Copulas and Correlation in Credit Risk

“Who will pay the difference?”

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Abstract
This paper is an extensive examination of the Gaussian copula for default correlation. We follow mathematical framework of Li for the Gaussian copula model. Three different aspects of the model are considered: The asset value model underlying the Li’s copula model. The accuracy of the assumptions underlying the Gaussian copula model. Lastly, the marginal transformations performed to rescale the marginal distribution to default distributions. We examine the how the simplifications of the model affect the default correlation estimates. We find that assuming that asset values and their correlations drive the dependence structure is insufficient and mitigates the dependence structure of the model since other risk factors are disregarded. Further, we argue that simultaneous defaults are extreme events and therefore a model must to be able to render strong dependence among simultaneous extremes. This is not the case with the Gaussian copula model as shown by the examination of the tail dependence coefficient. We contemplate the Student $t$-copula as an improvement since its tail dependence coefficients are dependent on its parameters $\rho$ and $\nu$ and are non-zero for $\rho > -1$ and $\nu < \infty$. Lastly, we discuss the impact of the marginal transformations on the dependence structure. We reveal that the dependence in the multivariate meta-distribution is positively dependent on the difference between the heavy-tailedness of the original and the meta-marginals. As a result, the Gaussian copula model extremely mitigates the dependence in the meta-distribution. Moreover, the default correlations seem static for variable credit quality. In consequence, we argue that the Gaussian copula model is effectively a one-parameter model. Again, the Student $t$-copula proves to be a more advantageous choice than the Gaussian copula. Finally, simulations of CDO tranche spreads confirm the results of our analysis and disclose the lower tranche spreads are more expensive and upper tranches are cheaper according to the Gaussian copula model than as estimated by the Student $t$-copula model.

Key words: Default correlation, correlation, dependence structure, copula, One-factor Gaussian model, $t$-copula, default times, CDS pricing, CDO pricing, synthetic CDO, hazard rate, marginal transformations, meta-distributions.

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To my mother

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The foundation of many principles in finance is the relationship between risk and return. The fundamental idea surrounding these principles states the greater the amount of risk, the greater the potential return. The dictionary definition of the word risk is “the possibility of incurring misfortune or loss;” this definition associates risk with chance or probability. In this view, it is intuitive that we try to explain risk through randomness. Andrey Kolmogorov\(^1\) established a modern axiomatic foundation for randomness in 1933 and ever since, these axioms have been one of the foundational pillars for the study of randomness in the field of stochastics\(^2\) and probability theory. The field of risk management is an applied subject of stochastics and probability theory. Therein many phenomena regarding risk are explain by the theory of probability and stochastics.

There are many types of risks in finance, the main two being market risk and credit risk. Market risk is the risk that a portfolio will decrease in value due to fluctuations in the underlying assets’ prices. Credit risk is risk due to credit events such as delay in repayments, debt restructuring, and default. This risk is mostly affiliated with credit and credit derivatives because prices are based on the credit risk involved. Examples of credit derivatives are basis-swaps, interest-rate swaps, first-to-default contracts, credit spread option, credit default swaps (CDSs) and synthetic collateralised debt obligations (CDOs). To give an idea of the size of the credit derivative market, the global CDS market alone grew in notional value from $1 to $62 trillion U.S. dollars from 2000 to 2008\(^3\). If we compare the latter value to the world’s Gross Domestic Product (GDP) equaling $60.6 trillion U.S. dollars in 2008\(^4\), we find that the value of the CDS market was larger. Products such as synthetic CDOs have created an enormous demand for CDSs, since 1987\(^5\). Figure 1.4 displays the growth of CDOs from 1988 to 2005.\(^6\) Before 2007, these relatively new products provided seemingly effortless and extremely secure means to assume the exact amount of credit risk desired. Since CDSs are standardized contracts and were extremely liquid\(^7\), many almost identical contracts could be securitized\(^8\) into one portfolio. This feature is perfect for a structure of a CDO. In Section 1.3, we explain the structures of different kinds of CDOs and the knowledge obtained there further clarifies the immense size of the CDS market.

Grasping the true nature of some credit derivatives and the risk involved fosters both theoretical and practical challenges. Therefore, bankers allocate large amounts of time and money in developing risk management strategies to help manage risks associated with their investment dealings in credit derivatives. A key component of the risk management process is the determination of the credit risks surrounding businesses or investments. Before 2000, statisticians, or “quants,” were monitored advisors for busy investment bankers, hired to solve these problematic enigmas about credit risk, for example assessing probabilities of default from real-world data or lay a sound scientific foundation for their decisions and strategies to estimate what return they should demand on certain risky investments. Simple models, such as Value-at-Risk (VaR) or Credit VaR, were mainly used to access these risks, though these models could not cope with complex structured derivatives such as CDOs. The number of obligors directly connected to a single CDO are so great that Counterparty risk becomes a high-dimensional problem. Quantifying the entire credit risk exposure of a CDO includes evaluating the interdependence of all possible credit events in the reference portfolio of a CDO. This dependence is mainly referred to as default dependence. Default dependence has been a topic of great

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\(^1\)See Kolmogorov\(^5\); he first published his work in German and thus the axioms are also known as Kolmogorov's Grundbegriffe.

\(^2\)The word “stochastic” originates from the Greek word “stokhastikos” which means “capable of guessing” and from “stokhazesthai” which translates as “to aim at, conjecture.”

\(^3\)In 1987, the first CDO was issued by bankers at Drexel Burnham Lambert Inc.

\(^4\)In Figure 1.4 the value of CDO issuance in 2006 is an estimation.

\(^5\)For example in late 2007, the monthly trading volume in CDS contracts of 18 global banks was on average 23 000 deals in July, more than 25 000 in August and 20 000 in September;\(^7\).

\(^6\)Securitization is discussed in Section 1.1.
interest within the banking and investment community, and as a result of the growing linkages in the financial markets, it is crucial to pricing credit derivatives of the new century. Default events of one obligor are non-predictable and thus simultaneous default events of many obligors can be viewed as random events. Therefore, stochastic dependence may provide methods to model default dependence in finance. We will learn that stochastic dependence is a complex mathematical concept and as a result sophisticated techniques and models are needed to represent default dependence.

In 1997, the RiskMetrics Group, a subsidiary at J.P. Morgan, developed a model, named CreditMetrics. It asserts the credit risk for individual instruments in a portfolio and takes credit migrations and correlations between credit events into account in order to model the overall credit risk needed to price convoluted products like CDOs \(^5\). One particular aspect of this model is default correlation which is meant to solve the dependence dilemma of the new credit derivatives. Since default is a rare event, not enough data is available to make any statistically sufficient conclusions of any kind. CreditMetrics uses asset values and their correlation to model default dependence through a joint Gaussian distribution. The theory of copulas and particularly the work of Abe Sklar inspired the Chinese mathematician David Xianglin Li to establish a statistical foundation for the distinguished feature of default dependence in the CreditMetrics model. The framework of this copula model, parameterized by a single correlation coefficient, presented a way to describe some of these convoluted default risk scenarios. Li was one of the forebearers in credit derivatives while working as the CDO Manager at the RiskMetrics Group. He has also worked at Barclays Capital and Citigroup where he managed the quantitative analytics team and advanced both companies into main players in the credit derivative business \(^15\). In an article entitled “On Default Correlation: A Copula Function Approach,” Li published the mathematical reasoning behind his copula model. The success of Li’s reasoning lead to the fast adoption of the CreditMetric model for pricing complex credit derivatives, such as CDSs and CDOs. Many mathematicians, including Li himself, argued that Li’s mathematical reasoning behind the CreditMetrics model was based on some substantial scenario assumptions. About his model, Li stated: “The most dangerous part is when people believe everything coming out of it;” \(^61\).

To be critical towards the results of a model, implies understanding it. In this paper, the main aim is to analyze the mathematics behind Li’s model and understand the gravity of the assumptions underlying it. Only then can we conclude why it was criticized by so many mathematicians and what undermined its usefulness so disastrously during the last couple of years. By familiarizing ourselves with Li’s model, we have to understand theories in two main areas, copula theory and stochastic dependence. There are three parts to the analytical examination of Li’s model. Each sheds light on a particular weakness of the model. At times, we compare Li’s Gaussian copula model to the t-copula model to illustrate the individual shortcomings.

The structuring of this paper is as follows: In the next chapter we consider specific financial products in more detail and provides insight into their pricing and evolution. Advantages and disadvantages of the products related to modeling are given and we explain the urge to quantify the dependence structure for assessing risks involved in these financial products.

In Chapter 2 we first lay a mathematical foundation for the theory of copulas. Section 2.1 serves as an introduction to copula functions and their important properties and characteristics. The main theorem is Sklar’s Theorem (Theorem 2.4) which is claimed to be central to the validity argument of Li’s model. In Section 2.2 the fundamentals of the term stochastic dependence are addressed. We establish a relation between stochastic dependence and its measurement. Two approaches are considered, correlation measures and dependence modeling through tail dependence. We discuss some crucial properties for dependence measures in order to reveal their ability to measure stochastic dependence, as this is of great significance for Li’s copula model. Three different correlation measures are introduced, Pearson’s or linear correlation and two rank correlation measures, Spearman’s rho and Kendall’s tau. Second, we define tail dependence which is distinct from correlation; it measures simultaneous extreme events. As default is an extreme event, this measure will prove to be valuable in examining default dependence.

In Chapter 3 we provide a critical view on Li’s model and analyze three crucial points. First, we discuss the underlying information for the correlation parameter as input for the joint distribution and the correlation measure employed for these calculations. We argue that this procedure is poor and insufficient to create a dependence structure for defaults. Secondly, we focus on the statistical tools to estimate the correlation parameter. In a Gaussian scenario, we find that linear correlation is an excellent tool to create a dependence structure. Li applies Sklar’s theorem with the assumption that one particular copula resembles the joint distribution, namely the Gaussian copula. Using the

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5Here, we refer to Sklar’s Theorem. This theorem is a key argument in this paper and is discussed in Section 2.1.
tail dependence coefficients, we demonstrate the main shortcomming of the Gaussian copula for modeling dependent defaults. To compare our results, we adopt the $t$-copula for the joint distribution and determine that heavy-tailed distributions pose a safer assumption. Lastly, we discuss the transformation of the marginals. This has an imminent effect on the overall dependence structure. Through an examination of the transformations with two distributions which differ largely in heavy-tailedness, we analyze the influence of the transformations on the dependence structure. An important effect is uncovered: the tail behavior of the marginals after the transformations corresponding to the heavy-tailedness of the copula may enhance or abate the dependence in the distribution. Moreover, we contemplate historical default probabilities from credit rating agencies (CRA) Moody’s and Standard & Poor’s and implied default distributions modeled from CDS data, to implement our analysis into the Gaussian copula model and examine the impact on the replication of a default dependence. We categorize default correlation in different credit quality groups using the CRAs’ rating systems and discover that the Gaussian copula seems insensitive to credit quality. Furthermore, we use a different copula model, the $t$-copula model, for comparative reasons. The $t$-copula model generally shows a negative relation between one entity’s credit quality and its default correlation with another entity. Although some inconsistencies in this relation are noticeable: the default correlation between a low-rated entity and high-rated entities (investment grade) seems to increase as the credit rating for the investment grade entity increases, thus showing a positive relation. We find that there is evidence that the credit quality of CRAs’ high-rated entities, i.e. investment grade entities, is inappropriately low in comparison with lower ratings and no distinctness of credit quality between these ratings can be found. This could explain the inconsistencies in the relation between one entity’s credit rating and its default correlation with a low-rated entity when using the $t$-copula model.

In conclusion, we summarize the result of our analysis in Section 4.1 and in Section 4.2, we model CDO tranche spreads with real-world market data and link these values to the results obtained in our previous analysis of the Gaussian copula model. Additionally, we model tranche spreads with the $t$-copula model to amplify the link between tranche spreads and dependence structure between default times.

Please note that throughout this paper definitions and proofs will be ended by $\perp$ and $\square$, respectively.
Chapter 1

Financial Products

To comprehend credit risk modeling, it is important to fully understand the structure of the products. In this section, we summarize the basics of collateralised debt obligations and credit default swaps. In doing this, we first explain the essence of securitization. Securitization is a financial process that underlies many financial products, such as CDOs, first-to-default contracts and other credit derivatives. Secondly, we briefly consider the structure of and a particular pricing model for a credit default swap (CDS). Further, we lay out the main principles of a CDO and provide insight on two different kinds of CDOs, ‘cash’ and ‘synthetic.’ Moreover, we explain the primary advantages and disadvantages of a CDO. Lastly, we summarize the connection between CDOs and default dependence.

1.1 Securitization

Securitization began in the 1970’s with structured pooling of mortgages. The U.S. Government created the Government National Mortgage Association (GNMA or Ginnie Mae) to sell securities backed by a pool of mortgages. Thereafter, the process of pooling financial assets into one portfolio was known as securitization. This process is a crucial part of the structure of CDOs. A reference portfolio of a CDO is a securitized pool of assets. In cases where these assets are loans, CDO belongs to the class of asset-backed securities (ABS). Later, we will learn about a different CDO structure which does not belong to the group of ABS.

Loosely speaking, one of the advantages of securitization is the reduction of credit risk. That is, one default represents a relatively small loss of the entire portfolio. This effect is conditional on the dependence of obligors in the reference pool. If for instance all obligors are perfectly positive dependent, one default results in the default of the entire reference pool. Hence, if poorly constructed, credit deterioration and immense losses in the reference portfolio are inevitable. Thus, the construction of a securitized pool of assets is crucial to the performance of the securities backed by these assets. In Section 1.3.3 we analyze this aspect in regard to collateralized debt obligations. More about securitization can be read in Lederman [35].

1.2 Credit Default Swap

A credit default swap is a derivative that provides insurance against default.

![Figure 1.1: Payment structure of a CDS.](image)

1Ginnie Mae should not be confused with Fannie Mae or Freddie Mac, which are government sponsored enterprises whereas Ginnie Mae is entirely owned by the U.S. government.
It is a contract between two parties, where the default risk of an underlying entity is transferred from one party to another. If the underlying party defaults, the protection seller is obligated to buy the protection buyer’s references for their principal value \( P \). The protection seller’s loss is called loss-given-default. This value is \((1 - R) \bullet P\), where \( R \) is the recovery rate of the reference entity. The protection buyer pays a periodic fee until maturity or occurrence of default, whichever occurs first. In Figure 1.1, we observe a diagram of a CDS structure. More about credit default swaps can be read in Hull [29], Chapter 21.

### 1.2.1 CDS Pricing

Here, we briefly lay out a risk neutral approach to pricing a CDS contract with maturity \( M \). Our pricing model cannot only be used to calculate the spread of the CDS but also to calibrate implied default probabilities necessary for Li’s model [4]. The main assumptions for this pricing model are the following:

1. The interest rate process incorporated in the risk-free discount factor \( DF \) and the survival process \( S \), which describes the probability of default are independent of each other. Moreover, default payments are settled immediately upon default.
2. There are two parts to the pricing model, one is the premium leg and the other is the protection leg. Suppose \( T \) is the random variable that describes the time of default of the reference entity. Let \( S(t) := 1 - F(t) = P(T > t) \) be the survival function, that is the probability that the reference entity defaults after time \( t \), then the premium leg is defined as follows,

\[
\text{PremLeg} = \sum_{i=0}^{K} E \left( \int_{t_{i-1}}^{t_{i}} s \gamma DF(t) S(t) dF(t) \right),
\]

where \( s \) is the spread, \( DF(t) \) is the risk-free discount factor for time \( t \) and \( K \) the number of payments during the life of the contract. Thus, \( 0 = t_0, \ldots, t_K = M \) is a partition of the time period from time zero until the maturity date \( M \). The factor \( DF(t) \) discounts the value of each future payment into its present value. The factor \( \gamma \) is called the day count and depends on how many payments are made per year [6]. The protection leg is defined as,

\[
\text{ProtLeg} = (1 - R) E \left( \int_{0}^{M} DF(t) S(t) dF(t) \right),
\]

where \( DF(t) \) is the risk-free discount factor and \( R \) the recovery rate. Likewise, we discount the loss-given-default value to its present value. The spread \( s \) can now be determined by setting (1.1) and (1.2) equal to each other. Hence,

\[
s = \frac{(1 - R) E \left( \int_{0}^{M} DF(t) S(t) dF(t) \right)}{\sum_{i=0}^{K} E \left( \int_{t_{i-1}}^{t_{i}} \gamma DF(t) S(t) dF(t) \right)}. \tag{1.3}
\]

The two legs do not contain the principal value \( P \), because the spread \( s \) is given as a percentage of \( P \). More about the risk neutral approach to pricing a CDS can be found in O’Kane and Turnbull [50].

### 1.3 Collateralised Debt Obligation

In general, a CDO is a structured security constructed on a reference portfolio of assets of credit derivatives. This portfolio may consist of different products (e.g. bonds, mortgages and credit default swaps) creating income in the form of periodic payments. Investors may buy into a CDO through tranches which is a special feature slicing the gains (and losses) generated by the reference portfolio. The gains are distributed by means of coupon payments according to seniority. One may think

---

2 The spread of a CDS is the equivalent to the periodic payments in Figure 1.1 expressed in basis points (bps) of the principal value \( P \). One basis point is \( \frac{1}{100} \) of a percent.
3 This procedure is laid out in Appendix B.1.
4 This process is laid out in Section 3.1.
5 We call this random variable Time-until-Default. A proper definition is given in Section 3.1.
6 If payments are made annually then \( \gamma = 1 \), semi-annually then \( \gamma = \frac{1}{2} \), quarterly then \( \gamma = \frac{1}{4} \), etc.
7 'Tranche' is the French word for slice.
8 Coupon payments are periodic payments at a fixed interest-rate received at previously stated times in the future. For instance an annual 5% coupon over a principal of $1000 U.S. dollars, pays $50 U.S. dollars once annually.
of a CDO as a fountain of cash flows running down several cascading trays; the top tray fills first, successively spilling into the lower trays. In a CDO contract, the tranche’s attachment and detachment points specify when a particular tranche is affected by the losses. The most junior tranche, or equity tranche, receives the residual cash flows and thus incurs the first losses. Basically, CDOs transfer the credit risk of a reference portfolio of (financial) assets. Investors in the most senior tranche are exposed to the least risk, whereas holders of the equity tranche are exposed to the most. In turn, riskier tranches are rewarded with higher coupon payments, thus yielding a better rate of return. This indicates that investing in more senior tranches need not always be the better investment strategy. More about credit risk allocation through CDOs can be read in Krahnen and Wilde [34].

Figure 1.2: Payment structure of a ‘cash’ CDO.

Figure 1.2 illustrates an example structure of a ‘cash’ CDO. Here, we observe four different tranches and a reference portfolio consisting of bonds. In the middle, the abbreviation SPV signifies Special Purpose Vehicle. Legally, an SPV is the creator of a CDO and holder of the reference portfolio. This means that the SPV essentially warehouses all credit risk and is highly leveraged by the underlying investment bank, in case it is required by the reference portfolio. An SPV might request funding, which means that investors have to deposit collateral with the SPV for potential loss-payments. More information about funded and unfunded CDO structures can be found in Meissner [45], page 20. Moreover, the SPV primarily specifies the attachment and detachment points in order for CRAs to award the most senior tranche a AAA-rating, whereas other tranches attain lower ratings. The equity tranche usually does not receive a rating.

In Figure 1.3, the structure of a ‘synthetic’ CDO is displayed. The difference between a cash and synthetic CDO is the reference portfolio. A cash CDO’s reference pool consists of bonds whereas a synthetic CDO’s consists of short positions in credit default swaps. In a synthetic CDO, loss-given-default values represent the losses of the CDO, and the incoming cash flows are the CDS spreads. One effect of a synthetic CDO is that the cash flows represented by the horizontal arrows in Figure 1.2 turn into CDS-like transactions. On the left, the SPV sells many swaps receiving periodic payments, and on the right, if all CDS have the same principals, the structure can be interpreted as many $k$th-to-default contracts, where $k$ varies according to the attachment and detachment points of each tranche. A synthetic CDO is always partially funded. These funds are invested by the SPV in risk-free assets (e.g. T-bills, money market), thus creating additional cash flows. A synthetic CDO is generally easier to manage for the SPV manager, since it does not have to raise any initial capital and does not legally own the assets in the reference portfolio. Further analysis of synthetic CDOs can be found in Gibson [?]. The full evolution of CDOs and many historical facts can be reviewed in Gibson [?] and Meissner [45].

In Figure 1.4 the growth of CDOs over a period of 18 years is illustrated. The quantity of 2006 is a growth estimate on the previous years, since these numbers were published in 2005. This graph shows the popularity of CDOs before the mortgage crisis of 2007-2008 in which CDOs suffered great losses. The notional value of outstanding CDOs in 2008 was estimated to be $1.2 trillion U.S. dollars, thereby decreasing about $700 billion U.S. dollars (Unmack [54]).
The real impact of the mortgage crisis can be seen in the issuance numbers. The yearly issuance number of CDOs peaked in 2006 when it reached $251 billion U.S. dollars. After the crisis, the issuance of CDOs fell to a modest value of $4 billion U.S. dollars in 2009, estimated by SIFMA [25]. More about CDOs and exotic CDO structures can be found in Adelson [1] and Lucas, et al. [41].

1.3.1 Advantages and Disadvantages of a CDO

Before the mortgage crisis, there seemed to be three reasons why CDOs were extremely attractive products to both creators and investors. The first reason was diversification: A basic financial principal is that a well diversified portfolio has minimum exposure to idiosyncratic risk and this benefits the yield-risk ratio. This is still true today, although as we have already seen in Section 1.1, the reduction of risk through securitization and thus diversification is subject to the dependence structure of the reference portfolio. Hence, the diversification argument is only advantageous if the reference portfolio is constructed correctly. Secondly, overcollaterisation is typically a characteristic of a CDO. This occurs because the reference portfolio usually has a larger value than the SPV’s liabilities to the tranches. Before the mortgage crisis, unsuspicious investors viewed this fact as an additional element of protection. Skeptics may be of the opinion that there is a reason for the SPV to create this layer of protection. Usually, (investment) bankers are not fond of overcollaterisation; "That is money making
Lastly, CDOs apply the principle of subordination; i.e. the lower tranches function as a risk buffer for the upper tranches. This feature is specific to CDOs and is the main advantage of the product. It gives investors the ability to manage their credit risk exposure and amplify yields without incurring high costs. However, given certain circumstances, CDOs are products that are greatly leveraged and may turn out to be a ticking time bomb of immense losses which may affect not only investors. A general view on advantages of business in CDOs can be read in Bluhm [5].

As history has shown, a CDO’s advantages may turn into disadvantages in an economic state of distress. Sudden losses may be lurking behind a change in the macroeconomic stability. Here are some examples of unexpected losses regarding CDO investments. The first example discusses a specific hedging situation and the relationship between tranche spreads and default correlation of the assets in the reference portfolio of the CDO. Intuitively, spreads and default correlation should be positively correlated. Although this is true for the upper tranches of a CDO, it is false for the equity tranche under the Gaussian copula model. High default correlation overstates the probability of many defaults (high losses) but also of extremely few defaults, or low losses. On the one hand, extremely low losses have no effect on the spreads of the upper tranches since these losses are absorbed by the residual tranche, however, increased probability of high losses has an increasing effect on their spreads. On the other hand, increased probability of extremely low losses decreases the spread of the equity tranche and high losses above the equity tranche’s detachment point have no influence on its spread. Figure 1.5 is adopted from Meissner [45] and displays the relationship between tranche spreads and default correlation.

Figure 1.5: Relationship between tranche spreads and default correlation in the Gaussian copula model. Source: Meissner [45].

