Evaluating the Black-Scholes option pricing model using hedging simulations

Wendy Günther

CKN : 6052088

Wendy.Gunther@student.uva.nl

June 24, 2012

Supervisor(s): Drona Kandhai, Dick van Albada
Signed:
Evaluating the Black-Scholes model

Abstract

Whether the Black-Scholes option pricing model works well for options in the real market, is arguable. To evaluate the model, a few of its underlying assumptions are discussed. Hedging simulations were carried out for both European and digital call options. The simulations are based on a Monte-Carlo simulation of an underlying stock. The influence of the rebalancing frequency of the portfolio and that of the volatility are discussed. The emphasis lies on delta hedging, but other ways of hedging, such as static hedging with a call spread, appear to work better for digital options. Finally, the Black-Scholes model is tested for European call options on actual data of a German stock. It can be concluded that, despite its flaws, the Black-Scholes option pricing model still works for European call options in the real market. However, hedging digital call options is, in general, difficult.
# Contents

1 Introduction &nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbs
7 Hedging Digital call options
   7.1 Delta hedging digital call options ........................................ 27
       7.1.1 Influence of volatility ............................................. 29
   7.2 Delta hedging as a call spread ........................................... 31
   7.3 Static hedging of digital call options .................................. 33
   7.4 Spread risk with other digital options .................................. 34

8 European call options with real data ................................. 35
   8.1 Delta hedging with implied volatility ............................... 35
   8.2 Delta hedging with historical volatility ............................ 37
   8.3 Transaction costs .......................................................... 40

9 Conclusion ........................................................................... 42

A Derivation of Black Scholes .............................................. 45

B Black-Scholes delta ............................................................. 48
CHAPTER 1

Introduction

In the world of option trading, the central question is how an option should be priced. An option is a financial contract that gives the owner the right to buy or sell a certain underlying asset at a price agreed. Trading options can be riskful and, therefore, option pricing models have been invented to be able to control this risk. When an option is priced correctly, it is possible to insure oneself against losses up to a certain level.

Several option pricing models are available nowadays and one of the most widely used ones is the so-called Black-Scholes model. In 1973, Fischer Black and Myron S. Scholes published their Black-Scholes equation. Robert Merton devised another method to derive the equation and generalized it. In 1997, Myron Scholes and Robert Merton received the Nobel price for their model. Fischer Black died in 1995, but he was mentioned as a contributor.

Even though the Black-Scholes equation is widely used to price options, its derivation is based on a number of assumptions of the market. The correctness of these assumptions and the way the model should be used are arguable.

This paper aims to evaluate the Black-Scholes option pricing model. This is done by first looking at the theory behind option trading, hedging and the Black-Scholes model itself. Experiments concerning the Black-Scholes model are done for different simulations of a stock price and the resulting hedging errors are discussed. These experiments are done for two kinds of options: European and digital call options. The influence of the rebalancing frequency on delta hedging and the importance of the volatility are discussed for both kinds. For digital options, more ways of hedging are discussed. Finally, experiments are done with real data of a German stock. The Black-Scholes model is tested on the evolution of this stock using two different estimates of volatility: implied volatility and historical volatility. The performance of the delta hedge is discussed for both estimates.
Many papers, lectures, articles and books about the Black-Scholes option pricing model can be found. The model has proven itself to be a rather popular subject of discussion. It is both criticized and supported. Some claim that the assumptions made to derive the Black-Scholes model are wrong and that, therefore, the model is not applicable when pricing options in the real market. They claim that the presence of transaction costs, the fluctuation of the volatility and the need to rebalance a portfolio continuously make the model inaccurate. Instead of avoiding the Black-Scholes model for these reasons, some people have suggested certain modifications of its parameters. An example is a modification of the volatility, discussed in the lectures of Myungshik Kim. According to him, this modification should reduce the risk when transaction costs are included. Others defend the model and claim that, despite the fact that its assumptions may not always hold, the model itself still works when pricing options in the real market. For example, Paul Wilmott claims that the Black-Scholes model is correct on average.

The Black-Scholes model has mostly been discussed for vanilla options, less for exotic options. Some books that do discuss the model for this kind of options were written by N. Taleb, who also addresses some problems with the Black-Scholes model for vanilla options, F. De Weert and A. Osseiran and M. Bouzoubaa. Another book that discusses several financial models and explains various terms concerning option trading and markets is John Hull’s. When one wants to know more about option trading, this book is certainly a recommendation.

In his book, John Hull also discusses the importance of hedging. Hedging is a general strategy, independent of any model. Therefore, it seems to have been discussed even more than the Black-Scholes model itself. Various hedging methods are available, one of which is delta hedging. The theory behind delta hedging is also discussed in John van der Hoek and Robert J. Elliott’s book, which is not even about the Black-Scholes model itself. Besides delta hedging, the theory behind static hedging is important, especially when trading digital options.
3.1 Financial assets

There are two types of financial assets: underlying assets and derivative assets. Examples of underlying assets are stocks and bonds. A stock represents the claim of the owner on a firm and can be traded in the stock market. A shareholder who owns a stock may be given the right to vote in some matters concerning the firm. A bond is a debt contract, issued by anyone who has borrowed money. It is a fixed-income instrument, for there is interest to be paid. It gives no corporate ownership privileges. Derivative assets are assets whose values depend on the value of the underlying asset. Examples of derivative assets are options and forwards \(^\text{(1)}\).

3.2 Options

An option is a financial contract that gives the owner the right to buy or sell a certain underlying asset at a price agreed. This price is called the strike price of the option and is included in the option contract. The owner of the option has the right, but not the obligation to actually buy or sell the asset. This in contrast with a forward contract, which is an agreement with the obligation to buy or sell the asset at a certain time at a certain price. A call option gives the owner the right to buy an asset at the strike price and a put option gives the owner the right to sell an asset at the strike price.

An option is traded at a certain price to compensate for later possible loss of the writer of the option. The writer of the option is the person who sells it. The price of the option at the time it is written is called the premium. An option contract has a certain time at which it expires. The time until the expiration time is called the time-to-maturity. At expiration, the option has a value, which is called the payoff.

There are all sorts of options. An important kind is the European option, which is an option that can only be exercised at the time it expires. European options are categorized as vanilla options, the kind of options that are common. Another kind of options is the exotic option, which is more complex than a vanilla option. The exotic option discussed in this paper is the cash-or-nothing digital option.

3.2.1 European options

European options are options that can only be exercised at expiration. The payoff of a European option depends on the price of the underlying asset at that time. This payoff is continuous, which means that it changes along with the price of the asset. This principle can easily be evaluated when looking at the equation of the payoff of a European call option, which is \(\max(S_T - K, 0)\).
In this equation, $S_T$ denotes the price of the underlying asset at the expiration time and $K$ denotes the strike price. It means that, if the price of the underlying asset turns out to be higher than the strike price, the holder of the contract can buy this asset at a price below the market price. That makes the price of the option at expiration, the payoff, equal to $S_T - K$. In this case, the option is said to be in-the-money. However, if the price of the underlying asset turns out to be lower than the strike price, it is useless for the holder of the contract to buy the underlying asset at a higher price than the market price. Therefore, the option will not be exercised and its payoff equals zero. The option is then said to be out-of-the-money.

For a European put option, the payoff is given by $\max([K - S_T], 0)$. In this case, if the price of the underlying asset turns out to be higher than the strike price at the expiration time, it is useless for the holder of the contract to sell the asset at a lower price. On the other hand, if the price of the underlying asset turns out to be lower than the strike price, the holder of the contract benefits from the fact that the asset can be sold at a price higher than the market price. Figure 3.1 shows the payoff of a European call and a European put option plotted against the price of the underlying asset.

![Figure 3.1: The payoff of a European put and a European call option, both with a strike price of €99.00, against the price of the underlying asset.](image)

### 3.2.2 Digital options

A digital option, also called a binary option, is an option of which the payoff at the time it expires either equals an amount agreed, or nothing at all. In the case of digital options, the strike price is the price that functions as the conditional price that needs to be met.

There are different kinds of digital options. If the price of the underlying asset ends up above the strike price at the expiration time, the payoff of a so-called asset-or-nothing digital call option is equal to the price of the underlying asset. In this case, the payoff of a so-called cash-or-nothing digital call option is equal to a fixed payoff, which is an amount of cash. The writer and the buyer of the option contract agree on this payoff at the time it is written. If the price of the underlying asset ends up below the strike price, both an asset-or-nothing call option and a cash-or-nothing call option have a payoff equal to zero. The buyer of the contract gains nothing and loses the premium paid for the digital call option. The holder of a digital put option would benefit from this situation, and will have the disadvantage if the price of the underlying asset ends up above the strike price. This paper focuses on cash-or-nothing digital options.

