Pricing of Bespoke CDOs

MSc. Thesis Stochastics and Financial Mathematics

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Abstract

In this research we investigate different bespoke CDO pricing methods and models. Four mapping methods rooted in the one factor Gaussian copula model and another bespoke CDO pricing model developed from an alternative Hull & White’s framework are of our great interest. For the four mapping methods, we analyzed them one by one to point out their advantages and drawbacks. For the Hull & White’s bespoke CDO pricing model, we first solve the calibration difficulty for the heterogeneous version of the model and then extended it based on the assumption that the two index jumps processes are independent from one another. We first test the two models with the no arbitrage conditions and find that the Hull & White’s model is by definition arbitrage free whereas the base correlation methodology under one factor Gaussian copula model is not. We next compare the performance of the Hull & White’s bespoke CDO pricing model with the moneyness matching method which we consider as a representative of the four mapping methods. The results show that the prices produced by moneyness matching method is close to the market consensus price but the price from Hull & White’s bespoke CDO pricing model deviates from the consensus price most prominently in equity and senior tranches.
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1 Introduction

1.1 Background

The credit derivatives market has experienced meteoric growth since 1998. The most popular instruments are credit default swaps (CDS). These provide a payoff when a particular single company defaults. However, in recent years portfolio credit derivatives have been attracting a lot of attention. These provide protection against the defaults experienced by a portfolio of companies. Statistics published by the Bank for International Settlements show that the outstanding notional principal for portfolio credit derivatives has grown from about 1.3 trillion in December 2004 (20% of the notional principal for all credit derivatives) to about $10.0 trillion in December 2006 (35% of the notional principal for all credit derivatives). The most popular portfolio credit derivative is a collateralized debt obligation (CDO). A CDO is a basket credit derivative that consists of multiple tranches, each having a different risk return profile.

Nowadays the industry has accepted the one factor Gaussian copula model [Li, 2000] to model the joint default behavior of the basket. One of the key assumptions of the one factor Gaussian copula model is that the asset values of all single name companies in a basket are correlated to a common factor with one correlation per single name. However with only a limited number of market quotes the model cannot be calibrated to the market with a unique correlation for each name as the number of single name companies is normally larger than the number of quotes. Therefore the market assumes that all companies have the same correlation with the common factor. This assumption can be tested by implying this correlation from market quotes of tranches on the same CDO basket, which is similar to the Black Scholes model where the volatility can be implied from the market. When true, this assumption would imply that we would get the same correlation for tranches on the same basket, however, practice learns that this is not the case as a correlation “smile”† is always observed.

The correlation smile brings challenges when a price is needed for a tranche that is not quoted in the market (bespoke tranche) as then it is unclear what correlation should be used. The reason for this is that the correlation is not only a function of the underlying basket but also of the tranche, which is inconsistent with the constant correlation assumption. The problem becomes even more complicated when the underlying portfolio consists of single name companies from more than one credit market or is of a non-standard maturity§. One natural proposal to cope with this is, based on the base correlation pricing methodology from one factor Gaussian copula model, to define certain measures of the riskiness between an index (equity) tranche and a bespoke (equity) tranche hence build up the equivalence between detachment points. Then pricing a bespoke tranche reduces to simply applying the base correlations assigned to the attachment and detachment points. In practice, various criteria of defining the riskiness of a tranche are attracting our interests, such as expected loss of the whole portfolio, probability of being wiped out, producing the same spread, etc.

Another recipe to overcome the difficulty of bespoke tranche pricing is to develop a model which can calibrate simultaneously to all tranches and the resulting characteristic parameters do not vary over tranches. These parameters are index-wise in the sense that they can be interpreted as the indicators of credit environment different for names that are from different index markets. When it comes to the pricing of a bespoke CDO tranche, we shall only link each single name company with its corresponding index-wise parameters and make use of a heterogeneous pricing routine.

1.2 Research targets

Based on the two options outlined above for pricing bespoke CDO tranches the targets of this research is to examine different base correlation mapping methods within the one factor Gaussian copula model and

†In other words, the implied correlation for the equity and senior tranche is higher than the implied correlation for mezzanine tranches.
§Non-standard maturities refer to those different from the index maturities which are of 3, 5, 7, and 10y.
implement one alternative dynamic model proposed earlier by Hull and White. Since in Hull & White’s paper, they only briefed the implementation procedure and calibration performance of the model, the first thing we need to do is to materialize the model and also to tackle the heterogeneous difficulty they did not solve.

After having both the one factor Gaussian copula model and the Hull & White model working, the next goal we aim to achieve is to set up comparisons among different bespoke CDO pricing methods. We will first check the theoretical advantages and drawbacks of different base correlation mapping methods within the one Gaussian copula framework. To this extent, we want to have insight into the question: is there an “optimal” mapping method out of many in the sense that it can be applied to a wide range of portfolios and fulfills most of the testing criteria?\footnote{Such as if the method is easy to implement, whether or not it takes into account the dispersive names within the portfolio, whether or not it generates continuous profit and loss in case of default.}

The second target of this research is to compare these four mapping methods\footnote{We choose one from them as the representative.} with an alternative bespoke CDO pricing model which is developed from a completely different framework than the one Gaussian copula model — the Hull & White’s bespoke CDO pricing model. Extended from the Hull & White’s framework without spoiling its integrity, this bespoke CDO pricing model incorporate two index jumps process based on the assumption that they are independent form one another. It also inherits Hull & White’s framework in that it has a clear financial interpretation. The comparisons between different bespoke CDO pricing methods/models are carried out with respect to disciplines such as no arbitrage conditions, spread curves, loss distributions, etc.

### 1.3 Organization

The rest of the thesis is organized as follows. Section 2 describes the structure of a CDO product and its valuation techniques. Following section 2, in section 3 and section 4 we elaborate on the settings of one Gaussian copula model and one alternative dynamic model proposed by Hull and White. We also detail the implementation procedure and show how the models can be calibrated to the market quotes. Next, in section 5 we focus on the pricing issue of the bespoke tranche. We examine in total four different base correlation mapping methods based on their own definitions of tranche’s riskiness and also extend the Hull & White model for use of bespoke CDO pricing. The results on the model calibrations and performance for bespoke pricing are presented in section 6. Finally we conclude our findings and observations in the last section.
2 Product outline and valuation

2.1 Collateralized Debt Obligations

A collateralized debt obligation (CDO) is a product that provides structured protections on a portfolio of debt instruments or credit derivatives like bonds or credit default swaps (CDS). A portfolio to be protected is sometimes called the reference pool and its underlying credit risky entities and related assets are often referred to as “names”, “companies” or “entities”. A reference pool defines the legal entities on which the risk is taken, and defines the asset, the size of the exposure, and how the loss is to be determined, should a credit event occurs. A reference pool can be a list of various types of underlying, for example, mortgages, bonds or CDSs, etc. Respectively, they are called a Mortgage Backed Security (MBS), Collateralized Bond Obligations (CBO) or synthetic CDO. In this research, we only consider synthetic CDO and in sequel by CDO we always mean a synthetic one.

The structure of a CDO is via “tranching” which define the magnitude of losses in case of the occurrence of credit events. A tranche on the reference pool with attachment and detachment points $[a\%, b\%]$ specifies losses to be protected between $a\%$ and $b\%$ on the total notional of the pool. For example, the $[3\%, 6\%]$ tranche on a EUR 1bn reference pool of 125 reference entities covers losses from EUR 30m up to EUR 60m related to that reference pool. The tranche technology provides access to customized risk by allocating the payout on the reference pool of assets to a collection of investors. Each investor will be exposed to losses at different levels and will therefore receive different levels of compensation for this risk. The cost of the tranche protection is paid as periodic coupons and measured in spread (basis points).

A CDO often consists of multiple tranches defined on a reference pool, see Figure 2.1. They are categorized as senior, mezzanine, and equity, according to their degree of credit risk. The first loss piece of a CDO is the $[0\%, b\%]$ tranche, also called equity tranche. It is the most risky position for the seller of protection as he must compensate for the first loss that occurs in the whole portfolio. The payment structure on the equity tranche is different from the other tranches. The seller of protection on the equity tranche receives part of the premium in advance. This part of premium, also called up-front premium, is calculated in percentage of the tranche notional such that a predefined spread (normally 500bps on the tranche notional, called running spread) is scheduled as the periodic coupon payments. A realization of such up-front premium payment therefore unloads the big risk exposure of the equity tranche. A mezzanine tranche starts in turn to absorb losses once the equity tranche is exhausted. Mezzanine tranches may also be called ‘junior’ (more risky) or ‘senior’ (less risky). The senior tranche is the least risky $[a\%, 100\%]$ tranche.

Pricing a CDO is therefore to determine the fair spread for each tranche (up-front percentage in case of equity tranche). By definition, each tranche involves two parties: a seller of protection on the tranche who receives the premium but must cover for the losses, and buyer of protection on the tranche who pays the premium but is compensated for the potential losses. The fair spread, or so called break-even spread is the one that equates the expected premium payment and expected loss of the tranche in future.

CDOs have quickly gained popularity during the last decade. It not only provides investors with opportunities of seeking for their desired risk exposure to a diversified portfolio, but also default protection, leveraged exposure, hedging tools, and relative value trading opportunities.

Default protection

Buying protection on an equity tranche provides protection against defaults, up to a certain limit. This limitation means that buying default protection via equity tranches may be less expensive than hedging against defaults using index portfolio.

Leverage

There are two types of leverages to the risk exposure an investor can take: leverage to risk exposure of portfolio losses and leverage to spread moves. To illustrate CDO’s leverage on exposure to the risk of
2.1 Collateralized Debt Obligations

portfolio losses, consider the seller of protection of the iTraxx index in comparison to the seller of the iTraxx [0, 3%] equity tranche. If there are no defaults, both sellers of protection will not bear any losses and will receive spread paid by buyers. However, in case of one credit event, the seller of the [0, 3%] tranche will lose 16% of their notional, while the seller of iTraxx protection will lose only 0.48%. Tranche exposure will also provide leverage to spread moves. Since this leverage refers to a tranches' sensitivity to underlying spreads, we also use the term “Delta” to refer to this type of leverage, i.e. the equity tranche always has the highest Delta.

![Figure 2.1 Tranching and the capital structure of a CDO](image)

**Relative value**
From a relative value point of view, tranches often provide higher spread for rating when compared with other credit investments.

**Hedging**
The synthetic tranche has become useful as a hedge against portfolio losses or spread moves in the underlying portfolio. From an outright trade perspective, investors with default risk against a portfolio of credits can now use tranches on bespoke portfolios to hedge against precisely the names in their portfolio. These hedges may be less expensive than using indices or options. And as tranches on the indices are more and more liquid, they have caught the attention of speculative traders, bank proprietary desks and hedge funds that may be interested in the risk on the other side of the hedge.

2.1.1 Index portfolios and standard CDO tranches

The motivation for developing credit derivative indices and standard CDO tranches which are simultaneously created on the indices is to meet the needs of professional institutions which require good view on markets in constructing credit solutions for their clients. They are currently the liquid data source for the market information.

The indices are defined as the reference portfolio comprising of the most liquid names in the CDS market selected from a range of industry sectors. All the names have equal weightings, i.e. they have the same notional amount to be protected. For example, the European largest CDS index, called the iTraxx Europe, consists of 125 European companies from six different industry sectors. The reference portfolio is redefined every six month, are of standard 5-, 7- and 10-year maturities, and are known as different ‘series’. A series is called ‘on-the-run’ during this six months and after that the series is replaced by a newly-issued series. The old series is then called ‘off-the-run’. Since the maturity of a CDO is longer then an on-the-run period, a CDO on an off-the-run series may contain some names that are not in the new series. There are two liquid indices, iTraxx — which we just introduced, and
CDX (CDX.IG and CDX.HY) which are North American portfolios. The standardized tranches have the following attachment and detachment points for both indices:

1. iTraxx portfolio: the standardized tranches are [0%, 3%], [3%, 6%], [6%, 9%], [9%, 12%], [12%, 22%]
2. CDX portfolio: the standardized tranches are [0%, 3%], [3%, 7%], [7%, 10%], [10%, 15%], [15%, 30%]

Table 2.1 iTraxx tranche quotes June 30, 2007

<table>
<thead>
<tr>
<th>Tranches</th>
<th>3-year</th>
<th>5-year</th>
<th>7-year</th>
<th>10-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% - 3%</td>
<td>8.75%</td>
<td>23.86%</td>
<td>34.75%</td>
<td>44.45%</td>
</tr>
<tr>
<td>3% - 6%</td>
<td>30.0</td>
<td>131.5</td>
<td>214.7</td>
<td>452.4</td>
</tr>
<tr>
<td>6% - 9%</td>
<td>15.0</td>
<td>61.8</td>
<td>111.7</td>
<td>210.5</td>
</tr>
<tr>
<td>9% - 12%</td>
<td>7.7</td>
<td>39.1</td>
<td>73.6</td>
<td>110.9</td>
</tr>
<tr>
<td>12% - 22%</td>
<td>5.4</td>
<td>24.8</td>
<td>37.9</td>
<td>59.9</td>
</tr>
</tbody>
</table>

Table 2.1 shows market quotes of tranches of maturities 3, 5, 7 and 10 years on June 30, 2007. Upfront premiums for the equity tranche are calculated as percentage and the running spreads for the rest tranches are calculated in basis points. 1bp = 1 × 10^{-4}.

Also there is a single tranche [0, 100%] CDO product, called index swap. In such a deal protection seller of an index swap has to cover all the losses between 0 and 100% but receives periodic premiums on the remaining notional only. Again, all names in the reference portfolio have equal weightings. Like the payment structure of the equity tranche, an up-front premium consisting of the difference between a predefined spread and the market spread. The standard maturities are 5-, 7-, and 10-year.

2.1.2 Bespoke CDOs

A tranche is ‘three dimensional’ — it depends on the underlying names of reference portfolio, attachment and detachment points and maturity. Tranches are termed as bespoke (tailor made) or non-standard ones if any of the following characteristics is met

1. it is defined on a non-standard portfolio, i.e. a portfolio with partially different composition or completely different compositions
2. it has non-standard attachment and detachment points, i.e. different from those on an iTraxx or CDX portfolio
3. it has a non-standard maturity, i.e. different from 5-, 7-, or 10-year maturity

Pricing bespoke CDOs is generally difficult. Because the market only provides information for those index portfolios and standard tranche structures and we have to cope with the non-standardization that arises from the three dimensions. The challenge is therefore to find a way to properly use the information from the standard products. Take the simplest bespoke CDO for example: suppose we want to calculate the up-front premium of an equity tranche on the iTraxx with detachment point 2%. We only have information at 3%, hence the common practice is to employ some interpolation or extrapolation techniques on the correlation or other intermediate quantities (e.g. expected loss). But this could sometimes end up with unrealistic prices (e.g. more senior tranche with higher premium) which exhibit arbitrage opportunities.

2.2 Valuation of Credit Derivatives

2.2.1 Introduction

The underlying contracts of a synthetic CDO are credit default swaps (CDSs). As the initial inputs to the CDO pricing model, the survival probabilities of these defaultable names are implied by their
corresponding CDS spreads. One common way of extracting these survival probabilities is via a so-called bootstrap process. Given a typical assumption of the function form of the survival probability, consequently an interpolation scheme, a bootstrap process is basically to solve a system of CDS valuation equations. In this research we assume the survival probability takes the reduced form and the intensity is a piece-wise function. For more details about the single name reduced form model, we refer the readers to Section 4.2. In the rest of this subsection we start by showing how a CDS contract can be valued and then a CDO tranche contract.