Many hedge funds enforced strategy to sell protection on equity tranches and hold a long position in mezzanine tranches to hedge the position in the equity tranche. This created an enormous positive carry since the spread of the equity tranche is higher than that of the mezzanine tranche. In 2005, the General Motors and Ford bonds, part of several CDO reference portfolios, were downgraded to junk bonds [10]. As a consequence, the reference portfolio of a CDO containing GM and Ford bonds became more heterogeneous. This resulted in decreased default correlations and the spread of the equity tranches increased and the spread of the mezzanine tranches decreased. The shift in spreads resulted in a loss in both positions. One can read more about this after-effect in Meissner [45], chapter 3. The following example addresses the leverage position SPVs take in constructing CDOs. In the mortgage crisis of 2007-2008, sub-prime mortgages were excessively issued and it was believed that the increase in real-estate value could secure mortgage payments. As mortgages were a popular reference product for CDOs and real-estate value started to stall in 2006/2007, interest rates of adjustable-rate mortgages increased significantly. Subsequently, many sub-prime borrowers could not service their loans. Huge losses lead to a near stop in CDO trading and many SPVs defaulted as highly leveraged

[10] Junk bonds are bonds rated BB or lower because of their high default risk.
positions had to be taken in order to construct CDOs. Another severe disadvantage of CDOs lies in pricing tranche spreads. We will learn that Li’s copula model is far from adequate to price complex structured products such as CDOs. A risk neutral pricing approach is laid out in the next section. Many other attempts have been undertaken. An examination of the most popular approaches to CDO pricing can be found in Meissner [45]. More examples and disadvantages surrounding CDOs can be read in Lucas, et al. [41] and Meissner [45], chapters 16 and 23.

1.3.2 CDO Pricing

For the pricing of a CDO tranche spread, let us assume that there are \( N \) entities in the reference portfolio. The total loss function \( L(t) \) of the reference portfolio at time \( t \) is defined by,

\[
L(t) := \sum_{i=1}^{N} (1 - R_i) P_i 1_{T_i < t}, \tag{1.4}
\]

where \( R_i \) is the recovery rate, \( P_i \) the principal and \( T_i \) the time-until-default of reference entity \( i \). Let the attachment and detachment points of a given tranche be represented by \( \Gamma \) and \( \Delta \), respectively, hence the cumulative loss \( L^{\Gamma, \Delta}(t) \) on a given tranche is defined by,

\[
L^{\Gamma, \Delta}(t) = \begin{cases} 
0, & \text{if } L(t) < \Gamma \\
L(t) - \Gamma, & \text{if } \Gamma \leq L(t) < \Delta \\
\Delta - \Gamma, & \text{if } L(t) \geq \Delta.
\end{cases}
\]

As we outlined for the CDS pricing model, we have two legs for the CDO tranche pricing model, the premium leg and the default leg. The premium leg is defined as the present value of the premium payments weighted by the outstanding capital at each payment date, that is,

\[
\text{PremLeg} = E \left( \sum_{i=1}^{K} \int_{t_{i-1}}^{t_i} s^{\Gamma, \Delta} \gamma DF(t) \min\{\max\{\Delta - L(t), 0\}, \Delta - \Gamma\} \ dt \right), \tag{1.5}
\]

where \( K \) are the amount of payments until maturity date \( M \), \( DF(t) \) is the risk-free discount factor, \( \gamma \) is a day count factor and \( s^{\Gamma, \Delta} \) is the spread of the given tranche. In addition, we can express the default leg as the expected value of the default payments, discounted from the time of default, that is,

\[
\text{DefLeg} = E \left( \int_{0}^{M} DF(t) dL^{\Gamma, \Delta}(t) \right). \tag{1.6}
\]

If we set (1.5) and (1.6) equal to each other, we can calculate the tranche spread as follows,

\[
s^{\Gamma, \Delta} = \frac{E \left( \int_{0}^{M} DF(t) dL^{\Gamma, \Delta}(t) \right)}{E \left( \sum_{i=1}^{K} \int_{t_{i-1}}^{t_i} \gamma DF(t) \min\{\max\{\Delta - L(t), 0\}, \Delta - \Gamma\} \ dt \right)}. \tag{1.7}
\]

More about CDO pricing can be read in Mortensen [48].

1.3.3 CDOs and Default Dependence

In a CDO, the losses of a tranche’s investors are directly dependent on the number of default occurrences in the reference portfolio and the attachment and detachment point of each tranche. The higher the attachment point of the tranche, the lower the effect of individual defaults on the losses of that tranche. This makes default dependence relevant for more senior tranches. Moreover, as we have observed in the example above, the lower the attachment point, the higher the effect of individual defaults. This makes default dependence significant for more junior tranches. In other words, the tranche’s price is reflected by the investors’ expectation of dependence in defaults of the underlying securities throughout the life of the CDO. This implies that in case of CDO pricing, default dependence in the reference portfolio is a crucial factor. Hence, in order to accurately measure and value the exposure of a CDO’s credit risk, an understanding and accurate measurement of the default dependencies is essential. More about financial products can be read in Meissner [45] and a different approach to pricing complex financial assets can be obtained in Lipton and Rennie [39].

\(^{11}\)Principal minus accumulated losses
Chapter 2

Copulas and Stochastic Dependence

Stochastic dependence refers to the relationship in distribution that may exist between two or more random variables. In the mathematical sense, we define random variables to be dependent if they do not fulfill the condition of stochastic independence. This means that if $X_1$ and $X_2$ are random variables, then $X_1$ and $X_2$ are independent if and only if, for all $x_1, x_2 \in \mathbb{R}$,

$$P(X_1 \leq x_1, X_2 \leq x_2) = P(X_1 \leq x_1) P(X_2 \leq x_2).$$

(2.1)

Hence, we speak of dependent random variables whenever (2.1) does not hold. This makes stochastic dependence a very broad and complex term, and quantifying stochastic dependence extremely difficult. There exist infinitely many possibilities in which (2.1) can fail, thus an all-encompassing dependence measure would require great intricacy and essentially a non-parametric statistical model to deal with the entire complexity of stochastic dependence. We will learn that copulas simplify this non-parametric question to more manageable proportions by assuming a parametric model to describe the dependence structure.

In the present chapter, we introduce copula theory and state the important relation between copulas and distribution functions. We prove that for continuous marginals, a multivariate distribution function has a unique copula. This result is called Sklar’s Theorem and is central to the validity of Li’s model. Further, we consider intuitively reasonable properties for dependence measures. We will see that some of these properties are contradictory. With this in mind, we discuss several dependence measures, such as Pearson’s correlation and two rank correlation measures and examine the properties they have. Moreover, we describe tail dependence coefficients for random variables and introduce some important properties for the analysis of Li’s copula model.

2.1 Introduction to Copulas

The word copula originates from Latin, meaning “tie, connection or link.” Hence through a copula, we are able to connect or couple marginal distributions into a multivariate distribution. This is the basic idea behind copulas. In loose terms, a copula is a marginally uniform representation of a multivariate distribution function. In this section, we present the theoretical basis of copulas. We give a formal definition and discuss important properties which allow us to investigate its use in determining the dependence structure of random variables. As mentioned in the previous chapter, dependence is a key element in pricing some convoluted financial products. Li utilizes a copula, specifically the Gaussian copula, to create a dependence structure among marginals. The central theorem here is Sklar’s Theorem (Theorem 2.4). This theorem provides a valuable result on the uniqueness of copulas for continuous random variables. Later in the section, we also introduce parametric families of copulas, namely the Gaussian and the $t$-copula.

Definition 2.1 (Copula of $X$). A copula $C$ of a random vector $X = (X_1, \ldots, X_n)'$ is the $n$-dimensional distribution function of the random vector $Y = (F_1(X_1), \ldots, F_n(X_n))'$, where $F_i$ are the marginal distribution function of $X_i$ with $i = 1, \ldots, n$.

The original definition of copulas can be found in Sklar [59], however, Nelson [49] provides an easier introduction to the theories of copulas. Nelson [49] derives the copula from an $n$-dimensional function with certain properties. Some of these properties are important, but if one knows the essentials of a
distribution function one may interpret a copula as in Definition 2.1. Our definition follows from the approach in Joe [31].

**Example 2.2.** A trivial example of a copula is the copula for independent random variables, which is represented by,

\[
C(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i.
\]

Other examples of copulas can be found in Joe [31], chapter 5.

Another useful copula function is the survival copula \( \hat{C} \). In the bivariate case, it is defined by,

\[
\hat{C}(u_1, u_2) := u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2).
\]  (2.2)

**Lemma 2.3.** Let \( X = (X_1, X_2)' \) be a random vector with distribution function \( H \) and marginals \( F_1 \) and \( F_2 \), then,

\[
\hat{H}(x_1, x_2) := P(X_1 > x_1, X_2 > x_2) = \hat{C}(\hat{F}_1(x_1), \hat{F}_2(x_2)),
\]

where \( \hat{F}_i(x_i) = 1 - F_i(x_i) \) for \( i = 1, 2 \).

**Proof.** Let \( u_i := F_i(x_i) \) for \( i = 1, 2 \). By (2.2), it follows that,

\[
\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2) = P(X_1 > x_1, X_2 > x_2).
\]

As a consequence of the standard uniform marginals of an \( n \)-dimensional copula function, its domain is the \( n \)-dimensional unit hypercube \([0, 1]^n\). Hence, the domain and the range of an \( n \)-dimensional copula function, or simply \( n \)-copula \( C \) are fixed:

\[
C : [0, 1]^n \rightarrow [0, 1].
\]

Furthermore, marginal uniformity implies that,

\[
C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i \quad \forall i = 1, \ldots, n.
\]

Directly derivable from our definition of a copula is another property: a copula separates the dependence structure of a multivariate distribution from its marginal distributions. As such, copulas form a convenient tool for model parameterization if the joint rather than the marginal properties are central to the problem at hand. Thus, a copula inherits the dependence structure of the random vector; this result is due to Abe Sklar. He proved that one can construct a multivariate distribution function with specified marginals. The original lay out of Sklar’s result can be found in Sklar [59], though no proof is given. In Nelson [49], Sklar’s result is divided into three theorems and serves as a good introduction to Sklar’s Theorem.

**Theorem 2.4** (Sklar’s Theorem). Let \( F \) be an \( n \)-dimensional distribution function with marginal distributions \( F_1, \ldots, F_n \). Then there exists an \( n \)-dimensional copula \( C \) such that for all \( x \in \mathbb{R}^n \),

\[
F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).
\]  (2.3)

The copula \( C \) is uniquely determined in (2.3) if the marginals \( F_1, \ldots, F_n \) are all continuous. Otherwise, \( C \) is only uniquely determined on \( \text{Ran}(F_1) \times \cdots \times \text{Ran}(F_n) \), where \( \text{Ran}(F_i) \) is the range of the function \( F_i \). Conversely, if \( C \) is a copula and \( F_1, \ldots, F_n \) are univariate distribution functions, then the function \( F \) defined in (2.3) is a multivariate distribution function with marginals \( F_1, \ldots, F_n \).

**Remark.** In the following proof we will only consider the case with continuous marginals and prove the existence and uniqueness of the copula. The converse is also proven for continuous marginals. A full proof can be found in Schweizer and Sklar [55] and Nelson [49].

**Proof.** Let \( X = (X_1, \ldots, X_n)' \) be a random vector with distribution function \( F \) and continuous marginals \( F_1, \ldots, F_n \). Since \( F_i \) are continuous for all \( i \), it holds that,

\[
P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(F_1(X_1) \leq F_1(x_1), \ldots, F_n(X_n) \leq F_n(x_n)),
\]
and further by Lemma A.1 we have,

$$F_i(x_i) \sim U(0, 1) \quad \forall i = 1, \ldots, n.$$  

By Definition 2.1, $(F_1(X_1), \ldots, F_n(X_n))'$ has a copula $C$ as its distribution function. This proves the existence. As for the uniqueness: Let $x_i = F_i^{-1}(u_i)$ for all $i = 1, \ldots, n$ then by the continuity of the marginal distributions we have,

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)).$$  \hfill (2.4)  

This is an explicit expression of $C$, thus $C$ must be unique. Conversely, let $C$ be a copula and $F_1, \ldots, F_n$ be continuous univariate distribution functions. Suppose the random vector $U = (U_1, \ldots, U_n)'$ has distribution function $C$. Let $X := (F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))'$, then by Lemma A.1,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(F_1^{-1}(U_1) \leq x_1, \ldots, F_n^{-1}(U_n) \leq x_n) = P(U_1 \leq F_1(x_1), \ldots, U_n \leq F_n(x_n)) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Although Sklar’s Theorem (Theorem 2.4) is a very general statement, its interpretation can be misleading. On account of Sklar’s Theorem, we can uniquely define a copula for a given multivariate distribution function $F$ if the marginals are continuous. That is, we have to know the joint distribution function of credit risks $X_1, \ldots, X_n$ to determine the copula function. The converse statement in Sklar’s Theorem specifies that with a copula one can define a multivariate distribution function $F$ with marginals $F_1, \ldots, F_n$ through Equation (2.3). That is to say that this implicit distribution function is an arbitrary distribution function and by no means the distribution of any random variable with marginals $F_1, \ldots, F_n$. This implies that Sklar’s Theorem does not provide any results on modeling the distribution for data with unknown multivariate distribution.

The next corollary proves that for continuous marginals, the copula $C$ is invariant under reparameterization of the covariates. It follows quickly from (2.4).

**Corollary 2.5.** Let $X = (X_1, \ldots, X_n)'$ be a random vector with continuous marginal distributions $F_1, \ldots, F_n$. Further let $h_1, \ldots, h_n$ be strictly increasing functions. If $C$ is the copula of $X$, then $C$ must also be the copula of the random vector $Y = (h_1(X_1), \ldots, h_n(X_n))'$.

The following theorem is called the Fréchet-Hoeffding’s Theorem and provides universal upper and lower bounds for copulas.

**Theorem 2.6** (Fréchet-Hoeffding). Let $C$ be a copula, then for every $(u, v) \in [0, 1]^2$,

$$W(u, v) := \max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\} =: M(u, v).$$  \hfill (2.5)  

**Proof.** Suppose $(u, v) \in [0, 1]^2$, then $C(u, v) \leq C(1, v) = v$ and $C(u, v) \leq C(u, 1) = u$. Hence, $C(u, v) \leq \min\{u, v\}$. Furthermore, by Lemma 2.3

$$1 - u - v + C(u, v) = \hat{C}(1 - u, 1 - v) > 0.$$  

Thus, $C(u, v) > u + v - 1$, which combined with the fact that $C(u, v) > 0$ gives $C(u, v) \geq W(u, v).$  \hfill \Box  

These upper and lower bounds also hold for higher dimensions, more about this and Fréchet-Hoeffding’s Theorem can be read in Nelson [49], Section 2.5. The following lemma can be found in McNeil, et al. [44].

**Lemma 2.7.** Let $(X_1, X_2)'$ be random vector with joint distribution function $F$ and continuous marginals $F_1$ and $F_2$, such that $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ for some unique copula $C$. Then, the conditional distribution of $C$ is given by,

$$C_{X_2|X_1}(u_2|u_1) = \frac{\partial}{\partial u_1} C(u_1, u_2),$$  

with $u_i = F_i(x_i)$ for $i = 1, 2$. 

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Proof. We know that \( C_{X_2|X_1}(u_2|u_1) = P(F_2(X_2) \leq u_2|F_1(X_1) = u_1) \) with \((u_1, u_2) \in [0, 1]^2\). Further we can write,

\[
P(U_2 \leq u_2|U_1 = u_1) = \int_{-\infty}^{u_2} f_{U_2|U_1}(t, u_1) dt,
\]

with \(U_i = F_i(X_i)\) for \(i = 1, 2\). Subsequently, we get \(f_{U_2|U_1}(t, u_1) = \frac{f_{U_2}(t, u_1)}{f_{U_1}(u_1)}\). Since \(U_1 \sim U[0, 1]\), we have \(f_{U_1, U_2}(t, u_1) = f_{U_1}(t, u_1)\) and,

\[
\int_{-\infty}^{u_2} f_{U_1, U_2}(t, u_1) dt = \frac{\partial}{\partial u_1} \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} f_{U_1, U_2}(t, s) ds dt,
\]

We can conclude that \(C_{X_2|X_1}(u_2|u_1) = \frac{\partial}{\partial u_1} C(u_1, u_2)\). \(\square\)

Now, let us introduce two parametric families of copula functions which serve as statistical models for dependence structures at a later stage. These copulas are the normal, or Gaussian, copula and the \(t\)-copula.

**Definition 2.8 (Gaussian copula).** The standard normal, or Gaussian, copula \(C^\nu_{\Sigma} \) is defined as follows,

\[
C^\nu_{\Sigma}(u_1, \ldots, u_n) := \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Sigma),
\]

where \(\Phi_n\) is the standard \(n\)-dimensional normal distribution function, \(\Phi^{-1}\) the inverse of the standard univariate normal distribution function and \(\Sigma_{ij} = E(\Phi^{-1}(U_i)\Phi^{-1}(U_j))\) for \(i, j \in \{1, \ldots, n\}\).

**Remark.** In Definition 2.8, \(\Sigma\) is a correlation matrix and contains the parameters for the Gaussian copula model. This correlation matrix is a common way to quantify dependence among covariates, which we discuss in Section 2.2.

As a consequence of Sklar’s Theorem (Theorem [2.4]), we may rewrite the Gaussian copula as follows,

\[
C^\nu_{\Sigma}(\Phi(x_1), \ldots, \Phi(x_n)) = \Phi_n(x_1, \ldots, x_1; \Sigma),
\]

(2.6)

where \(\Phi\) is the standard univariate normal distribution function. For the bivariate case, we write \(C^\nu_{\rho}\) for the Gaussian copula model.

**Definition 2.9 (\(t\)-copula).** The \(t\)-copula \(C^t_{\nu, \Sigma}\) with correlation matrix \(\Sigma\) and \(\nu\) degrees of freedom is defined as follows,

\[
C^t_{\nu, \Sigma}(u_1, \ldots, u_n) := t_n(t^{-1}_u(u_1), \ldots, t^{-1}_u(u_n); \nu, \Sigma),
\]

where \(t_n\) is the \(n\)-dimensional \(t\)-distribution with zero mean and \(t^{-1}_u\) the inverse of the univariate \(t\)-distribution with \(\nu\) degrees of freedom. Here the same holds as above: \(\Sigma_{ij} = E(\Phi^{-1}(U_i)\Phi^{-1}(U_j))\) for \(i, j \in \{1, \ldots, n\}\).

The result in (2.6) also holds for the \(t\)-copula, i.e. we have,

\[
C^t_{\nu, \Sigma}(t_\nu(x_1), \ldots, t_\nu(x_n)) := t_n(x_1, \ldots, x_1; \nu, \Sigma),
\]

where \(t_\nu\) is the univariate \(t\)-distribution with \(\nu\) degrees of freedom. For a bivariate \(t\)-copula, we write \(C^t_{\nu, \rho}\). More advanced discussions about copulas and dependence structures regarding copulas can be obtained in Joe [31]. For copula applications in finance and risk management, read McNeil, et al. [44] and Meissner, et al. [45].

### 2.2 Dependence Measures

As mentioned above, stochastic dependence is a non-parametric problem. Most dependence measures, especially one-dimensional measures, are only able to capture a fraction of the entire dependence. In the present section, we define basic properties for dependence and its measures in order to examine their ability to capture specific characteristics of the stochastic dependence between random variables. Dependence measures, in general, do not fulfill all the desirable properties and estimating these dependence measures in practice might be impossible due to the scarcity or absence of data. The former is a crucial limitation of all sensible dependence measures. The latter is a practical limitation, usually less ambiguous. Moreover, we introduce a few particular dependence measures later on in
this section. These dependence measures are Pearson’s or linear correlation, rank correlation and tail dependence. On account of the basic properties, we lay out restrictions of all these dependence measures. We will also see that some properties are contradictory. For instance, a dependence measure \( \delta \) cannot, at the same time, be invariant under reparameterization, that is with \( h \) a bijection,

\[
\delta(h(X), Y) = \delta(X, Y),
\]

and also satisfy the intuitively reasonable property,

\[
\delta(X, Y) = 0 \Leftrightarrow X \text{ and } Y \text{ are independent.}
\]

The proof of this contradiction follows below. Further, we will learn that the linear correlation coefficient usually only meets two of the basic properties. Below, we observe that generally the rank correlation coefficients fulfill more properties than the linear correlation coefficient. Different properties are established for the tail dependence coefficients; mainly tail dependence coefficients can be expressed explicitly in terms of the associated copula family.

### 2.2.1 Basic Properties for Dependence Measures

Here, we introduce intuitively reasonable properties for dependence measures to later ascertain which dependence measure meets which properties. First, we present two significant forms of dependence, comonotonicity and countermonotonicity. The concept of comonotonicity and countermonotonicity is based on a specific ordering of the outcomes of a random vector \( X \). As we may learn in Lemma 2.12 and Lemma 2.13 if a random vector \( X \) is comonotonic or countermonotonic then the joint distribution function can be rewritten in an expression of the marginal distributions.

**Definition 2.10 (Comonotonicity).** An \( n \)-dimensional random vector \( X = (X_1, \ldots, X_n)' \) is (strictly) **comonotonic** if there is a random variable \( Z \) and (strictly) increasing functions \( h_i \) for \( i = 1, \ldots, n \) such that,

\[
(X_1, \ldots, X_n) =_d (h_1(Z), \ldots, h_n(Z)).
\]  

**Definition 2.11 (Countermonotonicity).** The random vector \( X = (X_1, X_2)' \) is (strictly) **countermonotonic** if for some random variable \( Z \) it holds that,

\[
(X_1, X_2) =_d (h_1(Z), h_2(Z)),
\]

with \( h_1 \) (strictly) increasing and \( h_2 \) strictly deceasing functions, or vice versa.

**Lemma 2.12.** Let \( X = (X_1, \ldots, X_n)' \) be an \( n \)-dimensional continuous random vector with distribution function \( H \) and marginal distribution functions \( F_i \) for \( i = 1, \ldots, n \). The random vector \( X \) is strictly comonotonic if and only if,

\[
H(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\},
\]

for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \).

**Proof.** First we prove the ‘only if’ statement. Let \( X = (X_1, \ldots, X_n)' \) be a comonotonic vector, then by definition, there is a random variable \( Z \) and strictly increasing functions \( h_i \) for \( i = 1, \ldots, n \) such that,

\[
P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(h_1(Z) \leq x_1, \ldots, h_n(Z) \leq x_n) = P(Z \leq h_1^{-1}(x_1), \ldots, Z \leq h_n^{-1}(x_n)) = P(Z \leq \min_{1 \leq i \leq n} h_i^{-1}(x_i)) = \min_{1 \leq i \leq n} P(Z \leq h_i^{-1}(x_i)) = \min\{F_1(x_1), \ldots, F_n(x_n)\},
\]

\]
where we use the fact that the functions $h_i$ are one-to-one. For the ‘if’ statement, let $H(x)$ be as in (2.8) and $U \sim U(0, 1)$. Then,

$$
P(X_1 \leq x_1, \ldots, X_n \leq x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$$
$$= P(U \leq \min\{F_1(x_1), \ldots, F_n(x_n)\})$$
$$= P(U \leq F_1(x_1), \ldots, U \leq F_n(x_n))$$
$$= P(F_1^{-1}(U) \leq x_1, \ldots, F_n^{-1}(U) \leq x_n).$$

Since the marginal distributions $F_i$ are continuous, we conclude that (2.7) holds. □

Lemma 2.13. Let $X = (X_1, X_2)'$ be a continuous random vector with distribution function $H$ and marginal distribution functions $F_i$ for $i = 1, \ldots, n$. The random variables $X_1$ and $X_2$ are strictly countermonotonic if and only if

$$H(x_1, x_2) = \max\{F_1(x_1) + F_2(x_2) - 1, 0\},$$

for all $(x_1, x_2) \in \mathbb{R}^2$.