What happens if the price of the underlying asset turns out to be exactly the strike price at the expiration time, is agreed on by both parties and is written in the contract. Figure 3.2 shows the payoff of a digital call and a digital put option plotted against the price of the underlying asset.
Option trading Evaluating the Black-Scholes model

Figure 3.2: The payoff of a digital put and a digital call option, both with a strike price of €99.00, against the strike price of the underlying asset.

3.3 Arbitrage

The Black-Scholes model assumes that there are no arbitrage opportunities. An arbitrage-opportunity is the opportunity to gain profit, without any risk involved. For example, consider a stock of which the stock price in New York equals $152.00 and of which the stock price in London equals £100.00. Assume the exchange rate equals $1.55 per pound. An arbitrageur is now able to obtain a risk-free profit by buying a certain amount of shares in New York and, at the same time, selling them in London. The risk-free profit will then be equal to \( \text{amount of shares} \times (\text{($1.55 \times 100)} \ – \$152.00) \). In reality, an arbitrage-opportunity like this will never last long. The arbitrageurs themselves take care of it by buying more shares in New York, which causes the price of the stocks to rise there, and selling them in London, which causes the price to decline there \[12\].
CHAPTER 4

Valuation of options

Several option pricing models are available. At the moment, the model that is widely used for option pricing is the so-called Black-Scholes model. For an understanding of the model and its derivation, one should first look at the Binomial Tree model.

4.1 Binomial Tree model

The Binomial Tree model is an option pricing model that focuses on keeping a composed portfolio riskless. When a portfolio consisting of options and assets is riskless, it will neither cause a loss, nor will it make a profit. The portfolio on which the Binomial Tree model is based, consists of a long position in a call option contract and a short position in a certain number of shares of the underlying asset. A person that takes a long position belongs to the buying party, while a person that takes a short position belongs to the selling party. This means that one sells a call option contract and buys a number of shares of the underlying asset. This makes the value of the portfolio equal to \( \Delta S - f \), where \( S \) is the price of the underlying asset, \( f \) is the price of the option and \( \Delta \) is the number of shares bought.

The most important concept underlying the Binomial Tree model, and the reason why it got its name, is that it considers a world with only two moments: the moment at which the option contract is written (\( t = 0 \)) and the moment at which it expires (\( t = T \)). Since the price of an asset is moving, it can either go up or down. For the portfolio to be riskless, it should have the same value in both situations. This leads to the following equation:

\[
\Delta S_u - f_u = \Delta S_d - f_d. \tag{4.1}
\]

In this equation, \( u \) can be seen as the ratio with which the value of the asset goes up, while \( d \) can be seen as the ratio of it going down. From equation 4.1 it follows that

\[
\Delta = \frac{f_u - f_d}{S(u - d)}. \tag{4.2}
\]

In other words, for the portfolio to be riskless, \( \Delta \) shares have to be included in the portfolio. This forms the basis of delta hedging, the hedging method underlying the Black-Scholes model, which is explained in section 5.2. Since, following from equation 4.1 and 4.2, the values of the portfolio for both the up and down situation are the same, it is possible to determine the value of the portfolio at the time the option contract is written. This is made possible by two concepts called continuous discounting and continuous compounding. In short, continuous compounding means...
that a future value of money can be calculated by multiplying the current value by $e^{rT}$, where $r$ denotes the risk-free interest rate in decimals and $T$ denotes the time in years. Continuous discounting means calculating a past value by multiplying the present value by $e^{-rT}$. It follows that the value of the portfolio at the time it is written, for it to be risk-free, must be equal to

$$\Delta S - f = e^{-rT}(\Delta Su - f_u). \quad (4.3)$$

The most important equation leading to the Black-Scholes model is that of calculating the option price at the current time. It follows from equation (4.1) that this can be calculated as

$$f = -e^{-rT}\Delta Su + e^{-rT}f_u + \Delta S. \quad (4.4)$$

When substituting $\Delta$ of equation (4.2) into equation (4.4) this becomes

$$f = e^{-rT}(pf_u + (1-p)f_d), \quad (4.5)$$

where

$$p = \frac{e^{rT} - d}{(u - d)}.$$

Note that when $p$ is interpreted as the chance that the price of the underlying asset will go up, the price of the call option at the time the contract is written can be seen as the continuous discounted expectation of the payoff. In order to interpret $p$ as a chance, the condition $0 \leq p \leq 1$ should hold. Therefore the conditions $u \geq e^{rT}$ and $d \leq e^{rT}$ should hold for both the Binomial Tree model and the Black-Scholes model. Another important statement made for the derivation of the Black-Scholes model, is that the expected price of the underlying asset at the expiration time is then equal to

$$E(S_T) = pS_0u + (1-p)S_0d = S_0e^{rT}. \quad (4.6)$$

$$E(S_T) = S_0e^{rT}. \quad (4.7)$$

### 4.2 Black-Scholes model

The Black-Scholes model for option pricing was derived with the idea of delta hedging in mind. This way of hedging risk is further explained in section 5.2. For Fischer Black and Myron Scholes to have come to the Black-Scholes equation, a few assumptions were made, including:

- The price of the underlying asset is lognormally distributed, with a constant expected return and volatility.
- The underlying asset pays no dividend.
- There are no transaction costs attached to selling or buying underlying assets or the option contract.
- The risk-free interest rate is known and constant during the entire period.
- There are no arbitrage opportunities.

A variable that is lognormally distributed can take any value between zero and infinity [12].
4.2.1 Black-Scholes for European Options

As described in the previous section, when \( p \) is interpreted as the chance that the price of the underlying asset will go up, the price of the call option at the current time can be seen as the continuous discounted expectation of the payoff. For a European call option, this means that the option price \( c \) is equal to

\[
c = e^{-r(T)}E[max(S_T - K, 0)] = e^{-rT} \int_{K}^{\infty} (S_T - K)g(S)\,dS.
\]

In this equation, \( r \) denotes the risk-free interest rate in decimals, \( T \) the time-to-maturity in years, \( S_T \) the price of the underlying asset at the expiration time and \( K \) the strike price. Using some algebra, the Black-Scholes equation for pricing European call options turns out to be

\[
c = SN(d_1) - Ke^{-rT}N(d_2),
\]

where

\[
d_1 = \frac{\ln[S/K] + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
\]

and

\[
d_2 = \frac{\ln[S/K] + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.
\]

The derivation of this equation can be found in appendix [A]. The model depends on a couple of parameters. The price of the underlying asset at the current time, \( S \), and the risk-free interest rate, \( r \), in decimals, can easily be derived from the market. The strike price, \( K \), and the time-to-maturity \( T \), in years, are to be agreed on while the option contract is being written. In the equation, \( \sigma \), in decimals, denotes the percentage expected volatility. It is the only parameter that cannot immediately be derived from the market. The volatility is the intensity of the price-movement of the underlying asset and is further explained in section 4.3.

Using the assumptions underlying the Black-Scholes model, a relationship called put-call parity can be derived. This parity depends on the assumption that there are no arbitrage opportunities. Consider a portfolio \( A \), consisting of one European call option and an amount of cash equal to \( Ke^{-rT} \). At the expiration time, this amount of cash will be equal to \( K \). If, at that time, the price of the underlying asset is higher than the strike price, the asset will be bought at the price \( K \) and sold again at \( S_T \). The portfolio will then have taken the value of \( S_T \). However, if, at the expiration time, the price of the underlying asset is lower than the strike price, the call option will expire without being exercised and the value of the portfolio will be equal to \( K \). Now besides portfolio \( A \), consider a portfolio \( B \), consisting of one European put option on the same asset as where the call option of portfolio \( A \) is on, and one unit of this asset. If, at the expiration time, the price of the asset is higher than the strike price, the put option will expire without being exercised and the value of the portfolio will be equal to \( S_T \). However, if, at that time, the price of the asset is lower than the strike price, the value of the portfolio will be equal to \( K \). In conclusion, the values for both portfolios will be equal to \( max(S_T, K) \). Since one of the assumptions underlying the Black-Scholes model is that no arbitrage opportunities exist, the values of the portfolios at every time step have to be equal to each other, which leads to the relationship

\[
c + Ke^{-rT} = p + S,
\]

where \( p \) denotes the price of a European put option. Now, using this put-call parity, it follows that the Black-Scholes equation for the price of a European put option is
The only thing left to be able to delta hedge, the hedging method underlying the Black-Scholes model, is delta. This delta is the same as the $\Delta$ in the Binomial Tree model. In other words, it is the number of shares that has to be bought to keep the portfolio risk-free. It follows that delta is equal to the rate of change of the option price with respect to the price of the underlying asset. For European call options, this means that the analytic delta is equal to $N(d_1)$. The derivation of this delta can be found in Appendix [B].

