Jasmine & Lee's drawing ..................................................................................... 59
Figure 6.6 Stian & Simon's work ......................................................................................... 60
Figure 6.7 Julie & Isis's work ............................................................................................. 62
Figure 6.8 Victoria & Jacque's work ................................................................................... 63
Figure 6.9 Drawing on board ............................................................................................... 66
Figure 6.10 Worksheet ‘Corresponding Angles’ ................................................................ 69
Figure 6.11 Phoe & Cassie’s work ....................................................................................... 70
Figure 6.12 Worksheet ‘Co-interior Angles’ ....................................................................... 71
Figure 6.13 Haruki & Stian’s work ....................................................................................... 72
Figure 6.47 Ehsan’s worksheet on the 1st task ................................................................. 128
Figure 6.48 Ehsan’s worksheet on the 2nd task .............................................................. 131
List of Tables

Table 5.1 Quality of Proofs........................................................................................................46
Table 5.2 Kinds of Reasoning..................................................................................................47
Table 5.3 Instructional Scaffolding Strategies.........................................................................49
1 Introduction

Teaching mathematics to secondary school students comes with great challenges. Many students claim that mathematics is ‘hard’, ‘incomprehensible’ or ‘useless’. Students usually connect mathematics to routine procedures and algorithms that are meaningless and consequently make it hard for them to understand. Perhaps the reason for the latter relates to the way they are asked to learn mathematics. It could also be due in part to the lack of an appropriate and meaningful (to them) rationale for most of the things they are expected to learn.

Proof is a form of communication in terms of the mathematical language. When writing a proof, the goal is to convince of the truth of a statement. The nature and notion of proof should be used in the mathematics classroom for making mathematics reasonable and interesting to students and that is why proof plays an important role in the mathematics curricula.

In the NCTM Standards (2000), great emphasis is paid to reasoning and proof at all levels of schooling. According to the standards, students should be able to (as they appear in Hanna, 2000):

- Recognize reasoning and proof as fundamental aspects of mathematics;
- Make and investigate mathematical conjectures;
- Develop and evaluate mathematical arguments and proofs;
- Select and use various types of reasoning and methods of proofs.

However, requiring students to prove is not effective if the proving process provides little or no understanding. Hanna (2000) pointed out the challenge that the mathematics teacher faces when it comes to proving in the mathematics classroom:

Proof is an important part of mathematics itself, of course, and so we must discuss with our students the function of proof in mathematics, pointing out both its importance and its limitations. But in the classroom the key role of proof is the promotion of mathematical understanding, and thus our most important challenge is to find more effective ways of using proof for this purpose. 

(Hanna, 2000, p. 5)
In this research it was my aim to invite young students (13-14 years old) to prove and through their engagement in the activities, to promote the understanding of the nature of mathematics. Inviting students of this age to proving activities consists of an intervention, since the notion of mathematical proof is normally introduced to students in the later grades curricula. My interest in this research dates back to my school years and the difficulties I encountered then when working on geometrical proving tasks. It was my desire to develop design strategies which would allow me to investigate if I could eliminate the difficulties identified in my own and my friends’ experiences that I also expected present students to have. Because of the traditional teaching approach that mostly dominated my experiences as a student, I was not often encouraged or required to share my ideas and express my thoughts. As a result, in this research project I wanted to stress the need for students to share their thinking and express their reasoning, and pay special emphasis to the importance of becoming independent thinkers. My expectation was that different activities from the ones which they usually encounter, and which are challenging at the same time, would draw their attention. In fact, my expectation was met.

In addition to the above, a research project (Winter Project)\(^1\) conducted last year by another master student and me led to the present study. In our project we examined how young students engaged in proving mathematical relationships in Euclidean geometry. My purpose for the master study research reported in this thesis was to investigate further the students’ initial attempts to prove, their reasoning skills, and whether the integration of a dynamic learning environment would influence their progress. In addition to that, I aimed to find out how a teaching strategy can be used to support students’ learning.

In Chapter 2, I briefly discuss the literature on proofs, mathematical reasoning, the use of a dynamic learning environment for the teaching of proofs, and the idea behind instructional scaffolding. In Chapter 3, I present my research questions and the setting of the research; this is followed by my research design in Chapter 4. The framework I used to analyze my data is described in Chapter 5 and the analysis of my results is given in detail in Chapter 6. Finally, in Chapter 7 I discuss the findings of my research by answering the research questions and by recommending issues for future research.

\(^1\) Last year, my colleague and I developed and constructed a small research study in the Amsterdam International Community School. We taught two successive lessons concerning geometrical proofs to second-grade high school students and reported our research results in the Winter Project final report.
2 Theoretical Framework

In this chapter, I briefly discuss the research literature concerning proofs, reasoning and the use of a dynamic learning environment in the mathematics classroom. I continue by discussing how a learning environment can encourage the development of students’ proving and reasoning skills. Finally I outline the value of instructional scaffolding in establishing a good communication between the instructor and the students.

2.1 What is Proof?

Proof is commonly accepted as the cornerstone of mathematics. Rav (1999) invites us to realize that “proofs rather than the statement-form of theorems are the bearers of mathematical knowledge”. But what is a proof? Proof is an organized and logical sequence of arguments that lead to a general statement. Mathematicians tend to consider and value only the axiomatic proofs which are the basis for the construction of rigorous mathematics. However, when it comes to proving in the mathematics classroom, there is space for flexibility and emphasis on understanding rather than on rigor and formality. Students’ reasoning then comes to play an exceptional role in the development of deeper understanding of the concept of proof.

2.2 Proof and Reasoning in the Mathematics Classrooms

In the mathematics classroom proof is defined differently. Educators have given their own definitions and meanings of what a proof is and its role and position in the classroom. Bell (1976, p. 26) defines proof as follows:

Proof is a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or generally agreed facts or principles.

Stylianides’ perspective of proof is different. His view is socially extended and emphasizes the role of proof in the classroom. In his paper “Proof and Proving in School Mathematics” (2007, p. 291) he developed his notion of proof:

*Proof is a mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of the classroom community; and
3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community.

Mathematics educators recognize the importance of proving in the classroom. Opposed to the views of the past when only a strictly formal proof was accepted, in the past few decades more flexible views have appeared. Proof is nowadays “an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing” (Hanna, 1989). Convincing in the classroom must correspond to understanding. Consequently, proof must be included in the mathematics instruction in such a way that promotes understanding instead of creating an obstacle on the way towards it. Hanna (1990) made a distinction between proofs that prove and proofs that explain. Though both kinds of proofs are valid, the first show that a theorem is true while the second show why a theorem is true (Hanna, 1990), making it a meaningful tool for the students’ literacy in mathematics. Ross (1998) states: “Time need not be wasted on the technical details of proofs, or even entire proofs, that do not lead to understanding or insight” (p. 254). Tymoczko’s view on mathematical proof (as it appears in Hanna, 1991) best describes my own:

Mathematical proofs... generally have a certain openness and flexibility. They can be paraphrased, restated and filled out in various ways, and to this extent they transcend any particular formal system. We might say that an informal proof determines an open-ended class, or family, to use Wittgenstein’s term, of more specific proofs. (p. 57)

Convincing seems to be essential for the students, but Bell (1976) pointed out the danger that conviction may be reached by other means than through a logical proof. Educators must be really careful to maintain and underpin the real nature of proof and its mathematical meaning. De Villiers (1999) also shared his concern for limiting the meaning, purpose and usefulness of proof only to verifying the correctness of mathematical statements; he expressed the importance of showing to the students all functions of proof:
More than anything else, it seems that the fundamental issue at hand is the appropriate motivation of the various functions of proof to students. The question is, however, “what functions does proof have within mathematics itself which can potentially utilized in the mathematics classroom to make a proof a more meaningful activity?” (De Villiers, 1999)

Bell attributed three senses to the meaning of proof: verification or justification that is concerned with whether a statement is true, illumination, which is concerned with answering why a statement is true, while the third one reflects the formal mathematical notion of proof and it is systematization, i.e., “the organization of results into a deductive system of axioms, major concepts and theorems, and minor results derived from these” (Bell, 1976, p. 24). De Villiers (1999) built on these senses and expanded them into a useful list of the functions of proof:

- verification (concerned with the truth of a statement);
- explanation (providing insight into why it [the statement] is true);
- systematization (the organization of various results into a deductive system of axioms, major concepts and theorems);
- discovery (the discovery or invention of new results);
- communication (the transmission of mathematical knowledge);
- intellectual challenge (the self-realization/fulfillment).

Besides the importance of proof in learning to think mathematically and to move towards logical mathematical thinking, educators pay attention to the role of reasoning in students’ literacy in mathematics. Undoubtedly, proof and reasoning cannot be considered as separate streams. Instead, they should be considered as both necessary and interrelated tools for fostering students’ deeper understanding of mathematics. Ross (1998, p. 253-254) underlines the value of reasoning in mathematics’ instruction:

One of the most important goals of mathematics courses is to teach students logical reasoning. This is a fundamental skill, not just a mathematical one. […] It should be emphasized that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning. Thus the essence of mathematics lies in proofs, and the distinction among illustrations, conjectures, and proofs should be emphasized. It should be

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2 The page numbers from De Villiers (1999) are not available.
stressed that mathematical results become valid only after they have been carefully proved. [...] Construction of valid arguments or proofs and criticizing arguments are integral parts of doing mathematics. If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense.

Ball and Bass (2003) value mathematical reasoning and proof to be essential as both as an end and a means. They identify two functions of reasoning: the reasoning of inquiry that serves as a means for discovering and examining new ideas and the reasoning of justification through which mathematical claims are proved and justified. Ball and Bass (2003) define reasoning from a social perspective that matches the present study. According to them, “reasoning comprises a set of practices and norms that are collective, not merely individual or idiosyncratic, and rooted in the discipline”.

2.3 Proof Taxonomies

The role and importance of proof and reasoning in the mathematics classroom has tempted researchers to study the students’ conceptions of proof, their argumentation when asked to prove, and their responses to proving. In this section, I discuss the most predominant frameworks concerning proof and reasoning that are found in the literature.

Balacheff: Pragmatic versus Conceptual Proofs

Balacheff (1988) examined the process of how pupils were convinced of the validity of the proposed solution. For that, he created a social setting where pupils were expected and required to discuss with each other. He distinguished pragmatic from conceptual proofs and made clear that both are called proofs because they are recognized as such by the students and not because they are valid mathematical proofs. “Pragmatic proofs are those having recourse to actual action or showings, and by contrast, conceptual proofs are those which do not involve action and rest on formulations of the properties in question and relations between them.” (Balacheff, 1988, p. 217). Pragmatic proofs are the ones that are based on the use of examples whereas conceptual proofs are based on abstract and formal formulations. Pragmatic proofs include three types of justifications: naïve empiricism where the statement to be proven is verified by some examples; crucial experiment where the statement to be proven is checked in a specific and carefully chosen example; and generic example where the justification for the truth of an assertion is based on transformations or operations of an object
that is a characteristic representative of its own class. Conceptual proofs are distinguished in two types of justifications: *thought experiment* that involves action “by internalizing it and detaching itself from a particular representation”; and *symbolic calculations* where the justification is based on the use of symbolic expressions.

According to Balacheff (1988) the above forms of proof form a hierarchy. However, it is not clear what is meant by this. Are these categories from pragmatic to conceptual, steps that the student passes from or has to pass from in order to reach the desired ‘correct’ answer while proving a task? Or do these categories consist of the steps that build upon the students’ development in proving through a period of time? Whatever the case is, I consider these steps not necessarily a hierarchy in the sense that not all students need pass through every one of them. It is possible that a student jumps from the naïve empiricism to the generic example without ‘experiencing’ the crucial experiment phase. Moreover, when Balacheff (1988) considers the categories as a hierarchy, he does not leave open the possibility of moving through the forms of proof via another route. For instance, when a student finds her/himself ‘experiencing’ the thought experiment, Balacheff’s hierarchy excludes the possibility that s/he might return to the generic example stage.

**Harel & Sowder: Proof Schemes**

After observing students in the process of proving and examining the actions taken by them, Harel and Sowder (1998) classified the following categories (their completed framework on proof schemes is shown on Figure 2.1) of proof schemes that represent “a cognitive stage, an intellectual ability in students’ mathematical development”.

*External Conviction Proof Schemes*: “Proving […] depends (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (for example, proofs in geometry must have a two-column format), or (c) on symbol manipulations” (Harel & Sowder, 2007, p.809). In sum, Harel and Sowder recognize three subcategories of the external conviction proof schemes: *authoritarian, ritual* and, *symbolic* proof schemes.

*Empirical Proof Schemes*: Students rely either on (a) evidence from examples and testing the statement and their conjecture by simply measuring (*inductive* proof scheme) or (b) their perceptions (*perceptual* proof scheme).

*Deductive (Analytical) Proof Schemes*: In this category a conjecture is validated by means of logical deductions (Harel & Sowder, 1998). Two subcategories are classified here: *transformational* proof schemes that are characterized by generality, operational thought and
logical inference, and the *axiomatic* proof schemes with the above three characteristics plus more (Harel & Sowder, 2007).

**Figure 2.1 Harel & Sowder’s Proof Schemes (1998; 2001)**

Harel and Sowder’s proof schemes resulted from studying the concept of proof from different perspectives. Mathematical and historical-epistemological, cognitive and instructional-social factors all were taken into account for the construction of the proof schemes (Harel & Sowder, 2007). For mathematics educators and researchers, paying attention to all these factors is extremely important and necessary for understanding the students’ conceptions of proof and their understanding of this crucial subject. At the same time, however, it is very ambitious to include all these parameters in one framework. Harel and Sowder created a theoretical construct with a large number of different subcategories representing the cognitive
stages of students’ mathematical development after an extended series of studies. Also, they found points on this list (Figure 2.1) from many different students and at no point do they claim that the framework could represent any individual or even a single group of students. Therefore, using their theoretical construct on a case study or a small case study research (like the present master study) did not appear to be feasible.

Bell: Empirical and Deductive Justifications

Bell’s classification of students’ responses to geometrical tasks inspired the analytical framework of the present study. Bell (1976) distinguished empirical from deductive justifications in students’ responses to geometrical tasks. Empirical justifications are the ones where the use of examples is the means for conviction of the truth of a hypothesis, whereas deductive justifications are characterized by the appearance of a deductive element. For each of the two categories, Bell identified many subcategories that in the case of empirical answers describe the students’ awareness of the requirement to find a whole set of finite examples, and in the case of deductive response, the students’ abilities to complete a deductive sequence of reasoning.3

Summary

In all three frameworks above the same distinction is made. Balacheff, Harel and Sowder, and Bell have each distinguished two main categories according to the use of examples or approaches that contain deductive elements. For this study, I decided to analyze the students’ reasoning following the same scheme as Bell (1976) with two main categories: empirical and deductive. The hierarchy issue as discussed in Balacheff’s taxonomy discouraged me from talking about a hierarchy in my taxonomy. It is more likely and simultaneously desirable that the students move from the empirical to deductive reasoning, but the other way around may be observed as well.

2.4 Proof and the Dynamic Learning Environment

The use of dynamic learning environments in mathematics education has been investigated by several researchers. More specifically, findings when students engage in mathematical proofs with the use of such dynamic environments have been presented.

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3 Examples of subcategories are: extrapolation, non-systematic, partially systematic and so on for the empirical answers and relevant-general restatement, connected-incomplete, complete explanation etc. for the deductive answers.
Dynamic geometry software such as Cabri-géomètre or GeoGebra may positively contribute in students’ learning and understanding of geometrical properties. DGS (dynamic geometry software) offers students the possibility to experiment freely, to generate and test conjectures, and to discover general properties and mathematical relationships on their own. Indeed, “dynamic software has the potential to encourage both exploration and proof, because it makes it so easy to pose and test conjectures” (Hanna, 2000, p. 13). The dynamic nature of these learning environments allows students to visualize many different examples of a problem, something that does not happen with the traditional paper and pencil method. In fact, “in contrast to paper and pencil representation, the visual output of Cabri [Cabri-géomètre] is not a drawing of one instance of a geometry figure, but rather can be moved – or dragged – around the screen with its constructed properties preserved” (Healy & Hoyles, 2001, p. 1).

There are however researchers who express concerns regarding the use of DGS, supporting that it may promote the generation of empirical justifications and thereby hinder the process to formal proofs (Chazan, 1993; Hoyles & Jones, 1998).

Marrades and Gutiérrez (2000) conducted a research study with 4th grade students in secondary school (15-16 years old). A teaching experiment with the use of Cabri-géomètre was carried out for 30 weeks, and the authors presented a case-study describing the performance and progress of two groups of students. According to their results, digital software “may well help secondary school students understand the need for abstract justifications and formal proofs in mathematics” (Marrades & Gutiérrez, 2000).

Jones (2000) came to similar conclusions while investigating 12-year-old students’ abilities to conjecture and generalize on the classification of quadrilaterals. The students altered their thinking from “everyday” expressions to formal mathematical explanations as participants of a teaching unit focused on the use of a dynamic geometry software.

Given the similarity of GeoGebra to Cabri-géomètre or other dynamic software environments used in mathematics education (e.g. Geometer’s Sketchpad or Geometric Supposer), it may be assumed that students, when interacting with GeoGebra applets, are able to explore, make and test conjectures and explain geometrical properties and relationships. Despite some reservations (Chazan, 1993; Hoyles & Jones, 1998), though relatively few, concerning the use of DGS in the classroom, it is my hypothesis that DGS environments are more likely to facilitate students proceeding to formal proofs. Especially for younger students, DGS
promotes the generation of arguments and justifications, a way of thinking in which they are not used to engage. Furthermore, interacting in a dynamic learning environment “may be valuable in providing a foundation for further work on developing deductive reasoning” (Jones, 2000, p.57).

However, it goes without saying that for the use of such an environment in order to encourage the development of students’ mathematical understanding and the promotion of reasoning and proving abilities, other conditions must simultaneously hold: “a well-established and carefully nurtured classroom culture that values mathematical thinking, a sequence of carefully selected tasks, and a range of appropriate prompts from the researcher and teacher (and possibly other classmates)” (Hoyles & Jones, 1998, p. 126).

2.5 A Learning Environment Appropriate for Proving and Reasoning

In order to facilitate the students’ development in proving and making clear the requirement for reasoning, the teacher needs to create a learning environment appropriate for fostering these two interlinked aspects of classroom mathematics. In the traditional way of teaching, the teacher is the transmitter of knowledge, a lecturer in the classroom with an audience if not passive, then limitedly active. Proof in such an environment is likely to be accepted as a routine procedure which makes it difficult for students to explore meaning and develop independence when working with proving tasks. To the contrary, a learning environment where proof becomes a matter of discussion, an end to reach through explanations and contributions from all the classroom participants, is much more likely to strengthen students’ understanding of the notion of proof and consequently their deeper understanding of mathematics.

Such a learning environment is one in which the student is in the process of constructing his/her own knowledge. Ball and Bass (2003) argue that this happens through encouraging mathematical reasoning, thus making mathematics rational for the students as through the eyes of the mathematicians.

The ways in which students seek to justify claims, convince their classmates and teacher and participate in the collective development of publicly accepted mathematical knowledge have powerful resonances with mathematicians’ work. (Ball & Bass, 2003, p.29)
It is therefore important for students to participate in activities that highlight the nature of mathematics and make them members of an authentic mathematical community, i.e., similar to one of the mathematicians. The ‘laws’ of such a classroom community can be made explicit to students through the establishment of “social” and “sociomathematical” norms (Yackel & Cobb, 1996). Among the social norms that may well promote a student-centered instruction approach and constructivist learning environment is working in groups, participating in classroom discussions, providing explanations for their own solutions, listening, and criticizing the solutions of others.

Yackel and Cobb defined “sociomathematical norms” as the social norms that are “specific to the mathematical aspects of students’ activity” (Yackel & Cobb, 1996, p. 458). For example, the understanding that students should explain what is said in the classroom is a social norm, whereas the understanding of what is accepted as a mathematical explanation is a sociomathematical norm (Yackel & Cobb, 1996). Yackel and Cobb (1996) argue that the construction and adaptation of sociomathematical norms are extremely important and lead students to intellectual autonomy. It is therefore necessary for the teacher to make evident to the students the requirements and expectations of the mathematics classroom. The instructional practices that the teacher applies in the classroom and his/her way of communicating with the students may enable the development of reasoning and strengthen students’ proving abilities.

2.6 The Instructor’s Role in Establishing Good Communication: Instructional Scaffolding

The communication process between the teacher and the students must be studied carefully since good communication can be extremely helpful for developing both parts’ knowledge and understanding. Well-established communication between teacher and students and amongst students is beneficial for all participants in the sense that through dialogue or classroom discussions everyone’s thoughts become public and therefore a source of fruitful exchange of opinions. When students share their thoughts, then the teacher is able to trace both their current understanding and possible misunderstandings. As for the rest of the students, when they hear their classmates’ thinking, they receive stimuli that most probably lead to their own development. Through agreements or disagreements, the students have to support their opinions and therefore become responsible for their own learning. On the other
hand, it is of crucial importance for the teacher to reveal his/her own thoughts so that students receive feedback and are kept on track.

A healthy communication has its foundation in the constructivist view in which students are active participants in the classroom community and construct their knowledge rather than merely receive it. This comes as a result of meaningful effort on the part of the teacher. In the following discussion I focus on the communication between teacher and students and more specifically on the actions taken by the teacher that aim to create a setting where the students are encouraged and at the same time required to express their own thinking.

**Instructional scaffolding**

*Instructional scaffolding* is a “process in which a teacher supports students cognitively, motivationally, and emotionally in learning while helping them to further develop autonomy” (Meyer & Turner, 2002, p. 18). Differently stated, “it is the provision of guidance and support which is increased or withdrawn in response to the developing competence of the learner” (Mercer, 1995, p. 75). A way to provide scaffolding to the students is through prompting them into taking actions, asking them questions, confirming students’ statements.

Mercer (1995) writes that “one of their [the teachers’] aims is to guide the learning activity of their students along directions required by a curriculum, and to try to construct a joint, shared version of educational knowledge with their students” (p. 25). According to Mercer, teachers share some common techniques to guide learning. When they talk, teachers either *elicit relevant knowledge from students* through direct or cued elicitations, or *respond to things that students say* through confirmation, rejection, repetition, elaboration or reformulation, or *describe the classroom experiences that they share with students*. Through ensuing talk, teachers can realize what the students know and show what knowledge is commonly understood by both. Teachers also provide feedback and collect all students’ inputs for constructing generalized concepts. Finally through describing the experiences shared with the students, teachers make visible and pay attention to the importance of these experiences and the connection with the ones in the past and the ones that will come in the future so the continuity of the things taught is obvious to the learners. In order to succeed in this, the teachers may use techniques as ‘we’ statements, literal, and/or reconstructive recaps (Mercer, 1995).