Proof. For the ‘only if’ statement, let $X_1$ and $X_2$ be countermonotonic. Then,

$$P(X_1 \leq x_1, X_2 \leq x_2) = P(Z \geq h_1^{-1}(x_1), Z \leq h_2^{-1}(x_2)).$$

If $h_1^{-1}(x_1) \leq h_2^{-1}(x_2)$ then, $P(Z \geq h_1(x_1) or Z \leq h_2(x_2)) = 1$, so that,

$$1 = P(Z \geq h_1^{-1}(x_1)) + P(Z \leq h_2^{-1}(x_2)) - P(Z \geq h_1^{-1}(x_1), Z \leq h_2^{-1}(x_2)).$$

Hence, $P(X_1 \leq x_1, X_2 \leq x_2) = F_1(x_1) + F_2(x_2) - 1$. If $h_1^{-1}(x_1) > h_2^{-1}(x_2)$ then $P(Z \geq h_1^{-1}(x_1), Z \leq h_2^{-1}(x_2)) = 0$, so that,

$$0 = P(Z \geq h_1^{-1}(x_1)) + P(Z \leq h_2^{-1}(x_2)) - P(Z \geq h_1^{-1}(x_1), Z \leq h_2^{-1}(x_2)).$$

Hence, we get, $H(x_1, x_2) = W(F_1(x_1), F_2(x_2))$. To prove the ‘if’ statement, suppose $H(x_1, x_2) = W(F_1(x_1), F_2(x_2))$ and let $U \sim U(0, 1)$, then,

$$\max\{F_1(x_1) + F_2(x_2) - 1, 0\} = P(U \leq F_1(x_1), 1 - U \leq F_2(x_2))$$
$$= P(F_1^{-1}(U) \leq x_1, F_2^{-1}(1 - U) \leq x_2).$$

If we observe that $F_1^{-1}$ is strictly increasing and $F_2^{-1} \circ h$, with $h(y) := 1 - y$, is strictly decreasing, then $X_1$ and $X_2$ are strictly countermonotonic by Definition 2.11. □

More about comonotonicity and countermonotonicity with regard to distribution functions can be found in Nelson [19], chapter 2.4 and 2.5 and in McNeil, et al. [44], chapter 5.

In Table 2.1, we list a few basic properties for a dependence measure $\delta$. These properties are a selection of characteristics for solid dependence measures. A more extensive list of properties can be found in Hutchinson and Lai [30], Chapter 11.4.

<table>
<thead>
<tr>
<th>Property</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$\delta(X, Y) = \delta(Y, X)$ (Symmetry)</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-1 \leq \delta(X, Y) \leq 1$ (Boundedness)</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$\delta(X, Y) = 1 \Leftrightarrow X, Y$ are comonotonic. $\delta(X, Y) = -1 \Leftrightarrow X, Y$ are countercomonotonic.</td>
</tr>
<tr>
<td>$P_4$</td>
<td>For $h$ an strictly monotonic function on the range of $X$: $\delta(h(X), Y) = \begin{cases} \delta(X, Y) &amp; \text{if } h \text{ increasing,} \ -\delta(X, Y) &amp; \text{if } h \text{ decreasing.} \end{cases}$</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$\delta(X, Y) = 0 \Leftrightarrow X, Y$ are independent.</td>
</tr>
</tbody>
</table>

Table 2.1: Intuitively reasonable properties for correlation measures.
The notation of the dependence measure \( \delta \) in Table 2.1 may be misleading since one might interpret \( \delta(X, Y) \) to be a random variable. This is not the case, the dependence measure \( \delta \) is a function that maps the two-dimensional distribution space of \( X \) and \( Y \) onto \( \mathbb{R} \).

As already stated above, \( P_4 \) and \( P_5 \) are contradictory. We clarify this statement in the following theorem. We have adapted the proof from Embrechts, et al. [12].

**Theorem 2.14.** If \( \delta \) is a dependence measure then it cannot satisfy both \( P_4 \) and \( P_5 \) for all random vectors \((X_1, X_2)\).

**Proof.** Define \((X_1, X_2) := (\cos(U), \sin(U))\) with \( U \sim U(0, 2\pi) \), so that \((-X_1, X_2) \equiv \delta(X_1, X_2)\). Let \( \delta \) be a dependence measure and assume \( \delta \) satisfies \( P_4 \) and \( P_5 \). By definition of \((X_1, X_2)\),

\[
\delta(-X_1, X_2) = \delta(X_1, X_2).
\]

With \( T(x) := -x \), \( P_4 \) imposes,

\[
\delta(-X_1, X_2) = -\delta(X_1, X_2).
\]

So that \( \delta(X_1, X_2) = 0 \). Since \( X_1 \) and \( X_2 \) are dependent, this contradicts \( P_5 \).

As the properties in Table 2.1 are only a small selection of characteristics for dependence measures, we may already conclude that there is no universal dependence measure. All dependence measures satisfy a distinguishable set of characteristics. That is, each measure serves a different purpose and tells us something distinctive about the stochastic dependence between random variables. The user must decide which purposes have priority, keeping the endgoals of his analysis in mind.

### 2.2.2 Correlation

Here, we discuss the most popular measure for dependence, namely correlation. In practice, correlation is often taken as the representation for the entire dependence structure. A common abuse of correlation is the immediate conclusion that the random variables are independent when all that one knows is zero correlation. The fact that correlation merely measures a form of co-linearity is often unknown. In finance, correlation is a widespread measure for dependence since correlation measures, such as Pearson’s correlation defined in Definition 2.15 are mathematically simple and in practice very easy to compute. Though, Pearson’s correlation only accounts for the linear correlation and thus does not measure any higher-order dependence. We also address two different rank correlation coefficients, Spearman’s rho and Kendall’s tau. These correlation measures are based on the principle of concordance and discordance. With Table 2.1 in mind, we inspect which properties the correlation measures satisfy. For arbitrary distribution functions, we discover that Pearson’s correlation coefficient only meets two of the characteristics in Table 2.1. Although in the Gaussian case, linear correlation has great advantages over most correlation measures. It fulfills almost all properties in Table 2.1 and is directly dependent on the marginal distributions. We determine that Spearman’s rho and Kendall’s tau both satisfy \( P_1 \) through \( P_4 \), for continuous marginals. Furthermore, we prove that the rank correlation measures are exclusively dependent on the copula function. This characteristic can be used as an effective tool to compare dependence structure of copula families.

**Remark.** We consider bivariate distributions to discuss the properties of the various correlation measures, since only pairwise dependence is measured.

**Definition 2.15** (Pearson’s Correlation). Let \((X_1, X_2)'\) be a random vector with \( E(X_1^2), E(X_2^2) < \infty \), then **Pearson’s correlation** \( \rho_p \), or the linear correlation coefficient, is defined by,

\[
\rho_p(X_1, X_2) := \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1) \text{Var}(X_2)}},
\]

with,

\[
\text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2).
\]

**Theorem 2.16.** If \( H \) is a bivariate distribution function with marginals \( F_1 \) and \( F_2 \), then Pearson’s correlation \( \rho_p \) satisfies \( P_1 \) and \( P_2 \).

---

1 More about concordance and a definition of a measure of concordance can be found in Embrechts, et al. [13], Definition 3.3.
Remark. The linear correlation measure does not fulfill property $P_3$. The proof of property $P_2$ is based on the Cauchy-Schwarz inequality.

Since $E(X_1^2), E(X_2^2) < \infty$, we have for $a \in \mathbb{R}$ that,

$$0 \leq \text{Var}(X_1 + aX_2) = \text{Var}(X_1) + a^2\text{Var}(X_2) + 2a\text{Cov}(X_1, X_2). \quad (2.9)$$

Notice that the right hand side of Equation (2.9) is a quadratic function in $a$ and the coefficients of the quadratic and constant term are non-negative. Hence, the discriminant must be non-positive. That is,

$$4\text{Cov}^2(X_1, X_2) - 4\text{Var}(X_1)\text{Var}(X_2) \leq 0.$$

Proof. For the proof of property $P_3$, we refer to Theorem A.2. For property $P_4$ we give a counterexample. Suppose $X_1 \sim N(0,1)$ then we have $\rho_p(X_1, X_1) = 1$. Let $h(x) := x^3$. It is clear that $h$ is a strictly increasing function, though we have,

$$\rho_p(h(X_1), X_1) = \frac{E(h(X_1)X_1) - E(h(X_1))E(X_1)}{\text{Var}(h(X_1))\text{Var}(X_1)} = \frac{E(X_1^4)}{E(X_1^4) - [E(X_1^2)]^2}.$$

With the moments of $X_1$, $\rho_p(h(X_1), X_1) = \frac{1}{\sqrt{2}}$. We also give a counterexample to prove that Pearson’s correlation does not fulfill $P_3$. Assume that $X = (X_1, X_2)'$ is a random vector with $X_1 \sim N(0,1)$. Further, it is trivial to notice that the components of $X$ are clearly dependent. The covariance of $X_1$ and $X_1^2$ is zero because,

$$E(X_1^3) = 0 \quad \text{and} \quad E(X_1) = 0.$$

Thus, we have a linear correlation coefficient of zero, although the components are dependent.

Proof. For the proof of property $P_3$, we refer to Theorem A.2. For property $P_4$ we give a counterexample. Suppose $X_1 \sim N(0,1)$ then we have $\rho_p(X_1, X_1) = 1$. Let $h(x) := x^3$. It is clear that $h$ is a strictly increasing function, though we have,

$$\rho_p(h(X_1), X_1) = \frac{E(h(X_1)X_1) - E(h(X_1))E(X_1)}{\text{Var}(h(X_1))\text{Var}(X_1)} = \frac{E(X_1^4)}{E(X_1^4) - [E(X_1^2)]^2}.$$

With the moments of $X_1$, $\rho_p(h(X_1), X_1) = \frac{1}{\sqrt{2}}$. We also give a counterexample to prove that Pearson’s correlation does not fulfill $P_3$. Assume that $X = (X_1, X_2)'$ is a random vector with $X_1 \sim N(0,1)$. Further, it is trivial to notice that the components of $X$ are clearly dependent. The covariance of $X_1$ and $X_1^2$ is zero because,

$$E(X_1^3) = 0 \quad \text{and} \quad E(X_1) = 0.$$

Thus, we have a linear correlation coefficient of zero, although the components are dependent.

Remark. The linear correlation measure $\rho_p$ does not fulfill property $P_4$, although, it is invariant under strictly increasing linear transformations. That is, let $X_1$ and $X_2$ be random variables, then,

$$\rho_p(aX_1 + b, cX_2 + d) = \text{sgn}(ac) \rho_p(X_1, X_2),$$

with $a, b \in \mathbb{R} \setminus \{0\}$ and $b, d \in \mathbb{R}$, which follows directly from Definition 2.15.

Now we have determined that, in general, the linear correlation meets only two properties from Table 2.1 and a weaker form of property $P_4$. The next theorem shows another characteristic of Pearson’s correlation; it is directly dependent on the marginal distribution. Later, we observe that the rank correlations do not share this characteristic.

Theorem 2.18 (Hoeffding). Suppose $(X_1, X_2)'$ is a continuous random vector with distribution function $F$ and marginals $F_1$ and $F_2$. If $E|X_1X_2|, E|X_1|, E|X_2| < \infty$ holds, then,

$$\rho_p = \frac{1}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \iint_{\mathbb{R}^2} H(x_1, x_2) - F_1(x_1)F_2(x_2) \, dx_1dx_2. \quad (2.10)$$

Proof. Let $(Y_1, Y_2)'$ be an independent copy of $(X_1, X_2)'$. Then,

$$2(E(X_1Y_1) - E(X_1)E(Y_2)) = E((X_1 - X_2)(Y_1 - Y_2)) = E \iint_{\mathbb{R}^2} (1_{X_1 \geq u} - 1_{X_2 \geq u})(1_{Y_1 \geq v} - 1_{Y_2 \geq v}) \, du dv.$$

Since $E|X_1X_2|, E|X_1|, E|X_2| < \infty$, Fubini’s theorem justifies (2.10).

Corollary 2.19. Suppose $(X_1, X_2)'$ is a continuous random vector with unique copula $C$ and marginals $F_1$ and $F_2$. If $E|X_1X_2|, E|X_1|, E|X_2| < \infty$ holds, then

$$\rho_p = \frac{1}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \iint_{[0,1]^2} (C(u, v) - uv) \, dF_1^{-1}(u)dF_2^{-1}(v).$$
Let us now define the Spearman’s rho and examine the properties it satisfies.

**Definition 2.20 (Spearman’s rho).** Let \((X_1, X_2)\)' be a random vector, then Spearman’s rho \(\rho_s\) is defined by,

\[
\rho_s(X_1, X_2) = 3(P((X_1 - Y_1)(X_2 - Z_2) > 0) - P((X_1 - Y_1)(X_2 - Z_2) < 0)),
\]

where \((Y_1, Y_2)'\) and \((Z_1, Z_2)'\) are independent copies of \((X_1, X_2)'\). \(\blacksquare\)

**Remark.** Note that \(Y_1\) and \(Z_2\) are independent random variables.

**Theorem 2.21.** Let \(X_1, X_2\) be continuous random variables with copula \(C\). Then Spearman’s rho \(\rho_s\) for \((X_1, X_2)'\) is given by,

\[
\rho_s(X_1, X_2) = 12 \int \int_{[0,1]^2} uv \, dC(u,v) - 3. \tag{2.11}
\]

**Proof.** Let \((Y_1, Y_2)'\) and \((Z_1, Z_2)'\) be independent copies of \((X_1, X_2)'\) as in Definition 2.20. Since all these random variables are continuous, \(P((X_1 - Y_1)(X_2 - Y_2) > 0) = 1 - P((X_1 - Y_1)(X_2 - Y_2) < 0)\). Further, we know that,

\[
P((X_1 - Y_1)(X_2 - Z_2) > 0) = P(X_1 > Y_1, X_2 > Z_2) + P(X_1 < Y_1, X_2 < Z_2).
\]

Since \((Y_1, Y_2)'\) and \((Z_1, Z_2)'\) are independent copies of \((X_1, X_2)'\), it holds that,

\[
P(X_1 > Y_1, X_2 > Z_2) = P(X_1 < Y_1, X_2 < Z_2).
\]

Thus, we have,

\[
\rho_s(X_1, X_2) = 12P(Y_1 < X_1, Z_2 < X_2) - 3.
\]

As \(Y_1\) and \(Z_2\) are independent and \(C\) is the copula of the vector \((X_1, X_2)'\), we have,

\[
P(Y_1 < X_1, Z_2 < X_2) = \int \int_{\mathbb{R}^2} P(Y_1 < x_1, Z_2 < x_2) \, dC(F_1(x_1), F_2(x_2))
\]

\[
= \int \int_{\mathbb{R}^2} P(Y_1 < x_1) \, dC(F_1(x_1), F_2(x_2))
\]

\[
= \int \int_{\mathbb{R}^2} F_1(x_1) \, dC(F_1(x_1), F_2(x_2))
\]

\[
= \int \int_{[0,1]^2} uv \, dC(u,v),
\]

where \(F_1\) and \(F_2\) are the marginal distributions of \(X_1\) and \(X_2\), respectively, and in the last equality we used the transformation \(u := F_1(x_1)\) and \(v := F_2(x_2)\). \(\square\)

**Theorem 2.22.** Spearman’s rho \(\rho_s\) satisfies properties \(P_1, P_2, P_3\) and \(P_4\).

**Proof.** Properties \(P_1\) and \(P_2\) are easily verified. For \(P_3\) we first prove that \(\rho_s = 1 \iff X_1\) and \(X_2\) are comonotonic. By Theorem 2.21, it holds that \(\rho_s = 1 \iff C\) in (2.11) is maximal. By Fréchet-Hoeffding’s Theorem (Theorem 2.6), \(C\) is maximal \(\iff C(x_1, x_2) = \min(x_1, x_2)\). Now, Lemma 2.12 ends the proof. The implication \(\rho_s = -1 \iff X_1\) and \(X_2\) are countermonotonic can be proven by similar arguments and is left to the reader. To prove that Spearman’s rho satisfies \(P_4\) is trivial if we rewrite the expression of \(\rho_s\) into a function of \(C\) as in (2.11). \(\square\)

**Definition 2.23 (Kendall’s tau).** Let \((X_1, X_2)'\) be a random vector, then Kendall’s tau \(\tau_k\) is defined by,

\[
\tau_k(X_1, X_2) = P((X_1 - Y_1)(X_2 - Y_2) > 0) - P((X_1 - Y_1)(X_1 - Y_2) < 0),
\]

where \((Y_1, Y_2)'\) is an independent copy of \((X_1, X_2)'\). \(\blacksquare\)
Theorem 2.24. Let \(X_1, X_2\) be continuous random variables with copula \(C\). Then Kendall’s tau \(\tau_k\) for \((X_1, X_2)'\) is given by,

\[
\tau_k(X_1, X_2) = 4 \int \int_{[0,1]^2} C(u,v) \ dC(u,v) - 1.
\] (2.12)

Proof. This proof is almost identical to the proof in Theorem 2.21 and is therefore left to the reader. \(\square\)

Theorem 2.25. Kendall’s tau \(\tau_k\) satisfies properties \(P_1\), \(P_2\), \(P_3\) and \(P_4\).

The proof of Theorem 2.25 is almost identical to the proof of Theorem 2.22.

Remark. It is trivial to see that Spearman’s rho and Kendall’s tau do not satisfy \(P_5\). Moreover, it follows from Theorem 2.14.

For specific distributions, there are relationships between the rank correlations and Pearson’s correlation.

Theorem 2.26. If \((X_1, X_2)'\) is a normally distributed random vector, then the following equalities hold,

\[
\rho_s = \frac{6}{\pi} \arcsin \left( \frac{1}{2} \rho_p \right) \tag{2.13}
\]

and

\[
\tau_k = \frac{2}{\pi} \arcsin(\rho_p). \quad \tag{2.14}
\]

Proof. The proof of this theorem is beyond the scope of this paper and can be reviewed in McNeil, et al. \[44\], Theorem 5.36. \(\square\)

Under the assumption that the random vector \((X_1, X_2)'\) is normally distributed, we can plot the relation between the rank correlation measures and Pearson’s correlation, see Figure 2.1. However, there is no such equality if we deviate from elliptical distributions.

Figure 2.1: Pearson’s correlation plotted against Spearman’s rho and Kendall’s tau. The dashed line is the diagonal \(y = x\).

Theorem 2.27. If \((X_1, X_2)'\) is a normally distributed random vector with expectation \(\mu\) and covariance matrix \(\Sigma\), then Pearson’s correlation \(\rho_p\) satisfies \(P_1\), \(P_2\), \(P_3\) and \(P_5\).
Proof. Property $P_1$ and $P_2$ have been proven in Theorem 2.16. By Theorem 2.26, it holds that,

$$\rho_p = \sin \left( \frac{\pi}{2} \tau_k \right).$$

Theorem 2.25 states that $\tau_k$ satisfies $P_3$ thus we have,

$$X_1 \text{ and } X_2 \text{ comonotone } \iff \tau_k = 1 \iff \rho_p = \sin \left( \frac{\pi}{2} \right) = 1$$

and

$$X_1 \text{ and } X_2 \text{ countermonotone } \iff \tau_k = -1 \iff \rho_p = \sin \left( -\frac{\pi}{2} \right) = -1.$$

Hence, Pearson’s correlation fulfills $P_3$ for normally distributed random vectors. For property $P_5$, we have to prove Equation (2.1) for $\rho_p(X_1, X_2) = 0$. We know that,

$$P(X_1 \leq x_1, X_2 \leq x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{1}{2}(\mathbf{x} - \mu)'\Sigma(\mathbf{x} - \mu)\right) \, d\mathbf{x}_2 \, d\mathbf{x}_1,$$

where $\mathbf{x} = (x_1, x_2)$. If $\rho_p = 0$, then,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{1}{2}(\mathbf{x}_1 - \mu_1)(\mathbf{x}_2 - \mu_2)\right) \, d\mathbf{x}_2 \, d\mathbf{x}_1$$

$$= P(X_1 \leq x_1)P(X_2 \leq x_2).$$

The other implication is trivial. This proves property $P_5$. \( \square \)

Now, we have seen that the linear correlation measure is very beneficial in a Gaussian scenario. However, if the joint distribution is not Gaussian, we have seen that the linear correlation measure merely satisfies the first two properties in Table 2.1. This obviously would result in a very poor performance of Pearson’s correlation to describe dependence. Furthermore, we have seen in Theorem 2.21 and Theorem 2.24 that the rank correlation measures are directly dependent on the copula function. This provides a tool to compare the dependence structure of different copula families without the influence of marginal distributions. Other interesting results about dependence and correlation can be found in Lehman [36].

### 2.2.3 Tail Dependence

The measurement of dependence in the tails of multivariate distributions plays an important role in multivariate extreme value theory (MEVT). This field studies extreme events and behavior and characteristics of distributions under extreme deviations from the distribution’s median. An introduction to general theories of MEVT for risk management can be found in McNeil [43].

Here, we present coefficients of tail dependence to measure the limiting dependence between marginals of a bivariate distribution. Put differently, tail dependence coefficients measure the probability of joint extreme events in the upper and lower-quadrant tail of the distribution. The above mentioned credit and default risk are based on extreme and rare situations. Therefore, the tails of a model describing these risks must be tailored to fit the joint extreme events of the underlying risk factors. Let us first define the tail dependence coefficients.

**Definition 2.28** (Tail dependence). Let $X_1$ and $X_2$ be random variables with distribution functions $F_1$ and $F_2$, respectively. The upper and lower tail dependence coefficients are defined by,

$$\lambda_u = \lim_{q \downarrow 1} P(X_1 > F_1^{-1}(q) \mid X_2 > F_2^{-1}(q)) \quad \text{and} \quad \lambda_l = \lim_{q \downarrow 0} P(X_1 < F_1^{-1}(q) \mid X_2 < F_2^{-1}(q)),$$

respectively.
Since the definition of the tail dependence coefficients are probabilities, it is easy to observe that $0 \leq \lambda_u, \lambda_l \leq 1$. Further, it is said that two random variables $X_1$ and $X_2$ are asymptotically independent in the upper-tail if $\lambda_u = 0$ and $X_1$ and $X_2$ are asymptotically dependent in the upper-tail if $\lambda_u > 0$. The same hold for the lower tail dependence coefficient and the asymptotic dependence in the lower tail. The following two theorems enable us to rewrite the expression for the upper and lower tail dependence in terms of the copula $C$ for $(X_1, X_2)'$.

**Theorem 2.29.** Let $X_1, X_2$ be continuous random variables with joint distribution function $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ for some unique copula $C$ and marginals $F_1$ and $F_2$, respectively. Then the lower and upper tail dependence coefficient can be rewritten as follows,

$$
\lambda_l = \lim_{q \downarrow 0} \frac{C(q, q)}{q} \quad \text{and} \quad \lambda_u = \lim_{q \downarrow 0} \frac{\hat{C}(q, q)}{q},
$$

where $\hat{C}$ is the survival copula defined in (2.2).

**Proof.** We only give the proof of the upper tail dependence. By Definition 2.28, we have,

$$
\lambda_u = \lim_{q \uparrow 1} P(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)) = \lim_{q \uparrow 1} \frac{P(X_1 > F_1^{-1}(q), X_2 > F_2^{-1}(q))}{P(X_1 > F_1^{-1}(q))} = \lim_{q \uparrow 1} \frac{\hat{C}(1 - q, 1 - q)}{1 - q} = \lim_{q \downarrow 0} \frac{\hat{C}(q, q)}{q},
$$

Equation (2.15) follow as a consequence of Lemma 2.3.

The result in Theorem 2.29 is excellent for comparing the tail behaviors of different copulas; high tail dependence coefficients imply more probability of simultaneous extreme events.

