Noncommutative Crepant Resolutions and Toric Singularities

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Abstract

We give a brief introduction to the theory of noncommutative crepant resolutions, the analogue of crepant resolutions in noncommutative geometry. In particular, we give most of the technical background needed to understand the basics, and we focus on the relation to commutative algebraic geometry. After recalling the necessary facts from toric geometry, we apply this theory to toric affine Gorenstein surface singularities, in order to obtain crepant resolutions thereof.
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Introduction

In [B2], Bridgeland proved the 3-dimensional case of the following conjecture of Bondal and Orlov.

**Conjecture** (Bondal–Orlov). If $Y_1, Y_2 \to X$ are two crepant resolutions of $X$, then there is a triangulated equivalence $D^b \text{Coh}(Y_1) \cong D^b \text{Coh}(Y_2)$.

Van den Bergh realized that this derived equivalence could be established through a third derived category, namely that of a certain noncommutative ring. This motivated him to define so-called noncommutative crepant resolutions (NCCRs) in [vdB], and formulating the noncommutative version of the Bondal–Orlov conjecture.

**Conjecture** (NC Bondal–Orlov). If $\Lambda_1$ and $\Lambda_2$ are two NCCRs of a Gorenstein singularity $\text{Spec}(R)$, then there is a triangulated equivalence $D^b \Lambda_1\text{-Mod}^{f.g.} \cong D^b \Lambda_2\text{-Mod}^{f.g.}$.

If $Y$ is a crepant resolution of a Gorenstein ring $R$ over $\mathbb{C}$, then $Y$ is smooth and has trivial canonical sheaf, because it is the pullback of the canonical sheaf of $R$, which is trivial by definition. As such, $Y$ is a Calabi-Yau manifold, so that its derived category $D^b \text{Coh}(Y)$ describes the D-branes of the type B topological string theory on $Y$. The commutative Bondal–Orlov conjecture therefore allows us to compare D-branes on certain CY manifolds $Y_1$, $Y_2$ resolving $R$. But if we establish the equivalence $D^b \text{Coh}(Y_1) \cong D^b \text{Coh}(Y_2)$ through auxiliary NCCRs $\Lambda_1$ and $\Lambda_2$ of $R$, making the comparison becomes harder, as we also need to describe for example the equivalence of $D^b \Lambda_1\text{-Mod}^{f.g.}$ with $D^b \Lambda_2\text{-Mod}^{f.g.}$.

Due to their combinatorial nature, toric varieties should be expected to somewhat simplify the situation. Indeed, any two toric crepant resolutions of a toric Gorenstein threefold can be related by a finite sequence of so-called toric flops, and any two toric NCCRs of the same singularity are thought to be related by so-called toric mutations (see [B1]). Moreover, in that paper an algorithm is given for finding all toric NCCRs of a toric threefold.

In this thesis, we construct the toric NCCRs of an affine toric Gorenstein surface. This is very feasible for two reasons. Firstly, the existence of a minimal resolution for two-dimensional singularities leads one to expect that there is essentially only one NCCR to be found (this is true), and secondly, the affine toric Gorenstein surfaces are precisely the $A_n$ singularities.
1. Singularities

In this introductory chapter, we give a very brief account of the commutative theory of resolutions of singularities. The main use of this chapter is to introduce terminology, and to serve as a basis to compare the noncommutative theory from chapter 3 with. An excellent reference is [KM]. We work over an algebraically closed field $k$ of characteristic 0.

1.1. Resolutions of Singularities

The main definition is the following.

**Definition 1.1.** A morphism $f: X \to Y$ of varieties is called a resolution of singularities if $X$ is nonsingular and $f$ is proper and birational.

**Remarks 1.2.**

- If $Y$ is a curve, then the normalization $\tilde{Y} \to Y$ will be finite and birational. By part $R_1$ of Serre’s criterion for normality, $\tilde{Y}$ is non-singular, and hence a resolution of $Y$.

  Since normal varieties are a bit better behaved than non-normal ones, and because if we have a resolution of the normalization, we obtain a resolution of the original variety, we will generally assume our varieties to be normal. All toric varieties are normal, so for them no additional assumptions are required.

- Note that if we did not require the properness, the inclusion $X \setminus \text{Sing}(X) \to X$ would always be a resolution of singularities, where $\text{Sing}(X)$ is the singular locus of $X$.

As Hironaka’s theorem shows, resolutions of singularities always exist. As such, a large part of the theory consists of looking at resolutions that somehow behave better than the general resolution, for example rational resolutions. For these nicer resolutions, existence is no longer guaranteed by Hironaka’s theorem.

**Definition 1.3.** A resolution of singularities $f: X \to Y$ is called rational if $Rf_* \mathcal{O}_X[0] = \mathcal{O}_Y[0]$, and $Y$ is said to have rational singularities if such a resolution exists.

Recall that the canonical sheaf on a nonsingular variety $X$ is defined as the $n$th exterior power of the cotangent sheaf $\Omega^1_{X/k}$, where $n$ is the dimension of $X$. This can be extended to arbitrary normal varieties $X$ by pushing the canonical sheaf of the nonsingular locus forward to $X$. These sheaves are denoted $\omega_X$, and they play an important role in Grothendieck duality. It is of interest to compare the canonical sheaves of a singularity and its resolutions. The following result gives the first step in that direction.
Theorem 1.4. Let $Y$ be a normal variety over $k$. The following are equivalent:

- $Y$ has a rational resolution of singularities;
- all resolutions of $Y$ are rational;
- $Y$ is Cohen-Macaulay\(^1\) and for any resolution $f:X \to Y$ we have $Rf_*\omega_X[0] = \omega_Y[0]$.

The following definition is central to our interests, as we will discuss noncommutative crepant resolutions in chapter 3. The name crepant refers to the lack of discrepancy between the canonical sheaf of the singularity and that of its resolution.

Definition 1.5. A resolution $f:X \to Y$ of a normal variety is said to be crepant if $f^*\omega_Y = \omega_X$.

1.2. Dimension 2

In this section we describe the aspects of the two-dimensional theory which differ from the higher-dimensional theory.

Proposition 1.6. The rational Gorenstein\(^2\) singularities of dimension 2 are, up to analytic isomorphism, the following hypersurfaces in $\mathbb{A}^3_k$:

- $A_n : x^2 + y^2 + z^{n+1}$ $(n \geq 1)$
- $D_n : x^2 + y^2z + z^{n-1}$ $(n \geq 4)$
- $E_6 : x^2 + y^3 + z^4$
- $E_7 : x^2 + y^3 + yz^3$
- $E_8 : x^2 + y^3 + z^5$,

which are known as the rational double points, or the Kleinian singularities, or the Du Val singularities. These are in one-to-one correspondence with the finite subgroups of $SL_2(k)$, and are all obtained as the quotient of $\mathbb{A}^2_k$ by such a subgroup acting in the standard way. In particular, the $A_n$ singularity corresponds to the realization of $\mu_{n+1}$ as the subgroup of $SL_2(k)$ generated by the matrix $\begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix}$, where $\xi \in k$ is some primitive $(n+1)$th root of unity.

Given a surface singularity $Y$ with unique singular point $P$ and a resolution $f:X \to Y$, we call the fiber $f^{-1}(P)$ the exceptional divisor. It can be shown that $f^{-1}(P)$ is a union of projective curves. Taking these curves as vertices, and drawing an edge whenever two curves intersect, we obtain a graph known as the dual graph. For the rational double points, the dual graphs are precisely the simply laced Dynkin diagrams, giving a nice geometric interpretation of the ADE classification above.

A peculiar feature of surface singularities is the existence of a so-called minimal resolution, as described in the following theorem.

\(^1\)See section 3.1 for the definition of Cohen-Macaulay.

\(^2\)See section 3.1 for the definition of Gorenstein rings.
Theorem 1.7. Let \( Y \) be a two-dimensional normal variety. Then there exists a resolution \( X \to Y \) called the minimal resolution such that every other resolution \( X' \to Y \) uniquely factors as \( X' \to X \to Y \). Moreover, every crepant resolution of a surface is minimal.

Note that this implies that any two crepant resolutions of the same singularity are isomorphic. There is no higher dimensional analogue of a minimal resolution, but a weakened generalization of the uniqueness of crepant resolutions is conjectured to hold. Namely, for any two crepant resolutions \( Y_1 \) and \( Y_2 \) of the same affine normal Gorenstein variety, there is a triangulated equivalence \( D^b \text{Coh}(Y_1) \cong D^b \text{Coh}(Y_2) \). In dimension 2 this follows immediately from the fact that crepant resolutions are unique. In dimension three, this was proved by Bridgeland in [B2]. Bridgeland’s proof was in fact the main motivation for van den Bergh’s definition of noncommutative crepant resolutions in [vdB].
2. Toric Geometry

In this chapter we give the basic theory of toric varieties. Toric varieties are particularly nice cases of varieties defined by monomial equations, i.e., equations equating one monomial to another. They come with a natural torus action, and much of the equivariant geometry can be phrased in terms of combinatorial objects known as fans and convex rational polyhedral cones, which are discussed in appendix A. In the first section we describe affine toric varieties in terms of torus actions and in terms of convex geometry, and study their local properties. In the second section we expound on the global theory, and in the third we give the standard way to resolve a two-dimensional toric singularity. Some introductions to the subject can be found in [F], [O], and [KKMSD], which serve as our main references in this section. Any needed definitions and results pertaining to convex geometry can be found in appendix A. The reader is assumed familiar with the definitions therein.

We mostly work over an arbitrary algebraically closed field $k$, but ultimately we are interested in the case $k$ of characteristic 0, so we occasionally make assumptions in this general direction. Most of the theory works for fields which are not algebraically closed, as the combinatorial nature of toric varieties implies that base extension to an algebraic closure yields a toric variety. The assumption that $k = \bar{k}$ allows us to conflate a variety $X$ with its set of $k$-rational points $X(k)$, of which we freely make use without mention.

2.1. Affine Toric Varieties

Let $N$ be a lattice, and let $S$ be a submonoid of $N^\vee$ which generates $N^\vee$ as a group. With this monoid we can associate a $k$-scheme by taking the spectrum of the monoid-algebra $A_S = k[S]$, generated as a module over $k$ by the symbols $\chi^s$ for $s \in S$, with multiplication given by $\chi^s \chi^r = \chi^{s+r}$. The natural $N^\vee$-grading on $A_S$ and the $N^\vee$-homogeneous injection $A_S \to k[N^\vee]$ give an open immersion $T_N \to \text{Spec}(A_S)$ and a $T_N$-action which extends the action of $T_N$ on itself. The fact that this is an open immersion can be seen by writing a basis of $N^\vee$ as finite $\mathbb{Z}$-linear combinations of elements of $S$ (this can be done, as $S$ generates $N^\vee$ as a group), which shows that we can obtain $k[M^\vee]$ by localizing $A_S$ at (finitely many) suitable elements of the form $\chi^s$ with $s \in S$.

Conversely, suppose $X = \text{Spec}(R)$ is an integral $k$-scheme with a $T_N$-action and an open immersion $T_N \to X$ such that the action of $T_N$ on itself extends to that on $X$. Then by the characterization in B.2 of $T_N$ actions on affine schemes as $N^\vee$-gradings on the
coordinate ring, \( R \) is equal to the monoid-algebra of \( S = \{ n \in N^\vee \mid \chi^n \in R \} \). Here we have identified \( R \) with its image under \( i:R \to k[N^\vee] \), which we can do because the irreducibility of \( X \) shows that \( T_N \) is dense in \( X \), so the kernel of \( i \) is contained in \( \text{nil}(R) \), which is 0 by the reducedness of \( R \), so that \( i \) is injective. The fact that \( S \) generates \( N^\vee \) as a group follows easily from the fact that the function fields of \( X \) and \( T_N \) coincide.

It is clear that the monoid is finitely generated if and only if the associated scheme is of finite type over \( k \). Thus we have proved part of the following proposition.

**Proposition 2.2.** The affine varieties \( X \) with an open immersion \( T_N \to X \) and an equivariant open immersion \( T_N \to X \) are precisely those of the form \( \text{Spec}(A_S) \), where \( S \) is a finitely generated submonoid of \( N^\vee \) which generates \( N^\vee \) as a group.

Moreover, the variety \( \text{Spec}(k[S]) \) is normal if and only if \( S \) is saturated\(^2\) in \( N^\vee \).

**Proof.** The proof of the second part can be found in [KKMSD]. \( \square \)

The relevance of convex geometry to torus actions begins with the next result.

**Proposition 2.3.** The saturated finitely generated submonoids of \( N^\vee \) which generate \( N^\vee \) are precisely the ones of the form \( \sigma^\vee \cap N^\vee \), where \( \sigma \) is an \( N \)-rational strongly convex polyhedral cone.

**Proof.** If \( S \subseteq N^\vee \) is such a monoid, we can find finitely many generators \( x_1, \ldots, x_n \) of \( S \). Let \( \sigma \) be the intersection of the half-spaces in \( \mathbb{R} \otimes_{\mathbb{Z}} N \) defined by \( x_i \geq 0 \). Then \( \sigma \) is a rational convex polyhedral cone by duality and A.7, and \( S \subseteq N^\vee \cap \sigma^\vee \) by definition. In fact, every element of \( \sigma^\vee \) can be written as \( \sum_i r_i x_i \) with \( r_i \in \mathbb{R}_{\geq 0} \).

Suppose that \( \sum_i r_i x_i \in N^\vee \cap \sigma^\vee \). Then we can write the \( r_i \) as \( r'_i + \varepsilon_i \), where \( r'_i \in \mathbb{Q}_{\geq 0} \) and \( \varepsilon_i > 0 \). Multiplying \( \sum_i r'_i x_i \) by a large enough positive integer gives an element of \( S \), so because \( S \) is saturated we see that \( \sum_i r'_i x_i \in S \) for all nonnegative rational \( r'_i \). As the rationals are dense in \( \mathbb{R} \), we can make the \( \varepsilon_i \) arbitrarily small, so since \( \sum_i \varepsilon_i x_i = \sum_i (r_i - r'_i) x_i \in N^\vee \), we find that they have to be 0. It follows that \( \sum_i r_i x_i = \sum_i r'_i x_i \in S \), so \( \sigma^\vee \cap N^\vee = S \). Because \( S \) generates \( N^\vee \) as a group, \( \sigma^\vee \) cannot be contained in a hyperplane, so that \( \sigma \) must be strongly convex.

The converse follows from the results in appendix A, in particular proposition A.9 paired with A.3. Note that the strong convexity of \( \sigma \) dualizes to the statement that \( \sigma^\vee \cap N^\vee \) generates \( N^\vee \) as a group. \( \square \)

This motivates the following definition.

**Definition 2.3.** An affine toric variety is one of the form \( X_\sigma = \text{Spec}(k[\sigma^\vee \cap N^\vee]) \), where \( \sigma \) is a strongly convex rational polyhedral cone in the lattice \( N \). We use \( A_\sigma \) to denote \( A_{N^\vee \cap \sigma^\vee} \), the coordinate ring of \( X_\sigma \), and we use \( S_\sigma \) to denote the monoid \( \sigma^\vee \cap N^\vee \).

**Remarks 2.4.**

\(^2\)Here, saturated means that if \( nr \in S \) for \( n \in \mathbb{Z}_{>0} \) and \( r \in N^\vee \), then \( r \in S \).
• The preceding results show that we can alternatively define toric varieties in terms of torus actions, i.e., an affine toric variety is an affine normal variety with an open embedded torus whose self-action extends to the entire variety. This definition explains the name toric variety, and admits an obvious generalization to the non-affine case. We opt for the convex/combinatorial point of view rather than the toric one, so we postpone the full definition of a toric variety to section 2.2.

• Strictly speaking, the notation $X_\sigma$ is ambiguous. For example, when $N' \subseteq N$ is a finite index sublattice, and $\sigma$ is an $N'$-rational cone in $N \otimes \mathbb{Z} = N' \otimes \mathbb{Z}$, then $X_\sigma$ could be taken to mean either of $\text{Spec}(k[\sigma^\vee \cap N^\vee])$ and $\text{Spec}(k[\sigma^\vee \cap (N')^\vee])$. Where necessary, we avoid ambiguity by adding the lattice as an additional subscript: $X_{\sigma,N}$. Similar statements hold for $A_\sigma$ and $S_\sigma$.

• It will frequently be useful to consider the functor of points of $X_\sigma$. Given a $k$-algebra $R$, we denote by $R^\text{mult}$ its multiplicative monoid, which is obviously functorial in $R$. As a functor on $k\text{-Alg}$, $X_\sigma$ is $R \mapsto \text{Hom}_{\text{mon}}(S_\sigma, R^\text{mult})$, where the subscript mon denotes the fact that this is the hom-set in the category of monoids.

If $\tau = \sigma \cap u^\perp$ is a face of $\sigma$, where $u \in \sigma^\vee$, then by proposition A.6, $A_\tau$ is the localization $(A_\sigma)_\chi^u$, so $X_\tau$ is a distinguished open affine of $X_\sigma$. We will use this later to glue affine toric varieties in order to obtain general toric varieties.

For morphisms more general than open immersions, suppose we have a cone $\sigma$ in $N$, and a cone $\tau$ in $M$. Then a morphism from the pair $(\sigma, N)$ to the pair $(\tau, M)$ is a group homomorphism $f:N \to M$ such that $\text{id}_R \otimes f$ maps $\sigma$ into $\tau$. Dualizing, we obtain a homomorphism $A_{\tau,M} \to A_{\sigma,N}$, and hence $f_*:X_{\sigma,N} \to X_{\tau,M}$. As $f^\vee$ clearly preserves the gradings in the sense that the homogeneous degree $n \in M^\vee$ part of $A_{\tau,M}$ is mapped into the degree $f^\vee(m)$ part of $A_{\sigma,N}$, we find that $f_*$ is equivariant in the sense that $f_*(tx) = f_*(t)f_*(x)$ on points $t$ of $T_N$ and $x$ of $X_{\sigma,N}$.

The pairs $(\sigma, N)$ clearly form a category $\mathcal{C}$ ("C" as in "cone") with morphisms described as above. The duality theorem A.3 gives an equivalence $\mathcal{C} \to \mathcal{C}^{\text{opp}}$, so $\mathcal{C}$ is self-dual. If we have $(\sigma_1, N_1), (\sigma_2, N_2) \in \mathcal{C}$, then we can form their categorical product in the obvious way, namely $(\sigma_1 \times \sigma_2, N_1 \times N_2)$. It is easy to verify that this gives either the categorical product or coproduct, and then you get the fact that it is a coproduct or product for free by duality.

**Proposition 2.5.** The map $(\sigma, N) \mapsto X_{\sigma,N}$ gives a covariant functor from $\mathcal{C}$ into the category of $k$-varieties. This functor preserves products.

**Proof.** The only part we have not yet seen is the preservation of products. Let $(\sigma_1, N_1), (\sigma_2, N_2) \in \mathcal{C}$ with corresponding $k$-algebras $A_1$ and $A_2$, and let $A$ be the algebra of their product in $\mathcal{C}$. There is an obvious $k$-algebra homomorphism $\Phi:A_1 \otimes_k A_2 \to A$ given by sending $\chi^u \otimes \chi^v$ to $\chi^{u+v}$, where $u \times v$ is the map $N_1 \times N_2 \to \mathbb{Z}$ induced by $u$ and $v$. Because the elements of the form $\chi^u \otimes \chi^v$ give a $k$-basis of $A_1 \otimes_k A_2$, and because $\chi^{u+v} \neq 0$ for all such $u$ and $v$, we see that $\Phi$ is injective. Furthermore, any $w \in (\sigma_1 \times \sigma_2)^\vee$ gives rise to $u \in \sigma_1^\vee$ and $v \in \sigma_2^\vee$ by precomposing with the inclusions $n_1 \mapsto (n_1, 0)$ and $n_2 \mapsto (0, n_2)$. In this case $w = u \times v$, so because the elements of the
form $\chi^w$ span $A$, we see that $\Phi$ is surjective, so that $A \cong A_1 \otimes_k A_2$, which was to be shown.