Meyer and Turner (2002) underline the importance of scaffolding practices and highlight four types of interaction that support such practices: establishing a supportive classroom climate,
building competence and maintaining a focus on learning, providing opportunities for developing autonomy and supporting self-regulatory processes through shared responsibility. More closely to this study, researchers investigate instructional scaffolding that promote the development of reasoning and proving abilities. The teacher must create a classroom environment where the students are led and encouraged to make conjectures, make attempts towards generalization, justify their claims, unfold their way of thinking, and listen and reflect on their peers’ contributions. Through posing questions the teacher can build such an environment like that described by Martino and Maher (1999) who examined three types of questioning strategies: *questions that facilitate justification, questions offering the opportunity for generalization and questions that prompt students to make connections.*

[...] teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and/or generalization. Teacher questioning can invite students to reflect on their own ideas; further it can open doors for teachers and students to become more aware of each other’s ideas. (Martino & Maher, 1999, p. 75)

Blanton, Stylianou and David (2003) investigated the role of instructional scaffolding on undergraduate students’ understanding of proof. For this, they developed a framework for analyzing the teacher’s utterances in classroom discussion. Their framework\(^4\) contains the following utterance categories: transactive, facilitative, didactive and directed. Transactive utterances request *critiques, explanations, justifications, clarifications, elaboration* and *strategies* used or considered to use. Facilitative utterances are *revoicing* or *confirming* students’ contributions. *Didactive utterances* refer to the teacher’s interference about mathematical proof in order to help students understand its nature. Finally *direct utterances* are the ones where the teacher chooses to “tell directly rather than elicit the information indirectly” (Blanton, et al, 2003).

The framework created in this master study was used in order to analyze my instructional scaffolding strategies during whole-class discussions as well as my questions or comments while supervising the students when working in groups. My strategies match better with the

\(^{4}\) The framework presented in the Blanton, et al. paper is an extension of those from Krugger (1993) and Goos, Galbraith, and Renshaw (2002) (as cited in Blanton, et al., 2003). These frameworks are based on peer interactions; however, in this thesis I refer to and describe only those categories that Blanton, et al. defined for analyzing the teacher’s utterances to students.
transactive and facilitative utterances from Blanton, et al. (2003). However, the teaching strategies described in my framework are not distinguished with respect to whether they serve as facilitative or transactive utterances since each of them is aimed to facilitate students’ development of learning in the sense that they are prompted and encouraged to construct their own knowledge. Furthermore, every strategy aims at the transaction of knowledge in the classroom. My framework’s strategies, in this sense, serve both as facilitative and transactive utterances, but not didactive nor direct.
3 Research Questions and Research Setting

In this chapter, I present my research questions and my general expectations for my research study. In the second section, I share information concerning the research setting.

3.1 Research Questions and General Expectations

The present research was for me a chance to investigate young students’ first steps in proving geometrical statements. I aimed to see how students conceive the notion of proof and if their definition of proof matches the formal one of mathematicians. Because of their age (13-14 years old), the students were not familiar with proving tasks and thus I expected that their understanding of what constitutes a proof would not meet the formal notion of proof. Therefore, I was interested to identify the difficulties that they would encounter in the transition from empirical proofs (based on visual facts, measurement or intuitive arguments) towards the ones that are mathematically acceptable and the progress of this process across the lesson series.

Exploring students’ reasoning skills was also something that had my attention because I designed my instruction to include tasks that would require the students to share and explain their thoughts. I was keen to explore what kinds of arguments students use in order to convince someone of the truth of a mathematical statement and the role played by the integration of a dynamic mathematics learning environment in generating types of reasoning skills. In the literature there is evidence (e.g., Marrades & Gutiérrez, 2000; Jones, 2000) suggesting that the use of a dynamic learning environment contributes to the development of students’ proving abilities and their reasoning skills. However, reservations expressed by some researchers (e.g., Chazan, 1993; Hoyles & Jones, 1998) prompted me to investigate for myself the use of GeoGebra in the classroom.

Finally, I was interested to try out the characteristics of a teaching approach which would support and foster students’ efforts to prove geometrical statements. Are there some strategies that a teacher can adopt (or adapt) in order to create a classroom environment that inspires generation of conjectures and enables students to reason and prove?

My aims and interests led to the following research questions:
Research Question 1: How do students engage in proving mathematical relationships on the subject of angles, triangles and quadrilaterals?

Sub-question 1.1: What changes in students’ approach(es) to proving can be identified across the lesson series?

Sub-question 1.2: What difficulties do students encounter when working on geometrical proving?

Research Question 2: How do students reason when asked to convince someone of the truth of a statement on the subject of angles, triangles and quadrilaterals?

Sub-question 2.1: What kinds of reasoning do students use to convince someone of the truth of a statement?

Sub-question 2.2: Does the use of the dynamic learning environment (GeoGebra) influence students’ reasoning?

Research Question 3: How can the designed teaching strategy be used to support students’ initial attempts to prove?

Sub-question 3.1: How does the teaching strategy work in practice?

Sub-question 3.2: Does instructional scaffolding elicit desired kinds of responses from students?

Sub-question 3.3: What are the indications that students are starting to think for themselves?

In order to answer the above research questions I conducted an exploratory design project and analyzed the data collected qualitatively. I elaborate further on my expectations when discussing my aims for designing the study in Chapter 4, Research Design.
3.2 Research Setting

The research setting consists of the information concerning the school and the classroom where my research took place as well as information on the participants and my cooperating teacher.

The School

My research took place in the Amsterdam International Community School (AICS). The reason I chose an international school is because of the language of instruction, that is English. Since I do not speak or understand the Dutch language, teaching in a Dutch school would cause me difficulties. Though teaching in English in Dutch schools in the upper secondary levels is possible, in the lower secondary grades it is not recommended since the students are not expected to be fluent English speakers. Therefore, conducting my research in an English speaking community was beneficial for both me and the students and did not cause me any inconvenience during the data analysis procedure.

The Classroom

In the class where I conducted my research, there was a computer for the teacher and a beamer. Internet connection was also available in the classroom. That gave me the chance to present to the students through PowerPoint and have the GeoGebra applets, that were designed for the lessons, projected for all the classroom participants to see and therefore enable and facilitate the generation of classroom discussions. In the classroom there were no computers for the students and that is why, after consulting the cooperating teacher, I asked the students to bring their own laptops in the classroom for two of the four lessons for which the lesson design suggested the integration of the dynamic learning environment in group work through GeoGebra applets. Although I had the opportunity for my research to take place in the computer room of the school, I chose not to. The reason was that I had a teaching experience a year ago in the same computer room, again as a part of a research project, and realized that a computer room (at least that specific one) is not an appropriate learning environment. In the computer room students were situated in such a way that they were facing the computer screens and not the blackboard and therefore it was not easy for the teacher to draw their attention. Furthermore, the working place of each desk was very small, because of the space taken by the computer, making it hard for the students to write on their

5 The school’s webpage: http://www.aics.espritscholen.nl/
worksheets especially during group work. Also unlike laptops which are flexible to move for the convenience of all group partners, fixed hardware made it difficult for all group mates to have a good view of the screen and play with the applets. All the above contributed to the disorganization of the previous teaching and learning process.

The Students

The participants in this study were 21 second grade secondary students (13-14 years old). Of the 21 participants, 8 were males and 13 females. As students in an international school, the educational as well as cultural background of the students varied. Students’ home countries were the Netherlands, Japan, Brazil, Iran etc. It is customary for the students in an international school to change schools often or be educated in two schools at the same time. Therefore, the educational background of the students is very likely to differ. Consequently, it is possible for some students to be more familiar than others with some mathematical concepts.

The Cooperating Teacher

The regular teacher of the classroom was my cooperating teacher while I conducted my research. I consulted her, before entering the classroom, for the composition of the groups of students and for the use of the students’ laptops in the classroom. She informed me of the students’ cultural background and for the differences that may exist in the students’ educational background. The cooperating teacher was present in all my lessons and helped me by operating the data collection instruments (video and audio recorders) and also supervised with me the students while group work. In addition, her experience as a researcher, since she is a former student of the Master in Mathematics and Science Education, made our communication easier since she knew the demands of an educational research project.
4 Research Design

My aim for conducting the present research was to investigate young students’ conceptions of the notion of mathematical proof and how they engage in proving geometrical tasks. This consists of an intervention in the sense that 13-14 year old students are not familiar with proving tasks, therefore conducting the research would provide me with insight on their initial understanding of what *proof* means. I was interested to find out if *convincing* and *proving* have the same meaning for the students and how they would react to the use of these terms. In addition, the research was designed so as to examine the students’ reasoning as well as to investigate the effectiveness of instructional strategies aiming to support them while working on proving geometrical statements.

In this chapter I present my strategies for designing my research. I begin with the main principles that characterized my lessons in the classroom and continue with the detailed description of the lesson design. What follows is my rationale behind the design of the interviews and the chapter closes with the presentation of the data instruments and information of the implementation of the research project.

4.1 Design Strategies

My intention when designing the research was to create a classroom environment based on a student centred teaching approach that would promote the construction and discovery of knowledge. Opposed to the traditional way of teaching, my role as being the teacher of the classroom would not be the transmitter of knowledge, but rather would be the facilitator for enabling students to construct their own knowledge. Having this main principle in mind, I applied four strategies to promote beneficial learning opportunities for the students: group work, classroom discussions, instructional scaffolding, and integration of GeoGebra applets. The four strategies are discussed in detail below.

**Group Work**

Collaborative work is expected to create more learning opportunities for students who are trying together to solve a task. Peer-to-peer interactions can result in learning more effectively since students’ inputs within the group can lead to fruitful discussions. When students engage in an activity together they do try to express their thinking and at the same time, reflect on and respond to their partners’ contributions. Furthermore, when working with
one or more partners, students are likely to feel more comfortable and self-confident than when working alone since they can rely on their partners’ contributions to help them develop their own thinking. Especially in the case of proving tasks, within the groups, students make conjectures and are expected to support their thinking in order to convince their partners. Mathematical reasoning, therefore, becomes a sociomathematical norm (section 2.5) that should promote the development of proving abilities.

Students’ groups, however, if not composed carefully may bring the opposite undesirable results with respect to students’ learning and understanding. For this study, the groups were formed as suggested by the cooperating teacher (i.e., the regular teacher of the classroom) who regularly applies group work in her teaching. Therefore, the composition of groups was arranged in such a way so the group mates would work reasonably well together.

As will be clear later in the lesson design section, students would work in groups for all the tasks during the four lesson sequence. During group work, I as well as the cooperating teacher would supervise the groups in order to make sure that the students were on track and provide them our help when needed (see instructional scaffolding below).

**Classroom Discussions**

Classroom discussions had a central role in the lesson design because through classroom discussions everyone’s thoughts are brought to the surface and opinions are exchanged. Classroom discussions can replace the lecture by a teacher by giving opportunity to students to contribute their ideas and explain their own thinking. When encouraged to share their thoughts with their classmates, the students are likely to gain self-confidence and therefore become productive in the classroom. On the other hand, when students listen to their classmates’ inputs, then they are invited to reflect on those and their own thoughts concerning the issues that are discussed. When others’ ideas are in opposition to their own, they must try to resolve the conflict either by formulating their own argument more convincingly or by recognizing flaws in their thinking.

Besides the benefits that students get from classroom discussions, the latter are also very beneficial for the teacher. By generating classroom discussions, I would have the chance to estimate the students’ ‘status’ with respect to their current knowledge and their willingness to share their thoughts. As their teacher, I needed to know what they are thinking so I could guide them towards understanding and decrease their misunderstandings. Especially for a researcher who teaches in a classroom for a small period of time and interacts with the
students for the first time, it was thought that classroom discussions would be necessary for making both the researcher and the students familiar with the new classroom setting.

My thinking behind group work and classroom discussions is best described with the notion of *discourse-oriented teaching* that is defined as “actions taken by a teacher that support the creation of mathematical knowledge through discourse among students” (Williams & Baxter, 1996, p. 22). “Discourse-oriented teaching is an attempt to take into account the inherently social nature of teaching and learning and to provide a more natural social scaffolding for the production of knowledge” (Williams & Baxter, 1996, p. 26).

**Instructional Scaffolding**

When interacting with the students during classroom discussions or during their group work, it was my intention to provide scaffolding rather than showing them the way to move on with the task or directly answering their queries. In my opinion, this is a very important strategy for prompting students to construct their own knowledge and get to the solutions based on their own inputs and not based on an external source.

Instructional scaffolding was discussed extensively in the theoretical framework (section 2.6). It consists of guidance that an instructor provides to students for facilitating and encouraging their development, and its degree depends on students’ progress. My intention was to provide this kind of support during my interaction with the students by posing questions and making comments which would serve to encourage the students to unfold their thoughts, to give reasons for their claims, to make their thinking clear to me and their peers, and to reflect on their work. When working on proving tasks, instructional scaffolding becomes even more crucial as questions like “why do you say so?” or “could you explain what you mean?” promote mathematical reasoning and invite the students to think about their arguments and evaluate if their approaches are in fact convincing. Furthermore, comments like “that is very good, continue” or “that is correct” serve for giving feedback to the students, and in the case of classroom discussions serve also for informing their peers on what is/is not acceptable.

**Integration of GeoGebra applets**

The fourth main design strategy for this master study was the integration of a dynamic learning environment in the lessons. The advantages of the dynamic learning environment in the mathematics classroom were discussed in the theoretical framework chapter (section 2.4). Proving abilities can be fostered through the dynamic nature of figures on the screen.
Especially young students who experience proving tasks for the first time or are slightly familiar with such tasks, can be benefited from the integration of applets that encourage them to make conjectures and therefore experience the first steps to proving in mathematics.

Through GeoGebra I wanted to provide scaffolds to the students in order to get to the desired conclusions instead of giving them definitions. Through the applets the students would hopefully more easily memorize the definitions by bringing to mind the dynamic picture seen on the applet.

GeoGebra applets were designed to be used in the first two lessons. Taken together the applets were an introduction to the proving tasks that the students would work on. According to the design, students would firstly be convinced of a ‘fact’ on GeoGebra (e.g. vertically opposite angles are always equal); subsequently, they would be asked to prove what they had just experienced visually.

4.2 Lesson Design

Besides the main principles based on which I designed my lessons, I also integrated the ideas suggested by Ball and Bass (2003) in my instructional approach. According to Ball and Bass, in order to enable and facilitate mathematical reasoning, teachers should (a) design and use mathematical tasks, (b) make records of mathematical knowledge, and (c) make ideas public. The tasks were carefully designed so students could express their proving and reasoning skills. Besides choosing appropriate tasks, it is crucial for students to be reminded of the mathematical knowledge that is acquired in the classroom. For this purpose, a summary of the things taught in the classroom must be written on the blackboard and be updated with new elements of knowledge so students can consult them at every opportunity. Making ideas public is served through the classroom discussions where all participants are invited to share their thoughts.

My first lesson was the first geometry lesson for the year. I had then to introduce students to the new chapter concerning angles and polygons. For the purpose of my research I would discuss with the students the concepts of angles, triangles and quadrilaterals during my lesson sequence, concepts that are covered in 10:01-10:08 sections of the respective chapter in the students’ textbook (McSeveny, Conway, Wilkes & Smith, 2008). During my instructional sequence, I did not use the textbook except for assigning some homework tasks for the students. In my opinion, the exercises in the textbook do not invite students to develop their
reasoning skills and proving abilities. Furthermore, tasks that invite students to practice routine procedures (e.g. calculate the angles in a parallel lines setting) do not promote understanding and at the same time may underestimate the students’ abilities and therefore make mathematics lessons useless and boring in the eyes of the students. However, although I used the worksheets I designed in the classroom, I chose to assign the homework tasks from the textbook so the students could keep track of the discussed sections. The questions on the worksheets were carefully written so, throughout the lessons, students would identify the words prove and convince in the mathematics classroom. Besides that, I wanted to make clear to them that it is not only me, the teacher, who they should convince but also their peers.

I describe in detail my aims for the choices I made during working on the lesson design. I begin with an overview of each lesson and then proceed to discussing each part.

4.2.1 Lesson 1

Since the first lesson was the introductory one to “Angles” for the year, then the first thing should be a revision on what the students already knew concerning angles. After the revision, we would be able to proceed to tasks that promote deeper understanding of the concept of angle and at the same time invite students to share their thinking and reasoning for what they say. At the end of the lesson, the students would work on the first proving task in the lesson sequence. More specifically, the first lesson was designed in order to contain the following elements:

1. Whole-Class Activity: Introduction to Angles
2. Group Work: Size of Angles
3. Whole-Class Activity: Size of Angles
4. The Applet: Vertically Opposite Angles
5. Group Work: Vertically Opposite Angles

I prepared an introductory class-activity that is followed by a task aimed to having students thinking more deeply about the concept of angles.
Figure 4.1 Angles around us

The aim of projecting the pictures above on the board was to let students guess the “theme of the day”. The pictures were chosen in order to entertain students but also for underlying the fact that angles can be found everywhere; on embroideries, in architecture, our own body forms angles.

Questions like “what do you know about angles?”, “what is an angle?” etc. would initiate a discussion aimed at bringing students’ prior knowledge to the surface. Additionally, the discussion would reveal questions or misunderstandings of the students concerning angles. Subsequently, projecting Figure 4.2 was meant to encourage students to recall and think about different kinds of angles.

Figure 4.2 Kinds of angles
The students would then be given the following worksheet (Figure 4.3) to work within their groups and then share their work in a classroom discussion.⁶

![Worksheet Image]

Figure 4.3 Worksheet ‘Angles on the Grid’

The rationale behind this worksheet was to have the students think deeper into the concept of angle, its size and the fact that it does not depend on the length of the arms. Furthermore, it was hoped that the students would proceed to compare some of the angles or make conjectures about their sizes. During the classroom discussion, I wanted to make sure that all the above points would arise. However the “freedom” offered by the question on the worksheet would give the students an opportunity to express what came in their minds. The task would be the first push into students’ reasoning since the students were expected to give reasons for their statements.

The GeoGebra applet below consists of the next part in the lesson.

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⁶ All worksheets may be found in Appendix A.
By dragging the two red points (F and G) in the applet, the lines can be moved. The sizes of the angles are always visible so that students could realize that vertically opposite angles are always equal. My purpose for including the applet in the lesson and have the students play with it, was for them to explore several examples of intersecting lines and be able to conclude for themselves the equality of the pair of angles instead of trusting my words. It was important for all students to draw their own conclusions because, thereby, they become responsible learners who do not depend on the teacher’s words but on their own thinking.

The dynamic nature of GeoGebra offers the possibility to experience many examples of the same problem. It was my expectation that through interacting with the applets students would be given a push to think of a general way to express and explain what they could see on the screen. In other words, it was hoped that they would be led towards constructing a mathematical proof. After playing with the applet, the students would be given the following worksheet (see Figure 4.5).
The GeoGebra applet suggested that vertically opposite angles are equal. First, discuss the question below with your partner. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that vertically opposite angles are equal to each other?

**Explanation:**

Figure 4.5 Worksheet ‘Vertically Opposite Angles’

The question on the worksheet invites students to “convince someone” rather than prove that the equality of vertically opposite angles are equal. The wording was not randomly chosen. I wanted to see how the students perceived the word *convince*, and I also considered that the word *proof* might challenge students too much and prevent them from thinking and expressing themselves with more freedom. The word *proof* was introduced and used in the second lesson onwards.

For homework I wanted the students to bring pictures of angles around us. They could either use their own pictures or find pictures online.

4.2.2 Lesson 2

The second lesson was a two-period lesson and for the reader’s convenience I discuss it in two parts.

The last activity of the first lesson would give me an insight into what students considered to be convincing argumentation for the equality of vertically opposite angles. In the second lesson I wanted to show them what is considered convincing by mathematicians and by this, introduce the notion of ‘proof’. After the discussion, the students would work in proving geometrical statements, and I hoped that they would take steps to formulate valid
mathematical argumentation. As a result, the **first part** of Lesson 2 was designed in order to contain the following elements:

1. Whole-Class Activity: What is convincing in mathematics?
2. Whole-Class Activity: Why are the vertically opposite angles equal? The mathematical proof
3. The Applets: Alternate Angles – Corresponding Angles – Co-interior Angles
4. Group Work: Corresponding Angles – Co-interior Angles

A discussion on what is considered convincing in mathematics would take place in the beginning of lesson two, as suggested in my design. The rationale behind this discussion was to alert the students that visualization is not enough evidence for mathematicians and that convincing arguments linked in a logical sequence for supporting a statement are considered necessary. Together with the discussion, the proof for the equality of vertically opposite angles would be presented with help from students (since I expected that most of the students would not present a proof, but rather empirical arguments when working on the equality of vertically opposite angles in the previous lesson).

My expectations from students’ work on proving after the above discussion and our common attempt to reach the proof of vertically opposite angles, were higher than in the first lesson. What I expected to see in the first lesson’s worksheets was not a valid mathematical proof, because students were not accustomed to tasks on proving. I was hoping, however, that after the introduction of the second lesson, they could grasp the meaning of proof and its requirements.

Later on, the students would play with the designed GeoGebra applets (see Appendix B) in order to realize the relationships that hold concerning alternate, corresponding and co-interior angles between parallel lines. Following the order of the first lesson, they would later be given the following worksheets (Figures 4.6 and 4.7) to work within their groups (one worksheet per group):
By the help of the GeoGebra applets, we have recognized the following:

When two parallel lines are cut by a transversal,
- the alternate angles are equal.
- the corresponding angles are equal
- the co-interior angles are supplementary.

Let us consider the general case with angles $\alpha$ and $\beta$.

Imagine that from the above list only the rule of alternate angles is known to you. Prove that $\alpha = \beta$.

Figure 4.6 Worksheet ‘Corresponding Angles’
As may be observed, in the second lesson’s worksheets (Figures 4.6 and 4.7), the word ‘proof’ is used. However, the title also contains the word ‘convince’ with the purpose of making the two words equivalent in the mathematics classroom. After the introductory discussion on the meaning of proof, I expected now that students would understand better the requirements of a proving task.