In this section, we have examined two kinds of dependence measures, correlation and tail dependence. Although, individually they cannot represent the whole dependence structure between random variables, they are valuable tools in the examination of dependence structures. In the case of continuous marginals, the rank correlation and tail dependence coefficients are exclusively dependent on Sklar’s unique copula. More about correlation measures in general can be read in Embrechts, et al. [12] and Lehmann [36] and specifics about rank correlation can be reviewed in Lingskog, et al. [38] and Schweizer and Wolff [56]. A useful reference for advanced studies on rank correlation and their relationship is Fredricks and Nelsen [20]. Details about tail dependence and further properties can be found in Joe [31] and McNeil, et al. [44].

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\(^2\)For some distributions the upper and lower tail dependence coefficients are equal. These distributions are called radially symmetric and are discussed in Section 3.3.
Chapter 3

Li’s Model for Default Correlation

In this chapter, we discuss Li’s copula model in detail and give an analysis of its most prominent weaknesses. Li’s recognition of the copula framework behind the CreditMetrics model facilitated pricing methods for all kinds of credit based derivatives for all maturity dates. It is used to model default correlations for all time periods and examine dependent default risk in a portfolio of mortgages or other loans, relevant to CDO pricing. In short, the model utilizes correlations between asset values of different entities as parameters in order to construct a joint distribution through a Gaussian copula, and thus, propose a dependence structure. This is assuming that the asset values are jointly Gaussian distributed. Further, survival functions are then estimated to produce marginal default probabilities. This is done via a reduced-form approach. So called hazard rates are estimated through an implied calibration technique described in Appendix B.1. By rescaling the marginals of the Gaussian copula to resemble the survival functions, default correlations can be extracted by generating dependent survival times from this reproduced joint distribution through a Monte Carlo procedure.

This model is an extreme simplification of reality since one assumes a specific dependence structure between the underlying assets. Certainly not all assets have such similar dependence structures that one can assume one copula to model the dependence between all these assets. Furthermore, the model continues to assumes that this already simplified dependence structure between assets is the only driver of default dependence between the entities. The use of the linear correlation coefficient to render the default dependence in this model imposes another simplification. This is the common mistake that co-linearity is taken as representation for the full default dependence between entities.

In the Introduction, we provided statistics on the size of the credit derivatives over the last decade. Before 2000, the credit derivative market was relatively small because of its riskiness and the inability to accurately price these generally complex derivatives. An overwhelming change in these markets was lead by a misguided sense of security derived from the illusion that Li’s model provided all one needs to price such contracts. Within a very short amount of time, it was ubiquitous in the financial world and quickly adopted by everybody from investors and Wall Street bankers to credit rating agencies and even regulators. Its mathematical elegance and the ease with which Li modeled extremely complex default risks seemed remarkable and turned out to be misleading, to some extent.

3.1 Basic Ideas of Li’s Copula Approach

In this section, we present Li’s copula approach as discussed in Li [37]. We follow a slightly different notation than that used in Li’s paper, for consistency. We start with the characterization of the random variable time-until-default, or survival time, and the definition of the coveted default correlation $\rho_{AB}$.

Definition 3.1 (time-until-default). If $A$ is a financial entity, then $T_A$ is defined as time-until-default of entity $A$. $T_A$ is a random variable that measures the time until default occurs. The event $E = \{T_A < t\}$ means a particular entity or financial instrument $A$ defaults before $t > 0$.

Definition 3.2 (Default correlation). If $A$ and $B$ are financial entities, then the default correlation $\rho_{A,B}$ between $A$ and $B$ is defined as the linear correlation as in Definition 2.15 between their survival times $T_A$ and $T_B$.

$^1$Trivially, it holds that $T_A > 0$.  

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Consider a single financial entity $A$ with survival time $T \sim F_T$ and survival function $S_T(t)$. Assuming that $T$ has an absolutely continuous distribution, we can define the density function by taking the limit,

$$f_T(t) := -\frac{dS_T(t)}{dt} = \lim_{\Delta \to 0} \frac{P(t \leq T \leq t + \Delta)}{\Delta},$$

since $S_T(t) = 1 - F_T(t)$. Furthermore, the hazard rate function can be derived through,

$$P(t < T \leq t + \Delta | T > t) = \frac{F_T(t + \Delta) - F_T(t)}{1 - F_T(t)} \approx \frac{f_T(t) \Delta t}{S_T(t)}.$$  

**Definition 3.3** (Hazard Rate Function). A hazard rate function is the value of the conditional probability density of $T$ at exact age $t$, given survival to that time. That is, 

$$h(t) := \frac{f_T(t)}{S_T(t)} = -\frac{S_T'(t)}{S_T(t)} = -\frac{d}{dt} \log(S_T(t)), \quad (3.1)$$

with $S_T'(t)$ is the derivative with respect to $t$. 

Li uses the hazard rate as a means to obtain the instantaneous default probability for a financial instrument that has reached a certain age, $t$. From equation (3.1), it follows that,

$$S_T(t) = e^{-\int_0^t h(s) \, ds}. \quad (3.2)$$

The tools we use to obtain an expression for the survival function are methods originating from survival analysis and the full reasoning behind the features presented here can be found in Cox and Oake [10], Chapter 2. The joint survival function for two entities $A$ and $B$ based on their survival times $T_A$ and $T_B$ is denoted by,

$$S_{T_A, T_B}(t_1, t_2) := P(T_A > t_1, T_B > t_2).$$

Assume that the one year default probabilities for credits $A$ and $B$ are known and denote them $q_A$ and $q_B$, respectively. Applying a transformation enables us to attain standard normal quantiles for each default probability. If $Z_1$ and $Z_2$ are standard normal random variables then there exist quantiles $\zeta_A$ and $\zeta_B$ such that,

$$q_A = P(Z_1 \leq \zeta_A) \quad \text{and} \quad q_B = P(Z_2 \leq \zeta_B). \quad (3.3)$$

In the second step, we can model the joint default probability for credit $A$ and $B$ using a parameter $\rho$, as follows,

$$P(Z_1 \leq \zeta_A, Z_2 \leq \zeta_B) = \int_{-\infty}^{\zeta_A} \int_{-\infty}^{\zeta_B} \phi_2(x, y | \rho) \, dx \, dy = \Phi_2(\zeta_A, \zeta_B, \rho), \quad (3.4)$$

where $\phi_2$ is the standard bivariate normal density function with correlation coefficient $\rho$, and $\Phi_2$ is the standard bivariate normal distribution function.

Consequently, if the bivariate Gaussian copula function with a correlation parameter $\theta$ is used, and $T_A$ and $T_B$ are referred to as the survival times of $A$ and $B$, respectively, then the joint probability of default within one year can be calculated in the same manner as above, namely,

$$P(T_A \leq 1, T_B \leq 1) = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \theta), \quad (3.5)$$

where $F_A$ and $F_B$ are the distribution functions for survival times $T_A$ and $T_B$, respectively and $\Phi^{-1}$ the inverse of the standard normal distribution function. Now, if we see that,

$$q_i = P(T_i \leq 1) = F_i(1) \quad \text{with } i = A, B, \quad (3.6)$$

and the transformation to compute the quantiles is given by,

$$\zeta_i = \Phi^{-1}(q_i) \quad \text{for } i = A, B,$$

then (3.4) and (3.5) yield the same joint default probability over a one year period if $\theta = \rho$.

An crucial conclusion Li implies on theoretical framework is that if the the asset correlation is
employed for the parameter \( \rho \) in the model above, then we can use the bivariate Gaussian copula, and the marginal transformations to generate survival times of two entities. Moreover, Li states that the correlation parameter \( \rho \), meaning the asset correlation, is not the correlation coefficient between the two survival times, the coveted default correlation. He proclaims that this correlation coefficient between the survival times is much smaller than the asset correlation.

To emphasize the practical steps in Li’s copula model, we summarize it in several bullet points.

- Estimate survival time distributions \( S_1(t), \ldots, S_n(t) \);
- Simulate \( X_1, \ldots, X_n \) from an \( n \)-dimensional standard Gaussian distribution with correlation matrix given by the asset correlations \( \rho_{Asset}^{i,j} \) between the underlying credits;
- Obtain dependent time-until-default variables \( T_1, \ldots, T_n \) using the relation,
  \[
  T_i = F_i^{-1}(\Phi(X_i)) \quad \text{with} \quad F_i(t) = 1 - S_i(t) \quad \text{and} \quad i = 1, \ldots, n;
  \]
- Calculate default correlations by,
  \[
  \rho_{i,j}^{Def} = \rho(T_i, T_j) = \frac{\text{Cov}(T_i, T_j)}{\sqrt{\text{Var}(T_i)\text{Var}(T_j)}}.
  \]

To simplify the analysis of Li’s model below, we give a compressed introduction to the following three sections. We briefly discuss the reasoning and assumptions behind Li’s copula model. The attentive reader may have already recognized the two main assumptions, nevertheless, we provide a short layout:

Assumptions.

In order to create a dependence structure between the various risk factors, Li poses various underlying assumptions for the data. The main assumptions are the following.

**Assumption 3.4.** As we have seen, Li’s initial step in his model is based on asset value processes and the correlation between these processes. Thus, Li creates a dependence structure that is explicitly founded on the correlation between assets. Therefore, the model assumes that asset values of several financial entities are sufficient to fully model the default dependence.

**Assumption 3.5.** Li assumes that the Gaussian copula is a valid choice to capture the joint distribution function of the entities’ assets.

Below, we discuss these two assumptions and elaborate on the asset correlation and further effects of the Gaussian copula.

After modeling the dependence structure between the asset value processes with the Gaussian copula and a correlation coefficient, Li employs a transformation to rescale the marginals to fit the distribution function of the survival time or time-until-default. These transformations greatly influence the dependence structure between the credit risks. We discuss these transformation methods below and find that the dependence does indeed reduce. Furthermore, in a high-dimensional situation, i.e. if the dependence structure of many underlying risk factors is to be determined, another problem arises. The multivariate Gaussian distribution only exerts pairwise correlation coefficients and it is hard to comprehend the interdependence between all the credit risks. Therefore, higher-order dependence has to be considered in a model for default dependence and requires attention in model in higher-dimensional scenarios. This is extremely important in pricing CDOs, since CDOs are constructed of numerous mortgages, loans, CDS’s, or other financial instruments, whose pairwise correlation may fail to reflect the true risk, large-scale joint defaults.

Our approach to Li’s concept is divided into several parts to better analyze potential flaws and will be organized as follows: First, we acquire a good understanding of the correlation parameter in the copula model. For this we examine the asset valuation and the exertion of the asset correlation. We argue that asset values do not integrate all aspects of default and that asset correlation is insufficient to reflect all the information needed to model simultaneous default events. In addition, we take a closer look at the relation between Assumption 3.5 and the correlation coefficient used as parameter for the copula model. Regarding this relationship, two crucial fallacies are stated. Also, the use of Spearman’s rho or Kendall’s tau as parameters for the model is examined. It will be clear that there are many different correlation measures, however, none will be able to correctly reproduce default dependence in a parametric model between the asset values of the underlying financial entities.
Second, we argue that the adoption of Li’s Gaussian copula is deceptive and we refute the application of the Gaussian copula to model joint default probabilities. For this, we specifically focus on tail dependence and how well the bivariate normal copula models tail dependence. Here, the assumption of the Gaussian copula arguably suffers from many unrealistic properties that are forced upon the underlying dependence structure. The choice of a specific copula has great influence on the resulting dependence structure. We show that the $t$-copula is generally a more encompassing assumption because of its ability to specify probability of simultaneous extreme events through the parameter $\nu$. However, it will be evident that the mathematical problem of copula models does not lie in the type of copula one utilizes to model default risk, but in the copula model itself. This is the principle problem of a copula approach, because a copula generates a dependence structure independent from the underlying risk factors. High-dimensional extreme events of the true joint default distribution become too complex and any general choice of a parametric model is not able to cope with these variations.

Third, we discuss the rescaling techniques Li utilizes to rearrange the marginals to resemble survival distributions. We prove that the difference between the heavy-tailedness of the original marginal and the survival distribution is key to the transformation of the entire dependence structure. Through the categorization of the entities in the reference pool by credit ratings from Moody’s and Standard & Poor’s, we can determine the default correlation for entities of different ratings. Using default rates provided by the CRAs and implied default probabilities from CDS spread data, we may analyze the impact of the marginal transformations. It will become apparent that these transformations have a problematic effect on the dependence structure created by the Gaussian copula. Not only do they dramatically reduce the correlation in the model, but they are static for each credit rating. This is a flaw since we argue that for decreasing credit quality of the obligors, the default correlation increase. By comparing the results with simulated default correlations of the Gaussian copula with the $t$-copula, we find that for the $t$-copula default correlations are positively dependent on investment grade ratings and negatively dependent on junk ratings. This is a fundamental difference between the Gaussian and the $t$-copula model.

### 3.2 Correlations to create Dependence Structure in Li’s Model

In the present section, we discuss dependency measures in copula models, namely correlations. This is a very profound and consequential part of copula models. On the one hand, scarce data and the practical use of mathematical models in finance forces models to be simple. On the other hand, these models are required to be complicated enough to address complex and difficult questions about dependence structures. We discuss whether asset values and their correlation provide enough underlying information to model default dependence. Because of its simplicity, it is convenient to use asset valuation; models for asset valuation can be found in many books about finance, for example in Hull [29]. The correlation of asset values does not incorporate all dependence between financial entities, especially in the case of default dependence. Subsequently, we also discuss a few correlation measures, namely Pearson’s correlation coefficient $\rho_p$ (linear correlation), Spearman’s rho $\rho_s$ and Kendall’s tau $\tau_k$; all these were introduced in Section 2.2.2. As we have seen in Section 2.2, the use of correlation measures for the specification of dependence is open to discussion, and some properties of one-dimensional dependence measures are contradictory. For example, the properties $P_4$ and $P_5$ in Table 2.1. As we have verified in Section 2.2.2 different correlation measures fulfill different properties. In this section, we focus on which properties are relevant for Li’s model. We will see that linear correlation, as the input for the correlation parameter in the copula model, may be fatal due to the invalidity of Assumption 3.5. Further, we have shown that Spearman’s rho and Kendall’s tau can be expressed explicitly in terms of the copula function. This entails that these two correlation measures are not useful as input for the correlation parameter, because they are independent of the marginals. Although, this property proves to be an efficient tool to compare dependence structures created by different families of copulas.

#### 3.2.1 Asset Valuation and Correlation

Li’s assumption that default correlation can be modeled on the basis of the modest asset values can be problematic. First, we obtain a good understanding of the framework of the CreditMetrics’ assertion of asset values. In loose terms, the CreditMetrics model follows an extended version of the option pricing approach developed by Merton, for details see Merton [46]. Merton uses stochastic differential
equations (SDEs) to describe the process of asset values. Introduce $V^i_t$ as the asset value process of a financial entity $i$. It is defined by the SDE,

$$dV^i_t = V^i_t \left( (\mu_i - \frac{1}{2} \sigma^2_i)dt + \sigma_i dW^i_t \right) \quad \forall i \in \{1, \ldots, n\},$$  

(3.7)

where $\mu_i$ represents the mean of the rate of return on assets of $i$ and $\sigma_i$ the volatilities of returns on assets of $i$ and $W^i$ a standard real-valued Brownian motion. We know that $W^i_t - W^i_s \sim N(0, t-s)$ for $t > s$. Let $Z^i_t \sim N(0,1)$ a standard normal random variable dependent on $t$, then it holds that $\sqrt{t}Z^i_t = W^i_t - W^i_0 \sim N(0,t)$ for $t > 0$ and the solution to the SDE in (3.7) is given by,

$$V^i_t = V^i_0 e^{(\mu_i - \frac{1}{2} \sigma^2_i)t + \sigma_i \sqrt{t}Z^i_t} \quad \forall i.$$

(3.8)

Thus, $V^i_t$ follows a geometric Brownian motion and is log-normally distributed at a fixed time $t > 0$, that is $V^i_t \sim LogN(\ln(V^i_0) + (\mu_i - \frac{1}{2} \sigma^2_i)t, \sigma^2_i t)$ for $t > 0$. More details on asset value process and geometric Brownian motions can be read in Bielecki and Rutkowski [3], Hull [29] and Merton [46].

The asset value process $V^i_t$ for entity $i$ is then normalized as follows,

$$Z^i_t = \frac{\ln(V^i_t/V^i_0) - (\mu_i - \frac{1}{2} \sigma^2_i)t}{\sigma_i \sqrt{t}}.$$  

The random variable $Z^i_t$ are then used as the underlying marginal distributions for the multivariate Gaussian distribution, for some fixed $t > 0$.

Definition 3.6 (Asset correlation). Let $Z^i_t, Z^j_t$ be the normalized asset returns of financial entities $i, j$, respectively. The asset correlation $\rho_{i,j}^{\text{Asset}}$ of the financial entities $i$ and $j$ is defined by the linear correlation in Definition 2.15 between the normalized asset returns $Z^i_t$ and $Z^j_t$.

Individual entities’ defaults may be modeled using Merton’s model, however, modeling default correlation using Merton’s asset value approach results in severe simplifications of simultaneous default scenarios. One simplification is neglecting risk factors that influence default dependence. For example, common third party contracts among entities in the portfolio may affect default dependence. To illustrate this, we outline one specific scheme where ‘outside’ risk factors influence default dependence between two financial entities.

Example 3.7. For simplicity, let us assume that zero correlation implies independence. Suppose we have two financial entities $A$ and $B$ with zero asset correlation. Hence, the behavior of $R^A_t$ has no influence on the behavior of $R^B_t$, or vice versa. If both financial entities are insured through the same company, the financial entities are not default independent. In this particular example, the default dependence between financial entities $A$ and $B$ follows from the default probability of the insurance company, whereas the normalized asset returns of $A$ and $B$ would still be independent. Figure 3.1 illustrates the dependence more coherently.

Figure 3.1: Financial entities $A$ and $B$ are mainly insured through insurance company “AIG”.

\footnote{Note that zero asset correlation using the linear correlation coefficient does not always imply independence. We give a detailed discussion about this in Section 2.2.2.}

\footnote{This example proved to be a very realistic scenario in the Mortgage Crisis of 2007-2008 and one that cannot be ignored.}
For entities whose business is entrenched in the same country, the sovereign stability also reflects the same risk factor as described in Example 3.7. As a result of disregarding additional risk factors as in Example 3.7, it follows that if the comprehensive default event of the financial entity \(i\) is defined by \(D_i(t)\), the following strict inclusion holds,

\[
\{ V_i^T < F_i^T, V_j^T < F_j^T \} \subset D_i(t) \cap D_j(t). \tag{3.9}
\]

where \(F_k^T\) with \(k = i, j\) are time dependent default thresholds for the asset value processes.

Now that we have identified CreditMetrics’ model for asset correlations, we may conclude that asset returns render insufficient information on which to base dependent default events. One may consider Gersbach and Lipponer [23] and Erlenmaier and Gersbach [17], where a general upper boundary for the default correlation in terms of asset correlation is established and other weaknesses of asset returns with respect to dependent default events are laid out.

### 3.2.2 Correlation Measure for Li’s Model

An alternative question is how to measure the correlation between assets. Most often, the linear correlation coefficient is used. Other popular coefficients are rank correlation coefficients, such as Spearman’s rho and Kendall’s tau; these were introduced in Section 2.2.2. We briefly discuss their applicability in the CreditMetrics model and how they may contribute to the Gaussian copula model.

Over the years, the concept of correlation has become omnipresent and is a universal tool related to dependence in modern finance. However, on many occasions correlation is misunderstood. The dictionary definition of the word correlation is “a mutual or reciprocal relationship between two or more things;” this definition strongly points to a notion of dependence. To a mathematician, correlation is only one particular measure of dependence among many and generally, correlation is not an ideal one. Linear correlation, especially, is only a well-founded measure of dependence in the elliptical family of distributions. Unfortunately, distributions of real-world stochastics are rarely found in this family. Moreover, even for joint elliptical distributed random variables linear correlation can be troublesome. For example, consider the univariate \(t_2\)-distribution with 2 degrees of freedom. Here, the linear correlation coefficient is not defined due to infinite variances.

To clarify problems with linear correlation in the CreditMetrics model, we state two fallacies about linear correlation and elliptical distributions. These emphasize the gravity of Assumption 3.5. A more general and extended version about fallacies of linear correlation can be found Embrechts, et al. [11].

As a consequence of Assumption 3.5, the linear correlation coefficient is the preferred choice for the correlation parameter in the copula model. As the linear correlation in the joint Gaussian scenario satisfies the property \(P_1, P_2, P_3\), and \(P_5\) which we have seen in Theorem 2.27. This is, of course, only if we trust Assumption 3.5.

**Fallacy 3.8.** Marginal distribution and correlation determine the joint distribution.

If one is prepared to believe that certain risk factors are jointly elliptically distributed then this statement is verifiable.

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4See Black and Cox [4]

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**Figure 3.2:** Non-elliptical bivariate density with standard normal marginals and correlation zero.

Source: Embrechts [12], Example 1.
As described, this is rarely the case and with marginals and correlation given, there are infinitely many non-elliptical distributions that fit. Figure 3.2 illustrates a bivariate distribution with normal marginals and correlation zero very different from a bivariate normal distribution with zero correlation. This example demonstrates Fallacy 3.8 and stresses the fact that using the Gaussian copula in the CreditMetrics model is a very stringent assumption that does not do justice to the true complexity of the issue. Moreover, if Assumption 3.5 does not hold, then not only is the joint distribution in Li’s model incorrect, but the use of linear correlation is also unwarranted, as we have seen in Theorem 2.17. This has an enormous influence on the dependence structure in Li’s copula model since, in general, the linear correlation coefficient only satisfies $P_3$, if the multivariate distribution is non-elliptical. A more detailed discussion about this can be found in Embrechts, et al. [12]. The consequences of correlation estimates with a misspecified Gaussian copula model is laid out in Hamerle and Rösch [27]. Hamerle and Rösch [27] use a Gaussian framework to estimate the correlation parameter, while the ‘true’ distribution is Student $t$ with various degrees of freedom. Their estimation results are displayed in Table 1 in Hamerle and Rösch [27], the estimates constantly overestimate the ‘true’ correlation. If we compare this with the results of Figure 1.5 the CDO spreads of senior tranches are overestimated and the most junior tranches are underestimated according to the Gaussian copula, if the ‘true’ distribution is Student $t$.

Furthermore, we have the two rank correlations at our disposal. As already mentioned, these correlation measures are explicitly dependent on the copula function and thus invariant under marginal transformations. This characteristic makes them unsuitable for Li’s copula model, since asset correlation and default correlation would turn out to be identical.

Other dependence measures are discussed in Schweizer and Wolff [60], these dependence measures all satisfy property $P_3$. Although for practical use they are not interesting, since estimation procedures are unclear. This leaves us with the linear correlation coefficient. The fact that linear correlation is based on the moments of the underlying random variable is a very advantageous property in this instance, since it enables us to connect survival distributions in the marginal transformation procedure.

### 3.3 Tail Dependence in Copula Models

Asserting default risk of more than one financial entity, forces us to evaluate the asymptotic dependence between the entities involved. Tail dependence, which we introduced in Section 2.2.3 is thereby crucial to be elaborated on in the context of the Gaussian copula and the Gaussian approach to default risk. To better understand the tail dependence coefficients in copula models for default risk, we compare the Gaussian tail dependence coefficients with the copula’s tail dependence coefficients. We state a Lemma that shows that for radially symmetric distributions the upper and lower tail dependence coefficients are equal. Furthermore, we will derive a closed form expression for the two copula models at hand. Since a $t$-copula with infinite degrees of freedom is identical to the Gaussian copula, we can easily compare the two expressions for the tail dependence coefficients. To illustrate the concept of tail dependence in copulas, we give an example that points out the difference in tail dependence for different copulas.