### 4.2.2 Black-Scholes for digital options

The Black-Scholes equation for cash-or-nothing digital options is a little easier to derive than that for European options. After all, the payoff either equals a fixed payoff, or nothing at all. In the same way the Black-Scholes equation for European call options was derived, it is possible to derive the one for digital call options with

$$c_{\text{digital}} = \begin{cases} D & \text{if } (S_T > K) \\ 0 & \text{if } (S_T < K), \end{cases}$$

from which follows that

$$c_{\text{digital}} = De^{-rT}N(d_2), \quad (4.16)$$

where

$$d_2 = \frac{\ln[S/K] + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \quad (4.15)$$

In this equation, $N(x)$ denotes the cumulative normal function, $S$ the current price of the underlying asset, $K$ the strike price, $D$ the fixed payoff, $r$ the percentage expected volatility in decimals, and $T$ the time-to-maturity in years. The Black-Scholes equation for a digital put option is

$$p_{\text{digital}} = De^{-rT}N(-d_2). \quad (4.17)$$

The analytic delta that can be used to delta hedge digital call options is equal to

$$\frac{\delta c_{\text{digital}}}{\delta S} = \frac{De^{-rT}P(d_2)}{S\sigma\sqrt{T-t}}. \quad (4.18)$$

In this equation $P(x)$ denotes the derivative of the cumulative normal function: the standard normal probability density function, which is equal to $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

### 4.2.3 Black-Scholes delta

As mentioned before, delta (notation $\Delta$) denotes the rate of change of the option price with respect to the price of the underlying asset. This is the most important tool when delta hedging and it depends on various parameters.
Figures 4.1 and 4.2 show that delta does not only change when the stock price changes, but also when the time-to-maturity changes. This means that delta changes continuously. Because of that reason, for a delta hedge to work perfectly, it is necessary to rebalance the portfolio continuously. Also, when using the analytic delta, it is assumed that the price of the underlying asset can move in infinitely small steps. In reality, this is simply not possible. Therefore, it is also possible to delta hedge with a discrete delta, calculated as, for example

\[ \frac{c(S + \epsilon) - c(S - \epsilon)}{2\epsilon}, \]

where \( \epsilon \) is some small monetary value [16].

4.3 Volatility

An important term in option trading is the volatility. The volatility is the intensity of the price-movement of the underlying asset. If, for example, the underlying asset is a stock, then the stock has a high volatility when the exchange rate moves a lot. It is the standard deviation of the change in the stock price in one year. Some claim that the volatility is caused by the arrival of new information about the stock. Others claim that it is caused by trading [12].
When pricing options, the volatility to be concentrated on is the expected volatility. After all, the final payoff of the option depends on the stock price at the expiration time, so it depends on the way the stock price is expected to move in the future. There are different ways to estimate the expected volatility. The ones discussed in this paper are the implied volatility and the historical volatility.

### 4.3.1 Implied volatility

The implied volatility is the volatility that is implied by option prices of actively traded options on the same underlying asset. By observing these, one can calculate the volatility to be used as input in the Black-Scholes formula to match the market prices. For example, consider a European call option on a stock of which the current stock price is €120.00, with a strike price of €110.00. Suppose that the risk-free interest rate is equal to 6% and the time-to-maturity for this option contract is set to one year. The option has a value of €21.18. The volatility used to calculate this option price is unknown. One could find the value of the volatility, by looking for the volatility that, when substituted into the Black-Scholes equation with these parameters, gives a value equal to €21.18. This can be done using the Newton-Raphson root-finding algorithm on the function $f(\sigma) = c - c_{BS}(\sigma)$, where $\sigma$ denotes the volatility, $c$ the known option price and $c_{BS}(\sigma)$ the Black-Scholes equation on $\sigma$. This algorithm repeatedly uses equation

$$
\sigma_k = \sigma_{k-1} - \frac{f(\sigma_{k-1})}{f'(\sigma_{k-1})}
$$

and will converge to the zero point. The derivative $f'(\sigma)$ of $f(\sigma)$, with respect to the volatility, is equal to the negative of the Black-Scholes’ vega, so

$$
f'(\sigma) = -SP(d_1)\sqrt{T}.
$$

In this equation, $S$ denotes the current stock price, $P(x)$ the probability density function and $T$ the time-to-maturity in years.

### 4.3.2 Historical volatility

The Black-Scholes option pricing model assumes that the percentage changes in the stock in a short period of time are normally distributed with

$$
\frac{\delta S}{S} \sim \phi(\mu \delta t, \sigma \sqrt{\delta t}). \tag{4.19}
$$

The normal distribution is defined as $\phi(m, s)$, were $m$ denotes the mean and $s$ denotes the standard deviation. Also, $\delta S$ denotes the change in stock price, $\mu$ the expected return on the stock and $\sigma$ the volatility of the stock price.

To estimate the volatility using historical data, every time step, the daily return is defined as

$$
u_t = \ln \left( \frac{S_t}{S_{t-1}} \right).
$$

With this, the estimate of the standard deviation of the daily returns is defined as

$$
s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} u_t^2 - \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} u_t \right)^2}. \tag{4.20}
$$
In this equation, \( n \) denotes the number of preceding days on which the historical volatility should be based, including the current day. The model used also implies that

\[
\ln \frac{S_T}{S_0} \sim \phi \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \tag{4.21}
\]

This means that equation 4.20 estimates the value equal to \( \sigma \sqrt{T} \), where \( \sigma \) denotes the percentage volatility in decimals and \( T \) the time in years. This makes the equation for calculating the historical volatility equal to

\[
\sigma = \frac{s}{\sqrt{T}} \tag{4.22}
\]

In this equation, \( T \) should be measured in trading days. It is assumed that one year consists of 252 trading days \[12\]. This means that annualizing the volatility includes setting \( T \) at \( \frac{1}{252} \).
Hedging is an important concept when trading options. When one sells, for example, a European call option to someone else, a hedge portfolio is set up. At the expiration time, if the price of the underlying asset is higher than the strike price, the holder of the option will exercise the right to buy the asset at the strike price. Therefore, the hedge portfolio of the one who sold the option should have replicated an amount of cash equal to the payoff, to make up for the difference between the strike price and the price of the underlying asset. The asset has to be bought at its market price, before being able to sell it at the strike price again. In other words, the payoff of the call option should be hedged. One way of hedging, called static hedging, includes setting up a portfolio that does not have to be changed until the expiration time. Another way of hedging, called delta hedging, includes setting up a portfolio that has to be rebalanced frequently. This is a so-called dynamic hedging strategy.

5.1 Static hedging

Static hedging includes searching for a portfolio of options that replicates the value of the option at every time step, without having to rebalance the portfolio. It is assumed that, when this portfolio has the same value as the option price at the initial time, they have the same value at every time step. Therefore, at the expiration time, the payoff is replicated by just keeping the portfolio as it is. Section 7.3, for example, describes how a digital call option can be replicated using a static hedge with European call options.

5.2 Delta hedging

Delta hedging includes setting up a portfolio consisting of a long position in a number of shares of the underlying asset and a short position in writing a call option contract. At the time the option contract is written, the price of the underlying asset is known and the premium can be calculated using the Black-Scholes model.

A delta hedge aims to keep the value of this portfolio the same for the situation where the price of the underlying asset goes up, as for where it goes down. As known from the Binomial Tree model, the number of shares of the underlying asset bought then should be equal to delta. However, since the delta of the Black-Scholes model is changing with time and depends on the price of the underlying asset, hedging should be done dynamically.

Algorithm 1 shows in pseudocode the way this dynamic delta hedge works. The variable balance resembles an amount of money put on or borrowed from a bank, for which interest is paid. It is actually the negative of the portfolio that is set up, so it is equal to $-\Delta S + f$. This means that every time step, this balance plus the value of the number of shares held should be equal to the
option price. Therefore, every time step, the value of the hedge portfolio that should replicate the option price is defined as $balance + \Delta S_t$. For this paper, both the Black-Scholes model and the hedging methods are implemented in Java.

