Abstract
In this project we examine Luttinger liquids, a class of low-energy systems in one dimension. This class includes most systems of interacting bosons or fermions, like quantum wires or 1D Bose gases. The theory necessary to describe these systems is explained in detail, using the field-theoretical approach to bosonization. We consider the applications of the theory for 1D gases of bosonic cold atoms and discuss an interference experiment done by Hofferberth et al. The distribution function of the interference amplitude that they measured was found to be consistent with the predictions of Luttinger liquid theory.
“It would indeed be remarkable if Nature fortified herself against further advances in knowledge behind the analytical difficulties of the many-body problem.”

- Max Born, 1960

Summary

Consider a long row with hundreds of freely moving particles, each one exerting a force on all the others. Is it possible to make any predictions about such a complicated system?

It might seem like an abstract problem, but one-dimensional systems like this do actually exist. For example in nanotechnology, where you can have electrons in very thin conducting wires. Fortunately for the physicist working in these fields, the answer is often yes. In this project we explain how to calculate the properties of 1D systems at low temperature, using a special method called ‘Luttinger liquid theory’.

A normal fluid, like water, consists of tiny particles that all attract each other, but it can be described as a whole by hydrodynamics. The same is true for the so-called Luttinger liquids in 1D. However, when we work in just one dimension the effects of quantum mechanics start playing a large role, even if you have many particles. Luttinger liquid theory also describes the influence of these big quantum fluctuations.

To show how useful this Luttinger liquid theory is, we do the calculations for a specific example. It is a gas of Rubidium atoms, cooled down to just above the absolute zero. With strong magnetic fields, the gas is stretched out in one direction until the atoms form a line and can only move in that way. By describing the system as a Luttinger liquid, we can predict the outcomes of experiments involving these 1D gases.

In the report we describe one such experiment, where two 1D gases collide with each other and form an interference pattern. From this pattern we can measure the quantum fluctuations in the original gases and compare them to the predictions of Luttinger liquid theory. This provides an excellent test of the theory.
1 Introduction

Solving a system with a large number of interacting objects is not an easy task. Although there are only a select few cases for which analytical solutions exist, physicists are often forced to use approximations like mean-field or perturbation theory. This holds true for both classical systems as well as quantum mechanical ones, where effects like entanglement play an important role.

In this report, we look at the problem of interacting particles in a one-dimension quantum system, where both mean-field and perturbation theory fail. We examine a technique, known as Luttinger liquid theory, to describe the low-energy properties of such a system and we look at its applications in a real experiment.

We start this introduction with a note on why one-dimensional physics is interesting. Then we describe the concept of Luttinger liquids and their history. We also briefly touch on the subject of cold atoms, which will be important for the experimental setup in section 2, before finishing with the technical details of Luttinger liquid theory.

1.1 The relevance of one dimension

One-dimensional systems are often used only as a theoretical tool, a stepping stone towards more complicated models. When a physics student learns his first bit of quantum mechanics, he starts in 1D before gradually working his way up to three spatial dimensions. When a many-body quantum system remains unsolved, one will first try to solve it in one dimension.

But 1D systems are not only simpler than their higher-dimensional counterparts, their behavior is often fundamentally different. For one, when adding time-dependence to an 1D system\(^1\), fluctuations start to play a much larger role than usual. Additionally, the concept of particle statistics breaks down in one dimension, when one deals with interacting particles. There are many other peculiarities in one dimension, especially when looking at systems with spin, but these will not be discussed here (see for example [14]).

Fluctuations, whether originating from uncertainties in the quantum theory or from the thermodynamics, are usually small and of not much interest. In higher dimensions, the dynamics of the system can be described with a mean field theory, where a particle only feels the average interaction of all neighbors\(^2\). In higher dimensions, particles have many neighbors so that a fluctuation of one particle will have a very small effect on the average interaction. However, particles in a one-dimensional system have only two neighbors, one on each side. The result is that fluctuations have a very large effect on the total system. Long-range order, like one would find in a solid crystal, can simply not exist in one dimension, since the quantum fluctuations would break up the structure. Similarly, bosons in 1D can no longer condense into a true Bose-Einstein condensate, even at \(T = 0\) K. Most particles will be excited out of the ground state because of the effect of fluctuations.

Particle statistics deals with the exchange of identical particles and the result this has on the total system. Exchanging two particles twice in a row is equivalent to taking one particle and moving it in a loop around the other. In three (spatial) dimensions, this is topologically the same as not moving at all, meaning that a double exchange will leave the system invariant. So with a single exchange there are two possibilities: the system does not change at all (in this case the particles are called bosons) or the wavefunction gains a minus sign (fermions). When one goes down to two dimensions, a loop around a particle is no longer trivial. But since the physics remains the same, this can only result in an arbitrary phase change \(e^{i\theta}\). For \(\theta = 0\) one

\(^1\)In a sense time can be seen as extra dimension. That is why theorists often speak of 1+1 dimensions.

\(^2\)And in return, the particle contributes to this average.
would have bosons and for $\theta = \pi$ one has fermions. In all other cases the particles are called anyons.

In one dimension, looping two particles around each other becomes problematic. Since they are confined to move on a line, they cannot avoid collision. For free particles this is not a problem, but when one takes interactions into account it is no longer possible to separate particle statistics from particle interactions. This means that the division between bosons and fermions becomes even more fuzzy than in two dimensions: it depends on the specific microscopic model.

An example is the Tonks-Girardeau gas, a 1D system of bosons with an extremely strong repulsive interaction. Because the particles cannot move past each other, they show behavior that is very similar to that of a fermionic system. It turns out that there exists a one-to-one mapping between these impenetrable bosons and a kind of fermionic quasiparticles. Both descriptions, bosons and fermions, result in exactly the same physical system.

A transformation the other way around is also possible, describing a system of fermions in terms of bosons. This technique is called bosonization and is dealt with in great detail in Appendix B. It forms the mathematical basis of Luttinger Liquid theory.

1.2 Historical perspective

The class of systems known as Luttinger Liquids arose as a one-dimensional analogue to the Fermi liquid. Developed by Landau in 1956, Fermi liquid theory describes interacting fermions at low temperature in terms of free fermionic quasiparticles. The quasiparticles are very similar to the original electrons, having the same momentum, charge and spin. They can be visualized as an electron surrounded by a cloud of particle-hole excitations. Because the quasiparticles do not interact, they can easily be used to calculate all the low-temperature properties of the system. Examples of Fermi liquids are the conduction electrons in metals, or the liquid form of Helium-3.