**Example 3.10.** For the sake of simplicity, we are going to consider the two-dimensional case. Let $X = (X_1, X_2)'$ be standard normal distributed random vector and $Y = (Y_1, Y_2)'$ be standard Student $t$ distributed random vector both with $\rho = 0.7$. In Figure 3.3 we see that the $t$-copula $C^t_{\rho, \nu}$ $Y$ and the Gaussian copula $C^\rho_{\nu, \rho}$ of $X$ show different results in dependence structure. In particular, the dependence in the tails is very dissimilar. To refine the difference of the Gaussian copula and the $t$-copula, we present some contrasting examples of $t$-copula with distinct degrees of freedom. In Figure 3.4 it is noticeable that the tails of the $t$-copula become heavier if the number of degrees of freedom decreases and thus the dependence structure between the two random variables $Y_1, Y_2$ changes. If
lavishly big, for instance $\nu = 10^{15}$, then it becomes apparent that the $t$-copula and the Gaussian copula are very much alike.

Consequently, we suggest that the choice for one particular copula to reproduce the dependence structure between marginals is to be examined extensively and critically before it is applied. It is obvious from Example 3.10 that modeling with the $t$-copula gives us two parameters to capture the dependence structure of the underlying risk factors: the number of degrees of freedom $\nu$ and the correlation coefficient $\rho$.

Let us examine the tail dependence in the Gaussian copula further to give a more elaborate explanation of the results in Example 3.10.

**Definition 3.11** (Radial symmetry). A random vector $X = (X_1, \ldots, X_n)'$ is **radially symmetric about** $v$ if $X - v =^d v - X$.

**Remark.** We can also speak of a radially symmetric distribution function. In view of copulas, a copula is radially symmetric if $(F_1(X_1) - 0.5, \ldots, F_n(X_n) - 0.5) =^d (0.5 - F_1(X_1), \ldots, 0.5 - F_n(X_n))$. Thus we have $C = \hat{C}$.

**Lemma 3.12.** Let $X = (X_1, X_2)'$ be a random vector with radially symmetric distribution $F$, then,

$$\lambda_i = \lim_{q \downarrow 0} P(X_1 < F_i^{-1}(q)|X_2 < F_2^{-1}(q)) = \lim_{q \uparrow 1} P(X_1 > F_i^{-1}(q)|X_2 > F_2^{-1}(q)) = \lambda_u.$$

**Proof.** Since $X$ is radially symmetric, we have $(X_1, X_2) =^d (-X_1, -X_2)$ for $v = (0, 0)'$. This implies that,

$$P(X_1 > F_1^{-1}(q)|X_2 > F_2^{-1}(q)) = P(X_1 < -F_1^{-1}(q)|X_2 < -F_2^{-1}(q)) = P(X_1 < F_1^{-1}(q)|X_2 < F_2^{-1}(q)).$$

Now, the fact that $\lim_{q \uparrow 1} -F_i^{-1}(q) = \lim_{q \downarrow 0} F_i^{-1}(q)$ for $i = 1, 2$ proves Lemma 3.12.

**Definition 3.13** (Exchangeability). A random vector $X = (X_1, \ldots, X_n)'$ is **exchangeable** if for any permutation $\pi(1, \ldots, n)$ of $(1, \ldots, n)$,

$$(X_1, \ldots, X_n) =^d (X_{\pi(1)}, \ldots, X_{\pi(n)}).$$

And a copula $C$ is **exchangeable** if,

$$C(u_1, \ldots, u_n) = C(u_{\pi(1)}, \ldots, u_{\pi(n)}),$$

for any permutation $\pi$ of $(1, \ldots, n)$.
Lemma 3.14. Let $X_1, X_2$ be continuous random variables with an exchangeable joint distribution function $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ for some exchangeable copula $C$ and marginals $F_1, F_2$ of $X_1, X_2$, respectively. Then,

$$C_{X_2|X_1}(u_2|u_1) = C_{X_1|X_2}(u_2|u_1),$$

with $U_i = F_i(X_i)$ and $u_i = F_i(x_i)$ for $i = 1, 2$.

Proof. We know that $C_{X_2|X_1}(u_2|u_1) = P(U_2 \leq u_2 | U_1 = u_1)$. Let $\pi$ be a permutation on $\{1, 2\}$ defined by $\pi(1) = 2$ and $\pi(2) = 1$. By the definition of conditional probability and exchangeability it holds that,

$$P(U_2 \leq u_2 | U_1 = u_1) = P(U_2 \leq u_2, U_1 = u_1) = P(U_2 \leq u_2, U_{\pi(1)} = u_1) = P(U_2 \leq u_2 | U_2 = u_1)P(U_2 = u_1).$$

Since $P(U_1 = u_1) = F_{X_1}(F_{X_1}^{-1}(u_1)) = F_{X_2}(F_{X_2}^{-1}(u_1)) = P(U_2 = u_1)$ the lemma is proven.

The following two theorems provide an expression for the tail dependence coefficients in the context of the Gaussian and the $t$-copula. Sibuya [55] was the first to prove that the bivariate Gaussian copula was asymptotically independent, that is $\lambda_u = \lambda_l = 0$.

Theorem 3.15. Let $X_1, X_2$ be random variables with continuous distribution functions $F_1, F_2$, respectively. Further, let $C^G_{\rho}$ be the Gaussian copula. Then, the upper and lower tail dependence coefficients are defined as follows,

$$\lambda = \lim_{q \downarrow 0} C^G_{\rho, X_2|X_1}(q|q) = 0,$$

Figure 3.4: Two thousand simulated points from the $t$-copula with degrees of freedom $\nu = 2$, $\nu = 4$, $\nu = 20$ and $\nu = 10^{15}$.  

\[\begin{align*}
\text{I-Copula with 0.7 correlation and 2 degrees of freedom} & \quad \text{I-Copula with 0.7 correlation and 4 degrees of freedom} \\
\text{I-Copula with 0.7 correlation and 50 degrees of freedom} & \quad \text{I-Copula with 0.7 correlation and 1e+15 degrees of freedom}
\end{align*}\]
with \( \rho < 1 \). Thus, the Gaussian copula is upper and lower tail independent, or asymptotically independent.

**Proof.** As we know, the Gaussian copula \( C_{\rho}^G \) belongs to the elliptical family and is therefore radially symmetric. By Lemma 3.12 it follows that the upper and lower tail dependence coefficient are equal, that is \( \lambda_u = \lambda_l =: \lambda \). Thus, let us continue with the expression of the lower tail dependence in Theorem 2.29. By L'Hopital’s rule and Lemma 2.7 we have,

\[
\lambda = \lim_{q \downarrow 0} \frac{dC(q, q)}{dq} = \lim_{q \downarrow 0} C_{X_2|X_1}(q|q) + \lim_{q \downarrow 0} C_{X_1|X_2}(q|q).
\]

If we now apply Lemma 3.14 and see that \( C_{\rho}^G \) is exchangeable we get,

\[
\lambda = 2 \lim_{q \downarrow 0} C_{X_2|X_1}(q|q).
\]

Let us rewrite this expression as follows,

\[
\lambda = \lim_{q \downarrow 0} C_{X_2|X_1}(q|q) = \lim_{q \downarrow 0} P_{X_2}(U_2 \leq \Phi^{-1}(q) | U_1 = q) + \lim_{q \downarrow 0} C_{X_1|X_2}(q|q) = \lim_{y \to -\infty} P(Y_2 \leq y | Y_1 = y),
\]

where \( \Phi^{-1} \) is the inverse of the standard normal distribution function. Now, \( Y_1, Y_2 \sim N(0, 1) \) and in Theorem A.3 proven that \( Y_2|Y_1 = y \sim N(\rho y, 1 - \rho^2) \), where \( \rho = \rho(Y_1, Y_2) \). Finally if \( \rho < 1 \), we have,

\[
\lambda = 2 \lim_{y \to -\infty} \Phi\left( \frac{y - \rho y}{\sqrt{1 - \rho^2}} \right) = 0.
\]

We now compare this result with the tail dependence the \( t \)-copula \( C_{\nu, \rho}^t \).

**Theorem 3.16.** Let \( X_1, X_2 \) be random variables with continuous distribution functions \( F_1, F_2 \), respectively. Further, let \( C_{\nu, \rho}^t \) be the \( t \)-copula, then the upper and lower tail dependence coefficients are defined as follows,

\[
\lambda = 2 \lim_{q \downarrow 0} C_{\nu, \rho, X_2|X_1}(q, q) = 2 t_{\nu+1} \left( \frac{\nu + 1(1 - \rho)}{1 + \rho} \right),
\]

with \( \rho > -1 \). Here, \( t_{\nu+1} \) is the standard univariate \( t \)-distribution with \( \nu + 1 \) degrees of freedom.

**Proof.** The \( t \)-copula \( C_{\nu, \rho}^t \) belongs to the elliptical family and is thus radially symmetric. By Lemma 3.12 we have \( \lambda_u = \lambda_l = \lambda \). By the same arguments as in Theorem 3.15 it holds that,

\[
\lambda = 2 \lim_{q \downarrow 0} C_{X_2|X_1}(q|q),
\]

and

\[
\lambda = 2 \lim_{y \to -\infty} P(Y_2 \leq y | Y_1 = y),
\]

It is verified in Theorem A.4 that,

\[
\left( \frac{\nu + 1}{\nu + y^2} \right)^{1/2} \frac{Y_2 - \rho y}{\sqrt{1 - \rho^2}} \bigg| Y_1 = y \sim t_{\nu+1}.
\]

If \( \rho > -1 \) we can conclude that,

\[
\lambda = 2 t_{\nu+1} \left( -\frac{(\nu + 1)(1 - \rho)}{1 + \rho} \right). \tag{3.10}
\]
Figure 3.5 shows the upper tail dependence coefficient of the t-copula plot against degrees of freedom and correlation coefficient. For low degrees of freedom, the t-copula depicts a high degree of tail dependence, which is appropriate when modeling extreme event dependence.

![Figure 3.5: Upper tail dependence coefficient of the t-copula.](image)

Hence, from Theorems 3.15, 3.16 and Figure 3.5 we can deduce that the tail dependence of the two examined copulas are quite different. The tail dependence coefficient of the t-copula is directly dependent on the correlation coefficient $\rho$ and the degree of freedom $\nu$, whereas the Gaussian copula is asymptotically independent for non-extreme situations in which the correlation coefficient $\rho < 1$.

### 3.4 Marginal Transformations

In Li’s copula-based explanation of the CreditMetrics model, marginal transformations are performed. These transformations ensure that the marginals are the default probabilities of the different financial entities. Since the purpose of Li’s model is to determine default correlations, we have to examine the effect on the dependence structure if marginals are transformed. In statistics, a multivariate distribution with disparate marginals is called a meta-distribution, see McNeil, et al. [44]. In the copula model, we construct a meta-distribution with time-until-default marginals to produce dependent default times.

In this section, we give a proper definition of the marginal transformations and prove that for multivariate elliptical distributions, we can express the meta-density function in closed form. We determine that transformations from heavy-tailed marginals to light-tailed marginals result in large contraction toward the origin for initially extreme marginal values and minor ones for initially small values.\(^5\) As a contradiction, simultaneous extremes of this meta-distribution can be found in the vertices of a $[-c,c]^n$ hypercube, for some $c > 0$. If large values are contracted in the transformations, simultaneous large values of the meta-distribution should be rare. Conversely, transformations from light-tailed marginals to heavy-tailed marginals cause abundant amplification of initially extreme marginal values and a small amplification of initially diminutive marginal values.\(^6\) Here, we find a similar contradiction: simultaneous extremes of the meta-distribution are very rare. One would expect the amplification effect of the transformations to augment the simultaneous extreme values in the meta-distribution. The extreme values of this meta-distribution are located along the axes. Through asymptotic analysis we reveal the origin of this contradiction. We find that the expression of the meta-density function includes a specific fraction of the original and the meta-marginal density function. This fraction is highly dependent on the difference in heavy-tailedness of the densities in

\(^5\)See Figure 3.6a).

\(^6\)See Figure 3.6b).
question and dominates the effect of the marginal transformation itself.

Furthermore, we argue that depending on the difference in the marginals’ heaviness in the tails, the meta-distribution creates or reduces correlation among the marginals, especially in the upper and lower quantiles. As we are interested in the lower quantiles of the meta-distribution, the analysis of this characteristic is extremely valuable to the understanding of the copula model’s estimates. We use an implied calibration approach with CDS spreads and historical default data from studies by Standard & Poor’s and Moody’s to construct default distributions that serve as the meta-marginals for the copula model. Since the tail behavior of these default distributions is directly dependent on the credit quality, we use credit ratings to categorize the heavy-tailedness of the default distributions.

In addition, we contemplate default correlations according to the meta-distribution to differentiate them from those of Li’s Gaussian model.

Through a bootstrap procedure and regression analysis, we will find that default correlations change as the deviation in tail behavior between the original marginals and the time-until-default marginals become larger. We argue that true default correlations are inversely related to credit quality and examine if both models reproduce this relationship. By repeating the bootstrap procedure with implied default probabilities model by CDS spreads, we find interesting results concerning the validity of both the Gaussian and t-copula model. A main result is that the t-copula model shows some inconsistent default correlations regarding the relationship between default correlations and credit quality. This is explained by improper specification of credit ratings by CRAs. We conclude that regarding the marginal distributions both models show severe flaws in modeling default correlations and should be used with extreme care, if at all.

3.4.1 Meta-Densities and Tail Behavior

Here, we analyze the dependence structure of two particular meta-distributions. We contemplate Gaussian and t-distributions because of the dissimilarity in their heavy-tailedness. Specifically, we examine a meta-Cauchy distribution with Gaussian marginals and a meta-Gaussian distributions with Cauchy marginals to learn that the difference in heavy-tailedness of the original and the meta-marginals is a great influence on the dependence structure of the meta-distribution. To begin our examination, let us define the marginal transformations formally.

Definition 3.17. Let \( X \) be an \( n \)-dimensional random vector with distribution function \( F \) and continuous marginals \( F_i \). Further, let \( G_i \) with \( i = 1, \ldots, n \) be continuous univariate distributions on \( \mathbb{R} \) which are strictly increasing on \( \{ y \mid 0 < G_i(y) < 1 \} \). We define marginal transformation \( K \) as follows,

\[
K(y) := (K_1(y_1), \ldots, K_n(y_n)), \text{ where } K_i(y_i) := F_i^{-1}(G_i(y_i)) \text{ and } i = 1, \ldots, n.
\]

Remark. The marginal transformations that Li performs in the copula model are the inverse functions of \( K_i(y_i) \). We used this notation for simplicity regarding Theorem 3.18.

From Definition 3.17, we observe that the marginal transformation \( K \) is a component-wise function. In Figure 3.6 the red lines display the function \( K_i^{-1} = G_i^{-1} \circ F_i \) for both cases using the Gaussian and Student t-distribution. Since in both cases,

\[
\lim_{z \to \pm \infty} G_i^{-1} \circ F_i(z) = \pm \infty,
\]

neither of the marginal transformations are bounded. This affirms our statement above: for light-tailed meta-marginals, extreme original marginal values are subject to larger contractions, depicted in Figure 3.6(a) and conversely, for heavy-tailed meta-marginals, extreme original marginal values experience greater amplifications, as illustrated in Figure 3.6(b).

Balkema, et al. analyze the transformations by considering the density functions. Under certain assumptions, the meta-distribution has a continuous density. The following theorem provides an explicit expression.

---

7The lower quantile \( q \) of the meta-distribution is the benchmark for all the simultaneous defaults before the maturity date of the obligation.

8The calibration procedure to obtain implied default distributions from CDS spreads is discussed in Appendix B.1.

9The credit ratings we utilize from Standard & Poor’s are AAA, AA, A, BBB, BB, B, CCC. Moody’s administers the following credit ratings: Aaa, Aa, A, Baa, Ba, B, Caa.

10Bear in mind that a Cauchy distribution is a t-distribution with one degree of freedom.
Theorem 3.18. Suppose the $f$ is the density function of $n$-dimensional elliptical distribution $F$ and $g_i$ are positive densities of the continuous distributions $G_i$ with $i = 1, \ldots, n$. Then, the meta-distribution has a density $\tilde{f}$ with,

$$\tilde{f}(y) = f(K(y)) \prod_{i=1}^{n} \frac{g_i(y_i)}{f_i(K_i(y_i))}. \quad (3.12)$$

It is easily verifiable that the latter factor in (3.12) is the Jacobian for the re-coordination of the transformation in Definition 3.17. The proof of Theorem 3.18 can be found in Balkema, et al. [2].

Let us first consider the term $f \circ K$ in (3.12). We may observe the form of this term in Figure 3.7b) and 3.8b) for the meta-$t$ and meta-Gaussian distribution, respectively. In the case of Figure 3.7b), the function $K(y)$ increases faster to $\infty$ than $y$ as $y \to \infty$, since the univariate normal distribution $G_i$ contains an exponential factor and hence $f \circ K$ move faster to zero than $f$ itself. For this reason, the function $f \circ K$, in Figure 3.7b), is much steeper than the function $f$, in Figure 3.7a). Moreover, we may notice that the rectangular shape is achieved by the contractions along the axes.

The opposite holds for the marginal transformation in Figure 3.8. Since the meta-marginals are now Student $t$, $K$ involves a logarithmic term. Here, the term $K(y)$ diverges much slower than $y$ as $y \to \infty$, and hence the contour lines in Figure 3.8a) widen. Also, the amplifications along the axes can be seen in Figure 3.8b).

Though the contour lines are already deformed by the transformation $K$, the Jacobian term in (3.12) gives the contour lines an even more extraordinary shape. This can be seen in Figure 3.7c) and 3.8c).

---

Keep in mind that the functions in Figure 3.6 are $K^{-1}$. Further, they are inverse functions of each other.

See Figure 3.6a) and Equation (3.11).
We can asymptotically approximate the tail of a Student $t$ density by \( \frac{\alpha}{\kappa_1(y)} (1 - H_1(z)) \) and that of a Gaussian density by \( z(1 - H_2(z)) \) for \( z \to \infty \), where $H_1$ and $H_2$ are the Student $t$ and normal distribution functions, respectively. Here, $\alpha > 0$ governs the rate of decrease for the heavy-tailed density. By (3.11), it is clear that $K_i$ for all $i$ are increasing functions. Hence, if the original marginals $F_i$ are heavier-tailed, then asymptotically we find that,

\[
\frac{g_i(y)}{f_i(K_i(y))} \approx \frac{y(1 - G_i(y))}{\alpha \kappa_1(y) (1 - F_i(K_i(y)))} = \frac{yK_i(y)}{\alpha} \to +\infty,
\]

as \( y \to \infty \), if $\alpha < \infty$. As both of the marginal values become larger the Jacobian will be the dominant term, since in this case $K_i$ diverges exponentially to $\infty$ (See Figure 3.6)). The Asymptotic result in (3.13) implies that the contour lines of the meta-density $\tilde{f}$ shift most along the $2^n$ diagonal half-lines. Hence, meta-density $\tilde{f}$ is larger on the vertices of a hypercube $[c,c]^n$ than on its edges, for $c > 0$. This may be seen in Figure 3.7c) and in the first plot in Figure 3.9.

Consequently, if the original marginals are lighter-tailed, asymptotically it follows that,

\[
\frac{g_i(y)}{f_i(K_i(y))} \approx \frac{\alpha}{yK_i(y)(1 - G_i(K_i(y)))} = \frac{\alpha}{yK_i(y)} \to 0,
\]

as \( y \to \infty \), if $\alpha < \infty$. Now the opposite is true; the Jacobian term is small for simultaneous large marginal values. Therefore, the contour lines of the meta-density $\tilde{f}$ shift excessively along the axes and thus it is larger in the middle of the edges of a hypercube $[c,c]^n$ than on its vertices, for $c > 0$. This can be observed in Figure 3.8c) and in the second plot in Figure 3.9. Additionally, these deformative effects are amplified exponentially with $n$, the dimension of the dependence problem.

To illustrate the effect of the asymptotic results above on the sample cloud of the meta-distributions, we perform a linear regression analysis. In Figure 3.9, we display both meta-distributions with correlation parameter value (CPV) of zero. The red regression line represents the least square line of the 5% quantiles of each distribution. We notice that the correlation increases and decreases in the lower quantiles depending on the tail behaviour of the marginals. Underneath the plots in Figure 3.9, the equation of the linear regression lines and the corresponding correlations are given.

In conclusion, the modeled dependence structure of the meta-distribution is highly dependent on the deviation of heavy-tailedness between the marginal distributions $F_i$ and $G_i$, especially in the upper and lower quantiles. Through asymptotic results of the transformations and the meta-marginals, we have estimated the impact of tail behavior on the dependence structure. The convergence and divergence of the Jacobian term in (3.12) is of the utmost importance, as illustrated in Figure 3.9. More theoretical results regarding dependence structures of meta-distributions can be found in Balkema, et al. [2]. Further, an important discussion established in the analysis of Balkema, et al. [2] questions the preservation of asymptotic dependence of the original joint distribution in the meta-distribution. For the meta-$t$ distribution, it seems that though the copula of the original multivariate $t$-distribution is preserved by the marginal transformations, its tail dependence vanishes in the first order asymptotics of the meta-distribution. Moreover, it is stated that the reasons for the effect of the marginal transformations on the asymptotic dependence are unclear.
Figure 3.9: Meta-Cauchy distribution with Gaussian marginals and meta-Gaussian with Cauchy marginals. Each with correlation parameter zero.

The colored dots belong to the 5% quantiles of the meta-distribution and the linear regression lines are given for the entire distribution (Blue) and the 5% quantiles (Red).

3.4.2 Default Correlation and Tail Behavior of the Default Distributions

Here, we apply the results of Section 3.4.1 into Li’s copula model for default correlations. We examine the influence of the marginal transformations on the dependence structure through different default probability distributions. We use two estimates for these default distributions, a market implied default distribution modeled by CDS spreads\(^{13}\) and historical default rates by Standard & Poor’s [47] and Moody’s [6] to construct survival distributions \(S_i(t)\) for various ratings, according to (3.2). The default distributions are then defined as \(G_i(t) := 1 - S_i(t)\).

Let us first compare the heavy-tailedness of the default distributions with the standard Gaussian and Student \(t\) distribution. In Figure 3.10 we notice that for all default data sets, the default densities are heavier-tailed than the standard Gaussian density. To deduce the tail behavior between the default densities and the Student \(t\), we take the limit to infinity with respect to time. Since hazard rates are arduous to model for extensive time periods, we estimate the tail of the default distribution by an exponential density function with parameter \(\beta_i\), thus assuming a constant hazard rate for each financial entity \(i\). The heaviness of the exponential distribution’s tail is governed by the parameter \(\beta_i\), and we have, therefore, an inverse relationship between credit quality of entity \(i\) and the parameter \(\beta_i\). The parameter \(\nu\) regulates the tails of the Student \(t\) distribution and thus the tail behavior between estimates of the default distribution and the Student \(t\) distribution depends on the parameters \(\beta_i\) and \(\nu\). The tail of the Student \(t\) density will eventually exceed the exponential density since it is a polynomial with powers dependent on \(\nu\), whereas the exponential density contains an exponential factor.

Furthermore, we may observe the difference in heavy-tailedness of the modeled default densities in Figure 3.10. The dissimilarity between densities based on data from the CRAs can be explained by the time periods from which the data was taken; Standard & Poor’s default rates were estimated upon historical default data from 2009, and Moody’s dates back to 2007 when the credit crisis heavily impacted the historical default data. The implied default densities based on the average CDS spreads seem to be more pessimistic about survival probabilities than the CRAs\(^{1}\. We used average spreads of nearly 3500 CDSs categorized by Standard & Poor’s ratings of the reference entity to mitigate influences such as limited liquidity on single CDSs and other risk factors. Unfortunately, speculation is a large driver of CDS prices and the large deviations of the implied default densities from the historical default densities might partially be due to speculation in the market.

Let us now analyze the Jacobian term in (3.12) with exponential meta-marginals as estimates for the time-until-default marginals.

\(^{13}\) The specifications of the CDS data sets can be found in Appendix C.
Figure 3.10: Univariate densities of default distributions from Standard & Poor’s [47], Moody’s [6] and Implied Default Densities calibrated with CDS Spreads from Markit.