As mentioned before, at the time the option contract is written, the writer of the option contract receives the premium from the buyer and puts it on the balance. Delta shares of the underlying asset are bought with borrowed money to keep the portfolio risk-free. At the next time step, before anything else happens, the balance is compounded at the risk-free interest rate. This means that interest is paid, depending on the value of the balance. Next, since delta has changed for the current situation, the portfolio has to be rebalanced to remain risk-free. This is done by selling all the previous holdings of the underlying asset at the current market price and buying a new delta number of shares of the same underlying asset at the same price. Note that this is the same as buying or selling the difference in delta shares at that point. This is an important note when it comes down to transaction costs. Rebalancing is done every time step until the expiration time. At the expiration time, the payoff of the call option should be replicated. Selling the number of shares of the underlying asset at that time should therefore result in the value of the balance being equal to the option payoff.

### 5.3 Monte Carlo simulation

A variable that changes value over time in an uncertain way follows a stochastic process. These stochastic processes can be either discrete or continuous. It is assumed that stocks follow a stochastic process. More specifically, stocks follow a Markov process, which is a stochastic process where only the present value of the variable is important for the future value. The path of the stock price in the past is not important, even though the volatility is. A Monte Carlo simulation is a procedure for sampling random outcomes for a stochastic process. It can be used for simulating the movement of a stock price.\[12\]

The equation for the difference in stock price between two time steps is assumed to follow a brownian motion\[13\], i.e.

$$\frac{\delta S}{S} = \sigma \delta z + \mu \delta t.$$

In this equation, $\delta z$ is a random variable drawn from a normal distribution with a mean of zero and a variance of $\delta t$. Therefore, $\delta z$ can be written as $\delta z = \phi \sqrt{\delta t}$, where $\phi$ is a random variable drawn from the standard normal distribution. Also, $\mu$ is defined as the expected return, which, according to the Black-Scholes model, is equal to the risk-free interest rate. Since the value of the stock price at the next time step is equal to $S_{t+\delta t} = S_t + \delta S$, the stock price can be modelled every time step with

---

**Algorithm 1 Implementation of delta hedge**

$$balance = 0$$

$$balance - = \Delta_t S_t$$

$$balance + = \text{premium}$$

for $t \leftarrow dt$; $t < T$ do

$$balance + = (balance \times r \times \delta t)$$

$$balance + = \Delta_t S_t$$

$$t + = \delta t$$

end for

$$balance + = (balance \times r \times \delta t)$$

$$balance + = \Delta_t S_t$$
\[ S_{t+\delta t} = S_t + rS_t \delta t + \sigma_{stock} \phi S_t \sqrt{\delta t}. \] (5.1)

In this equation, \( S_t \) denotes the current stock price, \( \delta t \) a small period of time, \( r \) the risk-free interest rate in decimals and \( \sigma_{stock} \) the percentage volatility of the stock in decimals.
CHAPTER 6

Hedging European call options

In order to evaluate how well the Black-Scholes model for option pricing works, the delta hedge is implemented on a Monte Carlo simulation of a stock price. With the implementation, it is possible to repeat the delta hedge for many different simulations. For the draw of a random number out of the standard normal distribution, a random generator with a seed is used. Therefore, the experiments can easily be repeated on the same simulation, as long as the stock price is recalculated at the same frequency.

In this chapter, the experiments are concentrated on the evaluation of the Black-Scholes model for European call options. The influence of the rebalancing frequency will become clear, as well as the influence of a difference between the expected volatility and the actual volatility of the underlying stock price.

6.1 Delta hedging European call options

The experiments in this section are done for an initial stock price of €100.00, a strike price of €99.00, the expected volatility and the volatility of the stock price set at 20% per annum, a risk-free interest rate of 6% per annum and a random seed of 0. The time-to-maturity is set to one
year and varying rebalancing frequencies are considered. For the distributions, the experiments are repeated a thousand times for different simulations of the underlying stock price. The bins have a representative number. A hedging error is designated to the bin with the representative number it is closest to.

Figure 6.1 shows a delta hedge over time, where the portfolio is rebalanced at a daily frequency. The **hedging error** is defined as the value of the hedge portfolio at the expiration time, minus the payoff of the call option. This hedging error is inevitable, because, as mentioned before, the delta hedge underlying the Black-Scholes model requires continuous rebalancing of the portfolio. In practice, this is impossible of course. However, it is interesting to see exactly how big the hedging error is, considering varying rebalancing frequencies, and thus to what extent this hedging error can be reduced. It will become clear that the hedging error depends on a number of factors, including the rebalancing frequency, the evolution of the stock price and its volatility.

Figure 6.2 shows the distribution of the hedging error where, for every simulation, rebalancing of the portfolio is done at a daily frequency. In theory, rebalancing less frequent should result in
larger hedging errors. Figure 6.3, which shows the distribution of the hedging error where rebalancing is done at a weekly frequency, confirms this theory. The figure shows that the distribution of the hedging error reaches larger absolute values and has a larger standard deviation. Even though the mean of the distribution still lies close to zero, a more frequent rebalancing frequency seems to do better. In all the distributions in this section, the larger errors can be explained by the evolution of the underlying stock price. When the simulated stock price remains close to the strike price, delta hedging is less accurate because of the nature of delta, shown in figure 4.1. Figure 6.3 shows the distribution of the hedging error where rebalancing is done once every minute. According to the same theory as before, when rebalancing is done more frequently, the larger hedging errors should be reduced. Compared to figure 6.2, the hedging errors are indeed reduced substantially. Note that the errors shown in the distribution are hedging errors for the case where the stock price starts at €100.00 and the strike price equals €99.00. Increasing the value of these parameters will also increase the value of the hedging error. However, it is a good indication to see to what extent the rebalancing frequency influences the distribution of the hedging error, since the experiments are all done for the same initial stock price and the same strike price.

![Figure 6.4](image)

Figure 6.4: The distribution of the hedging error when delta hedging a European call option for a thousand different simulations of the underlying stock price, where rebalancing is done once every minute.

### 6.1.1 Influence of volatility

For the experiments in this subsection, varying volatilities and varying rebalancing frequencies are considered.

As mentioned before, an important term when pricing options is the volatility. In the equations used in the experiments, there are two kinds of volatility: the volatility of the stock price and the expected volatility. The volatility of the stock price is the intensity of the price-movement of the stock. This is the volatility used in the Monte-Carlo simulation. However, the volatility used in the Black-Scholes equation is the expected volatility. As long as the expected volatility equals the actual volatility of the stock, results like those in the previous section are obtained. However, it is interesting to see what happens when the expected volatility does not turn out to be the same as the actual volatility. Figure 6.6 shows the distribution of the hedging error where the volatility of the stock price is set at 20% per annum, the expected volatility used to calculate the option price is set at 40% per annum and rebalancing is done at a daily frequency. The figure shows that the hedging error remains positive for every simulation of the underlying stock price. The complete distribution lies on the positive side. To understand why this happens, take a look at figure 6.5, which shows the delta hedge on exactly the same simulation of the stock price as that of figure 6.1, but now the expected volatility is set at 40% per annum.
What happens is that the option is constantly assumed to be worth more than it actually is. The price of the European call option is higher when the expected volatility is higher. This can be explained by the fact that the chance that an option will be further in-the-money or further out-of-the-money is higher. In the case it gets further in-the-money, the holder of the option contract will benefit more. However, if the option gets further out-of-the-money, the holder of the option contract will only lose the premium. Therefore, the premium is higher when the volatility is higher. This is shown in figure 6.7. While the expected option price eventually equals the real option price, since they result in the same payoff, the value of the hedge portfolio causes a positive hedging error. The rebalancing frequency does not seem to influence the distribution of the hedging error anymore either. Even if one would be able to rebalance every minute, it would still be done with a wrong expected volatility, which keeps resulting in a positive hedging error. The hedging error will never be negative when the expected volatility is larger than the volatility of the stock, because the option price calculated using a higher volatility will always be at least the value calculated using a lower volatility, according to figure 6.7.
It could happen that the option price does not differ much from what it is supposed to be. This situation is shown in figure 6.8. This figure shows that the hedge portfolio does not differ much from the option price when the option is not overvalued and that this results in a small hedging error. In this case, the stock price is constantly higher than the strike price, rising from the start and ending at a value of €180,00. Figure 6.7 shows that the difference in volatility for stock prices far from the strike price, does not have much influence on the option price. The small hedging error is the result of the difference in volatility at the start, where the stock price is near the strike price.