2.2.2 Valuation of a CDS

A credit default swap (CDS) is an agreement in which one party buys protection against losses occurring due to a credit event of a reference entity up to the maturity date of the swap. The protection buyer pays a periodic fee for this protection up to the maturity date, unless a credit event triggers the contingent payment. If such trigger happens, the buyer of protection only needs to pay the accrued fee up to the day of the credit event (standard credit default swap), and deliver an obligation of the reference credit in exchange for par minus market value of the reference obligation in case of cash settlement or the face value of the reference entity in case of physical settlement. In this research, let’s always assume a cash settlement with a constant recovery rate.

Define the present value (PV) of the cash flows from the buyer to the seller as the fixed leg and the present value of the cash flows from the seller to the buyer as the default leg. Next we shall develop formulae for both two legs. First consider the fixed leg. Suppose that the premia are paid continuously and furthermore the evolution of interest rate and default probability are independent. If the premium per annum is denoted by \( p = s \cdot N \) (\( N \) is the notional amount and \( s \) is annum basis points), then the premium due in the interval \( t \) to \( t + \Delta t \) is \( p \times \Delta t \), has a present value of \( p \times \Delta t \times D(t) \) where \( D(t) \) is the discount factor. This premium will be paid only when the name survives up to that time, so the probability weighted value is \( p \times \Delta t \times D(t) \times S(t) \) where \( S(t) \) denotes survival probability up to \( t \) (so the default probability is \( 1 - S(t) \)). Therefore, by the conventional defining argument of an integral, the following holds

\[
PV(\text{fixed leg}) = p \int_0^T D(t) \cdot S(t) dt.
\]

In real world practice, instead of a continuous payment stream, the premia are rather paid quarterly in arrears with a proportion to the credit event date. In this case, a discrete variation of the above formula holds in stead

\[
PV(\text{fixed leg}) = p \left[ \sum_{i=1}^n \frac{t_i - t_{i-1}}{360} \cdot D(t_i) \cdot S(t_i) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{t - t_{i-1}}{360} \cdot D(t) \cdot d(1 - S(t)) \right],
\]

where \( t_i \)'s denote a sequence of date grids and 360 is a conventional number of days in a year. The second term on the right hand side is the accrual payment paid if the default occurs in the interval \( t_i \) to \( t_i + \Delta t \). In this case, the amount of premium is \( \Delta t \times p/360 \) and the probability of default happening in this interval is \( S(t_i) - S(t_i + \Delta t) \). Next consider default leg. We assume a known constant recovery rate denoted by \( R \) and physical settlement. The probability that the reference entity defaults in the interval \( t \) to \( t + \Delta t \) is given by \( S(t) - S(t + \Delta t) \). Hence the probability weighted payoff is \( (1 - R)N \cdot D(t) \cdot [S(t) - S(t + \Delta t)] \) and again by standard defining argument of an integral (in Stieltjes sense),

\[
PV(\text{default leg}) = (1 - R) \cdot N \cdot \int_0^T D(t) d(1 - S(t))
\]

The spread of a newly issued CDS is the spread that makes the present values of both legs equal. The fair value spread (so-called break-even spread) is given by

\[
s = \frac{(1 - R) \cdot \int_0^T D(t) d(1 - S(t))}{\sum_{i=1}^n \frac{t_i - t_{i-1}}{360} \cdot D(t_i) \cdot S(t_i) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{t - t_{i-1}}{360} \cdot D(t) \cdot S(t) \cdot d(1 - S(t))}. 
\]
When we assume that defaults can only happen on quarterly payment dates, there is no accrual payments and we can numerically approximate the integral on the numerator by
\[ \int_{0}^{T} D(t) d(1 - S(t)) = \sum_{i=1}^{n} D(t_i)[S(t_{i-1}) - S(t_i)]. \]

We obtain the following formula for the break-even spread
\[ s = \frac{(1 - R) \cdot \sum_{i=1}^{N} D(t_i)[S(t_{i-1}) - S(t_i)]}{\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{360} \cdot D(t_i) \cdot S(t_i)}. \]

Omission of intermediate default times and therefore the accrual generally has little impact. However for high spread names the modeling of accrual is important. Since the focus of this thesis is on the modeling of dependence of CDO portfolio and not the valuation of a CDS, we will just use the above simplified formulation for valuing a CDS.

### 2.2.3 Valuation of a CDO tranche

The key inputs for valuing a CDO are the loss distributions\(^1\) of the whole portfolio at series of date grids (e.g. coupon payment dates). A portfolio loss distribution function is the probability mass function (abbreviated pmf) that assigns probabilities to a set of possible (discrete) losses. Since we assume a common constant recovery rate across all names, having a set of discrete losses hence a pmf is always the case. Given a series of portfolio loss distributions, denoted by \( f(l; t) \) at each coupon date, we can calculate the expected losses on one particular tranche — the probability weighted average of the payoff function of the tranche. Suppose the lower and upper levels of the tranche be denoted by \( A = a_L \cdot N \) and \( B = a_H \cdot N \) respectively, where \( a_H \) and \( a_L \) are the attachment and detachment points and \( N \) is the portfolio notional. Therefore, the tranche notional is \( B - A \). Let \( m \) denote the total number of the underlying names. The ‘payoff function’ (tranche loss as a function of default unit) for this tranche is given by
\[
W(l) = \min \{ B - A, \max \{0, l - A\} \}.
\]

Tranches of a CDO can be thought of as options on portfolio losses, more specifically, a call spread with strikes at each attachment point. Figure 2.2 depicts payoff functions for various tranches.

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\(^1\)In this thesis, by loss distribution we actually mean a loss probability mass function (abbreviated pmf). A pmf can be seen as the probability density function for discrete random variable(s) with respect to the counting measure.
Having the payoff function, the expected loss of the tranche at time $t$ is given by

$$E[L_{A,B}(t)] = \int_0^{l_{\text{max}}} W(l) \cdot f(l; t) \, dl,$$

where $l_{\text{max}}$ is the maximum loss the portfolio can suffer. Again, this integral can be approximated using the knowledge that we only have a fixed number of discrete losses. In this research we always assume that the loss that may arise from each individual is the same and denoted as $\delta l$. It is nowadays the market common practice and the direct consequence of an equal notional and constant recovery rate across names. The above formula then becomes

$$E[L_{A,B}(t)] = \sum_{l=0}^{n \delta l} W(l) \cdot f(l; t) \cdot \delta l.$$

Once we have the expected tranche losses at coupon dates, together with the assumption that the evolution of interest rate and default probability are independent, the rest of the calculation shares a great similarity with the valuation of CDS which we just introduced. In fact, the present value of default leg is given by

$$\text{PV(}\text{default leg}) = \sum_{i=1}^{n} D(t_i) \cdot (E[L_{A,B}(t_i)] - E[L_{A,B}(t_{i-1})]).$$

The present value of the fixed leg is given by

$$\text{PV(}\text{fixed leg}) = s_{A,B} \cdot \sum_{i=1}^{n} D(t_i) \cdot \frac{t_i - t_{i-1}}{360} \cdot (B - A - E[L_{A,B}(t_i)]).$$

We finally arrive at the expression of the fair spread that equates the above two legs:

$$s_{A,B} = \frac{\sum_{i=1}^{n} D(t_i) \cdot (E[L_{A,B}(t_i)] - E[L_{A,B}(t_{i-1})])}{\sum_{i=1}^{n} D(t_i) \cdot \frac{t_i - t_{i-1}}{360} \cdot (B - A - E[L_{A,B}(t_i)])}.$$

The present value of a tranche $[A, B]$ $\text{PV}_{[A,B]}(s_0)$ for a given spread $s_0$ is given by

$$\text{PV}_{[A,B]}(s_0) = \sum_{i=1}^{n} D(t_i) \cdot (E[L_{A,B}(t_i)] - E[L_{A,B}(t_{i-1})]) - s_0 \cdot \sum_{i=1}^{n} D(t_i) \cdot \frac{t_i - t_{i-1}}{360} \cdot (B - A - E[L_{A,B}(t_i)]).$$
3 One factor Gaussian copula model

3.1 Portfolio loss distribution

As already addressed, the key input for valuing CDO tranches is to generate the portfolio loss distribution. There are basically two approaches to achieve this. One is to model the evolution of the portfolio loss distribution as a certain Markov chain/process (see e.g. Schönbucher [2005]) or so-called “top-down” model, whereas the opposite “bottom-up” is to model the marginal probability of each individual and combine them into a joint distribution. In this research, our interests are in those bottom-up type models. The reason for this is that although compared to bottom-up models, top-down models are theoretically sound and able to capture the default characteristics at a macro level, but the behavior of individual companies in the portfolio is not considered. Therefore the top-down models are useless to calculate any individual effect. For example, if we want to analyze the impact of one tranche price in response to the change of one CDS’s spread, a bottom-up is the only choice.

Consider first a homogeneous portfolio, i.e. a portfolio of names that have the same CDS spreads — which implies they all have the same probability of default at each coupon date. If a further assumption that all names’ occurrences of default are neutrally independent holds, then we know that the portfolio loss distribution is actually a binomial one. Indeed, denoting the common loss of each name $\delta l$, the total number of the names in the portfolio $m$ and the same probability of default of each name $p(t)$, we have

$$P(L = n \cdot \delta l) = \binom{m}{n} p(t)^n (1 - p(t))^{m-n}. $$

A slightly generalized case is that instead we have a heterogeneous portfolio, i.e. a portfolio of names that have different CDS spreads. Then there is not such a neat formulation as binomial distribution. A heuristic expression looks like the following

$$P[L(t) = n \cdot \delta l] = \sum_{x_1 + \cdots + x_m = n} \prod_{i=1}^{m} p_i(t)^{x_i} (1 - p_i(t))^{1-x_i},$$

where

$$x_i = \begin{cases} 0 & \text{if that name defaults} \\ 1 & \text{if that name does not default} \end{cases} \quad i = 1, 2, \ldots, m.$$

The heterogeneity makes it difficult to reduce the above formula. Constructing such a such a heterogeneous loss distribution may require a great deal of computational time, therefore some numerical recipes may be employed to do that. In this research, we apply the recursive algorithm by Andersen, Sidenius, and Basu [2003]. Andersen et al. mentioned in their research that this inventive method is computationally more efficient than the traditional approaches such as the Fourier Transformation approach. For more insight into the Fourier Transformation approach, we refer the readers to Debuysscher and Szego [2003].

To explain the recursive algorithm, assume again a common loss unit (equal weightings and recovery rates across all names) which we previously have denoted as $\delta l$. Then the possible losses are $l = \{ 0, \delta l, 2\delta l, \ldots, m\delta l \}$. The recursive algorithm always starts with an empty portfolio and adds one company into the portfolio at each step. At the end of this process, the number of the portfolio is equal to the total number of the portfolio and the loss density function is fully generated. The recursive relation for the value of the density function between consecutive two steps is given by

$$P^{k+1}[L(t) = n \cdot \delta l] = P^k[L(t) = n \cdot \delta l] \cdot (1 - p_{k+1}(t)) + p_{k+1}(t) \cdot P^k[L(t) = (n - 1) \cdot \delta l],$$

where $k$ is the number of the companies in the portfolio (i.e. the $k$th step for the recursive algorithm). The initial setting for the algorithm is intuitive,

$$P^0(L(t) = 0) = 1$$

$$P^0(L(t) = n \cdot \delta l) = 0 \quad \text{for} \quad n = 1, 2, \ldots, m$$
For example, in the one factor Gaussian Copula model, \( \Sigma \) is characterized as

\[
\Sigma = \begin{pmatrix}
1 & \rho_1 \rho_2 & \cdots & \rho_1 \rho_m \\
\rho_2 \rho_1 & 1 & \cdots & \rho_2 \rho_m \\
\cdots & \cdots & \cdots & \cdots \\
\rho_m \rho_1 & \rho_m \rho_2 & \cdots & 1
\end{pmatrix},
\]

where \( \rho_i \) is standard normal distribution function and \( \Phi \) is a multi-normal distribution with some predefined correlation matrix \( \Sigma \). For example, in the one factor Gaussian Copula model, \( \Sigma \) is characterized as

<table>
<thead>
<tr>
<th>No. of defaults</th>
<th>Losses</th>
<th>No. of names added in portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1 ( 1 - p_1 ) ( 1 - p_2 )</td>
</tr>
<tr>
<td>1</td>
<td>( \delta \ell )</td>
<td>( p_1 ) ( 1 - p_1 ) ( p_2 ) ( 1 - p_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2\delta \ell )</td>
<td>0 ( 0 ) ( p_1 ) ( p_2 )</td>
</tr>
</tbody>
</table>

One appealing feature about the Andersen recursive algorithm is that it is indifferent to the order of adding each individual default probability, i.e., processes with different orders of adding default probabilities all result in the same portfolio loss distribution. This is very useful in practice when calculating delta, the rate of change of one tranche’s price with respect to changes in one underlying CDS spread, because it is easy to unwind the last step of adding that particular underlying default probability without undergoing the whole procedure all over again.

### 3.2 Model setup

Nowadays the market-standard model is the one factor Gaussian copula model which is one of the bottom-up type. Its name indicates that it is the Gaussian copula we employ to formulate the multi-name credit derivative model. The one factor Gaussian copula model is the most widely used in the market due to its flexibility and tractability. The general form is as follows

\[
\begin{align*}
C(u_1, u_2, \ldots, u_m) &= \Phi(\Phi^{-1}_1(u_1), \Phi^{-1}_2(u_2), \ldots, \Phi^{-1}_m(u_m)); \\
\end{align*}
\]

where \( \Phi_i \) is standard normal distribution function and \( \Phi \) is a multi-normal distribution with some predefined correlation matrix \( \Sigma \). For example, in the one factor Gaussian Copula model, \( \Sigma \) is characterized as

\[
\Sigma = \begin{pmatrix}
1 & \rho_1 \rho_2 & \cdots & \rho_1 \rho_m \\
\rho_2 \rho_1 & 1 & \cdots & \rho_2 \rho_m \\
\cdots & \cdots & \cdots & \cdots \\
\rho_m \rho_1 & \rho_m \rho_2 & \cdots & 1
\end{pmatrix},
\]
where \( \rho_1, \rho_2, \ldots, \rho_m \) can be selected to reflect the correlation structure of the market.