In the second part of the lesson I wanted to investigate if students would be able to locate alternate, corresponding and co-interior angles in a complex figure, different from the one on which they worked in the first part of the lesson. I believe it is not easy for students to recognize images that they have already seen if they are in a more complex figure. At the same time it is important to be able to do this in order to handle more challenging geometrical tasks. To accomplish my goal and investigate the students’ reactions when encountering a relatively complicated figure, I intended to initiate a classroom discussion that would give students scaffolding so that they could work within their groups on a challenging geometrical
task after the discussion. More specifically, the second part of the lesson was designed in order to contain the following elements:

1. Whole-Class Activity: Tangram
2. Whole-Class Activity: Kinds of Quadrilaterals
3. Whole-Class Activity: Tangram & Congruent Triangles
4. Group Work: Compare Triangles

In order to initiate the classroom discussion, a slide (Figure 4.8) would be projected on the board. The pictures in the slide are made by putting together the 7 shapes of the Tangram\(^7\) game and using those would be a good starting point for initiating a discussion on how shapes can be seen from different perspectives. Take for example the colorful square on the low left corner of the slide. We can either look at it as a square or as a set of different shapes put together.

![Tangram](image)

**Figure 4.8 Tangram**

I also planned to hold in my hands actual shapes of the tangram puzzle so that I or the students could point out different ways to see a figure. I then wanted to make a connection with the first part of the lesson and everything discussed so far, so I planned to project the

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\(^7\) Tangram is a Chinese puzzle containing seven flat geometrical pieces that are put together to form shapes.
following parallelogram (see Figure 4.9) on the board. My aim was that students could recognize that angles $\alpha$ and $\beta$ are alternate and therefore equal.

![Figure 4.9 Tangram Triangles](image)

During the classroom discussion, it was also my intention to bring up the different kinds of quadrilaterals and so students would be reminded of their properties. Based on the projected figure (Figure 4.10), I wanted the students to contribute to a review of the shapes and to extend the discussion to more quadrilateral shapes.

![Figure 4.10 Quadrilaterals](image)

It was also my intention to discuss congruent triangles. Since the students were probably not familiar with this term, I planned to refer to them by saying “the one is the exact copy of the
other”. I wanted to have the students decide on the requirements for two triangles to be exactly the same. Having discussed all this, students would then be given a worksheet (Figure 4.11).

![Worksheet Image](image)

**Figure 4.11 Worksheet ‘Compare Triangles’**

By designing and including this task\(^8\) in my lesson sequence, it was my intention to challenge students and investigate if they would be able to proceed in comparing the elements of the two triangles that they were to work on. As a continuation to the preceding discussion, I wanted to find out if students could assimilate and apply in practice points from the discussion. I also wanted to know how they would tackle a rather complex geometrical figure.

For homework the students would be assigned some exercises from the textbook so as to become more familiar with alternate, corresponding and co-interior angles.

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\(^8\) The blue lines (FD and BC) in the figure are parallel as suggested from the arrows. The same holds for the green (FC and AC) and red lines (AB and DE).
4.2.3 Lesson 3

The third lesson was designed to begin with a review of the previous two with, if needed, a discussion about the last task of the second lesson (see Figure 4.11). The students would subsequently work on two proving tasks, one asking for proving why the angle sum of a triangle is 180 degrees and a second concerning the angle sum of a quadrilateral. Therefore, the third lesson was designed in order to contain the following elements:

1. Group Work: Angle Sum of a Triangle
2. Group Work: Angle Sum of a Quadrilateral

The following applet\(^9\) (Figure 4.12) was to be shown to students on the projector.

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By dragging any vertex, the triangle shape changed accordingly. On the right side, however, the students would be able to see that no matter the shape, the angle sum of the triangle was always constant and equal to 180 degrees. I planned to stress once more that visualization is not enough evidence for a mathematician, so students would then have to provide a convincing argumentation for what was suggested by the applet. My aim by the third lesson was that students would understand what is expected when asked to convince someone of the truth of a statement. To underscore this aim, they would be given the following worksheet to work on within their groups.

![Worksheet 1 / Proof: Convince me!](image)

The GeoGebra applet suggested that the angle sum of any triangle is $180^\circ$. First, discuss the question below with your group mates. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that the angle sum of any triangle is $180^\circ$?

![Figure 4.13 Worksheet ‘Angle Sum of Triangle’](image)

The rationale for this task was to see if students would take the initiative to draw a triangle on paper. Though obvious to teachers, it may not be obvious to students that drawing a figure discussed in a task can provide help and guidance to solve the task. I was also interested to see if the students would draw a scalene triangle or a specific kind of a triangle (right or isosceles and so on) and how their choice would contribute to their solution. Furthermore, my goal was to see if students would add to the figure by drawing extra lines which could provide them with insight for solving the task. If the students did not take such actions, I would encourage them by mentioning the usefulness of firstly drawing a figure and then ‘play’ with it by adding extra elements such as extra lines. I was once more interested to find out about students’ proving abilities and their reasoning skills in their attempts to convince for the truth of a statement.
After finishing work on this task, the students would be asked to prove that the angle sum of a quadrilateral is 360 degrees (see Appendix A). Again, students’ work would allow me to investigate if students took the initiative to draw a figure and interfere with it.

At the end of the third lesson, I was planning to assign some exercises from the textbook as a homework task for the students.

4.2.4 Lesson 4

Lesson four was the last one of my lesson sequence. For this lesson I designed two activities to serve as summary tasks which would allow me to see what the students had learned during the previous three lessons. The activities therefore were expected to give me insight into students’ understanding of the concepts discussed during the lessons and into how they dealt with these concepts in their attempts to work on proving statements. To sum up, the fourth lesson was designed in order to contain the following elements:

1. The Applet: Thales’ Theorem
2. Group Work: Thales’ Theorem
3. Group Work: Trapezium

![Figure 4.14 Applet ‘Thales’ Theorem’](image-url)
In order to proceed to working on Thales’ theorem (Figure 4.15), the students would firstly have to recall the basic properties of the circle. After a short review, they would then be asked to open the GeoGebra applet (a screenshot from the applet is shown in Figure 4.14) and conclude for themselves what is suggested from Thales’ theorem. After interacting with the applet, the students would be given the worksheet shown below (Figure 4.15).

![Worksheet 1: Thales' Theorem](image)

**Worksheet 1 / Thales' theorem**

The GeoGebra applet suggested that if A, B and C are points on a circle where the line AC is a diameter of the circle, then the angle ABC is a right angle. That is known as the Thales' theorem attributed to the famous Greek mathematician. First, discuss the question below with your group mates. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that Thales' theorem always holds?

![Diagram](image)

**Explanation:**

This task is a challenge for the students of this age. My expectations for this task were not very high, but I was interested to see if the students would be able to take a starting step with the task, and up to which point they could progress in the solving process. For solving the task it is crucial to draw the segment BO and proceed to conclusions concerning the two
triangles that are formed. If students could take these steps, I would be able to conclude that there was significant progress in their attempts to solve geometrical proving tasks.

The nature of the second worksheet (Figure 4.16) is different from the ones so far presented. It concerns a numerical task in which the students must calculate the missing angles. In order to solve the task, the students should recall the properties of a trapezium and then proceed to finding the missing elements by using the concepts we have discussed in the three previous lessons: vertically opposite angles, alternate, corresponding, co-interior angles between parallel lines, the angle sum of a triangle, and the angle sum of a quadrilateral. This task would give me the opportunity to see how able the students were in using some or all of the above concepts in a numerical task. I could examine once more students’ understanding of the discussed concepts during my teaching sequence.

Figure 4.16 Worksheet ‘Trapezium’
4.3 Design of Interviews

Conducting interviews with the students was already in my mind when I designed my research project. My main thinking on the interviews and the design of the interview tasks, however, was done during and after the end of the lesson sequence. The main reason was that I wanted to get in the classroom and meet the students first. In response to my observations on how my lesson plans unfolded and in alignment with the students’ abilities and progress I designed the interview tasks and decided on my interviewees.

Interviewing some students would provide me with more information on their way of thinking while working on proving geometrical statements. In addition, through the interviews I would acquire more information on students’ progress that would therefore give me the opportunity to reflect on my work both as a researcher and as a teacher.

The Interview Tasks

The main preparation for the interviews was designing the GeoGebra applets\textsuperscript{10} that I would use. After observing the students at work with the GeoGebra applets in the classroom I decided to include applets and not just use only paper and pencil tasks. The students’ way of thinking and their reasoning skills might be influenced by the dynamic nature of the environment. Through the interviews I would have the chance to investigate this further.

After having my interviewees play with the applet I wanted to encourage them to make conjectures and share their reasons. The students would also be given a worksheet on which to write their proofs on if they wished.

Selection of the Interviewees

Two groups of students (Jasmine & Julie, and Phoebe & Cassie) and an individual (Ehsan) were chosen to be interviewed because after observing the lessons, I noticed that these particular students were willing to talk and share their thoughts. They felt also more comfortable compared to their classmates in giving reasons for what they stated. Furthermore, the two groups were chosen after I saw that the partners were working well together with a collaborative spirit. At the end however, there was a change of plans since

\textsuperscript{10} The description of the applets is omitted here in the interest of saving space. Also this discussion here is of general nature and hence does not rely on the specific details of the applets. The description of the applets can be found in section 6.5
one group mate was absent. There were finally three interviews conducted, one with one group of students (Jasmine & Julie), one with Cassie, and one with Ehsan.

4.4 Summary of Data Sources and Implementation Timeline

In this section, I summarize my data instruments and afterwards I describe the implementation of the lessons compared to my designed plans. The chapter closes with the timeline of the research events.

Summary of Data Sources

I collected the data for my research through a variety of sources. All lessons were videotaped and also audio recorded so that the classroom discussions could be examined and analyzed. In addition, some of the groups were audio-recorded while working on the tasks. Because of a lack of enough audio recording devices, it was not possible to record all groups in the classroom. I also collected the worksheets of the students for analysis purposes. As for the interviews, they were audio recorded and I also collected the students’ worksheets where they had the chance to write in addition to sharing their thinking when talking.

Lesson Implementation

In general, the lesson implementation was done in the way it was planned in the research design. The first and second lesson unfolded exactly as intended during my research design. In the third lesson, however, there was a change of plans because a discussion on the last activity of the previous lesson (see Figure 4.11) lasted longer than expected. As a result, in the third lesson only one of the two activities took place (Group Work: Angle Sum of a Triangle) while the second one (Group Work: Angle Sum of a Quadrilateral) had to be moved to the fourth lesson. Because of this change in my lesson design, I had to think about how I wanted the fourth lesson to be implemented. After having spent time with the students and therefore having estimated their abilities in proving tasks, I decided to omit the first task (see Figure 4.15) of the fourth lesson suggested in my research design. I considered ‘Thales’ theorem to be a very challenging task for this group of students and therefore, the last lesson included the task on the angle sum of a quadrilateral (see Appendix A) and the task with the trapezium (see Figure 4.16).

Concerning the homework assignments the students did not work on the things they were asked to. Only at the end of the fourth lesson, some of the students brought pictures of angles
that I had asked them to in the first lesson. The students were apparently used to having a homework assignment only once a week and that is why they neglected doing the rest of the assigned tasks. This did not influence my research and the data collection procedure since homework assignments are not considered as a very reliable source to inform about the students’ progress. It is always safer for the researcher to analyze the work that takes place in the classroom since it is doable there to control the external help that the students receive while working on tasks.

As suggested from my design principles, the students were working in groups for all 4 lessons. Nine groups were formed in the classroom (6 groups of two and 3 groups of three students). The group members however did not stay constant for all the lesson series because of the students’ unwillingness to work with the same partners. Though I attempted to explain to the students that it was preferable the group composition stayed constant for all four lessons, I finally decided that ‘forcing’ them to work with partners they would not prefer might have influenced their willingness and focus to work.

**Implementation Timeline**

Below is the timeline of my lesson sequence and the days that the interviews took place.

<table>
<thead>
<tr>
<th>Date</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 9, 2012</td>
<td>1st lesson</td>
</tr>
<tr>
<td>February 13, 2012</td>
<td>2nd lesson</td>
</tr>
<tr>
<td>February 14, 2012</td>
<td>3rd lesson</td>
</tr>
<tr>
<td>February 16, 2012</td>
<td>4th lesson</td>
</tr>
<tr>
<td>February 20, 2012</td>
<td>Interviews with students</td>
</tr>
<tr>
<td>February 21, 2012</td>
<td>Interviews with students</td>
</tr>
</tbody>
</table>
5 Analytical Framework

In this chapter the scheme I used for analyzing my data is presented. Inspired by the work of other researchers (Bell, 1976; Mercer, 1995; Krugger, 1993; Blanton, et al., 2003), I designed the following analytical framework to describe the qualities of students’ proofs, their kind of reasoning and my instructional scaffolding strategies.

I analyzed the data lesson by lesson and then proceeded to the analysis of the interviews. I analyzed the students’ worksheets and when available, I listened to the dialogues of students when working within their groups. In the events where classroom discussions occurred, I listened to and analyzed the audio sources and consulted the video recording to verify a point as necessary. As for the interviews, I transcribed all three and proceeded then to the analysis of the transcripts using as verification the students’ worksheets when available.
5.1 Quality of Students’ Proofs

The Table below (Table 5.1) contains a description of all the categories used to classify the qualities of students’ proofs. After carefully examining a number of the students’ proofs, I designed this list of categories and used it for analyzing all the proofs from the students. The order of the categories indicates a hierarchy with *original* being the highest quality of proof and *no progress* being the lowest quality. *Original* is rated higher than *very good* because of the independence expressed from the student to handle and solve the geometrical task and the ability to approach the task with his/her own original ideas that differ from the ones suggested by the instructor.

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>Correct proof. The student follows a slightly or totally different route compared to the one suggested by the instructor. Organized way of thinking and solving the task. Clear conclusions.</td>
</tr>
<tr>
<td>Very Good</td>
<td>Correct proof. Arguments linked together in a logical sequence. Proof short and elegant.</td>
</tr>
<tr>
<td>Good</td>
<td>Correct proof. Redundancy of information, stating more things than needed. Lack of argumentation for the written statements. Wrong drawings that reveal possible misunderstanding.</td>
</tr>
<tr>
<td>Progress/No proof</td>
<td>Insufficient answer. The student may provide mathematical relationships and reasoning is evident but s/he does not use them to get to the solution. There is reasoning but it is hard to determine what the student thinks. Gets to the result but answer based on mistaken mathematical relationships.</td>
</tr>
<tr>
<td>No Progress</td>
<td>Answer limited to writing down the given data of the task.</td>
</tr>
</tbody>
</table>

Table 5.1 Quality of Proofs

5.2 Types of Students’ Reasoning

The two main categories used to analyze students’ reasoning were considered to be ‘empirical’ and ‘deductive’, similar to the work of Bell (1976). The ‘empirical’ category covers answers that are based on intuition, measurement, visual evidence, gestures, or proceed to generalizations from a few examples (Reiss, Heinze, Renkl & Grob, 2008;
Boero, 1999). On the other hand, in deductive reasoning a deductive element must be clearly visible. The subcategories are different to the ones identified by Bell; this is because the subcategories for analyzing my data take into account the students’ interaction with the dynamic learning environment. The subcategories are explained below and summarized in Table 5.2:

**Visual**: The student’s answer is based on visual evidence when looking on a static figure (i.e. “the two triangles look the same”), or when measuring with a protractor or a ruler. This subcategory belongs only under the ‘empirical’ main category while the following three consist of subcategories for both ‘empirical and ‘deductive’ main categories.

**Visual Dynamics**: The students’ answer is driven by something has been seen in the applet and is based on the dynamic figure.

**Finite Number of Examples**: Through a finite number of examples, the students come to the conclusion.

**Infinite Number of Examples**: The students present a number of examples that are representative of an infinite set of examples.

<table>
<thead>
<tr>
<th></th>
<th>EMPIRICAL</th>
<th>DEDUCTIVE</th>
</tr>
</thead>
<tbody>
<tr>
<td>VISUAL</td>
<td>EMP VIS</td>
<td></td>
</tr>
<tr>
<td>VISUAL DYNAMICS</td>
<td>EMP DYN</td>
<td>DED DYN</td>
</tr>
<tr>
<td>FINITE NUMBER OF EXAMPLES</td>
<td>EMP FIN EX</td>
<td>DED FIN EX</td>
</tr>
<tr>
<td>INFINITE NUMBER OF EXAMPLES</td>
<td>EMP INF EX</td>
<td>DED INF EX</td>
</tr>
</tbody>
</table>

Table 5.2 Kinds of Reasoning

11 This category was not found in students ‘answers.
5.3 The Instructional Scaffolding Strategies

A detailed explanation of each of the teaching strategies used in this research to provide scaffolding to students is given in Table 5.3. For building this analytical framework, I was inspired by Mercer’s work (1995), and included his ideas on how teacher talk can facilitate and foster student learning. Other contributions to my framework (see also section 2.6) came from Kruger’s framework (1993) for the analysis of transactive discussion, and its expansion from Blanton, et al. (2003).
<table>
<thead>
<tr>
<th>Questions or comments made by the teacher aiming at:</th>
<th>Explanation</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bringing on Track</td>
<td>Questions or comments that are made for mentioning that the students take the wrong direction.</td>
<td>BR-TR</td>
</tr>
<tr>
<td>Clarification</td>
<td>Questions or comments that require from the students to make their thinking clearer. They have to reformulate what they said in order to make their arguments clear and precise to their classmates. Questions or comments also that aim at summarizing and summing up what is discussed.</td>
<td>CLR</td>
</tr>
<tr>
<td>Confirmation</td>
<td>Questions or comments made to confirm that an answer is correct and the students may continue. That way the information in the classroom is continually updated and the students know what is accepted.</td>
<td>CONF</td>
</tr>
<tr>
<td>Encouragement</td>
<td>Questions or comments made for encouraging the students and abetting them to trust their own abilities. Comments to show them that the task is almost completed.</td>
<td>ENC</td>
</tr>
<tr>
<td>Explanation-Argumentation</td>
<td>Questions or comments that require from the students to explain further and argue aloud for their thinking, they have to defend their answers. In this way, what the students say becomes more understandable to themselves and their peers.</td>
<td>EX-AR</td>
</tr>
<tr>
<td>Independent Thinking</td>
<td>Questions or comments that provide hints to the students for helping them to get to the desired result. I do not reveal the steps to the solution but rather I give the direction towards it. Then the students find the way and gain self-confidence. Moreover, I make clear that there is no need to remember something but mention the ability to re-discover something from scratch. The students do not depend on my words or the textbook.</td>
<td>IND</td>
</tr>
<tr>
<td>Reflection</td>
<td>Questions or comments that prompt the students to reflect on their work. In this way, the students themselves become critical of their work.</td>
<td>REF</td>
</tr>
</tbody>
</table>

Table 5.3 Instructional Scaffolding Strategies
The coding of the research data according to the above analytical framework was tested with two experienced people. All three of us agreed on the analysis of samples of the data with respect to the quality of students’ proof, their kind of reasoning, and my instructional strategies. Below, an example is shown of two girls’ work on proving the equality of corresponding angles between parallel lines; this was characterized as *very good*.

![Figure 5.1 Phoebe & Cassie's work](image)

*Corresponding angles:*

Well, let's add in C'. We know for a fact that B = C as they are alternate angles. We also know that C and A are vertically opposite angles and we know vertically opposite angles are equal.
6 Analysis and Results

In this chapter I present the results that were obtained during the four lessons in the classroom, followed by the results from the interviews that I conducted with some of the students after the end of the learning sequence. Together with the results, I also provide the analysis according to the analytical framework described in detail in Chapter 5. In the excerpts that are included in this chapter, the letter R refers to me, the researcher and instructor, and the letters CT refer to the cooperating teacher who was also supervising the students while working within their groups.

The excerpts from the classroom discussions and the group discussions, as well as the works of the students which are presented, are chosen from among the data that were collected. The main criterion for choosing these particular data is that I find them interesting and illustrative with respect to the focus of this study.

In the analysis of the results that came out in the class work, I analyze lesson by lesson the students’ reasoning, the quality of their proofs, and the instructional strategies that I used when interacting with the students. At the beginning of each lesson, the reader will find an overview of the main classroom events. However, as the reader will find out, not all of the events are discussed in the analysis. The events which are not discussed did not offer insight to the questions of this study but they are mentioned so the reader can make the connection with the lesson design that is presented in Chapter 4. When a GeoGebra applet was used (lesson 1 & lesson 2), this is underlined for the convenience of the reader.

The analysis of the interviews follows on section 6.5.

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12 The coding for students’ reasoning is in green typeface, the one for the quality of proofs in blue, and that for teaching strategies is in red typeface.
6.1 Lesson 1

Lesson 1 encompassed the following elements:

1. Whole-Class Activity: Introduction to Angles
2. Group Work: Size of Angles
3. Whole-Class Activity: Size of Angles
4. The Applet: Vertically Opposite Angles
5. Group Work: Vertically Opposite Angles

After the introduction, the students worked within their groups in order to come up with three statements concerning the angles on the grid (see Figure 6.1). Afterwards, a classroom discussion took place where the students shared the statements they wrote on the worksheets. I begin the analysis of the first lesson by presenting two excerpts of the classroom discussion (element 3 in the list above).

**Whole-Class Activity: Size of Angles**

The grid with the angles (Figure 1) was shown on the board.

![Figure 6.1 Angles on the Grid](image)
Simon started the discussion by sharing the three statements he had written together with his partner Stean.

Simon: They are all acute angles!

R: That is correct.

Simon: They are all 45 degrees except for the n [η] and the weird 8 [θ]

Simon: And the third one. Most of them have the same angle but they end at different lengths.

R: What does this mean? Could you explain?

Simon: Well I don’t know how to explain it. Hmm well the lines at the ends they end up… He [referring to his partner Stean] can explain it!

Stean: Basically it means that it is not about the length of the triangle and how far it goes…, basically it is just about the angle itself at the corner.

Providing feedback to the students for what they say is important. Confirming what they state keeps the students on the correct path. Moreover, it is an important teaching strategy to be applied during the classroom discussion as it consists of an ‘indirect update’ of the information that is being shared in the class.

I asked for a clarification when Simon said something that was difficult to follow. I did this in order to encourage him to make his thoughts more precise. That way, I gave the opportunity to the student to reword the statement with the purpose of making it more understandable for him and his peers.

Simon had difficulty in explaining the third statement. The fact that he lacked the appropriate terminology probably made it hard for him to express himself. He gave up and asked his partner to explain. Stean, as Simon, lacked the terminology but his explanation was more concrete. Phrases as “it is not about the…” and “it is just about the…” reveal that what counts when comparing angles is only the size of the angle and not the length of the arms.

Ehsan raised his hand and contributed to the discussion with a new statement concerning the angles on the grid.
Ehsan: $\alpha$ and that... let’s just say $c$ [talking about $\zeta$], they are complementary.

R: Why do you say so?

Ehsan: Because they are both 45 degrees and they make a 90.

R: Why are they 45 degrees?

