Seismic Imaging

Kyle de Mos

August 4, 2018

Bachelor thesis Mathematics and Physics & Astronomy
Supervisor: dr. Chris Stolk, dr. Rudolf Sprik

Institute of Physics
Korteweg-de Vries Institute for Mathematics
Faculty of Sciences
University of Amsterdam
Abstract

In this thesis seismic imaging is studied. Seismic imaging is done by sending in waves into the subsurface and then measuring the reflections. Waves in the subsurface are mathematically described by the wave equation. Because of the varying wave speed in the subsurface of the earth, this wave equation has varying coefficients. In this thesis existence and uniqueness of solutions of such partial differential equations are discussed and continuous dependence of the solutions on those coefficients. After that a linearization is used to find approximate solutions given a function for the wave speed. This is called Born forward modelling. This Born model is then used to obtain migration formulas which construct images of the subsurface, given measurements of reflected waves. A specific example of a migration formula we look at is Reverse Time Migration or RTM. Finally some MATLAB programmes are examined which illustrate the principles of Born forward modeling and RTM.
Contents

1 Introduction 4

2 Mathematical preliminaries 7
   2.1 The wave equation ............................................ 7
   2.2 Fourier Transformation ....................................... 8
   2.3 The Helmholtz Equation ...................................... 9
   2.4 More on the Green’s functions ............................... 9

3 Uniqueness and existence of solutions in case of variable coefficients 11
   3.1 Sobolev Spaces .................................................. 11
   3.2 Rewriting the PDE as an ODE with values in a Hilbert space .... 13
   3.3 Existence and uniqueness of solutions ........................ 15

4 Born forward modeling 19
   4.1 In time domain ............................................... 19
   4.2 In frequency domain ......................................... 21
   4.3 Comparison of results ....................................... 21

5 Migration 23
   5.1 Derivation of migration formula from Born forward modeling .... 23
   5.2 MATLAB ......................................................... 25
      5.2.1 Finite difference method .................................. 26
      5.2.2 Born modeling ............................................. 27
      5.2.3 RTM ....................................................... 28
      5.2.4 Adjoint test ............................................... 30
   5.3 Comments on practicality .................................... 31

6 Conclusion 32

Populaire samenvatting 33

Bibliografie 35
1 Introduction

The goal in seismic imaging is to find information about the earth's substructure by sending in waves and measuring the reflections [5]. This is used mostly by the oil and gas industry to find petroleum deposits, but also by geophysicists for academic use[5].

The result we are looking for is an image such as in figure 1.1.

![Image of a seismic image](http://www.crs4.it/research/energy-and-environment/imaging-and-numerical-geophysics/seismic-depth-imaging/)

Figure 1.1: Example of a seismic image from (http://www.crs4.it/research/energy-and-environment/imaging-and-numerical-geophysics/seismic-depth-imaging/)

The experiment executed to obtain the data to create such images consists of so-called *sources*, which create seismic waves, and *receivers* which measure vibrations on the surface. This experiment can be done on land or on water. Figure 1.2 illustrates the experiment on land: a vibrator truck acts as source and creates seismic waves which reflect on a layer of the earth. These reflected waves are then measured by so-called geophones, which act as receivers in this experiment. Figure 1.3 illustrates the experiment on water. Here airguns act as sources and the receivers are hydrophones.

After collecting the data it needs to be interpreted. This means translating the data to an "image" of the subsurface. Wave propagation in the subsurface is described by the wave equation. So the data can be viewed as samples of solutions to the wave equation. In this case the wave equation has variable coefficients, specifically the wave speed, which varies in different materials in the subsurface. Finding the solutions when we know the coefficients is the forward problem for our partial differential equation. We are interested in the inverse problem: finding the coefficients when we know the wavefield at certain points.

In practice we look for approximate solutions. Geophysicists have created methods to
Figure 1.2: Illustration of the seismic imaginging experiment on land from the website (http://www.innoseis.com/seismic-surveying/)

Figure 1.3: Illustration of the seismic imaginging experiment above water from the website (http://clubofmozambique.com/news/oil-and-gas-exploration-seismic-survey-launched-in-nampula-and-zambezia/)
find such approximate solutions, named migration schemes [5]. However these were not
originally created as approximate solutions to the inverse problem of the wave equation,
but as methods to obtain images of the subsurface [5].

The aim of this thesis is to critically evaluate the mathematics behind imaging as
used in seismology and give a description of how commonly used migration formulas
follow from the wave equation.

To obtain migration formulas from the wave equation a couple of steps need to be
taken. First we find approximate solutions to the wave equation by linearizing it, fac-
tually taking into account only single scattering. This gives us an expression for the
approximate solution which can be viewed as a linear operator working on certain pa-
rameters from the wave equation. Since we are interested in finding these parameters
we then try to find something approximating the inverse of this operator, so that if we
apply this to the measured wavefield, it yields back the parameters. It turns out that the
adjoint of the mentioned operator is a good choice. An important result, which requires
too complicated mathematics to be proven in this thesis, is that of Beylkin [9]. He proves
that the discontinuities in the result gained by applying the adjoint to the data are a
subset of the discontinuities of the actual parameter function. When we think of a layered
subsurface, these discontinuities lie at the transitions between layers, which is exactly
what we would like to map. This also motivates a somewhat more precise definition
of an image: any function which has the same discontinuities as the parameter functions
describing the system, albeit perhaps with different amplitudes.

The structure of this thesis is as follows: In chapter 2 we will give some mathematical
preliminaries which are necessary for later chapters. In chapter 3 we will discuss
uniqueness, existence and continuous dependence on the parameters of the wave equa-
tion with variable parameters. In chapter 4 we will discuss the Born approximation;
which is the above mentioned linearization of the wave equation and will give us ap-
proximate solutions if we know the parameter functions. In chapter 5 we first discuss
migration formulas and how they follow from the Born approximation. After that we
examine some MATLAB programmes from [6] which illustrate the Born approximation
and migration, in particular Reversed Time Migration (RTM).
2 Mathematical preliminaries

In this chapter we will introduce some mathematics that is necessary for the following chapters. In particular we will discuss the wave equation, with constant and varying coefficients, the fourier transform, the Helmholtz equation and the concept of a Green’s function.