Intuitively, there is also a parallel “structural” interpretation for the one factor Gaussian Copula setting. It is first assumed that a default happens when an asset’s value of a defaultable entity drops below a critical level. Let \( V_i \) denote the value of the asset and \( v_i \) the critical level, and \( \tau_i \) the default time, then it holds that

\[
\tau_i < t \iff V_i < v_i \quad \text{with} \quad P(\tau_i < t) = P(V_i < v_i).
\]

Furthermore it is assumed that the value of the asset follows the standard normal distribution. Given the marginal probability of default \( p(t) \) implied by the market CDS spreads in combination with this distribution assumption of the asset’s value, the mapping between \( \tau_i \) and \( v_i \) is given by

\[
v_i = \Phi^{-1}(P(\tau_i < t)) = \Phi^{-1}(p(t)).
\]

Now the dependence between values of assets can be introduced. We assume

\[
V_i = \rho_i \bar{V} + \sqrt{1 - \rho_i^2} \epsilon_i,
\]

where \( \bar{V} \) is a common factor equal for all defaultable entities and \( \epsilon_i \) is individual factor assumed again to be standard normally distributed. Assume further that \( \epsilon_i \) and \( \bar{V} \) are all mutually independent random variables. Clearly, this is consistent with the correlation matrix we imposed earlier in the copula function, as \( \text{Cov}(V_i, V_j) = \rho_i \cdot \rho_j \). Our objective here is to calculate the joint default probability distribution at time \( t \),

\[
F(t, t, \ldots, t) = P(\tau_1 < t, \tau_2 < t, \ldots, \tau_m < t).
\]

We are generally interested in the probability distribution of the total number of defaults \( N \) at time \( t \), i.e., for \( n = 0, 1, \ldots, m \) (\( m \) is the number of all underlying names), we want to know \( P(N = n) \). An easy way to calculate this probability is to consider first the conditional probability of defaults given the value of the common factor \( \bar{V} \), we know that

\[
p^{\bar{V}}(t) = P(\tau_i < t|\bar{V}) = P(V_i < v_i|\bar{V}) = P(\rho_i \bar{V} + \sqrt{1 - \rho_i^2} \epsilon_i < \Phi^{-1}(p(t))|\bar{V})
\]

\[
= P\left( \epsilon_i < \frac{\Phi^{-1}(p(t)) - \rho_i \bar{V}}{\sqrt{1 - \rho_i^2}} \right) = \Phi\left( \frac{\Phi^{-1}(p(t)) - \rho_i \bar{V}}{\sqrt{1 - \rho_i^2}} \right).
\]

Given the value of the common factor, defaults events are independent. That is

\[
P[N = n|\bar{V}] = \sum_{x_1 + \cdots + x_n = n} \prod_{i=1}^{m} p^{\bar{V}}(1 - p^{\bar{V}})^{1 - x_i}.
\]

The above is a situation we have already encountered with — this conditional loss distribution can be constructed using the Andersen recursive algorithm which we introduced in previous subsection. Having obtained the conditional loss density function, the unconditional is obviously just the integral of the above over common factor. Finally we arrive at

\[
P(N = n \cdot \delta l) = \int_{-\infty}^{\infty} P(N = n \cdot \delta l|\bar{V}) f(\bar{V}) d\bar{V},
\]

where \( f(\bar{V}) \) is the density function of \( \bar{V} \).

The one factor Gaussian copula interpretation and the structural interpretation are equivalent. An illustrative proof in a simple bivariate Gaussian copula case is provided in the Appendix A.
3.2 Model setup

3.2.1 Role of correlation

As can be captured by the correlation matrix defined in the previous subsection, a typical reference portfolio generally has hundreds of correlation parameters — that is the most generalized one factor Gaussian copula model. For example, a 100-name portfolio involves 100 parameters (they are $\rho_i$'s) in its correlation matrix. Commonly simplifying assumptions are made to reduce the complexity of the model as well as the computations. For example, we could instead of specifying 100 correlation parameters use sector-based correlation parameters, that is to assume companies within the same industry sector share the same $\rho$. Taking this idea to the extreme, one can even eliminate the difference of $\rho$'s between sectors and make them all equal — that is the simplest version of the one factor Gaussian copula model in which the correlation between asset values, i.e. default times, is assumed to be the same for all single name companies. That is $\rho_i = \rho_j = \rho$.

Since a tranche specifies the amount of losses in the underlying portfolio, the correlation of the behavior of the reference entities in the event of portfolio losses is central to the value of a tranche. There is an intuitive explanation of the relation between the value of a tranche and the correlation, $\rho$ in case of one factor Gaussian model (the simplest version). First let’s look at Figure 3.1 which illustrates the behaviors of portfolio loss distribution with respect to different $\rho$'s. With low correlation level, e.g. $\rho = 0.2$, the portfolio loss distribution shows that portfolio losses are likely to remain confined and centralized below the more senior tranches. But, as correlation increases, the probability mass tends to move towards to both two tail areas. The extreme case, correlation of 1 across the portfolio, would result in one default within the portfolio causing the entire portfolio to default. In this case, the spreads across the tranches would all be equal.

Now we examine the relation between a tranche’s value and correlation. We first point out what the impact of correlation change on the expected loss is. In case of an equity tranche, the expected loss moves monotonically in reverse with the correlation. The trick is to look at both payoff functions of different tranches and the changes in the shape of the portfolio loss distribution which are due to different correlation inputs. The value of the expected loss of a tranche determined by its own payoff function together with corresponding probability weightings. The equity tranche is hit by the first losses in the portfolio and after a few defaults it is roughly wiped out. When the correlation level is low, the scenario of certain number of companies defaulting is most likely to happen as the probability mass is very much centralized round this number. Hence, probability weightings on the payoff function of the first a few defaults are relatively small, which is the reason for higher expected loss. However, when the correlation is high, likelihood of some extreme scenarios like all names defaulting or none of the names defaulting will increase — because companies more intend to move in the same direction. In this case, the probability weightings on the payoff function of the first a few defaults increase. This results in a drop in the expected value of the equity tranche.

For senior tranche the reverse holds. It is not or only partially hit when a few companies default, but are hit when a large range of companies default. The payoff function of a senior tranche is the very opposite to that of an equity tranche. As correlation increases, probability mass at right tail of the distribution increases, which explains the rise in the value of the expected loss senior tranche.

The last types of tranches are mezzanine tranches. Mezzanine tranches are between the equity and senior tranche and therefore inherit the sensitivity of spread to correlation of both these tranches. However, as the sensitivity of tranche spreads to correlation are opposite for an equity and senior tranche there is no monotonic relation between the correlation and the expected loss of a mezzanine tranche.

Next we point out that the tranche spread is a monotonic increasing function of expected loss. To see this, let’s once again examine the expression for the break-even spread of a tranche $[A, B]$. It reads

$$s_{A,B} = \frac{\sum_{i=1}^{n} D(t_i) \cdot (E[L_{A,B}(t_i)] - E[L_{A,B}(t_{i-1})])}{\sum_{i=1}^{n} D(t_i) \cdot \frac{t_{i-1} - t_{i-2}}{460} \cdot (B - A - E[L_{A,B}(t_i)])}.$$  

Apparently the expected loss plays a role both in the denominator and numerator. When the expected loss increases, the sum in the denominator decreases. Hence this leads to the increase of the overall
quantity and vice versa. However, the impact of expected loss in the numerator is not straightforward to capture as it is expressed as the sum of its differences between coupon dates. Instead, we approximate the numerator by assuming that all the discount factors \(D(t_i)\)'s are equal to 1. This is a rather realistic simplification as the impact of using different discount factors on the calculation of the fair spread is negligible. In this case, all the “time value” of the money is zero. Now we have a new numerator given by

\[
\sum_{i=1}^{N} \left( \mathbb{E}[L_{A,B}(t_i)] - \mathbb{E}[L_{A,B}(t_{i-1})] \right) = \mathbb{E}[L_{A,B}(T)],
\]

which is just the expected loss on the last coupon payment date — this expected loss is of course increasing when itself increases. We have justified for both denominator and numerator how they change with respect to the expected loss of a tranche. We now can draw the conclusions for three types of tranches: as their joint contribution, the spread of an equity, senior and mezzanine tranche is, respectively, decreasing, increasing or non-monotonic function of correlation.

![Figure 3.1](image)  
**Figure 3.1** Loss distribution of a portfolio of 100 names for various correlations

### 3.3 Calibrations to the market

#### 3.3.1 Implied correlation

The implied correlation of a tranche is based on the single correlation model. It is defined as the correlation number in one factor Gaussian copula model that makes the fair or theoretical value of a tranche equal to its market quote. If the assumption is true, i.e. all names are actually correlated with the same constant correlation parameter, then the implied correlation calibrated from market quotes of different tranches should be the same as each tranche is exposed to the same underlying portfolio of companies. However practice learns that this is not the case and a ‘smile’ pattern of correlation over tranches is usually observed. As shown in Figure 3.2, the implied correlation for the equity and senior tranche is higher than the implied correlation for mezzanine tranches.

Fitting a flat correlation structure is appealing because of its intuitive simplicity, but expressing a complex relationship in one number can often be inaccurate as it does not reflect the heterogeneity of a portfolio. This is somewhat analogous to the equity derivatives market, where the Black-Scholes model has gained universal acceptance. In the Black-Scholes model, the volatility parameter for underlying stock dynamics is assumed constant, but a translation of option prices to implied volatility shows that the volatility also depends on the strikes of the option with a similar smile pattern.
Despite the correlation smile contains useful market information on the correlation structure, the methodology itself has two major drawbacks: firstly, calculating an implied correlation is in general problematic for mezzanine tranches, as their spread is not a monotonic function of the correlation therefore ‘implied correlation’ for a mezzanine tranche is not unique or sometimes even does not exist. Figure 3.3 exhibits spreads for different tranches as functions of correlation. It can be seen that for the traded spread of 300bps, there are two possible correlations, one around 40%, and another around 90%; moreover, if the spread on this tranche were over 500bps, there would be no correlation that gives this spread, and hence no solution. Secondly, it is difficult to price bespoke tranches, (e.g. bespoke tranches with non-standard attachment and detachment points but on an index portfolio), because implied correlation is not a monotonic function of attachment and detachment points, therefore it is not clear how to interpolate or extrapolate without violating for example the no arbitrage conditions.

### 3.3.2 Base correlation

To avoid the difficulties associated with quoting correlation tranche-by-tranche, which can lead to meaningless implied correlations for mezzanine tranches, nowadays the market has well accepted an alternative methodology, an industry standard one, known as the base correlation method.

Base correlations are defined as the correlation inputs required for a series of fictive equity tranches that give the tranche values consistent with quoted spreads, using the one factor Gaussian copula model. We first point out that for expected losses on tranches \([0, K_1]\) and \([0, K_2]\) with \(K_2 \geq K_1\) at any time \(t\), the following relation holds,

\[
E[L_{0,K_2}(t)] = E[L_{0,K_1}(t)] + E[L_{K_1,K_2}(t)].
\]

In order to obtain the base correlations for each fictive equity tranche we need to use a **bootstrapping process**. To illustrate the bootstrapping process, consider a portfolio with tranche attachment points \(K_1 < K_2 < K_3, \ldots\) (e.g. 3%, 6%, 9%, \ldots in case of iTraxx portfolio), the process for calculating the expected loss for each fictive equity tranche is as follows:

\[
E[L_{0,K_1}(t)] = E[L_{0,K_{i-1}}(t)] + E[L_{K_{i-1},K_i}(t)]
\]

where \(E[L_{K_{i-1},K_i}(t)]\) comes from the market spread on the \([K_{i-1}, K_i]\) tranche and \(E[L_{0,K_1}(t)]\) is the
expected loss from the first observable equity tranche. Notice that for the (real) equity tranche, the base correlation is equal to the implied correlation.

Once we have expected losses for the sequence of fictive equity tranches, we can solve for the single base correlation for each tranche — applying our one factor Gaussian copula model. Because equity tranches are monotonic in correlation every fictive equity tranche has a unique base correlation. As shown in Figure 3.4, the base correlation curve calibrated from five market tranche quotes increases monotonically with respect to the detachment level and is a much smoother line at the equity and senior tranches. Now the correlation is a function of detachment level simple interpolation and extrapolation techniques can be employed to obtain base correlations for those bespoke tranches.
4 Hull and White’s Model

4.1 Motivations

The Gaussian copula model is a static model. A single normally distributed variable determines the
default environment for the whole life time of the model. When the variable has a low value, the
probability of each company defaulting during the life of the model is relatively high; when it has a
high value, the probability of each company defaulting is relatively low. The model does not describe
how the default environment evolves. Many alternatives to the Gaussian copula such as the t-copula,
the double-t copula, the Clayton copula, the Archimedean copula, the Marshall Olkin copula, and the
implied copula have been suggested. In some cases these models provide a much better fit to market
data than the Gaussian copula model, but they are still static models.

Default correlation is critical to the valuation of portfolio credit derivatives. The tendency of de-
faults to cluster has been studied by a number of researchers. One possible explanation is that de-
fault rates of all companies are influenced by one or more macroeconomic factors. Another is that
defaults are “contagious” in the sense that a default by one company may induce other corporate fail-
ures. Das, Duffie, and Saita [2007] argue that contagion accounts for some part of the default clustering
that is observed in practice. Modeling this well-observed contagion phenomenon becomes a new and
interesting research direction.

The availability of CDO data for multiple time horizons presents researchers with an interesting and
important challenge. This is to develop a dynamic model that fits market data and tracks the evolution
of the credit risk of a portfolio.

4.2 Single name reduced form model

The Hull and White’s dynamic model belongs to the class of multi-names reduced form model. Before
introducing the setting of the model let’s first review some basics on the single name reduced form model.
In a single name reduced form model, default is seen as an unexpected event that is modeled as a random
time until default. Let $\tau$ denote this default time and we can define the distribution function $F(t)$ of $\tau$
as
$$F(t) = P(\tau \leq t), \quad t \geq 0,$$
hence the survival function $S(t)$ is
$$S(t) = 1 - F(t) = P(\tau > t), \quad t \geq 0.$$ 

Now we can define the hazard rate function. A hazard rate function is the instantaneous default proba-
bility of a single name company which has survived up to time $t$. Mathematically, if $F$ admits a density $f$,

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < \tau \leq t + \Delta t | \tau > t)}{\Delta t} = \frac{1}{1 - F(t)} \cdot \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \frac{f(t)}{1 - F(t)},$$
or
$$h(t) = \frac{f(t)}{1 - F(t)} = -\frac{S'(t)}{S(t)}.$$

The above display characterizes the relation between the survival probability function and hazard rate
function. Suppose we know the form of the hazard rate function, then we can also derive $S(t)$ from $h(t)$.
In fact, by integrating both sides with respect to time, we get
$$S(t) = \exp \left( - \int_0^t h(u) \, du \right).$$
4.3 Model setup

Common specifications of hazard rate function are, for instance, that it is constant, linear or of quadratic form of time. In case it is constant the survival time \( \tau \) becomes an exponential distributed random variable with parameter \( h \) (intensity). In fact,

\[
F(t) = 1 - e^{-ht} \\
f(t) = he^{ht}.
\]

In this research we will assume a piece-wise constant form (between CDS maturities) for the hazard rate function. It is a simplification of the real-world situation, but has little impact on the valuation of a CDO in the later stage, as experiments have shown that the value of a CDO of \( T \)-year maturity is largely determined by the default probability at time \( T \), not the shape of the hazard function (Hull and White [2006]). Once the function form of hazard rate function is specified, an interpolation scheme is defined accordingly on those intermediate coupon payment dates. Then the constant hazard rates can be calibrated using market data of CDSs.

4.3 Model setup

To extend the single name reduced model to a multi-names default environment, the most natural thought would be to specify correlated diffusion processes for the hazard rates of the underlying companies. However, the experience of other researchers has shown that it is not possible to fit market data with this type of model. This is because there is a limit to how high the correlation between times to default can become. This has led researchers to include jumps in the processes for hazard rates. Duffie and Garleanu [2001] for example assume that the hazard rate of a company is the sum of an idiosyncratic component, a component common to all companies, and a component common to all companies in the same sector. Each component follows a process with both a diffusion and a jump component.