Unfortunately, the mathematics of Fermi liquid theory breaks down for one-dimensional systems. The increased effect of interactions makes it impossible to describe the complete system as non-interacting quasiparticles that are even remotely comparable to the original particles. However, one can make a mapping between 1D interacting fermions to a bosonic system that reproduces all of the system’s low-energy properties.

This was first realized by S. Tomonaga in 1950 and worked out mathematically by J. Luttinger in 1963. But the complete theoretical formalism was not constructed before 1981, by Duncan Haldane. He gave Luttinger liquid theory its current name and showed that it is applicable to almost any one-dimensional system at a low energy. Unlike with Fermi Liquid theory, also interacting bosons can be described as a Luttinger liquid. That is because the lack of a well-defined statistics makes the bosons display some behavior that is normally only associated with fermions, as seen in the Tonks gas mentioned earlier.

1.3 Cold Atoms

Luttinger liquid theory was originally constructed to describe electrons in one dimension. It did that perfectly, providing a theoretical framework for the low-energy properties of quantum wires, carbon nanotubes and other 1D conducting nanosystems. It also correctly described the edge effects of some two-dimensional systems, such as Quantum Hall conductors.

But unlike its higher dimensional analogue, the class of Luttinger liquids also includes bosonic systems. Because bosons are far easier to cool down to low temperatures, they can

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3While the Fermi gas describes free fermions, Fermi liquids occur at such a low temperature that interactions become important. Much like a normal liquid compared to its gaseous state.

4In literature, the systems are often also known as Tomonaga-Luttinger liquids.
provide an exceptionally 'perfect' quantum system. Since the first realized Bose Condensate in 1995, there has been an incredible advance in this field of cold atoms. Gases of bosonic atoms, usually Rubidium-78, confined in a magnetic or optical trap, can now be cooled down to the nanoKelvin range. Parameters like the interaction strength and the shape of the trap can be controlled with great accuracy. And since the atoms are only kept in place by a magnetic field or by light beams, there are no impurities.

These new advances in technology have made it possible to create Bose condensates that are effectively one-dimensional. By varying the trap, the condensate can be elongated until the transverse\(^5\) momentum states have an energy that is much higher than the thermal energy. So in a sense all the transverse degrees of freedom become 'frozen out' and what remains is a purely one-dimensional system. Because of the increased effects of quantum fluctuations, one-dimensional Bose gases cannot actually condensate into the ground state. At short distances the particles are relatively correlated, but at longer scales this order is broken. One can thus see the Bose gas as consisting of many domains, within which the particles have the same phase. Between different domains the phase is uncorrelated as a consequence of fluctuations. This total state is also known as a quasi-condensate.[12]

One could conceivably also use cold fermionic atoms to make such a system, but in practice this is not yet possible. Bose gases are generally cooled by a process called forced evaporative cooling, where one continuously allows the atoms with the highest energy to escape from the trap. This process does not work for fermions, since they hardly interact at low energies\(^6\) and hence do not redistribute their velocities. This makes bosonic atoms much easier to work with.

In section 2 we will describe an experiment done by Hoffenberth et al. \[7\] where we look at the interference between two 1D Bose gases. But first we will need a mathematical description of Luttinger liquid theory. We will be working mostly in the Second Quantization formalism, if you are not familiar with that I suggest you read Appendix A before continuing.

1.4 Luttinger Liquid Theory

Let us now consider a one-dimensional system of interacting bosons. Luttinger liquid theory uses long-range density and phase fluctuations to describe the system in a (different) bosonic basis \[3\]. For this we need to keep track of the particles’ locations and it is useful to introduce a labeling field \(\phi_l(x)\). This function simply increases by \(2\pi\) whenever it encounters a particle, from left to right. So if there is a particle occupying position \(x\), there will be a kink in \(\phi_l\) at \(x\) (see Fig. 1). This makes it natural to identify the gradient \(\nabla \phi_l(x)\) with the density operator, \(\rho(x)\). To be more precise, we can define \(\phi_l(x)\) as satisfying:

\[
\rho(x) = \sum_n \nabla \phi_l(x) \delta(\phi_l(x) - 2\pi n)
\]

where \(\delta\) is the Dirac delta function. This sum of delta functions is periodic in \(n\), so it can be written as a simple Fourier series:

\[
\rho(x) = \sum_p \frac{\nabla \phi_l(x)}{2\pi} e^{ip\phi_l(x)} = \left[ \rho_0 - \frac{1}{\pi} \nabla \phi(x) \right] \sum_p e^{i2p(\pi\rho_0 x - \phi(x))}. \tag{2}
\]

In the last equality we used the substitution \(\phi = \pi\rho_0 x - \frac{1}{2} \phi_l\), where \(\rho_0\) is the average density. The field \(\phi\) denotes the difference between a perfect system with equal spacing between the

\(^5\)Perpendicular to the axis of the trap.

\(^6\)To be more precise, fermions do not allow S-wave scattering because of their statistics. Their lowest partial waves that can scatter are P-waves, which are strongly suppressed at a low enough temperature.
Figure 1: Schematic graph of an example labeling operator $\phi_l(x)$. The circles beneath the x-axis represent the positions of particles. At each such a position, there is a kink in $\phi_l$ and the function reaches a multiple of $2\pi$.

particles\textsuperscript{7}, and the real system labeled by $\phi_l$. This field is more convenient to work with, despite the density operator looking very unwieldy this way. That is because its gradient, often denoted by $\Pi(x)$, now represents the long-range density fluctuations.

To construct the bosonic creation and annihilation operators, we need, apart from the density, also a phase $\theta(x)$.  

$$\Psi_b^\dagger(x) = \sqrt{\rho(x)} e^{-i\theta(x)} \tag{3}$$

We can now substitute equation (2) in this expression, but taking the square root of the sum makes it rather ugly. However, if we use equation (1) we see that the density hardly changes at all when taking the square root.

$$\sqrt{\rho(x)} = \left[ \sum_n \nabla \phi_l(x) \delta(\phi_l(x) - 2\pi n) \right]^\frac{1}{2}$$

$$= \sum_n \left[ \nabla \phi_l(x) \delta(\phi_l(x) - 2\pi n) \right]^\frac{1}{2}$$

$$= A \left[ \phi_l(x) \right]^\frac{1}{2} \sum_n \delta(\phi_l(x) - 2\pi n)$$

$$= A \left[ \rho_0 - \frac{1}{\pi} \nabla \phi(x) \right]^\frac{1}{2} \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)} \tag{4}$$