2.1 The wave equation

The governing equation is the scalar wave equation:

\[
\left( \frac{1}{c(x)^2} \frac{\partial^2}{\partial t^2} - \nabla \right) u(x,t) = f(x,t), \tag{2.1}
\]

where the wave speed \(c(x)\) is seen as a function \(c : \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}\) of position and \(f(x,t)\) is a forcing term. The requirements for existence and uniqueness of solutions of (2.1) will be discussed in chapter 3.

We assume zero initial conditions: \(u(x,0) = u_t(x,0) = 0\). We construct solutions by using a Green’s function, i.e. a solution \(g(x,t,x_s,t_s)\) to:

\[
\left( \frac{1}{c(x)^2} \frac{\partial^2}{\partial t^2} - \nabla \right) g = \delta(x-x_s)\delta(t-t_s). \tag{2.2}
\]

A solution for equation (2.1) with a general forcing term \(f(x,t)\) is then given by correlating \(f\) with the Green’s function [2]:

\[
u(x,t) = \int_0^{t^+} \int_{\mathbb{R}^3} g(x,t,x_s,t_s)f(x_s,t_s)d\mathbf{x}_sd\tau_s \tag{2.3}
\]

For constant \(c\) the Green’s function is given by [2]:

\[
g(x,t,x_s,t_s) = \frac{\delta(t-t_s-|x-x_s|/c)}{4\pi|x-x_s|}. \tag{2.4}
\]

In which case 2.3 simplifies to:

\[
u(x,t) = \int_{\mathbb{R}^3} \frac{f(x_s,t-|x-x_s|/c)}{4\pi|x-x_s|}d\mathbf{x}_s. \tag{2.5}
\]
2.2 Fourier Transformation

We define the Fourier transform of a function $f$ as follows:

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt,$$  \hspace{1cm} (2.6)

and the inverse Fourier transform of a function $g$ as:

$$\mathcal{F}^{-1}(g)(t) = \hat{g}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega.$$  \hspace{1cm} (2.7)

A direct calculation shows:

$$\mathcal{F}(f')(\omega) = -i\omega \mathcal{F}f(\omega).$$  \hspace{1cm} (2.8)

And if we define the convolution of two functions $f$ and $g$ as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy,$$

we have the following result:

$$\mathcal{F}(f * g)(\omega) = \hat{f}(\omega)\hat{g}(\omega).$$  \hspace{1cm} (2.9)

which we call the convolution theorem. More specifically if we fill in $t = 0$ we find:

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = (f * g)(0) = \mathcal{F}^{-1}(\hat{f}\hat{g})(0)$$  \hspace{1cm} (2.10)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega 0} \hat{f}(\omega)\hat{g}(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)d\omega.$$  \hspace{1cm} (2.11)

Another result about Fourier transforms we’ll need relates $\mathcal{F}^{-1}(\overline{\hat{f}})$ to $f$:

$$\mathcal{F}^{-1}(\overline{\hat{f}(\omega)}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \overline{\hat{f}(\omega)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega$$  \hspace{1cm} (2.12)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(-t)} \hat{f}(\omega) d\omega = \overline{\hat{f}(-t)} = \overline{f(-t)}.$$  \hspace{1cm} (2.13)

If we combine this with our previous result we find:

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega.$$  \hspace{1cm} (2.14)
2.3 The Helmholtz Equation

If we apply a Fourier Transform with respect to time to the wave equation we obtain:

\[ (\Delta + \frac{\omega^2}{c^2}) \hat{u}(x, \omega) = -\hat{f}(x, \omega). \]  \hspace{1cm} (2.15)

If we then define the wavenumber \( k = \frac{\omega}{c} \), write \( U \) for \( \hat{u} \) and \( h \) for \( \hat{f} \) we obtain the Helmholtz equation:

\[ (\Delta + k^2) U = -h. \]  \hspace{1cm} (2.16)

This problem can again be solved by means of a Green’s function. To make a distinction with the Green’s function for the wave equation we will call it \( G(x, x_s) \). This Green’s function is the solution to:

\[ (\Delta + k(x))^2 G(x, x_s) = -\delta(x - x_s). \]  \hspace{1cm} (2.17)

The solution to equation (2.16) then becomes [2]:

\[ U(x) = \int_{\mathbb{R}^3} h(x_s) G(x_s) dx_s \]  \hspace{1cm} (2.18)

When \( k \) is a constant we can write down an explicit form for \( G(x, x_s) \) [2]:

\[ G(x, x_s) = \frac{e^{ik|x-x_s|}}{4\pi|x-x_s|}. \]  \hspace{1cm} (2.19)

2.4 More on the Green’s functions

In this section we discuss some more properties of the Green’s functions described above. First we would like to note something about the Green’s function in the time domain. Because there are no constants varying with time in equation (2.1) we know that \( g \) must be of the form \( g(x, x_s, t, t_s) = \psi(x, x_s, t - t_s) \) for some \( \psi : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_\geq 0 \rightarrow \mathbb{R} \). The physical idea is that it should not matter whether we send a pulse now and measure it in 10 seconds or send the pulse at some other time and measure it 10 seconds after having sent it out then.

Now we would like to mention something about the relation between the Green’s functions in the time domain and in the Fourier domain and causality. If we Fourier transform equation (2.2) and compare it to equation (2.17) we find a relation between our two Green’s functions: \( \mathcal{F}(g)(x, x_s, \omega) = G(x, x_s)e^{i\omega t_s} \). Note that if \( G \) is a solution to (2.17) then so is \( \mathcal{F}(G) \). By using equation (2.13) we see that the inverse Fourier transform of \( \mathcal{F}(G)(x, x_s) \) is \( g(x, x_s, -t, 0) \), which is also a solution to equation (2.2) with \( t_s = 0 \). Here we need to make a distinction between the causal and the acausal Green’s function.