Considering the meta-\(t\) distribution created by the \(t\)-copula model, the asymptotic behavior of the Jacobian term in the multivariate meta-density function \(\tilde{f}\) is as follows,

\[
\frac{g_i(y)}{f_i(K_i(y))} \approx \frac{\beta_i e^{-\beta_i y}}{K_i(y)\left(1 - F_i(K_i(y))\right)} = \frac{\beta_i K_i(y)}{\alpha} \to +\infty, 
\]

as \(y \to \infty\), if \(\alpha < \infty\) and \(\beta_i > 0\). Hence, the \(t\)-copula model displays an amplification effect on the positive diagonal half-line; this is illustrated in Figure 3.11c.

Figure 3.11: Contour lines of a bivariate Student \(t\) density \(f\) with one degree of freedom, the function \(f \circ K\) and a meta-\(t\) density \(\tilde{f}\) with Time-until-Default marginals of Caa-rated entities based on ratings from Moody’s [6].

Now for the meta-Gaussian density constructed by the Gaussian copula model, the asymptotic behavior of the Jacobian term is as follows,

\[
\frac{g_i(y)}{f_i(K_i(y))} \approx \frac{\beta_i e^{-\beta_i y}}{K_i(y)\left(1 - F_i(K_i(y))\right)} = \frac{\beta_i}{K_i(y)} \to 0, 
\]

(3.16)
as \( y \to \infty \), if \( \beta_i > 0 \). Thus, the Gaussian copula model shows a contraction effect for initially larger values. In Figure 3.12), we observe that the Jacobian term shifts the contour lines towards the origin and generates a small amplification along the axes as we have already seen in Section 3.4.1.

Figure 3.12: Contour lines of a bivariate Gaussian density \( f \) with correlation zero, the function \( f \circ K \) and a meta-Gaussian density \( \tilde{f} \) with Time-until-Default marginals of Caa-rated entities based on ratings from Moody’s [6].

This is a fundamental difference between the Gaussian and the \( t \)-copula models. It may also be recognized in the regression analysis in Figure 3.14. The amplification effect of the \( t \)-copula model provides additional dependence between the random time-until-default variables. We recognize this in the third image of Figure 3.11 the contour lines are stretched out towards the upper right corner along the positive diagonal half-line. The extra dependence is governed by the parameter \( \alpha \), which is equal to the degrees of freedom of the \( t \)-copula.

Another source of dependence in the meta-distributions is the heavy-tailedness of the default distributions illustrated in Figure 3.10. The default distribution of an entity with better credit quality has heavier tails, implying a smaller \( \beta_i \) for the estimation and slower convergence of \( G_i(y) \approx 1 - e^{-\beta_i y} \) to one. Hence, for high credit quality the term \( g_i f_i \circ K_i \) converges or diverges much slower to zero and \( \infty \) for the Gaussian and \( t \)-copula model, respectively, since here \( K_i = F_i^{-1}(1 - e^{-\beta_i y}) \).

Figure 3.13: Illustration of the bivariate model for highly dependent asset value processes \( S_{t}^{CCC} \) and \( S_{t}^{A} \) (Red) with default threshold \( d \) and the distribution of \( S_{t}^{A} | S_{t}^{CCC} = s^{CCC} \) (green).
For the $t$-copula model, slower divergence of the Jacobian term restricts the amplification effect on the diagonal half-line and thus inhibits the creation of additional dependence caused by the meta-$t$ distribution. For the Gaussian model, slower convergence of the Jacobian term to zero results in less declivity of the contour lines of $f$ and generates a contraction for simultaneous large values as in Section 3.4.1 and diminishes the dependence embedded by the term $f \circ K$. This poses the question: Should, in practice, the default correlation between a high rated entity and a low rated entity be smaller than the default correlation between two low rated entities? Zhou [63] argues that the answer is yes. Intuitively, one can argue that for A-rated and CCC-rated financial entities with similar default thresholds $d$ and almost perfect positive asset correlation $\rho \approx 1$, the default of the CCC-rated entity caused by the asset value crossing the default threshold does not cause the A-rated entity to default at the same time. In terms of the Gaussian copula model, this is to say that if $S_{A}^{t}$ and $S_{CCC}^{t}$ are the asset values processes of the A-rated and CCC-rated entities, respectively, and $S_{A}^{t} | S_{CCC}^{t} = s_{CCC} < d$, then $S_{A}^{t} | S_{CCC}^{t} = s_{CCC} \sim N(\rho s_{CCC}, 1 - \rho^2)$ where $\rho$ is the asset correlation between the entities. If we have a large $\rho$, the variance of $S_{A}^{t} | S_{CCC}^{t} = s_{CCC}$ will be very small and thus $P(S_{A}^{t} < d | S_{CCC}^{t} = s_{CCC})$ is approximately zero. Figure 3.13 illustrates this argument.

Other reasons for the inverse relationship between credit quality and default correlations might be that for highly rated entities, default rates are not only rare but typically company-specific. As these are isolated incidents, company-specific defaults generally do not drive default correlations. On the other hand, defaults of low-rated financial entities are dependent on macroeconomic shocks. If the economy suffers under a downturn, low-rated entities will be more likely to default, resulting in higher default correlation. Therefore, taking the entire business cycle into account, default correlation and credit ratings should, in practice, be inversely related.

Figure 3.14: 500,000 simulated points from the Gaussian copula model and the $t$-copula model with correlation parameter 0.6 for financial entities with distinct credit ratings, based on default rates from Standard & Poor's [47]. Only the simultaneous default points are shown for a five year period.

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Note that in the reasoning the problem discussed in Section 3.2 are ignored.

For a proof see Theorem A.3 in Appendix A.
To illustrate the copula model’s change in dependence for different credit ratings, we have generated one random sample each from the bivariate Gaussian and bivariate \( t \) distribution and constructed two distinct meta-distributions according to each of the two copula models. We have used the same underlying sample from the Gaussian and \( t \)-copula to eliminate the effect of randomness and to solely exhibit the effect of the marginal transformations. The samples are depicted in Figure 3.14 and the results of a regression analysis underneath each plot exemplify the change in default correlation for each model. With a CPV of \( \rho = 0.6 \), the default correlation according to the Gaussian copula and \( t \)-copula model increases from 0.138 to 0.192, and from 0.144 to 0.403, respectively, as the rating decreases from \( A \) to \( B \). Thus, according to both models, default correlations and credit quality are inversely related. Notice the larger increase in default correlation for the \( t \)-copula model than for the Gaussian model. We return to this characteristic later in this section.

To further analyze the sensitivity of the models to the dissimilarity in underlying default distribution, we bootstrapped default correlations according to both models with different correlation parameters and hazard rates from the CRAs and average CDS spreads. The means of the bootstrap samples are delineated in Figure 3.15 to 3.18. Let us first discuss the results of the bootstrap based on the CRA’s data which we see in Figure 3.15 and 3.16. The Gaussian copula model seems not to exhibit any changes in default correlations as the underlying credit ratings change. We notice that for low and medium CPVs, the lines in Figure 3.15 and 3.16 are almost completely vertical. Only for extremely large CPV of 0.95 the Gaussian copula model shows an inverse relationship between default correlations and credit quality.

Homogeneous default correlations for all default rates infer that there is a exclusive relationship between the CPV and default correlation. Hence, the Gaussian copula mode is effectively a one-parameter model for default correlations where only asset correlations determine default correlations. Apart from the weakness of zero tail dependence discussed in Section 3.3, the inability to distinguish between various ratings of credit quality is a major deficiency of the Gaussian copula model.

For the \( t \)-copula model, declines of default correlations at specific rating reductions are noticeable. For instance in Figure 3.15, the \( t \)-copula model with CPV of 0.95 consistently produces shrinking default correlations for the investment grade ratings. Utilizing Standard & Poor’s estimates for default rates, the default correlations for investment grade ratings are approximately equal. Collectively, the
The model's default correlations for high ratings seem to be inconsistent with the other default correlation values. This would suggest that according to the model a high rated, say AA-rated, bank underwriting a CDS contract on low-rated companies, say CCC-rated, should generally charge less or approximately the same than for instance BBB-rated banks. Intuitively, this is nonsense since in most cases a guarantee of payment by a higher-rated bank on any reference entity should be more expensive than that of a lower-rated bank. There are three possible reasons for these unrealistic inconsistencies: the model incorporates the inconsistencies in the default correlations and shows a serious drawback, the underlying default probabilities are incorrectly estimated or the labeling of the CRA is inappropriate.

In the Section 3.4.1 and the asymptotic analysis of the Jacobian term in (3.15), we established that default dependence and heaviness of the meta-marginal tails are inversely related. This indicates that the model can not be the cause of the "correlation smile" in Figure 3.15 and 3.16.

The second potential cause for the smile in the default correlations of the model is a flawed estimation of the default distributions' tail. For this we briefly describe the estimation of the historical default rates by Standard & Poor's. The estimations are based on a 29 year period and the entire Standard & Poor's rating portfolio since 1980. Static pools of ratings are created for 15-year periods and in those periods two probabilities for default of the entities in each pool are taken into account: instantaneous default probabilities and credit migrations. The pools are static in the sense that their membership remains constant over the 15-year period. One year default rates are estimated through a bootstrap procedure, and for subsequent years the conditional default probabilities are estimated. That is for the second year, the probability of default conditional on survival in the first year are computed; cumulative default rates are calculated. A closer look reveals that the default thresholds for the probability estimations are doubtful. This is a reason to distrust the estimations for the default rates of investment grade ratings.

![Figure 3.16: Means of 25 bootstrapped Default Correlations from both models (sample size 500,000) for different correlation parameters based on the default rates from Standard & Poor's.](image)

In Standard & Poor's, default is defined as follows:

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\[16\] We do not refer to the correlation smile of implied tranche correlations of CDOs. Our default correlations are not implied correlations.
Definition 3.19 (Default). An entity defaults once the first occurrence of a payment default on any financial obligation is observed, a selective default on a specific obligation occurs or the obligor is under regulatory supervision.

This definition incorporates a crucial inconsistency. Entities that pose systemic risk and are rescued from default by either the government or another entity, do not fall under the last specification of a default. For instance, under the definition of Standard & Poor’s, AIG has not defaulted, it has only been downgraded. Other examples are the Royal Bank of Scotland and Lloyds TSB. Since exclusively high-rated entities have been saved from default, the default rates for those ratings are underestimated. Though this is a flaw in Standard & Poor’s estimation procedure, it does not cause the inconsistent default correlations of the $t$-copula model since the phenomena repeats itself using different estimates for default distributions. In Figure 3.17 and 3.18 we bootstrap default correlations based on implied default probabilities from CDS spreads and an even stronger correlation smile can be found for the $t$-copula model.

The third possible cause is the CRA’s rating system used to specify credit quality. Studies (e.g. Partnoy [52]) have shown that at times CRAs are under a lot of pressure to award financial entities investment grade ratings. Partnoy [52] asserts that for major CRAs, a conflict of interest influences the credit rating and results in corruption of its credit rating procedures. Moreover, the fact that CRAs are paid directly by the issuer of the rated entity or product, for instance the (investment) bank which is responsible for selling CDO tranches to investors, compels the agencies to incorrectly process harmful data. Earlier this year, evidence for these accusations was found by Segal [57] and McCool [42]. Standard & Poor’s and Moody’s are involved in lawsuits accusing the CRAs for wrongly ascribing investment grade ratings. As early as 2004, suspicion of inappropriately high ratings assigned by Moody’s was already confirmed by the Washington Post (see Klein [32]). Hence, the historical default probabilities for high ratings, i.e. AAA to BBB, may be misleading, as it is possible that defaulted entities included as investment grades in the CRAs’ estimation of historical default rates were actually of lower credit quality. This would lead to a collective overestimation of all default correlations for investment grade ratings and might be the cause of the irregularly high default correlations as displayed by the $t$-copula model in Figure 3.15 and 3.16. This would infer that the $t$-copula model reproduces reality better than the Gaussian copula model since it is more sensitive to changes in the underlying default rates.

<table>
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<th>A</th>
<th>Baa</th>
<th>B</th>
<th>Ba</th>
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Table 3.1: Standard Deviations of 25 bootstrapped Default Correlations from both models for different correlation parameters based on the default rates from Moody’s.

To elaborate on this fact, we consider an early empirical study in 1995 of default rates and default correlations by Lucas [40]. Lucas [40] determined default correlations empirically by focusing on all non-municipal issuers rated by Moody’s over a 23-year period. Though his estimation procedure is slightly different from ours, Lucas [40] finds almost identical results to the ones of the $t$-copula model. Default correlations between investment grade and lower-rated entities are positively related to credit quality while correlations between low-rated entities are inversely related to credit quality. However, the positive relationship between investment grade ratings and default correlations in Lucas [40] is rather small and comparable to the relationship in Figure 3.16. Unfortunately, Lucas [40] does not provide an explanation of this phenomenon. Since Lucas [40] used historical default rates categorized

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17This means that the entity purposely stalled payments on a specific obligation, but continues to pay other obligations.

18K. Ahern (Analyst at Standard & Poor’s), email communication 14 July, <kevin.ahern@standardandpoors.com>, 2010.
by Moody’s credit ratings, he might have incorporated the same bias of investment grade default rates in his empirical study.

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<th>BBB</th>
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<td>1.03%</td>
<td>0.64%</td>
<td>0.46%</td>
<td>0.52%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>3.60%</td>
<td>1.98%</td>
<td>0.79%</td>
<td>0.48%</td>
<td>0.34%</td>
<td>0.24%</td>
</tr>
<tr>
<td>$t$-Copula</td>
<td>0.32</td>
<td>5.05%</td>
<td>2.13%</td>
<td>2.26%</td>
<td>0.32%</td>
<td>0.48%</td>
<td>0.48%</td>
</tr>
<tr>
<td>Model</td>
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<td>4.19%</td>
<td>2.08%</td>
<td>0.80%</td>
<td>0.35%</td>
<td>0.24%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>16.36%</td>
<td>7.87%</td>
<td>6.13%</td>
<td>1.37%</td>
<td>0.61%</td>
<td>0.17%</td>
</tr>
</tbody>
</table>

Table 3.2: Standard Deviations of 25 bootstrapped Default Correlations from both models for different correlation parameters based on the default rates from Standard & Poor’s [17].

In Table 3.1 and 3.2 the standard deviations of the bootstrap samples can be found. Values for ratings Aa and A and high CPV, i.e. 0.63 and 0.95 in Table 3.1 are significantly larger than the values in Table 3.2, making the correlation smile in Figure 3.15 less pronounced. In Figure 3.16 the default correlations for investment grades are approximately equal. As a result, protection bought from entities rated inside the investment grade spectrum is equivalent. This suggests that the specifications of ratings within the investment grade ratings are highly equivocal and that the true credit quality within these ratings are equivalent. The joint overestimation of default correlations for investment grade ratings due to inappropriate labeling of ratings and the ambiguity in specifications depicted by equal default correlations seriously undermine both the practical value and credibility of the CRA’s investment grade ratings.

![Figure 3.17](image)

Figure 3.17: Means of 150 bootstrapped Default Correlations from both models (sample size 500,000) for different correlation parameters based on implied default rates from average CDS spreads categorized by ratings of Standard & Poor’s.

Source CDS data: Markit 18-12-2009.

Furthermore, we notice that for high ratings and extreme CPVs, i.e. $\rho = 0$ and $\rho = 0.95$, the standard deviations are irregularly high. Since the hazard rates are identical for the modeling of
each bootstrap value, the standard deviation must be the result of the underlying randomness of the bivariate distributions. For the \( t \)-copula model the larger standard deviations can be found for the default correlations of high credit ratings and high CPVs; the largest being 20.86% for the default correlation between Moody’s Aa and Caa-rating and CPV of 0.95. Larger standard deviation of the \( t \)-copula model can be explained by the fact that the underlying variances of the \( t \) distribution are governed by the parameter \( \nu \), whereas in the Gaussian case we have unit variances. For the Gaussian copula model, the extreme standard deviations can be found for low CPVs and high ratings; 13.65% being the highest standard deviation for CPV of zero and Moody’s Aa-rating. We may notice that abnormally large standard deviations are found for the default correlations that are not consistent with the statement that default correlations and credit ratings are conversely related. The standard deviations shed more light on the inconsistent default correlation of high ratings for the \( t \)-copula model.

**Figure 3.18:** Means of 150 bootstrapped Default Correlations from both models (sample size 500,000) for different correlation parameters based on implied default rates from average CDS spreads categorized by ratings of Standard & Poor’s. Source CDS data: Markit 19-03-2010.

As Li’s copula model uses implied default rates from CDS spreads, let us examine the default correlation based on these market implied default rates. We have utilized one, three, five, seven and 10 year CDS spreads on nearly 3500 reference entities to estimate average implied default rates for each credit rating by their spreads.\(^{19}\) The result of the bootstrap procedures with the implied default rates are depicted in Figure 3.17 and 3.18. The corresponding standard deviations of the bootstrap samples can be found in Table 3.3 and 3.4. The results for the Gaussian copula model are nearly the same. For medium CPVs, the model shows a slight increase in default correlations from investment grade to junk grade ratings. For the \( t \)-copula model, the implied default distributions reveal the same correlation smile in default correlations as the CRAs’ default rates. However, the correlation smile is more explicit for the implied approach to default distributions. In Figure 3.17 and 3.18 a clear correlation smile across all CPVs is visible. As we are still using Standard & Poor’s credit rating system, we inherit the

\(^{19}\)The estimation procedure is laid out in Appendix B.1.
labeling of credit quality to the reference entities. According to Fama’s efficient-market hypothesis (EMH), the implied default distributions resemble the true underlying credit quality of the reference entity. Let us consider a financial entity i with low credit quality, that has been assigned an investment grade rating by a CRA. By the inverse relationship between default correlation and the entity’s credit quality, the true default correlation according to the efficient market must be greater than that using the CRA’s investment grade credit rating assigned to entity i. Furthermore, the implied default distribution must be lighter-tailed than the default rates of the CRA’s assigned investment grade rating since entity i’s true credit quality is worse than the investment grade rating and lighter-tailed default distributions assign smaller survival probabilities to the underlying entity. Conforming to the results in the asymptotic analysis of the multivariate meta-\( t \) density with exponential marginals in (3.15), lighter-tailed meta-marginals create more dependence in the meta-\( t \) distribution. Furthermore, in both Figure 3.17 and 3.18 the default correlations for investment grade ratings are nearly equal, confirming the equivalence of the credit quality in each of the investment grade ratings.

What is striking about the \( t \)-copula model, however, are the negative signs of certain default correlations in Figure 3.17 and 3.18. The CPV and the default correlation should have equal signs since asset correlations are utilized as CPVs in both copula models. This is a clear weakness of the \( t \)-copula model. A reason for this could be the implied default distributions’ heavy-tailedness of investment grade ratings. The regression analysis in Figure 3.9 shows that even though the correlation parameter is zero, a regression line of a quantile could still have negative slope. The Gaussian copula model does not suffer from this flaw and, as such, exhibits more stability regarding the relationship between the CPV, or asset correlation, and the default correlation than the \( t \)-copula model.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \rho )</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
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<td>1.02%</td>
<td>0.79%</td>
<td>0.59%</td>
<td>0.40%</td>
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<tr>
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<td>0.58%</td>
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<td>0.29%</td>
</tr>
<tr>
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<td>1.45%</td>
<td>0.98%</td>
<td>0.55%</td>
<td>0.37%</td>
<td>0.32%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>28.31%</td>
<td>32.93%</td>
<td>7.13%</td>
<td>0.94%</td>
<td>0.32%</td>
<td>0.13%</td>
</tr>
<tr>
<td>( t )-Copula</td>
<td>0</td>
<td>2.19%</td>
<td>2.18%</td>
<td>1.32%</td>
<td>0.37%</td>
<td>0.38%</td>
<td>0.27%</td>
</tr>
<tr>
<td>Model</td>
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<td>2.44%</td>
<td>2.47%</td>
<td>1.51%</td>
<td>0.69%</td>
<td>0.38%</td>
<td>0.26%</td>
</tr>
<tr>
<td></td>
<td>0.63</td>
<td>3.10%</td>
<td>3.13%</td>
<td>1.96%</td>
<td>0.69%</td>
<td>0.39%</td>
<td>0.24%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>6.85%</td>
<td>6.78%</td>
<td>4.58%</td>
<td>1.37%</td>
<td>0.59%</td>
<td>0.12%</td>
</tr>
</tbody>
</table>

Table 3.3: Standard Deviations of 150 bootstrapped Default Correlations from both models for different correlation parameters based on implied default rates from average CDS spreads categorized by ratings of Standard & Poor’s.

Source CDS data: Markit 18-12-2009.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \rho )</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
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<td>1.23%</td>
<td>1.14%</td>
<td>0.86%</td>
<td>0.48%</td>
<td>0.44%</td>
<td>0.32%</td>
</tr>
<tr>
<td>Model</td>
<td>0.32</td>
<td>1.21%</td>
<td>1.08%</td>
<td>0.79%</td>
<td>0.65%</td>
<td>0.44%</td>
<td>0.31%</td>
</tr>
<tr>
<td></td>
<td>0.63</td>
<td>1.51%</td>
<td>1.49%</td>
<td>1.01%</td>
<td>0.62%</td>
<td>0.38%</td>
<td>0.30%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>NaN%</td>
<td>35.99%</td>
<td>9.22%</td>
<td>1.12%</td>
<td>0.38%</td>
<td>0.14%</td>
</tr>
<tr>
<td>( t )-Copula</td>
<td>0</td>
<td>2.88%</td>
<td>2.52%</td>
<td>1.59%</td>
<td>0.74%</td>
<td>0.38%</td>
<td>0.26%</td>
</tr>
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<td>0.41%</td>
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</tr>
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<td>3.57%</td>
<td>3.38%</td>
<td>1.95%</td>
<td>0.90%</td>
<td>0.45%</td>
<td>0.28%</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>9.11%</td>
<td>7.79%</td>
<td>4.45%</td>
<td>1.59%</td>
<td>0.59%</td>
<td>0.14%</td>
</tr>
</tbody>
</table>

Table 3.4: Standard Deviations of 150 bootstrapped Default Correlations from both models for different correlation parameters based on implied default rates from average CDS spreads categorized by ratings of Standard & Poor’s.

Source CDS data: Markit 19-03-2010.

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20 In short, the semi-strong EMH states that the market incorporates all publicly available information and that new information is immediately incorporated in the market prices. More about the other two versions of the EMH and the theory behind the EMH can be found in Fama [18] and Fama [19].