Remark 2.6. There is an obvious category $A$ of $k$-varieties with actions of tori, fibered over the category $T$ of $k$-tori (with algebraic group homomorphisms as morphisms). The category $C$ is fibered over the category $T$ of lattices (recall the equivalence of lattices and tori established in appendix B). The functor in the proposition above obviously factors through a functor of fibered categories $C \to A$. From the characterization of $T_N$-actions and $T_N$-equivariant maps as $N^\vee$ gradings and homogeneous maps, it is clear that this functor is fully faithful. It is shown in [KKMSD] that it is an equivalence of $C$ with the category of affine normal varieties $X$ containing an open torus whose self-action extends to all of $X$.

We can use the limits introduced in section B.3 to define some important distinguished points in toric varieties. As in appendix B, the lattice $N$ is identified with the group of one-parameter subgroups of $T_N$. So since $T_N$ is embedded in $X_{\sigma,N}$, we can associate with each element $a$ of $N$ a morphism $\lambda_a: \mathbb{G}_m \to X_{\sigma,N}$. Let $u_1, \ldots, u_r$ be generators of $S_{\sigma}$, so that $\chi^{u_1}, \ldots, \chi^{u_r}$ are global coordinates on $X_{\sigma,N}$. By proposition B.4, $\lim_{t \to 0} \lambda_a(t)$ exists if and only if the $\lim_{t \to 0} \chi^{u_i}(t)$ exist. On points, $\chi^{u_i}(a)$ is given by $t \mapsto t^{u_i}(a)$. The existence of $\lim_{t \to 0} \chi^{u_i}(t)$ is therefore equivalent to $u_i(a) \geq 0$, so $\lim_{t \to 0} \lambda_a(t)$ exists in $X_{\sigma,N}$ if and only if $a \in \sigma \cap N$. The distinguished points of $X_{\sigma,N}$ will be defined in terms of these limits.

We need the following lemma.

Lemma 2.7. An element $a$ of $N$ is in $\sigma$ if and only if $\lim_{t \to 0} \lambda_a(t)$ exists in $X_{\sigma}$, and $a, b \in \sigma \cap N$ give rise to the same limit $\lambda_a(0) = \lambda_b(0)$ if and only if $a$ and $b$ are in the interior of the same face of $\sigma$.

Proof. If $u_1, \ldots, u_r$ generate $S_{\sigma}$, then, as we saw above, the limit $\lambda_a(0)$ is given by

$$\chi^{u_i}(0) = \begin{cases} 1 & \text{if } u_i(a) = 0 \\ 0 & \text{if } u_i(a) \neq 0. \end{cases}$$

The smallest face containing $a$ is defined by the vanishing of those $u_i$ for which $u_i(a) = 0$, so it follows that this face is determined by $\lambda_a(0)$, proving the lemma.

This allows us to define the distinguished point of a face $\tau$ as $x_\tau = \lambda_a(0)$, where $a$ is any interior point of $\tau$. From the definition of the limit it immediately follows that $x_\tau$, viewed as a monoid homomorphism from $S_{\sigma}$ to $k^{\text{mult}}$, is given by

$$u \mapsto \begin{cases} 1 & \text{if } u \in \tau^\perp \\ 0 & \text{if } u \not\in \tau^\perp. \end{cases}$$

The importance of these points stems from the fact that they give a bijection between the faces of $\sigma$ and the orbits of the $T_N(k)$ action on $X_{\sigma}(k)$, as the following proposition shows.
Proposition 2.8. The map $\tau \mapsto \text{orb}(\tau)$, where $\text{orb}(\tau) = T_N(k) x_\tau$, is the $T_N(k)$-orbit of $x_\tau$, sets up a bijection between the faces of $\sigma$ and the $T_N(k)$-orbits of $X_\sigma(k)$.

Proof. From equation (2.1) and the description of $k$-points of $T_N$ as group homomorphisms $N^r \rightarrow k^\times$, we gather that distinct distinguished points are not in the same orbits, so the injectivity of $\tau \mapsto \text{orb}(\tau)$ follows.

We now describe the orbits of the $T_N$-action on $X_\sigma$ from the functorial point of view. For a face $\tau$ of $\sigma$, denote by $\text{orb}(\tau)$ the functor $R \mapsto \text{Hom}(\tau \cap N^r, R^\times)$, which is clearly representable by a torus. The surjection $A_\sigma \rightarrow k[\tau \cap N^r]$ given by mapping elements $\chi^u$ with $u$ outside of $\tau$ to $0$, presents $\text{orb}(\tau)$ as a closed subvariety of $X_\tau$. Composing this with the open immersion $X_\tau \rightarrow X_\sigma$ gives an immersion of $\text{orb}(\sigma)$ into $X_\sigma$.

We must show that $\text{orb}(\tau)$ is a $T_N$-orbit of $X_\sigma$ in the sense that $\text{orb}(\tau)(R)$ is an orbit in $X_\sigma(R)$ for all $R$. Since $X_\tau(R) \subseteq X_\sigma(R)$ is closed under the $T_N(R)$ action, we may as well show that $\text{orb}(\tau)$ is an orbit of $X_\tau$, so we assume $\tau = \sigma$. The action of $\phi \in T_N(R)$ on $\psi \in X_\tau(R)$ is given by restricting $\phi$ to $S_\sigma$ and multiplying it with $\psi$. The elements of $\text{orb}(\sigma)(R) \subseteq X_\sigma(R)$ are those $\psi$ which vanish outside of $\sigma^\perp$, from which it immediately follows that $\text{orb}(\sigma)(R)$ is closed under the $T_N(R)$-action, and that the $T_N(R)$ action is transitive on $\text{orb}(\sigma)(R)$, proving that $\text{orb}(\sigma)(R)$ is indeed an orbit (necessarily that of $x_\sigma$).

Let $\phi: S_\sigma \rightarrow k^{\text{mult}}$ be an element of $X_\sigma(k)$. With $\phi$ we associate the set $C = \{ u \in \sigma^\perp \mid \phi(u) \neq 0 \}$, and we let $\tau'$ be the cone in $\sigma^\perp \otimes \mathbb{R}$ generated by $C$. Now because $\phi$ is a homomorphism of monoids, we have $u(0) = 1 \neq 0$, so that $0 \in \tau'$. And if $u \in S_\sigma \setminus C$ and $v \in S_\sigma$, then $\phi(u + v) = \phi(u)\phi(v) = 0$, so that $u + v \notin C$. From proposition A.12 it follows that $\tau'$ is a face of $\sigma^\perp$, so that $\tau' = \sigma^\perp \cap \tau^\perp$ for some face $\tau$ of $\sigma$. Now by definition of $\tau$ we have $\phi \in \text{orb}(\tau)$, showing that the assignment $\tau \mapsto \text{orb}(\tau)$ is surjective onto the set of $T_N(k)$-orbits in $X_\sigma(k)$.

Due to their obvious relevance to the resolution of singularities, we now classify nonsingular affine toric varieties.

Proposition 2.9. An affine toric variety $X_{\sigma,N}$ is nonsingular if and only if $\sigma$ is generated by part of a basis for $N$. Moreover, if it is nonsingular, $X_{\sigma,N} \cong \mathbb{G}_m^{r-k} \times_k k^r$ for some $0 \leq r \leq \text{rk } N$.

Proof. If $e_1, \ldots, e_n$ is a basis of $N$, and $\sigma$ is generated by $e_1, \ldots, e_r$, then we can decompose $N$ into $N_1$ generated by $e_1, \ldots, e_r$, and $N_2$ generated by $e_{r+1}, \ldots, e_n$. Then $(\sigma,N) = (\sigma, N_1) \times (0, N_2)$, so $X_{\sigma,N} = X_{\sigma,N_1} \times_k X_{0,N_2}$, which is just the product of $k^r$ with $\mathbb{G}_m^{n-r}$, and hence nonsingular.

Conversely, suppose that $X_{\sigma,N}$ is nonsingular, and let $N_1$ be the sublattice in $N$ spanned by $\sigma \cap N$. Pick $N_2$ such that $N = N_1 \oplus N_2$, so we get $\sigma$ as a product of a cone $\sigma_1$ in $N_1$ with the zero cone in $N_2$. Now $X_{\sigma,N} = X_{\sigma_1,N_1} \times_k T_{N_2}$, so $X_{\sigma_1,N_1}$ is nonsingular, so we may assume that $\sigma \cap N$ generates $N$.

Let $u_1, \ldots, u_m \in S_\sigma$ be a minimal set of generators, and let $\phi$ be the surjective map $k[t_1, \ldots, t_m] \rightarrow A_\sigma$ given by $t_i \mapsto u_i$. Two non-constant monomials $t_1^{a_1} \cdots t_m^{a_m}$ and $t_1^{b_1} \cdots t_m^{b_m}$ map to the same element of $A_\sigma$ if $\sum_i a_i u_i = \sum_i b_i u_i$. An easy argument using
the fact that the $\chi^u$ are linearly independent, the fact that $0$ is a face of $\sigma^\vee$, and proposition A.12, shows that this can only happen if $\sum_i a_i \geq 2$ and $\sum_i b_i \geq 2$. From this one obtains that the kernel of $\phi$ is contained in $(t_1, \ldots, t_n)^2$.

Consider the distinguished point $x_\sigma$. From equation (2.1) it follows that $x_\sigma$ maps into the origin of $\mathbb{A}^m_k$ under the closed immersion $X_\sigma \to \mathbb{A}^m_k$ given by $\phi$. The induced map on cotangent spaces

$$(t_1, \ldots, t_m)/(t_1, \ldots, t_m)^2 \to T_{x_\sigma}^\vee X_\sigma$$

is surjective by definition of $\phi$, and injective by the fact that the kernel of $\phi$ is contained in $(t_1, \ldots, t_m)^2$. Therefore the $m$ equals the dimension of $T_{x_\sigma}^\vee X_\sigma$, which is $\dim X_\sigma = \text{rk} N$ by the nonsingularity of $X_\sigma$ and the fact that $S_\sigma$ generates $N^\vee$ as a group. Since the $u_i$ generate $S_\sigma$ and $S_\sigma$ generates $N$, it follows that the $u_i$ are a basis for $N^\vee$, from which it follows that $\sigma$ is of the required form.

The following theorem shows that toric varieties are Cohen-Macaulay\footnote{See section 3.1 for the definition of Cohen-Macaulay varieties.}. This is part of the reason why the theory of noncommutative crepant resolutions apply to them, as these only make sense for Cohen-Macaulay varieties.

**Theorem 2.10.** *Affine toric varieties are Cohen-Macaulay.*

**Proof Sketch.** Cohen-Macaulay rings are those rings for which all of Serre’s $S_n$ conditions hold. In the two dimensional case, the conditions $S_n$ for $n \geq 3$ are trivial, as there are no height $\geq 3$ primes. Being normal, toric surfaces satisfy $S_2$ by Serre’s criterion for normality, so in particular they are Cohen-Macaulay. In general, one notes that affine toric varieties can be written as a quotient of an open subset of some $\mathbb{A}^n_k$ by an action of a torus (see [C2]), which implies that they are Cohen-Macaulay by the Hochster-Roberts theorem.

**Remarks 2.11.**

- Being Cohen-Macaulay is a local property, so this theorem shows that general toric varieties are Cohen-Macaulay as well.

- An even stronger fact holds. Namely, all toric varieties have rational singularities. See [O] for a proof.

### 2.2. Toric Varieties

General toric varieties are obtained from so-called fans\footnote{The kind of fan used to induce airflow for the purpose of cooling oneself or others, after the French évantail, coined by Demazure.}, which we now define.

**Definition 2.12.** As usual, let $N$ be a finite rank lattice. A fan $\Sigma$ in $N$ is defined as a finite collection of strongly convex rational polyhedral cones in $N$ satisfying the following two conditions:
• if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of both $\tau$ and $\sigma$;
• if $\tau$ is a face of $\sigma \in \Sigma$, then $\tau \in \Sigma$.

A morphism of fans $f: (\Sigma, N) \to (T, M)$ consists of a lattice homomorphism $f: N \to M$ such that for each $\sigma \in \Sigma$, there exists $\tau \in T$ for which $(\text{id}_\mathbb{R} \otimes f)(\sigma) \subseteq \tau$. This defines the category $\mathcal{F}$ of fans, which is clearly fibered over the category $\mathcal{T}$ of lattices.

Remarks 2.13.

• As we noted earlier, these objects arise naturally when you consider normal varieties with an embedded torus whose self-action extends to the whole variety, but the equivalence of the two points of view in the non-affine case is more complicated. For a proof of this equivalence, see [KKMSD].

• The functor of points of a toric variety is slightly harder to describe in the non-affine case. In [C1], Cox gives a description of $X_\Sigma(Y)$ as a set of equivalence classes of certain families $\{L_\rho, u_\rho \in \Gamma(Y; L_\rho)\}_{\rho \text{ edge in } \Sigma}$ of line bundles and sections, required to satisfy compatibility conditions given in terms of the fan $\Sigma$. The nonsingularity comes into play in constructing a universal such collection in $X_\Sigma(X_\Sigma)$, where the Weil divisors associated to edges in $\Sigma$ give line bundles on $X_\Sigma$ (see the discussion right before theorem 2.16). In [AMRT], a generally valid description of $X_\Sigma(Y)$ is given in terms of sheaves of monoids on $Y$ whose stalks are isomorphic to the monoids associated with cones in $\Sigma$.

Example 2.14. A simple example of a fan is the fan $\Sigma(\sigma)$ associated to a cone $\sigma$. It consists of the faces of $\sigma$, and the fact that this does indeed give a fan follows from parts 2, 3, and 4 of proposition A.2. This embeds $\mathbb{C}$ in $\mathcal{F}$.

To construct a variety from a fan, recall that a face $\tau$ of $\sigma$ gives an open immersion $X_\tau \to X_\sigma$. As the intersection of two cones $\sigma_1, \sigma_2$ in $\Sigma$ is a face of both, we can glue $X_{\sigma_1}$ and $X_{\sigma_2}$ along the associated open subscheme. The result is an integral normal finite type $k$-scheme which we denote $X_\Sigma$ (or $X_{\Sigma, N}$ in cases where the lattice is not clear).

Proposition 2.15. The assignment $(\Sigma, N) \mapsto X_{\Sigma, N}$ gives a covariant functor from the category of fans into the category of $k$-varieties.

Proof. The functoriality follows by gluing the morphisms obtained by the functoriality for affine toric varieties.

The only thing left to prove is that $X_\Sigma$ is separated. If $\sigma_1, \sigma_2 \in \Sigma$ have nonempty intersection, then we can apply corollary A.11 to immediately conclude that the diagonal map $A_{\sigma_1} \otimes_k A_{\sigma_2} \to A_{\sigma_1 \cap \sigma_2}$ is surjective, i.e., $X_{\sigma_1 \cap \sigma_2} \to X_{\sigma_1 \times_k X_{\sigma_2}}$ is a closed immersion. This implies that $X_\Sigma$ is separated.\[\square\]

The $N^\vee$-gradings on the coordinate rings of the affine opens $X_\sigma$ for $\sigma \in \Sigma$ obviously agree, so we obtain an action of $T_N = X_0 \subseteq X_\Sigma$ on $X_\Sigma$. The torus action on $X_0 \subseteq X_\Sigma$ is just the action of $T_N$ on itself, so we have a normal variety with an open embedded torus
whose self-action extends to the whole variety. The converse is the hard part, which we will not show. As in the affine case (remark 2.6), the functor from $F$ into the category of varieties factors through a fully faithful fibered functor $F \rightarrow A$.

As the open affines $X_\sigma$ for $\sigma \in \Sigma$ are $T_N$-invariant, the description of the orbits in proposition 2.8 immediately gives us the $T_N$-orbits in $X_\Sigma$. That is,

$$\Sigma \ni \sigma \mapsto \text{orb}(\sigma) \subseteq X_\sigma \subseteq X_\Sigma$$

is a bijection between $\Sigma$ and the $T_N$-orbits of $X_\Sigma$. For $\sigma \in \Sigma$, the dimension of $\sigma^\perp \subseteq \mathbb{R} \otimes \mathbb{Z} N^\vee$ is $\text{rk } N - \dim \sigma$, by the rank-nullity theorem. It follows that

$$\dim \text{orb}(\sigma) = \text{rk } N - \dim \sigma = \dim X_\Sigma - \dim \sigma.$$ 

In particular, since the orbits are irreducible, the edges $\rho \in \Sigma$ (the cones of dimension 1) give rise to $T_N$-invariant Weil divisors $D_\rho := \text{orb}(\rho)$. It can be shown that all $T_N$-invariant divisors are $\mathbb{Z}$-linear combinations of these (see [F]). This allows us to describe the canonical divisor of a toric variety.

**Theorem 2.16 ([F], section 4.3).** The canonical sheaf $\omega_{X_\Sigma}$ of $X_\Sigma$ is $\mathcal{O}_{X_\Sigma}(-\sum \rho D_\rho)$, the sheaf of rational functions $f$ with $\text{div}(f) \geq \sum \rho D_\rho$. Here the sum ranges over all cones in $\Sigma$ of dimension 1. For affine toric varieties $X_\sigma$, this gives

$$\omega_{X_\sigma} = \bigoplus_{u \in \text{int}(\sigma^\vee) \cap N^\vee} k\chi^u,$$

where $\text{int}(\sigma)$ denotes the interior of $\sigma$.

In section 3.1 we give a homological definition of a local Gorenstein ring by requiring its injective dimension as a module over itself to be finite. This is closely tied to Grothendieck duality, which connects it to the canonical sheaf. In fact, for varieties, being Gorenstein is the same as having a line bundle as canonical sheaf. Thus we obtain the following corollary.

**Corollary 2.17.** An affine toric variety $X_\sigma$ is Gorenstein if and only if the ideal $\omega_{X_\sigma}$ is principal, that is,

$$\text{int}(\sigma^\vee) \cap N^\vee = u + S_\sigma$$

for some $u \in \sigma^\vee$.

We end with a combinatorial characterization of proper equivariant maps between toric varieties, the relevance of which to resolutions of singularities should be clear.

**Proposition 2.18.** Let $f:(T,M) \rightarrow (\Sigma,N)$ be a map of fans. The induced morphism $\phi:X := X_{T,M} \rightarrow X_{\Sigma,N} =: Y$ is proper if and only if $f^{-1}(\Sigma) = |T|$. 

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Proof. Suppose that \( \phi \) is proper, and let \( b \in |T| \cap M \) map to some \( a \in |\Sigma| \cap N \). If \( \lambda \) is the one-parameter subgroup of \( T_N \) corresponding to \( a \), then the limit \( \lim_{t \to 0} \lambda(t) \) exists in \( Y \) by lemma 2.7, i.e., \( \lambda \) extends to a map \( \mathbb{A}^1_k \to Y \). Now we have a commutative diagram

\[
\begin{array}{ccc}
G_m, k & \to & X \\
\uparrow \text{inclusion} & & \downarrow \\
\mathbb{A}^1_k & \to & Y
\end{array}
\]

where \( \gamma \) is the one parameter subgroup of \( T_M \) corresponding to \( b \), composed with the inclusion \( T_M \to X \). Proposition B.5 combined with lemma 2.7 now shows that \( b \in |T| \cap M \). From the rationality of the cones in \( T \) and \( \Sigma \) it follows that \( f^{-1}(|\Sigma|) = T \).