If \( g(x, x_s, t, t_s) \) is the causal Green’s function, which propagates forward in time, then \( \tilde{g}(x, x_s, t, t_s) := g(x, x_s, -t, -t_s) \) is the acausal Green’s function which propagates backward in time. If we have a constant wave velocity these are \( \delta(t - t_s - |x - x_s|/c) \) and
\[ \delta(t - t_s + |x - x_s|/c), \] see figure 2.1. It is illustrated how the causal Green’s function is a cone in positive time and the acausal Green’s function is a cone in negative time. To solve the wave equation we usually use the causal Green’s function because we do not allow the effect to come before its cause. We also call \( G(x, x_s) \) and \( \bar{G}(x, x_s) \) the causal and the acausal Green’s functions for the Helmholtz equation.

Figure 2.1: Figure from [1] illustrating the causal and acausal Green’s functions.
3 Uniqueness and existence of solutions in case of variable coefficients

In this chapter we will investigate for which functions \( c(x) \) equation (2.1) has a unique solution. We will also look at continuous dependence on \( c \) of such solutions. This is relevant, because in reality we never exactly know its values and if a small difference in \( c \) gives an entirely different solution, we have little hope of finding good approximations. The theory discussed is based on [4]. However as this is a dissertation the level exceeds that of a bachelor thesis. Therefore we will not derive all that is done in the chapter we are interested in, but rather highlight some proofs and ideas and discuss these in some more detail, while referring to [4] for more difficult but important results.

3.1 Sobolev Spaces

In this section we will give a discussion of Sobolev spaces. These function spaces are not only relevant for the generalized wave equation we are discussing, but in many partial differential equation problems [8]. The definitions and theorems discussed here are from [8].

For convenience we first introduce the so-called multi-index notation. Let \( x = (x_1, \ldots, x_n) \) be a point in \( \mathbb{R}^n \). We call the vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \) a multi-index of order \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). We then define \( x^\alpha \) as \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). If we have a function \( f: \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R} \) which is \( C^k(\Omega) \) we use multi-index notation to denote its partial derivatives:

\[
D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

Having this concept we can define so-called weak derivatives:

**Definition 3.1.1.** Let \( u, v \in L^1_{\text{loc}}(\Omega) \). We say \( v \) is the \( \alpha \)th weak partial derivative of \( u \) if for all \( \phi \in C^\infty_0(\Omega) \):

\[
\int_{\Omega} u D^\alpha \phi dx = (-1)^{|\alpha|} \int_{\Omega} v \phi dx,
\]

in which case we write \( D^\alpha u = v \).

Using these weak derivatives we define Sobolev spaces as follows:

**Definition 3.1.2.** The Sobolev space \( W^{k,p}(\Omega) \) consists of all \( u \in L^1_{\text{loc}}(\Omega) \) such that for each multi-index \( \alpha \) with \( |\alpha| \leq k \), \( D^\alpha u \) exists in the weak sense and belongs to \( L^p(\Omega) \). If \( p = 2 \) we usually write \( H_k(\Omega) \).
Sobolev spaces are Banach spaces if we equip them with the following norm [8]:

\[
||u||_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p(\Omega)} = \sum_{|\alpha| \leq k} \left( \int_\Omega |D^\alpha u(x)|^p dx \right)^{\frac{1}{p}}.
\] (3.2)

For \( p = 2 \) the Sobolev spaces are even Hilbert spaces if we equip them with the following inner product [8]:

\[
\langle u, v \rangle = \int \sum_{|\alpha| \leq m} (D^\alpha u) (D^\alpha \bar{v}) \, dx.
\] (3.3)

In case \( p = 2 \) and \( \Omega = \mathbb{R}^n \) we can give an alternative definition of Sobolev spaces using the Fourier transform. So far we have only defined the Fourier transform for functions of one variable. For functions of several variables we define the Fourier transform as follows:

**Definition 3.1.3.** If \( u \in L^1(\mathbb{R}^n) \) we define its Fourier transform as follows:

\[
\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} u(x) e^{i\xi \cdot x} \, dx,
\] (3.4)

where \( \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n \).

With this definition we have the following:

\[
\mathcal{F}(D^\alpha u) = (-i \xi)^\alpha \mathcal{F}(u).
\] (3.5)

Using Plancherel’s formula we then have:

\[
\int_{\mathbb{R}^n} |D^\alpha u(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |(-i \xi)^\alpha \hat{u}(\xi)|^2 d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi.
\] (3.6)

From this it follows:

\[
||u||_{H^k(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \sum_{|\alpha| \leq k} \xi^{2\alpha} \right) |\hat{u}(\xi)|^2 d\xi.
\] (3.7)

Now there is a constant \( C \) such that \((1 + |\xi|^2)^k \leq \sum_{|\alpha| \leq k} \xi^{2\alpha} \leq C(1 + |\xi|^2)^k \). Integrating this gives us:

\[
\int_{\mathbb{R}^n} (1 + |\xi|^k)^2 |\hat{u}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} \left( \sum_{|\alpha| \leq k} \xi^{2\alpha} \right) |\hat{u}(\xi)|^2 d\xi \leq C \int_{\mathbb{R}^n} (1 + |\xi|^k)^2 |\hat{u}(\xi)|^2 d\xi. \tag{3.8}
\]

Combining this with the previous equality gives us that \( ||(1 + |\xi|^2)^{k/2} \hat{u}||_{L^2(\mathbb{R}^n)} \) is an equivalent norm with \( ||u||_{H^k(\mathbb{R}^n)} \). This means that we could also have defined Sobolev spaces for \( \Omega = \mathbb{R}^n \) by saying \( u \) is in \( H^k(\mathbb{R}^n) \) if \( ||(1 + |\xi|^2)^{k/2} \hat{u}||_{L^2(\mathbb{R}^n)} < \infty \). This now allows us to define Sobolev spaces with non-integer order as follows:
Definition 3.1.4. Let $u \in L^2(\mathbb{R}^n)$. Then $u \in H_k(\mathbb{R}^n)$ if $(1 + |\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$.