In the same way, when the volatility of the stock price is actually higher than the expected volatility used to calculate the option price, the hedging error remains negative for every simulation of the underlying stock price. The distribution of the hedging error where the stock price is set at 40% per annum, the expected volatility at 20% per annum and rebalancing is done at a daily frequency, is shown in figure 6.9. Figure 6.10 shows the delta hedge, drawn from this distribution, that results in a hedging error of $-€20.45$. Every time step, the option is assumed to be worth less than it actually is, which results in a negative hedging error. Eventually, the option price is equal to what it should be, but due to a constant miscalculation of the option price, the hedge portfolio does not replicate the option payoff.

Figure 6.8: The evolution of the option price and the value of the hedge portfolio during a delta hedge of a European call option, where the stock price ends at €181.63 and the hedging error equals €2.65. The volatility of the stock price is set at 20% per annum, the expected volatility at 40% per annum and rebalancing is done at a daily frequency.
Figure 6.9: The distribution of the hedging error when delta hedging a European call option for a thousand different simulations of the underlying stock price. The volatility of the stock price is set at 40% per annum, the expected volatility at 20% per annum and rebalancing is done at a daily frequency.

Figure 6.10: The evolution of the expected option price, the real option price and the value of the hedge portfolio during a delta hedge of a European call option, where the stock price ends at €95.75 and the hedging error equals −€20.45. The volatility of the stock price is set at 40% per annum, the expected volatility at 20% per annum and rebalancing is done at a daily frequency.
CHAPTER 7

Hedging Digital call options

7.1 Delta hedging digital call options

The experiments in this section are done for an initial stock price of €100.00, a strike price of €99.00, the expected volatility and the volatility of the stock price set at 20% per annum, a risk-free interest rate of 6% per annum and a random seed of 0. The time to maturity is set to one year, the fixed payoff at €100.00 and varying rebalancing frequencies are considered. For the distributions, the experiments are repeated a thousand times on different simulations of the underlying asset.

Hedging a digital call option is more difficult than hedging a European call option. Therefore, the option is categorized as an exotic option. The distribution of figure 7.1, where rebalancing is done at a daily frequency, points this out.

Figure 7.1: The distribution of the hedging error when delta hedging a digital call option for a thousand different simulations of the underlying stock price, where rebalancing is done at a daily frequency.

Since the simulations of the underlying stock price of figure 7.1 are the same as those of figure 6.2, we can easily compare these two distributions. It appears that, when delta hedging a digital call option, the hedging error can become much larger than when hedging a European call option. Not only is the distribution for a digital call option more spread, but it also reaches extremely large hedging errors. One of the simulations of the stock price, on which the delta hedge is practised and an extremely large positive hedging error results, is shown in figure 7.2.
The figure shows that near the expiration time, the value of the hedge portfolio begins to differ substantially from the option price. It also shows that, at that time, delta begins to fluctuate more. Looking back at figure 4.2, this is an expected result. The curve of the Black-Scholes delta of a digital call option gets more peaked when closer to the expiration time. Important to note is that in figure 7.2, at time $T - dt$, which is the time step right before the expiration time, the stock price was equal to €102.04. This means that when the portfolio was rebalanced for the last time, the digital call option was in-the-money. That means that the hedge portfolio would contain a value almost equal to the payoff of the option. However, between the last two time steps, the stock price moved below the strike price, which made the option worth nothing. This results in a positive hedging error, so the person who wrote the digital call option actually made a profit. However, it could just as easily go the other way around. Figure 7.3 shows a simulation of the stock price on which the delta hedge is practised and an extremely large negative hedging error results.

In figure 7.3, at time $T - dt$, the stock price was equal to €97.48. In this case, when the portfolio was adjusted for the last time, the digital call option was out-of-the-money. Even though it was
so close to the strike price, the rebalancing made sure that the hedging error did not pay enough to make up for the payoff of the option. Unfortunately, in the last time step, the option moved in-the-money. The problem is that, from one moment to another, the payoff of a digital option can suddenly change from a 100% payoff to 0%, and the other way around.

Because of the nature of Black-Scholes’ delta for digital call options, delta hedging these options is extremely difficult. Increasing the rebalancing frequency may lead to better hedge results in most situations, but large hedging errors still occur. Not only does the nature of delta bring difficulties when trying to keep the hedge portfolio equal to the option price, but, very close to the expiration time, its value might even move to infinity. Trading such an amount of shares is impossible.

7.1.1 Influence of volatility

Figure 7.4: The distribution of the hedging error when delta hedging a digital call option, for a thousand different simulations of the underlying stock price. The volatility of the stock price is set at 20% per annum, the expected volatility at 40% per annum and rebalancing is done at a daily frequency.

Figure 7.5: The distribution of the hedging error when delta hedging a digital call option for a thousand different simulations of the underlying stock price. The volatility of the stock price is set at 40% per annum, the expected volatility at 20% per annum and rebalancing is done at a daily frequency.
For the experiments in this subsection, varying volatilities and varying rebalancing frequencies are considered. As with the experiments for European call options, the influence on the hedging error when the expected volatility differs from the actual volatility of the stock are discussed.

Figure 7.4 shows the distribution of the hedging error, where the volatility of the stock price is set at 20% per annum, the expected volatility used to calculate the option price is set at 40% per annum and rebalancing is done at a daily frequency. Figure 7.5 shows the distribution where rebalancing is done at the same frequency, but now the volatility of the stock price is set at 40% per annum and the expected volatility is set at 20% per annum.

![Figure 7.4: The evolution of the option price of a digital call option against the price of the underlying asset for varying expected volatilities, with the time-to-maturity set to one year.](image)

Figure 7.6 explains more or less why this happens. A difference in volatility when delta hedging digital call options can result in overvaluing the option, as well as undervaluing it. Overvaluing the option causes the hedge portfolio to be worth more than the option, which results in a positive hedging error.

![Figure 7.6: The evolution of the option price of a digital call option against the price of the underlying asset for varying expected volatilities, with the time-to-maturity set to one year.](image)

The distributions in figures 7.4 and 7.5 both show an odd distribution of the hedging error. They both show large and many hedging errors on the positive, as well as on the negative side. Figure 7.6 explains more or less why this happens. A difference in volatility when delta hedging digital call options can result in overvaluing the option, as well as undervaluing it. Overvaluing the option causes the hedge portfolio to be worth more than the option, which results in a positive hedging error.
hedging error. Undervaluing the option causes the hedge portfolio to be worth less than the option, which results in a negative hedging error. During the delta hedge, a digital call option can move from being undervalued to being overvalued, which sometimes neutralizes the hedging error. This situation is shown in figure 7.7. Not only does the fact of overvaluing or undervaluing have influence during the delta hedge, but the peaking of delta near the expiration time and near the strike price still remains a big problem.

7.2 Delta hedging as a call spread

The experiments in this section are done for an initial stock price of €100.00, a strike price of €99.00, the expected volatility and the volatility of the stock price set at 20% per annum, a risk-free interest rate of 6% per annum and a random seed of 0. The time to maturity is set to one year and the fixed payoff is set at €100.00.

A different way of looking at a digital call option, is by seeing it as a call spread, or, more specifically, a bull spread. A bull spread is created by taking a long position in a European call option with a certain strike price and taking a short position in a European call option on the same underlying asset, but with a higher strike price. Both options must have the same time-to-maturity. Figure 7.8 shows the payoff obtained by this kind of call spread.

![Payoff of a bull spread](image)

Figure 7.8: The payoff of a bull spread composed by a long position in a European call option with strike \( K - \epsilon \), and a short position in a European call option with strike \( K \).

Comparing figure 7.8 with figure 3.2 shows that the price of a digital call option is actually the same as an infinitely small call spread. What kind of call spread depends on the nature of the digital option \[16\]. For example, if the payoff equals some amount of cash when the stock price is at least the strike price at the expiration time, the digital call option is the same as an infinitely small call spread given by

\[
c_{\text{digital}} = D \times \left( c_{\text{European}}(K - \epsilon) - c_{\text{European}}(K) \right). \tag{7.1}\]

If the payoff of a digital call option would be some amount of cash when the stock price ends up higher than the strike price, instead of at least the strike price, the spread would look like

\[
c_{\text{digital}} = D \times \left( c(K) - c(K + \epsilon) \right). \tag{7.2}\]

In these equations, \( D \) denotes the fixed payoff, \( \epsilon \) some monetary value and \( c(K) \) the price of a European call option with strike price \( K \), according to the Black-Scholes equation. Note that this also means that the price of a digital call option is actually the same as the negative derivative of the price of a European call option with respect to the strike price.
The biggest problem when delta hedging a digital call option is delta. According to figure 4.2, delta changes extremely fast when close to the expiration time and close to the strike price. Therefore, it is extremely difficult to delta hedge in situations like that. However, if there were some way to smooth delta, this problem would be partly solved. This smoothing can be done by using the delta of a call spread where

\[ c_{\text{digital}} = D \times \left( \frac{c(K - \epsilon) - c(K + \epsilon)}{2\epsilon} \right) \tag{7.3} \]

and thus the delta used is

\[ \Delta_{\text{spread}} = D \times \left( \frac{\Delta_{\text{European}}(K - \epsilon) - \Delta_{\text{European}}(K + \epsilon)}{2\epsilon} \right) \tag{7.4} \]

Figure 7.9 shows how, with a time-to-maturity of one day, this delta is smoother than that of the delta calculated with the Black-Scholes formula.