Now assume that \( f^{-1}(|\Sigma|) = |T| \). We will check the (discrete) valuative criterion for properness for \( \phi \). Let \( R \) be a dvr with field of fractions \( K \) and valuation \( v \), and suppose we are given a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \to & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \to & Y
\end{array}
\]

To show the properness of \( \phi \), we need to extend \( \gamma \) to \( \text{Spec}(R) \). By the proof of tag 0894 of [SP], we may assume that \( \gamma \) lands in \( T_M \subseteq X \), and since properness is local on the base, we may assume \( Y = \text{Spec}(A_\sigma) \). Therefore we are faced with the commutative diagram of rings

\[
\begin{array}{ccc}
K & \leftarrow & k[M^\vee] \\
\uparrow & & \uparrow g \\
R & \leftarrow & A_\sigma
\end{array}
\]

The morphism \( k[M^\vee] \to K \) corresponds to a homomorphism of monoids \( h : M^\vee \to K^\times \). Since \( A_\sigma \to K \) maps into \( R \), it follows that \( vhg \) is nonnegative on \( S_\sigma \). This means that \( vhg|_{\sigma \cap N^\vee} \) is in \( \sigma^\vee \), and because \( g \) is given by \( f \), it follows that \( vhg \) is the image of \( vh \in M \) under \( f \). By the assumption on \( f \), there exists a \( \tau \in T \) such that \( f(\tau) \subseteq \sigma \) and such that \( vh \in \tau \). But this means that the image of \( A_\tau \to k[M^\vee] \to K \) is in \( R \), which gives us the desired extension of \( \gamma \) to \( \text{Spec}(R) \).

\[
\square
\]

2.3. Toric Resolutions

We now turn to combinatorial methods to resolve toric singularities. As toric varieties are normal, their singular loci are of codimension \( \geq 2 \), so the one-dimensional case is trivial (in fact, it is easy to see that the only 1-dimensional toric varieties are \( \mathbb{A}^1 \), \( \mathbb{G}_m \), \( \mathbb{G}_a \), etc.).
and \( \mathbb{P}^1 \). Therefore we start with the two-dimensional case, which is still manageable, despite being non-trivial.

Let us first classify the cones whose toric varieties are nonsingular and two-dimensional. We consider a cone \( \sigma \) in \( \mathbb{Z}^2 \). If \( \dim \sigma \neq 2 \), then it is clear that \( \sigma \) is either 0 or a half-line, which correspond to \( \mathbb{G}^2_m \) and \( \mathbb{G}_m \times \mathbb{A}^1_k \), respectively. The remaining cases are of the form \( \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2 \) with \( v_1, v_2 \) a basis of \( \mathbb{Z}^2 \) (these correspond to \( \mathbb{A}^2_k \)). We claim that \( v_1, v_2 \in \mathbb{Z}^2 \) is a basis if and only if the area of the triangle spanned by 0, \( v_1 \), and \( v_2 \) is 1/2. Clearly this is true for the standard basis \( e_1, e_2 \). Any basis can be transformed into any other using an element of \( g \in \text{GL}_2(\mathbb{Z}) \). Such \( g \) have unit determinant in \( \mathbb{Z} \), so they preserve the area of the triangle spanned by 0, \( v_1 \), and \( v_2 \). The claim follows.

**Proposition 2.19.** The equivariant isomorphism classes of affine two-dimensional toric varieties with singularities are in bijection with the set

\[
S = \{(a, b) \in \mathbb{Z}^2 \mid 0 < a < b \text{ and } \gcd(a, b) = 1\}.
\]

**Proof.** Let \( \sigma = \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2 \subseteq \mathbb{R}^2 \) be a singular cone, with \( v_1, v_2 \in \mathbb{Z}^2 \) primitive in the sense that \( \mathbb{Z} v_1 \) is saturated in \( \mathbb{Z}^2 \). As \( v_1 \) is primitive, we may assume it is \( e_1 = (1, 0) \) (we can extend \( v_1 \) to a basis \( v_1, v'_1 \) of \( \mathbb{Z}^2 \), and then the map \( v_1 \mapsto e_1 \) and \( v'_1 \mapsto e_2 \) gives an isomorphism of \( X_\sigma \). Let \( v_2 = (a, b) \). The isomorphism \( e_i \mapsto (-1)^{i+1} e_i \), which preserves \( v_1 \), shows that we may assume that \( b > 0 \). The characterization of bases of \( \mathbb{Z}^2 \) shows that \( b > 1 \), since otherwise the area of the convex hull of \( v_1, v_2 \), and 0 would be 1/2 (and vice versa). The matrix \( \begin{pmatrix} 1 & n \vspace{1.5ex} \\ 0 & 1 \end{pmatrix} \) maps \( v_2 \) to \( (a + nb, b) \), which shows that adding any multiple of \( b \) to \( a \) leaves the area invariant, so we may assume \( 0 \leq a \leq b \). The fact that \( v_2 \) is primitive means that \( \gcd(a, b) = 1 \) (which of course implies \( 0 < a < b \)).

This shows that mapping \( (a, b) \) to the toric variety of \( \sigma_{(a,b)} := \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(a,b) \) gives a surjective map from \( S \) to the set of isomorphism classes of two-dimensional toric varieties with singularities.

To show injectivity, note that \( v, w \in S \) map to the same isomorphism class if there exists \( g \in \text{GL}_2(\mathbb{Z}) \) such that \( \text{id} \otimes g \) maps \( \sigma_v \) onto \( \sigma_w \). As such a \( g \) leaves the area of the parallelogram spanned by \((1,0)\) and \( v =: (a,b) \) invariant, and because this area is \( b \) (base times height), we are done.

Duality of cones gives an involution \( S \to S \). Dualizing the cone \((a,b) \in S \) yields the cone spanned by \((0,1)\) and \((b,-a)\). Now applying the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) \) shows that the cone \( \mathbb{R}_{\geq 0}(b,-a) + \mathbb{R}_{\geq 0}(0,1) \) corresponds to \((b-a,b)\in S \). Therefore, duality is given on \( S \) as \((a,b) \mapsto (b-a,b)\).

Under the assumption\(^5\) that \( k \) is algebraically closed and has characteristic coprime to \( b \), we now describe these varieties \( X := X_{\sigma_{(a,b)}}N \) as quotients of \( \mathbb{A}^2_k \) by an action by a finite cyclic group. The trick is to consider a lattice \( M \subseteq N \) in which \( \sigma_{(a,b)} \) becomes nonsingular. Take \( M \) to be the lattice based by \((1,0)\) and \((a,b)\), or equivalently,

\(^5\)This assumption guarantees that \( \mu_b \) is isomorphic to its Cartier dual (the constant group scheme over \( k \) associated to \( \mathbb{Z}/b\mathbb{Z} \)), giving us the easy characterization of a group action of \( \mu_b \) on an affine \( k \)-scheme \( \text{Spec}(R) \) as a group homomorphism \( \mathbb{Z}/b\mathbb{Z} \to \text{Aut}_k(R) \).
\(M = \mathbb{Z} \oplus b \mathbb{Z} \subseteq N\). As \((1,0)\) and \((a, b)\) are precisely the primitive vectors along the two edges of \(\sigma(a, b)\), we find that \(\sigma(a, b)\) is non-singular w.r.t. \(M\), so \(Y := X_{\sigma(a, b), M}\) is isomorphic to \(\mathbb{A}^2_k\). The inclusion \(M \to N\) therefore gives us a morphism \(Y \to X\), which will be the quotient map.

Since \(M\) is a subgroup of \(N\) of index \(b\), the quotient will be isomorphic to \(\mathbb{Z}/b\mathbb{Z}\), and is generated by \(\overline{e_2} = e_2 + M\). Dually, we have an inclusion \(N^\vee \to M^\vee\) \((M \to N\) is an epimorphism as in proposition B.2, so its dual is a monomorphism), the cokernel of which is also isomorphic to \(\mathbb{Z}/b\mathbb{Z}\). Viewing both \(N^\vee\) and \(M^\vee\) as subgroups of \(N^\vee \otimes \mathbb{R} = \mathbb{R} e_1^\vee \oplus \mathbb{R} e_2^\vee\) in the natural way, where \(e_1^\vee, e_2^\vee\) is the dual basis of \(e_1, e_2\), we have \(M^\vee = \mathbb{Z} e_1^\vee \oplus b \mathbb{Z} e_2^\vee\), so the quotient \(M^\vee/N^\vee\) is generated by \(\frac{1}{b} e_2^\vee + N^\vee\). It follows that there is a natural bi-additive pairing \(\langle \cdot, \cdot \rangle\): \((M^\vee/N^\vee) \times (N/M) \to \mathbb{Z}/b\mathbb{Z}\) given by \((\phi + N^\vee, v + M) \mapsto b\phi(v) = \langle \phi, v \rangle\). Now we have an action of \(\mu_b = \text{Spec}(k[x]/(x^b - 1))\) on \(Y\) given by

\[
\chi^\phi \mapsto \chi^\phi \otimes x^{\langle \phi, g \rangle},
\]

where \(g\) is some generator of \(N/M\) (we take \(g = \overline{e_2}\)). As this action clearly preserves the \(M^\vee\)-grading, the ring of invariants \(k[\sigma^\vee \cap M^\vee]^{\mu_b}\) is generated by monomials. For \(u \in \sigma^\vee \cap M^\vee\), the monomial \(\chi^\phi\) is \(\mu_b\)-invariant if and only if

\[
\chi^u \otimes x^{\langle u, g \rangle} = \chi^u \otimes 1,
\]

i.e., if and only if \(bu(e_2)\) is an integer multiple of \(b\). As \(u\) is in \(M^\vee\), it can be written as \(me_1^\vee + \frac{n}{b} e_2^\vee\) for some \(m, n \in \mathbb{Z}\), so the statement \(bu(e_2) \in \mathbb{Z}\) just means that \(m/b \in \mathbb{Z}\), which is the same as \(u \in N^\vee\). This shows that \(k[\sigma^\vee \cap M^\vee]^{\mu_b}\) is precisely \(k[\sigma^\vee \cap N^\vee]\), so we’ve exhibited \(X\) as a quotient of \(\mathbb{A}^2_k\) by a \(\mu_b\)-action.

Taking the generators \((e_1^\vee - \frac{b}{2} e_2^\vee, \frac{1}{b} e_2^\vee)\) of the dual of \(\sigma(a, b)\) w.r.t. \(M\) as the standard coordinates \((x, y)\) on \(Y\), and letting \(\xi \in \mu_b(k)\) be a primitive \(b\)th root of unity, we find that the \(\mu_b\) action is given on points by \(\xi \cdot (x, y) = (\xi^{-a} x, \xi y)\).

**Proposition 2.20.** If \(k\) has characteristic 0, then the 2-dimensional affine toric Gorenstein singularities are precisely the rational double points of type \(A_n\), where the \(A_n\) singularity is realized by the cone \(\sigma_{A_n} := \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(1, n+1)\) in \(\mathbb{Z}^2\).

**Proof.** It is immediately clear from the description above of the \(\mu_b\)-action on points that these toric surfaces are indeed the rational double points of type \(A_n\).

Let \(\sigma\) correspond to \((b-a, b) \in S\), and let \(\omega\) be its canonical module, as described in corollary 2.17. We identify \(\sigma^\vee\) with the cone associated to \((a, b) \in S\). Consider the parallelogram \(P\) spanned by \((0,0)\), \((a, b)\) and \((1,0)\). As \((1,1) = \frac{1}{2}(a, b) + (1 - \frac{b}{2})(1,0)\), we always have \((1,1) \in P\). The interior of \(P\) is clearly contained in \(\omega\), so if \(\omega\) is principal, it follows that \((1,1)\) must be its generator.

According to Pick’s theorem, the number of interior lattice points of \(P\) is equal to \(A - \frac{1}{2}B + 1\), where \(A\) is the area of \(P\), and \(B\) is the number of lattice points on the
boundary of $P$. Since $a$ and $b$ are coprime, the lattice points on the boundary of $P$ are precisely $(0,0), (1,0), (a,b)$, and $(a+1,b)$, so $B = 4$. The area of a parallelogram is its base times its height, so $A = b$, from which it follows that $P$ has $b - 1$ interior lattice points.

It is easy to verify that $(x,x)$ is in the interior of $P$ if and only if $0 < x < \frac{b}{b-a}$, so $(1,1)$ through $(b-1,b-1)$ are in the interior of $P$ if and only if $b - a = 1$. When $b - a > 1$, Pick’s theorem therefore provides us with an integral element $(x,y)$ of the interior of $P$ which is linearly independent of $(1,1)$.

We know that $(1,1)$ generates $\omega$, and $(x,y) \in \omega$, so there is an element $u \in \sigma^\vee \cap N^\vee$ such that $u + (1,1) = (x,y)$. It is clear that the only elements $u \in \sigma^\vee \cap N^\vee$ for which $u + (1,1) \in P$ is possible are the $u$ in $P \cap \omega$. So from $u + (1,1) = (x,y)$ we obtain that $u$ is linearly independent from $(1,1)$, and inside $P \cap \omega$. The coordinates of $u$ are necessarily smaller than those of $(x,y)$, so by taking $(x,y)$ minimal we arrive at a contradiction.

It follows that the only affine toric surfaces which could be Gorenstein are the $A_n$ singularities (namely the points $(b-a,b) \in S$ satisfying $b - a = 1$). Of course, one can invoke proposition 1.6 to prove that these are indeed Gorenstein, but it easily verified by hand that $(1,1)$ generates $\omega$ in this case.

We now set out to resolve the $A_n$ singularities in an equivariant (or toric) manner. The idea is to subdivide $\sigma = \sigma_{A_n}$ in $N = \mathbb{Z}^2$ into smaller cones which are nonsingular.

The result will be a nonsingular fan in $N$, and the identity on $N$ will give a map $f$ from $X_\Sigma$ into $X := X_\sigma$. As it comes from the identity on $N$, $f$ will be the identity on the dense torus inside $X_\Sigma$, so $f$ is birational. Because $|\Sigma| = |\sigma|$, proposition 2.18 shows that $f$ is proper, and therefore a resolution of singularities. This idea works in greater generality, of course, and the two-dimensional case is worked out in [KKMSD]. The higher dimensional case works essentially the same way, but is harder to work out explicitly because of the simple fact that the difficulty of convex geometry increases steeply as we go beyond the two-dimensional case.

Consider the two-dimensional fan $\Sigma$ in $N$ spanned by the edges $(1,0), (1,1), \ldots, (1,n+1)$. The two-dimensional cones in $\Sigma$ are $\sigma_i = \mathbb{R}_{\geq 0}(1,i) + \mathbb{R}_{\geq 0}(1,i+1)$ for $0 \leq i \leq n$. Since $(1,i), (1,i+1)$, and $(0,0)$ form a triangle of area $1/2$, the discussion at the beginning of this section shows that $\sigma_i$ is a nonsingular cone, so that $Y = X_\Sigma$ is nonsingular. It follows from the previous paragraph that the natural map $Y \to X$ is an equivariant resolution of singularities.
3. Noncommutative Crepant Resolutions

In this chapter, we outline the basics of noncommutative crepant resolutions (NCCRs), as first defined by van den Bergh [vdB], focusing on toric singularities and dimension two in particular. Our main references are Iyama and Wemyss’ [IW2], and the lecture notes [W2]. Due to the technical nature of the subject, we start off with a section of algebraic preliminaries.

Throughout this chapter, $R$ is a commutative Noetherian $k$-domain, where $k$ is an algebraically closed field of characteristic 0. Furthermore, to avoid confusion of the concepts of finitely generated $R$-modules and finitely generated $R$-algebras, we call an $R$-module $\Lambda$ finite if it is finitely generated as an $R$-module, and we call it finitely generated if it is finitely generated as an $R$-algebra (if it is an $R$-algebra in the first place).

3.1. Algebraic Preliminaries

We gather here, mostly without proofs, commutative algebraic and homological results needed for the theory of noncommutative resolutions. We focus in particular on the notions of Cohen-Macaulay rings and modules, Gorenstein rings, and the relation between these and regularity. As we are interested in surface singularities, we prove certain theorems in the special case of dimension 2, where simpler arguments can be given. The references for this preliminary section are [BH], [W1], and [E], and proofs for most, if not all, results can be found there.

3.1.1. Cohen-Macaulay and Gorenstein Rings

Definition 3.1 (Depth, Cohen-Macaulay, Gorenstein). Let $(R, m)$ be a local ring, of necessarily finite Krull dimension $d$, and let $M \neq 0$ be a finite $R$-module. An $M$-regular sequence is a sequence of elements $x_1, \ldots, x_r \in m$ such that for all $i$ left-multiplication by $x_i$ is an injective map on $M/(x_1, \ldots, x_{i-1})M$. The depth\(^1\) of $M$ is defined as the maximal length of such a sequence.

If depth $R = d$, then $R$ is said to be a Cohen-Macaulay (CM) ring, and if the injective dimension of $R$ as an $R$-module is finite, then $R$ is called Gorenstein.

Remarks 3.2.

• Though it is not apparent from the definition, Gorenstein implies CM.

\(^{1}\)Prof$M$ in some sources, as in the French word for depth, profondeur.
• These definitions can be globalized by calling a general Noetherian ring CM (resp. Gorenstein) if its local rings are CM (resp. Gorenstein).

• Factoring out a simultaneous non-zerodivisor \( x \) of \( R \) and \( M \) yields a ring with strictly lower Krull dimension, namely \( \dim R/(x) = \dim R - 1 \) (a similar statement holds for the depth and dimension of a module \( M \)). An immediate consequence, proved by induction, are the inequalities

\[
\text{depth } M \leq \dim M \leq \dim R,
\]

for all finite \( R \)-modules \( M \).

**Definition 3.3.** An \( R \)-module \( M \) is called maximal Cohen-Macaulay (MCM for short) if \( \text{depth } M_p = \dim R_p \) for all \( p \) (which is as high as it can get, as the remarks show), and \( \text{Supp } M = R. \)

Despite its purely commutative-algebraic definition, the following theorem shows that depth admits a homological interpretation, allowing us to exploit techniques of homological algebra.

**Theorem 3.4 (Rees, [BH] theorem 1.2.8).** If \( (R, \mathfrak{m}) \) is a local ring, and \( M \neq 0 \) is a finite \( R \)-module, then

\[
\text{depth } M = \inf \{ n \mid \text{Ext}^n_R(R/\mathfrak{m}, M) \neq 0 \}.
\]

So far, we have not considered the depth of the zero module. Rees’ theorem and the definition in terms of regular sequences give two competing suggestions, namely \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \). The latter choice seems to preserve the more theorems than the former (for example \( -\infty = \text{depth } 0 \leq \dim 0 = -\infty \)), so we will set \( \text{depth } 0 = -\infty \).

**Example 3.5.** If \( M \) is a simple module over \( (R, \mathfrak{m}) \), then any nonzero \( m \in M \) gives a surjection \( R \to M \), so it follows that \( M \) is of the form \( R/I \) for some ideal \( I \). If \( I \subseteq J \) for some ideal \( J \), then \( J/I \) is a submodule of the simple module \( R/I \), so that either \( J = R \) or \( J = I \). It follows that \( R/\mathfrak{m} \) is the only simple module of \( R \). In particular, Rees’ theorem shows that depth \( M = 0 \) for any simple \( M \), and more generally any module containing a simple one has depth 0. Since any module of finite positive length contains a simple module, it follows that finite length modules have depth 0.

**Proposition 3.6.** If \( 0 \to M' \to M \to M'' \to 0 \) is an exact sequence of finite modules over a local ring, then

- \( \text{depth } M' \geq \min\{\text{depth } M, \text{depth } M'' + 1\} \);
- \( \text{depth } M \geq \min\{\text{depth } M'', \text{depth } M'\} \);
- \( \text{depth } M'' \geq \min\{\text{depth } M' - 1, \text{depth } M\} \).

**Proof.** Each of these inequalities follows immediately by applying Rees’ theorem to the long exact sequence obtained from \( 0 \to M' \to M \to M'' \to 0 \) and \( \text{Ext}^n_R(R/\mathfrak{m}, -) \).
We end this section with two celebrated outcomes of the interaction of homological algebra with commutative algebra, namely the Auslander-Buchsbaum formula relating projective dimension to depth, and Serre’s homological criterion for regularity.

**Theorem 3.7** (Auslander-Buchsbaum, [W1] theorem 4.4.15). If $R$ is a local ring, and $M$ is a finite $R$-module of finite projective dimension. Then

$$\text{depth } R = \text{depth } M + \text{pdim } M$$

**Theorem 3.8** (Serre, [W1] theorem 4.4.16). A local ring $R$ is regular if and only if $\text{gl.dim } R < \infty$. And if $R$ is regular, then

$$\text{depth } R = \text{dim } R = \text{gl.dim } R.$$

An immediate consequence is that regular rings are Gorenstein.