We would like to note that there are also ways to define Sobolev Spaces of fractional order when the underlying space is not all of $\mathbb{R}^n$, this is discussed in [10].

An important theorem states that if a function has enough weak derivatives, it has a certain degree of smoothness. To explain this we first define Hölder spaces:

Definition 3.1.5. A function $u : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\gamma$ if there is $c > 0$ such that $|u(x) - u(y)| < c||x - y||^\gamma$ for all $x, y \in \Omega$. The space $C^{k,\gamma}(\Omega)$ consists of all functions whose partial derivatives up to order $k$ exist (in the classical sense) and whose $k$th partial derivatives are Hölder continuous with exponent $\gamma$.

With this definition we can now state the so-called Sobolev embedding theorem:

Theorem 3.1.6. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, with a $C^1$ boundary. Assume $u \in W^{k,p}(\Omega)$.

1. if $k < n/p$, then $u \in L^q(\Omega)$ where $1/q = 1/p + 1/n$.
2. if $k > n/p$, then $u \in C^{k-[n/p]-1,\gamma}$, where $\gamma = [n/p] + 1 - n/p$ if $n/p$ is not an integer and $\gamma$ being any number lesser than 1 is $n/p$ is an integer.

Here $[x]$ means $x$ rounded to an integer, where $1/2$ is rounded to 0. So for example if $u \in H_1(\Omega)$ for $\Omega \in \mathbb{R}$, then $u$ also lies in $C^{0,1/2}$.

3.2 Rewriting the PDE as an ODE with values in a Hilbert space

We are looking to solve the following generalized wave equation

$$-\sum_i \frac{\partial}{\partial x_i} \frac{1}{\rho(x)} \frac{\partial}{\partial x_i} u(x,t) + \kappa(x) \frac{\partial^2}{\partial t^2} u(x,t) = f(x,t),$$

(3.9)

where $u$ is the pressure, $\rho(x)$ the mass density and $\kappa(x)$ the compressibility. Often we try to model a system of layers of different material in the earth’s subsurface, which implies that our coefficients $\rho(x), \kappa(x)$ can vary discontinuously. Because of this we will interpret equation (3.9) in a distributional sense, that is:

$$\int \left( -\sum_i \frac{v(x,t)}{\rho(x)} \frac{\partial}{\partial x_i} \frac{1}{\rho(x)} \frac{\partial}{\partial x_i} u(x,t) + v(x,t)\kappa(x) \frac{\partial^2}{\partial t^2} u(x,t) \right) dxdt = \int v(x,t)f(x,t)dxdt,$$

(3.10)

for all test functions $v \in C^\infty_0(\mathbb{R}^n \times [0, T], \mathbb{C})$. If we apply partial integration this becomes:
\[
\int \left( \sum_i \frac{1}{\rho(x)} \frac{\partial u}{\partial x_i}(x,t) \frac{\partial \bar{v}}{\partial x_i}(x,t) - \bar{v}(x,t) \kappa(x) \frac{\partial u}{\partial t}(x,t) \frac{\partial \bar{v}}{\partial t}(x,t) \right) dx dt = \int \bar{v}(x,t) f(x,t) dx dt.
\]

(3.11)

for all test functions \( v \in C^\infty_0(\mathbb{R}^n \times [0,T], \mathbb{C}) \). Now comes the point at which we need to choose our space of solutions. Equation (3.11) does not require \( u \) or \( f \) to be smooth. We will choose our solutions \( u \in H_1(\mathbb{R}^n \times [0,T], \mathbb{C}) \) and our forcing term \( f \in L^2(\mathbb{R}^n \times [0,T], \mathbb{C}) \). These choices turn out to give us uniqueness and existence of solutions to (3.11).

We choose initial conditions:

\[
u(x,0) = u_0(x),
\]

(3.12)

\[
\frac{\partial u}{\partial t}(x,0) = u_1(x),
\]

(3.13)

with \( u_0 \in H_{(1)}(\mathbb{R}^n, \mathbb{C}) \) and \( u_1 \in L^2(\mathbb{R}^n, \mathbb{C}) \). A priori we do not know whether \( u(x,0) \) and \( \frac{\partial u}{\partial t}(x,0) \) are well defined, because general \( L^2 \)-functions cannot be restricted to a hyperplane. In our case they can be; this will be discussed below equation (3.19).

We are going to view \( u(x,t) \) as a function \( t \mapsto u(t) \) with values \( u(t) : x \mapsto u(x,t) \) in a Hilbert space of functions on \( \mathbb{R}^n \). Fubini’s theorem says that if a function \( h : X \times Y \to \mathbb{C} \) is absolutely integrable, we can calculate the integral over \( X \times Y \) as a repeated integral, that is:

\[
\int_{X \times Y} hd(x,y) = \int_X \int_Y h(x,y) dy dx.
\]

(3.14)

In particular this means we can identify the space \( L^2(X \times Y, \mathbb{C}) \) with \( L^2(X, L^2(Y, \mathbb{C})) \).

If we apply this to the situation as described above we obtain \( u \in H_1(\mathbb{R}^n \times (0,T], \mathbb{C}) \) if and only if \( u \in L^2((0,T], H_1(\mathbb{R}^n, \mathbb{C})) \) and \( u' \in L^2((0,T), L^2(\mathbb{R}^n, \mathbb{C})) \). For convenience we will define \( V = H_1(\mathbb{R}^n, \mathbb{C}) \) and \( H = L^2(\mathbb{R}^n, \mathbb{C}) \).