Under this circumstance, Hull and White proposed in their paper a reduced form model in which the hazard rate of a company follows a deterministic process that is subject to periodic impulses. This leads to a jump process for the cumulative hazard rate (or equivalently for the logarithm of the survival probability). Define the dynamics for the (cumulative) survival probability up to time \( t \), \( S(t) \), as,

\[
S(t) = e^{-X(t)} \\
\frac{dX(t)}{dt} = \mu(t)dt + dJ(t),
\]

or equivalently,

\[
S(t) = \exp\left\{-\int_0^t \mu(u)du - J(t) \right\}.
\]

where \( J(t) \) is chosen as the correlated jump process which is usually specified as a (non-homogeneous) Poisson process given a realization of the intensity process \( \lambda(t) \) and the jump size \( H(J) \) depends on the number of jumps so far. The drift term \( \mu(t) \) is just a deterministic function of time.

It is assumed that most of the time the default probabilities of companies are independent of one another. Periodically there are economy-wide shocks to the default environment. When a shock occurs each company has a nonzero probability of default. As a result there are likely to be one or more defaults at that time. It is these shocks and their size that create the default correlation. The jumps in the model can lead to several companies defaulting at the same time. For example, suppose that \( S \) decreases from 1 to \( 1 - q \) as a result of a jump at time \( t \). There was no chance of default before the jump, but each company has a probability \( q \) of defaulting by time \( t \).

The model is a simplification of reality. In practice shocks to the credit environment do not cause several companies to default at exactly the same time. The defaults arising from the shocks are usually spread over several months. However, the model’s assumption is reasonable because it is the total number of defaults rather than their precise timing that is important in the valuation of most portfolio credit derivatives. Another simplification is that shocks to the credit environment affect all companies. In
practice they are liable to affect just a subset of companies in the portfolio. However, as an approximation we can think of a shock affecting a subset of companies (or companies in the same industry sector) as being equivalent to a smaller shock affecting all companies as far as its effect on the number of defaults is concerned.

The model can also be formulated as a top-down model. A calibration method is utilized such that credit default swap spreads are exactly matched and CDO tranche quotes are matched as closely as possible (e.g. application of the least squared optimization).

Now we describe how to value a CDO analytically using this model. Let’s first list some useful formulae and notations which are relevant for later calculations. Assuming that the correlated jump process is a (non-homogeneous) Poisson process, the probability of \( J \) jumps between time zero and time \( t \) is

\[
P(J, t) = \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!}
\]

\[
\Lambda(t) = \int_0^t \lambda(u)du,
\]

where \( \lambda(t) \) denotes the Poisson intensity function. The conditional value of \( S(t) \) given that there have been \( J \) jumps is

\[
S(t|J) = \exp(-M(t) - \sum_{j=0}^J H_j)
\]

\[
M(t) = \int_0^t \mu(u)du,
\]

where \( \mu(t) \) is the drift function and \( H_j \) is the jump size at the \( j \)th jump impulse. Therefore computing the (unconditional) survival probability becomes straightforward,

\[
S(t) = \sum_{J=0}^{\infty} S(t|J)P(J, t).
\]

If an extra assumption that the portfolio is homogeneous holds, which we have also seen from the previous section, we can form the probability of \( n \) defaults out of \( m \) names as a binomial one,

\[
\Psi(n, t|J) = \binom{N}{n} (1 - S(t|J))
\]

However, a general non-homogeneity or heterogeneity assumption for the portfolio has to be made, when it comes to the pricing of bespoke CDO tranches. In such case each company has a different CDS spread and for one particular company in the portfolio the jump size and jump intensity are assumed to be the same as that for the representative company (index-wise). The deterministic drift, \( \mu(t) \), on the other hand, is adjusted to match the individual CDS spread. The binomial distribution above then must be replaced by an iterative procedure such as the Anderson recursive algorithm to create the the conditional loss distribution of the whole portfolio. Once this has been done, the rest of the formulations are similar — an expression for expected tranche loss at time \( t \) is easy to get,

\[
E(t|J) = \sum_{n=0}^{N} \Psi(n, t|J)W(n)
\]

\[
E(t) = \sum_{J=0}^{\infty} P(J, t)E(t|J),
\]

where \( W(n) \) is the payoff function of one particular CDO tranche we have defined in the previous section.
Having derived the expected tranche loss at any arbitrary time $t$, we can give the values of fixed leg and default leg by (everything is presented in absolute notional values)

$$\text{PV(\text{fixed leg})} = s_{A,B} \int_0^T D(u)(B - A - \mathbb{E}(u))du$$

$$\text{PV(\text{default leg})} = \int_0^T D(u)d\mathbb{E}(u)$$

Notations are consistent with their definitions in previous sections, e.g. $B - A$ denotes the tranche notional. Note that the above are only interests for the continuous-time computation. In practice, various assumptions about the schedule of the spread payment and possible default times will give rise to slightly different formulae, but generally they only have little impact. For the sake of simplicity and our modeling purpose, we will assume that coupons are quarterly paid and defaults can only happen at these coupon dates. It means that we can reach an approximation by

$$\text{PV(\text{fixed leg})} \approx s_{A,B}\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{360} \cdot D(t_i)(B - A - \mathbb{E}(t_i))$$

$$\text{PV(\text{default leg})} \approx \sum_{i=1}^{n} D(t_i)[\mathbb{E}(t_i) - \mathbb{E}(t_{i-1})]$$

We therefore obtain a break-even spread given by

$$s_{A,B} = \frac{\sum_{i=1}^{n} D(t_i)[\mathbb{E}(t_{i-1}) - \mathbb{E}(t_i)]}{\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{360} \cdot D(t_i)\mathbb{E}(t_i)}.$$

### 4.4 Model implementation

#### 4.4.1 Introduction: Two versions

We have described the general setting of the Hull and White model but still left a few details unspecified. For example, we did not assume any particular form for the jump size $H_j$ at each jump impulse. This gives a great flexibility in the application procedure of the model. In practice, many combinations of the model assumptions have been proposed and each of them corresponds to a “version” of the model. In this subsection, we will present two versions and discuss calibration procedures for each version accordingly.

#### 4.4.2 Version A: Zero Drift; Constant Jumps; Time-Dependent Intensity

The simplest version of the model would be the case in which $\mu(t) = 0$ and the jump size is constant. Then there is just one free parameter—the jump size $H$. This is somewhat analogous to one factor Gaussian copula model where there is just one free correlation parameter $\rho$. More consistency between the two lies in the fact that when the jump size approaches zero the default correlation approaches zero and when the jump size becomes large the default correlation approaches one. Therefore the jump size can also be seen as a measure of default correlation.

Now we discuss the calibration procedure for this version of the model. We will assume a deterministic recovery rate of 40%. Denote by $0 = T_0 < T_1 < \ldots < T_M$ all the CDS maturities and $t_i$’s the intermediate payment dates. An asterisk sign (*) is adopted to denote the quantity that is obtained by interpolation. The calibration procedure then goes as follows.

1. For given value of $H$, the jump intensities $\lambda(T_i)$’s are chosen to meet the term structure of CDS spreads.
2. Search for a value of $H$ such that the model prices back the market spread of one particular tranche (of a particular maturity).
Let’s see how some simplification can be made in this version of the model. First of all, the (unconditional) survival probability \( S(t) \) can be derived explicitly. It follows that,

\[
S(t) = \sum_{J=0}^{\infty} S(t|J) P(J, t) \\
= \sum_{J=0}^{\infty} \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!} \cdot \exp(-JH) \\
= e^{-\Lambda(t)} \sum_{J=0}^{\infty} \frac{(\Lambda(t)e^{-H})^J}{J!} \\
= e^{(e^{-H}-1)\Lambda(t)}
\]

We numerically approximate the integral \( \Lambda(t) \) by

\[
\Lambda(t_i) \approx \sum_{j=1}^{m-1} (T_j - T_{j-1}) \lambda(T_j) + \lambda^*(t_i)(t_i - T_{m-1}), \quad T_{m-1} < t_i < T_m.
\]

where this \( \lambda^*(t_i) \) is determined by some interpolation method. We now illustrate how to calibrate all the \( \lambda(T_i) \)'s using again a bootstrap process\(^1\). One common assumption about the form of function \( \lambda(t) \) is that it is piece-wise constant. Given this simplification assumption, the log-linear interpolation scheme can be employed. It is defined as,

\[
S^*(t_i) = \exp \left( (e^{-H} - 1)[- \sum_{j=1}^{m-1} (T_j - T_{j-1}) \lambda(T_j) - (t_i - T_{m-1}) \lambda^*(t_i)] \right)
\]

Put the second equation in another way, it actually means “linear interpolation in log value”:

\[
\log S^*(t_i) = \frac{T_m - t_i}{T_m - T_{m-1}} \log S(T_{m-1}) + \frac{t_i - T_{m-1}}{T_m - T_{m-1}} \log S(T_m), \quad T_{m-1} < t_i < T_m.
\]

Notice that we do not need to calculate each \( \lambda^*(t_i) \) at each coupon payment date but only \( S^*(t_i) \), because in the end we only need those survival probabilities to plug into the CDS pricing formula. To be exact, we solve for CDS of maturity \( T_m \):

\[
\text{PV}_{\text{CDS}} = (1 - R) \sum_i D(t_i)[S^*(t_{i-1}) - S^*(t_i)] - S_{T_m} \sum_i \frac{t_i - t_{i-1}}{360} D(t_i)S^*(t_i) = 0.
\]

So for an input vector of CDS spreads and a trial value of \( H \), we can obtain the vector of implied \( \lambda(T_i) \)'s (and thus all \( S^*(t_i) \)'s by interpolation) by solving the above bootstrapping system. Then we can proceed further and search for an \( H \) value such that it gives the market spread of one particular tranche. Observe that there is an infinite summation over all positive integer jumps in calculating the expected tranche loss and unfortunately no analytic expression exists. In this case, we propose to approximate it only up to some big \( J \). We want to solve

\[
\text{PV}_{\text{CDO tranche}} = \sum_i D(t_i)[E^*[L(t_i)] - E^*[L(t_{i-1})]] - S_{T_m} \sum_i \frac{t_i - t_{i-1}}{360} D(t_i)E^*[L(t_i)] = 0,
\]

with

\[
E^*[L(t_i)] = \sum_{j=1}^{J} \frac{\Gamma^*[j, t_i]}{\sum_{n=0}^{N} \Psi(n, t, J)W(n)}.
\]

\(^1\)The bootstrapping process defined here is in general the same as bootstrapping of default probabilities which we use as input to the Gaussian copula function. Given a typical assumption of the functional form, in this case piece-wise constant \( \lambda(t) \), the process is basically an iterative process of solving a system of equations.
4.4.3 Version B: Non-Zero Drift; Non-constant Jumps; Constant Jump Intensity

Empirical evidence suggests that default correlation increases in adverse credit conditions. This motivates the use of an alternative model where the jump sizes are larger in adverse market environment which might fit market data better than a constant jump size model. To test this hypothesis we relax the constant jump size assumption and presume the following function form for the jump size at the jth jump:

\[ H(j) = H_0 e^{\beta j} \]

Clearly the jump size increases exponentially with the number of jumps so far. More jumps that have occurred so far indicate a worse credit environment and leads to a bigger jump size for the next jump and therefore bigger decrease in the survival probability. Such decreases in the survival probability among all names in the reference portfolio correspond to a higher default correlation. For the rest of the parameters, we assume a non-zero drift \( \mu(t) \), and a constant intensity \( \lambda \). The model has three free parameters \((H_0, \beta, \lambda)\) and the calibration procedure looks a bit different from the preceding version.

1. First choose trial values of \( H_0, \beta, \lambda \);
2. Solve \( \mu(T_i) \)'s such that the term structure of CDS spread is matched — given a typical specification of the form of \( \mu(t) \);
3. Calculate the sum of squared differences between the model and the market spreads (upfront premium in case of the equity tranche);
4. Determine the values of \( H_0, \beta, \lambda \) such that the squared sum of spread differences is minimized.

The sum of squared spread differences is one objective function we can choose to minimize, but a significant drawback of this approach is that it implicitly put much bigger weights for tranches other than the equity tranche. This is because an equity tranche’s upfront premium is measured in percentage not basis points. Therefore, 1% error in equity upfront premium is actually 100 basis points difference. For this “unfairness” reason, one could choose to minimize the sum of squared PV values instead. Recall on the issuing date of a CDO contract a fair spread should set the present value of each tranche zero. So our objective is to search for the values of \( H_0, \beta, \lambda \) such that all the tranches’ present values are approximately zero. To eliminate the effect of senior tranches having more notional amount\(^1\), we standardize each tranche’s notional the same, e.g. 100 currency units.

We will look at these steps in more details. Let’s also consider the (unconditional) survival probability \( S(t) \) first. Similarly,

\[
S(t) = \sum_{J=0}^{\infty} S(t|J)P(J, t) = \sum_{J=0}^{\infty} \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!} \cdot \exp \left( -M(t) - \sum_{j=0}^{J} H_0 e^{\beta j} \right)
\]

\[
= e^{-M(t)} \sum_{J=0}^{\infty} \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!} \exp \left( -H_0 \frac{1 - e^{\beta(J+1)}}{1 - e^{\beta}} \right)
\]

Our objective here is to calibrate \( M(t) \) given each set of values of \( H_0, \beta, \lambda \). Notice that the each element added in the summation is of doubly exponential form which converge to zero extremely fast.