### 3.1.2. Reflexive Modules

Let $R$ be a ring, and $M$ a finite $R$-module. Then we define the dual $M^\vee$ of $M$ to be $\text{Hom}_R(M, R)$. There is an obvious map $M \to M^{\vee\vee}$, which is an isomorphism in the case that $R$ is a field. In general this homomorphism need not even be injective, leading to the following definitions.

**Definition 3.9.** A finite $R$-module $M$ is called torsionless if the canonical homomorphism $M \to M^{\vee\vee}$ is injective. If this map is an isomorphism, then $M$ is called reflexive.

We have the following characterization of reflexive modules:

**Proposition 3.10** ([BH], proposition 1.4.1). A finite $R$-module $M$ is reflexive if and only if the following two conditions hold for all $p \in \text{Spec}(R)$:

(i) $M_p$ is reflexive if $\text{depth } R_p \leq 1$;

(ii) $\text{depth } M_p \geq 2$ if $\text{depth } R_p \geq 2$.

Using Serre’s criterion for normality, we find that reflexive modules are fairly well-behaved over normal rings.

**Proposition 3.11.** If $R$ is normal, then the category of finite reflexive modules over $R$ is closed under extensions and kernels. In particular, if $M$ and $N$ are finite $R$-modules with $N$ reflexive, then $\text{Hom}_R(M, N)$ is a reflexive $R$-module.

**Proof.** Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite modules. From proposition 3.6 it follows that if $M$ and $M''$ (resp. $M'$ and $M''$) satisfy part (ii) of proposition 3.10, then $M'$ (resp. $M$) does.

Suppose that $\text{depth } R_p \leq 1$. The $S_2$ part of Serre’s criterion shows that $\text{ht } p \leq 1$. From part $R_1$ we then get that $R_p$ is regular of dimension $\leq 1$, so it is a PID (it is either a...
field or a dvr, both of which are PIDs). By theorem 3.8, the global dimension of \( R_p \) is \( \leq 1 \), so given a finite \( R_p \)-module \( M \), we have a free resolution

\[
0 \to P_1 \to P_0 \to N \to 0,
\]

with \( P_0 \) and \( P_1 \) of finite rank. Applying \( \text{Hom}_{R_p}(\cdot, R_p) \) gives \( 0 \to N^\vee \to P_0 \to P_1 \) (we have \( P_1^\vee = P_1 \) since \( P_1 \) is free of finite rank). Since \( R_p \) is a PID, submodules of free modules of finite rank are again free of finite rank, so \( N^\vee \) is free of finite rank. It follows that \( N^{\vee\vee} \) is also free of finite rank, proving that the finite reflexives over a PID of global dimension \( \leq 1 \) are precisely the finite rank free modules. As the category of such modules is closed under extensions and kernels, it follows that if \( M \) and \( M'' \) (resp. \( M' \) and \( M''' \)) satisfy (i) of proposition 3.10, then \( M' \) (resp. \( M'' \)) does.

To obtain the last statement, take a finite presentation \( R^b \to R^a \to M \to 0 \) and apply \( \text{Hom}_R(\cdot, N) \). Then we see that \( \text{Hom}_R(M, N) \) the kernel of the morphism \( N^a \to N^b \).

As \( N^a \) and \( N^b \) are reflexive, we find that \( \text{Hom}_R(M, N) \) is.

**Remark 3.12.** The exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \) over \( \mathbb{Z} \) (manifestly normal) shows that quotients of reflexives by reflexives are not in general reflexive (or even torsionless), as \( (\mathbb{Z}/2\mathbb{Z})^\vee = 0 \).

**Proposition 3.13.** If \( M \) is finite and reflexive over a normal ring \( R \), then \( M_p \) is free for all height one \( p \in \text{Spec}(R) \).

**Proof.** By Serre’s criterion for normality and proposition 3.10, it suffices to show that finite reflexives over a dvr are free. Consider a surjection \( R^a \to M^\vee \to 0 \). Dualizing yields an exact sequence \( 0 \to M \to R^a \to N \to 0 \) for some finite \( R \)-module \( N \). As \( R \) is regular, we have \( \text{gl.dim } R = \dim R = 1 \) by Serre’s theorem, so it follows that \( \text{pdim } N \leq 1 \), from which we obtain that \( M \) must be projective. Since projectives over a Noetherian local ring are free, this proves the desired result.

Consider a finite \( R \)-algebra \( \Lambda \) (not necessarily commutative). Then we denote by \( \text{ref}_R \Lambda \) the category of finite \( R \)-reflexive \( \Lambda \)-modules, and we use \( \text{proj} \Lambda \) to denote the category of finite projective \( \Lambda \)-modules.

**Proposition 3.14** (Reflexive Equivalence). Let \( M \) be a finite reflexive over a normal ring \( R \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\text{ref}_R R & \sim & \text{ref}_R \text{End}_R(M)^\text{opp} \\
\uparrow & & \uparrow \\
\text{add } M & \sim & \text{proj} \text{End}_R(M)^\text{opp} \\
\end{array}
\]

where the horizontal arrows are equivalences given by \( \text{Hom}_R(M, \cdot) \), and the vertical arrows are just inclusions.
Proof. Let $\Lambda = \text{End}_R(M)^{\text{op}}$. The fact that $\text{Hom}_R(M, -)$ lands in $\text{ref}_R \Lambda$ follows from the second part of proposition 3.11, and $\text{Hom}_R(M, N)$ being a left $\Lambda$-module by $\phi \cdot \psi = \psi \circ \phi$ for $\phi \in \Lambda$ and $\psi : M \to N$.

The commutativity of the diagram and the bottom equivalence do not depend on the normality of $R$ and are easy to verify (see proposition II2.1 in [ARS]). The top equivalence is harder, and can be found as proposition 2.4(2)(i) in [IR].

Proposition 3.15. Over a normal CM ring, MCM modules are reflexive.

Proof. Let $M$ be an MCM module over a normal CM ring $R$. For all $p$ we have depth $M_p = \text{depth} R_p = \dim R_p$, so part (ii) of proposition 3.10 is trivially satisfied.

Since $R$ is CM, $p \in \text{Spec}(R)$ with depth $R_p \leq 1$ are of height $\leq 1$, so that by part $R_1$ of Serre’s criterion, $R_p$ is regular of dimension $\leq 1$ (and in particular a PID). To show that $M_p$ is reflexive, we need to show that it is torsion-free, since finite torsion-free modules over PIDs are free, and hence reflexive.

For $p$ of height 0, $R_p$ is a field, so $M_p$ is trivially torsion-free.

If $p$ is of height 1, then the fact that $M$ is MCM states that $\text{Hom}(k(p), M_p) = 0$, where $k(p)$ denotes the residue field at $p$. The set of zero-divisors on $M_p$ is the union of all associated primes of $M_p$. Since $R_p$ is a dvr, the only prime ideals are 0 and $p R_p$, so it follows that either $M_p$ is torsion-free (no non-zero associated primes), or $p R_p$ is an associated prime. The latter case is impossible, because then $p R_p$ is the annihilator of some non-zero element $m \in M_p$, giving a non-zero homomorphism $1 \mapsto m$ from $k(p)$ to $M_p$, contradicting the fact that $M$ is MCM. Therefore $M_p$ is reflexive for all $p$ with depth $R_p \leq 1$, proving that $M$ is reflexive by proposition 3.10.

Reflexives over two-dimensional rings

We now specify to our case of interest, namely that of two-dimensional rings. Here some more can be said about MCMs.

Proposition 3.16. Over a local normal ring $R$ of dimension 2, the MCM modules are precisely the second syzygies. That is, a module $M$ is MCM if and only if there exist finitely generated projectives $P_1$ and $P_0$ and a finitely generated module $N$ with an exact sequence

$$0 \to M \to P_1 \to P_0 \to N \to 0.$$

In particular, $R$ itself is CM, and because reflexives are second syzygies, reflexives are MCM.

Proof. First note that Serre’s criterion for normality, the fact that regular rings are CM, and $\dim R = 2$ immediately show that $R$ is CM. Therefore we can apply the preceding proposition.

If $M$ is MCM, then it is reflexive by the previous proposition, so we can take a presentation $R^b \to R^a \to M \to 0$ with $a$ and $b$ finite. Dualizing yields an exact sequence $0 \to M \to R^a \to R^b \to B \to 0$, so $M$ is a second syzygy.
Conversely, suppose we are given such an exact sequence (note that the finitely generated projectives on a Noetherian local ring are free). The sequence can be split into $0 \to M \to R^a \to A \to 0$ and $0 \to A \to R^b \to B \to 0$. The fact that $\text{depth } R^a = \text{depth } R^b = 2$ and proposition 3.6 imply that $\text{depth } M = 2$, so second syzygies are MCM.

Here’s a vastly simplified version of a duality theorem which holds in much greater generality.

**Proposition 3.17** ([E], theorem 21.21). If $(R, m)$ is a local Gorenstein ring of dimension 2, then the functor $(-)^\vee$ gives a duality on the category of MCM modules (i.e., it is an idempotent equivalence which preserves exact sequences).

**Proof.** Let $M$ be an MCM, and pick a presentation $R^b \to R^a \to M \to 0$. Dualizing yields $0 \to M^\vee \to R^a \to R^b$, so $M^\vee$ is a second syzygy, and therefore MCM (the normality was not used in the second part of the proof of the preceding proposition). It follows that $(-)^\vee$ restricts to a functor from the category of MCMs to its opposite, and the reflexivity of MCMs shows its idempotence.

If $0 \to A \to B \to M \to 0$ is exact, with $M$ MCM, then to show the exactness of the dualized sequence, we need $\text{Ext}^1(M, R) = 0$. Since $R$ is Gorenstein, $\text{idim } R = \dim R = 2$ ([W1], corollary 4.4.10). Let $(x, y)$ be a maximal length regular $M$-sequence. Then we have exact sequences $0 \to M \to M \to M/xM \to 0$ and $0 \to M/xM \to M/xM \to M/(x, y)M \to 0$, which induce exact sequences

$$\text{Ext}^1(M, R) \to \text{Ext}^1(M, R) \to \text{Ext}^2(M/xM, R)$$

and

$$\text{Ext}^2(M/xM, R) \to \text{Ext}^2(M/xM, R) \to \text{Ext}^3(M/(x, y)M, R).$$

First consider the second sequence. Now the third Ext-group vanishes because $\text{idim } R = 2$, so multiplication by $y$ on the second Ext-group is surjective. But $y \in m$, so Nakayama’s lemma implies $\text{Ext}^2(M/xM, R) = 0$. A repetition of the same argument applied to the first sequence gives $\text{Ext}^1(M, R) = 0$, which was to be shown.

**Remark 3.18.** Note that the second part of the proof admits an obvious extension to higher dimensional Gorenstein local rings.

### 3.2. Noncommutative Crepant Resolutions

In the current context, it is common and convenient to denote by $\text{End}_R(M)$ the $R$-algebra $\text{End}_R(M)^{\text{opp}}$, and we adopt this practice (this is the only section where we do so).

**Definition 3.19.** A noncommutative crepant resolution (NCCR for short) of $R$ is an $R$-algebra $\Lambda$ of the form $\text{End}_R(M)$ with $M$ reflexive and finite, and such that the following two conditions hold:

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• $\Lambda$ is a finite maximal Cohen-Macaulay module over $R$;
• for each $p \in \text{Spec}(R)$, $\text{gl.dim} \Lambda_p = \text{gl.dim} R_p$.

In appendix B, it is shown that torus actions and equivariance of morphisms can be rephrased in terms of the lattice of one-parameter subgroups of the torus. In chapter 2, it is shown that the category of toric varieties can be embedded into the category of varieties with torus actions and equivariant maps. This motivates the following modification of the definition above.

**Definition 3.20.** Let $R$ be a Gorenstein affine toric variety, with dense subtorus $T_N$. A toric NCCR of $R$ is an NCCR which is $N$-graded as an $R$-algebra.

The definition of an NCCR requires some motivation. Firstly, the requirement that $\text{gl.dim} \Lambda_p = \text{gl.dim} R_p$ is a noncommutative analogue of nonsingularity (cf. Serre’s theorem 3.8). However, one would expect this analogue to simply be $\text{gl.dim} \Lambda_p < \infty$, so more clarification is in order.

Let $\Lambda$ be an arbitrary finite MCM $R$-algebra which satisfies $\text{gl.dim} \Lambda_p < \infty$ for all $p \in \text{Spec}(R)$. We can consider the $\Lambda_p$-module $S := \Lambda_p/(x_1, \ldots, x_r)\Lambda_p$, where $x_1, \ldots, x_r \in p R_p$ is a regular sequence for $\Lambda_p$. Since the $x_i$ form a maximal regular sequence, the depth of $S$ as an $R_p$-module is 0, so by Auslander-Buchsbaum (3.7) $\text{pdim}_{R_p} S = \dim R_p$.

Since $\Lambda$ is MCM, depth$_{R_p} \Lambda_p = \dim R_p$, so Auslander-Buchsbaum implies $\text{pdim}_{R_p} \Lambda_p = 0$. It follows that

$$\text{pdim}_{\Lambda_p} S = \text{pdim}_{R_p} \Lambda_p + \text{pdim}_{\Lambda_p} S \geq \text{pdim}_{R_p} S = \dim R_p,$$

where the inequality is the general change of rings theorem 4.3.1 from [W1]. This shows that

$$\text{gl.dim} \Lambda_p \geq \dim R_p, \quad (3.2)$$

so the condition in the definition is the lowest possible global dimension we can hope for.

The next curious thing about the definition is that we require $\Lambda$ to be Cohen-Macaulay as an $R$-module and of the form $\text{End}_R(M)$ with $M$ reflexive. The motivation for this lies slightly deeper, and is given in the form of the following theorem:

**Theorem 3.21** ([IW3], section 4). Let $f: Y \to \text{Spec}(R)$ be projective and birational, with $Y$ and $R$ normal Gorenstein varieties of the same dimension. If $Y$ is derived equivalent to an $R$-algebra $\Lambda$ (in the sense of a triangulated equivalence $D^b \text{Coh}(Y) \cong D^b(\Lambda-\text{Mod}^{f.g.})$), then $f^*\omega_Y \cong \omega_Y$ if and only if $\Lambda$ is CM as an $R$-module. Furthermore, in this situation $\Lambda$ is of the form $\text{End}_R(M)$ with $M$ reflexive and finite over $R$.

**Proof Sketch.** The proof relies, among other things, on the existence of a so-called tilting complex $V \in D^b \text{Coh}(Y)$ of vector bundles inducing the derived equivalence of $Y$ and $\Lambda$ (see [R1]). That is, $\Lambda = \text{End}_{D \text{Coh}(Y)}(V)$, $\text{Hom}_{D \text{Coh}(Y)}(V, V[i]) = 0$ for all $i \neq 0$, and $V$

\[\text{In the Gorenstein case, one can show that gl.dim} \Lambda_p < \infty \text{ is in fact sufficient, see for example [W2] or [vdB].}\]
generates $D^b \text{Coh}(Y)$ by taking cones and direct summands (compare Morita equivalence, which is induced by finite projective generators). This complex $V$ is given as the image of $\Lambda$ under the equivalence $D^b(\Lambda\text{-Mod}^{f.g.}) \rightarrow D^b \text{Coh}(Y)$.

From the conditions on the Hom-sets for $V$, we get

$$\Lambda = R \text{Hom}(V, V) = Rf_* R\text{Hom}(V, V),$$

where the final object is the sheaf Hom, and we use the fact that the global sections functor on $Y$ is essentially given by $f_*$. Because $f_*$ is proper it preserves coherents, and $R\text{Hom}_Y(V, V)$ is coherent because $V$ is a bounded complex of vector bundles, so it follows that $\Lambda$ is a finite $R$-module.

When $\Lambda$ is MCM, it is reflexive by proposition 3.15, so in particular its support is all of $\text{Spec}(R)$ (reflexive implies torsion-free). From this it follows that the localized complexes $V_p$, for $p \in \text{Spec}(R)$ of height one, are nonzero (localized in the sense that we restrict to the fiber product $\text{Spec}(R_p) \times_R Y$). Pushing forward to $\text{Spec}(R_p)$ we obtain a complex of the form $R^p_{\mathbb{Z}}[b]$. From the fact that $R_p$ is a dvr ($p$ is height one and $R$ is normal), and that $g: \text{Spec}(R_p) \times_R Y \rightarrow \text{Spec}(R_p)$ is projective and birational, it follows that $g$ is an isomorphism, so we see that $\Lambda_p = \text{End}(R^p_{\mathbb{Z}})$, which is Morita equivalent to $R_p$. This and the fact that $\Lambda$ is reflexive, characterizes $\Lambda$ of the form $\text{End}(R)$ with $R$ reflexive and finite.

To see that the crepancy of $f$ implies that $\Lambda$ is MCM, we need that crepancy of $f$ is characterized by $f^! \mathcal{O}_{\text{Spec}(R)} = \mathcal{O}_Y$ (lemma 4.7 in [IW3]), where $f^!$ is the right adjoint of $Rf_*$ provided by Grothendieck duality, i.e.,

$$Rf_* R\text{Hom}_Y(F, f^! M) \cong R\text{Hom}_R(Rf_* F, M).$$

By applying the derived hom-tensor adjunction and the fact that $\mathcal{O}_Y$ is a tensor unit in the derived category, one sees that $R\text{Hom}_Y(V, V) \cong R\text{Hom}_Y(R\text{Hom}_Y(V, V), \mathcal{O}_Y)$. By applying $Rf_*$, the fact that $f^! \mathcal{O}_{\text{Spec}(R)} = \mathcal{O}_Y$, and Grothendieck duality, we find

$$R\text{Hom}_Y(V, V) \cong R\text{Hom}_R(R\text{Hom}_Y(V, V), R).$$

As $V$ is a tilting object, $R\text{Hom}_Y(V, V) = \Lambda$ (the right-hand side is to be interpreted as a complex concentrated at degree 0), so $\Lambda \cong R\text{Hom}_R(\Lambda, R)$. Taking homology, we find $\text{Ext}_R^i(\Lambda, R) = 0$ for all $i \neq 0$, so $\Lambda$ is MCM over $R$.

The converse, namely that $\Lambda$ being MCM implies crepancy of $f$, is proved using relative Serre functors, which are outside the scope of this text. \hfill \Box

If $M$ is a finite reflexive $R$-module, and $R$ is a normal domain, then proposition 2.4(3) in [IR] shows that $\Lambda := \text{End}_R(M)$ is isomorphic to $\text{Hom}_R(\Lambda, R)$ as a $(\Lambda, \Lambda)$-bimodule. Using the hom-tensor adjunction, it follows that

$$\text{Hom}_\Lambda(-, \Lambda) \cong \text{Hom}_R(\Lambda \otimes_\Lambda -, R) \cong \text{Hom}_R(-, R)$$

on the category of $\Lambda$-modules.
**Theorem 3.22.** If $R$ is a normal local Gorenstein ring, $\Lambda$ is an NCCR of $R$, and $M$ is a finite $\Lambda$-module, then

$$\text{depth}_R M + \text{pdim}_\Lambda M = \dim R.$$  

**Proof.** The discussion preceding the theorem shows that $\text{Hom}_\Lambda(-, \Lambda) = \text{Hom}_R(-, R)$ on the category of finite $\Lambda$-modules, so if $M$ has depth equal to dim $R$, then the proof of proposition 3.17 shows that $\text{Ext}^i_\Lambda(M, \Lambda) = \text{Ext}^i_R(M, R) = 0$ for $i > 0$. When $n = \text{pdim}_\Lambda M$ there holds $\text{Ext}^n_\Lambda(M, \Lambda) \neq 0$, so we find that $M$ is projective, as desired. Continuing by induction, let $\text{pdim}_\Lambda M = n$, and let $0 \to P_n \to \ldots \to P_0 \to M \to 0$ be some minimal $\Lambda$-projective resolution. We can split this up into short exact sequences, to which we can apply proposition 3.6 to conclude that depth $M \geq \dim R - n$. Another application of the same proposition shows that the $(\dim R - \text{depth} M)$th syzygy of $M$ is CM, so the base case shows that it is $\Lambda$-projective. This implies $\dim R - \text{depth} M \geq n$, proving the theorem. \(\square\)

We might ask ourselves when two NCCRs are to be considered geometrically the same (we temporarily abbreviate this relation as $\sim$). Firstly, there is the obvious choice of taking $\sim$ to be isomorphism. Secondly, a result of Rickard [R1] says that derived equivalent rings have isomorphic centers, so going by the criterion that the restriction of $\sim$ to affine schemes ought to be isomorphism shows that $\sim$ could also be Morita equivalence or derived equivalence (the former trivially implies the latter). When we pass to arbitrary schemes, derived equivalence no longer implies isomorphism (cf. the existence of non-isomorphic crepant resolutions in dimension three, which have to be derived equivalent by the Bondal-Orlov conjecture [B2]). However, the Gabriel-Rosenberg reconstruction theorem states that quasi-separated schemes with isomorphic categories of quasicoherents are isomorphic. We therefore take $\sim$ to be Morita equivalence.