Ehsan: Because they go... If they were to be 90 there would be one straight line and one horizontal line. In this one, one line crosses between the two [he is showing with his fingers] so it is a 45 degrees angle.

By applying continuous questions, I here sought for explanation and argumentation for what Ehsan stated. It was my intention to make visible the need for arguing for what we claim in the mathematics classroom. Giving reasons gradually becomes a requirement for all the participants in the classroom.

Ehsan, when asked to explain why “the angles are 45 degrees” provided an answer without hesitating. He had already written on his worksheet that the angle $\alpha$ is obviously 45 degrees, but he was ready to explain further. He explained how the angle would have looked if it were a right angle: then it would have been formed by a straight line (meaning a vertical straight line) and a horizontal one. In this case though, the one arm is horizontal and the other is the diagonal of the square. Though he had not used appropriate terminology, he tried his best to support the point he made by also using his fingers (seen in video 1) to clear it up for me and his classmates. What is also worth mentioning is the fact that he combined two of the elements on the grid in order to make a statement. He ‘transposed’ angle $\alpha$ (or angle $\zeta$) in his mind into making the two angles adjacent to each other and concluded that they are complementary. His reasoning is characterized as deductive since he deduced the conclusion from a known statement (a 90 degree angle would have been like this, so half of it must be like that). I interpret this to be the category of finite examples since he could probably handle similar cases in the same way. Considering also his gesture, it may indicate the case of a prototypic example but there is no further evidence of this.
After I summed up the statements and the important points that were suggested by the students, and after I defined the vertically opposite angles on the blackboard, I asked the students to open the corresponding applet on their laptops.

**The Applet: Vertically Opposite Angles**

The applet consisted of two intersecting lines that the students could move individually but not both simultaneously. From the dynamic picture they could conclude that the vertically opposite angles are always equal to each other (Figure 6.2).

![Applet for Vertically Opposite Angles](image)

**Figure 6.2 Applet for Vertically Opposite Angles**

I did not expect the students to get anything else from the applet, but apparently the dynamic nature of the environment influenced some of the students’ reasoning and guided their argumentation. When I asked them to share their conclusions after playing with the applet, what I expected to hear was that vertically opposite angles are always equal. However, some students provided reasoning that was rooted in dragging the points in the applet and observing the movement of the lines. Here is what Ehsan got from using the applet:

Ehsan: When you move one of them [one of the lines] the other one stays the same. So the other one is constant but the other one is moving, the other one is changing the same in both vertically opposite angles so they both kind of change in the same way.
Ehsan described in his own words what he saw on the screen. Moreover, he made an attempt to explain why the vertically opposite angles are the same. His reasoning is empirical and driven by the dynamic nature of the applet that prompted him to generalize what is happening with the vertically opposite angles. The fact that only one of the lines could be dragged and moved each time (the two lines could not be moved simultaneously) is a built-in constraint of the applet that has apparently influenced Ehsan’s reasoning.

**Group Work: Vertically Opposite Angles**

The students were given the lesson’s worksheet on which they were asked to write an explanation for the equality of vertically opposite angles. The worksheet is shown in Figure 6.3.

![Worksheet 2 / Convince me!](image)

**Worksheet 2 / Convince me!**

The GeoGebra applet suggested that vertically opposite angles are equal. First, discuss the question below with your partner. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that vertically opposite angles are equal to each other?

**Explanation:**

---

I discuss here the work of some of the student teams. Excerpts from discussions within groups, or mine with the students, are also presented. Again, what is chosen for analysis and discussion are the most illustrative cases with respect to students’ reasoning, the qualities of their proofs, and the strategies used by the teacher.

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13 This constraint was planned. If both lines were moving simultaneously, there would be no way to restrict them only to intersect, so the applet was designed in such a way that only one line could be moved at a time.
Haruki & Sean

The students drew two intersecting perpendicular lines (like a cross; see Figure 6.4), in which case all the angles are equal to 90 degrees. They explained that as soon as you start moving the lines then the following applies: if one of the angles decreases then the opposite one must decrease as well. Using another shape when the angles were now named as $x$, $z$, $y$ and $a$, they wrote that if an angle decreases the opposite one must decrease by the same amount.

Figure 6.4 Haruki & Sean's work

From the students’ drawings, shown in Figure 6.4, as well as from what they have written, I conclude that the dynamic nature of the GeoGebra applet has influenced their reasoning as it was based on the movement of lines. It is interesting that they used two perpendicular lines as a starting point. They began with a geometric setting in which the statement is clear; in the case of a ‘cross’ the vertically opposite angles are equal to 90 degrees (lines are perpendicular). Then they described that the equality of the angles must hold as the lines start to move. The fact that they started from something known and accepted, i.e., from a specific case, and from there moved towards generalizing the hypothesis to be proved, makes their reasoning, in my opinion, deductive.

Jasmine & Lee

The girls mentioned that a line is 180 degrees. They were moving towards a solution based on mathematical relationships. When I approached them, I encouraged them to write what they were thinking in mathematical terms.
R: So why is that angle equal to that one? [Showing the vertically opposite angles on the figure they had drawn]

Lee: Because it’s a straight line. That’s 180 and that’s 180 and this is 90… Oh no no and this is the same… [showing the vertically opposite angle]

R: That is what you want to show, you don’t know yet.

Jasmine: No matter how you move it this line will always be 180 degrees and so the other line will always stay the same.

R: Try to write that down. How would you write that down in mathematics? What if you name these angles? We don’t know the exact degrees. So, this one and that one… You already know something about these [supplementary angles], right?

Although having recognized an important clue in the figure that could lead them to the solution, the two girls were having difficulty in going further. They mentioned that a line is 180 degrees and that is the starting point for the mathematical proof. Lee got confused when looking at the static figure and assumed for a moment the hypothesis to be proven and that is the reason I interfered to bring her on track.

Before I left them to continue their attempt to get to the solution, I tried to guide them into writing down mathematical relationships. I advised them to give names to the angles and prompted them to think about how they could use their observation that a line is 180 degrees. My intention was not to tell the students what to do next. Rather, I gave them hints to help them find the steps by themselves.

In spite of my attempt to guide them towards a solution, the girls did not proceed further in writing down some mathematical relationships. They gave, however, names to the angles (see Figure 6.5) as suggested. Interestingly, they gave names to the lines as well!
From the discussion that follows, it is clear to me that the girls were now trying to give an explanation based on the movement of the lines in the GeoGebra applet.

Jasmine: We have to explain why two vertically opposite angles are equal. What I was trying to tell you [was] that no matter how or where we move line $f$, line $g$ will always stays still. And so angle $a$ and $b$ always make 180 degrees and so if you move it, it will always remain 180.

Lee: Ye she is right. [Talking to the other group opposite to them]

Jasmine: And then $c$ and $a$ are always opposite so they will always be the same!

Lee: Ye!

Jasmine: Why doesn’t she understand that? Why do we have to explain it to her? She is the math teacher here!

The girls seemed to be satisfied with their explanation. They focused on the fact that supplementary angles always add up to 180 degrees but they did not proceed to a valid argumentation for the equality of the vertically opposite angles. They finally wrote on their worksheet:

No matter how or where we move line $f$, line $g$ will always remain the same and if one of the lines is the same then both angles will be equal to each other. [Jasmine & Lee, worksheet]
Their reasoning is empirical and without a logical link between the statements they make. However, they are convinced and satisfied by their explanation, and they do not see the need to explain further. The constraint of the applet that was mentioned earlier (see footnote 13 on p. 56) again influenced the girls’ attempt to convince for the equality of the vertically opposite angles.

Stian & Simon

The boys just drew two intersecting lines (see Figure 6.6) and stated that the vertically opposite angles are equal. Furthermore, they took “a non-vertical angle” as they named it by drawing an intersection of a straight and a curved line. “Vertical”, on the contrary, is their own terminology for the angle with straight arms.

Stian: Dude, this is epic! Even if one of the lines aren’t straight, they [the vertically opposite angles] are still the same!

[Stian & Simon, group discussion]

![Figure 6.6 Stian & Simon’s work](image)

Neither student felt the need to explain why the angles are equal, they found it convincing to draw the lines and state the visually obvious. Moreover, it may be that the boys have an alternative angle conception where the arms of the angle do not have to be straight lines. Their approach is original. It is like they left themselves free to express whatever came to
their minds. Besides originality, this also manifests independence. Although their reasoning is limited to showing distinct cases on paper for convincing someone for the equality of the vertically opposite angles, I conclude that these cases consist of prototypes for a set of infinite number of examples.

It is interesting here to comment on the boys’ use of terminology of a ‘vertical’ angle and speculate about the origin of their thinking. It seems that there is a misunderstanding here that rises probably from the term ‘vertically opposite angles’. Vertically here means that we are talking about the angles that share the same vertex. It is possible and rational, however, that Stian and Simon connect the term ‘vertically’ to the term ‘vertical’ that is opposite to ‘horizontal’. ‘Vertical’ brings a straight line to their minds and as a result a ‘non-vertical’ is a curved line that leads them to a richer conception of what an angle can be.

Julie & Isis

The two girls drew more than one pair of intersecting lines (see below in Figure 6.7) to show that the angles are equal.

The crucial point in Julie & Isis’s work is the phrase they have written on their worksheet (see Figure 6.7):

No matter how many examples you use they [the vertically opposite angles] will always be equal. [Julie & Isis, worksheet]

From the beginning of the phrase I interpret this to be a generic example. They have drawn three examples where the vertically opposite angles are equal but with their statement “no matter how many examples you use” they made it clear that the three cases consist of representative cases of the set of an infinite number of examples. However, their reasoning is empirical; they support their explanation by what is visually obvious and not by linking together logical arguments.
The girls’ phrase makes their reasoning more emphatic compared to the reasoning of Simon & Stian who drew two different cases to convince for the statement (see Figure 6.6). In the girls’ case we can be certain for the extension of their reasoning to an infinite number of examples whereas in the case of the boys this is not that obvious.

Victoria & Jacque

Instead of providing a convincing argument for the equality of the vertically opposite angles, Victoria and Jacque actually verified the fact via an example. However, for them that was considered as a way to convince someone as shown from their introductory sentence (see Figure 6.8).
The girls made clearer what is meant from the statement to be proven. They use this example but it seems that they consider it as a representative one of a set of infinite examples (see $\alpha=\beta$). Moreover, in their work a deductive element can be traced when they conclude (looking at the angle of revolution) that the other pair of vertically opposite angles is $140$ degrees each.

Sophia & Risako

Finally, for Sophia and Risako the way to convince someone of the equality of the vertically opposite angles was

By measuring the degrees with protractor.

[Sophia & Risako, worksheet]
The girls haven’t taken steps towards the requirements of a proof like the notion of generalization. It makes absolute sense to them to use the protractor and measure the angles in order to convince someone that they are equal. Though logical, their reasoning is in a premature stage. It is empirical visual since the girls depend on the evidence shown on the protractor for arguing for the equality of angles.

**Analysis Overview of Lesson 1**

The first lesson and more specifically the “vertically opposite angles” activity that required the students to “convince someone” of a mathematical statement, can be considered like a pre-test to inform a researcher about the students’ conception of what counts as a convincing argument in mathematics. In this first lesson, not even one group of students was able to present a valid mathematical proof. However it is still possible to comment on the quality of their work with respect to their conception of proving. Though none of the students came up with a series of logical arguments that would lead to the conclusion, we saw cases where some students did get to the desired conclusion. Haruki and Sean, for example, followed a clear route from a specific case to the general one. Julie and Isis presented three distinct cases but as representatives of the general situation. On the other hand, in Victoria and Jacque’s work we detect a ‘circular’ type of reasoning that leads nowhere but to making clearer what the statement is about. In the case of Jasmine and Lee, the girls concluded the desired result but there was no logical connection between their arguments.

In some students’ work, however there were deductive elements revealing literacy towards being able to prove. Moreover, the use of an infinite number of examples either through the dynamics of the GeoGebra applet or through mentioning a number of prototypes as representatives of an infinite set, reveals a realization of what we consider convincing in mathematics. Furthermore, some groups of students used letters for naming the angles, (and the lines in the case of Jasmine & Lee), a fact that allows me to deduce the direction of their thinking as moving from specific to general.

An observation that was made concerning the term ‘vertically opposite angles’ (see the analysis of Stian & Simon’s work) should be repeated here as a recommendation to teachers. The terminology used for describing the pair of opposite angles that are formed when two lines intersect may cause confusion in the students. The word ‘vertically’ may be related to the word ‘vertical’ and not the word ‘vertex’ from which its meaning stems. There are
textbooks that refer to these angles simply as ‘opposite angles’, and perhaps this is the best choice of term to avoid possible misunderstandings that were met in this study.

In general, however, the students’ reasoning was empirical. As it has already been mentioned, the students for this first attempt at a mathematical proof were asked to “convince someone” rather than “prove” the equality of vertically opposite angles. Their answers might have been influenced by this phrase, but it is also possible that the word “prove” might have restricted them from taking an intuitive approach, possibly to end up with more mathematically rigorous answers.
6.2 Lesson 2

Lesson 2 was a two period-lesson and consequently the analysis is separated because I consider it easier for the reader to follow it in two parts.

The first part of Lesson 2 encompassed the following elements:

1. Whole-Class Activity: What is convincing in mathematics?
2. Whole-Class Activity: Why are the vertically opposite angles equal? The mathematical proof
3. The Applets: Alternate Angles - Corresponding Angles – Co-interior Angles
4. Group Work: Corresponding Angles – Co-interior Angles

After a small presentation on what is and what is not considered convincing in mathematics, we returned to the last activity of the first lesson (Group-Work: Vertically Opposite Angles); a whole classroom discussion led to the mathematical proof of the equality of vertically opposite angles. An excerpt from the classroom discussion (element 2 of the list above) is firstly presented.

Whole-Class Activity: Why are the vertically opposite angles equal? The mathematical proof

After making a drawing of two intersecting lines on the board, I asked the students what to do to make sure that we could all understand to which of the four angles we were referring. Two students suggested to name or color the angles (see Figure 6.9).
Jasmine and Ehsan made attempts to explain the equality of vertically opposite angles:

Jasmine: It... whenever you change the left side, it always ends up the same [she is referring to the movement of the lines as seen in the GeoGebra Applet in the first lesson]... I don’t know how to explain that... Can we look it up in our books?

R: No, we will find it ourselves.

Ehsan: So, just why they are always equal?

R: Yes, give me something to move on.

Ehsan: Because when one of them [one of the lines] changes, it changes the same in both, so... they both [the vertically opposite angles] change in the same way so it’s like adding two from both, two to both of them or out of them, it’s basically the same they will always be the same.

R: That is good. How would I write that down?

Ehsan: One is constant and the other one changes the same in both.

Both students provided explanations similar to the ones they attempted in the previous lesson after working the GeoGebra applet. Jasmine realized however that her explanation was not sufficient and asked for permission to look in the textbook. My refusal was accompanied by stating that we would find the proof ourselves. That way I strengthened the belief that we were able and would accomplish our goal with no ‘help’ from the textbook. Such a strategy sets the basis for independent thinking and trust in one’s own ideas. Moreover, after Ehsan’s attempt to reason about the equality of vertically opposite angles, I applauded his answer but I asked at the same time for a way to write that down, intending by this to ask for a formulation of what the student said using mathematical relationships. I did not make this clear enough, and Ehsan did not change his approach the way I intended him to.

Subsequently, I used the ruler for mentioning the one of the two straight lines of the drawing on the board (shown on Figure 6.9 above). The following discussion took place.
R: What would you say about that angle $\alpha$ and that one $[x]$?

Several students: They are supplementary.

Jasmine: Ohh I know how to explain it!!

R: Tell me what to write.

Jasmine: $\alpha$ plus $x$ equals 180 right?

R: Why?

Simon: Because it’s on the same line!

Jasmine: Because it’s a straight angle.

R: We are almost there. **Error! Bookmark not defined.**

Jasmine: $\beta$ plus $x$ also equals 180 degrees! So $\alpha$ must equal $\beta$! Because $x$ is the same so if $\beta$ plus $x$ equals 180 and $\alpha$ plus $x$ equals 180 then they must be the same $[\alpha$ and $\beta]$.

**Very good**

At every opportunity, I was prompting the students to think more and independently for making progress on the task, and asking them to give reasons for what they stated. It is interesting here to see the answers that Jasmine and Simon gave after I asked them to reason why “$\alpha$ plus $x$ equals 180”. Both students answered the same thing but by using a different terminology which is a benefit of a classroom community where a variety of vocabulary and expressions can be shared. My comment “We are almost there” served to encourage the students and confirm that they were on the correct track.

After I asked them about the relationship that holds for two angles on a straight line, Jasmine finally presented the proof. Her approach is the mathematically accepted one, since it consisted of arguments linked together in a logical sequence. Jasmine’s reasoning was altered in the two lessons. At first, she focused on the movement of the lines that she saw in the GeoGebra applet; here she failed to present satisfactory reasoning. She was mostly describing what happens when the lines are moving and from that concluded that vertically opposite angles are equal. In the above excerpt, however, it seems that a hint or a push was required for her to present a different kind of reasoning, deductive reasoning, and build the mathematical proof. It appears that the use of the ruler for mentioning the one line and the two angles on it helped Jasmine to isolate just this one line and reach the conclusion that
angle $\alpha$ and $x$ are supplementary. Then, she recognized that angle $x$ is also lying on the other line and is also supplementary with angle $\beta$. She finally concluded that $\alpha$ must be equal to $\beta$.

I wrote Jasmine’s proof on the board, which all students seemed to understand. I proceeded to give the definitions of alternate, corresponding and co-interior angles. Most of the students were already familiar with the pairs of angles. They were then invited to play with the GeoGebra applets (see Appendix B) to determine the relationship of these pairs of angles when they are formed between parallel lines. Again the students had to drag points and move the figure. No matter where the points were dragged, the students could realize that the alternate angles are equal, and the corresponding as well and the co-interior angles are supplementary (the lines always stayed parallel). After this, students were given the worksheets and this time asked to prove (rather than convince someone as in the first lesson) the relationships of the angles between parallel lines.

**Group Work: Corresponding Angles – Co-interior Angles**

Half of the groups were asked to prove that in the case of parallel lines the corresponding angles are equal (see Figure 6.10).
Figure 6.10 Worksheet ‘Corresponding Angles’

Phoebe & Cassie

The girls did not have difficulty in writing down the proof for the equality of corresponding angles (Figure 6.11).
The girls apparently did not hesitate naming another angle they would use. They recognized that $c$ and $\beta$ are alternate and used this fact since it was known (it is written on the question in the worksheet). They recognized then that $c$ and $\alpha$ are vertically opposite angles, and they easily came to a result. Their proof is clear, and they argued for every step they took. Besides the clarity, the girls’ proof is also characterized by its elegant and short presentation.

What follows is the analysis of the work of the students who were asked to prove that the co-interior angles are supplementary (see the worksheet in Figure 6.12).
Figure 6.12 Worksheet ‘Co-interior Angles’
It seems that the boys did not have any difficulty in proving that the co-interior angles are equal (see Figure 6.13 below).

The boys explained every step of their argumentation and additionally they put their words in mathematical formulas. They could translate their thinking into mathematical relationships. They also showed their reasoning on the drawing. Moreover, what is interesting in their proof is the choice to name the two ‘new’ angles with the same name as the ones that are equal to them. I would say that their choice actually ‘saved’ them from an extra ‘trouble’ that they
would face if they had given different names to the angles involved. Using the numbers 1 and 2 (to refer to the relationships that they have also marked on the drawing) saved them from possible misunderstandings due to the use of the same letters.

The fact that they stated more things than needed since their proof would be complete after mentioning the equality of only, for example, the “two \( b \) angles” makes their proof ‘good’ and not ‘very good’.

\[ Ehsan \]

Ehsan’s approach can be considered special as he took the initiative to draw an extra line, and he used the angle sum of two triangles to get to the conclusion (see his worksheet below on Figure 6.14). It consists of an original approach to the problem. It may be the case, though, that someone had already shown it to him, in which case it was not something that he had just thought of. In any case, however, Ehsan presented three proofs that are consistent, short and elegant. The student did not have any difficulty in writing down mathematical relationships. He used the arrows to justify the written conclusions and mention the logical links between the statements he made. Though he was asked to work only on one proof (the co-interior angles are supplementary), he also handed in the proofs for the equality of alternate and corresponding angles.

\[ 14 \text{ Students in the International school come from different backgrounds, and they may have already been taught some concepts with a different approach.} \]
Figure 6.14 Ehsan’s work
Jasmine & Julie

For the two girls it was not easy to come to the proof. It is interesting to see how they did it while discussing with each other within their group.\textsuperscript{15}

At the beginning, Jasmine and Julie became confused and took for granted that the co-interior angles are supplementary though this is what they needed to prove:

Julie: Ok, let’s start from the beginning.

Jasmine: We know that $\alpha$ plus $\beta$ equals 180 degrees [that is what they are asked to prove]

Julie: Ok.

Jasmine: We also know that $\alpha$ plus $d$ equals $\beta$ plus $c$

They continued by repeating the above and they were unable to get somewhere until Jasmine made progress, but she could not complete her thought and finalize the proof:

Jasmine: Oh I know it! I think. I know it. So we need to use that, look. So, one thing that we know is that these two [vertically opposite angles] are the same, right?

Julie: Ye

Jasmine: And then I think there is a rule about these two…

Julie: Corresponding

Jasmine: Corresponding angles are equal! Miss! Corresponding angles are equal?

Julie: I think so

Jasmine: Ye, so these two are equal. Where am I getting at? Now I have confused myself but I had the idea, I had it!

\textsuperscript{15} The excerpts presented here are not continuous. In between, discussions within the group took place but it was not possible to transcribe (because of classroom noise or other reasons).
Jasmine was missing one more step to conclude that the angles $\alpha$ and $\beta$ are supplementary. After repeating again (thinking aloud and looking at the drawing) what they have discovered so far, Julie got to the desired result:

Julie: These two are the same… And these two are the same (a different pair of angles)

Julie: And this is…I have got it!!

Julie: Well these, these two [alternate angles] are the same right? And this is 180 so this – these are the same – so this must also be 180!

Jasmine could not understand what her partner was saying so Julie had to explain again by reformulating her words. Jasmine repeated the arguments, confirmed them and shouted “We have it!” with a sentiment of joy and relief.