We can move the square root within the sum because the delta functions do not overlap, i.e. for any $x$ only one of the terms contributes. Then we use the fact that $\sqrt{\delta(x)} = A\delta(x)$, where the proportionality constant $A$ depends on the exact definition of the Dirac delta (but note that it does not depend on $x$). This gives us the following creation operator, up to an undetermined prefactor:

$$\Psi_b^\dagger(x) \propto \left[ \rho_0 - \frac{1}{\pi} \nabla \phi(x) \right]^\frac{1}{2} \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)} \tag{4}$$

If this operator is to create bosons, it should satisfy the commutation relations (A.2) and this imposes special requirements on the relation between $\phi$ and $\theta$. Looking only at long-range density fluctuations, we can neglect the oscillating terms in equation (2), finding simply

\textsuperscript{7}Where we would have: $\phi_l(x) = 2\pi\rho_0 x$
\[ \rho(x) = \rho_0 - \frac{1}{\pi} \nabla \phi(x). \]

In this low-energy approximation, the bosonic commutation relations are automatically satisfied if \( \theta \) and \( \frac{1}{\pi} \nabla \phi \) are canonical conjugates, so if \( \theta \):

\[
\left[ \frac{1}{\pi} \nabla \phi(x), \theta(x') \right] = -i \delta(x - x')
\]  

This will turn out to be useful when doing calculations with the two fields.

Now let us look at the Hamiltonian. Plugging Eq. (3) into the kinetic Hamiltonian (A.5) and making the approximation \( \rho(x) = \rho_0 \) yields:

\[
H_{\text{kin}} \propto \frac{\rho_0}{2m} \int_0^L (\nabla \theta(x))^2 \, dx
\]  

Similarly, the leading term of the interaction Hamiltonian (A.6) will become:

\[
H_{\text{int}} \propto \int_0^L (\nabla \phi(x))^2 \, dx
\]

provided that we consider low-energies and relatively short-range interactions. The specific proportionality constants depend on the exact form of the interaction potential, as well as other higher corrections. We can include them by writing the total Hamiltonian as:

\[
H = \frac{u}{2\pi} \int_0^L \left[ K(\nabla \theta(x))^2 + \frac{1}{K}(\nabla \phi(x))^2 \right] \, dx.
\]  

This is the famous Luttinger Hamiltonian, derived in this form by Haldane. The two parameter \( u \) and \( K \) specify the proportionality constants in Eqs. (6) and (7). Dimensional analysis shows that \( u \) is a velocity and can be interpreted as the speed of the excitations. \( K \) is a dimensionless constant and denotes the relative weight of the density and the phase terms. For a non-interacting Bose gas, one would have to take the limit \( K \to \infty \) so that \( H_{\text{int}} \) vanishes. For the Tonks gas of impenetrable bosons, as described in the introduction, one would have \( K = 1 \).

With the equations for the single-particle operator (4) and the Hamiltonian (8), all low-energy properties of a one-dimensional system can be calculated. To completely describe the system, we only need to know the values of \( u \) and \( K \). These can be measured experimentally, by examining thermodynamical quantities like the specific heat and the compressibility of the system.

Another important factor are the boundary conditions of the system, which will affect the precise form of the correlation functions. The easiest to work with are periodic boundary conditions, where the two end point are connected to each other.\(^9\) Open boundary conditions, corresponding to particles in a box, are also possible. However, other types of boundaries, like an external harmonic potential, usually make the calculations much more difficult (see [3]).

The technique we used here to describe a Luttinger liquid mathematically is known as the `field-theoretical approach'. Another way, usually called the `constructive approach', explicitly calculates the bosonic creation operators. This approach is more commonly known as bosonization and is discussed in Appendix B. For more on this, see [14] [6] [15].

2 Methods

To test the validity and reach of the Luttinger liquid theory, many different kinds of experiments can be conducted. In this section we will describe a specific setup, where two 1D Bose show interference.
One-dimensional Bose gases are very clean quantum systems, meaning that they show a strong wave-like behavior. It should thus be possible for them to interfere with each other. Interference experiments can measure the phase differences between two systems, even though phases usually do not correspond to a physical quantity. This makes them a powerful tool to analyze 1D quantum systems.

2.1 Experimental setup

We consider two parallel 1D Bose gases, each one confined in a separate magnetic trap. They are aligned horizontally along the x-axis and positioned above each other (on the y-axis). The quasi-condensates are prepared independently, so they are in no way correlated.

If we now turn off the magnetic traps, the Bose gases will no longer be confined. They will fall down due to gravitation and start to expand thermally, the atoms radiating outwards. As soon as they begin to overlap, one will see a two-dimensional interference pattern in the xy-plane of the two condensates. By sending a light beam perpendicular to this plane, it is possible to measure the absorption and use that to analyze the density pattern (see Fig. 2).

Because of thermal and quantum fluctuations, different parts of the condensate can have a different phase. Since the positions of the fringes in the interference pattern depend on the relative phase between the two condensates, they can also change for different points on the x-axis. The consequence is that the interference fringes are not straight lines, but ‘wiggly’. To take these fluctuations into account, we need to integrate our pattern along the x-axis. For that we define an integration length $L$ that marks the distance we integrate over (see Fig. 3). This integration gives us a density profile $\rho_L(y)$. For small values of $L$ the fringes are relatively straight and the effect of the fluctuations is small. For large $L$ the fluctuations are more prominent and the lines are more wiggly. In that case, the contrast between the peaks and troughs of the density profile will be reduced. This contrast is measured by the complex interference amplitude $A(L)$. It can be extracted from the density profile by finding the

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Figure 2: Experimental setup of the interference experiment. Two 1D quasi-condensates are confined in elongated magnetic fields below an atom chip. When the magnetic field is removed, the condensates expand and show interference. The interference pattern is then measured by absorption imaging.[7]
Figure 3: An interference pattern obtained by Hofferberth et al. and integrated along various values of the integration length L. Blue areas have a low density and red denotes a high density. Note that the interference pattern is ‘wiggly’ because different parts of the condensate can have a different phase. Shaded parts are outside of the integrated area. The resulting density profile for each value of L can be seen in the bottom row.\[7\]

wavelength of the interference pattern and calculating the corresponding (complex) Fourier coefficient. This quantity turns out to be very suited to analyze the fluctuations in the system with Luttinger liquid theory.