The following gives a noncommutative analogue of the uniqueness of a minimal resolution for surface singularities, which implies uniqueness of crepant resolutions by theorem 1.7.

**Theorem 3.23.** Any two NCCRs of a local two-dimensional normal Gorenstein ring are Morita equivalent.

**Proof.** Consider two NCCRs $\Lambda = \text{End}_R(M)$ and $\Gamma = \text{End}_R(N)$, and let $P$ be an $R$-reflexive finite $\Lambda$-module. Then depth $P \geq 2$ by proposition 3.16, so by the Auslander Buchsbaum formula 3.22, $P$ is $\Lambda$-projective. Conversely, all finite direct sums of $\Lambda$ with itself are $R$-reflexive, because $\Lambda$ is MCM, and hence reflexive (proposition 3.15). Since direct summands of reflexives are reflexive, it follows that all finite $\Lambda$-projective modules are $R$-reflexive. The same arguments work for $\Gamma$, of course.

Proposition 3.14 now shows that $\text{add} M = \text{add} N$. In particular, we obtain that $P = \text{Hom}_R(M, N)$ is a finite projective $\Lambda$-module, whose endomorphism ring is isomorphic to the endomorphism ring of $N$ in $R\text{-Mod}$, i.e., $\Gamma^{\text{opp}}$. Furthermore, since $M \in \text{add} N$, we can find $m \in \mathbb{Z}_{\geq 0}$ and finite $B \in R\text{-Mod}$ such that $N^m = M \oplus B$, so

$$P^m = \text{Hom}_R(M, N)^m \cong \text{Hom}_R(M, M \oplus B) \cong \Lambda \oplus \text{Hom}_R(M, B),$$

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showing that $P$ is a progenerator in $\Lambda$-mod, therefore inducing a Morita equivalence of $\Lambda$ with $\Gamma$, as was to be shown.

Remarks 3.24.

- To get a connection of NCCRs with geometry, we need the singularity to be Gorenstein. In the context of the minimal resolution for surface singularities, this manifests itself through the fact that surfaces which admit NCCRs have finite CM type (only finitely many indecomposable MCM modules), as is shown in the first section of [W2]. It is proved in chapter 11 of [Y] that the only surface singularities over an algebraically closed field of characteristic 0 which satisfy this condition are the ADE singularities described in proposition 1.6, which are precisely the Gorenstein singularities in dimension 2. In the running example, we show the existence of an NCCR for the $A_n$ singularity over an algebraically closed field of characteristic 0. Similar techniques apply to the remaining rational double points, so this gives us a noncommutative analogue of the existence of the minimal resolution of a surface singularity.

- As in the commutative case, the existence of the minimal resolution does not generalize to higher dimensions. However, as in the commutative case, there is a Bondal-Orlov conjecture, namely that any two NCCRs of the same affine Gorenstein variety are derived equivalent. This is known to be true in dimensions $\leq 3$, see [IW1]. The case of dimension 2 of course follows from the theorem above, as Morita equivalence trivially implies derived equivalence.

Definition 3.25. Let $G$ be a finite group, and let $\Lambda$ be an $R$-algebra with a homomorphism $\rho: G \to \text{Aut}_{R-\text{Alg}}(\Lambda)$. The smash product (or skew group ring) $\Lambda\#G$ is the free $\Lambda$-module $\bigoplus_{g \in G} \Lambda g$, with product structure given by

$$(s_1 g_1) \cdot (s_2 g_2) = s_1 (\rho(g_1) s_2) g_1 g_2,$$

extended linearly.

Lemma 3.26. Let $\Lambda\#G$ be a skew group ring. If the characteristic of $k$ does not divide $|G|$, then $\text{gl.dim} \Lambda\#G \leq \text{gl.dim} \Lambda$.

Proof. If $M$ and $N$ are $\Lambda\#G$-modules, then we have an representation $\pi$ of $G$ on $\text{Hom}_\Lambda(M, N)$ given by $\pi(g)(\phi)(m) = g\phi(g^{-1}m)$. A map $\phi: M \to N$ is $G$-invariant if and only if it commutes with every $g \in G$, so it follows that $\text{Hom}_{\Lambda\#G}(M, N) = \text{Hom}_\Lambda(M, N)^G$. Any $\Lambda\#G$-projective resolution of $M$ will be a $\Lambda$-projective resolution, because $\Lambda\#G$ is a free $\Lambda$-module. Taking $G$-invariants is clearly left-exact, and Maschke’s theorem shows that $\text{gl.dim} k[G] = 0$, so that taking $G$-invariants is exact. It immediately follows from these assertions that $\text{Ext}_\Lambda(M, N)^G = \text{Ext}_{\Lambda\#G}(M, N)$, so that $\text{gl.dim} \Lambda\#G \leq \text{gl.dim} \Lambda$. 

Suppose that $\Lambda$ is an NCCR of $R$ (better yet, suppose it satisfies the final two conditions in the definition of an NCCR). Since $\Lambda\#G$ is a direct sum of MCM $R$-modules
(namely $\Lambda$), we also see that $\Lambda^#G$ is MCM, so we can apply inequality (3.2) to get $\operatorname{gl.dim}(\Lambda^#G)_p \geq \dim R_p$ for all $p \in \operatorname{Spec}(R)$. The homomorphism $\rho: G \to \operatorname{Aut}_{\text{R-Alg}}(\Lambda)$ induces for each $p \in \operatorname{Spec}(R)$ a homomorphism $G \to \operatorname{Aut}_{R_p\text{-Alg}}(\Lambda_p)$, so we can form the smash products $\Lambda_p^#G$. From the R-linearity of the automorphisms $\rho(g)$, it follows that $\Lambda_p^#G = (\Lambda^#G)_p$. Combining this with the lemma above applied to $\Lambda_p^#G$, we get $\operatorname{gl.dim}(\Lambda^#G)_p \leq \dim R_p$ for all $p \in \operatorname{Spec}(R)$. Therefore $\Lambda^#G$ satisfies the final two conditions in the definition of an NCCR.

**An example: $A_n$ singularities**

Consider the $A_{n-1}$ singularity $R = k[U^n,UV,V^n]$, which is 2-dimensional, toric, and Gorenstein, as is shown in chapter 2. Written as such, we realize this singularity as the quotient of $\Lambda = k[U,V]$ by the group action $\rho: \mu_n \to \operatorname{Aut}_k(k[U,V])$, $\rho(x)U = \xi U$ and $\rho(x)V = \xi^{-1}V$, where $\xi$ is a primitive $n$th root of unity in $k$, and $x$ is a generator of $\mu_n$. The smash product $\Gamma := \Lambda^#\mu_n$ is presented as

$$
\Gamma = k\langle U,V,x \rangle/(x^n - 1, UV - VU, xU - \xi Ux, xV - \xi^{-1}Vx),
$$

where we use $k\langle U,V,x \rangle$ to denote the free $k$-algebra generated by $U,V$, and $x$.

We will show that $\Gamma$ is a toric NCCR of the $\Lambda$ (not the opposite thereof). We temporarily drop the convention that $\operatorname{End}_R(\Lambda)$ denotes the opposite of the $R$-endomorphism algebra of $\Lambda$.

We have a natural $R$-linear map $\Phi: \Gamma \to \operatorname{End}_R(\Lambda)$ given by $\Phi(rg)(s) = r\rho(g)(s)$. It is easy to verify that $\Phi$ is a homomorphism of $R$-algebras. We first show that $\Phi$ is injective.

Let $\phi:S \to S$ be a homogeneous element of degree $(p,q)$. We will show that $p$ and $q$ are
nonnegative. Let $i$ and $j$ be such that $\phi(U^i V^j) = \lambda U^{i+p} V^{j+q} \neq 0$. Without loss we may assume that $i > j$, and the $R$-linearity of $\phi$ shows that we may assume $i < n$. Then we get

$$\lambda U^{i+p} V^{j+q} = \phi(U^i V^j) = (UV)^j \phi(U^{i-j}),$$

so $\phi(U^{i-j}) = \lambda U^{i-j+p} V^q$, since $\Lambda$ is a domain. From the fact that $\Lambda$ does not contain any negative powers of $V$ it follows that $q \geq 0$. Now we multiply with $(UV)^{n-i+j}$ to get

$$\lambda U^{n+p} V^{n-i+j+q} = \phi(U^n V^{j-i+n}) = U^n \phi(V^{j-i+n}),$$

so that $\phi(V^{j-i+n}) = \lambda U^p V^{n-i+j+q}$, showing that $p \geq 0$.

Now we show that $\phi$ is of the form $\Phi(\sum c_i U^p V^q x^i)$ for some $c_i \in k$. For $0 \leq i < n$, let $r_i$ be the unique element of $\Lambda$ such that $U^p V^q r_i = \phi(U^i)$ (note that by degree considerations, $r_i \in k \subseteq \Lambda$). From the definition of $\Phi$, it follows that we need to find $c_i$ in $k$ such that the Vandermonde matrix of $(1, \xi, \ldots, \xi^{n-1})$ applied to $(c_0, \ldots, c_{n-1})$ is $(r_0, \ldots, r_{n-1})$. By linear algebra this can be done, and therefore we find an element $\gamma = \sum c_i U^p V^q x^i$ of $\Gamma$ such that $\Phi(\gamma)$ and $\phi$ agree on the $U^i$, for $0 \leq i < n$. Aside from this, for $0 \leq i < n$, we have

$$(UV)^{n-i} \phi(V^i) = \phi(U^{n-i} V^n)$$

$$= V^n \phi(U^{n-i})$$

$$= V^n \Phi(\gamma)(U^{n-i})$$

$$= \Phi(\gamma)(U^{n-i} V^n) = (UV)^{n-i} \Phi(\gamma)(V^i),$$

so that $\phi(V^i) = \Phi(\gamma)(V^i)$ as well. Since the $U^i$ and $V^i$, $0 \leq i < n$, generate $\Lambda$ as an $R$-module, it follows that $\Phi(\gamma) = \phi$, proving that $\Phi$ is surjective, and therefore an isomorphism of $R$-algebras, which was to be shown.

**Remark 3.27.** The results of this example (and the one in the next section) hold in a more general context, with $\mu_n$ replaced by certain finite subgroups of $\text{GL}(2, \mathbb{C})$, and $\rho$ replaced by the obvious actions of such groups on $A_k^2$. See for example chapter 10 of [Y], where the general case is proved using number-theoretic techniques.
4. Quivers

In this chapter we give the basics of quivers and their representations. We need quivers and their representations in the next chapter to pass from NCCRs to commutative crepant resolutions. These representations are parameterized by certain moduli spaces, so by writing an NCCR as a quotient of a quiver, we can associate a variety to an NCCR, which turns out to be a commutative crepant resolution under suitable circumstances. The material in this chapter is well-known, and can be found in many books, for example [ARS].

4.1. Quivers

Recall that quiver is synonymous with directed graph. For a quiver $Q$, we denote by $Q_0$, $Q_1$ the finite sets of its vertices and edges (often called arrows in the context of quivers), and by $h, t : Q_1 \to Q_0$ the head and tail functions which define the orientation of $Q$.

**Definition 4.1.** The path algebra $kQ$ associated to a quiver $Q$, is generated as a $k$-algebra by $Q_1$ and $Q_0$, and has relations

- $v^2 = v$ for $v \in Q_0$;
- $vw = 0$ for distinct $v, w \in Q_0$;
- $h(a)t(a) = a$ for $e \in Q_1$.

A relation in $Q$ is a $k$-linear combination of paths of length at least 2 with the same start and endpoints, where a path of length $n$ is an element of $kQ$ of the form $a_1 \ldots a_n$ with $a_i \in Q_1$, and $h(a_i) = t(a_{i-1})$. A quiver with relations is a quiver $Q$ with a collection $R$ of such relations, to which we can associate the algebra $kQ/R$, abusively denoting by $R$ the two-sided ideal in $kQ$ generated by $R$.

**Example 4.2.** The path algebra of the $n$-loop quiver with one vertex is the free $k$-algebra on $n$ generators.

The ubiquity of quivers in the theory of finitely generated algebras is explained as follows. Consider a system of orthogonal idempotents $e_0, \ldots, e_r$ in a $k$-algebra $R$, i.e., $e_i^2 = e_i$ and $e_ie_j = 0$ when $i \neq j$. We may assume that $\sum_i e_i = 1$, as it is easy to verify that $1 - \sum_i e_i$ is an idempotent orthogonal to the $e_i$. We construct a quiver $Q$ by taking the idempotents $e_i$ as its vertices. The set of arrows from $e_i$ to $e_j$ is obtained by picking a set of generators $\{x_1, \ldots, x_n\}$ for $R$, and taking $\{e_jx_ie_i\}^n_{r=1}$ as the set of arrows from $i$ to $j$. Then, using $r = (\sum_i e_i) r (\sum_i e_i) = \sum_{i,j} e_jre_i$, it is easy to see that we have a
surjective morphism of \(k\)-algebras \(kQ \to R\), presenting \(R\) as a quiver with relations. By the preceding example, a presentation of \(R\) as a quotient of a free algebra is a special case of this procedure. These more general quiver presentations are more useful than presentations by free algebras in the sense that whenever \(Q\) has more than one vertex, such a quiver presentation allows us to visualize certain computations in the algebra.

**Example 4.3.** As an example, let us present the \(A_{n-1}\) NCCR \(\Gamma := \mu_n \# k[U, V] \) as a quiver with relations, using the procedure outlined above. This will be of use to us in the next chapter, where we use this presentation to associate a commutative crepant resolution of the \(A_{n-1}\) singularity to \(\Gamma\). Recall that we take \(k\) algebraically closed and of characteristic 0.

In the group algebra \(k[\mu_n] = k[x]/(x^n - 1)\) we have the complete system of idempotents

\[
e_i = \frac{1}{n} \sum_{m=1}^{n} \xi^{im} x^m,
\]

with \(i\) ranging from 1 to \(n\). Of course, these \(e_i\) also give us a complete system of idempotents for \(\Gamma\) and hence a presentation of \(\Gamma\) as a quiver with relations.

It is easy to verify that in \(\Gamma\), we have \(e_i U = U e_{i+1}\), \(e_i V = V e_{i-1}\), and \(x e_i = \xi^{-i} e_i\), where the subscripts are to be read modulo \(n\). We will apply the procedure outlined at the end of section 4.1 to the generators \(U, V, x\) of \(\Gamma\). Being an element of \(\mu_n\), \(x\) commutes with the idempotents \(e_i\), so we can omit \(x\) in constructing the arrows of \(Q\). By the idempotence and orthogonality of the \(e_i\), we have

\[
e_i U e_j = \begin{cases} U e_j & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
e_i V e_j = \begin{cases} V e_j & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}.
\]

Therefore we must take \(Q\) to be

\[\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccendarray}\]

and the surjection \(\pi : kQ \to \Gamma\) is given by mapping the vertices \(e_i\) to the idempotents \(e_i\), and the arrows \(u_i\) and \(v_i\) to \(U e_i\) and \(V e_i\). It is easy to verify that \(\pi(u_i v_{i-1} - v_i u_{i+1}) = 0\),

\[\text{One way to arrive at these idempotents is to take the sums } e_V = \frac{\dim V}{\dim \mu_n} \sum_{g \in \mu_n} \chi_V(g) g^{-1}, \text{ where } V \text{ is an irreducible } k\text{-representation of } \mu_n. \text{ It is easy to see that } e_V \text{ is in the center of } k[\mu_n], \text{ so it acts as a scalar on all irreducible representations. By the orthogonality of characters, } e_V \text{ acts as the identity on } V, \text{ and as 0 on all other irreps. Maschke’s theorem gives the decomposition } k[\mu_n] = \bigoplus_V k[\mu_n] e_V.\]
so $\pi$ factors through a map $kQ/I \to \Gamma$, where $I$ consists of the relations $u_i v_{i+1} - v_i u_i$ (recall that subscripts are to be read modulo $n$).

Using the presentation of $\Gamma$ in equation (3.3), we can construct an inverse $\Gamma \to kQ/I$ by defining $k\langle U, V, x \rangle \to kQ/I$ as $U \mapsto \sum_i u_i$, $V \mapsto \sum_i v_i$, and $x \mapsto \sum_i \xi^{-i} e_i$. The fact that this factors through $\Gamma$ and gives an inverse to $kQ/I \to \Gamma$ is routine (we need the relations $I$ to get the commutator $[\sum_i u_i, \sum_j v_j]$ to vanish). It follows that $\Gamma \cong kQ/I$.

4.2. Representations of Quivers

One reason why quivers are nice is because the modules of their path algebras have a very concrete interpretation. Recall the following definition.

**Definition 4.4.** A $k$-representation $(V_i, f_a)$ of a quiver $(Q_0, Q_1)$ is a collection of finite-dimensional $k$-vector spaces $V_i$ indexed by $i \in Q_0$ and $k$-linear maps $f_a = f_{h(a), t(a)}: V_{t(a)} \to V_{h(a)}$ for all arrows $a \in Q_1$. A morphism $\phi:(V_i, f_a) \to (W_i, g_a)$ consists of a collection of $k$-linear maps $\phi_i: V_i \to W_i$ for all $i \in Q_0$ such that

\[
\begin{array}{ccc}
V_{t(a)} & \xrightarrow{f_a} & V_{h(a)} \\
\downarrow{\phi_{t(a)}} & & \downarrow{\phi_{h(a)}} \\
W_{t(a)} & \xrightarrow{g_a} & W_{h(a)}
\end{array}
\]

commutes for all $a \in Q_1$. This defines the category $\mathbf{fRep}(Q)$ of finite-dimensional representations of $Q$.

A representation of a quiver with relations $(Q, R)$ is a representation $V = (V_i, f_a)$ of $Q$ such that the relations in $R$ hold for the $f_a$. These give a full subcategory $\mathbf{fRep}(Q, R)$ of $\mathbf{fRep}(Q)$. The dimension vector $\dim(V): Q_0 \to \mathbb{Z}$ for such a representation is defined as the function $i \mapsto \dim_k V_i$.

**Remark 4.5.** Viewing the quiver $Q$ as a category\(^2\), we have an obvious equivalence of $\mathbf{fRep}(Q)$ with $(k\mathbf{-Mod}^{f.d.})^Q$, the category of functors from $Q$ to the category of finite-dimensional vector spaces. By general principles (section 1.6 from [W1]), $\mathbf{fRep}(Q)$ is an abelian category, and we obtain an explicit description of its (co)kernels as $(\text{co})\ker(\phi)_i = (\text{co})\ker(\phi_i)$ for all $i \in Q_0$ (indeed, this holds for all other finite limits and colimits). From this description it is clear that $\mathbf{fRep}(Q, R)$ is closed under all finite (co)limits, so from [W1] lemma 1.6.2, we obtain that $\mathbf{fRep}(Q, R)$ is abelian as well.

\(^2\)More precisely, we consider the image of $Q$ under the left adjoint of the forgetful functor from the category of small categories to the category of (not necessarily finite) quivers, which exists for example by an application of Freyd’s adjoint functor theorem.
The relation of representations of \((Q, R)\) with modules over the path algebra \(kQ/R\) is given in the following proposition. Of course, a similar statement holds for arbitrary modules and representations, but we are mainly interested in finite-dimensional representations, so we state it for this case.