\(^1\)PV also depends on the notional amount of a tranche. If one tranche has a larger notional amount than the others, the same unfairness would occur again.
This means we can reach a good approximation of it as long as the integer we sum up to is fairly big, i.e.

\[ S(t) \approx e^{-M(t)} \sum_{J=0}^{J_0} \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!} \exp \left( -H_0 \frac{1-e^{\beta(J+1)}}{1-e^{\beta}} \right). \]

However, one adjustment must be made if we want to apply the same two calibration schemes to \( \mu(T_i) \) as we did for \( \lambda(T_i) \). More specifically, we require \( S(0) = 1 \). But at time zero, it holds for the calibration schemes that \( \Lambda(0) = 0 \) and \( M(0) = 0 \), hence

\[ S(0) = 1 \cdot \exp(-H_0) = \exp(-H_0) \neq 1 \quad \text{unless} \quad H_0 = 0. \]

To fix this, we can add an additional \( H_0 \) into the exponential function, this will come down to the following display,

\[ S(t) \approx e^{-M(t)} \sum_{J=0}^{J_0} \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!} \exp \left( H_0 \left[ \frac{1-e^{\beta(J+1)}}{1-e^{\beta}} \right] \right). \]

### Formulation of Bootstrap as a polynomial equation system

Rewrite the (unconditional) survival probability above as

\[ S(t) \approx e^{-M(t)} \cdot C(t; H_0, \lambda, \beta), \]

where \( C(t; H_0, \lambda, \beta) \) is the short notation for the Poisson weighted summation in the previous display. Again \( M(t) = \int_0^t \mu(u)du \) and \( \Lambda(t) = \lambda t \) and we denote by \( 0 = T_0 < T_1 < \ldots < T_M \) all the CDS maturities and \( t_i \)'s the intermediate payment dates. Recall the first step in the calibration procedure is to bootstrap \( \mu(T_i) \)'s such that the term structure of underlying CDS spreads are matched. That is, \( \mu(T_i) \)'s depend on the choice of different parameter set \( H_0, \lambda \) and \( \beta \). Then together with these three independent parameters, \( \mu(T_i) \)'s eventually enter the CDO pricing formula too. In contrast with one factor Gaussian Copula model, this dependency structure gives rise to a great complexity in the calibration of heterogeneous version of the model, since for each update of the parameters all underlying realizations of \( \mu(T_i) \)'s have to be solved all over again. Figure 4.4 below illustrates the difference between the two pricing mechanisms. Solving hundreds of \( \mu(T_i) \)'s of underlying names requires a huge amount of the calculation time at each update of the parameters and therefore becomes computationally intractable in the later calibration procedure — especially when a numerical solver is utilized in solving \( \mu(T_i) \)'s.
4.4 Model implementation

However, if one sticks to the assumption (again a typical assumption) that \( \mu(t) \) is a piece-wise constant function (between CDS maturities), a log-linear interpolation scheme can be adopted for \( e^{-M(t)} \) at each coupon payment date. Then after a careful reformulation, the bootstrap procedure can be expressed in an analytical way — a system of polynomial equations. Solving a system of polynomial equations with pre-specified coefficients are much less time-consuming than implicitly defined system where some numerical solver is invoked. Hence we could expect that it will (partially) reduce the computational complexity and hence make the heterogeneous model available again. Now let’s see how this can be done in details. Recall that

\[
PV_{\text{CDS}} = (1 - R) \sum_{i=1}^{M} D(t_i)[S(t_{i-1}) - S(t_i)] - s \sum_{i=1}^{M} \alpha(t_i, t_{i-1})D(t_i)S(t_i)
\]

\[
= (1 - R)D(t_1)S(0) + (1 - R) \sum_{i=1}^{M-1} (D(t_{i+1}) - D(t_i))S(t_i)
\]

\[
- (1 - R)D(t_M)S(t_M) - s\alpha \sum_{i=1}^{M} D(t_i)S(t_i)
\]

\[
= (1 - R)D(t_1)S(0) + \sum_{i=1}^{M-1} [(1 - R)(D(t_{i+1}) - D(t_i)) - s\alpha D(t_i)]S(t_i)
\]

\[
- [(1 - R) + s\alpha]D(t_M)S(t_M) = 0.
\]

Now if we assume that \( D(0) = 0 \) and \( D(T_{M+1}) = 0 \), then the above display becomes

\[
PV_{\text{CDS}} = \sum_{i=0}^{M} [(1 - R)(D(t_{i+1}) - D(t_i)) - s\alpha D(t_i)]S(t_i) = 0.
\]

Consider the first time period \([0, T_1]\). We have \( S(t_i) = e^{\mu(T_1)} = S(t_1)^{t_i/T_1} \). If we assume further that coupon payment dates are all quarterly ones, this simply implies: \( S(t_i) = e^{\mu(T_1)} = S(t_1)^{t_i} \). Therefore, the above expression can be further reduced to

\[
PV_{\text{CDS}} = \sum_{i=0}^{M} [(1 - R)(D(t_{i+1}) - D(t_i)) - s\alpha D(t_i)]C(t_i; H_0, \lambda, \beta) \cdot S(t_1)^{t_i} = 0.
\]
where \( C(t_i; H_0, \lambda, \beta) \) is the Poisson weighted summation in the previous display. Solving \( S(t_1) \) is just equivalent to solving a polynomial equation.

Next consider the second or any arbitrary time period \([T_{m-1}, T_m]\) in general. Similarly, assuming \( D(t_{M_1}) = 0 \) and \( D(t_{M_2+1}) = 0 \), we have

\[
\text{PV}_{\text{CDS}} = (1 - R) \sum_{i=1}^{M_1} D(t_i)[S(t_{i-1}) - S(t_i)] - s \sum_{i=M_1+1}^{M_2} \alpha(t_i, t_{i-1}) D(t_i) S(t_i)
\]

\[
= + (1 - R) \sum_{i=M_1+1}^{M_2} D(t_i)[S(t_{i-1}) - S(t_i)] - s \sum_{i=M_1+1}^{M_2} \alpha(t_i, t_{i-1}) D(t_i) S(t_i)
\]

\[
= \text{PV}_1 + \sum_{i=M_1}^{M_2} [(1 - R)(D(t_{i+1}) - D(t_i)) - s\alpha D(t_i)] S(t_i) = 0.
\]

Here we use the log-linear interpolation scheme for \( S(t_i) \), i.e.,

\[
\frac{S(t_i)}{S(T_{m-1})} = \left( \frac{S(T_m)}{S(T_{m-1})} \right)^{\frac{t_i - T_{m-1}}{T_m - T_{m-1}}}, \quad T_{m-1} < t_i < T_m
\]

\[
= \left[ \frac{S(t_i)}{S(T_{m-1})} \right]^i
\]

Therefore, we arrive at

\[
\text{PV}_{\text{CDS}} = \text{PV}_1 + \sum_{i=M_1}^{M_2} [(1 - R)(D(t_{i+1}) - D(t_i)) - s\alpha D(t_i)] C(t_i; H_0, \lambda, \beta) S(T_{m-1})^i \cdot \left[ \frac{S(t_i)}{S(T_{m-1})} \right]^i = 0.
\]

\( \text{PV}_1 \) enters the above polynomial as an additional constant term. Solving \( S(t_1) \) is just equivalent to solving a polynomial equation (given the value of \( S(T_{m-1}) \)).
5 Pricing bespoke CDOs

5.1 Introduction

Having developed the two models and the detailed implementation procedure, in this section we discuss how to apply both models in particular to price bespoke CDOs. Since we only have information from those index portfolios and standard tranche structures — for example, we have implied correlation and base correlation skews in one factor Gaussian copula model and calibrated parameters in Hull & White’s model, therefore how to define ways to properly translate, or map these information to a non-standard situation is a tricky task. For this purpose, we will analyze feasibilities of various options under each model framework and find out for each of them what the advantages and drawbacks are.

5.2 Base correlation mapping methods

5.2.1 Introduction

In this subsection we tackle the problem of pricing bespoke CDOs under the one factor Gaussian copula framework. Our starting point is to assume we already have the calibrated base correlation curves of from the index quotes. The key to pricing bespoke CDO tranches is to determine which base correlation parameter $\rho$ to input into the model (for one bespoke tranche). Since we have already built up the base correlation curves, this would mean to find the equivalent detachment point for a bespoke tranche. Once this equivalence has been set up, the pricing of the bespoke tranche is easy. It simply consists in applying the one factor Gaussian copula pricer to the bespoke portfolio with the base correlations of the equivalent index tranche.

For bespoke CDO tranches of a non-standard maturity determining the equivalent tranche also requires first having the base correlation surface for each index. A base correlation surface is a 3-D plot that combines correlation skew and term structure of correlation into one consolidate view. Usually some interpolation and extrapolation technique is presumed in generating such a surface. For example, suppose we want to price 6y [4%, 5.8%] iTraxx tranche, we first need to interpolate base correlation skews of 5y and 7y maturity and obtain the interpolated base correlation skew of 6y maturity, then search for the equivalent tranche therefore the equivalent base correlation.

Since finding the base correlation input for a bespoke CDO tranche of a non-standard maturity is nothing more than determining the equivalent tranche on an interpolated base correlation skew (i.e. base correlation surface), we will, in the rest of this subsection, focus on the main difficulty — defining possible criteria that map the equivalent tranche $[0, K]_{\text{index}}$ for a bespoke equity tranche $[0, K]_{\text{bespoke}}$ and assume the bespoke CDO has one of the standard maturities.

5.2.2 Bespoke portfolio with one reference index

We start by showing solutions when names comprising a bespoke portfolio are from only one reference tranche market (iTraxx or CDX, for example). Furthermore, we assume the composition of the bespoke portfolio is completely or partially different from that of the index portfolio. In this case, the number of names comprising the bespoke portfolio might even vary from the standard 125. Common practice to set up the equivalence between detachment points is via the view of the “riskiness”, in other words, an index tranche and a bespoke tranche are equivalent if they both have the same riskiness. There are many ways people can define their own criteria on the riskiness of a tranche, for example, expected loss, probability of being wiped out, etc. In this research, we will review the analysis in Turc et al. [2006] and highlight four possible methods to find the equivalent tranche $[0, K]_{\text{index}}$ for a bespoke equity tranche $[0, K]_{\text{bespoke}}$.

- Moneyness matching: the bespoke and index equivalent tranches have the same moneyness defined as the ratio between the attachment point and the expected loss of the portfolio. For example, a
5.2 Base correlation mapping methods

[0.8%] bespoke tranche is equivalent to a [0, 4%] index tranche if the bespoke portfolio expected loss is twice as wide as the index expected loss.

- Probability matching: the bespoke and index equivalent tranches have the same probability to get wiped out.
- Equity spread matching: the bespoke and index equivalent equity tranches [0, K] have the same spread.
- Expected loss ratio matching: the expected loss of the two equivalent equity tranches represents the same percentage of the expected loss of their respective portfolios.

Moneyness matching

The rationale behind the approach is that if portfolio A is twice as risky than portfolio B, then, e.g., a 4% detachment point in portfolio A is equivalent to a 2% detachment point in portfolio B. So here the measure of “riskiness” is the expected loss of the portfolio. Following this thought, one can define the “moneyness” of a equity tranche as the ratio between the attachment point and the portfolio’s expected loss. Therefore the method is to find \( K^{\text{index}} \) such that it equates the following

\[
\text{Moneyness}^{\text{index}} = \frac{K^{\text{index}}}{\mathbb{E}(L^{\text{index}}[0,100\%])} = \frac{K^{\text{bespoke}}}{\mathbb{E}(L^{\text{bespoke}}[0,100\%])} = \text{Moneyness}^{\text{bespoke}},
\]

where \( \mathbb{E}(L^{\text{index}}[0,100\%]) \) and \( \mathbb{E}(L^{\text{bespoke}}[0,100\%]) \) denote the expected loss of the index portfolio and bespoke portfolio respectively. Using this method, base correlations of equivalent tranches/detachment points are the same across portfolios. Computing equivalent \( K^{\text{index}} \) requires calculation of both expected loss of bespoke portfolio and index portfolio. Once this has been done, the bespoke base correlation skew is determined.

There are two ways of calculating expected loss of a portfolio. One is to first calculate expected loss of each individual company and then sum them up as the portfolio expected loss. In this case, each CDS spread is utilized as an instrument. The alternative is to calculate the expected portfolio loss based on the spread of the index swap. Recall an index swap is a single tranche [0, 100%] CDO. It can be seen as an index-based CDS contract and its spread defined by the portfolio can be interpreted as the average expected loss per year. The expected loss calculated using two approaches above are approximately equal — there is always some small discrepancy and in practice it is usually adjusted by multiplying a certain spread adjustment factor.

The main advantage of this method is that it does not depend on any correlation assumption because no model is involved in calculating the expected losses of both portfolios. Therefore it is very easy to implement and the result is almost immediate. It nevertheless has three major drawbacks:

- Firstly, the method does not work very well when the bespoke portfolio spread is very tight or very wide compared to the index spread. In case the bespoke portfolio spread is very tight, \( K^{\text{index}} = \frac{\mathbb{E}(L^{\text{index portfolio}})}{\mathbb{E}(L^{\text{bespoke portfolio}})} \cdot K^{\text{bespoke}}, \)

and the expected loss ratio on the right hand side of above display would be very large. Then any bespoke tranche senior to some level (e.g. 10%) has an equivalent index detachment point that is higher than 22% which is the most senior quoted tranche symbolfootnote[2]22% is the most senor quoted tranche of an index CDO, therefore beyond this tranche there is no information at all. A similar situation happens when the expected loss of the bespoke portfolio is very large so that the ratio becomes small, which likely results in an equivalent detachment point lower than the most junior quoted tranche.

- More importantly, the moneyness approach does not take into account the dispersion of the bespoke portfolio but only its expected loss. Therefore, it does not distinguish between a 45bp homogeneous portfolio (all CDS trading at 45bp) and a portfolio with all names trading tighter (say at 30bp) except one CDS trading close to default (say at 10000bp) as both portfolios have the same expected loss. This
is a problem for equity tranches, because an equity tranche in the second case is much more risky than an equity tranche in the first case (homogeneous portfolio).

Thirdly, the P&L (profit and loss) of a bespoke tranche is discontinuous in case of default using the moneyness approach. P&L is a party-definitive concept. For example, the P&L of a CDO tranche for the protection buyer is the immediate income due to default plus the present value of a CDO tranche. Recall that the present value is defined as

\[
PV_{CDO \text{ tranche}[A,B]} = PV(\text{default leg}) - PV(\text{fixed leg})
\]

\[
= \sum_{i=1}^{n} D(t_i) \cdot (E[L_{A,B}(t_i)] - E[L_{A,B}(t_{i-1})])
- s_{A,B} \cdot \sum_{i=1}^{n} D(t_i) \cdot \frac{t_i - t_{i-1}}{360} \cdot (B - A - E[L_{A,B}(t_i)]).
\]

The spread \(s_{A,B}\) in the above display is the break-even spread or contract spread which have been agreed on by both parties at the beginning. When a default occurs in the bespoke, the protection buyer still has to pay this spread. However the new break-even spread may already change very significantly due to the sudden change of ratio of expected losses of the bespoke and index portfolio. As a result, although the protection buyer receives a partial amount of income, it could be the case that the present value of the tranche is very negative, which results in the discontinuity.

**Probability matching**

For the probability matching approach, the equivalent tranche is the index tranche that has the same probability of being eliminated as the bespoke tranche. Mathematically,

\[
P(L^{\text{bespoke}} \leq K^{\text{bespoke}}, \rho^{\text{index}}) = P(L^{\text{index}} \leq K^{\text{index}}, \rho^{\text{index}}).
\]

Notice that \(\rho^{\text{index}}\) is a function of \(K^{\text{index}}\) (i.e. the index base correlation skew), we can rewrite the above formula as

\[
P[L^{\text{bespoke}} \leq K^{\text{bespoke}}, \rho(K^{\text{index}})] - P[L^{\text{index}} \leq K^{\text{index}}, \rho(K^{\text{index}})] = 0.
\]

The \(K^{\text{bespoke}}\) is given, so the above becomes an equation with only one unknown \(K^{\text{index}}\) which is just the equivalent tranche detachment point on the index portfolio. Unlike the moneyness matching approach, probability matching does require a model assumption on the correlation parameter which its self depends on the equivalent tranche, therefore it is not completely straightforward and always numerical procedures are required.

This method takes into account the portfolio dispersion. It is also continuous in case of default of the risky name, as the equivalent detachment points of the pre-default and post-default tranches are very close. But on the other hand, this method has two drawbacks:

- Computing equivalent detachments is numerically difficult when using deterministic recovery rates. Because of this assumption, the cumulative loss distribution function is not continuous and subtle numerical schemes are required to create a continuous loss distribution. For example, if all names have a recovery rate of 40% and there are 100 names in the portfolio, the possible losses are multiples of 0.6%. As a result, the cumulative loss distribution function is piecewise-constant with discontinuities for each multiple of 0.6%. Figure 5.1 shows this type of cumulative loss distribution created by the model. Due to this fact, solving the above equation could result in more than one equivalent tranche as it could be the case that multiple \(K^{\text{index}}\)'s are mapping to the same cumulative probability. Furthermore, although we can produce a continuous loss distribution function, this method is quite time-consuming.