![Figure 7.9: The delta of a call spread where $\epsilon = \text{€}5.00$ versus that of a digital call option calculated with Black-Scholes, with a time-to-maturity of one day.](image)

The way of hedging remains the same, but the number of shares used to rebalance the portfolio now depends on a different delta. It would be the same as dynamically delta hedging a call spread, though the premium and the fixed payoff of the corresponding digital call option are used. The question is how large $\epsilon$ should be. Taking $\epsilon$ too small, would just imitate the delta of a digital call option. Therefore, to smooth delta, $\epsilon$ should be relatively large. However, taking $\epsilon$ too large can be problematic.

Figure 7.10 shows the delta hedge using Black-Scholes’ delta for digital call options, which results in an extremely large error of 103.23. Figure 7.11 shows the delta hedge on the same simulation of the underlying stock price, but now delta is calculated using equation 7.4. For this delta hedge, $\epsilon$ is set at €5.00.

The most extreme positive hedging error is reduced substantially, from €103.23 to €56.47. Apparently, delta starts to differ enough from the actual delta near the expiration time. However, in some situations, the hedging error, when using the delta of the spread, is larger than when just using Black-Scholes’ delta for digital call options. Figure 7.11 shows the distribution of the hedging error when delta hedging with the delta of a call spread, where rebalancing is done at a daily frequency. These experiments are done on the same simulations as those of figure 7.2. The most extreme hedging errors are reduced, but, using the delta of a European call spread, has led to increased hedging errors in other situations.
Figure 7.10: The evolution of the option price, the stock price, the delta of the call spread, and the value of the hedge portfolio during a delta hedge on a digital call option, where the stock price ends at €98.83 and the hedging error equals €56.47.

Figure 7.11: The distribution of the hedging error of delta hedging a digital call option, for a thousand different simulations of the underlying stock price, where the delta is that of a call spread with \( \epsilon = €5.00 \). Rebalancing is done at a daily frequency.

### 7.3 Static hedging of digital call options

Since a digital call option is equal to an infinitely small bull spread of European call options, it is also possible to perform a static hedge of a digital call option. At the time the digital call option is written, the hedge portfolio can be set up by, for example, buying \( \frac{D}{\epsilon} \) European call options with strike price \( K - \epsilon \) and selling \( \frac{D}{\epsilon} \) European call options with strike price \( K \). Since the European call option sold is worth less than the European call option bought, setting up the portfolio initially requires a loan. However, this can be prevented by pricing the digital call option exactly the same as the costs it takes to set up the call spread. The premium received for the digital call option then pays up for the costs of setting up the hedge portfolio. Eventually, at the expiration time, the payoff of the call spread should replicate the payoff of the digital call option. However, this is not always the case. Whether this static hedge replicates the payoff depends on the final value of the stock price. If the stock price ends up between the two strike prices...
Hedging Digital call options

prices, the long position in the European call option pays \( D \times (S_T - (K - \epsilon)) \) and the short position in the European call option costs nothing. On top of that, the digital call option sold with strike price \( K \) has a payoff of zero. Therefore, the claim of the digital call option is overreplicated.

Hedging errors like these can be eliminated by using a smaller \( \epsilon \). However, the smaller \( \epsilon \) gets, the more European call options have to be bought and sold to replicate the payoff of the digital call option. In reality, this is impossible. Not only is it hard to find an option with a strike price so close to another strike price, but the market makes it impossible to trade in such a large amount of European call options.

7.4 Spread risk with other digital options

The previous sections in this chapter showed that hedging a digital call option can be very difficult, if not impossible. Therefore, instead of trying to hedge a digital option using delta hedging or static hedging, one could try to spread the risk involved by trading in other digital options. For example, taking a long position in a digital call option can be riskful, since the option can end up out-of-the-money and the entire premium would be lost. However, taking a long position in a digital put option on the same asset with a higher strike price could offset this loss somehow. In case the stock price ends up between the two strike prices, an even bigger profit can be made than when one would only own the digital call option. The drawback is that the costs of buying both a digital put and a digital call option are larger and, if this requires a loan, more interest has to be paid.

Consider a long position in a digital call option with a certain strike price. If one would be able to take a short position in a digital call option with the exact same strike price on the same underlying asset, the digital call option sold would be perfectly replicated. The option is simply hedged using another digital option. In reality, this may not be possible, but the idea of spreading risks with other digital options is certainly interesting. A digital option is much like a bet \([16]\). Therefore, instead of concentrating digital options on a single strike price, one could take positions on multiple strike prices in order to reduce the risk of losing a large amount of money. Instead of trying to hedge, one would just spread the risk using other digital options.
In this chapter, delta hedging and the Black Scholes model are tested on the actual data of a European DAX-option and its underlying stock, where DAX stands for the Deutscher Aktien Index, or the German stock index. The data are delivered in Excel format and can be read in Java using the Java Excel API. The option is a European call option with a strike price of €4828.87. It expires on 3 July 2015 and was written on 21 July 2005, when it was at-the-money. This means that at the time the option was written, the underlying stock was worth the strike price. The data used to delta hedge are the value of this option reaching from 21 July 2005 until 9 May 2012, and the value of its underlying stock reaching from 4 February 2005 until 9 May 2012. In reality, trading can only take place on trading days, which means there are no values available during the weekends.

In the original copy of the data acquired, only the strike price, the price of the option and the corresponding stock price at certain dates are given. Since the Black-Scholes model has to be applied, the risk-free interest rate, the time-to-maturity and the volatility have to be set. For the experiments in this chapter, the risk-free interest rate is set at 3% per annum. The time to maturity is calculated in Microsoft Excel by subtracting the expiration date and the date on which rebalancing takes place, and dividing the result by 365. In the same way, $dt$, which is used to compound with the risk-free interest rate, is calculated in Microsoft Excel by subtracting two following time steps and dividing the result by 365.

Section 4.3 described two ways of estimating the volatility: the implied volatility and the historical volatility. The delta hedges in this chapter are performed using both types.

8.1 Delta hedging with implied volatility

As explained in section 4.3, the implied volatility is calculated using the price of an actively traded option on the same underlying asset. This means that, every time step, the volatility that makes the option price calculated with the Black-Scholes equation equal to the option price acquired from the data at that time, has to be calculated. In the implementation, this is done with the Newton-Raphson root-finding algorithm, with a marge of €0.001. With the risk-free interest rate set at 3% and the portfolio rebalanced on every day the price of the DAX-option is given, the delta hedge shown in figure 8.1 is obtained. Even though the expiration time is long but reached for this option contract, it is good to see how the delta hedge performs on the dates given.

Figure 8.1 shows that until the beginning of the year 2010, the hedge portfolio does not replicate the option price. This is probably caused by the one and only uncertain parameter of the Black-Scholes equation: the volatility. Figure 8.2 shows the evolution of the implied volatility during this delta hedge.
Figure 8.1: The evolution of the option price, the value of the hedge portfolio and the stock price during the delta hedge of the DAX-option, from 21 July 2005 until 9 May 2012. The volatility is calculated as the implied volatility and the portfolio is rebalanced on every day the price of this option is given. On 9 May 2012, the hedge portfolio differs from the option price with €68.62.

Figure 8.2: The evolution of the implied volatility during the delta hedge of the DAX-option, where the portfolio is rebalanced on every day the price of this option is given.

The first thing that should be noted is that the volatility is not constant, even though the Black-Scholes model assumes it to be. To test whether this is the cause of the hedging error, the delta hedge is repeated using a constant volatility. It appears that delta hedging with a constant volatility of 25% does extremely well. Since figure 8.2 shows that the implied volatility is more than 30% per annum for most of the time, undervaluing the option does not seem to be the reason for the difference between the hedge portfolio and the option price. More precisely, the option seems to be overvalued, which is probably the reason that the hedge portfolio is closer to the option price at later time steps. Therefore, the error shown in figure 8.1 during the first time steps of the delta hedge, appears to be the result of the fluctuation of the implied volatility. Apparently, the hedge portfolio cannot keep up with the change in the option price, when the option price is suddenly priced with a different volatility.