2.2 Calculating the amplitude distribution

When the atoms are released from their trap, they will start to spread out, moving away from the condensates as a standard cylindrical wave. If we only consider the spread in the y-direction\[10\], then the operator that creates a particle in this total system will become\[13\]

$$\Psi_{tot}^\dagger(x,y) = \Psi_1^\dagger(x) e^{-i(Q_1(y) (y-\frac{d}{2}) - Q_1^2 t/2m)} + \Psi_2^\dagger(x) e^{-i(Q_2(y) (y+\frac{d}{2}) - Q_2^2 t/2m)}$$

(9)

where the operators $\Psi_1(x)$ and $\Psi_2(x)$ create a particle at position $x$ in condensates 1 and 2 respectively, before the expansion. $d$ is the initial distance between the condensates and $t$ is the time after the release of the gases. $Q_1$ and $Q_2$ are the momenta (in the y-direction) that particles from condensate 1 and 2 need to reach the position $y$ after time $t$. These can easily be calculated as being $Q_1(y) = \frac{m}{t} (y-\frac{d}{2})$ and $Q_2(y) = \frac{m}{t} (y+\frac{d}{2})$. So the first term in the exponents represents the wavefunction of the particles radiating outwards from the condensates, while the second term denotes the energy of the particles, gained from the release of the trap\[11\]. We can use this expression to find the density operator for the interference pattern:

$$\rho_{tot}(x,y) = \Psi_{tot}^\dagger(x,y) \Psi_{tot}(x,y) = \rho_1(x) + \rho_2(x) + \Psi_2^\dagger(x) \Psi_1(x) e^{-iQy} + \Psi_1^\dagger(x) \Psi_2(x) e^{iQy}$$

(10)

where $Q = Q_2(y) - Q_1(y) = md/t$ is the wavevector of the interference pattern. $\rho_1(x)$ and $\rho_2(x)$ are the densities of the original condensates.

\[10\] So we take $z = 0$. Because the two condensates are located at the same $z$-position, there will be no interference in this direction. Also note that any expansion in the x-direction is negligible, because the initial condensates are very weakly confined along the x-axis.

\[11\] This is just the $e^{-i\omega t}$ factor that represents the time-development of a stationary solution of the Schrödinger equation. In this case, $\omega = Q^2/2m$. 

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To acquire the interference amplitude, we need to integrate this along the x-direction and then find the Fourier mode of wavelength $Q$. It is clear that only the last term Eq. (10) will contribute to this Fourier mode, since the other terms are either constant or harmonic with another wavelength. This will yield us the simple expression:

$$A(L) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-L/2}^{L/2} \Psi_1^\dagger(x)\Psi_2(x) e^{iQy} e^{-iQy} \, dx \, dy$$

(11)

Because the condensates are independent, we can write:

$$\langle \Psi_1^\dagger(x)\Psi_2(x) \rangle = \langle \Psi_1^\dagger(x) \rangle \langle \Psi_2(x) \rangle = 0 \quad (12)$$

From this we see immediately that $\langle A(L) \rangle = 0$. The physical reason is that $A(L)$ has a random complex phase, representing the positions of the peaks in the density profile. A more interesting quantity would thus be the absolute value. It would have a finite average, but still vary between experiments because of the fluctuations. Looking at the distribution of these variations could tell us something about the nature of the quantum noise.

That is why it is convenient to introduce the normalized$^{12}$ variable:

$$\alpha(L) = \frac{|A(L)|^2}{\langle |A(L)|^2 \rangle} \quad (13)$$

We can now examine the distribution function $W(\alpha)$ which simply denotes the chance that an experiment will result in that specific value of $\alpha$. Note that $W(\alpha)$ can be measured experimentally, by repeating the interference experiment a large number of times and finding the distribution of $\alpha$. Because it is a probability density function, it can be expressed in terms of its so-called moments $\langle \alpha^m \rangle$ where $m$ is a natural number.$^{13}$

$$\langle \alpha^m \rangle = \int_{-\infty}^{\infty} W(\alpha) \alpha^m \, d\alpha \quad (14)$$

Using this equation and some Fourier transformations, we can find an explicit expression for $W(\alpha)$ as a function of all its moments:

$$\mathcal{F}^{-1}[W(\alpha)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha t} W(\alpha) \, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \int_{-\infty}^{\infty} W(\alpha) \alpha^m \, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \langle \alpha^m \rangle$$

(15)

$$W(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha t} \left[ \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \langle \alpha^m \rangle \right] \, dt$$

Hence by calculating all the moments of $\alpha$, one can find the full probability distribution.

$^{12}$I.e. the average is equal to 1.

$^{13}$Since $\alpha$ is always positive, $W(\alpha) = 0$ for negative values of $\alpha$. So we could just as well set the lower bound to 0, but I choose not to since this notation makes it more clear why the Fourier trick in Eq. (15) is warranted.
This is easier said than done, however. In Appendix C we will show how to compute $\langle |A(L)|^2 \rangle$ using Luttinger liquid theory. This term is important because it appears in the denominator of every moment. We use Eqs. (11) and (4) to express $A(L)$ in terms of $\phi$ and $\theta$. Known relations between these fields yield us the correlation functions we need. But calculating the higher moments requires some advanced quantum field theory that is beyond the scope of this project. For that derivation we refer to Imambekov, Gritsev and Demler [8] [5], who used a mapping between the 1D Bose gases and a 2D statistical system. In the next section we will compare their results with the experimental data of Hofferberth.

3 Discussion

The experiments and calculations described in the previous section were done by Hofferberth, Lesanovsky, Schumm, Imambekov, Gritsev, Demler and Schmiedmayer. For more details on their experiment, see [7]. Here we will report their results and discuss the implications for Luttinger liquid theory. We also mention some different kinds of experiments that can or have been done with Luttinger liquids.

3.1 Comparison of results

Hofferberth et al. measured 370 instances of interference between two Bose gases. 170 times at $T = 31nK$ and 200 times at $T = 60nK$. The other parameters, like the density and size of the condensates or the time before making the image, were kept the same for all experiments. The Bose gases they used had a Luttinger parameter $K = 46$. For every interference pattern they computed the amplitude $A(L)$, which together yields the average $\langle |A(L)|^2 \rangle$ and thus the normalized variable $\alpha(L)$ for each pattern, as defined in Eq. (13). This full analysis was done at four different values of the integration length $L$. The result is eight histograms of the distribution of $\alpha(L)$, which can be seen in Figure 4.

In the same figure, we see the outcome of the theoretical calculation of $W(\alpha)$. The shape of the distribution function that the theory predicts seems to be very consistent with the experimental data. For small values of $L$, the distribution tends to be sharply peaked. This was to be expected, since at such short distances the effect of fluctuations will be small. For a non-interacting Bose gas, where the whole condensate can be fully correlated, the distribution would approach the shape of a delta function. At larger integration lengths, the function spreads out and approaches the shape of a Poisson distribution. In that case one can view the condensates as consisting of many independent domains, within which the particles are completely correlated. In between these limiting cases we see two separate peaks, one for the short-ranged quantum fluctuations and one for thermal fluctuations. The good agreement between the theoretical and experimental data shows that Luttinger liquid theory is a powerful tool to describe these 1D systems.