- The probability matching method does not work well for bespoke portfolios with wide spreads. Wide spreads imply names in the bespoke portfolio are relatively risky, in other words, they are expected to have lower probabilities to survive. In contrast, if the index portfolio have tighter spreads, it often holds that

\[
P[L^{\text{index}} \leq K^{\text{index}}, \rho(K^{\text{index}})] \geq P[L^{\text{index}} \leq 0, \rho(0)] = c,
\]
where \( c \) is a certain small but positive number. Therefore if for the bespoke tranche with wide spreads it is the case that
\[
P[L_{\text{bespoke}} \leq K_{\text{bespoke}}, \rho] < c, \quad \forall \rho,
\]
then there is no equivalent detachment point for this bespoke tranche. A concrete example would be a 5y portfolio with 100 names all trading at 100bp. For the iTraxx index portfolio, the probability for the loss to be zero after five years was 14%, according to the calculations on 30 June 2007. On the other hand, for any correlation assumption, the probability that bespoke detachment points below 1.3% are not hit is lower than 14% because of the wide spread of the bespoke portfolio. As a result, there isn’t any equivalent detachment point for bespoke detachment points lower than 1.3%.

**Equity spread matching**

The equity spread matching methodology consists in finding the equivalent index equity tranche \([0, K_{\text{index}}]\) that has the same spread as the bespoke equity tranche \([0, K_{\text{bespoke}}]\). The one factor Gaussian copula model with a correlation assumption is needed to compute the bespoke tranche spread and this correlation assumption itself depends on the equivalent tranche. Therefore, as is the case with the probability-matching approach, numerical solving is required.

This method takes dispersion into account as the existence of a very risky name in the portfolio increases the equity spread. It nevertheless has three major drawbacks:

- It does not work very well when the bespoke portfolio spread is tight. In this case, for senior tranches, the tranche spread is quite tight too and generates an equivalent detachment point higher than most senior quoted tranches. For example, for an European bespoke portfolio of 100 names trading at 15bp, any detachment point higher than 8.6% has an equivalent detachment point higher than 22%. If the tranche is too senior, there may be no equivalent tranche at all. Indeed, if the bespoke tranche spread is below the index portfolio spread (which is the spread of the tranche \([0, 100\%]\)), then no index tranche matches the spread of the bespoke tranche. This case is rare though: for an European portfolio of 100 names trading at 15bp, this happens only for detachment points higher than 47% which are unlikely to be useful in practice.

- It does not work well for junior tranches on bespoke portfolios with wide spreads. There is a maximum possible spread for an index tranche. For example, a \([0, 0.01\%]\) tranche on the iTraxx index was priced at 3210bp on 30 June 2007. If the bespoke tranche is too risky, there may not be any index tranche with a spread as tight as this.
equivalent tranche with the same spread. For example, for a bespoke portfolio with 100 names trading at 100bp, any detachment point lower than 2.7% does not have any equivalent detachment point.

- P&L is not continuous using equity matching in case of default.

**Expected loss ratio matching**

Two tranches are equivalent if they have the same expected loss ratio, defined as the ratio between the expected loss of the equity tranche and the expected loss of the portfolio. Therefore the bespoke tranche $K_{\text{bespoke}}$ and the equivalent tranche $K_{\text{index}}$ verify:

$$\frac{\mathbb{E}(L_{\text{bespoke}}[0, K_{\text{bespoke}}])}{\mathbb{E}(L_{\text{bespoke}}[0, 100\%])} = \frac{\mathbb{E}(L_{\text{index}}[0, K_{\text{index}}])}{\mathbb{E}(L_{\text{index}}[0, 100\%])}.$$ 

The expected loss ratio methodology works well in practice for most bespoke portfolios, either tight or wide. First, it always finds a solution as the expected loss ratio of the bespoke tranche is between 0 and 1 and any ratio between 0 and 1 corresponds to one index tranche. Furthermore, it gives equivalent detachment points that are most of the time inside the quoted tranches on indices. For example, for a European bespoke portfolio with 100 names trading at 15bp, all detachment points below 18% have equivalent detachment points below 22%, which is the most senior quoted tranche for iTraxx indices.

This methodology has two drawbacks: it takes dispersion into account but in a counterintuitive way. For example, if one name widens significantly inside a portfolio, the equivalent detachment points given by this method do not change much while they move more with the other methods. Another problem is again that P&L is not continuous in case of default.

**Summary**

The table below summarizes the analysis we made in this subsection.

<table>
<thead>
<tr>
<th>Method</th>
<th>Moneyness</th>
<th>Wiped-out probability</th>
<th>Equity spread</th>
<th>Expected loss ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Easiness</td>
<td>+</td>
<td>-</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td>View of dispersion</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Availability tight portfolios</td>
<td>=</td>
<td>=</td>
<td>=</td>
<td>+</td>
</tr>
<tr>
<td>Availability wide portfolios</td>
<td>+</td>
<td>=</td>
<td>=</td>
<td>+</td>
</tr>
<tr>
<td>P&amp;L continuity</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

We assign a “+” sign to a method if it works well in terms of one of those listed merits. A method with a “=” means it is only a mediocre method for that merit and in general works in less extreme cases. In the light of Table 5.1, we conclude that there is no such an “optimal” method in the sense that none of them works well universally for every bespoke portfolio and each has their own limitation.

When comparing all equivalent strike methods, it is also important to look at the range of detachment points for which each method works well. In the table below, we computed the bespoke detachment points for which each method gives an equivalent strike between 0% and 22%. This gives the range of detachment points for which pricing a bespoke tranche is possible. We looked at two examples: one European portfolio with 100 names at 15bp and one European portfolio with 100 names at 100bp.

<table>
<thead>
<tr>
<th>Detachment range equivalent to index [0, 22%] (European portfolio, June 2007)</th>
<th>Moneyness</th>
<th>Wiped out probability</th>
<th>Equity spread</th>
<th>Expected loss ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 name at 15bp</td>
<td>[0, 10.5%]</td>
<td>[0, 13.4%]</td>
<td>[0, 8.6%]</td>
<td>[0, 18%]</td>
</tr>
<tr>
<td>100 name at 100bp</td>
<td>[0, 66%]</td>
<td>[1.3%, 42%]</td>
<td>[2.7%, 60%]</td>
<td>[0, 31%]</td>
</tr>
</tbody>
</table>

The wider the range, the greater the amount of information the model is able to extract from the quotations of index tranches. A narrow range means that model prices for bespoke CDOs are determined by very few, or just one, index tranches. The table shows that the expected loss ratio matching approach...
is the most flexible method and that the equity and senior spread matching approaches are quite limited for wide portfolios (detachment points lower than 2.7% or 4.9%, respectively, do not have any equivalent detachment point).

5.2.3 Bespoke portfolio with two reference indices

Often it is the case that a bespoke portfolio consists of names that come from different index regions (for example European and US names). Then the detachment point mapping methodology discussed in the single reference index context needs to be extended to take into account both correlation inputs. For the sake of simplicity, we only look at a Europe/US portfolio and consider the separate weighted average method.

The separate weighted average method is relatively easy to implement. The idea is to first consider the bespoke portfolio as a 100% European one and a 100% US one respectively and find the equivalent correlation in each case (denoted by $\rho^{\text{Traxx}}$ and $\rho^{\text{CDX}}$). Finding both equivalent correlations in this step is the subject covered by the previous subsection. Next define some weighting criterion and determine the weights $\alpha^{\text{Traxx}}$ and $1 - \alpha^{\text{Traxx}}$ for the correlations or other fundamental quantities calculated using equivalent correlations as input therefore combine them into a weighted average. Finally use this weighted average to come up with a fair spread of the bespoke tranche. In practice, there is great flexibility in defining the weighting criterion and choosing which quantity to apply the weights. Below we present a few variations for both procedures.

The most straightforward approach to determine weights is to look at the percentage of portfolio that is comprised of European names. If each name of the bespoke portfolio has equal notional this becomes even simpler — just calculate how many names out of the total names are from the European index region. Another option to set weights is to use the expected loss ratio between sub-portfolio of European names and overall portfolio. Again, this procedure does not require any model assumption as the expected loss is simply the sum-up of each individual contribution.

Next we discuss the possible choices of the quantities to apply the weights. First of all, we could simply determine a bespoke correlation as the weighted average of the equivalent correlations from the two indices. That is

$$\rho^{\text{bespoke}} = \alpha^{\text{Traxx}} \rho^{\text{Traxx}} + (1 - \alpha^{\text{Traxx}}) \rho^{\text{CDX}}.$$ 

Once we have this bespoke correlation parameter, the rest is to use it as the input and run the model. A second, more sophisticated, choice is to apply the weights for the present values calculated using both the equivalent correlations. Mathematically, we have

$$s_{A,B} = \frac{\alpha^{\text{Traxx}} \text{PV}^{\text{Traxx}}(\text{default leg}) + (1 - \alpha^{\text{Traxx}}) \text{PV}^{\text{CDX}}(\text{default leg})}{\alpha^{\text{Traxx}} \text{PV01}^{\text{Traxx}}(\text{fixed leg}) + (1 - \alpha^{\text{Traxx}}) \text{PV01}^{\text{CDX}}(\text{fixed leg})},$$

where $\text{PV01(\text{fixed leg})}$ denotes the present value of fixed leg for 1bp on the survival notional, that is

$$\text{PV01(\text{fixed leg})} = \sum_{i=1}^{n} D(t_i) \cdot \frac{t_i - t_{i-1}}{360} \cdot \left( B - A - \mathbb{E}[L_{A,B}(t_i)] \right).$$

The latter mixing method is the same as to apply the weights for the survival probabilities of the bespoke tranche calculated using different equivalent correlations, because a PV value is linear in the survival probability, in fact,

$$\text{PV(\text{default leg})} = (B - A) \sum_{i=1}^{n} D(t_i) \cdot \left\{ 1 - \frac{\mathbb{E}[L_{A,B}(t_{i-1})]}{B - A} \right\} \frac{t_i - t_{i-1}}{360} \cdot \left( 1 - \frac{\mathbb{E}[L_{A,B}(t_i)]}{B - A} \right) \frac{t_i - t_{i-1}}{360}.$$

$$\text{PV01(\text{fixed leg})} = (B - A) \sum_{i=1}^{n} D(t_i) \cdot \left\{ 1 - \frac{\mathbb{E}[L_{A,B}(t_{i-1})]}{B - A} \right\} \frac{t_i - t_{i-1}}{360} \cdot \left( 1 - \frac{\mathbb{E}[L_{A,B}(t_i)]}{B - A} \right).$$
The separate weighted average method with its many variations is only a practical procedure which lacks important theoretical support. For example, viewing a mixed Europe/US portfolio as a 100% European portfolio and then a 100% US portfolio is certainly not satisfying.

5.3 Hull & White bespoke pricing model

In this subsection we consider the extended Hull and White’s model to price bespoke tranches. If the bespoke portfolio only consists of names from one index region, this is already a solved problem as we just need to run the heterogeneous version of the model with the calibrated parameters as input.

Now suppose we have a bespoke portfolio with mixed names from two index regions. For any particular company in the portfolio the jump size and jump intensity are assumed to be the same as that for the representative company of its corresponding index portfolio. This means we have to incorporate two jump processes simultaneously into the bespoke CDO pricing model. Due to the lack of market information to characterize the dependency between the two jump processes, one common practice is to assume the two are independent from one another. This would result in a two-dimensional jump process and the evolutions of them are independent from each other. For example, it could be the case that at one time there are 3 jumps occurred in the jump process of iTraxx sub-portfolio but 5 in the jump process of CDX sub-portfolio. Take $J_{\text{max}}$ as the maximum number of jumps that could occur for both sub-portfolios, then we have to distinguish $J_{\text{max}} \times J_{\text{max}}$ scenarios in total.

Now we detail the implementation of the extended bespoke pricing model. Our starting point is that we have two sets of calibrated parameters from both iTraxx and CDX indices. Suppose we have a bespoke portfolio of $m$ names out of which there are $m_1$ names linked to iTraxx and $m - m_1$ linked to CDX. Without loss of generality let’s say they are sorted — the first $m_1$ are iTraxx sub-portfolio and the next $m - m_1$ are CDX sub-portfolio. A pair of sets of parameters calibrated from iTraxx and CDX are denoted by $\{H_{01}, \lambda_1, \beta_1\}$ and $\{H_{02}, \lambda_2, \beta_2\}$ respectively. Given the assumption that the evolutions of both jump processes are independent, the probability of joint occurrence of $k$ jumps in iTraxx jump process and $j$ in the CDX jump process is the product of probabilities of the two events. That is

$$ P(J_{\text{Traxx}} = k, J_{\text{CDX}} = j) = P(J_{\text{Traxx}} = k) \cdot P(J_{\text{CDX}} = j), \quad k, j = 0, 1, \ldots, 31 $$

We already know that each jump process is of Poisson type with $\lambda_1$ and $\lambda_2$ as their each intensity, therefore the above probability is easy to calculate. Next consider the conditional survival probabilities of names from the iTraxx sub-portfolio and the CDX sub-portfolio, given that there are $k$ jumps in the iTraxx jump process and $j$ in the CDX jump process. We actually have a vector of conditional survival probabilities as follows

$$ \{S(t|k)_1, S(t|k)_2, \ldots, S(t|k)_{m_1}, S(t|j)_{m_1+1}, \ldots, S(t|j)_m\}^T $$

with

$$ S(t|k) = \exp(-M(t) - \sum_{i=0}^{k} H_i) $$

$$ = \exp\left(-M(t) - H_{01} \cdot \left(1 - \frac{1 - e^{\beta_1 (k+1)}}{1 - e^{\beta_1}}\right)\right) \quad \text{for iTraxx names}, $$

and

$$ S(t|j) = \exp(-M(t) - \sum_{i=0}^{j} H_i) $$

$$ = \exp\left(-M(t) - H_{02} \cdot \left(1 - \frac{1 - e^{\beta_2 (j+1)}}{1 - e^{\beta_2}}\right)\right) \quad \text{for CDX names}. $$
As before, the drift term of each individual name is chosen to match their own CDS spread. Now we can once again convert this vector of conditional probabilities into a portfolio loss distribution using Andersen recursive algorithm. Let’s denote this loss distribution as $\Psi(l, t|k, j)$. Recall previously we have $\Psi(l, t|J)$ where we only take into account one-dimensional jump process. Having known the conditional portfolio loss distribution together with the probability assigned to it, the rest formulations are the same except transforming conditional loss distribution to unconditional one is now weighting along two dimensions.
6 Results

6.1 Introduction

In this section we show the results from this research. We first look at some general characteristics of the data we have drawn from March 2007 to March 2008. In the next subsection a comparison is set up between the portfolio loss distributions constructed by the one factor Gaussian copula model and the Hull and White’s model. The following subsection will be devoted to the analysis of the calibration of both versions of Hull and White’s model. There we highlight the impact of changing the objective function during the calibration procedure. The last subsection is our main focus, that is bespoke tranche pricing using both model frameworks.