Figure 8.3 shows the delta hedge where the portfolio is rebalanced once every month, starting 21 July 2005. If in a month the twenty-first happens to be in a weekend, rebalancing is done on the closest trading day. The values printed are only those on the day rebalancing took place, which means that the stock price, the hedge portfolio and the option price are shown in a sampled way compared to figure 8.1. It appears that the rebalancing frequency plays only a small role when delta hedging the call option on this stock. The difference between the hedge portfolio and the
option price on 20 April 2012, where rebalancing is done almost every day, is equal to €65.40. When rebalancing is done once every month, this difference goes up to €103.19.

Figure 8.3: The evolution of the option price, the value of the hedge portfolio and the stock price during the delta hedge of the DAX-option, from 21 July 2005 until 20 April 2012. The volatility is calculated as the implied volatility and the portfolio is rebalanced once every month. On 20 April 2012, the hedge portfolio differs from the option price with €103.19.

8.2 Delta hedging with historical volatility

Figure 8.4: The evolution of the option price, the value of the hedge portfolio and the stock price during the delta hedge of an option on the DAX-stock, from 21 July 2005 until 9 May 2012. The volatility is calculated as the historical volatility, based on the stock prices of the preceding ten days. The hedge portfolio is rebalanced on every day the price of the DAX-option is given. On 9 May 2012, the hedge portfolio differs from the option price with −€532.79.

Figure 8.4 shows the delta hedge using the same data of the stock as that of figure 8.1, but now the volatility is calculated using the historical data of the stock price. The way this is done was explained in subsection 4.3.2. The data used to calculate this historical volatility are the data of the preceding ten days with stock prices given. This includes the stock price on the current date. Figure 8.5 shows the evolution of the volatility during this delta hedge. The option price
is no longer equal to that of the acquired data. In this case, the volatility fluctuates much more than it does when it is calculated as the implied volatility. The fluctuation of the volatility also causes a bigger fluctuation of delta. Hedging with digital call options showed that delta hedging with a fluctuating delta can be problematic.

Figure 8.5: The evolution of the historical volatility during the delta hedge of an option on the DAX-stock, calculated using the stock prices of the preceding ten days. The portfolio is rebalanced on every day the price of the DAX-option is given.

What would probably make the historical volatility fluctuate less, is using a much larger history. Figure 8.6 shows the delta hedge using the historical volatility based on the stock prices of the preceding 252 days with stock prices given. Again, this includes the stock price on the current date. Because, for such a large history, not enough data are available to start the hedge on 21 July 2005, the delta hedge is started on 21 July 2006. Figure 8.7 shows the evolution of the volatility during this delta hedge.

Figure 8.6: The evolution of the option price, the value of the hedge portfolio and the stock price during the delta hedge of the an option on the DAX-stock, from 21 July 2006 until 9 May 2012. The volatility is calculated as the historical volatility, based on stock prices of the preceding 252 days. The portfolio is rebalanced on every day the price of the DAX-option is given. On 9 May 2012, the hedge portfolio differs from the option price with −€701.61.

Using a larger history indeed results in a less fluctuating volatility, and, therefore, a better performing delta hedge. However, when the stock price is close to the strike price and the volatility rises, the delta hedge performs badly. It seems that, even though the volatility fluctuates less
when a larger history is used, large movements in the volatility can still result in large hedging errors.

Looking back at figure 6.7, the influence of a difference in volatility is larger when the stock price is close to the strike price. This is exactly what happens during the delta hedge shown in figure 8.6. Also, comparing the historical volatility with the delta hedge with a constant volatility, the option seems to be undervalued.

Figure 8.7: The evolution of the historic volatility during the delta hedge of an option on the DAX-stock, calculated using the stock prices of the preceding 252 days. The portfolio is rebalanced on every day the price of the DAX-option is given.

Figure 8.8: The evolution of option price, the value of the hedge portfolio and the stock price during the delta hedge of the DAX-option, from 21 July 2006 until 20 April 2012. The volatility is calculated as the historical volatility, based on the stock prices of the preceding 252 days, and the portfolio is rebalanced once every month. On 20 April 2012, the hedge portfolio differs from the option price with −€647.65.

Reducing the rebalancing frequency actually makes up for a small part of the difference between the hedge portfolio and the option price that resulted from the fluctuation of the volatility, but this difference is still very large.
For as well the delta hedges that use the implied volatility as the ones that use the historical volatility, the rebalancing frequency does not play a very large role. The main problem seems to be the volatility.

8.3 Transaction costs

For the derivation of the Black-Scholes option pricing model, it is assumed that no transaction costs are involved. However, since this chapter looks at real data, this section discusses the influence of transaction costs on the Black-Scholes model. When buying or selling a stock, one pays transaction costs to the broker, who actually makes the deal.

The pseudocode in Algorithm 1 in section 5.2 showed that, every time step, the previous holdings of the stock are sold and a new delta number of the shares is bought. An important note is that this is the same as just buying or selling the difference in delta. This is important, because the first method brings more transaction costs than the second, since buying and selling both bring transaction costs.

Transaction costs can be expressed as a percentage of the stock price. Every time step, they have to be subtracted from the balance of Algorithm 1. That means that, every time step, $\tau \delta S$ has to be subtracted, where $\tau$ denotes the decimal representation of the percentage transaction costs per one share of stock, $\delta$ the number of shares traded to rebalance the portfolio, and $S$ the current stock price.

Figure 8.9 shows the total transaction costs during the delta hedge on the German stock using the implied volatility, where the portfolio is rebalanced on every day the price of the DAX-option is given and transaction costs are set at 0.1% of the stock price.

![Figure 8.9: The evolution of the total transaction costs during the delta hedge of the DAX-option. The volatility is calculated as the implied volatility and the portfolio is rebalanced on every day the price of this option is given.](image)

The amount of money lost in comparison with a delta hedge without transaction costs, is not just the total amount of transaction costs. Since the transaction costs are subtracted from balance every time step, the amount of interest that has to be paid also differs. This also contributes to a larger hedging error.

The interesting thing about what happens to the delta hedge of figure 8.1 when these transaction costs are included, is that, at the latest time steps, the value of the hedge portfolio is actually closer to the option price. This is because, at those time steps, the value of the hedge portfolio is actually higher the option price. Therefore, including transaction costs makes up for this positive error. However, when the delta hedge without transaction costs would be close to perfect or when the value of the hedge portfolio is lower than the option price, the introduction of transaction costs will only make the hedging error increase.

Wendy Günther
Transaction costs also have influence on the advantages of increasing the rebalancing frequency. Figure 8.10 shows the total transaction costs during the delta hedge, using the implied volatility, where rebalancing is done once every month. It shows that the the lower the rebalancing frequency, the lower the total transaction costs.

The total transaction costs also depend on the fluctuation of delta. When delta does not fluctuate much, the difference in delta between time steps does not result in large transaction costs. However, when delta fluctuates much, the number of shares that have to be bought or sold to rebalance the portfolio is larger, which results in more transaction costs. When the historical volatility is used, based on the stock prices of the preceding ten days, delta fluctuates a lot. The total amount of transaction costs that comes with delta hedging using this historical volatility, where the portfolio is rebalanced on every day the price of the DAX-option is given, is shown in figure 8.11.

Figure 8.11: The evolution of the total transaction costs during the delta hedge of an option on the DAX-stock. The volatility is calculated as the historical volatility and the portfolio is rebalanced on every day the price of the DAX-option is given.
Trading options can be riskful. Option pricing models have been invented to be able to control this risk. The derivation of the Black-Scholes option pricing model is based on a number of assumptions that do not always hold in the real market.

In theory, the Black-Scholes model should work perfectly for European options, if one would be able to rebalance a portfolio of derivatives continuously and if the expected volatility would be a constant equal to the volatility of the stock price. The experiments for European options show that, even though rebalancing cannot be done continuously in reality, the risk that comes with trading European call options can still be controlled. Rebalancing as frequent as every day would still prevent large hedging errors from occurring and does well on average. However, constantly hedging with a wrong expected volatility can result in large hedging errors. From the trader’s perspective, overvaluing an option can be profitable, but undervaluing an option can cause a loss.

The experiments for digital options show that the Black-Scholes model does not work that well for this kind of options. Even though delta hedging a digital option can work well for some simulations of the underlying stock price, large hedging errors can occur because of the nature of Black Scholes’ delta. Smoothing delta may reduce the most extreme hedging errors, but it may increase hedging errors in other situations. When the expected volatility does not match the volatility of the stock, unmanageable situations and hedging errors result. In the case of digital options, a static hedge may work, using Black Scholes for pricing the European call spread. However, the market makes the setup of such a spread to replicate digital call options impossible. Therefore, digital options should be considered and treated as normal bets.