It was not easy for the girls to get to the proof. At first, the problem was that they assumed the hypothesis that they needed to prove. Afterwards, it seems that they understood their mistake, but they were still struggling with the task. The two girls, however, finally got to the proof with no external help. The two of them alone managed to overcome their confusion and proceed after they were stuck. Both girls contributed to the solution since they each provided the crucial elements that were required for the proof. In their worksheet, they gathered and wrote down the absolute necessary statements (accompanied with the appropriate argumentation) they needed, resulting in presenting a ‘very good’ proof. They managed after a long discussion to just grasp the things they needed and write down a well-structured and short proof (as shown in Figure 6.15).
\[ \alpha + \theta = 180^\circ \text{(straight angles)} \]
\[ B = d \text{ (alternate angle)} \]
if \( B \) is equal to \( d \), then \( B + \theta \) also equals 
\[ B + \theta = 180^\circ \text{ is the same as } D + \alpha = 180^\circ \]

Figure 6.15 Jasmine & Julie’s work
The students’ work is shown in the Figure 6.16 below.

**Figure 6.16 Simon & Sean’s work**

It is not easy to follow the boys’ thinking. They named $\alpha$ and $\beta$ the angles around the ones the task is about. Although we cannot be sure, they probably used the same names for mentioning the equality of the angles. They wrote down two relationships concerning the angles $\alpha$ and $\beta$ but without giving any arguments for these statements. In their explanation, they wrote:

> They are co-interior because both [are] in the inside of the line and the [they] are also, the angles are not equal. Unlike alternate and corresponding angles, those are both equal.

[Simon & Sean, worksheet]

Simon & Sean misunderstood the task they were asked to solve. They did not suggest an explanation for why the co-interior angles are supplementary. Although they have written the relation that holds for the angles $\alpha$ and $\beta$ on their worksheet, nothing suggests that they knew
where it comes from. What they wrote as an explanation is a reason why the two angles of
the task are co-interior; they actually explained the term that is used for the pair of angles.
They also stated that the co-interior angles are not equal as the alternate or corresponding
angles, and it seems that this was their way of proving the task. The boys took no steps
towards a mathematical proof.

**Analysis Overview of the 1st part of Lesson 2**

A comparison between the first and the second lesson with respect to students’ reasoning and
the quality of their proof is inevitable. While in the first lesson the students based their
explanations on their intuition or their visualization of the facts, in the second lesson there
was a change in their way of thinking. This alteration is probably a result of the presentation
on what is considered convincing in mathematics and the classroom discussion that led to the
proof of the equality of the vertically opposite angles. It is likely that the students were also
influenced by the word “prove” rather than “convince” that was used in the first lesson. After
observing the proof of the equality of vertically opposite angles, some of the students were
able to proceed similarly in the new task and use a deductive approach to prove the equality
of the corresponding angles or the fact that the co-interior angles are supplementary.
However, there were students who did not come up with arguments to support the
hypotheses.
The second part of Lesson 2 encompassed the following elements:

1. Whole-Class Activity: Tangram
2. Whole-Class Activity: Kinds of Quadrilaterals
3. Whole-Class Activity: Tangram & Congruent Triangles
4. Group Work: Compare Triangles

The second part of the second lesson started with a discussion on the kinds of quadrilaterals. To introduce the discussion I used the Chinese game ‘Tangram’ focusing on the idea that when different kinds of shapes are put together a variety of figures can be formed. After the students mentioned the properties of the different kinds of quadrilaterals, I used two equal triangle shapes from the tangram and put them together to form a square, a triangle, a parallelogram. Afterwards, on the board the students could see the parallelogram below (Figure 6.17) and they were asked what holds concerning the angles $\alpha$ and $\beta$. The majority of the students recognized the alternate angles and the fact that they are equal because they are formed between parallel lines. Apparently, it was not difficult for them to ‘locate’ the pair of angles in a context slightly different from the one they were used to ‘seeing’ the pairs of alternate, corresponding and co-interior angles (i.e., drawing on the worksheet in Figure 6.12).

![Figure 6.17 Tangram Triangles](image)

Using again the two tangram triangles, I initiated a classroom discussion on congruent triangles. However I did not use the term ‘congruent’, a term that is introduced in later grades. Instead of “congruent triangles” I was referring to the triangles with the phrase “the one is the exact copy of the other”.

In the excerpt below Jasmine and Ehsan were trying to figure out what is needed in order to show that two triangles are congruent.

R: The triangles then are exactly the same [holding the two tangram shapes]. The one is the copy of the other. What does this mean about the angles of the triangles?

Jasmine: They are exactly the same.

R: What does this mean about the sides of the triangle?

Jasmine: They are the same.

R: So, if I ask you to show that a triangle is a copy of another, what do you need to show?

Jasmine: The same angles.

R: Ok, you think that if you just show me that the three angles are equal, is that enough?

Ehsan: Yes.

Jasmine: They have to be in the same position, the same spot.

Ehsan: Oh wait. That doesn’t prove that there is the same length of the sides [showing with his hands].

R: Let me ask once more. Do you think that is enough to show that the three angles are equal?

Jasmine: No, you still have to find the length of the lines, where they are positioned.

Students were asked questions to mobilize them into thinking for themselves what it means for a triangle to be a copy of another. They were asked a clarifying question aimed at repeating once more their thoughts and summing up the conclusion for the whole classroom. Additionally, I kept posing questions to the students prompting them to rethink and reflect on their answers. Jasmine pointed out that the corresponding angles and not just a random pair of angles must be equal when comparing the triangles. Moreover, Ehsan, after rethinking his answer, realized that the equality of angles alone does not prove the equality of the triangles. At the end of the discussion, I asked a ‘summary’ question aiming for the students to repeat and formulate an integrated answer. In this way it was hoped to decrease any misunderstandings concerning the discussion.
For strengthening the point of discussion (i.e. the equality of angles does not suggest the equality of the triangles), the students were shown the two triangles below (Figure 6.18) where all three corresponding angles are equal but not the triangles.

![Figure 6.18 The equality of the angles alone does not suggest congruence](image)

I asked the students why the third angle was not shown. Ehsan tried to explain descriptively that we know for sure that the third angle must be the same in both triangles:

Ehsan: If two are equal the third has to be equal. […] Because they are meeting each other at the same angle. So… there is really no other way to…

Cassie presents a different explanation:

Cassie: Both triangles have to have a total angle sum of 180 degrees and if… if the other two angles are equal then it means the third must be equal as well.

Ehsan and the rest of the students seemed to accept Cassie’s explanation.

Both students provided a kind of reasoning for explaining what they were asked. However, Ehsan’s reasoning is limited to an intuitive argument that it is hard for him to express. It is empirical reasoning and since I consider that he would reason the same for any pair of triangles, his reasoning is placed under the category of infinite number of examples. For the same reason, Cassie’s reasoning is placed under the infinite examples category. Moreover, Cassie’s reasoning is mathematically correct and well stated and as a result it is characterized as deductive.

For the rest of the lesson period, the students were asked to work on a task (Figure 6.19) within their groups.
**Group Work: Compare Triangles**

The students were asked to compare two triangles. They were actually asked to prove that a triangle is the copy of another as is shown on the worksheet (Figure 6.19) they were given. The lines of the same colour are parallel as indicated by the arrows.

![Worksheet 'Compare Triangles'](image)

*Figure 6.19 Worksheet ‘Compare Triangles’*

What follows is the analysis of the students’ work.
The girls have written down that the angle sum of both triangles must be 180 degrees. Recognizing that the lines are parallel they tried to find a way to proceed. They wrote down that the angles $f$ and $g$ (the names they gave in the drawing to $D$ and $C$ ($\triangle BCE$) respectively, after the suggestion of the cooperating teacher) are equal because they are corresponding...
angles. As for the sides of the triangles they argued that “lines are all parallel to each other so they are equal”, something that indicates a confusion. When I approached them, I made clear that a pair of sides is equal because they are the opposite sides of a parallelogram and encouraged them to argue similarly for the other sides. The girls finally wrote on their worksheet: “blue lines are equal (parallelogram), green lines are equal (parallelogram), red lines are equal (trapezium). Their argumentation is deficient. They did not mention the parallelograms they were talking about and in the case of the red lines their argumentation is either wrong (they missed the trapezium properties) or misleading.

As for the angles of the triangles, though they have realized the equality of $D$ and $C$ (
Though they took the initiative to work on the problem and made some progress, they did not manage to get to the solution of the task.

Victoria & Jacque

The girls have named the angles as $a = \measuredangle D\hat{A}C$, $\beta = \measuredangle A\hat{C}B$, $c = \measuredangle C\hat{B}E$ as it is shown on their worksheet (Figure 6.21).

![Figure 6.21 Victoria & Jacque’s work](image)

Victoria and Jacque began by pointing out that the angles $\alpha$ and $\beta$ are equal though they did not recall how to characterize them (alternate angles). The cooperating teacher (CT)
approached them and after reminding them of the terminology, the discussion continued as seen below. Jacque did not participate in any dialogue with the cooperating teacher.

CT: Alternate [reminding the terminology]. That is the first step. You want to show that these two triangles, the white ones [talking for the triangles ADC and BCE] are the same.

Victoria: So, $b$ equals $c$.

CT: Why? Ye, why?

Victoria: Because they are also...

CT: Aha, so what is your conclusion from these two relations that you wrote about the triangles that interest you?

Victoria: That they are all equal

CT: What is equal?

Victoria: $a$ equals $b$ and equals $c$.

CT: So, ye.

Victoria: So and we are done…?

CT: That is the first step. So you have proven that these two angles are equal but these two triangles have other angles as well. Do it for the other angles now. Until you prove that the two triangles are equal.

Through the dialogue between Victoria and the cooperating teacher, the teacher applied different teaching strategies either for prompting the girls to think independently or requiring them to reason for what they said. She also asked for clarification in order to receive a more specific answer from the girls concerning the equality of the angles. Confirming Victoria’s answer aimed at providing feedback and give the ‘permission’ to the girls to proceed further with the task.

The girls pointed out the equality of alternate angles as soon as they recognized the parallel lines. However they did not connect their finding to the task that they had to solve, as it
seems from their question “so and we are done?” The girls took the initiative to start working on the task but apparently they did not have a ‘plan’ for getting to the proof. They asked for guidance as they did not seem to recognize whether they had gotten a satisfactory answer. The cooperating teacher encouraged them to draw conclusions for the other angles of the triangles and reminded them once more of the requirement of the task. Without having made any progress, they called the cooperating teacher again:

Victoria: Miss, can you help us?

CT: I helped you enough and you did it yourself actually. So, you showed these two, now do it for the other angles of the triangles.

Victoria: This and this [probably showing d and e]?  

CT: Ye for example. Which ones seem equal to you?

Victoria: Ok… and what is the explanation?

CT: Similar, like this (the first two angles). Look at the lines.

Victoria: They are all alternate angles

CT: I don’t know, look. Try and figure it out together

The girls stated what they could conclude from looking at the drawing but could not proceed to further findings that would give them an integrated answer. The cooperating teacher encouraged them once more to examine what holds for the other angles of the triangles, but the girls seemed unwilling to proceed on their own, they continuously depended on the teacher in order to move on, in spite of the teacher’s attempts to prompt them into thinking independently. Aside from the push given by the cooperating teacher, the girls did not make more progress other than writing down that the angles d and e are equal but without explaining the reason why. However, the girls did write on the worksheet the reasons why the first two pairs of angles are equal. Moreover, they deduced from these equalities (α=β and β=c) that α=c, a fact that allows me to conclude that there were some progress in their attempts to prove.
When the group was handed the worksheet, the following dialogue took place between Ehsan and Sofia. It should be noted that the third member of the group, Risako did not contribute in the solution of the task at all.

Sofia: Let’s prove that the triangle $ADC$ is a copy of the triangle $BCE$ [reading the question]

Ehsan: $ABC$…these are the triangles. Oh I am sorry no. This triangle and this triangle [showing the triangles $ADC$ and $BCE$]

Sofia: Well they have the same sizes, the same lengths

Ehsan: Ye but you can’t just measure in algebra. In algebra you have to prove that it is always this case, the case is always the same. You have to find a proof.

Sofia: They have the same degrees

Ehsan: Well ye but how do you know that? You can’t just measure it with a protractor. You have to prove it.

Sofia: By looking

Ehsan: You are never really accurate by looking. So these two are definitely the same, it’s Z angles, alternate angles […]

Sofia suggested that the two triangles are the same because they look the same, but Ehsan explained to her that this is not an adequate argument in algebra. “You have to prove it.” Ehsan was trying to help Sofia to understand what it means to reason mathematically. Sofia, however, insisted in expressing an empirical reasoning based on visual evidence and she was not convinced from her partner’s words. After a while, Ehsan lost patience and started working on the task on his own in order to come up with mathematically valuable argumentation.

Ehsan justified everything he wrote (see Figure 6.22), but he argued for the equality of the triangles $ABC$ and $BCE$ and not for the requested triangles $ADC$ and $BCE$ that he was expected to.
His approach was as follows: he firstly established the equality of two pairs of angles and for the third one he stated that “must be equal because the others are”. Before he reached the conclusion for the third pair of angles he tried to find a similar way, to prove their equality, similar to what he did for the other two pairs (parallel lines, alternate, corresponding) and finally he said, addressing to his partner:
Ehsan: I am wasting my time all this time!

Sofia: Mmm?

Ehsan: Nothing, I just did not see something really obvious. I don’t need to prove \( A \) and \( E \), if I know the two other angles the third has to be equal, I said it myself!

Sofia: What?

Ehsan: I don’t need to prove that this and this are equal. Because if I have these two, I can already say for sure that these have to be equal as well. It’s just the rule. So I will call this \( N \) and that one \( G \). These are also equal. Now all I need to do is… \( BC \)… [writing on the paper] … Done! We proved it! Miss! I finished!

As for the sides of the triangles he wrote that “\( BC \) is also equal in both triangles because they share it.” And he finished by writing that “If two triangles share 2 angles and a side, they must be equal.”

Ehsan did not argue for the triangles he was asked by the task. I consider this mistake to be an oversight. The comparison of the two triangles he chose, however, is easier than the one that was asked in the task. I overlook his mistake and analyze his reasoning since it is interesting to be studied. Opposite to the two groups of girls we saw earlier (Jasmine & Julie and Victoria & Jacque), Ehsan had a more “open-minded” approach. He was capable of thinking in a wider range for proving the task by using his knowledge on triangles, angles and parallel lines. He also depended on his own logic, his state of mind when he reached his final conclusion to support that the one triangle is a copy of the other. Moreover, when stuck he managed to get ahead without any help from the teachers or his classmates.

Ehsan’s proof is original since the student solved the task and provided a slightly different approach from the one suggested by me when he concluded the equality of the third pair of angles using the angle sum of a triangle. Moreover, he made a “big jump” into making a generalization that is stated in the last sentence he wrote.
The two boys were in the same group but they each wrote on the worksheet their own personal explanation for the task. On Figure 6.23, Haruki’s explanation is shown:

Figure 6.23 Haruki’s work

Haruki’s explanation escaped the equality of angles using the parallel lines and the equality of the sides of the two triangles. He argued using the fact that the diagonal (in his words “the transversal”) of the parallelogram divides it into two equal triangles. He noticed that a triangle is a half part of more than one parallelogram and that is how he reached his conclusion. Being familiar with the properties of the parallelogram and being able to “decompose” the figure, he presented a valid and original mathematical proof. He used arguments that are very different from the ones that were suggested by me. Furthermore, his solution is organized and well-written.

To the contrary, Stian’s answer is characterized by erroneous, rather incomplete reasoning.

Stian: We can see that all the same colored lines are parallel which means that no matter the size of the triangle they [probably meaning the colored lines] are equal. [Stian, worksheet]

It is hard for me to follow Stian’s thinking. He did make an attempt to answer the question, and I believe that he was quite satisfied with the answer he gave.
Lee & Isis

Lee and Isis have simply written down what I or the cooperating teacher actually told them. They have given new names to the angles as suggested by the cooperating teacher and they have written down the equality of two pairs of angles that was pointed out to them. They could not proceed or conclude something when working alone. Though they have written some mathematical relationships on their worksheet, when listening to the audio-recording I realized that everything that was written was concluded from comments from the teacher and not from the students. Therefore, concerning the quality of their work, there is no progress.

Ines & Miriam & Hayley

In the case of Ines, Miriam and Hayley, they have written down the givens of the task (the existence of parallel segments). Besides mentioning the equality of the segments AC and BE and without giving a reason for this, they did not get to any further arguments which would lead to the solution.

Figure 6.24 Ines & Miriam & Hayley’s work
What they wrote is enough for them to conclude the congruence of the triangles ADC and BCE. It is likely, however, that they understood that this is not an appropriate or even convincing proof, but they felt ‘they had’ to end up to the statement to be proved.

Analysis Overview of the 2nd part of Lesson 2

With the exception of one student (Haruki, Figure 6.23), none of the rest managed to solve the task.\textsuperscript{17} Apparently, the task was difficult for them, and one of the possible reasons was the complexity of the drawing. Though the required elements for getting to the proof were discussed in the day’s lesson and also in previous lessons, it was hard for the students to ‘discover’ these elements due to the complex drawing. It was hard for them for example to locate alternate angles.

Apparently, it was also hard for them to focus on and in a way ‘isolate’ the two triangles discussed in the task among the triangles that appear in the drawing. Moreover, for some of the elements of the triangles the comparison was not direct. On the contrary, two steps were required in order to reach the equality of some sides or angles of the discussed triangles\textsuperscript{18}, something that made the task even harder for the students. However, Haruki (Figure 6.23) and Ehsan (Figure 6.22) (though the task he worked on was not as challenging as the original one) managed to find a way and overcome the above difficulties and present correct answers.

After analysing the students’ work on the task, I also realized that they were having difficulties with tasks where the question is not specific enough. Some evidence for this may be found in the work of Victoria & Jacque, Stian, Ines, Miriam & Haley, and others. Based on such evidence and on my impressions of students’ progress, it is fair to say that the majority of the students in the classroom would have performed better if the question of the task was enriched as follows:

“Let’s prove that the triangle \textit{ADC} is a copy of the triangle \textit{BCE}! Let’s prove then that:

•
Such a question would have made the goal more specific to the students. Gradually, after working on similar tasks, the students could probably encounter with more ‘abstract’ questions that would promote their independent thinking and they would be able to realize for themselves what it is that they have to in order to get to the solution.

As a last recommendation to teachers and researchers concerning the design of this task, I would like to report here one more observation concerning the task. The communication among the students as well as mine or the cooperating teacher’s with the students was made difficult because of the absence of names of the angles in the drawing. When designing the task I decided to name the vertices and let the students name the angles using three vertices. Apparently, that was an extra difficulty as part of a challenging task.
6.3 Lesson 3

Lesson 3 encompassed the following elements:

1. Whole-Class Activity: Compare Triangles
2. Group Work: Angle Sum of a Triangle

The third lesson began with the last task of the previous lesson. The students had to prove that a particular triangle is a copy of another (see the worksheet in Figure 6.19). I presented a proof using the theory of parallel lines and the angles formed between them, and the properties of parallelograms to prove that the two triangles are congruent. I mentioned to the students that rotating the paper may be of help since conclusions concerning parallel lines are easier to make when the lines are horizontal. Besides that, I drew on the blackboard isolated parts of the figure to make the conclusions for the angles clearer to students. They were then shown an applet on the interactive board in which the vertices of a triangle could be dragged so that the triangle would change. No matter the change, the angles of the triangle summed up to 180 degrees. Subsequently, the students were given the worksheet.

Group Work: Angle Sum of a Triangle

The students worked on the following worksheet:

![Worksheet Image](image)

Figure 6.25 Worksheet ‘Angle Sum of Triangle’
Jasmine & Lee

The girls started by drawing a right triangle on their worksheet and discussed:

Lee: This is 90 degrees [meaning the right angle of the triangle]

Jasmine: And these two [meaning the other two angles] together should equal to 90 degrees

Lee: Yes

Jasmine: But how do you prove that?

Drawing a right triangle made sense to the two girls probably because they already knew the degrees of one of the three angles. However, they got nowhere with that approach, and proceeded then to drawing a random triangle.

Figure 6.26 Jasmine and Lee’s worksheet
They had then the idea to draw a line passing through one of the vertices and parallel to the opposite side. Before that, however, they were looking online so the idea to draw the line may not be theirs but they came up with it after having seen it on their computer.

The arguments that are written in the parentheses for the equalities of the angles were completed after my request to do so. The girls had not written that the line passing through angle $a$ is parallel to the opposite side of the triangle. That proves the fact that for them the drawing itself is convincing enough for making conclusions. After this, the girls completed their proof with the arguments that were missing. Their proof is good.

Stian & Sean

The boys followed the same approach as Jasmine and Lee:

Very Good
angles are the same. The boys gave arguments for everything they wrote and so presented a very good proof.

Ehsan

Ehsan’s work is shown on Figure 6.28:

![Ehsan's work](image)

**Figure 6.28 Ehsan’s work**

Once more, the student followed the approach that we have met so far. Ehsan drew the extra line coming through one of the vertices and parallel to the opposite side of the triangle. The use of arrows probably suggests that the lines are parallel. Besides that, probably for strengthening the fact that the extra line is not randomly drawn, he drew two extra segments that must be parallel (marked with the use of the arrows again). He wrote down the proof but neglected to give reasoning for his statements (e.g. why $e = d$ and $b = a$). Ehsan has interestingly labeled angle $c$ to be 90 degrees. However, it is difficult to interpret the reason

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19 An approach that was followed on a task in one of the previous lessons from a group of which Stian was again a participant. See lesson 2, Haruki & Stian’s work.
since his proof is not based on this fact. His proof is not limited to a right-angle triangle. It is possible that he had started working on the task by drawing a right-angle triangle but then managed to prove the statement without having to mention the 90 degrees angle.

Haruki & Simon

For Haruki and Simon the approach is totally different:

The boys’ work is an interesting approach regarding their interpretation. They have ‘cut’ the angles of a triangle, placed them next to each other and ended up with a straight line. This ‘cut & paste’ approach is also shown in the textbook. It is probably not something that the boys came up with on their own, and they considered it convincing enough for proving that the angle sum of a triangle is 180 degrees. They do not see it as just a way to test or validate the hypothesis. Though what they have done is just explaining further the statement to be proven, their work is for them a convincing argument that holds for any triangle. They have made some progress in working with the task but they have not presented a proof. Their work consists of a transformation of something they have adopted from an external source (probably their textbook) and trusted without verification.
The first thought of the group came from Isis:

**Isis:** Shouldn’t we examine the angles? Mmm

[…]

**Isis:** We can say like no matter how long the angle [probably meaning how big the triangle is] is, it will always be 180 degrees and then like give like random triangles, draw random triangles and we can put the degrees to 180.