6.2 Data

The data we use are monthly quotes of tranches on iTraxx and CDX portfolios from 30 March 2007 to 31 March 2008. Since every year on March 20th and September 20th new series of iTraxx and CDX are issued, this sample period actually includes two series of index portfolios. This selection should have little impact on our investigation, because the change in the compositions of both indices is limited (10 out of 125 to be exact) and the parameters calibrated to indices are not series-specific but indicators of each market environment. This period is an interesting one to look at, as it contains a part of the on-going subprime mortgage crisis which started last July. During the crisis, credit conditions of both index market collapsed and therefore increase in default correlation is expected to be reflected by our model.

![Figure 6.1 Tranche spreads and upfront premium of iTraxx](image)

Figure 6.1 depicts the dynamics of the spreads of different tranches on both index portfolios over the sample period. Note that the equity tranche is quoted in upfront premium rather than running spread (the running spread of an equity tranche is standardized as 500bp). As clearly exhibited by the picture, there is a huge jump in the spreads of all tranches between June and July which implies the start of the crisis. A similar pattern is also observed for the CDX tranche spreads and upfront premium.
6.3 Calibration to indices

6.3.1 Implied jump v.s. implied correlation

We first estimate the single jump size parameter $H$ in the Hull & White’s model version A. As stated in the previous section this implied jump size is a measure of default correlation. As the jump size approaches zero, the default correlation approaches zero. As the jump size becomes large, the default correlation approaches one. Figure 6.2, 6.3 and 6.4 compare the implied jump sizes with the implied correlations using the one factor Gaussian copula model for different maturities on June 30 2007. It can be seen that the two exhibit very similar patterns. Results are similar for CDX index CDOs.

Instead of calculating implied jumps from tranche quotes as shown in the pictures, “implied base jumps” can be calculated in the same manner as users of the Gaussian copula model calculate base correlations.

\[ \text{Implied correlation and jump of iTraxx 5y on June 30 2007} \]

**Figure 6.2** Implied jump and correlation of iTraxx 5y

6.3.2 The optimal triple $H_0, \lambda, \beta$

Next we see the estimation of the three parameters in Hull & White’s model version B. We have carried out calibrations for the following each case:

- Homogeneous model and Heterogeneous model
- Two different objective function (sum of squared PVs and spread differences)
- 5y, 7y, 10y and simultaneous the three maturities
The homogeneous model is a simplified version of the Hull & White’s model. It uses the spread of the index swap (i.e. [0, 100%] CDO) as the average spread and assumes hypothetically that all names take this average spread instead. This procedure avoids a great complexity of the model, but at the price of a huge ignorance of information implied by the heterogeneity of the portfolio. Therefore the optimal triple obtained using the homogeneous model is only a rough approximation and in practice we use it as the starting solution for the later heterogeneous calibration. The optimal triples found using a homogeneous and a heterogeneous model are indeed close and this is an uniform observation regardless of the choice of objective function and maturity. Table 6.1 below summarizes this point.

Table 6.1 Estimations of parameter triple iTraxx

<table>
<thead>
<tr>
<th></th>
<th>Homogeneous</th>
<th></th>
<th>Heterogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_0$</td>
<td>$\lambda$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>5y</td>
<td>0.0036</td>
<td>0.1259</td>
<td>0.9539</td>
</tr>
<tr>
<td>7y</td>
<td>0.0021</td>
<td>0.1650</td>
<td>0.9387</td>
</tr>
<tr>
<td>10y</td>
<td>0.0024</td>
<td>0.1892</td>
<td>0.8181</td>
</tr>
<tr>
<td>5,7,10y</td>
<td>0.0026</td>
<td>0.1627</td>
<td>0.8994</td>
</tr>
</tbody>
</table>

Another observation during the calibration is that as the maturity increases, the fitting of the model to the market quotes gets worse. This means that the explanatory power of the model weakens when maturity of the tranche grows. A possible explanation for this behavior is that approximation error accumulates as the time horizon becomes longer.

Last but not least, it has been found that with three independent parameters we can always achieve a perfect calibration to three market quotes (at most). This is somewhat similar to the situation we encountered when fitting the implied correlation to one single tranche quote. Perfect calibration means the model-shaped loss distribution produces the exact expected losses for these tranches therefore can be trusted as a good approximation to the true distribution within the area of these tranches.

### 6.3.3 Risk of changing objective function

As already addressed, there are two possible choices of objective functions in order to determine the optimal triple: one is the sum of squared differences between the model and the market spreads (upfront premium in case of the equity tranche); the other is the sum of squared PVs of tranches with standardized notional amount (e.g. 100 currency units). The latter is considered as a fair objective, as it attaches equal weights to various tranches. The two are actually equivalent if we, using our model, transform the upfront premium of the equity tranche into running spread for the former objective function.

However, using the sum of squared PVs of tranches as our objective function results in a different dilemma. Due to the extreme riskiness of an equity tranche, its equivalent running spread is normally 10 times as big as the second risky mezzanine tranche. Consequently, the sensitivity of its spread’s move in accordance with the parameters is much bigger than the other tranches. Therefore, making use of squared PVs of tranches as our objective function is no different than putting much bigger weight in the equity tranche. Below we collect some typical calibration results for each objective function (SD is short for sum of squared spread differences — objective function 1; PV is the sum of squared tranche PVs — objective function 2; error is calculated as the model spread minus the market spread).
Table 6.2 Spread errors resulted from the two objectives CDX

<table>
<thead>
<tr>
<th></th>
<th>Equity</th>
<th>Mezz junior</th>
<th>Mezz</th>
<th>Mezz senior</th>
<th>Senior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tranche spread</td>
<td>5y</td>
<td>29.50%</td>
<td>127.6875</td>
<td>28.5156</td>
<td>12.9375</td>
</tr>
<tr>
<td></td>
<td>7y</td>
<td>46.72%</td>
<td>311.125</td>
<td>67.0556</td>
<td>29.1111</td>
</tr>
<tr>
<td></td>
<td>10y</td>
<td>54.91%</td>
<td>630.2778</td>
<td>173.0417</td>
<td>76.9306</td>
</tr>
<tr>
<td>Error obj. SD</td>
<td>5y</td>
<td>5.0212</td>
<td>0.0205</td>
<td>0.9374</td>
<td>0.8373</td>
</tr>
<tr>
<td></td>
<td>7y</td>
<td>6.957</td>
<td>0.2736</td>
<td>0.3574</td>
<td>0.9596</td>
</tr>
<tr>
<td></td>
<td>10y</td>
<td>4.6097</td>
<td>1.1539</td>
<td>-0.2862</td>
<td>1.3781</td>
</tr>
<tr>
<td>Error obj. PV</td>
<td>5y</td>
<td>0.1142</td>
<td>-3.0194</td>
<td>12.6432</td>
<td>12.8096</td>
</tr>
<tr>
<td></td>
<td>7y</td>
<td>0.6758</td>
<td>-7.7598</td>
<td>30.6294</td>
<td>24.1147</td>
</tr>
<tr>
<td></td>
<td>10y</td>
<td>2.1686</td>
<td>-10.2213</td>
<td>17.9115</td>
<td>7.0729</td>
</tr>
</tbody>
</table>

As revealed by Table 6.2, when we adopt the SD as the objective function the model spreads are much better matched to the market quotes for mezzanine and senior tranches than the equity tranche, for the average error is below 1bp. In this case, junior mezzanine tranche becomes the one whose spread is most sensitive to the parameters’ change. On the other hand, when we instead use PV as the objective function to minimize, the opposite situation stands.

![Loss distributions using two objective functions](image)

Figure 6.3 Loss distributions using different objective functions CDX 5y

Figure 6.3 shows the portfolio loss distributions corresponding to the two different objective functions. The one obtained using SD as the objective function is more valid above the 3% loss level as the expected losses (hence the spreads) it produces are close to the market quotes. However, it fails to correctly shape the distribution in \([0, 3%]\) compared to the other distribution. This figure is also consistent with Table 6.2 in that the two distributions shift around and the one resulted from the PV objective function gives more weighting in \([0, 3%]\), which leads to the decrease of the expected loss of this area.

6.3.4 Time series of the parameters

We carried out a successive calibration of the three parameters for both indices from March 30 2007 to December 18 2007 to see their evolution over time\(^1\). We discover that the jump size parameters \(H_0\) and \(\beta\) experience significant change in their values around the start of the subprime crisis in July. However, the intensity \(\lambda\) remains almost intact during the same period. What most attracts our attention is that

---

\(^1\)For this successive calibration, we choose the SD as our objective function.
the moves in the values of $H_0$ and $\beta$ show a clearly offset pattern: before the crisis, $H_0$ holds a relatively high value and $\beta$ is at a lower level; during the crisis, there is a huge dive in $H_0$’s value but a huge rise in $\beta$. Below Figure 6.4 and 6.5 exhibit this behavior of both indices.

![Figure 6.4 Jump size parameters and intensity of iTraxx](image)

An explanation for this is to look at the jump size function we introduced earlier. Recall we have justified that the jump size should increase as the number of jumps increases to reflect a downturn credit environment. The proposed form of the function is $H(j) = H_0 e^{\beta j}$. In this function, $H_0$ and $\beta$ affect the jump size in different ways. $H_0$ is the constant coefficient therefore measures the linear impact of the number of jumps so far on the jump size; $\beta$ is in the exponential term so it measures the exponential impact instead. The speed of an exponential increase is far more substantial than that of a linear one. Therefore the observed offset change in values of $H_0$ and $\beta$ would indicate a different development path of the jump size — either showing a linear tendency or exponential tendency. To illustrate this point, we collect a few calibrated jump size parameters from the pre-crisis and post-crisis period.
6.3 Calibration to indices

Figure 6.5 Jump size parameters and intensity of CDX

Table 6.3 Calculate the development path of jump sizes iTraxx

<table>
<thead>
<tr>
<th>J</th>
<th>March ( H_0 = 0.003, \beta = 0.87 )</th>
<th>June ( H_0 = 0.003, \beta = 0.90 )</th>
<th>August ( H_0 = 0.0003, \beta = 2.02 )</th>
<th>December ( H_0 = 0.0007, \beta = 1.77 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0068</td>
<td>0.0063</td>
<td>0.0023</td>
<td>0.0039</td>
</tr>
<tr>
<td>2</td>
<td>0.0163</td>
<td>0.0155</td>
<td>0.0172</td>
<td>0.0232</td>
</tr>
<tr>
<td>3</td>
<td>0.0390</td>
<td>0.0381</td>
<td>0.1305</td>
<td>0.1362</td>
</tr>
<tr>
<td>4</td>
<td>0.0932</td>
<td>0.0936</td>
<td>0.9877</td>
<td>0.7985</td>
</tr>
<tr>
<td>5</td>
<td>0.2229</td>
<td>0.2301</td>
<td>7.4743</td>
<td>4.6813</td>
</tr>
<tr>
<td>6</td>
<td>0.5326</td>
<td>0.5655</td>
<td>56.5586</td>
<td>27.4441</td>
</tr>
</tbody>
</table>

We see in Table 6.3 that the jump size of March and June (the pre-crisis period) initiates at a relatively high level but grows more or less in a linear speed. In contrast, the jump size of August and December (post-crisis period) starts at a slightly low level but increases extremely fast. The development path of jump sizes can be interpreted as the market view on the chain effect of future credit environment. In this sense, a development path of jumps that starts high but grows slow suggests a minor impact of the current market condition on the next — or the contagion level is low. On the contrary, a development path of jumps that starts low but grows fast suggests a major impact of the current market condition on the next — or the contagion level is high. This contagion or chain effect can also be seen as the
correlation, though in an implicit way.

6.4 Portfolio loss distribution

In this subsection we will see the empirical portfolio loss distributions estimated using both the one factor Gaussian copula model and the Hull and White’s model (heterogeneous version B). Figure 6.6 shows a comparison between the loss distributions from the two models. We use the calibrated parameters of iTraxx 5y on June 30 2007. In case of the one factor Gaussian copula model, each tranche has a different loss distribution using the distinctive implied correlation input. Therefore, distributions are only approximately valid in the sense that they produce the correct expected loss for their each particular tranche. In this context, our observation on the changing shapes with respect to different tranches is consistent with the theory we developed for correlation’s role in the previous section, as we apply the smiled implied correlations as the inputs for various tranches. Another one generated by Hull & White’s model is an simultaneous approximation for all 5 tranches, but, very inaccurate for the equity tranche, for we have bigger spread difference using the squared tranche spread differences as our objective function during the calibration. The shape of the distribution can be very different using the PV objective — we have seen this issue in the previous subsection.

![Graph of empirical portfolio loss distributions](image)

**Figure 6.6** Various portfolio loss distributions of iTraxx 5y

One distinguishing feature we found about the various loss distributions is that, for those created by Hull & White’s model, there are some small bumps in the tail area whereas it is smoothly converging to zero for those from the one factor Gaussian copula model. Figure 6.7 give us some idea about this. This fat-tailed feature is consistent with the observation that for senior tranches relatively high spreads are paid. Actually expected losses of the senior tranches are nearly zero, but since nobody would take the risk for a spread of nearly zero, the expected losses implied in the market spreads are fat tailed. To highlight the tail area of the loss distributions from the Hull & White’s model, we present Figure 6.8 for more insight.

6.5 Bespoke CDO pricing

We now look at the performance of the bespoke CDO pricing model under Hull & White’s framework in comparison with the moneyness matching method under one factor Gaussian copula framework. We provide results in the following three categories. The first category we look into is the pricing of “thin tranches” or “tranchelets”. Tranchelets are those bespoke tranches with 1% thickness defined on an index
portfolio. Pricing tranchelets under the one factor Gaussian copula model does not involve any of the mapping methods but requires some interpolation and extrapolation techniques\(^\dagger\). The second category is pricing of bespoke portfolio with one index reference. For this purpose, we choose the portfolio called “Europe old” (old index) having total 100 European names\(^\ddagger\). The third category is pricing of bespoke portfolio with two index references and we select a global portfolio named “Global 75” consisting of 75 names from either Europe (37) or North America (38). Choosing these two portfolio is also out of the consideration to minimize the influence of non-standard recovery rates, for not all of the portfolio consist of names that have the same recovery rate (e.g., 40%) and this could count for a big source for the inaccuracy of the price. Tranches on these two all have standard 3, 5, 7, and 10y maturities therefore we do not need to trouble with the time dimension difficulty in this case.

There exist no “true” prices for bespoke tranches because of their less liquidity and lack of market information. The best people can do is to collect all the independent pricing sources from each market participant and come up with a so-called “consensus” price. In this research, the consensus price we

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\(^\dagger\)In practice, interpolation or extrapolations can be applied in many quantities such as base correlation skew, expected loss, etc. Also there are difference between each interpolation and extrapolation scheme. The results generated in this subsection are direct application of linear interpolation of the base correlation skew.

\(^\ddagger\)There are 7 names of Europe old that have different recovery rate other than 40%. We consider this as negligible.
6.5 Bespoke CDO pricing

reference to is from the Totem Service™. Totem data is made up of a consensus of mid-market prices from the leading market makers in each product along with information on the range of submissions and where the client is an outlier. Therefore, this consensus price can be seen as one objective price as it counterbalances most of the mispricing effect (underprice or overprice). We use this consensus price as our judge on the model performance.