Let the operators \( A,C \) be defined as:

\[
A = -\sum_i \frac{\partial}{\partial x_i} \frac{1}{\rho(x)} \frac{\partial}{\partial x_i},
\]

(3.15)

\[
C = \kappa(x).
\]

(3.16)

Then we have certain symmetric forms associated with \( A \) and \( C \):

\[
a(u,v) = \int \sum_i \frac{1}{\rho(x)} \frac{\partial u}{\partial x_i}(x) \frac{\partial \bar{v}}{\partial x_i}(x) dx
\]

(3.17)

\[
c(u,v) = \int \kappa(x) u(x) \bar{v}(x) dx
\]

(3.18)
and with these definitions our differential equation becomes:
\[
\int_0^T \left[ a(u(t), v(t)) - c(u(t), v(t)) \right] dt = \int_0^T \langle f(t), v(t) \rangle dt.
\] (3.19)
for all \( v \in C_0^\infty([0, T], V) \), which can be explained as the weak form of:
\[
Au(t) + C \frac{\partial^2 u}{\partial t^2} = f(t).
\] (3.20)
The initial conditions above can then be written as:
\[
\begin{align*}
    u(0) &= u_0, \\
    u'(0) &= u_1,
\end{align*}
\] (3.21)
(3.22)
with \( u_0 \in V, u_1 \in H \). Again, since \( t \mapsto u(t) \) is only \( L^2 \), \( u(0) \) is not defined a priori. But because \( u' \in L^2((0, T), H) \) we have \( u \) lying in a version of a Sobolev space, only with values in a Hilbert space in stead of \( \mathbb{R} \). If we accept that such Sobolev spaces have the same Sobolev embedding theorems, we find that \( u \) is Hölder continuous with exponent \( \frac{1}{2} \), and we can restrict a continuous function to a hyperplane. Also, since \( u \) satisfies the above differential equation, we know that \( u'' \in L^2(0, T), V' \). So by the same argument we know that \( u' \) is Hölder continuous with exponent \( \frac{1}{2} \) and can be restricted to a hyperplane.

We will require \( \frac{1}{\rho(x)} \) and \( \kappa(x) \) to be measurable and bounded away from zero, in particular we require there to be constants \( \alpha, \beta \geq 0 \) such that
\[
\begin{align*}
    \frac{1}{\rho(x)} &\geq \alpha, \\
    \kappa(x) &\geq \beta.
\end{align*}
\] (3.23)
(3.24)
We would like to mention that since the wavespeed is given by \( \sqrt{\kappa/\rho} \), [5], this corresponds to \( c \) being measurable and bounded away from zero.

This gives us for \( \lambda > \alpha \):
\[
\begin{align*}
    c(u, u) &\geq \beta \| u \|_H^2, \\
    a(u, u) + \lambda \| u \|_H^2 &\geq \alpha \| u \|_V^2.
\end{align*}
\] (3.25)
(3.26)
For convenience we will from now on write \( | \cdot | \) for \( \| \cdot \|_H \) and \( \| \cdot \| \) for \( \| \cdot \|_V \).

### 3.3 Existence and uniqueness of solutions

Now that we have our ordinary differential equation we will show uniqueness of solutions and continuous dependence on parameters. To do this we define the energy of the system as follows:
\[
E(t) = \frac{1}{2} (a(u(t), u(t)) + c(u'(t), u'(t)) + \lambda |u(t)|^2).
\] (3.27)
Now we can prove a lemma about our energy:
Lemma 3.3.1. Suppose $a,c,u,f$ are as above and $u$ satisfies (3.19). Then there is a subset $I_0$ of $(0,T)$ that differs from $(0,T)$ by a set of measure zero, such that the energy as defined above is Hölder continuous of order $\frac{1}{2}$ on $I_0$ and it obeys

$$E(t) \leq C \left( E(s) + \int_0^T |f(\sigma)|^2 d\sigma \right),$$

(3.28)

for all $s,t \in I_0$, $C = C(\alpha, \beta, \lambda, T)$.

Proof. In order to proof the theorem we will use a regularizing sequence, that is, a sequence $(\rho_m)$ of functions in $C_0^\infty((0,T))$, which satisfies

$$\rho_m(t) \geq 0,$$  

(3.29)

$$\int \rho_m(t) dt = 1,$$  

(3.30)

$$\text{supp} \rho_m \to \{0\}.$$  

(3.31)

Now if $h \in L^p((0,T))$, then $h \ast \rho_m(s) \in C_0^\infty((0,T))$ with $n^{th}$ derivative $h \ast \rho_m^{(n)}(s)$ and we have $h \ast \rho_m(s) \to h(s)$ in $L^2$-norm if $m \to \infty$ [3]. If we now define $v(t) = \rho_m(s-t)(\rho_m' \ast u)(s)$, then $v \in C_0^\infty([0,T], V)$. Because of this $u$ satisfies equation (3.19) with this choice of $v$. However $\int_0^T \rho_m(s-t)u(t)dt$ equals $\rho_m \ast u(s)$, thus:

$$a(\rho_m \ast u(s), \rho_m' \ast u(s)) + c(\rho_m'' \ast u(s), \rho_m' \ast u(s)) = (\rho_m \ast f(s), \rho_m' \ast u(s)).$$

Next we define:

$$u_m(s) = \rho_m' \ast u(s),$$  

(3.32)

$$f_m(s) = \rho_m \ast f(s),$$  

(3.33)

$$E_m(s) = \frac{1}{2}(a(u_m(s), u_m(s)) + c(u_m'(s), u_m'(s)) + \lambda |u(s)|^2).$$  

(3.34)

Now, we want to differentiate $E_m$ to find bounds for $E_m(t)$ and then let $m \to \infty$ to obtain the statement in the lemma. The derivative of the inner product of two functions can be found as follows

$$\frac{d}{ds} \langle u(s), v(s) \rangle = \lim_{h \to 0} \frac{(u(s+h), v(s+h)) - (u(s), v(s))}{h}$$

(3.35)

$$= \lim_{h \to 0} \langle u(s+h), \frac{v(s+h) - v(s)}{h} \rangle + \langle \frac{u(s+h) - u(s)}{h}, v(s) \rangle$$