Isis suggested drawing several random triangles and stating that for all holds that the angle sum is 180 degrees. What she did is actually a confirmation of what she saw on the applet. In the applet, the size of the triangle and its elements change when dragging the vertices. Her phrase “no matter how long the angle is” is probably a result from seeing the change in the applet. She explained, therefore, what holds, no matter the triangle, but without arguing appropriately. For Isis, drawing random triangles and stating that the angle sum is 180 degrees is convincing. Her reasoning is premature, empirical and drawn from the dynamic nature of the applet that was shown in the classroom. Julie was not convinced, and they abandoned the idea. The two girls were seated opposite to Jasmine and Lee (the first group that was discussed) and they were asking them to help them with the solution. Finally, they have written on the worksheet what the other group explained to them. They did nothing independently.

The third approach that was observed in the students’ work was the argument that a triangle is half of a square. The group below was chosen as the representative of the approach (one more group argued in the same way).
The argument is correct though it does not consist of a valid mathematical proof. The girls have presented a specific case even though they considered it general. Dividing any quadrilateral in two would have made their approach more severe and mathematically acceptable though the question “why is the sum of a quadrilateral 360 degrees?” would still exist.

Figure 6.30 Ines, Miriam, Hayley & Risako’s work
Analysis Overview of Lesson 3

Proving that the angle sum of a triangle is 180 degrees was not a challenge for the students. Even though not all groups came to a strictly mathematical proof, most of them could work on the task and argue about it. Isis & Julie argued empirically though, and could not proceed to constructing a proof, while Haruki and Simon’s answer consists of a verification of the statement to be proven.

Three groups drew an extra line coming through one of the vertices of the triangle and parallel to the opposite side. My intention for including the task was to see if the students ‘intervened’ in the drawing of a triangle by adding new elements (i.e., lines) that would help them see through the solution. However, the students proceeded to drawing an extra line probably because they had seen the proof before (perhaps last year).

It was observed that some students drew a right-triangle to begin solving the task. That is not very surprising since a number (in this case 90 degrees angle) consists of an element that takes away the ‘abstraction’. It makes sense to start with numbers since the question is to end up with a number. However, that starting point was not an obstacle to the students since they found a way to proceed and to refer to a random triangle.
6.4 Lesson 4

Lesson 4 encompassed the following elements:

1. Group Work: Angle Sum of a Quadrilateral
2. Group Work: Trapezium

In the last lesson I wanted to omit classroom discussions and leave the students to work within their groups on two tasks. As in the previous three lessons, during the fourth lesson the cooperating teacher and I were again supervising the student groups, but our interference was kept to a minimum.

Both tasks (especially the second one) consist of summary tasks that require knowledge of everything that has been taught and discussed in the previous lessons. Students’ work on both tasks, therefore, would, once again, reveal their understanding of properties concerning angles, triangles and quadrilaterals and their progress in proving geometrical tasks.

Following is the analysis of the students’ work.

**Group Work: Angle Sum of a Quadrilateral**

The students were given the following worksheet:

![Worksheet 2 / PROOF: Convince us!!](image)

**Figure 6.31 Worksheet ‘Angle Sum of a Quadrilateral**
The majority of the students (with the exception of one group) in order to prove that the angle sum of a quadrilateral is 360 degrees, divided it into two triangles.

Victoria & Jacque

![Image](76x707 to 86x717)

**Figure 6.32 Victoria & Jacque’s work**

The girls have correctly stated that the angle sum of any quadrilateral is 360 degrees. To prove it, they drew a random quadrilateral (not a rectangle like some of the groups did as we will see next) and drew the diagonal. They also added the definition of a quadrilateral. Though their thinking, their approach to divide the quadrilateral into two triangles and end up to 360 degrees is correct, what they wrote down is not correct. Though they drew the diagonal they haven’t divided the angles $b$ and $d$ in two parts. On the contrary, they have included angles $b$ and $d$ in the relationships they wrote concerning the triangles. The girls did not realize – maybe because of not paying attention to the task – that angles $b$ and $d$ are not angles of the triangles shown and that makes their proof incorrect.
Ehsan and Phoebe & Cassie have taken the same approach as Victoria & Jacque but they presented correct proofs:

**Ehsan**

![Figure 6.33 Ehsan’s work](image)

\[
\alpha + b + c = 180 \\
d + e + f = 180 \\
180 + 180 = 360
\]

*The diagonal in every quadrilateral turns into two equal triangles.*

Ehsan drew the diagonal of the quadrilateral and named all the angles that are formed. He wrote down the correct relations that hold for the triangles and got to the desired result. It seems, however, that his conception of what a quadrilateral is, is limited to rectangles. I draw this conclusion from both his drawing (where angles $f$ and $c$ are marked as right angles) and his final sentence “the diagonal in every quadrilateral turns into two equal triangles”. Besides his misconception (that may be due to inattention), his proof is not restricted to rectangles.
Phoebe & Cassie

Phoebe and Cassie have not drawn a random quadrilateral; their drawing reminds us of a rectangle. However, nothing reveals that they shared Ehsan’s misconception.

Figure 6.34 Phoebe & Cassie’s work

The girls have done a very good proof where all the steps are described and explained sufficiently. They have drawn two quadrilaterals for showing that the diagonal divided the two angles $a$ and $d$ into $a_1$, $a_2$, $d_1$, and $d_2$ respectively. Although their drawing reminds us of a rectangle, their proof would work for any quadrilateral. They did not have any difficulty in
translating the words into mathematical relationships and ended up in presenting a formal mathematical proof.

Jasmine & Julie

Jasmine and Julie also took a quadrilateral and divided it in two triangles:

Figure 6.35 Jasmine & Julie’s work
The relationships that the girls wrote down are not strictly mathematical. They have combined words and symbols for getting to the conclusion. They have drawn a rectangle as it becomes obvious from the right angle symbol on the drawing, as well as from the multiplication of four right angles. They have added, however, an example of a random quadrilateral and using the same argument, proved that its angle sum is 360 degrees. That consists of a creative way to generalize, since they chose a non-trivial example for it.

The choice of the two drawings reveals a possible misunderstanding – the same as the one Ehsan (see Figure 6.33) had – that a quadrilateral’s diagonal divides it into two congruent triangles. Though non-trivial, it is likely that the second quadrilateral was drawn in such a way that the two triangles were congruent.

The approach we saw so far (proving what the angle sum of a quadrilateral is by dividing it into two triangles) was also followed by three more groups. In their case, they have not given names to the angles. They drew several examples of quadrilaterals and their diagonals and stated that two triangles make a quadrilateral and proved the task (See Ines, Hayley & Miriam’s work below).
What distinguishes this proof from the ones we saw earlier is a lack of formality because of lacking names for the angles and providing mathematical relations for proving the task. Drawing more than one quadrilateral, however, shows a step towards a proving competency that is more formal. Except for drawing several random quadrilaterals, the phrase “2 triangles always make up any quadrilateral” witnesses the girls’ understanding of the requirement to argue for any kind of quadrilateral, and therefore, their realization of what ‘proving’ means. In spite of its lack of formality, their proof is labeled as ‘good’ since it is correct and the statements are valid, well argued and follow a logical sequence.

Below is the work of the group that presented a totally different work for proving the angle sum of a quadrilateral. Haruki & Stian were the only students who took a different approach in solving the task.
Haruki & Stian

The boys followed the same approach as the one Haruki and his partner at the time previously used for proving that the angle sum of a triangle is 180 degrees (See Haruki & Simon’s work, Figure 6.29).

Haruki and Stian ‘cut and pasted’ the four angles of a quadrilateral and made adjacent angles after placing them next to each other. They did it firstly by cutting and pasting the four right angles of a rectangle that sum up to an angle of revolution. They continued by placing next to each other the four angles of a (probably) random quadrilateral and got again to the same conclusion. In the case of a rectangle, they are convincing enough since adding together four right angles makes 360 degrees. Their attempt though to generalize with the random quadrilateral does not consist of a proof but rather a test for validating the statement to be proven by using a pair of scissors. Though they have taken an initiative and made some progress with the task, Haruki and Stian’s work is not a proof.

Figure 6.37 Haruki & Stian’s work
The students’ attempt to apply the method that was used in proving that the angle sum of a triangle is 180 degrees (see Figure 6.29) to quadrilaterals is worth mentioning. Although they did nothing more than verifying the hypothesis, the students managed to apply a methodology they saw in the textbook in a different context from the one presented, something that reveals that they took responsibility for their action. Their idea of generalizing a methodology builds on their probable conception of mathematics.

**Group Work: Trapezium**

The students were asked to work on the following task:

![Worksheet 'Trapezium'](#)

For solving the task, the students should trace back everything we have done during the previous four lessons. The students, firstly, had to bring to mind the properties of a trapezium and conclude that sides AB and CD are parallel. Subsequently, by locating alternate, corresponding, co-interior angles, and by applying the angle sum of a triangle is 180 degrees or that of a quadrilateral is 360 degrees, they could proceed in finding all the missing angles.
and solve the task. As a result, my aim for including this task at the end of my learning sequence was to find out what the students got from the lessons and whether they could apply everything we discussed to a numerical task. Because of the nature of the task – the students are not asked to prove but solve – the students’ work will not be analyzed as I did so far, but I will rather discuss the students’ reactions to this ‘summary’ numerical task.

All students could start with the task and proceed to a certain extent.

None of the groups paid attention to the hypothesis that the quadrilateral is a trapezium. There was no one who wrote on the worksheet the argument that I expected to meet that AB and CD are parallel because ABCD is a trapezium. Most of them however took for granted that the sides AB and CD are parallel only by looking the figure.

In general, the students did not present a well organised or easy-to-read worksheet. There are calculations on the paper, as for example on the worksheet below by Phoebe and Cassie (Figure 6.39), but without explanation for their answers. What is missing, therefore, from their work is the argumentation for the steps they are taking. An explanation for this can be that the task gives and asks for numerical data. In such a case writing the numbers seems to be enough for the students, as can be seen in Figure 6.39.

Below, I discuss some of the students’ work.
Ehsan was calculating the missing angles by applying the angle sum of a triangle. After a while, he got stuck, since he had not considered that the sides AB and CD are parallel.

Ehsan: Wait [talking to his partner]. We have got too little things
Miss. Miss! We have got too little information to work it out.

[Audio recording]

After the cooperating teacher mentioned the existence of parallel lines, Ehsan continued his work by using alternate, corresponding and co-interior lines.

Figure 6.39 Ehsan’s worksheet
Phoebe & Cassie

The girls solved the task without difficulty. They calculated all the missing angles. However, there is no argumentation for their actions.

Figure 6.40 Phoebe & Cassie’s worksheet
Stian & Haruki handed in an organised and carefully written worksheet (Figure 6.41). What distinguishes their work from all the other groups is the organised way of presenting the results (they missed though writing down angle ‘$h$’) enables me to follow their argumentation. They have also written on the right side what they used for getting to the results. I can not, however, see the connection between their statements and their work except for their first argument that describes the way they reached in calculating angles ‘$c$’ and ‘$e$’. Like the rest of the students, they also did not mention the existence of parallel lines.
Jasmine & Julie’s worksheet is not well-presented. However, the girls solved the task and after my prompting, they included reasons for what they did but in some cases insufficient (see ‘z angle’).

![Figure 6.42 Jasmine & Julie’s worksheet](image)

In general, the students calculated the missing angles using everything we have discussed in the previous lesson and therefore my intention for including the task matched with practice. The students used the angle sum of a triangles, the supplementary angles, the vertically opposite angles, the alternate, corresponding and co-interior angles and in different ‘combinations’.
Analysis Overview of Lesson 4

In the first task that required the students to come up with a proof that the angle sum of a quadrilateral is 360 degrees, all students had a correct idea for proving it. The idea was to divide a quadrilateral into two triangles and therefore two triangles make 360 degrees. Some students could present a mathematical proof, whereas the rest’s work was more informal and it is this which distinguishes the qualities of proofs we saw. In Ehsan’s case (Figure 6.33) the proof is characterized as ‘good’ since it is correct and clear. However, when comparing it with Phoebe & Cassie’s proof, I see that his proof is missing the formality of the girls’ proof. Phoebe & Cassie have presented a formal proof and their ability to move from one relationship to the next, ending with ‘a+b+c+d=360’ shows a ‘superiority’ in deduction.

It was observed that some students began their work by drawing a rectangle but that did not restrict their argumentation (their work was generalizable in every quadrilateral). It is, however, safer always to mention to the students to draw a random shape in order to make easier the transition from the ‘specific’ to general, consequently pave the road for literacy in proving.

As for the second and last task of the lesson sequence, it was a numerical task and the students were much more comfortable than with the proving tasks. That shouldn’t surprise us since numerical tasks (tasks requiring calculations) are not as abstract as the proving tasks and the students must have had a lot of practice. Concerning the lack of arguments on their worksheet (the property of the trapezium and more), it does not necessarily mean that the students could not support their work but only neglected to write down their reasons. My insistence for reasoning is crucial for the students of this age (who are just starting working on proving tasks) to be thorough, for understanding the requirement of the nature of mathematics that nothing is self-evident.
6.5 The Interviews

As described in Section 4.3, the interviews took place after the end of the learning sequence in the classroom. I interviewed one group of students and two individuals. The analysis of the interviews comes after the description of the interview tasks.

The Tasks in the Interviews

The following two GeoGebra applets were designed for the interviews.

1. Intersecting Lines

![Interview Applet ‘Intersecting Lines’](image)

Three lines intersect as shown in Figure 6.43. The lines can be moved by dragging the red points (J, R, N), but no matter where the points are dragged the lines always intersect. In the applet the sizes of the angles are always visible as shown in the screenshot.

When shown the applet and after playing with it, I expected the students to recognize the vertically opposite angles and point out that the sum of the marked angles is 180 degrees, no matter where you move the lines.

My question to the students went no further than “What do you observe when dragging the points?” My intention was to let the students share whatever came in mind when playing with the applet and not to restrict their view with a question like “What could you say about the angles?”
2. **Perpendicular Bisector**

This applet was more complex than the first one and the construction of it was shown to the students. Here are the steps for the construction of a perpendicular bisector:

1. Drawing a segment AB;
2. Taking C as the middle point of the segment AB;
3. Drawing the perpendicular bisector of the segment AB;
4. Taking point D on the perpendicular bisector;
5. Drawing the segments AD and BD.

What was shown on the screen after making the last step is seen on the following Figure.

![Figure 6.44 Interview Applet ‘Perpendicular Bisector’](image)

When dragging point D, it was moving on the perpendicular bisector and therefore segments AD and BD were changing accordingly.

After I finished constructing the figure as described above, I asked the students to drag point D (red point) and describe what was happening. Once more, I did not pose a more specific question concerning the applet (like “what holds for the segments AD and BD?”) for allowing the students share freely their thoughts after observing what is happening when moving point D.

My intention for designing this task was that it would allow students to make conjectures concerning the elements of the figure (angles, sides). I expected statements like “whenever I
move point D, AD and DB are always the same” and I wanted to investigate their attempts to explain their statements. This task is much more complex and demanding compared to the first one since the students had to pay attention to the construction of the geometrical figure in order to take steps towards a mathematically valued proof.

**Analysis of the Interviews**

I begin with the analysis of the interview conducted with the group of two girls and discuss both tasks and continue with the interviews conducted with the two individuals.

**Jasmine & Julie**

*1st task*

Jasmine’s reactions at the beginning show that she is ‘limiting herself’ to searching for things we had discussed in the classroom. The first thing she did was to point out that there were no parallel lines on the applet so that excluded the possibility to comment concerning alternate, corresponding and co-interior angles. Furthermore, she pointed out the existence of vertically opposite angles and showed pairs of supplementary angles on the figure. Her ‘insistence’ on looking for the things we had discussed in the classroom prevented her from seeing the figure as a whole. Instead, she isolated parts of the figure and drew conclusions. On the other hand, Julie observed the figure as a whole picture and got to the conclusion that I expected to hear concerning the marked angles:

Julie:  All together [the marked angles] would be 180, right? 
        'Cause they are opposite from those and then that would be 180 cause it’s a triangle.

Jasmine agrees and adds:

Jasmine: Yes because these two, all of them [showing the marked angles and the vertically opposite] are vertically opposite and if they are vertically opposite that means they are equal.

Afterwards, I asked the girls to write down their proof. Julie started writing on the paper: “All together they sum up to 180 because they are vertically opposite…” Jasmine interrupts her with a critical tone and asked her “What’s they?”. The girls then named all angles and continued their work. Jasmine realized that there was something unclear in what Julie was writing and immediately mentioned the need to be more specific. Her observation led to naming the angles and coming up with a more formal proof.
After they finished writing their proof in a sentence, I asked them to write it down in mathematics language. Jasmine wrote the mathematical relationships that accompany the girls’ initial answer. The girls worked in a very collaborative spirit and presented a very good proof based on what they both contributed.
2nd task

I started constructing the figure and meanwhile I was making sure that the girls understood the terms I was using (e.g., midpoint, perpendicular bisector etc.). After finishing the construction, I asked them to drag point D and tell me what was happening. Jasmine answered first by pointing out what stays the same and what changes. She mentioned that the segment $AB$ stays the same and later on added that what changes is “the length of the lines $[AD$ and $DB]$ and the way they are positioned”. She continued by adding that “no matter what happens the right angle always remain the same because we are not moving the line $AB$ or the line $CD$”. Julie jumped in and claimed that the angles $A$ and $B$ are equal but she had trouble in justifying it. Jasmine then argued that:

Jasmine: Oh they mirror each other so actually they are going to be the same!

When I asked them to provide a proof for the equality of $AD$ and $DB$, Jasmine tried to explain something by mentioning the movement of the lines. After I asked them to write something down, the girls drew an extra line on the paper and tried to proceed by looking for relationships between alternate, corresponding and co-interior angles. The fact that they interfered in the figure by drawing an extra line suggests progress in their proving abilities in geometrical tasks. Instead of looking at the figure, they chose to take action that could lead them to the solution. Besides drawing the parallel line to $AB$, they also moved on extending the segment $AB$ with the hope of getting some insight and more relationships to use.
The girls did not get as far as I had hoped but the task was really demanding, especially for an interview task. The statement Jasmine made concerning the mirror image is already a good argument for the equality of the sides $AD$ and $DB$. My expectation to proceed in comparing the elements of the two triangles was not met, but it could have been met if there had been more time and the task to solve was clearer for the students.
Cassie

1st task
I asked Cassie to share any conclusions she got from dragging the points in the applet. She started by explaining what was happening when a line was moving:

Cassie: Whenever you pull this line and this line changes, it will change, and these points would change so whatever this is, whatever this is, this must equal to 180 degrees, straight angle but then this would change as well 'cause it’s on the same line.

Cassie explained that when moving the line, the marked angle was changing and its supplementary angle must change as well so they both always summed up to 180 degrees. Then she saw that the sum of the three marked angles must be 180 degrees because they are vertically opposite to the interior angles of a triangle. However, she firstly wondered if the angles were indeed vertically opposite:

Cassie: […] But is this a vertically opposite angle [wondering if one of the marked angles is vertically opposite to the interior of the triangle]?

R: When do we have vertically opposite angles?

Cassie: Well… intersecting lines

R: Is that the case here?

Cassie: Yes. Which means if that one and that one and that one since it’s vertically opposite which means these, well, added together [to] equal 180 degrees ’cause it’s a triangle, ye.

Though she hesitated for a moment to recognize the vertically opposite angles, after my question she reminded herself when vertically opposite angles are formed and subsequently she phrased the desired conclusion.
2nd task

After I finished with the construction of the figure, Cassie was moving point $D$ and commented on what was changing and what stayed the same:

Cassie: Well the line stays perpendicular because they are the same [the 90 degree angles]. It changes the height. But these angles change so that…

R: What angles?

Cassie: All the angles. $A$, angle $A$ and angle $B$ and angle $D$ will change 'cause the higher it is [the higher point $D$ goes] the thinner this gets [the smaller angle $D$ gets] and the others increase to compensate for them and the lower it gets the wider the $D$ angle gets and the smaller the $A$ and $B$ angles get again to compensate for the increase in size. But the perpendicular bisector never changes, it stays at 90 degrees because only the top angle moves so … the base of the triangle does not change.

Cassie focused on the change of the angles as she was moving the figure. She was commenting on the relationship between the angles because they consist of the three angles of the triangle $ADB$. So she was explaining that as $D$ changes, then $A$ and $B$ should change accordingly so the angle sum of the triangle always stays 180 degrees. After my question asking for a conjecture concerning the elements of the figure, Cassie claimed that angles $A$ and $B$ are equal but she was not sure:

Cassie: Ye I think [laughing]. They [angles $A$ and $B$] are definitely looking to [be equal], I don’t know.

When I confirmed that the angles are indeed equal, she repeated it with confidence and after my request she attempted to reason for the equality. According to her, the triangle is isosceles and so $AD$ and $DB$ must be the same, and the same for angles $A$ and $B$.

R: Could you maybe explain it somehow? Why are they [angles $A$ and $B$] the same? Why they are equal?

Cassie: Because… the top angle, both angles extend the same so no matter if the top angle changes the two lines will always change with that and will always change the same and so they will always be equal.
Her reasoning was based on the dynamic figure. She did not attempt to answer using mathematical language and deductive arguments. However, she was not asked to in a clear way. Subsequently, I asked her to make a conjecture concerning the two right triangles $DAC$ and $DBC$ and without hesitation she answered that they are the same because “they are like mirror image of each other”. Like Jasmine and Julie, Cassie also recognized that the perpendicular line is a line of symmetry and the two triangles are congruent. I insisted on asking for further reasoning hoping to guide Cassie to take steps for constructing a proof. Apparently, Cassie did attempt to isolate the elements of the two triangles and somehow compare them:

Cassie: So they both share the same point $D$. 
so that “they fit into one another”. She returned to empirical reasoning and she seemed to believe that her argumentation was convincing enough.

Ehsan

1st task

Ehsan played with the applet and his first and immediate conclusion was that the sum of the marked angles must be 180 degrees since they are “the opposite angles of the ones inside”.

Figure 6.47 Ehsan’s worksheet on the 1st task

Very good
He then wrote the proof (Figure 6.47) and that caused him no difficulty. He firstly named all the angles and marked them on the figure and then proceeded to writing down the mathematical relationships.

2nd task
Ehsan was moving point $D$ on the perpendicular bisector and firstly pointed out that the two triangles on the figure would always be right triangles. He continued by saying that angles $A$ and $B$ would always be equal because the triangle is isosceles. After a while, he reached an impasse after attempting to argue for the equality of the angles $A$ and $B$, and the sides $AD$ and $BD$. He found himself reasoning in a circle, because for proving that $A=B$, he needed to prove that sides $AD=BD$. He then also realised that
Ehsan: Oh Miss! I think I can prove it. If I draw this, this I mean parallel to this one [drawing a line passing through $D$ and parallel to $AB$], I am just kind of not thinking ahead.