In fact the consistency is somewhat surprising, considering the amount of approximations and simplifications that were made in the theoretical approach. For one, Luttinger liquid theory cannot describe short-range fluctuations. Although at these low temperatures and relatively long distances, they are unlikely to have significant impact. Secondly, the correlation functions required to find $W(\alpha)$ were calculated on the assumption of periodic boundary conditions. While the effect of the boundary conditions is probably small, this is simply incorrect. An 1D system

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\[14\] This is known as a Wick rotation, a technique often used in QFT. They did not actually use Eq. (15).

\[15\] Long relative to the the size of the correlated domains. In the described experiment this so-called healing length was $\xi_h = 0.3 \mu m$ while the smallest value of $L$ was 10 $\mu m$. 

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Figure 4: Comparison of experimental and theoretical data, by Hofferberth et al. The eight histograms show the measured probability distributions of \( \alpha(L) = |A(L)|^2 / \langle |A(L)|^2 \rangle \). The top row is the combined result of 170 experiments done at a temperature of 31 nK. For the bottom row 200 experiments were used, at a temperature of 60 nK. Overlayed as dashed lines are the theoretical predictions. The blue and black lines are the upper and lower error bounds, taking into account errors in the experimental parameters, like the density, length and temperature.

with periodic BCs is possible, but it would look more like a circle. However, taking into account proper boundaries of the magnetic trap would make the mathematics a lot more complicated.

3.2 Other experiments

Interference experiments are particularly suited to study quantum fluctuations, because they are sensitive to long-range phase differences. There are other ways to study 1D systems, like directly measuring density fluctuations in an expanding Bose gas, but these often look at short range fluctuations and are thus less appropriate for testing Luttinger liquid theory. Recently there have been experiments where an 1D condensate interferes with itself as a result of the spin dynamics [10]. Kitagawa et al. analyse the distribution of the interference contrast, similar to Hofferberth’s experiment, but can also measure the time evolution of the distribution function. Alternative Luttinger liquid experiments where spin plays a role can include two-component Bose gases [11] or of course many different kinds of fermionic systems, like carbon nanotubes [9].

Another factor that we have not taken into account is the possibility to work in a lattice. With carefully aligned laser beams, one can create a periodic potential that serves as an optical
lattice. The presence of such a lattice can have some serious effects on the system, like the existence of a so-called Mott transition between a conducting and an isolating phase [2] [14].

Lastly, one can consider bosonic systems other than cold atoms. Cooper pairs, the couples of two electrons that cause superconductivity\(^{16}\), are bosons and can also be described with Luttinger liquid theory, provided that they are confined to an 1D superconductor. An example is the Josephson junction array, an line-up of superconducting dots which can be seen as the sites of a lattice. The Cooper pairs can hop from site to site and show the behavior of a Luttinger liquid [4].

4 Conclusion

We have looked at many-body quantum systems in one dimension and at one of the techniques to describe them mathematically. This method, known as Luttinger liquid theory, turns out to be exceptionally powerful to calculate the low-energy properties of 1D systems. The theory makes use of the ambiguity between bosons and fermions in 1D, as well as the prominent role of fluctuations.

To compare the predictions of Luttinger liquid theory with experimental data, we discussed an experiment performed by Hofferberth et al. [7] in 2008. They used an elongated gas of bosonic atoms, cooled down to 31 nK. At this low temperature, the gas becomes effectively one-dimensional and can be described very well as a Luttinger liquid. Hofferberth let two of such gases expand and interfere with each other. The analysis of the resulting interference patterns can tell us much about the long-range fluctuations in 1D systems, and shows a remarkable agreement with the theoretical prediction.

The real strength of Luttinger liquid theory lies in its enormously wide range of applications. There are analytical solutions to the 1D interacting Bose gas, like the Lieb-Liniger model and the Analytic Bethe Ansatz, but their use is often fairly restrictive. For example, the presence of any external potential will make the Bethe Ansatz method inadequate. In comparison, Luttinger liquid theory works for any 1D system, as long as its excitations have a linear spectrum without gaps in it [3].

With the ongoing advance in technology, the realization of one-dimensional systems will continue to become easier. The field of cold atoms is still in its early stages and each year nanotechnology becomes more prominent in everyday use. The theory of Luttinger liquids is invaluable for a complete understanding of these systems, and it will be the subject of many more experiments in the future.

\(^{16}\)This is of course only the case for Type I superconductors. The exact mechanism behind Type II SCs is as of yet unknown.
Appendix A: Second Quantization

The name ‘second quantization’ refers to the quantization of field operators and is somewhat of a misnomer. It is a formalism particularly suited to describe many-body systems, since it treats the total number of particles as a variable. This so-called number representation uses creation and annihilation operators to add or remove particles in a specific quantum state. The operators, usually denoted $c^\dagger$ (creation) and $c$ (annihilation) are completely equivalent to the raising and lowering operators in algebraic formulation of the quantum harmonic oscillator. Instead of adding or removing energy quanta, they now change the number of particles, moving the total system to a different state within the Fock space.

A particle created by $c^\dagger$ can be in any quantum state, but usually the operators are defined so that they create a particle either at a specific location $x$ or possessing a specific momentum $k$. The two are related by the Fourier transform

$$c^\dagger(k) = \int e^{-ikx} c^\dagger(x) \, dx. \quad (A.1)$$

If $c^\dagger(x)$ is well-defined on every point in space, it is also called a field operator and is usually denoted by $\Psi^\dagger(x)$. We will use this notation from here on. The mathematical properties of the creation/annihilation operators are completely defined by their (anti-)commutation relations. For bosons the relations are the same as for the raising and lowering operators of the harmonic oscillator:

$$[c(k_1), c^\dagger(k_2)] = \delta(k_1 - k_2)$$
$$[c(k_1), c(k_2)] = [c^\dagger(k_1), c^\dagger(k_2)] = 0 \quad (A.2)$$

while for fermions the anticommutator is used instead. This also makes it clear that $c$ and $c^\dagger$ are in fact hermitian conjugates of each other, so that the notation with the daggers is warranted.