(3.36)

$$= \langle u(s), v'(s) \rangle + \langle u'(s), v(s) \rangle,$$  

(3.37)

where we used the continuity of the inner product to interchange the innerproduct and the limit in the last step. If we apply this to $u = v$ we find:

$$\frac{d}{ds} \langle u(s), u(s) \rangle = \langle u(s), u'(s) \rangle + \langle u'(s), u(s) \rangle = 2 \text{Re}(u(s), u'(s)).$$  

(3.38)

16
Using this relation we can differentiate $E_m(s)$ to obtain:

\[
\frac{dE_m}{ds} = \text{Re}(a(u_m(s), u_m'(s)) + c(u''_m(s), u'_m(s)) + \lambda \langle u_m(s), u'_m(s) \rangle) \quad (3.39)
\]

\[
= \text{Re}\left(\langle f_m(s), u'_m(s) \rangle + \lambda \langle u_m(s), u'_m(s) \rangle\right), \quad (3.40)
\]

where we used equation (3.3) in the second step. If we send $m \to \infty$ $u_m, u'_m$ and $f_m$ go to $u, u'$ and $f$ respectively in the appropriate $L^2$-norms, therefore all the inner products and the energy $E_m$ go to their unconvolved version in $L^1([0, T])$. So if we let $m \to \infty$ equation (3.40) becomes:

\[
\frac{dE}{dt} = \text{Re}\left(\langle f, u \rangle + \lambda \langle u, u' \rangle + \lambda \langle u, u' \rangle\right). \quad (3.41)
\]

This means $\frac{dE}{dt} \in L^1((0, T))$ and thus $t \mapsto E(t)$ is continuous and bounded. Because of (3.25), (3.26) it follows that $||u(t)||, |u'(t)|$ are bounded. It follows:

\[
\frac{dE}{dt} \leq |f(t)||u(t)| + \lambda |u(t)||u'(t)| \leq C_1|f(t)| + C_2. \quad (3.42)
\]

Integrating this we obtain:

\[
|E(t) - E(s)| \leq \int_s^t (C_1|f(\sigma)| + C_2d\sigma)
\]

\[
= C_1 \int_0^T 1_{[s, t]}|f(\sigma)|d\sigma + C_2(t - s) \quad (3.43)
\]

\[
= C_1 \sqrt{t-s} \left[ \int_0^T |f(\sigma)|d\sigma \right]^{1/2} + C_2(t - s). \quad (3.44)
\]

It follows that $E(t)$ is Hölder continuous of order $\frac{1}{2}$ on $I_0$.

For any inner product the following inequalities hold:

\[
\langle a - b, a - b \rangle = |a|^2 - 2 \text{Re}(\langle a, b \rangle) + |b|^2 \geq 0 \quad (3.46)
\]

\[
\text{Re}(\langle a, b \rangle) \leq \frac{1}{2}(|a|^2 + |b|^2) \quad (3.47)
\]

If we integrate equation (3.40) and apply the above inequality we find:

\[
E_m(t) \leq E_m(s) + \int_s^t \frac{1}{2} (|f_m(\sigma)|^2 + (1 + \lambda)|u'_m(\sigma)|^2 + \lambda |u_m(\sigma)|^2). \quad (3.48)
\]

Using the boundedness of $\frac{1}{\rho(x)}$ and $\kappa(x)$ we find:

\[
E_m(t) \leq E_m(s) + \int_0^T \frac{1}{2} |f(\sigma)|^2 d\sigma + C \int_s^t E_m(\sigma)d\sigma. \quad (3.49)
\]
Applying Gronwall’s lemma we obtain:

$$E_m(t) \leq C \left( E_m(s) + \int_0^T |\frac{1}{2} f(\sigma)|^2 d\sigma \right),$$

with $C$ depending on $\alpha, \beta, \lambda, T$. If we now take the limit $m \to \infty$ we obtain (3.3.1). □

In [4] they now continue to proof that $u$ and $u'$ are continuous after modifications on a set of measure zero. We will omit these proofs and refer to [4] for them as they do not fall within the scope of this bachelor thesis.

We will now state the final result on existence and uniqueness of solutions to the problem (3.11),(3.12),(3.13). The proof for existence again falls outside the scope of this bachelor thesis, but we will give the proof of the uniqueness of the solutions and their continuous dependence on parameters.

**Theorem 3.3.2.** The problem as described above has a solution $u \in C([0,T],V) \cap C^1([0,T],H)$. The solution $u$ is unique and depends continuously on $u_0$, $u_1$ and $f$.

**Proof.** Uniqueness

Suppose $\tilde{u}$, $\bar{u}$ are both solutions to the problem. Then $u = \tilde{u} - \bar{u}$ is a solution to the differential equation with $f = 0$, $u_0 = 0$ and $u_1 = 0$. Since $u(t)$ is continuous in $V$ and $u'(t)$ in $H$ [4], we can use estimate (3.3.1) and find:

$$E(t) \leq C \left( E(0) + \int_0^T |f(\sigma)|^2 d\sigma \right)$$

$$= C \left( ||\tilde{u}_0||^2 + ||\bar{u}_1||^2 + \int_0^T |f(\sigma)|^2 d\sigma \right) = 0.$$

Thus by (3.25), (3.26) $u = 0$ and $\tilde{u} = \bar{u}$

Continuous dependence on parameters

Now suppose we have two problems, one with $\tilde{u}_0, \tilde{u}_1, \tilde{f}$ and one with $\bar{u}_0, \bar{u}_1, \bar{f}$. Then $u_0$ solves the differential equation with $u_0 = \tilde{u}_0 - \bar{u}_0$, $u_1 = \tilde{u}_1 - \bar{u}_1$, $f = \tilde{f} - \bar{f}$. Equation (3.3.1) gives us:

$$E(t) \leq C \left( ||\tilde{u}_0 - \bar{u}_0||^2 + ||\tilde{u}_1 - \bar{u}_1||^2 + \int |f(\sigma)|^2 d\sigma \right),$$

which implies continuity of the problem with respect to $u_0$, $u_1$ and $f$. □


4 Born forward modeling

In this chapter we construct approximate solutions to the wave equation given the wave speed function. We do this by linearizing the wave equation. This is called Born modeling or the first order perturbation approximation. We will perform the same linearization for the Helmholtz equation to obtain approximate solutions in the Fourier domain. Finally we will show how the results from the time domain and frequency domain agree. The results of this chapter will be the starting point for obtaining migration formulas in the next chapter.