R: You don’t have to.

Ehsan: Then, then this and this… [looking on his paper and thinking]

He found himself in an impasse again and apologized: “I don’t think I can do this one. I am sorry I just don’t think I can prove it”. I then returned to our previous discussion and insisted on finding out what we already knew concerning the elements of the two triangles.

Ehsan: So the 90 is definitely equal and the line [meaning $CD$] is shared.

R: Ye and there is something more, let me show you how I started [the construction]. There was the segment $AB$, the first thing I did…

Ehsan: But the segment is cut so this dot [meaning point $C$] is equal… so this line $AB$?

I did the construction from the beginning and Ehsan was reminded that $AC=CB$. Forgetting the steps of the construction and therefore not being able to make clear what needed to be proved or not was something that I should expect. After showing the construction for the second time, Ehsan immediately shared a conclusion:

Ehsan: Oh so we can just oh ok so that’s two sides oh ye so that’s two sides and an angle we have already got and with two sides and an angle the triangles have to be equal!

R: You think?

Es: Yes ’cause if there is an angle and two sides the third one has to be, there is no other way for the third one [he forms a right triangle with his fingers to stress his point]. So that is the way we have to do it so it’s $AC$ equals $BC$ [writing on paper] and $CD$ is…

Ehsan took a step to generalization in comparing the two triangles.

He had done the same before, when working in an activity in the classroom (see section 6.2, 2nd part of the lesson, group work: compare triangles). Ehsan –clearly excited for his solution– wrote his proof on his worksheet:
$90 + D_1 + B = 180$
$90 + D_2 + A = 180$

$AC = BC$
$CD$ is common
$C_1 = C_2$

Thus the triangles are equal.

Figure 6.48 Ehsan’s worksheet on the 2nd task
Analysis Overview of the Interviews

The first task was not difficult for the students. All of them got to the desired conclusion that the sum of the highlighted angles must be 180 degrees since they are vertically opposite to the interior angles of a triangle. In the case of the girls’ group, Jasmine started by pointing out that there were no parallel lines so there were no conclusions to make about alternate, corresponding or co-interior angles. She continued by isolating parts of the figure and pointing out the relationships of pairs of angles until Julie jumped in and shared the conclusion I expected. Cassie hesitated for a while for the existence of vertically opposite angles, but after she got convinced, she immediately expressed the desired conclusion. Finally, Ehsan argued for the sum of the marked angles, as soon as he started dragging the figure. Thus, it was for all students a task that they could handle and in which they had no difficulty in proving it. With the exception of Jasmine, the students could see the figure as a whole. The fact that they made the conjecture concerning all three angles and did not limit themselves to isolated facts of the figure suggests a progress in their thinking skills and in their approach of manipulating a figure. The dynamic nature of the GeoGebra applets contributes to this suggestion, by the movement of the figure, that conclusions concerning the whole setting of the geometrical figure could be drawn.

When it comes to the second task, the conclusions to be drawn from its use in the interviews are different since overall it was a difficult task for the students to work on. However, they all worked on it and had ideas to try. Besides their difficulty to handle the task, all students could recognize the symmetry in the figure and use it for arguing for the comparison of the elements.

Jasmine and Julie made conjectures for the angles and sides in the figure and attempted to reason by referring to what was happening when moving the figure. After my request to proceed to a more mathematical style of reasoning, they augmented the figure by drawing an extra line on their worksheet such that they created a set of parallel lines and attempted to proceed by searching for pairs of alternate, corresponding and co-interior angles. Cassie, besides her attempt to support her conjectures on the movement of the lines and to share her intuitive reasoning, also took steps towards a proof when she compared the two triangles by using the angle sum of a triangle. Ehsan, struggled with the task as well. He did not give reasons referring to the movements when dragging point D and he had on his mind that he should present a mathematically acceptable explanation for the conjectures he made. At the
end, he managed to reach to an integrated answer and generalize his conclusion for the equality of the two triangles.

The second task was very challenging for the students. Though I constructed the figure in front of their eyes, the students did not seem to remember the steps and therefore retain the givens of the task in their mind. With respect to their kind of reasoning, which was influenced by the applet in the sense that the arguments they used would not have been given in the absence of a dynamic environment, it all began, except for Ehsan’s, as empirical. This does not necessarily mean that the students could not reason deductively, but that the applet itself suggested a way to support their conjectures. With this point, I want to stress the need to inform the students about the purpose of applets, which is to help them make a conjecture by seeing different examples of the same problem. When it comes to giving their reasoning for their conjectures, then stress the need for mathematical language and for the sequence of logical arguments.
7 Conclusions and Discussion

In this chapter, I summarize the findings of my research and discuss the conclusions through answering my three research questions:

**Research Question 1:** How do students engage in proving mathematical relationships on the subject of angles, triangles and quadrilaterals?

**Research Question 2:** How do students reason when asked to convince someone of the truth of a statement on the subject of angles, triangles and quadrilaterals?

**Research Question 3:** How can the designed teaching strategy be used to support students’ initial attempts to prove?

I firstly begin by answering the sub-questions and proceed then to answering the main question. Some student data are used to illustrate particular points. I follow how the class as a whole progressed using the phrase “the students”, and I summarize the findings that were discussed in Chapter 6.
7.1 The First Research Question

Sub-question 1.1: What changes in students’ approach(es) to proving can be identified across the lesson series?

The first lesson of the learning sequence was considered as a pre-test to inform about the students’ conception of what constitutes a convincing argument in mathematics. In this first lesson, none of the students presented a valid mathematical proof. To the contrary, the students’ answers were based on empirical arguments and intuitive feelings. The students verified the statement to be proven mainly by drawing a number of figures and supporting their reasoning on visual evidence. Jasmine (see also section 6.1) referred to the lines that were moving in the applet for convincing someone for the equality of vertically opposite angles:

Jasmine: No matter how you move it this line will always be 180 degrees and so the other line will always stay the same.

Despite my effort to prompt the group of Jasmine and Julie to use mathematical language, like proceeding in naming the angles and writing down mathematical relationships, they were not convinced. Jasmine’s approach was convincing to them as she made clearer with her comment “Why doesn’t she understand that? Why do we have to explain it to her? She is the math teacher here!”

In the second lesson, however, an obvious change in Jasmine’s and other students’ approaches in proving took place. Most students (4 groups) presented a formal mathematical proof by writing down mathematical relationships that followed a logical sequence. Although not all students wrote a mathematically acceptable proof, all of them used mathematical language by writing down mathematical relationships and not just empirical arguments. The observed change can be interpreted as a result of the discussion that took place in the classroom at the beginning of the second lesson. During that time, I explained to the students what is considered convincing for mathematicians, therefore what a proof is. In addition, students contributed to constructing the proof for the equality of vertically opposite angles that made clearer the requirements of a geometrical proving task. When working on proving statements for the rest of the lesson sequence, students seemed to generally have grasped the notion of a mathematical proof, and they were using mathematical language instead of intuitive or empirical arguments to get to the solution. Despite difficulties they encountered, most students were able to apply their knowledge of the discussed concepts (kinds of angles,
angle sum of a triangle etc.) while working on the activities and therefore showed progress in the use of mathematical language and their proving abilities. Apart from that, an indication of the development of students’ understanding of the notion of proof and their progress in proving geometrical statements was their action in adding something to a figure by drawing, for instance, extra lines that would help them to make conclusions for the task. Taking action instead of staring at a figure suggests progress in their thinking concerning working on a geometrical task. It suggests that students become active and have ideas about proceeding in a proving task, as in the case of Jasmine & Julie during the interview (see also section 6.5). While working on the second task, they decided to enrich the drawing with an extra line, an initiative that led them to a fruitful mathematical discussion and strengthened their attempts to attain a proof in the task.

In general, after analyzing the students’ work in the classroom and that during the interviews, I found out that compared to the first lesson, students made progress in their efforts to prove. This was shown in two ways: most of them came to realize what is expected of a mathematical proof, and most also developed independence in working on proving tasks.

**Sub-question 1.2: What difficulties do students encounter when working on geometrical proving?**

That the students would encounter difficulties was expected since they were attempting initial steps in proving geometrical statements. The following difficulties were detected in students’ work:

- **Circular argumentation**: Victoria & Jacque (see also section 6.1) found themselves thinking in a circle while working on the proof for the equality of vertically opposite angles. This led them nowhere though they were able to clarify the statement to be proven.
- **Assuming the hypothesis to be proven**: Sometimes students found themselves at an impasse by assuming at the start the statement which they wanted to prove.
- **Validating the hypothesis**: For proving that the angle sum of a triangle is 180 degrees, Haruki & Simon (see also section 6.3) described a cut & paste approach by cutting the angles of a triangle and pasting them next to each other, suggesting in this way that all angles added together form a straight angle. Testing the hypothesis was for them equivalent to proving the hypothesis.
• **Difficulties because of complex figure:** Evidence from students’ work on the “compare triangles” activity in the second lesson showed that the students had a hard time in making conclusions because of the complex figure that was given in the task. Jasmine and Julie (see also section 6.2), for instance, were struggling to prove the relationships between a pair of angles, but an angle in-between blocked their attempts:

> Jasmine: That doesn’t make sense! It’s not $\angle CAD$ and $\angle CBE$ ’cause there is another angle lying here that interrupts all this. That ruins the balance!

• **Difficulty because of not considering a random shape to represent a category of shapes:** Having had in mind a rectangle or a square instead of a random quadrilateral influenced some students’ attempts to prove that the angle sum of a quadrilateral is 360 degrees. Ines, Miriam, Hayley & Risako (see also section 6.4) failed to present a valid mathematical proof because of their limited conception of a quadrilateral.

**Research Question 1:** How do students engage in proving mathematical relationships on the subject of angles, triangles and quadrilaterals?

I expected that during the four lesson intervention the students would at least partially grasp the meaning of mathematical proof and would be able to engage in proving mathematical statements. In fact, most students, mainly from the second lesson and on, could make progress in proving geometrical statements. Students took initiatives in drawing figures and giving names to the elements in the figures, thus facilitated the writing of mathematical relationships. As the lessons progressed, my questions, comments and hints were needed much less often because students showed increasing independence in working in their groups and demonstrated greater trust in their own thinking while communicating mathematically. All this indicated they were making progress in working towards formal mathematical proofs.

In sum, concerning the quality of their proofs, students could proceed to a certain extent in proving tasks even though not all managed to present an integrated answer. There were students who showed originality in their proving solutions by taking a different route from the one suggested by the lesson, showing thus independence when engaging in proving. However, as expected, there were difficulties in their attempts to proving as evidenced by, for example, the equivalence of proving and verifying or validating the statement to be proven.
7.2 The Second Research Question

**Sub-question 2.1:** *What kinds of reasoning do students use to convince someone of the truth of a statement?*

In the first lesson a majority of students used empirical reasoning in order to convince of the truth of a statement. That was expected since because of their age, the students were not familiar with proving tasks and deductive argumentation. They reasoned either by referring to visual evidence (e.g., “the two triangles look the same”) or by using examples. In some cases, however, where the students used examples to reason for the truth of the statement, it was apparent that they were implying an infinite number of examples, something that suggested a more advanced type of empirical reasoning and a direction towards generalization. This type of reasoning could be attributed to the dynamic nature of the GeoGebra applets which enabled the generation of general conjectures.

Deductive reasoning also appeared in the first lesson, in the sense that, at least one deductive element was detected in the students’ attempts (2 groups) to convince someone that a particular statement holds. In the case of Haruki & Sean (see also section 6.1), for example, the students began from something known and moved towards the hypothesis to be proven. After the second lesson, when students had started to grasp the requirements of proving tasks (mainly the use of mathematical language) their reasoning was mostly deductive.

In sum, students mostly began by using empirical reasoning but ended using deductive reasoning.

**Sub-question 2.2:** *Does the use of the dynamic learning environment (GeoGebra) influence students’ reasoning?*

The use of the dynamic learning environment did influence students’ reasoning. Evidence for this was found, for example, when, after using the GeoGebra applets, some students expressed reasoning based on the movement of the figure, as in the case of Cassie when she was playing with the ‘perpendicular bisector’ applet (see also section 6.5):
Cassie: All the angles. A… angle A and angle B and angle D will change ’cause the higher it is [the higher point D goes] the thinner this gets [the smaller angle D gets] and the others increase to compensate for them, and the lower it gets the wider the D angle gets and the smaller the A and B angles get again to compensate for the increase in size. But the perpendicular bisector never changes, it stays at 90 degrees because only the top angle moves so … the base of the triangle does not change.

The dynamic figure offered Cassie the opportunity to develop her understanding of the angle sum of a triangle, and based on that fact she explained what was happening when dragging one of the vertices of the triangle. Her reasoning, though empirical, is actually more complex since it suggests a transition to more abstract ways of conjecturing. Furthermore, the interaction with the applets may have prompted students into a type of reasoning that, though based on examples, suggested the appearance of a generic example. An example of this comes from Julie & Isis (see also section 6.1) who drew three figures to convince for the equality of the vertically opposite angles and accompanied their work with the phrase:

No matter how many examples you use, they [the vertically opposite angles] will always be equal. [Julie & Isis, worksheet]

The girls made it clear with the phrase above that the examples they found were but representative of a set of an infinite number of examples. The generation of such reasoning is very likely to have been influenced by the dynamic figure (applet) in which different examples of intersecting lines could be made, and once made would suggest such a conjecture.

**Research Question 2:** How do students reason when asked to convince someone of the truth of a statement on the subject of angles, triangles and quadrilaterals?

For convincing someone of the truth of a statement, students expressed different kinds of reasoning. Students reasoned empirically by using examples and trusted that their intuition would be convincing argumentation. According to what they said, they indicated that when visualizing GeoGebra applets, students based their reasoning on the movement of the dynamic figures but extended their thinking by considering an infinite number of examples to be convincing, and so moved towards generalization and therefore towards proof. Evidence was found for deductive elements in students’ reasoning; some of these even in the first
lesson with more coming in the second lesson when students took the first steps in proving mathematically. Students were found to reason empirically when talking while they interacted with the applets, and when invited to write down their proof, students were definitely moving to a deductive approach by writing mathematical relationships and attempting to connect them logically.

7.3 The Third Research Question

Sub-question 3.1: How does the teaching strategy work in practice?

The teaching strategy is characterized using four principles: group work, classroom discussions, instructional scaffolding, and integration of a dynamic learning environment. The combination of all four was expected to support students’ first attempts to prove. In fact, the students in general benefited from working in groups since their dialogue generated demands for explanations and reflection on their shared thoughts. Similarly, classroom discussions invited students to express their thinking, be spontaneous in making conjectures and reflecting on what they said by listening to the contributions of their peers. Instructional scaffolding, especially my questions and comments to students’ remarks and thoughts, was aimed to encourage the students to construct their own knowledge by prompting them to unfold their thoughts and get to the desired results by using their own inputs. Through group work, classroom discussions, and my instructional scaffolding strategies, a socio-mathematical norm was made explicit to the classroom participants; this norm was the expectation that everyone would provide explanation for each utterance. In this way, the nature and requirements of the proving activities were made clear throughout the lesson sequence. Also, the integration of GeoGebra applets offered students additional tools for making conjectures, thereby influencing them in their attempts to reason.

Sub-question 3.2: Does instructional scaffolding elicit desired kinds of responses from students?

Instructional scaffolding consisted of questions and comments made in response to students’ sharing of ideas. In general, the students responded to my planned scaffolding as expected. After a request to do so, students attempted to explain their claims and even reworded statements so as to clarify their thoughts to me and their peers. My comments also invited students to reflect on their thoughts and therefore find for themselves an error in their thinking, as Ehsan did during the classroom discussion on the comparison of triangles (see
also section 6.2). After he said that the equality of the angles alone suggests that two triangles are congruent, he then reflected on his answer and realized there was something missing:

R: Ok, you think that if you just show me that the three angles are equal, is that enough?

Ehsan: Yes.

Jasmine: They have to be in the same position, the same spot.

Ehsan: Oh wait. That doesn’t prove that there is the same length of the sides [showing with his hands].

In general, instructional scaffolding was found to be effective in supporting the students’ first attempts at proving and also contributed to setting the socio-mathematical norms in the classroom. During the interview, Jasmine having realized the need for making everything explicit in order to communicate in mathematics “demanded” from her partner to be clear about what angles she meant when she wrote “they” on the paper. Instructional scaffolding did elicit desired and desirable responses from students. Even in cases when students were wrong, the strategies contributed both to making ideas public, therefore providing me with insight into students’ misunderstandings, and also to letting students reflect on their mistakes. Another rationale behind the instructional scaffolding was to encourage students to share their thoughts even when they were not very willing to do so:

R: Is that enough?

Cassie: Hmm

R: It may be, whatever you think

Cassie: Ye I think it would because if the angles are equal they are the same. ’Cause if they have the same angles then they are the same triangle basically.

Cassie was not sure if the equality of the three angles suggests the equality of two triangles. My encouragement to share her opinion, even though it was not quite correct, was intended to eliminate the possibility that students ‘feared’ sharing a wrong thought.
**Sub-question 3.3: What are the indications that students are starting to think for themselves?**

The teaching strategy was aimed to enable students to think and work independently, and in fact there was evidence that after awhile some students could work on some tasks with no external help. Ehsan managed to move on his own after being stuck while working on the task concerning the comparison of two triangles (see also section 6.2). Not only did he find his own way into solving the task, he also exhibited independence when he stated a generalization concerning the comparison of any two triangles. Jasmine and Julie also managed to prove that co-interior angles are supplementary after struggling for some time with the task and being confused with what was given and what was not (see also section 6.2). Without receiving any help from me or the cooperating teacher, they finally proved the statement with an apparent sentiment of joy and relief. My instructional scaffolding, especially, contributed to students’ development of independent thinking. When working on the first task of the interview, Cassie wondered for a moment if the angles she was looking at were vertically opposite. Avoiding answering her question directly, I encouraged her to think and get to the answer by herself:

Cassie:  [...] But is this a vertically opposite angle?

R:  When do we have vertically opposite angles?

Cassie:  Well… intersecting lines

R:  Is that the case here?

Cassie:  Yes. Which means if that one and that one and that one since it’s vertically opposite which means these, well, added together [to] equal 180 degrees ’cause it’s a triangle, ye.

The teaching strategy encouraged the students’ independent thinking since my role as a teacher was limited to providing scaffolding and therefore opening room and inviting the students to depend on their own thinking. In fact, as the lessons progressed, my interference in students’ work, including giving help, became more limited. Especially in the third and fourth lessons, the students received much less help from either me or the cooperating teacher while working on tasks.

In sum, to a large extent, my scaffolding strategies not only encouraged students to think for themselves but also encouraged confidence building. The former was evidenced if we look at the change in the quality of the questions asked by the students (for example, see lesson 1 and
lesson 3). The latter could be seen through students’ willingness to tackle some kinds of questions they had never seen before or at least to which they were unaccustomed.

**Research Question 3:** *How can the designed teaching strategy be used to support students’ initial attempts to prove?*

The teaching strategy was designed with the purpose of showing to students new to proving how they should formulate their first mathematical proofs and do this semi-independently of the teacher by working in small groups. In fact, the teaching strategy contributed as desired to students’ initial attempts to prove. The combination of group work, classroom discussions, and instructional scaffolding strategies, as well as the integration of a dynamic learning environment in the lesson sequence supported the students while they worked to prove statements. This was accomplished by encouraging them to make conjectures and to discuss them with their peers. My scaffolding strategies were found to be effective by requiring students continually to provide explanations for their conjectures and therefore by making apparent the socio-mathematical norms in the classroom. More generally, my responses to students’ questions was limited to a minimum, in the sense that I did not give direct answers, but rather I provided hints for them to get to the answer by themselves. As a result of the teaching strategy, the indications were that the students, while proving geometrical statements, became more independent thinkers and took initiative steps.

### 7.4 Discussion

This master study was a small case study research with the purpose to investigate young students’ initial attempts to prove geometrical statements. In order to do that, I designed a teaching strategy to encourage their proving abilities and their reasoning skills. Throughout the lesson series the students worked on proving tasks, and despite their unfamiliarity with proofs, they could, in general, engage in the proving process. Similar to the results found in other studies, for example Jones (2000), students moved from empirical “everyday” reasoning to formal mathematical expressions. Also, as Marrades and Gutiérrez (2000) found with their students when they learned geometry in a dynamic computer environment, the GeoGebra applets used in this study prompted students to make generalizations and abstract justifications. As for the instructional practices I used with the students, the findings supported Mercer’s (1995) suggestion that instructional scaffolding may as well help students to develop autonomy and independence in learning.
The use of mathematical language in the classroom is very crucial. Especially in cases when students encounter the notion of proof and proving activities, and they are encouraged and required to express their reasoning, it is the instructor’s role to make sure that the appropriate mathematical language is used by everyone in the classroom, and that it is understood by everyone. During the time of my research, there was evidence that some terms used sometimes caused difficulties in students’ communication. For instance, doubts about what angles are called corresponding or alternate created confusion to students while discussing within their groups. Another example has to do with the misunderstanding that a group of students had concerning the term ‘vertically opposite angles’ when they connected the term ‘vertically’ to ‘vertical’ though the former refers to the word *vertex*. The misunderstanding misled the students and influenced their approach to convince for the particular statement.

Especially in engagement with proofs, when students are actually invited to learn a new way of communicating, appropriate and clear language is extremely important. This is the reason why, during my lessons, I kept posing continuous questions in order to for the students to make clear what they meant by what they said, to which elements they referred when sharing a thought (e.g. , “this and this have to be equal”). It is crucial for the students to understand the need for clarification as a requirement for a healthy and fruitful conversation in a mathematics classroom. The decrease in my questions by the end of the lesson series shows that students had started to realize and respect the sociomathematical norm in our classroom.

The findings of this study suggested that the integration of a dynamic learning environment influenced students’ reasoning. My aim for including the GeoGebra applets in the lesson design was to facilitate students in making conjectures. The interaction with the applets was seen as an introduction to the proving tasks, and I stressed to students, in any opportunity, that visualization is not enough argumentation to convince for the truth of a statement.

While playing with the applets, students used empirical reasoning based on the movement of the figures. Though empirical, their reasoning can be characterized as one of an advanced level. This is because students went further than arguing that ‘this is true because you can see it on the screen’ or arguing by presenting a number of examples. The dynamic nature of the environment and the change of the figures in front of their eyes have driven them in adopting the view of an infinite number of examples. This consists of a step to generalization, thereby to proof. Such an advanced, superior type of empirical reasoning would have probably not occurred in the absence of the dynamic learning environment.
In this research, the dynamic learning environment was used for the generation of conjectures and the proof itself was considered as a separate task from the students. The moment they had a paper in front of them, the students realized that they were required to write down mathematical relationships. It would be of great importance to investigate the use of the dynamic learning environment in the proving process itself.