The experiments with the DAX-option and its underlying stock showed that the accuracy of the Black-Scholes model is highly influenced by the volatility. In reality, estimates of the volatility are not constant, even though the model assumes it to be. The best way to calculate the volatility turns out to be the implied volatility, since this kind of volatility shows no sudden large movements. The implied volatility is on average higher than the historical volatility.

In the real market, the assumptions underlying the Black-Scholes model do not hold. However, the only truly uncertain parameter in the Black-Scholes equation is the volatility. When the right volatility is used, the Black-Scholes model works relatively well for European call options, even when continuous rebalancing is not possible. The risk involved with option trading is still reduced when using the Black-Scholes option pricing model with delta hedging.
Bibliography

The equation of the current price of a European call option is
\[ c = e^{-r(T)}E[\max(S_T - K, 0)]. \] (A.1)

The stock price is assumed to be lognormally distributed. Therefore, the expectation of the payoff can be written as
\[ \int_{-\infty}^{\infty} (S_T - K) g(S) dS. \] (A.2)

When the stock price is lognormally distributed, it means that the logarithm of the stock price is normally distributed. The properties of the lognormal distribution [6] state that
\[ E[S] = e^{m + \frac{1}{2}s^2}. \]

In this equation, \( m \) denotes the mean, which equals
\[ m = \ln(E[S]) - \frac{1}{2}s^2. \]

It is known that when a variable \( Z \) is normally distributed, then \( \frac{Z - m}{s} \) is standard normally distributed. Therefore, in the following equations, \( Q \) follows a standard normal distribution with
\[ h(Q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}}: \]
\[ Q = \frac{\ln(S) - m}{s}, \quad S = e^{Qs + m}. \] (A.3) (A.4)

Substituting \( S \) of equation (A.4) into equation (A.2) gives
\[ \int_{\ln K - m}^{\infty} (e^{Qs + m} - K) h(Q) dQ, \]
which is equal to
\[ \int_{\ln K - m}^{\infty} e^{Qs + m} h(Q) dQ - K \int_{\ln K - m}^{\infty} h(Q) dQ. \] (A.5)
The first integral of equation A.5 can be transformed into
\[
e^{Q_s + m}h(Q) = \\
\frac{1}{\sqrt{2\pi}} \times e^{-\frac{Q^2}{2}} \times e^{Q_s + m} = \\
\frac{1}{\sqrt{2\pi}} \times e^{-\frac{Q^2 + Q_s + m}{2}} = \\
\frac{1}{\sqrt{2\pi}} \times e^{-\frac{(Q - s)^2 + 2m + s^2}{2}} = \\
\frac{1}{\sqrt{2\pi}} \times e^{m + \left(\frac{s^2}{2}\right)} e^{-\frac{(Q - s)^2}{2}} = \\
e^{m + \left(\frac{s^2}{2}\right)} \times h(Q - s).
\]

The equation then becomes
\[
e^{m + \left(\frac{s^2}{2}\right)} \int_{\ln K - m}^{\infty} h(Q - s) \, dQ - K \int_{\ln K - m}^{\infty} h(Q) \, dQ. \tag{A.6}
\]

The cumulative normal function, \(N(x)\), shown in figure A.1, defines the chance that a variable with a standard normal distribution is smaller than \(x\). This makes the first integral of equation A.6 equal to
\[
1 - N\left[\ln K - m \over s\right] - s = N\left[\left(-\ln K + m\right) \over s\right] + s = \\
N\left[\left(-\ln K + \ln[E(V)]\right) \over s\right] - s = \\
N\left[\left(-\ln K + \ln[E(V)]\right) \over s\right] + \frac{s}{2} = \\
N\left[\left(-\ln K + \ln[E(V)]\right) + \frac{s^2}{2}\right],
\]
which is equal to \(N(d_1)\), where
\[
d_1 = \frac{\ln[E(S)/K] + \frac{s^2}{2}}{s}.
\]
The second integral then equals
\[ 1 - N\left(\frac{\ln K - m}{s}\right) = N\left(\frac{-\ln K + m}{s}\right) = \right] 
\[ -N\left[\ln K + (\ln[E(S)] - \frac{s^2}{2})\right], \]
which is equal to \( N(d_2) \), where
\[ d_2 = \frac{\ln[E(S)/K] - \frac{s^2}{2}}{s}. \]

Now the equation of the expectation of the payoff has become
\[ E[\max(S - K, 0)] = e^{m + \frac{s^2}{2}} N(d_1) - K N(d_2) = \]
\[ e^{\ln[E(S)] - \frac{s^2}{2} + \frac{s^2}{2}} N(d_1) - K N(d_2) = \]
\[ E(S)N(d_1) - KN(d_2). \]

Since it is known that \( E(S) = Se^{rT} \) (equation 4.6) and that \( s = \sigma \sqrt{T} \) (equation 4.21), the Black-Scholes equation for European call options, derived from equation A.1, is equal to
\[ c = SN(d_1) - Ke^{-rT} N(d_2), \]
where
\[ d_1 = \frac{\ln[S/K] + (r + \frac{s^2}{2})T}{\sigma \sqrt{T}} \]
and
\[ d_2 = \frac{\ln[S/K] + (r - \frac{s^2}{2})T}{\sigma \sqrt{T}}. \]
Black-Scholes delta

The Black-Scholes equation for calculating the price of a European call option is known as

\[ c = SN(d_1) - Ke^{-rT}N(d_2). \]

In this equation, \( N(x) \) is the cumulative normal function, of which the derivative is equal to \( P(x) \), the probability density function, which is equal to

\[ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \]

The derivative of the call price with respect to the price of the underlying asset is now defined as

\[ \frac{\delta c}{\delta S} = S'N(d_1) + S[N(d_1)]' - Ke^{-rT}[N(d_2)]' + [Ke^{-rT}]'N(d_2) = N(d_1) + S[N(d_1)]' - Ke^{-rT}[N(d_2)]'. \]

The derivative of \( N(d_1) \) with respect to the price of the underlying asset, notated as \([N(d_1)]'\) is equal to

\[ P(d_1) \frac{1}{S} \frac{1}{\sigma \sqrt{T}}. \]

The derivative of \( N(d_2) \) with respect to the price of the underlying asset is in the same way equal to

\[ P(d_2) \frac{1}{S} \frac{1}{\sigma \sqrt{T}}. \]

This makes the entire equation for the delta of European call options equal to

\[ \frac{\delta c}{\delta S} = N(d_1) + S \left( P(d_1) \frac{1}{S} \frac{1}{\sigma \sqrt{T}} \right) - Ke^{-rT} \left( P(d_2) \frac{1}{S} \frac{1}{\sigma \sqrt{T}} \right). \]
The right part of the equation can be simplified, knowing that $d_2 = d_1 - \sigma \sqrt{T}$:

$$Ke^{-rT}P(d_2) = Ke^{-rT}P(d_1 - \sigma \sqrt{T}) =$$

$$Ke^{-rT} \frac{1}{2\pi} e^{-\frac{(d_1 - \sigma \sqrt{T})^2}{2}} =$$

$$Ke^{-rT} \frac{1}{2\pi} e^{-\frac{(d_1^2 - 2d_1 \sigma \sqrt{T} \sigma^2 T)}{2}} =$$

$$Ke^{-rT} P(d_1)e^{\left(\frac{(d_1^2 - 2d_1 \sigma \sqrt{T} \sigma^2 T)}{2}\right)} =$$

$$Ke^{-rT} P(d_1)e^{(d_1 \sigma \sqrt{T} - \frac{\sigma^2 T}{2})} =$$

$$Ke^{-rT} P(d_1)e^{\left(\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2 \sigma^2}{2}\right)T\right)} =$$

$$Ke^{-rT} P(d_1)e^{\left(\ln(S/K) + rT - \frac{\sigma^2 T}{2}\right)} =$$

$$Ke^{-rT} P(d_1)e^{\left(\ln(S/K) + rT\right)} =$$

$$KP(d_1) \frac{S}{K} =$$

$$SP(d_1).$$

Substituting this in the previous equation gives

$$\frac{\delta C}{\delta S} = N(d_1) + SP(d_1) \frac{1}{S} \frac{1}{\sigma \sqrt{T}} - SP(d_1) \frac{1}{S} \frac{1}{\sigma \sqrt{T}} = N(d_1).$$