4.1 In time domain

Let \( s(x) \) be the slowness field defined by \( s(x) = 1/c(x) \). Then the wave equation (2.1) can be recast as:

\[
\left( s(x)^2 \frac{\partial^2}{\partial t^2} - \Delta \right) u(x,t) = f(x,t). \tag{4.1}
\]

The idea is to write the slowness field \( s(x) \) as the sum of a "slowly varying", smooth background field \( s_0(x) \) and a "rapidly varying", perhaps even discontinuous, perturbation field \( \delta s(x) \). We also decompose our solution \( u(x,t) \) into a background solution \( u_0(x,t) \), which solves equation (4.1) with \( s(x) = s_0(x) \), and a perturbation \( \delta u(x,t) \). Furthermore, we let \( g_0(x,x_s,t,t_s) \) be the background Green’s function which solves equation (4.1) with \( s(x) = s_0(x) \) and \( f(x,t) = \delta(x-x_s)\delta(t-t_s) \), so that we can write \( u_0(x,t) \) as:

\[
u_0(x,t) = \int \int g(x,x_s,t,t_s)f(x_s,t_s)dt_sdx_s. \tag{4.2}\]

If we now plug in \( u(x,t) = u_0(x,t) + \delta u(x,t) \) and \( s(x) = s_0(x) + \delta s(x) \) into equation (4.1) and ignore second order terms (and thereby linearize the problem) we obtain:

\[
\left( s_0^2 \frac{\partial^2}{\partial t^2} + 2s_0\delta s \frac{\partial^2}{\partial t^2} - \Delta \right) (u_0 + \delta u) = f, \tag{4.3}\]

\[
\left( s_0^2 \frac{\partial^2}{\partial t^2} - \Delta \right) \delta u + 2s_0\delta s \frac{\partial^2}{\partial t^2} u_0 = 0, \tag{4.4}\]

\[
\left( s_0^2 \frac{\partial^2}{\partial t^2} - \Delta \right) \delta u = -2s_0\delta s \frac{\partial^2}{\partial t^2} u_0. \tag{4.5}\]

Notice that this is the wave equation for \( \delta u \) with forcing term \(-2s_0\delta s \frac{\partial^2}{\partial t^2} u_0 \). We interpret \( 2s_0\delta s \) as an approximate reflection distribution. Then equation (4.5) can be interpreted.

19
as follows: whenever the background wave is at a reflector (that is: a place where \(2s_0\delta s\) is not zero) it serves as a source for the perturbation field. In this sense the perturbation solution \(\delta u\) can be viewed as a reflected wave, taking into account only first order reflections.

Because equation (4.5) is just the wave equation with forcing term \(-2s_0\delta s \frac{\partial^2}{\partial t^2} u_0\) we can write its solution using the background Green’s function. Since the operators \(s_0^2 \frac{\partial^2}{\partial t^2} - \Delta\) and \(-\frac{\partial^2}{\partial t^2}\) commute we can also solve the problem with a forcing term \(2s_0\delta su_0\) and then apply \(-\frac{\partial^2}{\partial t^2}\). This gives:

\[
\delta u(x,t) = -\frac{\partial^2}{\partial t^2} \int \int 2s_0(x')\delta s(x')u_0(x',t')g_0(x,t,x',t')dx'dt'. \tag{4.6}
\]

A case of special interest is \(f(x,t) = \delta(x - x_s)\delta(t - t_s)\), so that \(u_0(x,t) = g_0(x,x_s,t,t_s)\) and \(u(x,t) = g(x,x_s,t,t_s)\), in which case \(\delta u(x,t)\) can be viewed as a perturbation on the system’s Green’s function, we will call it \(\delta g\). Plugging this in gives:

\[
\delta g(x,x_s,t,t_s) = -\frac{\partial^2}{\partial t^2} \int \int 2s_0(x')\delta s(x')g_0(x',t',x_s,t_s)g_0(x,t,x',t')dx'dt'. \tag{4.7}
\]

If we take the background slowness field \(s_0\) to be constant, we know the background Green’s function and can therefore give a more explicit formula for \(\delta g\):

\[
\delta g(x,x_s,t,t_s) = -\frac{\partial^2}{\partial t^2} \int \int 2s_0\delta s(x') \frac{\delta(t - t' - |x - x'|/c)}{4\pi|x - x'|} \frac{\delta(t' - t_s - |x' - x_s|/c)}{4\pi|x' - x_s|} dx'dt'. \tag{4.8}
\]

If we evaluate the \(t'\)-integral this further simplifies to:

\[
\delta g = -\frac{\partial^2}{\partial t^2} \int_{\mathbb{R}_+^2} 2s_0\delta s(x') \frac{\delta(t - t_s - |x - x'|/c - |x' - x_s|/c)}{16\pi^2|x' - x_s||x - x'|} dx'. \tag{4.9}
\]

This formula can be interpreted as follows: for a fixed location \(x\) and time \(t\) the perturbed Green’s function \(\delta g(x,t,x_s,t_s)\) is given by weightedly integrating the reflection distribution over the ellips \(|x' - x_s| + |x - x'| = c(t - t_s)\), see figure 4.1.