21 The default correlation using the CRA’s credit rating mentioned here is the default correlation based on the ‘theoretical’ default distributions according to the CRA’s criteria of that particular rating.
In this section, we discussed the marginal transformations that Li undertakes in his copula model. Through analysis of the separate terms of the multivariate density $f$ of the meta-distribution, we established a grave distinction between Li’s Gaussian copula model and the $t$-copula model. For the latter model, the marginal transformations bring forth another element of dependence, whereas for the former the transformations reduce the tail dependence in the model. Regression analysis of random samples and a bootstrap procedure confirm the results produced by the analysis. Regarding the $t$-copula model, an inconsistency in the relationship between default rates and default correlations is discovered. Explanations for this are, on the one hand, a possible bias in the upper credit ratings utilized to categorize credit quality and on the other hand, unclear and hazy specifications of ratings in the investment grade spectrum. This seriously undermines the usefulness of the CRA’s rating system and may have severe consequences, such as misspecification of the risk exposure of pension funds. By law, pension funds are only allowed to deal in the investment grade spectrum without insurance. For junk grade products, a pension fund has to buy protection from an investment grade entity on this product. According to our analysis, protection bought from entities with investment grade ratings may have hidden default risk in higher default correlations with junk grades, and thus unknowingly increase the risk exposure. Furthermore, we determined that the default correlations modeled by the Gaussian copula do generally not distinguish between credit ratings; this is a major flaw of the Gaussian copula model. This results in an exclusive determination of default correlations by the CPV, or asset correlation. The bootstrap procedure based on the implied default distributions reveals that at times the $t$-copula model with positive CPVs produces negative default correlations. This characteristic is unattractive for default correlation modeling and indicates that the $t$-copula suffers from the effects discussed in the regression analysis of Figure 3.9. These deficiencies indicate that both models should be applied with care. Further engagement in analysis of the relationship between default correlations and the dissimilarity of the entities’ credit ratings is needed. The true nature of this change in default correlation cannot be analyzed easily due to discrepancies in the credit ratings used to categorized default distributions. An empirical investigation of the validity and proportions of major CRA’s credit ratings may give us a clear answer to conduct a more conclusive analysis of default correlations.
Chapter 4

Conclusion and Discussion

In this chapter, we present our conclusion and final discussion about the Gaussian copula model. In Section 4.1, we provide a summary of the Gaussian copula model and its analysis in Chapter 3. Furthermore, we state numerical results of modeled CDO tranche spreads based on real market data in Section 4.2. We utilize the five-year i-Traxx Europe CDS index as the reference pool of a synthetic CDO containing six tranches. We conclude that the characteristics of the Gaussian copula model examined in Chapter 3 are directly accountable for the behavior in tranche spreads of the different models. Our main result is that the spread for the Super Senior tranche turns out to be cheaper and the Equity tranche’s spread more expensive according to the Gaussian model than for the t-copula model. This is due to the difference in numbers of defaults within the five-year period produced by the two copula models. The t-copula model generates extreme numbers of defaults; either very few or excessively many defaults within the five-year period whereas the Gaussian copula model creates almost no extreme default numbers. The heavy-tailedness and the difference between tail behavior of the copula and the default marginals are the cause of more extreme numbers of default in the t-copula model. Interestingly, on average, the Gaussian copula model generates more defaults within the five-year period than the t-copula model. As a result, the Gaussian produces higher spreads for most tranches, i.e. junior and mezzanine tranches. This link between the numerical results and the discussion in Chapter 3 will enable us to fully understand the criticisms on the Gaussian copula model stated in Chapter 4.

We conclude this chapter with a discussion about regulating credit derivatives like CDSs, synthetic CDOs, kth to default contracts, etcetera. A different regulation approach would level the purpose of these products in reality and mathematical pricing models. It would reduce the purpose of, for example credit default swaps, to solely insurance related matters.

4.1 A Critical View on Dependence Modeling with Copulas: Summary

Throughout this paper, we discussed multivariate dependence and the stochastic dependence modeling approach through copulas utilized by the CreditMetrics model. In particular, we focused on the Gaussian copula model for default correlation in Li [37]. We introduce the structure behind some of the financial products where the risk involved is influenced by the default dependence between two or more financial entities, one of these products is a collateralized debt obligation, or CDO. In Chapter 2, we discuss the mathematical tools used to model default correlations in the CreditMetrics model. These tools originated from copula theory and the theory of stochastic dependence. In Chapter 3, we used the previously acquired knowledge about stochastic dependence to examine the Gaussian copula model for default correlations. This model is a parametric model with two input parameters: asset correlations and hazard rates. Through the Gaussian copula, a dependence structure between random variables $T_1, \ldots, T_n$, called time-until-default, is created. Three main problems of this model are:

1. Linear asset correlation is the exclusive input parameter for dependence in the model, thus assuming that asset correlation is a sufficient driver of simultaneous defaults (this practice arises}

\footnote{The simulation procedure is depicted in Appendix B.2.}
as a result of the almost complete absence of relevant data),

2. The assumption that the distribution of simultaneous defaults can be approximated by a Gaussian distribution (this results in an oversimplified parametric model),

3. The effect of the marginal transformations on the dependence structure of the Gaussian distribution. Here, undesirable imprints on the dependence structure by the transformations lead to equal default correlations for different marginal transformations (this implies that different credit qualities of the financial entities have the same impact on the value of the default correlations).

Let us explain and discuss the model’s deficiencies mentioned above. In order to calculate the asset correlations, the CreditMetrics model uses Merton’s option pricing approach to model asset values and computes the linear correlation between the normalized asset values. We found that using asset correlation as base for default correlation creates some deficiencies. In Example 3.7, we illustrated a scenario where zero asset correlation still results in positive default correlations, concluding that the following strict inclusion holds,

\[ \{V_i^t < F_i^t, V_j^t < F_j^t\} \subset D_i(t) \cap D_j(t), \]

where \( D_i(t) \) and \( D_j(t) \) are comprehensive default events of entity \( i \) and \( j \), respectively, and \( F_k^t \) default thresholds for the asset value processes \( V_k^t \) with \( k = i, j \).

Moreover, asset correlations are calculated using the linear correlation coefficient. We found that for a multivariate Gaussian scenario, linear correlation is the best choice since in this case, it satisfies many intuitively reasonable characteristics. For a non-elliptical scenario, however, the linear correlation cannot specify the entire dependence structure of the underlying data. We show this using two fallacies about linear correlation and elliptical distributions. The first fallacy states that marginals and correlation coefficients specify the joint distribution. In a non-elliptical scenario, there are infinitely many different distribution with identical marginals and correlation coefficients. The second fallacy asserts that for given marginals, all linear correlation values in \([-1, 1]\) can be obtained by certain specifications of the joint distribution. Hence, for non-elliptical joint distributions with \( \rho \neq \pm 1 \), marginals might still be counter- or comonotonic, implying that the linear correlation coefficient underestimates the correlation of the underlying data.

The second inadequacy of the Gaussian copula model is the assumption of the Gaussian dependence structure. Since dependence is an infinite dimensional concept, the only adequate models must be non-parametric. Using parametric models to capture the dependence is accompanied by grave assumptions on the underlying data and enormously simplifies the problem of assessing dependence. In our case, modeling dependence with the Gaussian copula may instill an illusion of understanding very complex stochastic dependence. It offers only a very simple (one parameter) model of an extremely convoluted reality of dependent defaults. An improvement would be the addition of further parameters, or covariates, to model more aspects of this complex default dependence between the financial entities. Examples for this include the BB family of copulas in Joe or the Gumbel and Clayton copula. More about the use of Archimedean copulas for risk management can be found in McNeil, et al. We want to stress that this possible improvement does in no way perfect the model, because these copula models continue to be parametric approximations to a non-parametric concept.

Another reason why the Gaussian copula model does not reflect the entire dependence structure realistically can be observed by examining the correlation matrix \( \Sigma \). Recall that multivariate Gaussian distributions are fully determined by the expectation vector \( \mu \) and the correlation matrix \( \Sigma \) and exhibit only pairwise correlations as measure of dependence. Moreover, the Gaussian copula does not display any dependence for simultaneous extreme events like defaults. Rare simultaneous extreme events underestimate the default correlation. This can also be seen in our comparison of the Gaussian with the \( t \)-copula in Section 4.3.2. Thus, a copula showing positive tail dependence, as the \( t \)-copula, or a model incorporating a higher-order dependence measure would be needed to encompass the entire default dependence among financial entities. This is a central point in pricing CDOs, considering they consist of many mortgages or other loans. The incorporation of such a dependence measure in a CDO

\[^2\text{See Merton [46].}\]
pricing model would lead to higher spreads for senior tranches and lower spreads for equity tranches, as seen in Figure 1.5.

Last not least, we analyzed the impact of the marginal transformations on the dependence structure and ascertain further crucial flaws of the Gaussian copula that uncover its true simplicity. As mentioned by Li [37], default correlations turn out to be consistently smaller than the asset correlations according to the Gaussian copula model. In practice, this might not always be true since we have found other default drivers. In Example 3.7, we showed a scenario where two uncorrelated entities share the same insurance company, and hence, show default correlation through the third party. Furthermore, through a bootstrap procedure, we find that default correlations for the Gaussian copula model are static with respect to changes in credit quality of the reference entities. That is to say that default correlations only change if the asset correlation changes. This leads to the conclusion that, though the Gaussian copula model has two input parameters (i.e. correlation and hazard rates), it embodies in effect only a one-parameter model. In consequence, assuming a Gaussian dependence structure for defaults effectively results in an one-to-one relationship between asset and default correlation. We argued that default correlations and credit quality are inversely related in practice. Since the Gaussian copula does not reflect this, we conclude that this extreme simplification of default dependence reveals the true impropriety of using the Gaussian copula model. Moreover, the $t$-copula model exhibits the effect more truly, because it yields higher default correlations for lighter-tailed meta-marginals. This characteristic makes the $t$-copula model more attractive to reproduce default correlations. However, the $t$-copula model also remains a parametric model for default dependence.

In addition, we discover an interesting result for the default correlations between two entities with different credit rating modeled by the $t$-copula model. A bootstrap procedure in Section 3.4.2 shows that default correlations between a low-rated entity and entities with an investment grade rating are approximately equal. This is an inconsistency in the relationship between credit rating and default correlations for the $t$-copula model since for default correlations between a CCC-rated entity and junk rated entities are increasing as the rating of the junk rated entity decreases. Since the $t$-copula model shows an inverse relation between default correlations and the heavy-tailedness of the underlying default distributions, we argue that reasons for the inconsistencies are a possible bias in the upper credit ratings of credit rating agencies and/or ambiguous definitions/criteria for ratings in the investment grade spectrum. This seriously undermines the practical value and credibility of the rating agencies’ rating systems.

4.2 CDO Pricing with Li’s Gaussian Copula Model: Numerical Results

As Li’s Gaussian copula model has been used by many major players in the credit derivative and ABS market to model default correlations and value CDO tranche spreads, we state numerical results of the CDO pricing through the Gaussian and the $t$-copula model in this section. We examined and discussed the main practical and theoretical deficiencies of the Gaussian copula model, and here, we display the effects of these deficiencies on CDO tranche spreads. We illustrate the numerical calculation of synthetic CDO tranche spreads based on real market data using Li’s Gaussian copula model. We explain that results about the tail dependence coefficients and the effect of the tail behavior of the original and the meta-marginals on the dependence structure in the copula models in Section 3.3 and 3.4.2, respectively, are directly linked to the values of the modeled tranche spreads. Further, we find that the tranche spreads modeled by the Gaussian copula model are more desirable to the CDO issuer than those of the $t$-copula model since the Gaussian copula model assigns extremely high spreads to junior tranches and low spreads to senior tranches.

To numerically approximate CDO tranche spreads, we have to estimate two parameters for the copula models: the asset correlation and the hazard rates. We estimate the asset correlation by the linear correlation of daily equity values from the period June 9th, 2009 until June 9th, 2010 and model the survival functions based on real-world CDS spreads taken on June 9th, 2010. A popular estimation method for hazard rates is calibration. In Appendix B.1, we briefly describe the calibration technique for the implied hazard rate functions used by Li’s copula model. The CDO pricing model is illustrated in Section 1.3.2 and the estimation procedure is sketched in Appendix B.2. This estimation of CDO tranche spreads incorporates a Monte Carlo simulation of correlated default times using Li’s Gaussian copula model. It is the principal step in the CDO pricing model in Appendix B.2.
The reference portfolio of our synthetic CDO is the five-year i-Traxx Europe CDS index. The attachment and detachment points of our CDO tranches are depicted in the first column of Table 4.1.

Figure 4.1 and Table 4.1 display the numerical results of the spread simulations. We notice that the Gaussian copula model shows higher spreads than the t-copula model for the junior tranches and a lower spread for the Super Super Senior tranche. The reason for this is that the simulations of the correlated default times using the t-copula model generate more extreme numbers of defaults in the reference pool; either extremely few defaults or excessively many defaults. For an extreme low number of defaults, the spread for the junior (and also senior) tranches decline since no defaults implies low risk even for the Equity tranches. For an excessively large number of defaults, the spread for the senior tranches increases because a large number of defaults implies a great amount of risk even for the senior tranches.

For the junior tranches, a medium amount of defaults results in the same risk exposure as a high amount of defaults, thus spreads of junior tranches do not rise excessively for extremely large numbers of defaults. To illustrate this more coherently, let us consider the following example.

**Example 4.1.** Let us assume all entities in the reference portfolio are perfectly positively correlated. Hence, all copula models are equal by Lemma 2.12. Perfect positive dependence implies that either all entities or none of the entities default. Consequently, there is an equal amount of risk for all tranches and the spread lines in Figure 4.1 must be constant.

From Example 4.1, we can conclude that higher correlation or more dependence implies a less steep decline in the spread line. Hence, the t-copula model exhibits more dependence among default times. If we look at all the pairwise default correlations modeled by the Gaussian and the t-copula model then we find that approximately 89% of the 775 default correlations are higher according to the t-copula model than according to the Gaussian copula model. The reason for this is the effect of the marginal transformations on the dependence structure. In Section 3.4.2, we show that if the orginal marginals are heavier-tailed than the meta-marginals the copula model creates more dependence.

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3Specifications of the data used for the simulation can be found in Appendix C.
4Here, a default implies an entity’s default within the five-year period of the CDO contract.
among its marginals. Also, the copulas’ tail dependence coefficients show its effect in the slope of the spread lines. Default correlations are calculated for a five-year horizon. This is a small quantile for the meta-distributions created by each copula. Since the Gaussian copula exhibits zero tail dependence, correlation below these quantiles are extremely small. The \( t \)-copula model has very heavy tails and shows positive tail dependence coefficients for asset correlation \( \rho^{\text{Asset}} > -1 \) and degrees of freedom \( \nu < \infty \) and this is another reason for higher default correlations, and hence, less decline in the tranche spread line modeled by the \( t \)-copula model.

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Gaussian Copula Model</th>
<th>( t )-Copula Model</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity (0%-3%)</td>
<td>2575.9</td>
<td>1466.7</td>
<td>1109.2</td>
</tr>
<tr>
<td>Mezzanine Junior (3%-6%)</td>
<td>1862.9</td>
<td>1222.6</td>
<td>640.2</td>
</tr>
<tr>
<td>Mezzanine (6%-9%)</td>
<td>1340.4</td>
<td>964.6</td>
<td>375.8</td>
</tr>
<tr>
<td>Senior (9%-12%)</td>
<td>1050.3</td>
<td>818.1</td>
<td>232.2</td>
</tr>
<tr>
<td>Super Senior (12%-22%)</td>
<td>582.4</td>
<td>518.1</td>
<td>64.3</td>
</tr>
<tr>
<td>Super Super Senior (22%-100%)</td>
<td>65.8</td>
<td>106.5</td>
<td>-40.7</td>
</tr>
</tbody>
</table>

Table 4.1: Synthetic CDO spreads (in bps) modeled by 50,000 Monte Carlo simulations of the Gaussian and \( t \)-copula with the i-Traxx EUR CDS 13 series index as underlying portfolio.

The difference in spreads from both models for each tranche is depicted in the last column of Table 4.1. The spread difference for the most junior tranches is very large. For the Equity tranche, the spread is approximately 11% higher according to the Gaussian copula model than for the \( t \)-copula model. Thus, the Gaussian copula model generates an immensely enhanced return for the junior tranches in comparison with the returns from the \( t \)-copula model. The issuer of a CDO often keeps all or part of the most junior tranches to ensure confidence among other investors to buy into the CDO. From the issuer’s point of view, it is more advantageous to use the Gaussian copula model to price the tranche spreads than other copulas that reflect more dependence among default times since those copulas would not generate as high of a return for the most junior tranches as the Gaussian copula does. Moreover, the most senior tranche yields less return according to the Gaussian copula model than according to the \( t \)-copula model. As junior tranches are mostly kept by the issuer, the senior tranches are sold off to other investors. Using the Gaussian copula, the issuer has to pay less to the safest tranche – the Super Super Senior tranche. In addition, this tranche has the largest principle (78% of the reference portfolio), and thus, generates a huge income for the issuer in terms of fees. Hence, the Gaussian copula generates the most favorable spreads for the CDO issuer. Embrechts quoted an investment banker with whom he had discussed the use of the \( t \)-copula model: “Who will pay the difference?” The investment banker implied that the spreads of the \( t \)-copula model would be too costly for the CDO issuer.

The numerical results in the section imply that the deficiencies of the Gaussian copula in modeling extreme events such as defaults have impact in real-world investment dealings. It has become apparent that a copula model generating more default dependence among obligors creates a flat tranche spread line. The problem of modeling the spread line that reflects the true underlying risk structure of the CDO remains. A \( t \)-copula model with few degrees of freedom might overestimate the spread for the Super Super Senior tranche and underestimate the spread of the junior tranches. Since tail dependence of the copula impact the slope of the spread line and the tail dependence coefficient in (3.10) is dependent on the correlation coefficient \( \rho \) and the degrees of freedom \( \nu \), using the \( t \)-copula with an improved estimation procedure for \( \rho \) and a new procedure to estimate the parameter \( \nu \), which incorporates other risk factors than correlation in asset values, could be an interesting modification in the CDO pricing model. A discussion about a possible model for estimating the parameter \( \nu \) of the \( t \)-copula is beyond the scope of this thesis and is a suggestion for possible further research.

### 4.3 Conclusions

Mathematical modeling is an important tool in understanding the complexity of reality. It uses logic and stochastics to quantify the uncertainty of incomprehensible concepts and provides us with an explanation of the phenomenon. In the context of CDO pricing, using copulas to model the dependence among assets is crucial for determining the spreads. The Gaussian copula model is less suited for this purpose due to its inability to account for heavy tails and extreme events. The \( t \)-copula model, on the other hand, captures these aspects better and provides a more realistic representation of the underlying risk structure.

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5Professor P. Embrechts, personal communication, September 7th, 2009.
apprehensible simplification of a complex reality. However, models should not be simplified to the point where they misrepresent the underlying data. Einstein [16] said: “It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience.” This idealization is often re-stated as “Everything should be made as simple as possible, but no simpler.”

Table 4.2 gives all the simplifications we have made in our analysis. In first column practical simplifications are represented and the second column illustrates mathematical simplifications of the Gaussian copula model.

<table>
<thead>
<tr>
<th>Practical Simplifications</th>
<th>Mathematical Simplification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset correlation is the only driver of default dependence</td>
<td>No higher-order dependence is included, only pairwise co-linearity</td>
</tr>
<tr>
<td>Capital structure in modeling asset values</td>
<td></td>
</tr>
<tr>
<td>Market CDS spreads render marginal default distributions</td>
<td>Piecewise constant hazard rates</td>
</tr>
<tr>
<td>Static default correlations for variable marginal distributions</td>
<td>Defaults have a Gaussian dependence structure (no tail dependence)</td>
</tr>
</tbody>
</table>

Table 4.2: Simplifications of modeling default dependence with the Gaussian copula model.

In this thesis, we have come to understand that the overwhelming nature of these simplifications shed doubt on the adequacy of Li’s Gaussian copula model to render correct default correlations and cause us to doubt the values of the tranche spreads, in Figure 4.1 and Table 4.1 modeled by the Gaussian copula. The Gaussian copula model does, in fact, inadequately represent the underlying data because of the implications of the simplifications in Table 4.2, which are discussed in Section 4.1 and 4.2. Hence, the Gaussian copula model should be used with the utmost caution, if at all!

An additional result obtained in Section 3.4.2 shows an inconsistency in CRAs credit rating system. The practice of major CRAs, like Standard & Poor’s and Moody’s, seem to indicate a certain misperception of high ratings. Investment grade ratings seem to be awarded to financial entities too often. The manner in which the CRAs do their business seems to lack the separation between the analysis of ratings and their own incentives⁶. By Fama’s EMH, an efficient CDS market is a valid substitution for CRA. Implied default distributions would correctly identify the underlying credit quality of the reference entity. Unfortunately, CDSs are highly speculative products and the CDS market has proven to be illiquid at times, particularly in times of crisis. These facts pose liquidity risks reflected in the CDS spreads and manipulations that corrupt the use of CDS spreads to model implied default distributions which efficiently reflect the underlying credit quality. The fact that an investor can buy protection on an obligation that he/she is not entitled to, gives investors the ability to speculate on changes in default rates of the underlying entity. If CDS were to be regulated in such a manner that only the holders of the underlying reference entity would be able to buy protection through a CDS, it would not make sense for investors to speculate on changes of default rates. The investor would receive the payment in either case; if the reference entity defaults or survives. This would force investors to use CDS exclusively for insurance purposes and not for pure speculation, thus restoring market efficiency. Of course, this would imply that liquidity risk would rise immensely since the trading volume of CDS would be forced to drop dramatically. As we need CDS spreads for multiple maturity dates, liquidity risk represents a major problem with respect to modeling implied default distributions. For instance, at the time of writing⁷ five-year maturity transactions in the CDS market are relatively liquid where as the two to four-year transactions are extremely illiquid. This implies that creating a CDS market which is fully representative for the default distribution of the reference entities is almost impossible. Even if we would be able to create a liquid and efficient CDS market without the manipulations of speculators and the implied default distributions would not suffer the present manipulations, the determination of the correct prices for the CDSs represents a problem. Since these should be based on the default distributions that are rendered by the price of

⁶See Klein [32] and McCool [42].
⁷Summer 2010
the CDS. This is a vicious circle! Regarding this vicious circle and the application of implied default distributions for CDO pricing, Balkema stated: “This has a large component of Baron Münchhausen in it,” referring to the anecdote of the Baron who pulled himself out of a swamp by his bootstraps. Unfortunately, there are no other obvious substitutes for the CRA’s credit rating regarding the reflection of credit quality.

For interested readers the following paper, which we have not cited above, are worth a read. These articles provide a great addition to our examination of copulas and enhance the understanding of the evolution of copulas in finance. It is best to read them in the following order: Genest and Neslejová [22], Genest and Favre [?], Embrechts and Donnelly [14], Mikosch [53] with rejoinder and discussions, and Embrechts [15].

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*Professor G. Balkema, personal communication, January 22nd, 2009.*
Appendix A

Lemmas and Theorems

Lemma A.1. Let $F$ be a continuous distribution function. Then,

$$X \sim F \iff F(X) \sim U(0, 1).$$

Proof. Since $F$ is continuous it has an inverse, denoted by $F^{-1}$. Suppose $X \sim G$, then,

$$P(F(X) \leq u) = P(F^{-1}(F(X)) \leq F^{-1}(u)) = P(X \leq F^{-1}(u)) = G(F^{-1}(u)).$$

If now $G = F$, then $G(F^{-1}(u)) = u$. Further, if $F(X) \sim U(0, 1)$, then,

$$u = P(F(X) \leq u) = G(F^{-1}(u)),$$

hence $G = F$.

Theorem A.2. Let $(X_1, X_2)'$ be a random vector with a non-elliptical distribution $H$ and marginals $F_1$ and $F_2$. Further, assume that the variances of $X_1$ and $X_2$ are positive and finite. Then

- The set of all possible correlations is a closed subinterval $[\rho_{\text{min}}, \rho_{\text{max}}]$ of $[-1, 1]$ with $\rho_{\text{min}} < 0 < \rho_{\text{max}}$.
- The extremal correlation $\rho = \rho_{\text{min}}$ is attained if and only if $X_1$ and $X_2$ are countermonotonic.
- The extremal correlation $\rho = \rho_{\text{max}}$ is attained if and only if $X_1$ and $X_2$ are comonotonic.
- $\rho_{\text{min}} = -1$ if and only if $X_1$ and $-X_2$ are of the same type.
- $\rho_{\text{max}} = 1$ if and only if $X_1$ and $X_2$ are of the same type.

Two random variables $X_1$ and $X_2$ are of the same type if there is an $a > 0$ and $b \in \mathbb{R}$ such that $X_1 = aX_2 + b$. A proof of Theorem A.2 can be found in Embrechts, et al. [12], Theorem 4.

Theorem A.3. If $(X_1, X_2)'$ be a standard normal distributed random vector with correlation $\rho = \rho(X_1, X_2)$, then $X_2 | X_1 = x \sim N(\rho x, 1 - \rho^2)$.