**Proposition 4.6.** Let \((Q, R)\) be a quiver with relations. There is an equivalence of \(\text{fRep}(Q, R)\) with \(kQ/R\)-Mod\(^{f.d.}\), the category of finite-dimensional modules over \(kQ/R\).

**Proof Sketch.** With a finite-dimensional module \(V\) of \(kQ/R\) we associate a representation \((V, f_a)\) by setting \(V_i = iV\) for all \(i \in Q_0\). The arrows are obtained by noting that for \(a \in Q_1\), we have \(at(a)V = h(a)at(a)V \subseteq h(a)V\), so the action of \(a\) on \(V\) induces a map \(f_a: V_t(a) \to V_h(a)\). As we factor out the relations \(R\) in the path algebra, it follows that \((V, f_a)\) is in \(\text{istRep}(Q, R)\). The inverse is given by mapping \((V, f_a)\) to the direct sum \(\bigoplus_{i \in Q_0} V_i\). Arrows \(a\) act as \(f_a\), giving us a \(kQ/R\)-module.

Let \(M_n(k)\) be the algebra of \(n \times n\) matrices with entries in \(k\). As an illustration of the use of quivers, we give a quick proof that \(M_n(k)\) is Morita equivalent to \(k\), by presenting \(M_n(k)\) as a quiver with relations, and studying the representations thereof.

**Example 4.7.** The complete system of idempotents \(e_i\) (1 on position \((i, i)\), 0 everywhere else) give a presentation of \(M_n(k)\) as the quiver

\[
Q = e_1 \xleftarrow{x_1} e_2 \xrightarrow{x_2} e_3 \cdots e_{n-1} \xrightarrow{x_{n-1}} e_n ,
\]

with \(y_{i+1}x_i = e_i\) and \(x_iy_{i+1} = e_{i+1}\) as its relations \(R\) \((x_i\) is mapped to the matrix with 1 at position \((i, i + 1)\) and 0s everywhere else, and \(y_i\) to the matrix with 1 at \((i - 1, i)\)).

Now we can apply the analogue of proposition 4.6 for modules of arbitrary dimension to learn more about \(M_n(k)\)-Mod.

If \(M = (V, x_i, y_i)\) is a representation of \((Q, R)\), then the relations in \(R\) are precisely the statement that \(x_i\) is inverse to \(y_{i+1}\) for all \(i\). In particular, we can assume that \(V := V_i = V_j\) for all \(i\) and \(j\). In fact, if \(N\) is the representation \((V, \text{id}_V, \text{id}_V)\), then we can define a morphism \(\phi: N \to M\) by taking \(\phi_1 = \text{id}_V\), \(\phi_2 = x_1\), \(\phi_3 = x_2x_1\), and so on. Since the \(x_i\) are isomorphisms, \(\phi\) is an isomorphism, so \(M \cong N\). As \(N\) clearly decomposes into a direct sum of representations of the form \((k, 1, 1)\), the endomorphism ring of which is \(k\), we obtain an equivalence of \(M_n(k)\)-Mod with \(k\)-Mod, so \(k\) and \(M_n(k)\) are Morita equivalent.
5. From NCCRs to Commutative Crepant Resolutions

We are interested in parameterizing the representations of a given quiver with representations. The dimension vector gives an obvious discrete parameter. In order to find continuous parameters, we take the representations of fixed dimension vector, and factor out a group which acts with isomorphisms. In order to do this effectively we need geometric invariant theory (GIT), which we discuss in the first section. This allows us to define moduli spaces of representations in the second section, which we then use to link NCCRs to commutative crepant resolutions in the third section.

5.1. Preliminaries on GIT

Let $C$ be a category with finite products, let $G$ be a group object in $C$ (defined in the obvious way), and let $X$ be an object of $C$ with a group action $\rho: G \times X \rightarrow X$ (also defined in the obvious way). A reasonable first attempt at defining the quotient $X/G$ is the following:

$$X/G = \text{colim} \left( \begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\pi_2 & & \\
X & & 
\end{array} \right),$$

where $\pi_2$ is the projection onto the second factor. This is what is known as the categorical quotient, and the canonical morphism $X \rightarrow X/G$ is called the quotient map. More explicitly, it is a $G$-invariant morphism $q: X \rightarrow X/G$ (i.e., $q\pi_2 = q\rho$) through which all other $G$-invariant morphisms with domain $X$ uniquely factor. It coincides with the intuitive notion of quotient in for example the category of topological spaces.

We are primarily interested in the case where $C$ is the category of varieties over an algebraically closed field $k$ of characteristic 0. Here we have an existence result, the hard part of which is the fact that $R^G$ is of finite type over $k$, which is a theorem of Nagata.

**Theorem 5.1.** Let $G$ be a reductive group acting on an affine variety $X = \text{Spec}(R)$, and let $R^G \subseteq R$ be the subalgebra of $G$-invariant functions on $X$. Then $\text{Spec}(R^G)$ is the quotient $X/G$ in the category of k-schemes, and $R^G$ is finitely generated over $k$, so that $X/G$ exists in the category of k-varieties as well.

**Remark 5.2.** A reductive group is a linear algebraic group $G$ if every finite-dimensional rational representation of $G$ decomposes as a direct sum of simple ones. We will only
encounter essentially only one example of such a group, namely $GL_n$ and products thereof.

Although this existence theorem is nice, and we certainly want any concept of quotient to be a categorical quotient, the notion of categorical quotient is not adequate in algebraic geometry. One of the reasons is that we intuitively expect $X/G$ to be in some sense the set of $G$-orbits in $X$, and that the fibers of $X \to X/G$ are the orbits. This would imply that the $G$-orbits in $X$ are all closed (being the fiber over a point), which is not always satisfied, as the action of $G_m$ on $k^1$ shows (in fact, most toric varieties give counterexamples). One is lead to the following notion of quotient.

**Definition 5.3.** Let $G$ be a group scheme over $k$ acting on a $k$-scheme $X$. Then a morphism $q: X \to Y$ is called a geometric quotient if the following hold:

(i) $q$ is a categorical quotient;

(ii) $q$ is a quotient map in the sense of topology;

(iii) the image of $(g, x) \mapsto (gx, x)$ from $G \times X \to X \times X$ is precisely $X \times_Y X$;

(iv) $\mathcal{O}_Y \subseteq q_* \mathcal{O}_X$ is the subsheaf of $G$-invariant sections.

**Remark 5.4.** Parts (iv) and (ii) imply that if $U \subseteq X$ is a $G$-invariant open affine, then $U \to \pi(U)$ is the categorical quotient of $U$ by $G$ (so it is the affine scheme $k[\mathcal{O}_X(U)^G]$). In explicit computations this is useful, because it allows us to compute geometric quotients by computing the quotients of some open affines, and then gluing.

A $G$-linearized line bundle $\mathcal{L}$ on $X$ is a line bundle equipped with an action $G \times \mathcal{L} \to \mathcal{L}$ such that $g \in G$ linearly maps the fiber $\mathcal{L}_x$ into $\mathcal{L}_{gx}$. This allows us to define (semi)stability of points in $X$. The action of $G$ induces actions on the tensor powers $\mathcal{L}^\otimes n$ in the obvious way, which gives us a notion of $G$-equivariant sections of $\mathcal{L}^\otimes n$ over $X$. The space of such sections is denoted $\Gamma(X; \mathcal{L}^\otimes n)^G$.

**Definition 5.5.** A point $x \in X$ is called $\mathcal{L}$-semistable if there exists $f \in \Gamma(X; \mathcal{L}^\otimes n)^G$ for some $n > 0$ with $f(x) \neq 0$ and such that $X_f = \{p \in X \mid f(p) \neq 0\}$ is affine. Such a point is called $\mathcal{L}$-stable if furthermore the action of $G$ on $X_f$ is closed, and the stabilizer $G_x$ is finite. The set of $\mathcal{L}$-semistable (resp. stable) points is denoted $X^{ss}(\mathcal{L})$ (resp. $X^s(\mathcal{L})$).

**Remark 5.6.** When the line bundle is clear, we frequently drop it from our notations, so $\mathcal{L}$-(semi)stable becomes (semi)stable, and $X^{ss}(\mathcal{L})$ becomes $X^{ss}$.

The use of these definitions is the following theorem.

**Theorem 5.7** ([MFK], Theorem 1.10). If $G$ is a reductive group acting on a $k$-variety $X$, and $\mathcal{L}$ is a $G$-linearized line bundle, then there exists a good categorical quotient\(^1\) $\pi: X^{ss} \to X^{ss}/G$. There is an open subset $U \subseteq X^{ss}/G$ such that $\pi^{-1}(U) = X^s$ and $X^s \to U$ is a geometric quotient.

\(^1\)“Good” meaning, among other things, that the quotient map maps disjoint closed $G$-invariant sets to disjoint closed sets.
Now consider an affine variety $X = \text{Spec}(R)$ and an action of a reductive group $G$ on $X$. A character $\chi: G \to \mathbb{G}_m$ gives a linearization of the trivial bundle $\mathcal{L} = X \times \mathbb{A}^1_k \to X$ by the action $g \cdot (x, t) = (g \cdot x, \chi(g)t)$ on points. A section of $\mathcal{L}^\otimes n$ is just a function $f \in R$, and such a function is equivariant if and only if $f(g \cdot x) = \chi^n(g) f(x)$ on points. The set of equivariant sections of $\mathcal{L}^\otimes n$ (as a subset of $R$), is denoted $R^G_{X^n}$, the elements of which are called semi-invariants of weight $\chi^n$. Note that $\bigoplus_{n \geq 0} R^G_{X^n}$ is a graded ring.

**Definition 5.8.** In the situation above, we define the GIT quotient of $X$ by $G$ to be

$$X_{\chi} G := \text{Proj} \left( \bigoplus_{n \geq 0} R^G_{X^n} \right).$$

**Remarks 5.9.**

- The graded ring above is the ring of invariants of the action of $G$ on $X \times \mathbb{A}^1_k$ by $g \cdot (x, t) = (g \cdot x, \chi(g)^{-1}t)$ on points. Nagata’s theorem therefore shows that the GIT quotient is a variety if $X$ is.

- For any graded ring $S = \bigoplus_{n \geq 0} S_d$, there is a natural map $\text{Proj}(S) \to \text{Spec}(S_0)$. Therefore we obtain a map from $X_{\chi} G$ to the naive quotient $\text{Spec}(R^G)$, which is the degree 0 part of $\bigoplus_{n \geq 0} R^G_{X^n}$.

The GIT quotient gives us precisely the good quotients from theorem 5.7, for affine varieties.

**Proposition 5.10 ([M], Chapter 6).** In the situation of the definition above, there is a canonical quotient map $\pi: X_{ss} \to X_{\chi} G$, so by the uniqueness of categorical quotients, we obtain an expression for the quotients in theorem 5.7. Two points $x$ and $y$ in $X_{ss}$ are identified under $\pi$ if and only if the closures of $Gx$ and $Gy$ intersect. As every orbit closure contains a closed orbit, it follows that $X_{\chi} G$ parameterizes the closed orbits in $X_{ss}$.

### 5.2. Moduli of Representations

Our main use for quiver representations is to define moduli spaces thereof, which will allow us to associate a commutative resolution to a noncommutative one in section 5.3. General representations would yield too unwieldy a space, so we focus on representations of fixed dimension vector $\alpha$. Picking bases for the representations shows that a $(Q, R)$ representation of dimension $\alpha$ is a collection of matrices $(M^a \in k^{\alpha(h(a)) \times \alpha(t(k))})_{a \in Q_1}$ satisfying the relations given by $R$. Therefore, a coarse first attempt at defining our moduli space is given by the finitely generated $k$-algebra

$$A_\alpha := k[x_{mn}^a | a \in Q_1, 1 \leq n \leq \alpha(t(a)), 1 \leq m \leq \alpha(h(a))]/I_R,$$

where we use $I_R$ to denote the ideal generated by the relations between the $x_{mn}^a$ induced by $R$. We set $\text{Rep}_\alpha(Q, R) = \text{Spec}(A_\alpha)$, or just $\text{Rep}_\alpha(Q)$, if the relations are trivial. A $k$-point of $\text{Rep}(Q, R)$ is a collection of matrices as above, or a representation of dimension
α with a choice of basis.

There are more isomorphisms to factor out. These consist of a $GL(\alpha) = \prod_{i \in Q_0} GL(\alpha(i))(k)$-action on $(M^a)_{a \in Q_1} \in \text{Spec}(A_\alpha)(k)$, where $(g_i)_{i \in Q_0} \in GL(\alpha)$ acts by conjugating $M^a$, i.e., $(g_i)(M^a) = (g_{ia}M^ag_{ia}^{-1})$. Note that the subgroup $\Delta \cong k^\times$ of diagonal matrices acts trivially. To obtain our desired moduli space, we need some suitable way to define the quotient $\text{Spec}(A_\alpha)/GL(\alpha)$ (or, to apply the definitions from the first section directly, the quotient by $\text{PGL}(\alpha) := GL(\alpha)/\Delta$). This was done by King in [K], of which we now give an outline.

The fundamental notion is the following.

**Definition 5.11.** Let $(Q, R)$ be a quiver with relations, and $\theta: Q_0 \to \mathbb{Z}$ some function. For a representation $V = (V, f) \in f\text{Rep}(Q, R)$, we define

$$\theta(V) = \sum_{i \in Q_0} \theta(i) \dim_k V_i.$$ 

Such $V$ is said to be $\theta$-semistable if $\theta(W) \geq 0$ for all subrepresentations $W \subseteq V$. If, in addition, all proper nonzero $W \subseteq V$ satisfy $\theta(W) > 0$, then $V$ is called $\theta$-stable. If the notions of $\theta$-stability and $\theta$-semistability coincide, then $\theta$ is called generic.

The subset of $\theta$-(semi)stable points in $\text{Rep}_\alpha(Q, R)$ is denoted $\text{Rep}_{s\theta}^\alpha(Q, R)$.

**Remark 5.12.** Note that the group of characters on $GL(\alpha)$ is isomorphic to $\mathbb{Z}^{Q_0}$, mapping $\theta \in \mathbb{Z}^{Q_0}$ to $\chi_\theta := \prod_{i \in Q_0} \det^{\theta(i)}_{\alpha(i)}$ (see B.4). This will allow us to match up King’s notion of (semi)stability with the notion of (semi)stability from GIT.

The following theorem is proved in [K] for quivers without relations, but his proof still works if we replace the quiver by a quiver with relations.

**Theorem 5.13.** When $\chi_\theta(\Delta) = \{1\}$, the notion of $\theta$-(semi)stability for $\alpha$-dimensional representations of $(Q, R)$ coincides with the (semi)stability$^2$ associated to the character $\chi_\theta$ as in definition 5.8, when viewing points of $\text{Spec}(A_\alpha)$ as representations of $(Q, R)$.

The condition that $\chi_\theta$ is trivial on $\Delta$ means that for $\lambda \in \Delta \cong \mathbb{G}_m$, we have

$$1 = \chi_\theta(\lambda) = \prod_{i \in Q_0} \lambda^{\alpha(i)\theta(i)} = \lambda^{\sum_i \alpha(i)\theta(i)},$$

i.e., $\sum_i \alpha(i)\theta(i) = 0$. This motivates the following definition.

**Definition 5.14.** If $\theta \in \mathbb{Z}^{Q_0}$ satisfies $\sum_i \theta(i)\alpha(i) = 0$, then the moduli space of $\theta$-semistable representations of $(Q, R)$ is the GIT quotient

$$\mathcal{M}^{\text{ss}}_{\alpha, \theta}(Q, R) = \text{Spec}(A_\alpha)_{/\chi_\theta} GL(\alpha).$$

$^2$actually, we modify the stability condition to require that the stabilizers have dimension $\text{dim} \Delta = 1$, which is as low as it can get, considering the fact that $\Delta$ acts trivially on $\text{Spec}(A_\alpha)$.
By definition, this variety is the Proj of the graded ring $\bigoplus_{n \geq 0} A_{\lambda_n}^n$, where $A_{\lambda_n}^n$ consists of the functions $f$ on $\text{Spec}(A_n)$ satisfying $f(gx) = \lambda_n^g f(x)$ for all points $g$ of $G$ and $x$ of $\text{Spec}(A_n)$. Such functions are called $\chi$-semi-invariants of weight $n$.

We use $\mathcal{M}_{\alpha,\theta}^s(Q,R)$ to denote the open subset of $\mathcal{M}_{\alpha,\theta}^{ss}(Q,R)$ as given in theorem 5.7, i.e., the geometric quotient of $\text{Rep}_{\alpha,\theta}(Q,R)$ by $\text{GL}(\alpha)$.

**Example 5.15.** One choice of stability which works regardless of what dimension $\alpha$ we consider, is $\theta = 0$. It is clear that all representations are semistable, and that a representation is stable if and only if it is simple. By proposition 3.2 of [K], an orbit $\text{GL}(\alpha)M$ in $\text{Rep}_{\alpha}(Q)$ is closed if and only if $M$ is semisimple. It follows from theorem 5.10 that the moduli space $\mathcal{M}_{\alpha,0}^{ss}(Q)\times \mathcal{M}_{\alpha,0}^{ss}(Q,R)$ parameterizes the semisimple representations of $Q$, and $\mathcal{M}_{\alpha,0}^s(Q,R)$ the simple ones. We denote these moduli spaces $M_{\alpha}^{ss}(Q)$ and $M_{\alpha}^{ss}(Q,R)$, respectively (the ss stands for semisimple, and the s for simple). As $\text{Rep}_{\alpha}(Q,R)$ is closed in $\text{Rep}_{\alpha}(Q)$, similar remarks hold for $(Q,R)$, and we denote by $M_{\alpha}^{ss}(Q,R)$ and $M_{\alpha}^{ss}(Q,R)$ the resulting moduli spaces of \(\alpha\)-dimensional (semi)simples. Because $\text{Rep}_{\alpha,0}^{ss}(Q,R) = \text{Rep}_{\alpha}(Q,R)$, the moduli space $M_{\alpha}^{ss}(Q,R)$ is just the quotient $\text{Rep}_{\alpha}(Q,R)/\text{GL}(\alpha)$, so for any $\theta$ we have a map $\mathcal{M}_{\alpha,\theta}^{ss}(Q,R) \to M_{\alpha}^{ss}(Q,R)$ by the remark right after definition 5.8.

We collect some geometric properties of these moduli spaces in the next proposition.

**Proposition 5.16.**

(i) The stable moduli space $\mathcal{M}_{\alpha,\theta}^s(Q)$ is nonsingular;

(ii) the semistable moduli space $\mathcal{M}_{\alpha,\theta}^{ss}(Q,R)$ is projective over $M^{ss}(Q,R)$.

**Proof Sketch.** For part (i), consider representations of a quiver $Q$ without relations. The stabilizer in $\text{GL}(\alpha)$ of an $\alpha$-dimensional representation $M$ is precisely the automorphism group of $M$. Since this is the principal open subvariety of the affine space $\text{End}_Q(M)$ given by the determinant, it follows that it is irreducible and dense in $\text{End}_Q(M)$. If $M$ is stable, then its stabilizer has dimension 1, so that $\text{End}_Q(M)$ must be one-dimensional, and hence equal to $k \text{id}_M$, which shows that $\text{Aut}_Q(M) = k^{\times} \text{id}_M$. The stabilizer of $M$ in $\text{PGL}(\alpha)$ is $\text{Aut}_Q(M)/(k^{\times} \text{id}_M)$, and therefore trivial when $M$ is stable. It follows that $\mathcal{M}_{\alpha,\theta}^s(Q)$ is nonsingular regardless of our choice of $\theta$.