To count the number of particles occupying a given state, one can define the density operator $\rho(x) = \Psi^\dagger(x) \Psi(x)$. Since it is a combination of a creation and an annihilation operator, $\rho$ does not change the number of particles in the system. It merely counts them. The density can also be written in terms of $c^\dagger(k)$ and $c(k)$, by using another (inverse) Fourier transform:

$$\rho(x) = \sum_{k, k'} \frac{1}{\Omega} e^{i(k-k')x} c^\dagger(k') c(k) = \sum_{k, q} \frac{1}{\Omega} e^{-iqx} c^\dagger(k+q) c(k) \quad (A.3)$$

where $q = k' - k$. Here we use summation instead of integration because we generally look at finite systems where momentum is quantized. This requires us to introduce the volume $\Omega$ of the system as a normalization constant. This is not to be confused with the momentum-space density $\rho(k) = c^\dagger(k) c(k)$, which simply counts the number of particles occupying momentum state $k$.

Using these density operators, we can construct the most simple Hamiltonian for a system of free particles.

$$H = \sum_k \epsilon_k \ c^\dagger(k) \ c(k) \quad (A.4)$$

where $\epsilon_k$ is the dispersion relation, specifying the energy of a single particle with momentum $k$. A more general form for the kinetic energy in terms of the field operators is given by:

$$H_{kin} = \frac{1}{2m} \int_0^L \partial_x \Psi^\dagger(x) \partial_x \Psi(x) \, dx. \quad (A.5)$$
To add interaction to the system, it is the easiest to define a two-point potential \( \nu(x-x') \) that depends only on the distance between two particles. The interaction term in the Hamiltonian would then become:

\[
H_{\text{int}} = \frac{1}{2} \int_0^L \Psi(x) \Psi(x') \nu(x-x') \Psi(x) \Psi(x') \, dx
\]  

(A.6)

Appendix B: Bosonization

Using second quantization (see appendix A) one can describe a many-body system using creation and annihilation operators. For the same system it is often possible to use different sets of operators, resulting in descriptions using different kinds of particles. Transforming between these descriptions corresponds to a change of basis within the Fock space. This is one of the most used tools in condensed matter theory, since it allows one to choose a basis in which the Hamiltonian is diagonal, resulting in an easily solved system.

In one dimension, the separation between bosons and fermions is not as clear-cut as usual. In fact it turns out to be very natural to transform a system of fermions into a bosonic system, where the bosons correspond to long-range density fluctuations of the fermions. This process is called \textit{bosonization} and it can be used to describe almost any one-dimensional system. It is even applicable to systems that are originally bosonic, though the name ‘bosonization’ is somewhat strange in this case. We just transform one system of bosons into another one. So while the following derivation will be for fermions, one should keep in mind that exactly the same can be done for interacting bosons\(^{17}\).

Let us start with a one-dimensional system of free fermions of finite length \( L \). We ignore the presence of spin, for the sake of simplicity\(^{18}\). The Hamiltonian is given by Eq. (A.4). For free particles, the dispersion relation is of the form:

\[
\epsilon_k \propto -\cos(k)
\]

At temperature \( T = 0 \)K, all states up to a certain energy are occupied, the so-called Fermi energy. This energy corresponds to two Fermi points on the dispersion curve, at \( k_F \) and \( -k_F \) (see Fig. 5a). In the low-temperature regime, all excitations occur around these Fermi points. That means we can apply a first order Taylor approximation around the two points, effectively linearizing the dispersion relation (see Fig. 5b). However, there are still two different linear branches, one for positive and one one for negative \( k \). To solve this, one can split the fermions into two different kinds of particle, right-moving \( (k > 0) \) and left-moving \( (k < 0) \) fermions, each with a completely linear dispersion relation. The two types of fermions get their own creation operators, \( c_R^\dagger(k) \) and \( c_L^\dagger(k) \), and are in every way considered as distinct particles.

Unfortunately, the linear dispersion predicts an infinite number of negative energy states. These need to be filled with a ‘Dirac Sea’ of particles with a negative energy, similar to Dirac’s interpretation of antimatter. Though this will not come up in this report, it is necessary to introduce a cut-off \( \alpha \), to get rid of infinities in the math.

With this new dispersion relation, the Hamiltonian will be given by:

\[
H = \sum_{k_{\gamma=R,L}} v_F(a_{\gamma}k - k_F)c_{\gamma}^\dagger(k)c_{\gamma}(k)
\]  

(B.1)

\(^{17}\)This is because 1D bosons do in fact have a dispersion relation and a Fermi level, which is all that is needed for bosonization.

\(^{18}\)For example,
Figure 5: **a.** Dispersion relations of interacting fermions. Typically the dispersion behaves like a (negative) cosine. Where the dispersion curve reaches the Fermi energy $E_F$, there are two Fermi points at $k_F$ and $-k_F$.

**b.** Linearized dispersion relation. Approximation of dispersion relation **a.** to describe the physics around the Fermi points. Since we now have an infinite number of negative energy states, we need a ‘Dirac sea’ to fill them up.

with $a_r = +1$ for right-moving particles and $a_r = -1$ for left-movers. $v_F$ is the velocity corresponding to momentum $k_F$.

The linear dispersion relation means that one can have particle-hole excitations with a well-defined energy and momentum. If you annihilate a particle with momentum $k$ and then create one at $k + q$, then the energy of the excitation will simply be proportional to the difference $q$, independent of the original $k$. These excitations can thus be seen as quasi-particles, and they will form the foundation of our bosonic description.

Now let us look at the density operator defined in Eq. (A.3). With another Fourier transform, it can be written in terms of $q$:

$$\rho_{R,L}(q) = \sum_k c^\dagger_{R,L}(k + q) c_{R,L}(k)$$  \hspace{1cm} (B.2)

where we attach the label R or L to distinguish between right- and left-moving particles. This expression is a function of a momentum $q$, but it still represents the density in position space\(^{19}\).

The operator, which can be seen as a density fluctuation with momentum $q$, is simply a superposition of particle-hole excitations, meaning that we can interpret it as well-defined quasiparticle.