6.5.1 Pricing of tranchelets

The absence of reference prices limits us in that we cannot compare model outputs by means of the pricing error to validate a model. Instead we can compare methods by means of no arbitrage conditions and smoothness. Pricing of tranchelets provides a good point of view to examine these qualities of a model. In term of expected loss, therefore tranche spread†, we can define our no arbitrage conditions as

1. Expected loss of any tranche is a non-decreasing function of time:

\[
E[L^T_{K_1, K_2}(t_i)] \geq E[L^T_{K_1, K_2}(t_{i-1})]
\]

2. Expected loss is a non-decreasing function of tranchelet:

\[
E[L^T_{K_1, K_2}(t)] \geq E[L^T_{K_2, K_3}(t)], \quad K_0 < K_1 < \cdots < K_n \text{ are detachment points of tranchelets}
\]

The second validation criterion is the smoothness of the spread curve as a good model should have a smooth bespoke spread curve. Figure 6.9 provides help in visually showing how the curves look like using each model.

![Figure 6.9 Pricing of tranchelets by the two models](image)

The graph above to the left is the spread curve calculated using the Hull & White’s model for iTraxx portfolio on June 30 2007 (the blue curve) and December 18 2007 (the red curve) respectively. It is clear the model is arbitrage free according to the definition we established as the spread is monotonically decreasing with respect to the detachment point of tranchelet. However, it is not the case for base correlation methodology embedded in the one factor Gaussian copula model as some small bumps are observed in its spread curve (tranchelets on iTraxx June 30 2007) to the right. The reason for this phenomenon is that when applying the base correlation methodology, two different correlation parameters corresponding to the attachment and detachment point are input to calculate the expected loss of a tranche. This means we actually utilize two different loss distributions inconsistent with each other. Therefore the expected loss produced by this methodology may not be a monotonically decreasing function over tranchelets unless some subtle inter- or extrapolation scheme is invoked in the first place. In

†Recall the relation between the expected loss of a tranche and its spread discussed in section 3.
contrast, the Hull & White’s model determines one unique loss distribution for one triple parameters, hence the expected loss is monotonically decreasing over tranchelets which is the direct consequence of weighting of the payoff functions of the tranchelets.

### 6.5.2 Pricing of tranches with single reference

Next we look at the pricing of the bespoke tranches with single reference. Figure 6.10 shows the spread curves of tranches on Europe old portfolio calculated by moneyness matching method and Hull & White’s heterogeneous model. Clearly, the consensus price and the moneyness matching price are essentially close as their curves are almost overlapping with each other. It is not quite a surprise because now the industry standard pricing model is still the one Gaussian copula model and it is likely that most pricing institutions adopt the same or similar methodologies under this framework (e.g. mapping methods), therefore the difference between prices is limited. Compared to the consensus price, Hull & White’s model gives higher price for the equity and the junior mezzanine tranche but lower price for the senior tranches. This deviation sometimes can be very large and more significant in December than June, for example, the price of the equity tranche is 3800.5bp according to Hull & White’s model but the consensus price is only 2211.6bp.

![Figure 6.10 Pricing bespoke tranches on Europe old 5y (June and Dec. 2007)](image)

### 6.5.3 Pricing of tranches with two references

Similar pattern of the results are observed when pricing bespoke tranches on Global75 (both on June 30 and December 18 2007). First of all, the consensus price is close to the moneyness matching price and relative error are within 1%. Second of all, Hull & White’s model always deviates significantly in both equity and senior tranche. This deviation from the consensus price becomes more observable in December. Figure 6.11 summarizes the discoveries.

One possible reason for the behavior that the Hull & White’s model overprices in equity tranche but underprices in senior tranche is that it has relatively big calibration errors for equity tranche and senior tranche as these two types of tranche are the most non-sensitive ones to the parameters’ change. Therefore the loss distribution for these two tranches are likely to be the most misshaped and in case of the equity tranche the model produces some extra probability mass and the reverse holds for the equity tranche.

### 6.5.4 Impact of changing objective function

It has also been noticed that different input parameters resulted from the two objective functions produce different pricing results, though the difference due to this is limited. Below are some typical results.
Figure 6.11 Price bespoke tranches on Global 75 5y (June and Dec. 2007)

### Table 6.3 Prices difference objective (Europe old 5y)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Consensus</th>
<th>SD objective</th>
<th>PV objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 – 60%</td>
<td>1.14</td>
<td>0.06</td>
<td>1.55</td>
</tr>
<tr>
<td>30 – 45%</td>
<td>2.67</td>
<td>0.27</td>
<td>5.18</td>
</tr>
<tr>
<td>15 – 30%</td>
<td>4.07</td>
<td>1.42</td>
<td>11.29</td>
</tr>
<tr>
<td>10 – 15%</td>
<td>10.39</td>
<td>6.66</td>
<td>11.63</td>
</tr>
<tr>
<td>7 – 10%</td>
<td>25.79</td>
<td>15.16</td>
<td>20.07</td>
</tr>
<tr>
<td>5 – 7%</td>
<td>61.97</td>
<td>39.83</td>
<td>44.02</td>
</tr>
<tr>
<td>3 – 5%</td>
<td>196.78</td>
<td>130.80</td>
<td>93.09</td>
</tr>
<tr>
<td>2 – 3%</td>
<td>535.02</td>
<td>493.12</td>
<td>375.86</td>
</tr>
<tr>
<td>Equity</td>
<td>1610.62</td>
<td>1988.30</td>
<td>1880.00</td>
</tr>
</tbody>
</table>

### Table 6.4 Prices difference objective (Global 75 5y)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Consensus</th>
<th>SD objective</th>
<th>PV objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 – 60%</td>
<td>4.13</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>30 – 45%</td>
<td>9.54</td>
<td>0.02</td>
<td>0.42</td>
</tr>
<tr>
<td>15 – 30%</td>
<td>20.42</td>
<td>1.32</td>
<td>9.79</td>
</tr>
<tr>
<td>9 – 15%</td>
<td>92.42</td>
<td>17.99</td>
<td>36.31</td>
</tr>
<tr>
<td>6 – 9%</td>
<td>315.93</td>
<td>205.26</td>
<td>194.04</td>
</tr>
<tr>
<td>3 – 6%</td>
<td>924.31</td>
<td>1098.70</td>
<td>1030.90</td>
</tr>
<tr>
<td>Equity</td>
<td>2,633.26</td>
<td>3676.60</td>
<td>3599.60</td>
</tr>
</tbody>
</table>

The PV objective function puts equal weight to each tranche during the calibration and therefore the equity tranche becomes the most sensitive one to the parameters’ change. As a result, with little error arising from the equity tranche, it produces more accurate (lower in this case) price for the equity tranche. Recall the loss distribution linked to the PV objective function shifts towards more junior tranches (Figure 6.3). This is why the spread calculated using PV objective function is lower for the equity tranche but higher for the senior tranches.
7 Conclusion

In this research we have investigated different bespoke tranche pricing methods. These include four mapping methods which are rooted in the one factor Gaussian copula model and another bespoke CDO pricing model developed from an alternative Hull & White’s framework. For the four mapping methods, we analyzed them one by one to see their advantages and drawbacks. We conclude that there is no “optimal” solution in the sense that none of them works well universally for every bespoke portfolio and each has its own limitation. Either it is not suitable for a portfolio with wide spreads or tight spreads or it does not take into account the dispersions, etc. In addition, in less extreme occasions the results they produce are indifferent. For the Hull & White’s bespoke CDO pricing model, we first solved the calibration difficulty for the heterogeneous version of the model by reformulating the CDS bootstrapping procedure as a polynomial equation system. Then, for the purpose of bespoke pricing, we extended their model based on the assumption that the jumps processes from the two index regions are independent from one another.

In the result section, we first test the two models with the no arbitrage conditions by plotting the spreads of the tranchelets on an index portfolio. We found that the Hull & White’s model is by definition arbitrage free as we only utilize one unique loss distribution corresponding to a triple parameters. The base correlation methodology under the one factor Gaussian copula model, on the other hand, does not exclude the arbitrage opportunities as bumps in its tranchelet spread curves are often observed unless some subtle inter- and extrapolation scheme is adopted. Next we compare the performance of the Hull & White’s bespoke pricing model with the moneyness matching method which we consider as a representative of the four mapping methods. We took two test portfolios from many real bespoke portfolios: one is one-reference, the other is two-reference. We discovered that the prices obtained by the moneyness matching method is always close to the market consensus price as they are constructed under the same industry standard model, which is still one factor Gaussian copula model. The Hull & White’s bespoke CDO pricing model, despite its richness in the financial interpretation, deviates from the consensus price most prominently in equity and senior tranches. Another phenomenon is that different input parameters resulted from the two objective functions produce different pricing results, though the difference due to this is limited. We reason this behavior as due to the imperfect calibration of the Hull & White’s model and PV objective function brings relatively more accuracy in the equity tranche and resulting loss distribution shifts towards the equity tranche. As a result, it produces slightly balanced prices — less expensive for the equity tranche and more expensive for the senior tranche.

Along the development of Hull & White’s bespoke pricing model, we also looked at one simple version which only involves one free jump size parameter $H$. We have justified that this jump size parameter can be seen as a measure of default correlation, which has a similar feature as the correlation input for the one factor Gaussian copula setting. We calibrated this simple version of the model to the market quotes and discovered a similar “smile” pattern which is consistent with the correlation smile. Instead of calculating implied jumps from tranche quotes, the “implied base jumps” can be calculated in the same manner as users of the Gaussian copula model calculate base correlations.
APPENDIX

A A small derivation for Gaussian Copula

For ease of explaining the idea, we consider a simple portfolio with only two names. Suppose the following equivalence between default time and asset value holds (two events).

\[ \tau_i < t \iff V_i < v_i \quad \text{with} \quad P(\tau_i < t) = P(V_i < v_i) \]

Then it follows further that

\[ P(\tau_i < t_i, \tau_j < t_j) = P(V_i < v_i, V_j < v_j) \]

Under the one-factor Gaussian copula framework, we impose that the value of assets is standard normally distributed with common correlation parameter \( \rho^2 \) (\( \rho_i = \rho_j = \rho \)). Therefore, the above becomes

\[ P(\tau_i < t_i, \tau_j < t_j) = P(V_i < v_i, V_j < v_j) = \int_{-\infty}^{v_i} \int_{-\infty}^{v_j} \phi(x, y | \rho^2) \, dx \, dy \]

where \( \phi(x, y | \rho^2) \) is the density function of a bivariate normal distribution with correlation matrix depending solely on \( \rho^1 \). We know \( v_i \) and \( v_j \) through the relation

\[ P(\tau_i < t_i) = P(V_i < v_i) \quad \text{and} \quad P(\tau_j < t_j) = P(V_j < v_j) \]

or

\[ v_i = \Phi^{-1}(P(\tau_i < t_i)) = \Phi^{-1}(p_i) \quad \text{and} \quad v_j = \Phi^{-1}(P(\tau_j < t_j)) = \Phi^{-1}(p_j) \]

where \( \Phi(x) \) denotes the cumulative distribution function of a standard normal. Combining the above two steps, we arrive at the following expression

\[ P(\tau_i < t_i, \tau_j < t_j) = \int_{-\infty}^{v_i} \int_{-\infty}^{v_j} \phi(x, y | \rho^2) \, dx \, dy \]

\[ = \Phi(\Phi^{-1}(p_i), \Phi^{-1}(p_j); \rho^2) \]

which shows the exact the bivariate Gaussian copula transformation from marginal to joint distribution. Next we will discover the equivalence between the above Gaussian copula expression and the traditional “structural interpretation”. In this structural interpretation of the model, we first consider the conditional joint distribution given the value of the common factor. Because of the independence structure, it is just the product of each distribution function. Namely,

\[ P(\tau_i < t_i, \tau_j < t_j | \bar{V}) = \Phi \left( \frac{\Phi^{-1}(p_i) - \rho \bar{V}}{\sqrt{1 - \rho^2}} \right) \cdot \Phi \left( \frac{\Phi^{-1}(p_j) - \rho \bar{V}}{\sqrt{1 - \rho^2}} \right) \]

Then the computation of unconditional joint distribution becomes easy—just integrate the conditional over the common factor. With slight abuse of notation, we denote \( \Phi^{-1}(p_i) = \Phi^{-1}_i \) and \( \Phi^{-1}(p_j) = \Phi^{-1}_j \).

\[ ^1 \text{That is } \Sigma = \begin{pmatrix} 1 & \rho^2 \\ \rho^2 & 1 \end{pmatrix}. \]
It then follows that,

\[
\int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(p_i) - \rho V}{\sqrt{1-\rho^2}} \right) \cdot \Phi \left( \frac{\Phi^{-1}(p_j) - \rho V}{\sqrt{1-\rho^2}} \right) \cdot \phi(V) dV
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot \Phi^{-1} \cdot \exp \left[ -\frac{1}{2(1-\rho^2)}(x - \rho V)^2 \right] dx
\]

\[
-\frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot \Phi^{-1} \cdot \exp \left[ -\frac{1}{2(1-\rho^2)}(y - \rho V)^2 \right] dy \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{V^2}{2}} dV
\]

\[
= \Phi^{-1}(p_i) \cdot \int_{-\infty}^{\infty} dx \cdot \int_{-\infty}^{\infty} dy \cdot \frac{1}{(2\pi)^{3/2}(1-\rho^2)}
\]

\[
\cdot \exp \left\{ -\frac{1}{2(1-\rho^2)}[x^2 + y^2 + 2\rho^2 V^2 - 2\rho(x + y)V + (1 - \rho^2)V^2]\right\} dV
\]

Focus on the most inner integral, we have

\[
\int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2(1-\rho^2)}[x^2 + y^2 + 2\rho^2 V^2 - 2\rho(x + y)V + (1 - \rho^2)V^2]\right\} dV
\]

\[
= \frac{1}{(2\pi)^{3/2}(1-\rho^2)} \cdot \exp \left[ -\frac{1}{2(1-\rho^2)}(x^2 + y^2) \right] \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2(1-\rho^2)}[(1 + \rho^2)V^2 - 2\rho(x + y)V]\right\} dV
\]

\[
= \frac{1}{(2\pi)^{3/2}(1-\rho^2)} \cdot \exp \left[ -\frac{1}{2(1-\rho^2)}(x^2 + y^2) \right] \cdot \int_{-\infty}^{\infty} \exp \left\{ -\frac{1 + \rho^2}{2(1-\rho^4)}[(x^2 + y^2)(1 + \rho^2) - \rho^2(x + y)^2]\right\} dV
\]

\[
= \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left[ -\frac{1}{2(1-\rho^4)}[(x^2 + y^2)(1 + \rho^2) - \rho^2(x + y)^2]\right\}
\]

\[
= \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left[ -\frac{1}{2(1-\rho^4)}[(x^2 + y^2)(1 + \rho^2) - \rho^2(x + y)^2]\right\}
\]

Combine the results from previous two steps, we arrive at

\[
\mathbb{P}(\tau_i < t_i, \tau_j < t_j) = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(p_i) - \rho V}{\sqrt{1-\rho^2}} \right) \cdot \Phi \left( \frac{\Phi^{-1}(p_j) - \rho V}{\sqrt{1-\rho^2}} \right) \cdot \phi(V) dV
\]

\[
= \int_{-\infty}^{\Phi^{-1}(p_i)} \int_{-\infty}^{\Phi^{-1}(p_j)} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^4)}[x^2 + y^2 - 2\rho^2 xy]\right\} dxdy
\]

which coincides with the one factor Gaussian copula expression we derived earlier.
References