I would also like to add here more recommendations for future research that emerged from the implementation of the present study. Research on students’ talk within their groups while working on proving geometrical statements, would inform on the influence of collaboration on students’ engagement in proofs, and on their progress in their reasoning skills. Apart from that, a research study on the progress of proving skills in individuals, their way of thinking while proving could be examined in other studies, as here I examined the progress of the whole classroom. This could confirm the usefulness of my method for others also and, results from such studies could throw additional light on the findings reported here.

With this I would like to express my hope that this master study has slightly contributed to the research literature on students’ engagement in proving, their reasoning skills and the effectiveness of instructional strategies that aim to foster independent thinking. The small number of participants in this exploratory study does not allow me to generalize, but it is my belief that my research design and analytical framework could be used in future research so its use could be verified and then, in the long run, acquire general conclusions. My role as a teacher and researcher could be considered as a limitation because of my close involvement to the research. However, having both roles simultaneously allowed me to apply my research aims with punctuality, make choices for my strategies, and observe students with the eyes of a teacher receiving insight that perhaps I would have missed by being just an observer in the classroom.
References


Appendix A Worksheets
Write down three statements concerning the angles above!
Names:

WORKSHEET 2 / Convince me!

The GeoGebra applet suggested that vertically opposite angles are equal. First, discuss the question below with your partner. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that vertically opposite angles are equal to each other?

Explanation:
By the help of the GeoGebra applets, we have recognized the following:

When two parallel lines are cut by a transversal,

- the alternate angles are equal.
- the corresponding angles are equal
- the co-interior angles are supplementary.

Let us consider the general case with angles $\alpha$ and $\beta$.

Prove that the angles $\alpha$ and $\beta$ are supplementary by using alternate angles or corresponding angles!
Names:

**WORKSHEET / PROOF: Convince your partner!**

By the help of the GeoGebra applets, we have recognized the following:

When two parallel lines are cut by a transversal,

- the alternate angles are equal.
- the corresponding angles are equal
- the co-interior angles are supplementary.

Let us consider the general case with angles $\alpha$ and $\beta$.

Imagine that from the above list only the rule of alternate angles is known to you. Prove that $\alpha = \beta$. 
Let's prove that the triangle ADC is a copy of the triangle BCE!

Explanation:
The GeoGebra applet suggested that the angle sum of any triangle is 180°. First, discuss the question below with your group mates. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that the angle sum of any triangle is 180°?

**Explanation:**
Names:

WORKSHEET 2 / PROOF: Convince us!!

First, discuss the question below with your partner. Hereafter, write down the explanation of your group.

Discussion Question: We saw that the angle sum of any triangle is $180^\circ$.
Let's move on finding out the sum of any quadrilateral! Go!

Explanation:
The GeoGebra applet suggested that if \( A \), \( B \) and \( C \) are points on a circle where the line \( AC \) is a diameter of the circle, then the angle \( ABC \) is a right angle. That is known as the Thales' theorem attributed to the famous Greek mathematician. First, discuss the question below with your group mates. Hereafter, write down the explanation of your group.

**Discussion Question:** How would you convince someone that Thales' theorem always holds?

**Explanation:**
The quadrilateral below is a trapezium. Find the size of all the unknown angles!

Give your reasons!!

Explanation:
Appendix B Applets
Vertically Opposite Angles

Drag points F and G. What do you observe concerning the vertically opposite angles?
Alternate Angles

Drag points D and H. What do you observe concerning the alternate angles \( \alpha \) and \( \beta \)?
Corresponding Angles

Drag points D and H. What do you observe concerning the corresponding angles $\alpha$ and $\beta$?
Co-interior Angles

Drag points D and H. What do you observe concerning the co-interior angles $\alpha$ and $\beta$?

$\beta = 119.94^\circ$

$\alpha = 60.06^\circ$
Thales’ Theorem

Drag point B. What do you observe?
Intersecting Lines

Perpendicular Bisector
Appendix C Interview Transcripts
Interview 1: Jasmine – Julie

1st task

R: You can drag the points, the red points and you can tell me what is the conclusion you can make.

Jasmine: Of what?

R: Generally, the angles, the lines.

Jasmine: Oh ok. Well I don’t see any parallel lines. These two angles would be, because they are vertically opposite they would be equal.

Julie: And these two.

Jasmine: Ye those two as well. Those two, no matter what, no matter where you put the lines they will always be equal. This one and that angle will always equal 180 degrees so that, no matter where you move it. [Julie agrees, Jasmine goes on].

Jasmine: Well you can’t have any Z angles or alternate angles or co-interior angles cause there are no parallel lines. All together these four angles will make 360 degrees.

R: Why?

Jasmine: Cause it’s a revolutionary angle and no matter what they will… Is there anything specifically that we should pointing out?

R: Why do you think that the sizes are written? The sizes of the angles?

Jasmine: The sides?

R: The sizes

Jasmine: Oh right the size. So we can determine, well cause then knowing these ones we can find a lot more angles. Cause now we know what this angle, what that angle is, what that angle, what these angles. Well it gives you the base lines for a lot more angles. [Julie agrees]

R: Ok. Maybe you can make a conclusion about those three.

Julie: All together would be 180, right? Cause they are opposite from those and then that would be 180 cause it’s a triangle. [Jasmine agrees]

R: Ok. You can also verify that here right?

Jasmine: Yes because these two, all of them are vertically opposite and if they are vertically opposite that means they are equal.
Julie: Ye that’s what I just said [laughing]

Jasmine: I am just repeating it [laughing]

R: You are a good team.

Jasmine: I like my voice [and she sings]

R: Ok, you want to write something down?

Jasmine: Ok. Should we, do we write down what we just said?

Julie: Mmm

R: You said about the sum of these three angles, right? If you want to generally prove it, how could you write that down?

[They are writing]

Jasmine: Is it possible to say angle $r$ and $j$ for instance?

R: Whatever you want, you can say whatever you want. You can name them as you want

Jasmine: What’s they? [talking to Julie]

Julie: Mm?

Jasmine: What’s they?

[Julie wrote “they” on the paper and then specifies it by writing $(a, b, c)$]

Jasmine: And together they form a triangle and a triangle is equal to 180 degrees. We have to prove why a triangle is equal to 180 degrees?

R: No, I believe you can though huh?

[Julie keeps writing and Jasmine helps her, they are actually doing it together. So far they have written their proof in a sentence]

R: Could you write that down in mathematics? What you just wrote in a sentence?

Jasmine: Oh can I try?

Julie: Yes sure [So far Julie was writing]

[pause-Jasmine is writing]

Jasmine: I don’t know how to write vertically opposite in mathematical..

R: Let me see. This is why?
Jasmine: Cause it’s a triangle.
R: Ok and you want to show that one right?
Jasmine: Mmm
R: How can you show that one? You know that $f$, where is $f$? That one? Is vertically opposite to $b$
Jasmine: How do you get, they are similar vertically opposite, they are equal? Oh I get it! $f$ equals $b$
R: Because they are vertically opposite angles. What else? And the other angles as well right?

[And she continues writing]

2nd task

R: Let me show you something else. You both see huh? I have this segment AB and then I take point C it is the midpoint. What does this mean?
Jasmine: That’s the intersect, that’s…
R: What’s a midpoint?
Jasmine: $A$ plus $B$ equals no $C$ plus, what?
Julie: Emm
Jasmine: $C$ plus $B$ equals
R: $C$ is the midpoint of $AB$, of the segment $AB$.
Jasmine & Julie: Ye
R: So?
Jasmine: So it’s the intersection
R: Intersection?
Julie: Oh no intersection is a line, right? Like going through it.
Jasmine: Ye if you draw a line it would be the intersection but…
R: It’s a point on the segment $AB$, right? And if there is a line ok we have an intersection. But it’s a midpoint so that means that it is in the middle.
Jasmine & Julie: Mmm
R: It’s exactly in the middle. So $AC$ equals $CB$. 
R: Then I draw the perpendicular line…

Jasmine & Julie: Ye

R: that goes through C. What does this mean?

Jasmine: So A… this angle is equal to that angle cause they are vertically opposite and…

Julie: And they are both 90 ye…

Jasmine: …they are all right angles

Julie: …both 90 degrees

R: Perpendicular then…

Jasmine: So C, the angle C is equal to 360 degrees

R: …perpendicular means that the angle is 90 degrees, it’s not, it’s, it’s not like that [I show with my hands].

Jasmine & Julie: Ye

R: Then I take a point D on the perpendicular line

Jasmine: Mmm

Julie: Ye

R: And I draw the two segments AD…

Jasmine: What are you trying to do with this?

R: …and BD

Jasmine: they turn umm form a triangle.

R: You can now drag point D and tell me what is happening.

Julie: Oh

Jasmine: So line AB, ACB is huh remains the same.

Julie: No don’t move it

R: Ok, what does change?

Jasmine: It huh changes the length of the lines and the way they are positioned.

R: What lines?
Jasmine: The length of the lines
R: \(AD\) and \(DB\)
Jasmine: \(AD\) and \(DB\) and it changes the position they are in because huh by making it shorter point \(D\) goes lower
Julie: And these angles change.
Jasmine: But no matter what happens the right angle always remain the same because we are not moving the line \(AB\) or the line \(CD\)… It still equals 180 degrees
Julie: I know and these will also be the same.
Jasmine: I said that at the beginning.
Julie: Aha
R: What these?
Jasmine: \(ACB\)
Julie: The \(A, B\) will always be the same like they are both
R: \(AB\)? The segment?
Julie: No the angles will be, will always be the same.
Jasmine: No not necessarily.
Julie: They are both the same.
Jasmine: No see? This got smaller
Julie: Yeah but no but like they will always like…
Jasmine: Oh ye ye
Julie: …\(A, B\) always like be the same angle.
Jasmine: Why??
R: Why?
Julie: Because…
Jasmine: I shouldn’t have gotten us into this! [both laughing] Because…
R: It is true but why?
Julie: Well the lines are like equal length and
R: What lines?
Julie: Like $AD$…
Jasmine: Oh they mirror each other so actually they are going to be the same!
Julie: Ye
R: Mmm. You said something about the lines $AD$ and $DB$. What about those?
Julie: Well they are always the same length.
R: Ok so as you drag point $D$
Jasmine: Then the lines $AB$, $AD$ and $DB$ will be the same
Julie: Ye
R: Why?
Jasmine: Because when dragging the point...
R: Could you somehow prove it?
Jasmine: Yes because if you have got two lines and you have got not tie them together, and if you move them then both lines will be moved because they are not as holding both of them and the same length, the same length and there is not as holding them well then no matter where you move those two lines will always be the same because they are tied to it.
R: If you had to write that down how would you explain it?
Jasmine: I would try not!
Jasmine & Julie: Well mm
Jasmine: Is there some kind of angle to investigate?
Julie: There is only one line that goes…, we could use a parallel line
[They are drawing the parallel line]
R: So what is it that you want to show?
Jasmine: We have to try..
Julie: That angles $A$ and $B$ are always the same
R: Ok
Jasmine: So how come angles $A$ and $B$ are always the same?
[They are trying with alternate, corresponding, supplementary angles and name the angles on the worksheet]

Jasmine: So together plus this one they should equal 180 degrees because it’s a straight line. We know that this angle and this angle should be equal because they are alternate angles. And then that angle…should be equal to that angle. If you call this $g$, $e$, $f$, $x$. That’s a that’s b together they sum up … So we know that e equals x because they are Z angles, how do you call them? Corresponding?

Julie: No, alternate angles.

Jasmine: And goes that $g$ equals f… We also know that $g$ plus this angle plus $e$ should equal 180 degrees. But this…What’s this angle then? If you find out these two angles then you could pretty much find out what that angle is. …So these two…

Julie: Well this is 90 and

Jasmine: No so these two, if you add those two together then the remaining thing would, would be plus this angle would make it 180 degrees so this…wait I am confusing myself

Julie: Ok wait

R: Let her help you

Julie: So $e$ equals $x$ cause they are alternate angles. Then $g$ equals $f$. No wait $g$ equals $f$ and $e$ equals $x$

Jasmine: $x$ plus $b$ equals

Julie: Well we know that $b$ equals…should we give these names to? The…we don’t have $y$ and…

Jasmine: But what if we are looking in the wrong spot? Maybe it has something to do with…we know that this 90 degrees, cause we need it this is the only number we have.

Julie: Ye

Jasmine: So we know that these four together they should equal 360 degrees

Julie: Ye

Jasmine: Then this one, so this is also automatically 90 degrees I think, right? Cause it's a right angle

Julie: Ye
Jasmine: So if this is 90 degrees then 90 plus this plus should also equal to 180 degrees. So together this triangle should theoretically be 360

R: The triangle? How can that be?

Jasmine: Because there are two triangles.

R: Ok, so two triangles are 360 degrees

Jasmine: Ye but we could also look at it as 180 degrees cause it can also be this one triangle with a line between. But anyhow we know that this is 90 degrees, that’s 90 degrees.

Julie: And we know that… we know that b equals y plus huh whatever this was called

Jasmine: Is there an angle we could use…? Co-interior angles! These two are equal because they are co-interior no it’s not…

[The girls couldn’t proceed further]

**Interview 2: Cassie**

1st task

R: You can drag the points, the red points and perhaps you can get to some conclusions

[Cassie is playing with the applet]

Cassie: Whenever you pull this line and this line changes, it will change, and these points would change so whatever this is, whatever this is, this must equal to 180 degrees, straight angle but then this would change as well ’cause it’s on the same line.

R: Ok very nice.

Cassie: Same with this one. But is this a vertically opposite angle [wondering if one of the marked angles is vertically opposite to the interior of the triangle]?

R: When do we have vertically opposite angles?

Cassie: Well… intersecting lines

R: Is that the case here?

Cassie: Yes. Which means if that one and that one and that one since it’s vertically opposite which means these, well, added together [to] equal 180 degrees ’cause it’s a triangle, ye.
2nd task

R: Ok! Let me show you something else. I have this segment $AB$ and then I take point $C$ as the midpoint of the segment, you know what that means?

Cassie: Yes

R: What does it mean?

Cassie: huh…half-way point?

R: Mmm, it’s exactly in the middle. And then I draw the perpendicular line that…

Cassie: Bisects the

R: What does perpendicular mean?

Cassie: When it forms, when it meets another line at 90 degree angle.

R: Yes. So now that is the perpendicular bisector…

Cassie: Yes

R: …because it goes through the midpoint and that is also

Cassie: It forms

R: 90 degrees

Cassie: Yes

R: And then I take a point $D$ on the perpendicular bisector and finally I draw these two segments $AD$ and $DB$

Cassie: Yes

R: You may now drag the red point and tell me what is happening

[Cassie is playing with the applet]

Cassie: Well the line stays perpendicular because they are the same [the 90 degree angles]. It changes the height. But these angles change so that…

R: What angles?

Cassie: All the angles. $A$, angle $A$ and angle $B$ and angle $D$ will change ’cause the higher it is [the higher point $D$ goes] the thinner this gets [the smaller angle $D$ gets] and the others increase to compensate for them and the lower it gets the wider the $D$ angle gets and the smaller the $A$ and $B$ angles get again to compensate for the increase in size. But the perpendicular bisector never
changes, it stays at 90 degrees because only the top angle moves so … the base of the triangle does not change.

R: \(AB\) huh?

Cassie: Yes \(AB\).

R: Perhaps you can make a conjecture about the sides of the triangle? Or the angles that you mentioned? Is there a relation between some angles?

Cassie: Angle \(A\) and angle \(B\) are the same? They look the same when you move it…. \(A\) plus \(B\) plus \(D\) always equal 180 degrees cause it’s a triangle and if \(D\) changes then the two constantly stay, well they don’t stay the same but they are equal to each other.

R: Mmm

Cassie: Ye I think [laughing]. They [angles \(A\) and \(B\)] are definitely looking to [be equal], I don’t know.

R: They are equal.

Cassie: Yes \(A\) and \(B\) are equal [as soon as she heard it from me, it became certain and she repeats it] which means…

R: Could you maybe explain it somehow?

Cassie: …it’s an isos [she wants to say isosceles], is it?

R: Mmm

Cassie: an isosceles yes.

R: What does this also mean?

Cassie: That these lines \([AD\) and \(DB]\) are the same and the two angles here are always the same

R: Could you maybe explain it somehow? Why are they [angles \(A\) and \(B\)] the same? Why they are equal?

Cassie: Because… the top angle, both angles extend the same so no matter if the top angle changes the two lines will always change with that and will always change the same and so they will always be equal.

R: What about the angles?

Cassie: They will always be equal. \(A\) and \(B\) will always be equal.

[pause]
R: How many triangles do you see?

Cassie: You can see two because there is $DCB$ and $DCA$ because that bisects in the middle and.. they are right-angle triangles

R: What can you say about those two?

Cassie: They are equal

R: They are the same huh?

Cassie: They are like mirror image of each other

R: So how could you show that they are exactly the same? Could you show it on the paper? You can write down something if you wish

[She is thinking]

R: How would you show to someone that those two are the same? What would you say to convince someone? Whatever comes in your mind

Cassie: So they both share the same point $D$. 

R: The triangles?
Cassie: Yes
R: So if you wanted to show someone that this is the same as that one you would take the angles huh?
Cassie: Ye
R: And show that they are all the same huh?
Cassie: Ye
R: Is that enough?
Cassie: Hmm
R: It may be, whatever you think
Cassie: Ye I think it would because if the angles are equal they are the same. ‘Cause if they have the same angles then they are the same triangle basically
R: What about the sides?
Cassie: Hmm ye
R: They must be the same as well huh?
Cassie: Ye… we could look at the lengths, the angles and the lengths would have to be the same.
R: So let’s see if they are the same. Choose one to start
Cassie: Hmm this side [pause] You could, another one is you could turn it and show if it fits with the other one and then you could show, you know turn it, or flip over and then you could show that they fit into one another.
R: Could you do it with the paper?
Cassie: The paper? We could cut them out and fit them over each other or fold it and then it will be the same…
R: Yes exactly
Cassie: Ye cause it’s a mirror image of the other one, the same as folding
R: Ok
Cassie: Or you could measure and redraw it from the top but ye.
Interview 3: Ehsan

1st task

R: You may drag the points
Ehsan: Ok
R: and let me know if you have any conclusions

[He is playing with the applet]
Ehsan: The outer angles huh… the sum of all outer angles is also 180
R: What angles you said?
Ehsan: The outer angles
R: The ones…
Ehsan: Ye. Well basically because they are the opposite angles of the ones inside so.
R: Ok, could you write that down? Somehow prove it?
Ehsan: Ok, can I use the pen?
R: Yes, sure. So it always holds huh? No matter what the angles are.
Ehsan: Yes.

[He is writing on the worksheet]
Ehsan: Ok, I just have to name a, b, c, d, e, f. a plus b plus c is 180 [as he writes it down]
R: Why?
Ehsan: Eh [confused], so can we just take that for granted? Say that we already know that the inner angles of a triangle…
R: Ye ye I am just asking you again. Because it’s a triangle huh?
Ehsan: Ok

[He continues writing]
2nd task

R: Mmm very nice. Let me show you something else.

R: I have a segment ok? AB

Ehsan: Ok

R: And I take point C as the midpoint…

Ehsan: Yes

R: …of AB. Do you know what that means?

Ehsan: Yes.

R: Could you tell me?

Ehsan: It’s in the middle.

R: Ok. Then I draw the perpendicular line.

Ehsan: Mm

R: What does this mean?

Ehsan: It cuts it in half huh and it’s huh it comes… it comes directly down. Oh! It creates two 90 degrees angles

R: Ok very nice. Here, so it’s 90 degrees [I make it visible on the applet]

Ehsan: Yes

R: Then I take point D on this perpendicular bisector. Bisector is? Because it cuts

Ehsan: Yes it cuts yes, bisects

R: And then I also draw these two segments AD and DB

Ehsan: Mmm

R: You may now drag point D and see what happens

Ehsan: So…the point is that… this bisector is always obviously can…these triangles are always obviously going to be 90

R: Mmm What did you…

Ehsan: Ok ye these two will always be equal.

R: What two?

Ehsan: Ehh angle B and angle A
R: Ok, why do you say so?

Ehsan: Well… a short explanation would be that this is an isosceles triangle that means these two lines are equal plus these two angles also have to be equal but to prove it mathematically… ehh

R: Why is it an isosceles?

Ehsan: Isosceles because these two lines are equal but then I have to prove that too. Ok so I ‘ll just say it like this. Can I just? Is there a paper for that?

R: Here

Ehsan: Ok thanks

[He writes on the paper]

Ehsan: Do I have to prove that
Ehsan: Yes that this is the bisector so that these two must be equal. Would that be enough if I do that?

R: Yes if you could show how to do it I believe it would be enough. But I need to see exactly what you mean.

Ehsan: I am just going to try another way

R: Ye. Maybe something simple. Don’t think strictly about proving it. So you want to show that AD is equal to DB or that
R: Which one?

Ehsan: This line [showing CD]

R: Ok, what else?

Ehsan: Emm these

R: We always start from the thing that we know

Ehsan: Ok so these two angles 90’s and this line because they share it and these two because you know

R: Is there something more?

Ehsan: And another thing they share is…

R: It is also something else that you know for sure

Ehsan: Oh Miss! I think I can prove it. If I draw this, this I mean parallel to this one [drawing a line passing through D and parallel to AB], I am just kind of not thinking ahead

R: You don’t have to

Ehsan: Then, then this and this… [looking on the paper and thinking]

[pause]

Ehsan: I don’t think I can do this one. I am sorry I just don’t think I can prove it

R: Tell me what else is the same. It’s pretty difficult don’t worry. It’s tough. The good thing to do is to start with what is given, what we already know.

Ehsan: So the 90 is definitely equal and the line [meaning CD] is shared

R: Ye and there is something more, let me show you how I started [the construction]. There was the segment AB, the first thing I did…

Ehsan: But the segment is cut so this dot [meaning point C] is equal… so this line AB?

R: We are talking about two triangles right?

Ehsan: Yes

R: What about AC and CB?

Ehsan: Oh so we can just oh ok so that’s two sides oh ye so that’s two sides and an angle we have already got and with two sides and an angle the triangles have to be equal!
R: You think?

Ehsan: Yes 'cause if there is an angle and two sides the third one has to be, there is no other way for the third one [he forms a right triangle with his fingers to stress his point]. So that is the way we have to do it so it's AC equals [writing on paper] and CD is… Miss what…

R: Common.

Ehsan: Common ye I keep forgetting that word, common.

R: Or shared as you said.

[He continues writing on the paper]