The hermitian adjoint of $\rho(q)$, which would destroy such an excitation, is simply given by\(^{20}\):

$$\rho^\dagger(q) = \sum_k c^\dagger(k) c(k + q) = \sum_{k'} c^\dagger(k' - q) c(k') = \rho(-q)$$

as long as $q \neq 0$. Here $k' = k + q$ and the subscripts R and L have been left out for simplicity. Destroying a density fluctuation with momentum $q$ is the same as creating one with momentum $q$. So to examine the behavior of these excitations, one should consider the commutation $[\rho(q), \rho(-q')]$. Computing this is not trivial, because it depends on the expectation value of the vacuum, where we have to take the filled negative energy states into account. The result is\(^{14}\):

$$[\rho_r(q), \rho_{r'}(-q')] = -\delta_{r,r'}\delta_{q,q'} a_r a^{\dagger}_r q' L \frac{2\pi}{2\pi}.$$  \hspace{1cm} (B.3)

This relation is similar, but not equal, to the bosonic commutation relations (A.2). To define a proper bosonic creation operator, we need to get rid of the constant factor. It would also be

\(^{19}\)Do not confuse it with the density in momentum-space, given by $\rho(k) = c^\dagger(k) c(k)$

\(^{20}\)Remember that $(AB)^\dagger = B^\dagger A^\dagger$
nice to have a single operator for both left- and right-going fluctuations. This can be done in the following manner:

\[ b^\dagger(q) = \begin{cases} \left( \frac{2\pi}{L|q|} \right)^{1/2} \rho_L(q) & \text{if } q < 0 \\ \left( \frac{2\pi}{L|q|} \right)^{1/2} \rho_R(q) & \text{if } q > 0 \end{cases} \]  

(B.4)

Here we have identified right-moving particles with a positive momentum, and left-moving ones with a negative \( q \). This should be fairly intuitive. Note that \( b^\dagger(q) \) is not defined for \( q = 0 \). It is now easy to check that this expression, combined with Eq. (B.3), satisfies Eq. (A.2). Hence \( b^\dagger \) creates bosonic quasiparticles that can be used to describe the originally fermionic system.

To make this description complete, we will also need to express the original fermionic creation operators \( \Psi^\dagger_R(x) \) and \( \Psi^\dagger_L(x) \) in terms of these new \( b^\dagger \) and \( b \). The easiest way to do that is to look at the commutators. For instance, let us take:

\[
[\rho_r(q), \Psi^\dagger_r(x)] = \frac{1}{\sqrt{\Omega}} \sum_{k,k'} e^{ik'x} \left[ c^\dagger_r(k + q) c_r(k) c^\dagger_r(k') c_r(k + q) c_r(k) \right] \\
= \frac{1}{\sqrt{\Omega}} \sum_{k,k'} e^{ik'x} \delta_{k', k + q} c^\dagger_r(k) = e^{iqx} \Psi^\dagger_r(x) 
\]

(B.5)

If we can find an expression in terms of the bosonic operators that satisfies the same relation, then it will behave mathematically the same as \( \Psi^\dagger_r \) and we can identify it with the creation operator. One such expression is:

\[
\Psi^\dagger_r(x) \propto e^{\sum_q \frac{2\pi e}{\Omega} e^{-iqx} \rho_r(q)} 
\]

(B.6)

which is easy to check, using Eq. (B.3) and the fact that \([A, e^B] = [A, B]e^B\). As before, \( a_r = +1 \) for \( r = R \) and \( a_r = -1 \) for \( r = L \). The proportionality constant is an important operator, known a Klein factor and often denoted by \( U_r \). It is need to change the total number of fermions in the system.

The same trick can be used with Eq. (B.1) to find the Hamiltonian, resulting in the simple expression\[14\]:

\[
H \propto \sum_{q \neq 0} v_F |q| b^\dagger(q) b(q). 
\]

(B.7)

Clearly the bosons also behave like free particles with a linear dispersion relation. The advantage of this bosonic description is that adding interactions of the form (A.6) to your original system will only influence the bosonic Hamiltonian in small ways. To see this, let us make the final connection to section 1.4. From the the boson operators, we can define the two fields:

\[
\phi(x) = -(N_R + N_L) \frac{\pi x}{L} - \frac{i\pi}{L} \sum_{q \neq 0} \left( \frac{L|q|}{2\pi} \right)^{1/2} \frac{1}{q} e^{-\alpha|q|/2} e^{-iqx} (b^\dagger(q) + b(-q)) \\
\theta(x) = (N_R - N_L) \frac{\pi x}{L} + \frac{i\pi}{L} \sum_{q \neq 0} \left( \frac{L|q|}{2\pi} \right)^{1/2} \frac{1}{q} e^{-\alpha|q|/2} e^{-iqx} (b^\dagger(q) - b(-q)) 
\]

(B.8)

where \( N_R \) and \( N_L \) are the total number of right- respectively left-going fermions. Furthermore, \( \alpha \) is a small cutoff length, introduced to get rid of infinities. These two fields \( \phi \) and \( \theta \) are equivalent to the ones defined in Eqs. (1) and (3). Equation (B.8) provides the link between the constructive and the field-theoretical approach to Luttinger liquid theory.
Appendix C: Correlation functions

In this appendix we show how to calculate \(|A(L)|^2\) where \(A(L)\) is the interference amplitude defined in Eq. (11). Using that formula, we find:

\[
\langle |A(L)|^2 \rangle = \langle A^\dagger(L)A(L) \rangle = \int_{-L/2}^{L/2} \langle \Psi^\dagger_2(x)\Psi_1(x)\Psi^\dagger_1(x')\Psi_2(x') \rangle \, dx \, dx' \\
= \int_{-L/2}^{L/2} \langle \Psi^\dagger_1(x)\Psi_1(x') \rangle \langle \Psi^\dagger_2(x')\Psi_2(x) \rangle \, dx \, dx' + N \tag{C.1}
\]

where we have split the average because the two condensates are independent. The extra term \(N\), which appears because \(\Psi^\dagger_1\) and \(\Psi_1\) do not commute, denotes the total number of particles in either condensate.

If the two condensates are prepared identically, then we can say that:

\[
\langle \Psi^\dagger_1(x)\Psi_1(x') \rangle = \langle \Psi^\dagger_2(x)\Psi_2(x') \rangle.
\]

These expressions are called correlation functions. They play an important role in many-body systems and as we will see, Luttinger liquid theory is very suited to calculate these functions.

Plugging this into Eq. (C.1) yields:

\[
\langle |A(L)|^2 \rangle = \int_{-L/2}^{L/2} \langle \Psi^\dagger(x)\Psi(x') \rangle^2 \, dx \, dx' + N = L \int_{-L/2}^{L/2} \langle \Psi^\dagger(x)\Psi(0) \rangle^2 \, dx + N \tag{C.2}
\]

where we have dropped the indices. To acquire the second equality, we make the assumption that \(\langle \Psi^\dagger(x)\Psi(x') \rangle\) only depends on \(x - x'\) and not on the two coordinates separately, which greatly simplifies the integral.