Part (ii) is a fact that holds for GIT quotients in general. \qed

**Example 5.17.** Let $(Q,R)$ be the second Beilinson quiver,

$$
\begin{array}{c}
\begin{array}{c}x_0, y_0, z_0 \quad e_0 \quad x_1, y_1, z_1 \quad e_1 \quad x_2, y_2, z_2 \quad e_2,
\end{array}
\end{array}
$$

with relations $y_1x_0 = x_1y_0$, $z_1x_0 = x_1z_0$, and $z_1y_0 = y_1z_0$. As dimension vector we take $\alpha = (1, 1, 1)$, and $\theta = (-2, 1, 1)$ as stability parameter. A dimension $\alpha$ representation is semistable if and only if it does not have subrepresentations of dimension $(1, 0, 0)$, $(1, 0, 1)$, and $(1, 1, 0)$. A trivial check shows that $\theta$ is generic.

We first consider the moduli of representations of the quiver $Q$ without the relations. Let $(a_0, b_0, c_0, a_1, b_1, c_1) \in \mathbb{A}_k^6 = \text{Rep}_\alpha(Q)$ be a representation. A dimension $(1, 0, 0)$
representation exists if and only if \( a_0 = b_0 = c_0 = 0 \), so \( \text{Rep}^{ss}_{\alpha, \theta}(Q) \subseteq \mathbb{A}^6_k \setminus \{(0) \times \mathbb{A}^2_k \} \). Similarly, the nonexistence of a \((1,1,0)\)-dimensional representation shows that \( \text{Rep}^{ss}_{\alpha, \theta}(Q) \subseteq \mathbb{A}^6_k \setminus \left( \{(0) \times \mathbb{A}^2_k \} \cup (\mathbb{A}^2_k \times \{0\}) \right) \). The nonexistence of a \((1,0,0)\)-dimensional representation implies that there are no \((1,0,1)\)-dimensional representations, so it follows that

\[
\text{Rep}^{ss}_{\alpha, \theta}(Q) = (\mathbb{A}^3_k \setminus \{0\})^2.
\]

The action of \( \text{GL}(\alpha) = \mathbb{G}^3_m \) on \( \text{Rep}_{\alpha}(Q) = \mathbb{A}^6_k \) is given by

\[
(s, t, u)(a_0, b_0, c_0, a_1, b_1, c_1) = \left( \frac{t}{s}a_0, \frac{t}{s}b_0, \frac{t}{s}c_0, \frac{u}{t}a_1, \frac{u}{t}b_1, \frac{u}{t}c_1 \right),
\]

so the \( \mathbb{G}^3_m = \text{PGL}(\alpha) \)-action is \( (s, t) \cdot (x, y) = (sx, ty) \), where \( x, y \in \mathbb{A}^3_k \setminus \{0\} \), and \( (s, t) \) is the coset of the point \((s^{-1}, 1, t)\). It follows that

\[
\mathcal{M}^s_{\alpha, \theta}(Q) = \left( (\mathbb{A}^3_k \setminus \{0\}) / \mathbb{G}^3_m \right)^2 = \mathbb{P}^2_k \times \mathbb{P}^2_k.
\]

Alternatively, we can simply apply the definition of the GIT quotient to compute \( \mathcal{M}^s_{\alpha, \theta}(Q) \). From equation (5.1), it follows that a function \( f \) on \( \mathbb{A}^6_k \) is a \( \chi_\theta \)-semi-invariant of weight \( n \) if and only if it has degree \( 2n \) in \( a_0, b_0, c_0 \) and degree \( n \) in \( a_1, b_1, c_1 \). From [H] exercise II.5.11 we therefore obtain

\[
\mathcal{M}^s_{\alpha, \theta}(Q) = \text{Proj} \left( k[a_0, b_0, c_0]^{[2]} \right) \times \text{Proj} \left( k[a_1, b_1, c_1] \right) \cong \mathbb{P}^2_k \times \mathbb{P}^2_k,
\]

where, for a graded ring \( S \), \( S^{[d]} \) denotes the graded ring \( \bigoplus_{n \geq 0} S_{dn} \), which satisfies \( \text{Proj}(S^{[d]}) \cong \text{Proj}(S) \).

To obtain \( \mathcal{M}^s_{\alpha, \theta}(Q, R) \), we simply take the subvariety of \( \mathcal{M}^s_{\alpha, \theta}(Q) = \mathbb{P}^2_k \times \mathbb{P}^2_k \) cut out by \( a_0b_1 = a_1b_0, a_0c_1 = a_1c_0, \) and \( b_0c_1 = c_0b_1 \). Some explicit computations show that this is just the diagonal, so we find \( \mathcal{M}^s_{\alpha, \theta}(Q, R) \cong \mathbb{P}^2_k \).

If we take \( \theta = (0,0,0) \) instead, then every representation is \( \theta \)-semistable, so \( \mathcal{M}^s_{\alpha, \theta}(Q) \) is the categorical quotient of \( \mathbb{A}^6_k \) by the \( \text{GL}(\alpha) \)-action in equation (5.1), i.e., \( \text{Spec}(\mathcal{A}^{\text{GL}(\alpha)}) \), where \( A \) is the coordinate ring of \( \mathbb{A}^6_k \). From the definition of the action it is clear that this is Spec\( (k) \), illustrating the impact of the choice of stability parameter on the form of the moduli space. Using the equations given by \( R \), we obtain \( \mathcal{M}^s_{\alpha, \theta}(Q, R) = \text{Spec}(k) \) as well. Being \( \theta \)-stable is the same as being simple, and there are no simple representations of dimension \( \alpha = (1,1,1) \) for either \( Q \) or \( (Q, R) \), so the stable moduli spaces are empty, in sharp contrast with the previous stability parameter, for which the stable and semistable moduli spaces coincided.

5.3. From NCCRs to Commutative Resolutions

In [vdB], van den Bergh proves the following theorem.
Theorem 5.18. If $R$ is a 3-dimensional terminal Gorenstein singularity, then a commutative crepant resolution exists if and only if an NCCR exists. If $Y \to \text{Spec}(R)$ is a crepant resolution, and $\Lambda$ is an NCCR, then there is a triangulated equivalence $D^b \text{Coh}(Y) \cong D^b \Lambda \text{-Mod}^{ss}$.

In particular, if $Y_1$ and $Y_2$ are crepant resolutions of the same singularity in this setup, then there is an NCCR $\Lambda$, and we have a string of equivalences

$$D^b \text{Coh}(Y_1) \cong D^b \Lambda \text{-Mod}^{ss} \cong D^b \text{Coh}(Y_2),$$

proving the commutative Bondal-Orlov conjecture in dimension 3.

Given the existence of an NCCR $\Lambda$, he proceeds by presenting it as a quiver with relations $(Q,I)$ in order to compute its moduli space of representations for a suitable dimension vector $\alpha$ and stability parameter $\theta$. He shows that $M^s_{\alpha}(Q,I)$ is precisely $\text{Spec}(R)$, hence obtaining a resolution $\mathcal{M}_{\alpha,\theta}^s \to \text{Spec}(R)$. He proves that $\mathcal{M}_{\alpha,\theta}^s(Q,I)$ and $\Lambda$ are derived equivalent, so for reasons similar to the proof of theorem 3.21, this resolution is crepant (or see corollary 4.15 of [IW3]). This is the direction we are interested in. We finish by explicitly computing the associated crepant resolution of the $A_n$ singularity.

**An example: $A_n$ singularities**

Recall the presentation $(Q,I)$ of the $A_{n-1}$ NCCR given in example 4.3. We consider the representations of $(Q,I)$ of dimension $\alpha = (1,\ldots,1)$, and we take $\theta = (1 - n, 1, \ldots, 1)$ as the stability parameter. An easy verification shows that $\theta$ is generic, and that an $\alpha$-dimensional representation is semistable if and only if it has no subrepresentations of dimension $(1, \alpha')$, where $\alpha'$ is an $(n-1)$-tuple of 0s and 1s with at least one 0.

We claim that the semistable representations are precisely those for which there exists $-1 \leq i \leq n - 2$ such that $v_0 \cdots v_i \neq 0$ and $u_0u_{n-1}u_{n-2} \cdots u_{i+3} \neq 0$ (when $i = -1$, the former is vacuous, when $i = n - 2$, the latter is). If a representation $M$ does not satisfy any of these conditions, then there is an $i \neq 0$ such that the two arrows pointing into $M_i$ are both 0. Then the representation $N$ obtained from $M$ replacing $M_i$ and the arrows coming out of it by 0 is a subrepresentation of $M$. Now $\theta(N) = -1$, so $M$ is not semistable. Conversely, if $M$ does satisfy one of these conditions, then an argument similar to the one in example 5.17 shows that $M$ is semistable.

The condition $v_0 \cdots v_i \neq 0$ and $u_0u_{n-1}u_{n-2} \cdots u_{i+3} \neq 0$ defines a $\text{GL}(\alpha)$-invariant open subset $U_i$ of $\text{Rep}_{\alpha,\theta}(Q,I)$. The relations $I$ imply that $v_{i+2} = v_{i+1}u_{i+2}/u_{i+3}$, so by induction we obtain $v_j = v_{i+1}u_{i+2}/u_{j+1}$ for all $j \geq i + 2$. Similarly, $u_j = v_{i+1}u_{i+2}/v_{j-1}$ for all $j \leq i + 1$. It follows that the map $U_i \to \mathbb{G}^{i+1}_m \times \mathbb{A}^2 \times \mathbb{G}^{n-i-2}_m$ given by mapping a representation to $(v_0, \ldots, v_i, v_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_n)$ is an isomorphism. Under this isomorphism, the action of $s = (s_0, \ldots, s_{n-1}) \in \text{GL}(\alpha)$ on $U_i$ is given by

$$s \cdot (v_0, \ldots, v_i, v_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_n) = \left(\frac{s_{j+1}}{s_j} v_{j}, \frac{s_{j-1}}{s_{\ell}} u_{\ell}\right),$$

where on the right-hand side $j$ runs from 0 to $i + 1$, and $\ell$ runs from $i + 2$ to $n$. Therefore, acting with $s$ on a monomial $v_0^{a_0} \cdots v_i^{a_i}v_{i+1}^{m_{i+1}}u_{i+2}^{m_{i+2}}u_{i+3}^{m_{i+3}} \cdots u_n^{m_n}$ multiplies that monomial
by a factor

\[ s_0^{-m_n-l_0} s_1^{-l_1} \cdots s_{i-1}^{-l_{i-1}} s_i^{-\ell_i} s_{i+1}^{m_i+3+\ell_i} \cdots s_{i+3}^{m_{i+3}} \cdots s_{n-1}^{m_{n-1}}. \]  

(5.2)

For this monomial to be invariant under the \( GL(\alpha) \)-action, we therefore need \( \ell_0 = -m_n, \ell_1 = \ldots = \ell_i, m_{i+3} = m_{i+4} = \ldots = m_n \), and \( \ell_0 = \ell - m \) (the vanishing of the exponent of \( s_{i+2} \) is automatic). Since the \( GL(\alpha) \)-action preserves the \( \mathbb{Z}^{i+1} \times \mathbb{Z}_{\geq 0}^2 \times \mathbb{Z}^{n-i-2} \)-grading on the coordinate ring \( A_i \) of \( U_i \), it follows that

\[ A_i^{GL(\alpha)} = k \left[ \frac{v_0 \cdots v_i}{u_0 + 3 \cdots u_{n-2}} \frac{u_{i+1} + \cdots u_n}{v_0 \cdots v_i} \right]. \]

We use the left (resp. right) generator as the \( x \) (resp. \( y \)) coordinate of the plane \( \mathbb{A}^2_k = U_i/GL(\alpha) \), which we denote \( x_i \) and \( y_i \).

We can now glue these affine pieces to compute the geometric quotient \( \mathcal{M}^{\alpha,\theta}_{a,q}(Q,R) \).

The intersection of \( U_{i-1} \) and \( U_i \) \((i \neq -1)\) consists of representations with \( u_0 \cdots v_i \neq 0 \) and \( u_{i+2} \cdots u_n \neq 0 \). From the relation \( u_{i+1} v_i = v_{i+1} u_{i+2} \) we gather that with respect to the isomorphisms \( U_i \cong \mathbb{G}_m^{i+1} \times \mathbb{A}^2_k \times \mathbb{G}_m^{n-i-2} \) and \( U_{i-1} \cong \mathbb{G}_m^i \times \mathbb{A}^2_k \times \mathbb{G}_m^{n-i-1} \), the gluing isomorphism is given by

\[ V := \mathbb{G}_m^{i+1} \times \mathbb{A}^1_k \times \mathbb{G}_m \times \mathbb{G}_m^{n-i-2} \longrightarrow \mathbb{G}_m^i \times \mathbb{G}_m \times \mathbb{A}^1_k \times \mathbb{G}_m^{n-i-1} =: W \]

\[ (v_0, \ldots, v_{i+1}, u_{i+2}, \ldots, u_n) \mapsto (v_0, \ldots, v_i, v_{i+1} u_{i+2}, u_{i+2}, \ldots, u_n) \]

The image of \( V \) under the quotient map \( U_i \to \mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{G}_m \), the image of \( W \) under \( U_{i-1} \to \mathbb{A}^2_k \) is \( \mathbb{G}_m \times \mathbb{A}^1_k \), and from the gluing map \( V \to W \) we obtain the gluing isomorphism

\[ (x_i, y_i) \mapsto \left( \frac{1}{y_i}, y^2 x_i \right) \]

(5.3)

If we intersect \( U_i \) with \( U_j \), \( j \neq i \pm 1 \), then we get representations in \( U_i \) with either \( v_{i+1} \) or \( u_{i+2} \) nonzero (because one of these two has to be nonzero in \( U_j \)), plus some additional conditions. It follows that \( U_i \cap U_j \) is contained in one of \( U_i \cap U_{i+1} \) and \( U_{i-1} \cap U_i \). Another intersection that is covered by the intersections above is that of \( U_{i-1} \) with \( U_{i-2} \), because in this intersection we have \( u_2 \neq 0 \) and \( v_1 \neq 0 \) by definition, so \( u_2 v_1 = u_1 v_0 \) implies \( u_1 \neq 0 \neq v_0 \). Similarly, \( u_0 \neq 0 \neq v_{n-1} \), so it follows that \( U_0 \cap U_{n-1} \) is contained in any of the intersections \( U_i \cap U_{i-1} \) with \( i > -1 \). It follows that the gluing isomorphisms (5.3) completely describe the moduli space \( X := \mathcal{M}^{\alpha,\theta}_{a,q}(Q,I) \).

The description of \( X \) above immediately shows that \( X \) is nonsingular (it is covered by copies of \( \mathbb{A}^2_k \), and from proposition 5.16 we obtain a projective morphism

\[ \pi : X \longrightarrow M^{ss}_{\alpha,\theta}(Q,I) =: Y, \]

which must be a crepant resolution by corollary 4.15 of [IW3].

What remains to be shown is that \( Y \) is actually the \( A_{n-1} \) singularity \( k[U^n, UV, V^n] \) we started out with. We need to compute the quotient of

\[ \text{Rep}^{ss}_{\alpha,0}(Q,I) = \text{Spec}(k[u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}]/I) =: \text{Spec}(S), \]

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where we abusively denote by $I$ the ideal of $\text{Rep}_\alpha(Q)$ induced by the quiver relations $I$, by the action of $s = (s_i) \in \text{GL}(\alpha) = \mathbb{G}_m^n$ given by

$$(s_i)(u_j, v_\ell) = \left(\frac{s^{-1}_{j}u_j}{s^1_{j}}, \frac{s_{\ell+1}v_\ell}{s_{\ell}}\right).$$

Under the action of $s$, a monomial $v_0^{\ell_0} \cdots v_i^{\ell_i} v_{n-1}^{\ell_{n-1}} u_0^{m_0} \cdots u_{n-1}^{m_{n-1}}$ in $S$ picks up a factor similar to the one in equation (5.2). Similar reasoning shows that $m_i + 1 - \ell_i = m_i - \ell_i - 1$ for all $i$ is necessary and sufficient. The monomials $u_{i+1} v_i$ clearly satisfy this relation. Suppose $p = v_0^{\ell_0} \cdots v_i^{\ell_i} v_{n-1}^{\ell_{n-1}} u_0^{m_0} \cdots u_{n-1}^{m_{n-1}}$ is another $\text{GL}(\alpha)$-invariant monomial, not divisible by any of the monomials $u_{i+1} v_i$. Without loss $\ell_0$ is nonzero (rotate the quiver and invert its arrows, if necessary). It follows that $m_1 = 0$, because otherwise $u_1 v_0$ would divide $p$. From $m_2 - \ell_1 = -\ell_0$ we obtain that $\ell_1$ is nonzero, so because $p$ is not divisible by $u_2 v_1$, $m_2$ must be 0. Hence $\ell_0 = \ell_1$. Proceeding by induction, we find that $p$ is a power of $v_0 \cdots v_{n-1}$. Therefore the invariant ring $S^{\text{GL}(\alpha)}$ is the subring of $S$ generated by the $u_{i+1} v_i$, $v_0 \cdots v_{n-1}$, and $u_0 \cdots v_{n-1}$.

Define $k[u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}] \to k[U, V]$ by $u_i \mapsto U$, $v_i \mapsto V$. This map factors through $S \to k[U, V]$, and precomposing with the inclusion we obtain surjection $S^{\text{GL}(\alpha)} \to R$ (our description of the generators of $S^{\text{GL}(\alpha)}$ shows that it lands in $R$). Comparing the homogeneous components with respect to the $\mathbb{Z}_2^2$-grading on both sides, we find that $S^{\text{GL}(\alpha)} \cong R$. Therefore we have found a crepant resolution $\mathcal{M}_{\alpha, \theta}^a(Q, I) \to \text{Spec}(R)$ of the $A_{n-1}$ singularity.
A. Some Convex Geometry

Because of its relevance to the theory of toric varieties, we summarize in this appendix some basic convex geometry, omitting insubstantial proofs. All of the material here can be found in [F] and [O]. Let $N$ be a finite rank lattice, and let $V = \mathbb{R} \otimes \mathbb{Z} N$.

**Definition A.1.** A convex polyhedral cone $\sigma$ is a subset of $V$ of the form

$$\{r_1v_1 + \cdots + r_sv_s \mid r_i \geq 0 \text{ for all } i\},$$

where the $v_i$ are some elements of $V$, called the generators of $\sigma$. The dual $\sigma^\vee$ is defined as the set of $\phi \in V^\vee$ such that $\phi|_\sigma$ is nonnegative. The faces of $\sigma$ are defined as the sets of the form $\sigma \cap u^\perp$, with $u \in \sigma^\vee$. The dimension of $\sigma$ is defined as the dimension of $\sigma + (-\sigma)$ as a real vector space. A facet is a codimension 1 face. A strongly convex cone is one which does not contain a linear subspace of $V$.

**Proposition A.2** (Basic facts about faces). Let $\sigma$ be a convex polyhedral cone, generated by $v_1, \ldots, v_s \in V$. Then there holds

1. the faces $\sigma \cap u^\perp$ of $\sigma$ are again convex polyhedral cones, generated by those $v_i$ such that $\langle u, v_i \rangle = 0$,
2. there are only finitely many faces of $\sigma$,
3. $\sigma \cap (u + v)^\perp = (\sigma \cap u^\perp) \cap (\sigma \cap v^\perp)$, so the intersection of faces is again a face,
4. the face of a face of $\sigma$ is again a face of $\sigma$,
5. every proper face of $\sigma$ is contained in a facet of $\sigma$,
6. every proper face is the intersection of the facets containing it.

**Proof.** For fact 4, suppose that $\tau = \sigma \cap u^\perp$ is a face of $\sigma$, and that $\rho = \tau \cap v^\perp$ is a face of $\tau$, where $u \in \sigma^\vee$ and $v \in \tau^\vee$. Due to the fact that $u \in \sigma^\vee$ and the linearity of $u$ and $v$, we can find a large positive real number $p$ such that $v + pu \in \sigma^\vee$. Now $\rho = \sigma \cap (v + pu)^\perp$.

For 5, let $U = \sigma + (-\sigma)$ and $W = \tau + (-\tau)$. Then we can perturb the hyperplane $H \in \mathbb{P}(U/W)^\vee$ associated to $\tau$ in such a way that it contains one more of the generators of $\sigma$. The perturbed hyperplane defines a face of $\sigma$ which contains $\tau$, and which has strictly lower codimension than $\tau$.