Proof. By Bayes’ law, we can express the density function of $X_2 | X_1 = x$ as follows,

$$f_{X_1 | X_2 = x}(x_1) = \frac{f_{X_1, X_2}(x_1, x)}{f_{X_2}(x)}.$$  

Thus, we have,

$$f_{X_1 | X_2 = x}(x_1) = \frac{1}{2\pi \sqrt{1 - \rho^2}} e^{-\frac{x_1^2 - 2\rho x_1 x + x^2}{2(1 - \rho^2)}}$$

$$= \frac{1}{\sqrt{2\pi(1 - \rho^2)}} e^{-\frac{x_1^2 + x^2 - 2\rho x_1 x}{2(1 - \rho^2)}} e^{x^2}$$

$$= \frac{1}{\sqrt{2\pi(1 - \rho^2)}} e^{-(x_1 - \rho x)^2 / 2(1 - \rho^2)}.$$  

The last equality proves the theorem.
Theorem A.4. If $(X_1, X_2)'$ be a bivariate $t$ distributed random vector with correlation coefficient $\rho = \rho(X_1, X_2)$ and $\nu$ degrees of freedom, then conditional on $X_2 = x$, it holds that,

$$Z := \left( \frac{\nu + 1}{\nu + x^2} \right)^{1/2} \frac{X_1 - \rho x}{\sqrt{1 - \rho^2}} \sim t_{\nu+1}.$$  

Proof. By Bayes’ rule, we can express the density function of $X_1 | X_2 = x$ as follows,

$$f_{X_1 | X_2 = x}(x_1) = \frac{f_{X_1, X_2}(x_1, x)}{f_{X_2}(x)}.$$  

Let,

$$c_1(\nu) := \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu^2 \pi}} \quad \text{and} \quad c_2(\nu, \rho) := \frac{\Gamma \left( \frac{\nu+2}{2} \right) \Gamma \left( \frac{\nu+1}{2} \right)}{\nu \sqrt{1 - \rho^2} \Gamma \left( \frac{\nu}{2} \right)}.$$  

Then,

$$f_{X_1 | X_2 = x}(x_1) = \frac{c_2(\nu) \left( 1 + \frac{x_1^2 + x^2 - 2\rho x_1 x}{\nu(1 - \rho^2)} \right)^{-\frac{\nu+2}{2}}}{c_1(\nu) \left( 1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+1}{2}}} = \frac{\Gamma \left( \frac{\nu+2}{2} \right)}{\sqrt{\nu^2 \pi(1 - \rho^2)}} \left( 1 + \frac{x_1^2 + x^2 - 2\rho x_1 x}{\nu(1 - \rho^2)} \right)^{-\frac{\nu+2}{2}} \left( 1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+2}{2}}.$$  

Let now,

$$z := \left( \frac{\nu + 1}{\nu + x^2} \right)^{1/2} \frac{X_1 - \rho x}{\sqrt{1 - \rho^2}} \quad \text{and} \quad c(\nu, \rho) := \left( \frac{\nu + 1}{\nu + x^2(1 - \rho^2)} \right)^{1/2}.$$  

Then, the density of $Z | X_2 = x$ is given by the following,

$$c(\nu, \rho) f_{Z | X_2 = x}(z) = \frac{\Gamma \left( \frac{\nu+2}{2} \right)}{\sqrt{\nu^2 \pi(1 - \rho^2)}} \left( 1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+2}{2}} \left( 1 + \frac{z^2}{\nu + 1} \right)^{-\frac{\nu+2}{2}} \left( 1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+1}{2}} = \frac{\Gamma \left( \frac{\nu+2}{2} \right)}{\sqrt{\nu^2 \pi(1 - \rho^2)}} \left( 1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+2}{2}} \left( 1 + \frac{z^2}{\nu + 1} \right)^{-\frac{\nu+2}{2}}.$$  

This completes the proof.

$\square$
Appendix B

MatLab Simulations

Here, we describe the algorithms for the hazard rate calibration and the CDO tranche spread calculations. In Section B.1 and B.2, we explain the calculation methods and calculation techniques used to model CDO tranche spreads. The valuation formula for CDSs in Section 1.2.1 is used to calibrate the hazard rates and model the survival function as in Cox and Oake [10]. Further, we lay out an algorithm including a Monte Carlo simulation to model CDO tranche spreads with the pricing model presented in Section 1.3.2. The MatLab codes can be found in the attached CD.

B.1 Implied Hazard Rate Calibration with CDS spreads

The implied approach to default distributions is most often based on CDS spreads of different maturity dates. Here, we lay out our procedure to obtain hazard rates \( h \) to calibrate survival distributions according to Cox and Oake [10]. In Section 1.2.1, we discussed the pricing equation for a CDS contract. Through this equation, we can solve for a piece-wise constant hazard rate function by postulating three minor assumptions. The assumptions are:

1. the interest rate process and default process are independent,

2. there is a finite set of discrete default dates \( 0 = t_0, \ldots, t_m = M \), where \( M \) is the maturity date,

3. default payments are settled immediately upon default.

The premium leg in (1.1) can be estimated by,

\[
\widehat{\text{PremLeg}} = s K \sum_{i=1}^{K} \gamma DF(t_i) \left( S(t_{i-1}) - S(t_i) \right),
\]

(B.1)

where \( S(t) = P(T > t) \) the survival probability, \( K \) is the amount of premium payments, \( DF(t) \) the risk-free discount factor and \( s \) is the spread. Here, \( \gamma \) is a day count factor, that is, if premiums are paid annually \( \gamma = 1 \), semi-annually \( \gamma = \frac{1}{2} \), et cetera. Further, in (B.1) the default occurs between two premium dates, thus half the premium is used for the accrued premium for that period. Since (B.1) only takes the accrued premium of each period into account, a better estimation of the premium leg is,

\[
\widehat{\text{PremLeg}} = s \sum_{i=1}^{K} \gamma DF(t_i) \left( S(t_i) + \frac{1}{2} (S(t_{i-1}) - S(t_i)) \right).
\]

(B.2)

This not only includes the accrued premium of the period \([t_{i-1}, t_i] \) if the reference entity defaults in this period, but also the full premium of that period if the entity defaults after \( t_i \). The estimation in (B.2) is utilized in our model. This estimation procedure is more extensively explained in O’Kane and Turnbull [51]. For the protection leg in (1.2), the estimation is as follows,

\[
\widehat{\text{ProtLeg}} = (1 - R) \sum_{j=1}^{m} DF(t_j) (S(t_{j-1}) - S(t_j)),
\]

(B.3)
where \( R \) is the recovery rate. The survival function \( S(t) \) is defined in (3.2) and by assuming a piece-wise constant hazard rate \( h \), we can approximate the integral in \( S(t) \) by a sum of constant hazard rates \( h_1, h_2, \ldots \). Generally, we have spreads for one to five year and seven and 10 year contracts. Thus, for a one year CDS contract, we have \( \int_0^t b(s) \, ds \approx b_1 t \), where \( t \leq 1 \). This implies that if the one year spread \( s \) is known, we can estimate \( h_1 \) by setting (B.2) and (B.3) equal as in (1.3), that is,

\[
L(e^{-h_1 t_{i1}}, \ldots, e^{-h_1 t_{iK}}) = (1 - R) \sum_{j=1}^m DF(t_j)(e^{-h_1 t_{j1}} - e^{-h_1 t_{jK}}).
\]  

(B.4)

If \( h_1 \) is obtained by (B.4), we can calculate \( h_2 \) by using the two year CDS contract on the same reference entity and estimating the integral in (3.2) by \( h_1 t \), if \( t \leq 1 \), and \( h_1 + h_2 t \), if \( 1 < t \leq 2 \). By continuously applying (B.4), we can find all the hazard rates and hence estimate the default distribution \( F(t) = 1 - S(t) \).

B.2 Algorithm for Pricing CDO Tranche Spreads

In the present section, we introduce the algorithm used to simulate CDO tranche spreads from real market data. We utilize CDS spreads to calibrate default distributions, as in Appendix B.1 and time series of equity values to calculate the linear correlation coefficients. In section 1.3.2, we introduce the theoretical model to price CDO tranche spreads. In this algorithm, we include a Monte Carlo simulation procedure for \( \Gamma \) and \( \Delta \) as attachment and detachment points for the tranches. The following steps describe the simulation procedure for \( N \) financial entities in the reference portfolio:

1. Calculate linear asset correlations \( \rho_{i,j}^{asset} \) by time series of equity values of entities \( i \) and \( j \) and calibrate hazard rates \( h_i \) for each entity \( i \) in the reference portfolio.

2. For each simulation \( n \) repeat the following routine:

   (a) Use the asset correlations and hazard rates to generate an \( n \)-dimensional vector of correlated default times by using the Monte Carlo simulation of the model described in bullet points in Section 3.1.

   (b) Sort the \( n \) default times in ascending order and define the vector \( T_n^{def} := (T_1^{def}, \ldots, T_n^{def})' \), such that \( T_j^{def} \leq T_i^{def} \) for all \( j, i \) such that \( j < i \leq n \).

   (c) Calculate the premium leg for each tranche by the following routine:

      i. Calculate the accumulated loss function \( L^n(t_k) \) in (1.3) for all the premium payment dates \( t_1, \ldots, t_K \) based on the specific realization \( T_n^{def} \) and compute the premium leg by,

      \[
      \text{PremLeg}^n(\Gamma, \Delta) = \sum_{k=1}^K \gamma DF(t_k) \min\{\Delta - L^n(t_k), 0\}, \Delta - \Gamma,
      \]

      where \( \gamma \) is the day-count, \( DF(t) \) is the discount factor and \( \Gamma \) and \( \Delta \) the attachment and detachment points of the tranche.

   (d) Calculate the protection leg for each tranche by the following routine:

      i. Calculate the accumulated loss function \( L^n(M) \) in (1.4) based on \( T_n^{def} \).

      ii. If \( L^n(M) < \Gamma \), then set the protection leg to zero. That is \( \text{ProtLeg}^n(\Gamma, \Delta) = 0 \).

      iii. If \( \Gamma \leq L^n(M) < \Delta \), define the lower trigger \( t^n_{\eta} := \inf\{t > 0 \mid L^n(t) \geq \Gamma\} \) and the vector \( T_n^{def, \Gamma, \Delta} := (T_{\eta}^{def}, \ldots, T^n_{\eta})' \), such that \( T_j^{def, \Gamma, \Delta} \geq t^n_{\eta} \) for all \( \eta \). For each component of \( T_n^{def, \Gamma, \Delta} \) calculate the discounted default value \( DV_i^n \) given by \( DV_i^n := (1 - R_i)DF(T_i^n) \).
and finally add all the discounted default values, that is

\[ \text{ProtLeg}^n(\Gamma, \Delta) = \epsilon_1 DV^n_\eta + \sum_{\xi=\eta+1}^{\kappa} DV^n_\xi. \]

iv. If \( L^n(M) \geq \Delta \), define the upper trigger \( \tau^n_\kappa := \inf\{t > 0 | L^n(t) \geq \Delta\} \) and the vector \( \mathbf{T}_{\text{def.f.}, \Gamma, \Delta} := (T^n_\eta, \ldots, T^n_\kappa)' \), such that \( \tau^n_\eta \leq T^n_i \leq \tau^n_\kappa \quad \forall i \in \{\eta, \ldots, \kappa\} \). For each component of \( \mathbf{T}_{\text{def.f.}, \Gamma, \Delta} \) calculate the discounted default value \( DV^n_i \) and finally add all the discounted default values, that is

\[ \text{ProtLeg}^n(\Gamma, \Delta) = \epsilon_1 DV^n_\eta + \sum_{\xi=\eta+1}^{\kappa} DV^n_\xi + \epsilon_2 DV^n_\kappa. \]

3. Compute the arithmetic average \( \text{PremLeg}(\Gamma, \Delta) \) and \( \text{ProtLeg}(\Gamma, \Delta) \) of \( \text{PremLeg}^n(\Gamma, \Delta) \) and \( \text{ProtLeg}^n(\Gamma, \Delta) \) for each CDO tranche and determine the tranche spread by,

\[ s_{\Gamma, \Delta} = \frac{\text{ProtLeg}(\Gamma, \Delta)}{\text{PremLeg}(\Gamma, \Delta)}. \]

---

1 We multiply the discounted default value of the first entity that breaks the tranche’s losses by a factor \( \epsilon_1 \in [0, 1] \). Only the losses of this defaulter that exceed the threshold \( \Gamma \) should be taken into account. For instance, if the lower default threshold \( \Gamma \) is 6%, the relative defaulter’s default value is 0.5% and the cumulative losses are 6.25%, then only half of the defaulter’s losses have to be taken into account for that tranche. The fact that \( L^n(M) < \Delta \) ensures that all other defaulter’s losses stay within the tranche’s boundaries.

2 As above, we multiply the defaulter that breaks the tranche’s losses by \( \epsilon_1 \in [0, 1] \). Furthermore, we multiply the defaulter for which the cumulative losses start to exceed the tranche’s detachment point \( \Delta \) by \( \epsilon_2 \in [0, 1] \), since part of this defaulter’s losses might not belong to that tranche.
Appendix C

Data Specifications

The tables below depict the data specifications of the data sets used for the analysis in this paper. Table C.1 displays the CDS data used the model the implied hazard rates for the marginal transformations in the bootstrap procedure in Section 3.4.2. For Figure 3.17 and 3.18 we used nearly 3500 names for each of the two dates.

<table>
<thead>
<tr>
<th>Specifications</th>
<th>Hazard rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>One, three, five, seven and 10-year CDS spreads of all nearly 3500 different reference entities</td>
</tr>
<tr>
<td>Time period</td>
<td>18-12-2009 and 19-03-2010</td>
</tr>
<tr>
<td>Source</td>
<td>Markit</td>
</tr>
</tbody>
</table>

Table C.1: Data table for Section 3.4.2

For the CDO tranche spread simulation, we utilized the i-Traxx Europe index\(^1\) as a reference portfolio for a synthetic CDO. Further, we used daily equity values of all 125 entities for a period of one year to calculate asset correlations. Table C.2 contains the data specifications of the i-Traxx Europe index and the equity values.

<table>
<thead>
<tr>
<th>Specifications</th>
<th>Correlation coefficients</th>
<th>Hazard rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>Daily equity values of entities in the five-year i-Traxx EUR CDSI Series 13 index</td>
<td>One to five, seven and 10-year CDS spreads of entities in the five-year i-Traxx EUR CDSI Series 13 index</td>
</tr>
<tr>
<td>Time period</td>
<td>09-06-2009 until 09-06-2010</td>
<td>09-06-2010</td>
</tr>
<tr>
<td>Source</td>
<td>Bloomberg</td>
<td>Markit</td>
</tr>
</tbody>
</table>

Table C.2: Data table for Section 4.2

A list of all 125 names of CDSs in the five-year i-Traxx Europe index is given in alphabetical order in Table C.3.

---

\(^1\)Ticker: itraxx EUR CDSI S13 Y5.
<table>
<thead>
<tr>
<th>A-D</th>
<th>E-P</th>
<th>R-Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adecco SA</td>
<td>E.ON AG</td>
<td>Reed Elsevier PLC</td>
</tr>
<tr>
<td>Aegon NV</td>
<td>Edison SpA</td>
<td>Repsol YPF SA</td>
</tr>
<tr>
<td>Akzo Nobel NV</td>
<td>EDP-Energias de Protugal SA</td>
<td>Rolls-Royce PLC</td>
</tr>
<tr>
<td>Allianz SE</td>
<td>EDF SA</td>
<td>RWE AG</td>
</tr>
<tr>
<td>Alstom SA</td>
<td>EnBW Energie Baden-Wuerttemberg</td>
<td>SABMiller PLC</td>
</tr>
<tr>
<td>Anglo American PLC</td>
<td>Enel SpA</td>
<td>Safeway Ltd</td>
</tr>
<tr>
<td>ArcelorMittal</td>
<td>Experian Finance PLC</td>
<td>Sanofi-Aventis SA</td>
</tr>
<tr>
<td>Assicurazioni Generali SpA</td>
<td>Fimmecanica SpA</td>
<td>Siemens AG</td>
</tr>
<tr>
<td>Aviva PLC</td>
<td>Fortum OYJ</td>
<td>Societe Generale</td>
</tr>
<tr>
<td>AXA SA</td>
<td>France Telecom SA</td>
<td>Sodexo</td>
</tr>
<tr>
<td>BAE Systems PLC</td>
<td>Gas Natural SDG SA</td>
<td>Solvay SA</td>
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<tr>
<td>Banca Monte dei Paschi de Sien</td>
<td>GDF Suez</td>
<td>STMicroelectronics NV</td>
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<tr>
<td>Banco Bilbao Vizcaya Argentari</td>
<td>Hannover Ruckversicherung AG</td>
<td>Swedzucker International Fin.</td>
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<td>Banco Espirito Santo SA</td>
<td>Group Auchan SA</td>
<td>Swiss Reinsurance Co Ltd</td>
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<tr>
<td>Banco Santander SA</td>
<td>Henkel AG &amp; Co KGaA</td>
<td>Technip SA</td>
</tr>
<tr>
<td>Bank of Scotland PLC</td>
<td>Holcen Ltd</td>
<td>Telecom Italia SpA</td>
</tr>
<tr>
<td>Barclays Bank PLC</td>
<td>Iberdrola SA</td>
<td>Telefonica SA</td>
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<tr>
<td>BASF SE</td>
<td>Imperial Tobacco Group PLC</td>
<td>Telekom Austria AG</td>
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<tr>
<td>Bayer AG</td>
<td>J Sainsbury PLC</td>
<td>Royal Bank of Scotland</td>
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<tr>
<td>Bayerische Motoren Werke AG</td>
<td>JTI UK Finance PLC</td>
<td>Royal Bank of Scotland</td>
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<td>Bertelsmann AG</td>
<td>Intesa Sanpaolo SpA</td>
<td>Royal Bank of Scotland</td>
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<tr>
<td>BNP Paribas</td>
<td>Imperial Tobacco Group PLC</td>
<td>Royal Bank of Scotland</td>
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<td>Bouygues SA</td>
<td>Intesa Sanpaolo SpA</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>BP PLC</td>
<td>J Sainsbury PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>British American Tobacco PLC</td>
<td>JTI UK Finance PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>British Telecommunications PLC</td>
<td>Koninklijke DSM NV</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Cadbury Holdings Ltd</td>
<td>Koninklijke Philips Electronic</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Carrefour SA</td>
<td>Air Liquide SA</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Casino Guichard Perrachon SA</td>
<td>LVMH Moet Hennessy Louis Vuit</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Centrica PLC</td>
<td>Lanxess Finance BV</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Commerzbank AG</td>
<td>Linde AG</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Cie de St-Gobain</td>
<td>Marks &amp; Spencer PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Cie Financiere Michelin</td>
<td>Metro AG</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Compass Group PLC</td>
<td>Muenchener Ruckversicherungs</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Credit Agricole SA</td>
<td>National Grid PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Credit Suisse Group AG</td>
<td>Nestle SA</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Daimler AG</td>
<td>Next PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Danone</td>
<td>Pearson PLC</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Deutsche Bahn AG</td>
<td>Portugal Telecom International</td>
<td>Royal Bank of Scotland</td>
</tr>
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<td>Deutsche Bank AG</td>
<td>PPR</td>
<td>Royal Bank of Scotland</td>
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<tr>
<td>Deutsche Post AG</td>
<td>Publicis Groupe SA</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Deutsche Telekom AG</td>
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<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Diageo PLC</td>
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<td>Royal Bank of Scotland</td>
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</table>

**Table C.3:** Names of entities in the five-year i-Traxx Europe CDSI Series 13 index.

Source: Markit.
Appendix D

List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$=$</td>
<td>Identical in distribution</td>
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<tr>
<td>$\dagger$</td>
<td>Transpose or derivative</td>
</tr>
<tr>
<td>$\perp$</td>
<td>End of a definition</td>
</tr>
<tr>
<td>$\square$</td>
<td>End of a proof</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The extended real line, $[-\infty, \infty]$</td>
</tr>
<tr>
<td>$DF$</td>
<td>Risk-free Discount factor</td>
</tr>
<tr>
<td>$R$</td>
<td>Recovery rate</td>
</tr>
<tr>
<td>$s, s^\Gamma, \Delta$</td>
<td>(CDO Tranche) spread</td>
</tr>
<tr>
<td>$M$</td>
<td>Maturity date</td>
</tr>
<tr>
<td>$K$</td>
<td>Amount of premium payments throughout the life of an financial product</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Day count factor</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of financial entities in a CDO reference portfolio</td>
</tr>
<tr>
<td>$L(t), L^\Gamma, \Delta(t)$</td>
<td>Cumulative (tranche) loss function</td>
</tr>
<tr>
<td>$DV_i$</td>
<td>Default value of entity $i$</td>
</tr>
<tr>
<td>$U, V, X, Y$</td>
<td>Random variables</td>
</tr>
<tr>
<td>$Z$</td>
<td>Random variable or quantile</td>
</tr>
<tr>
<td>$T$</td>
<td>Time-until-default (random variable)</td>
</tr>
<tr>
<td>$\tau_\eta, \tau_\kappa$</td>
<td>Upper and lower default trigger times</td>
</tr>
<tr>
<td>$T$</td>
<td>Time-until-default (random variable)</td>
</tr>
<tr>
<td>$V_t$</td>
<td>Asset value process</td>
</tr>
<tr>
<td>$W_t$</td>
<td>Standard Brownian motion</td>
</tr>
<tr>
<td>$E_t$</td>
<td>Equity</td>
</tr>
<tr>
<td>$R_t$</td>
<td>Normalized asset returns</td>
</tr>
<tr>
<td>$U, X, Y$</td>
<td>Random vectors</td>
</tr>
<tr>
<td>$q$</td>
<td>Quantile</td>
</tr>
<tr>
<td>$\rho_P$</td>
<td>Pearson’s correlation coefficient</td>
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<tr>
<td>$\rho_\kappa$</td>
<td>Spearman’s rho</td>
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<tr>
<td>$\tau_\kappa$</td>
<td>Kendall’s tau</td>
</tr>
<tr>
<td>$\rho_{Def}$</td>
<td>Default correlation</td>
</tr>
<tr>
<td>$\rho_{Asset}$</td>
<td>Asset correlation</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Tail dependence coefficient</td>
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<tr>
<td>$\mu$</td>
<td>Expectation</td>
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<tr>
<td>$\sigma^2$</td>
<td>Variance</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>Covariance matrix</td>
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<tr>
<td>$\nu, \alpha$</td>
<td>Degrees of freedom</td>
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<tr>
<td>$\beta$</td>
<td>Parameter of exponential distribution</td>
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<tr>
<td>$C$</td>
<td>Copula</td>
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<tr>
<td>$\hat{C}$</td>
<td>Survival copula</td>
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<tr>
<td>$C_\rho^{Ga}$</td>
<td>Gaussian copula with correlation $\rho$</td>
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</tbody>
</table>
\( C_{ν,ρ} \)  
\( F, H \)  
\( F_i \)  
\( S_i \)  
\( f \)  
\( Φ_n \)  
\( Φ \)  
\( φ_n \)  
\( φ \)  
\( t_n \)  
\( t_ν \)  
\( h \)  
\( δ \)  
\( ρ, θ \)  
\( D_i(t) \)  
\( c, ϵ \)  

t-copula with correlation \( ρ \) and \( ν \) degrees of freedom  
Multivariate distribution function  
Univariate distribution function or threshold value  
Univariate survival function  
Density function  
\( n \)-dimensional standard normal distribution function  
Univariate standard normal distribution function  
Univariate standard normal density function  
\( n \)-dimensional standard normal density function  
Univariate standard \( t \)-distribution function  
Univariate standard \( t \)-distribution with \( ν \) degrees of freedom  
Function or hazard-rate function  
Dependence measure  
Correlation coefficient  
Comprehensive default event  
Constants