So it all reduces to computing the correlation function \(\langle \Psi^\dagger(x)\Psi(0) \rangle\). Let us start by inserting the Luttinger Liquid expressions for \(\Psi(x)\) (see Eq. (3)). As there can be no long-range order in 1D systems, we expect the correlation function to fall off with distance. Thus to find the leading terms, we can make the approximation \(\rho(x) = \rho_0\), since \(\rho(x)\) only changes slowly.

\[
\langle \Psi^\dagger(x)\Psi(0) \rangle = \langle \rho_0 \, e^{-i[\theta(x) - \theta(0)]} \rangle = \rho_0 \, e^{-\frac{1}{2}[\theta(x) - \theta(0)]^2} \tag{C.3}
\]

Here we used the standard equality \(\langle e^{A^\dagger} \rangle = e^{\frac{1}{2}\langle A^2 \rangle}\). To simplify the expression that we must average, we can expand the square and use a Fourier transform to find:

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \langle \theta(x)^2 + \theta(0)^2 - 2\theta(x)\theta(0) \rangle = \frac{1}{(\beta\Omega)^2} \sum_{k_1,k_2} \langle \theta(k_1)\theta(k_2) \rangle e^{i(k_1 + k_2)x} \tag{C.4}
\]

where as usual \(\theta(k)\) is the Fourier transform of \(\theta(x)\).

So now we are to calculate correlation functions like \(\langle \theta(k_1)\theta(k_2) \rangle\). In statistical mechanics one would write this as:

\[
\langle \theta(k_1)\theta(k_2) \rangle = \frac{1}{Z} \sum_n \theta(k_1)\theta(k_2)e^{-\beta E_n}
\]

where \(Z\) is the partition function, \(n\) ranges over all possible states and \(E_n\) is the energy corresponding to state \(n\). Unfortunately, in our case we do not quite know the states and corresponding energies, so we will have to involve the Luttinger liquid Hamiltonian (8). Furthermore, the
fields $\theta$ can also be time-dependent, which will have a significant effect on the averages. The way to write this in Quantum Field Theory is:

$$\langle \theta(k_1)\theta(k_2) \rangle = \frac{1}{Z} \int D\phi(x,\tau) \ D\theta(x,\tau) \ \theta(k_1,\omega)\theta(k_2,\omega) \ e^{-S}. \quad (C.5)$$

This is known as a functional integral [1] and it can be seen as integrating over uncountably many variables, i.e. all the values that $\phi(x)$ and $\theta(x)$ can take. $\tau$ is the imaginary time and $\omega$ is its conjugate variable after the Fourier transform. $S$ is the action, which can be derived from the Hamiltonian of the system. The action depends on $\phi$ and $\theta$ as well as the Luttinger parameter $K$.

These types of integrals are too advanced to discuss in this thesis and here we will simply give the answer. For the derivations, see [14]. Equations (C.4) and (C.5) result in:

$$\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{K} \log \left[ \frac{x}{\xi_h} \right] \quad (C.6)$$

where $\xi_h$ is the so-called healing length, which depends on the short-range cutoff of the Luttinger model. With that, Eq. (C.3) reads:

$$\langle \Psi^\dagger(x)\Psi(0) \rangle = \rho_0 \ e^{-\pi \frac{\log [x]}{\xi_h}} = \rho_0 \ e^{\log \left( \frac{\xi_h}{x} \right) \frac{\pi}{2}} = \rho_0 \ \left( \frac{\xi_h}{x} \right)^{\frac{1}{2K}} \quad (C.7)$$

So the two-particle correlation function falls off as $x^{-\frac{1}{2K}}$. Such a non-integer power law is very unusual and can never be acquired with perturbation theory, unless you take the order of perturbation to infinity. This is yet another oddity of 1D physics.

Plugging the above expression into Eq. (C.2) yields the formula we want:

$$\langle |A(L)|^2 \rangle \propto \rho_0^2 L^2 \left( \frac{\xi_h}{L} \right)^{\frac{1}{2K}} + N \quad (C.8)$$

References


\[ \text{21} \text{i.e. the } \alpha \text{ in Eq. (B.8) \[} \]


Koude atomen in de file

Moos van Caspel

Het is maar goed dat we niet in één dimensie leven, in plaats van drie. Met zijn allen beperkt tot een enkele lijn, zouden we elkaar niet kunnen passeren zonder te botsen. Geef je buurman een duw, en honderd meter verderop voelen ze hem ook. Net alsof je vierentwintig uur per dag in de file staat. Kortom, het leven zou een stuk minder interessant zijn.

Toch kan je een theoretisch natuurkundige heel erg blij maken met een één-dimensionale wereld. Problemen die in drie dimensies onoplosbaar zijn, kunnen in 1D soms zomaar uitgerekend worden. Bovendien treden er in 1D vreemde verschijnselen op, die in hogere dimensies niet te zien zijn. Door het 'file-effect', oftewel het niet langs elkaar kunnen en het ver doorgeven van beweging, gaat de kwantum-mechanica zich heel anders gedragen.

Maar hebben zulke één-dimensionale kwantum effecten nog wel iets te maken met de echte wereld om ons heen? Zeker wel, met de opkomst van de nanotechnologie wordt het steeds gemakkelijker om één-dimensionale systemen te maken in een laboratorium. Koolstof nanobuisjes, 50.000 maal dunner dan een mensenhaar maar net zo lang, zijn bij lage temperaturen heel goed te beschrijven als een 1D object. Ook gassen kunnen zo lang worden uitgerekt dat ze 1D gedrag gaan vertonen. Door de atomen extreem af te koelen tot vlak boven het absolute nulpunt, kan met magneetvelden de vorm van het gas worden veranderd.


Het mooie aan deze experimenten is dat er precieze theorieën bestaan over het gedrag van 1D kwantum gassen. Daarmee kunnen we uitlezen wat voor interferentiepatronen er kunnen ontstaan en hoe vaak we welk patroon kunnen verwachten. De experimenten vormen dus een nauwkeurige test voor de onderliggende theorie, en lijken deze goed te bevestigen. En dat is waar het uiteindelijk om draait in de wetenschap.

Figure 6: Twee één-dimensionale kwantum gassen, weergegeven door de rode cigaren, worden op hun plek gehouden onder een 'atoomchip'. De chip maakt hele precieze magneetvelden, die worden gebruikt om de gassen te vervormen. Als de chip wordt uitgeschakeld, zetten de gassen uit en ontstaat het interferentiepatroon onderaan. De vorm en intensiteit van de slierten kan ons veel vertellen over de kwantummechanische eigenschappen van de gassen. Afbeelding aangepast van Hofferberth et al (2008).