In the case that $\tau$ has codimension two, $\mathbb{P}(U/W)^\vee$ is just $S^1$, so one obtains exactly two facets of $\sigma$ containing $\tau$ (by rotating $H$ in the two separate directions). It follows that $\tau$ is the intersection of two distinct facets, proving the base case for fact 6. \qed
We state the following important result without proof (a proof may be found in [R2]).

**Theorem A.3 (Duality theorem).** If $\sigma$ is a convex polyhedral cone, then $\sigma^\vee$ is a convex polyhedral cone, and under the natural identification $(V^\vee)^\vee = V$, we have $(\sigma^\vee)^\vee = \sigma$.

**Remark A.4.** The fact that the dual of a convex polyhedral cone is again a convex polyhedral cone is commonly known as Farkas’ theorem.

**Corollary A.5.** The topological boundary of $\sigma$ in $\sigma + (-\sigma)$ is the union of its facets.

**Proposition A.6.** If $\tau = u^\perp \cap \sigma$ is a face of $\sigma$, then $\tau^\vee$ is $\sigma^\vee + \mathbb{R}_{\geq 0}(-u)$.

**Proof.** By duality, we only need to show that the dual of $\tau$ is $(\sigma^\vee + \mathbb{R}_{\geq 0}(-u))^\vee$, and this is trivial, because the dual of a sum is the intersection of the duals. \qed

This proof shows the power of duality, as it allows us to conclude from $(\sigma_1 + \sigma_2)^\vee = \sigma_1^\vee \cap \sigma_2^\vee$ (which is easy) that $(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$ (which is hard).

When $\sigma + (-\sigma) = V$, we have an injective map

$$\{\text{facets of } \sigma\} \hookrightarrow P V^\vee,$$

mapping a facet $\tau$ to the associated hyperplane $[u_\tau]$. The representative $u_\tau$ can be, and is always taken from $\sigma^\vee$. The half-space $\{\langle u_\tau, v \rangle\}$ is denoted by $H_\tau$.

**Corollary A.7 (dual description of convex polyhedral cones).** If $\sigma$ spans $V$, then we have

$$\sigma = \bigcap_{\text{facets } \tau} H_\tau.$$

If $\sigma = V$, one can construct a generating set of $\sigma^\vee$ as follows. Solve $\langle u, v_i \rangle = 0$ for any set of $n - 1$ independent generating vectors of $\sigma$. If $\pm u \in \sigma^\vee$, $\pm u$ is added to the generating set of $\sigma^\vee$. If $\sigma$ does not span $V$, one needs to add generators of $\sigma^\perp \subseteq V^\vee$.

**Definition A.8.** A convex polyhedral cone is called rational if it can be generated by elements from $N$. As $N$ is countable, and generating sets are finite, the majority of convex polyhedral cones are not rational.

**Proposition A.9 (Gordan’s lemma).** If $\sigma$ is a rational polyhedral cone, then $\sigma^\vee \cap N^\vee$ is a finitely generated semigroup.

**Proof.** Take a finite number of elements in $N^\vee$ which generate $\sigma^\vee$ (as a cone), and intersect $N^\vee$ with the convex set spanned by these generators. The resulting set generates $\sigma^\vee \cap N^\vee$, and is finite because $N^\vee$ is discrete, and the convex set spanned by the generators is compact. \qed

**Proposition A.10 (Separation lemma).** If $\sigma$ and $\sigma'$ are two convex polyhedral cones such that $\sigma \cap \sigma'$ is a face of both, then there exists $u \in \sigma^\vee \cap (\sigma')^\vee$ for which

$$\sigma \cap \sigma' = \sigma \cap u^\perp = \sigma' \cap u^\perp.$$
Corollary A.11. If $\sigma$ and $\sigma'$ intersect in a common face, then

$$\sigma^\vee \cap N^\vee + (\sigma')^\vee \cap N^\vee = N^\vee (\sigma \cap \sigma')^\vee.$$ 

Proof. As $(\sigma \cap \sigma')^\vee = \sigma^\vee + (\sigma')^\vee$ (see the remark after the duality theorem), $(\sigma^\vee \cap N^\vee) + ((\sigma')^\vee \cap N^\vee) \subseteq (\sigma \cap \sigma')^\vee \cap N^\vee$. For the converse, we use the separation lemma to find $u \in N^\vee$ such that $u \in \sigma^\vee$, $-u \in (\sigma')^\vee$, and $\sigma \cap \sigma' = \sigma \cap u^\perp = \sigma' \cap u^\perp$. Then by proposition A.6, $(\sigma \cap \sigma')^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-u)$, so the claim follows. $\square$

The following result is proposition A.9 in [O].

Proposition A.12. The faces of a convex polyhedral cone $\sigma$ are precisely the subsets $\tau \subseteq \sigma$ such that $v \in \sigma \setminus \tau$ and $w \in \sigma$ implies $w + v \notin \tau$, and $0 \in \tau$. 

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B. Lattices and Algebraic Tori

Let $k$ be an algebraically closed field. In this appendix we collect some basic results about algebraic groups. In particular show that tori over $k$ are equivalent to lattices, which is of importance to the theory of toric varieties. We also define and give basic properties of maps from $\mathbb{G}_m$ into a variety, and we compute the characters of a general linear group, which we use in the construction of moduli spaces of quiver representations.

B.1. Tori

Let $N$ be a lattice, i.e., a finite-rank free abelian group. The associated torus is defined as $T_N = \text{Hom}_\mathbb{Z}(N^\vee, \mathbb{G}_m)$, where $\mathbb{G}_m$ is the multiplicative algebraic group $\mathbb{C}^*$, and $N^\vee$ is the dual of $N$. The one-parameter subgroups of $T_N$ are given by the map $N \to \text{Hom}(\mathbb{G}_m, T_N)$, $v \mapsto \rho_v$, defined by $\rho_v(t)(u) = t^{\langle u, v \rangle}$ (the hom-set is taken in the category of algebraic groups). This map is clearly injective, and from $\text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ we obtain its surjectivity. The latter fact can be seen by observing that if $\phi: \mathbb{G}_m \to \mathbb{G}_m$ is a homomorphism, then we have $\phi^*(T)\phi^*(T') = (\phi^* \otimes \phi^*)\Delta T = \Delta \phi^*(T) = \phi^*(TT')$, where $\Delta$ is the comultiplication associated to the multiplication $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$, with which $\phi^*$ commutes because $\phi$ is a homomorphism. It follows that $\phi$ must be of the form $z \mapsto z^k$ for some $k \in \mathbb{Z}$. The group of characters of $T_N$ is isomorphic to $N^\vee$, by $N^\vee \to \text{Hom}(T_N, \mathbb{G}_m)$, $u \mapsto \chi^u$, given as $\chi^u(\phi) = \phi(u)$. Composition of characters with one-parameter subgroups gives a bilinear pairing $N \otimes \mathbb{Z} N^\vee \to \mathbb{Z}$, which is easily seen to be the usual duality pairing.

**Proposition B.1.** The mapping $N \mapsto T_N$ gives a fully faithful functor from the category of finite-rank lattices into the category of algebraic groups.

**Proof.** As it is just the composition of the functors $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$ and $\text{Hom}(-, \mathbb{G}_m)$, it is clear that $N \mapsto T_N$ is indeed a functor.

By choosing bases, a morphism $N \to N'$ can be identified with a matrix $M$, and the spaces $T_N$ and $T_{N'}$ can be identified with $\mathbb{G}_m^{\text{rk}N}$ and $\mathbb{G}_m^{\text{rk}N'}$. The morphism $T_N \to T_{N'}$ associated to $M$ is given by $(t_i) \mapsto \left( \prod_{i=1}^{\text{rk}N'} t_i^{M_i} \right)$. 

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from which we can obviously recover $M$, showing that $N \to T_N$ is faithful. Conversely, every morphism $T_N \to T_N'$ is of this form because $\text{Hom}(G_m, G_m) \cong \mathbb{Z}$, so $N \to T_N$ is full. 

**Proposition B.2.** The epimorphisms in the category of lattices consist of the homomorphisms $f: N_1 \to N_2$ whose image has finite index in $N_2$, or equivalently such that $\text{id}_k \otimes f$ is surjective. The monomorphisms are precisely the injective maps. The epimorphisms in the category of tori are precisely the surjective homomorphisms.

**Proof.** If $f$ is mono, then $f_*: \text{Hom}(\mathbb{Z}, N_1) \to \text{Hom}(\mathbb{Z}, N_2)$ is injective by definition. But $\mathbb{Z}$ obviously represents the forgetful functor into the category of sets, so we find that $f$ is injective. The converse is obvious.

Let $f: \mathbb{Z}_k^\ell \to \mathbb{Z}_r^\ell$ be an epimorphism, represented by a matrix $A$. Then, by duality, the transpose of $A$ gives a monomorphism in the category of lattices, which shows that it has trivial kernel. As a consequence, the matrix $A$ represents a surjective linear map $\mathbb{Q}_k^\ell \to \mathbb{Q}_r^\ell$, which is obviously equivalent to $f(\mathbb{Z}_k^\ell)$ being of finite rank in $\mathbb{Z}_r^\ell$.

For the final part, let $\phi: T_{2\ell} \to T_{2\ell}$ be an epimorphism of lattices. We prove the case with algebraically closed $k$, from which the general case follows by standard arguments. The induced map $(\mathbb{Z}_k^\ell)^\vee \to (\mathbb{Z}_r^\ell)^\vee$ between the lattices of characters must be mono, hence injective, so it follows that $r \geq \ell$. Since it is surjective as a map between $\mathbb{Q}$-vector spaces, the matrix $A: \mathbb{Z}_r^\ell \to \mathbb{Z}_k^\ell$ admits a rational section $B$, i.e., an $r \times \ell$ matrix whose entries are rational, and such that $BA$ is the identity matrix. Let $s = (s_1, \ldots, s_\ell)$ be a $k$-point of $\mathbb{G}_m$. Then the $k$-point $(t_1, \ldots, t_r)$ of $\mathbb{G}_m$ given by $t_i = \prod_j s_j^{B_i j}$ obviously gets mapped to $s$ under $\phi$. Here we use $k = \overline{k}$ to make sense of the fractional powers of the $s_i$ occurring in the definition of the $t_i$. 

**B.2. Torus Actions**

The action of a $k$-torus $T_N$ on an affine $k$-scheme $X = \text{Spec}(R)$ is defined as a morphism $T_N \times X \to X$ satisfying the obvious axioms. As we are working with affine schemes, we can equivalently describe this as a map $\rho: R \to R \otimes_k [N^\vee]$ such that $(\text{id} \otimes \Delta)\rho = (\rho \otimes \text{id})\rho$ and $(\text{id} \otimes \varepsilon)\rho = \text{id}$, where $\varepsilon$ and $\Delta$ are the coalgebra structure maps on $k[N^\vee]$. If $r \in R$, we can decompose $\rho(r)$ as $\sum_{m \in N^\vee} r_m \otimes \chi^m$, so that

$$\sum_{m \in N^\vee} r_m \otimes \chi^m \otimes \chi^m = (\text{id} \otimes \Delta)\rho(r) = (\rho \otimes \text{id})\rho(r) = \sum_{m \in N^\vee} \rho(r_m) \otimes \chi^m.$$

So from the linear independence of the $\chi^m$ over $k$, we obtain $\Delta(r_m) = r_m \otimes \chi^m$. From $(\text{id} \otimes \varepsilon)\rho = \text{id}$ we obtain $r = \sum_{m \in N^\vee} r_m$, so that $R = \sum_{m \in N^\vee} R_m$, where $R_m = \{ r \in R \mid \rho(r) = r \otimes \chi^m \}$. Since the $\chi^m$ are linearly independent over $k$, we see that this is in fact a direct sum, and because $\rho$ is an algebra homomorphism, the multiplication on $R$ maps $R_m \times R_n$ into $R_{m+n}$. Therefore we obtain what is known as an $N^\vee$-grading on $R$: a direct sum decomposition $R = \bigoplus_{m \in N^\vee}$ such that multiplication maps $R_m \times R_n$ into $R_{m+n}$.
Conversely, if $R = \bigoplus_{n \in \mathbb{N}} R_n$ is an $\mathbb{N}$-graded $k$-algebra, then it is easy to verify that $\rho(\sum_{m \in \mathbb{N}} r_m) = \sum_{m \in \mathbb{N}} r_m \otimes \chi^m$ defines a $T_N$-action on $\text{Spec}(R)$.

It is clear that a map between two affine schemes with $T_N$-action is equivariant if and only if the corresponding map on rings is a homogeneous degree 0 map (in the obvious sense). Thus, using the fact that $\mathbb{N}$ is the group of characters on $T_N$, we have the following proposition.

**Proposition B.3.** Let $T$ be a $k$-torus. The category of affine $k$-schemes with a $T$-action and $T$-equivariant morphisms is equivalent to the category of $\text{Hom}(T, \mathbb{G}_m)$-graded rings and homogeneous degree 0 $k$-algebra homomorphisms.

### B.3. Limits without Epsilons

Given a separated $k$-scheme $X$ and morphism $\gamma : \mathbb{G}_m \to X$, we say the limit $\lim_{t \to 0} \gamma(t)$ exists if $\gamma$ can be extended to a map $\mathbb{A}_k^1 \to X$. Since $X$ is separated, $\mathbb{A}_k^1$ is reduced, and $\mathbb{G}_m \subseteq \mathbb{A}_k^1$ is a dense open subset, this extension is unique. So given the existence of $\lim_{t \to 0} \gamma(t)$, the image of $0 \in \mathbb{A}_k^1(k)$ under the extension is a well-defined $k$-rational point of $X$, which we will denote by $\lim_{t \to 0} \gamma(t)$.

**Proposition B.4.** If $X = \text{Spec}(R)$ is an affine finite type $k$-scheme and $\gamma : \mathbb{G}_m \to X$, then for any global set of coordinates $(x_1, \ldots, x_n) = R = \text{Hom}_k(X, \mathbb{A}_k^1)$, the limit $\lim_{t \to 0} \gamma(t)$ exists if and only if $\lim_{t \to 0} x_i \circ \gamma(t)$ exists for every $i$.

**Proof.** The implication from left to right is immediate from the definitions. For the converse, the existence of $\lim_{t \to 0} \gamma(t)$ is equivalent to the ring map $\gamma^* : R \to k[t, t^{-1}]$ factoring through the inclusion $k[t] \to k[t, t^{-1}]$, which is equivalent to $\gamma^*(x_i)$ being a polynomial in $t$ for all $i$, since $R = (x_1, \ldots, x_n)$. But the existence of $\lim_{t \to 0} x_i \circ \gamma(t)$ means that the image of $t \in k[t]$ under $(x_i \circ \gamma)^*$ lands in $k[t]$, that is, $\gamma^*(x_i) = \gamma^* x_i^*(t) \in k[t]$, so we’re done.

From the proof, we see that the existence of $\lim_{t \to 0} \gamma(t)$ for $\gamma : \mathbb{G}_m \to \mathbb{A}_k^1$ means that $\gamma^*(t)$ is a polynomial $\sum_{i \geq 0} c_i t^i$, and that $\lim_{t \to 0} \gamma(t)$ is $c_i$ in this case. This shows that for $k = \mathbb{C}$, this notion of limit coincides with the analytical one.

Proper morphisms of finite presentation lift limits in the following sense.

**Proposition B.5.** Suppose $X \to Y$ is a proper finite presentation morphism of $k$-schemes, and we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{G}_m, & \longrightarrow & X \\
\text{inclusion} \downarrow & & \downarrow \\
\mathbb{A}_k^1 & \longrightarrow & Y
\end{array}
$$

Then $\lim_{t \to 0} \gamma(t)$ exists, and maps to the limit of $\mathbb{G}_m, \to Y$ under $X \to Y$. 47
Proof. Immediate consequence of proposition 15.5 in [GW].

Note that the proposition in particular implies that proper morphisms of varieties lift limits.

B.4. Characters of the General Linear Group

The Hopf algebra of $\text{GL}_{n,k}$ is the localization $R$ of $k[x_{11}, x_{12}, \ldots, x_{nn}]$ at the determinant polynomial $\det$, with the comultiplication and counit given by

$$\Delta(x_{ij}) = \sum x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}.$$ 

It is possible to write down the antipode using the formula for the inverse of a matrix in terms of its adjugate and determinant, but we will not need this.

We wish to determine the characters of this group, i.e., the algebraic homomorphisms $\chi: \text{GL}_{n,k} \rightarrow \mathbb{G}_m,k$. These are in natural bijective correspondence with the Hopf algebra morphisms from $k[t, t^{-1}]$ into $R$. A $k$-algebra morphism $k[t, t^{-1}] \rightarrow R$ is just an invertible element $u$ of $R$, and $u$ is group-like and satisfies $\varepsilon(u) = 1$ if and only if the corresponding map is a morphism of counital bialgebras. The fact that $\det$ is irreducible shows that the invertibles of $R$ are precisely the integer powers of $\det$ times an element of $k^\times$. Such $u$ get mapped to 1 by $\varepsilon$ if and only if they are of the form $\det^k$. This shows that all characters on $\text{GL}_{n,k}$ are powers of the determinant, and from linear algebra we know that each of these is in fact a character.

Furthermore, the integer $k$ can be recovered from $\det^k$, so we obtain an isomorphism

$$\mathbb{Z} \rightarrow \text{Hom}_{\text{alg.grp./}k}(\text{GL}_{n,k}, \mathbb{G}_m,k)$$

$$k \mapsto \det^k.$$
Populaire Samenvatting

Algebra en meetkunde zijn zeer sterk aan elkaar gerelateerd. Bijvoorbeeld, het bovenstaande citaat komt uit Germain's *Pensées Diverses*, waar ze dit stelt in de context van krommen en hun vergelijkingen, die respectievelijk duidelijk aan de meetkunde en algebra toebehoren. In de eeuwen die daarop volgden, is het citaat niet minder waar geworden, mits we het over *commutatieve* algebra hebben. De dualiteit wordt dan doorgaans gegeven door aan een meetkundig object $M$ de ring van functies $\mathcal{O}(M)$ op $M$ toe te kennen (bijvoorbeeld, als $M$ een topologische ruimte is, kan $\mathcal{O}(M)$ de ring van reëelwaardige continue functies op $M$ zijn). Deze dualiteit is de grondslag van de *algebraïsche* meetkunde, waar aan elke commutatieve ring $\mathcal{R}$ een meetkundig object Spec($\mathcal{R}$) wordt toegekend.

In de jaren tachtig, gemotiveerd door de natuurkunde$^1$, vond Alain Connes de niet-commutatieve meetkunde (NCM) uit, waarin niet-commutatieve algebraïsche concepten als meetkundig geïnterpreteerd worden (het moet opgemerkt worden dat er in de NCM evenveel verschillende richtingen zijn als in de meetkunde, met evenveel verschillende grondleggers). In deze scriptie beschouwen we een variant op de niet-commutatieve algebraïsche meetkunde, in het bijzonder resoluties van singulariteiten. Een resolutie van een singuliëre variëteit $X$ is een gladde variëteit $Y$ met een afbeelding $Y \rightarrow X$ met compacte vezels, en die een bepaald open deel van $Y$ isomorf afbeeldt op een open deel van $X$. Zoals boven genoemd is zo een variëteit $X$ duaal aan een ring $\mathcal{R}$, dus de niet-commutatieve versie van een resolutie van $\mathcal{R}$ is een niet-commutatieve $\mathcal{R}$-algebra $\Lambda$ die in zekere algebraïsche zin glad is.

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$^1$De zwaartekracht wordt beschreven door kromming van ruimtetijd (meetkunde), en in de kwantummechanika is er het Heisenberg onzekerheidsprincipe, hetgeen geïnterpreteerd kan worden als een soort niet-commutatieve algebra (eerst impuls meten en dan positie is niet hetzelfde als andersom). Om de twee samen te brengen is er dus een notie van niet-commutatieve meetkunde nodig.
References