![Figure 4.1: Illustration of the integration domain for equation (4.9).](image-url)
4.2 In frequency domain

Here we will repeat the process from the last section, but then in the frequency domain. That is, we will linearize the Helmholtz equation in a similar way as we did for the wave equation. We let $k(x) = \omega s(x)$ and again write $s(x) = s_0(x) + \delta s(x)$ where $s_0(x)$ is the background slowness field and $\delta s(x)$ a perturbation on the slowness field. Furthermore we let $U_0$ be the solution to the Helmholtz equation with $k_0(x) = \omega s_0(x)$ and $G_0(x, x_s)$ the corresponding Green’s function. We write $U(x) = U_0(x) + \delta U(x)$ and $\delta k(x) = \omega \delta s(x)$. Plugging this all into the Helmholtz equation and ignoring second order terms we obtain:

\[
(\Delta + k_0^2 + 2k_0 \delta k(x))(U_0 + \delta U) = -h(x),
\]
\[
(\Delta + k^2)\delta U = -2k_0 \delta k U_0 = -2\omega^2 s_0 \delta s U_0.
\]

Using the background Green’s function we obtain a solution:

\[
\delta U(x) = \omega^2 \int 2s_0(x')\delta s(x')U_0(x')G_0(x, x')dx'.
\]

If the source is $f(x) = \delta(x - x_s)$ then $U(x) = G(x, x_s) = G_0(x, x_s) + \delta G(x, x_s)$ and:

\[
\delta G(x, x_s) = \omega^2 \int 2s_0(x')\delta s(x')G_0(x', x_s)G_0(x, x')dx'.
\]

And if we take the background slowness to be constant this gives us:

\[
\delta G(x, x_s) = \omega^2 \int 2s_0\delta s(x') \frac{e^{ik_0|x' - x_s|}}{4\pi|x' - x_s|} \frac{e^{ik_0|x - x'|}}{4\pi|x - x'|} dx'.
\]

4.3 Comparison of results

In this section we will show that the results of the Born approximation in time domain and in frequency domain agree. First we will show that equations (4.14) is the Fourier transform of equation (4.8). Since the amplitude factors agree I will focus on the part concerning time dependence. Let $f_1(t) = \delta(t - |x - x'|/c)$, $f_2(t) = \delta(t - t_s - |x - x_s|/c)$. Then we find:

\[
(f_1 * f_2)(t) = \int \delta(t - t' - |x - x'|/c)\delta(t' - t_s - |x - x_s|/c)dt',
\]

which we recognize as the time dependent part of equation (4.8). The Fourier transforms of $f_1$ and $f_2$ are $\hat{f}_1(\omega) = e^{i\omega|x-x'|/c}$ and $\hat{f}_2(\omega) = e^{i\omega|x-x_s|/c}e^{i\omega t_s}$. If we now apply the convolution theorem (2.9) we find:

\[
\hat{f}_1 * \hat{f}_2(\omega) = \hat{f}_1(\omega)\hat{f}_2(\omega) = e^{i\omega|x-x'|/c}e^{i\omega|x-x_s|/c}e^{i\omega t_s}.
\]

The factor $e^{i\omega t_s}$ is a phase shift we don’t have in the frequency domain. However if we Fourier transform equation (2.2) we would also get this term, which would then carry
to the solution. The second time derivative in equation (4.8) will turn to a factor \(-\omega^2\) in the frequency domain. If we plug in \(k = \frac{\omega}{c}\) and ignore the phase term (that is, assume the pulse was sent at \(t = 0\)) we find that the Fourier-transform with respect to time of (4.8) is (4.14).

We will now show a more general version of this. That is, we will show that equation (4.12) is the Fourier transform of (4.6) if \(h = \hat{f}\). In that case we have \(\hat{u}_0 = U_0\) and \(\hat{g}_0 = G_0 e^{i\omega t s}\). In general we know that the time dependence of the Green’s function for the wave equation with parameters independent of time to be of the form \(g(x, t, x_s, t_s) = \psi(x, x', t - t_s)\), see section 2.4. If we then again look at the time-dependent part of equation (4.6) we can define \(f_1(t) = g_0(x, x', t, 0) = \psi(x, x', t)\) and \(f_2(t) = u_0(x', t)\) and find:

\[
(f_1 \ast f_2)(t) = \int f_1(t - t') f_2(t') dt' = \int \psi(x, x', t - t') u_0(x', t') dt' = \int g_0(x, x', t, t') u_0(x', t') dt',
\]

which we recognize as the time dependent part of equation (4.6). If we now apply a Fourier transform in the time domain and use the convolution theorem we obtain

\[
(\hat{f_1} \ast \hat{f_2})(\omega) = \hat{f}_1(\omega) \hat{f}_2(\omega) = G_0(x, x') U_0(x')
\]

which is exactly what we find in equation (4.12). The \(-\frac{\partial^2}{\partial t^2}\) in (4.6) becomes the \(\omega^2\) in (4.12) and so we see that equation (4.12) is indeed the Fourier transform of equation (4.6).
5 Migration

This chapter consists of three parts. In the first part we give an argument based on [1] why we should apply the adjoint of the operator that turns the reflectivity function into the approximate solution to the data to obtain an image of the subsurface. A much stronger result was proven by Beylkin in [9]; he proved that the resulting adjoint operator gives back the discontinuities of the wave speed function, but to prove this we would need quite some mathematics that is far beyond the scope of this bachelor thesis. We also give some interpretation of the resulting formulas. In the second part we illustrate the ideas of Born forward modeling and migration with a MATLAB programme from [6]. In the third part we discuss some practical issues of migration.

5.1 Derivation of migration formula from Born forward modeling

In this section we will use the Born Forward Model derived in the last chapter to obtain a method for imaging. This is based on [1].

We will, for now, work in the frequency domain. The experiment consists of sources \( s \) which cause waves in the subsurface, which are then measured by geophones \( g \). We will be interested in the wavefield due to a specific source and will hence use the following notation: \( U_0(x|s) \) will denote the background wavefield due to source \( s \) and \( \delta U(x|s) \) will denote the perturbation wavefield due to source \( s \). In the time domain we will use the notation \( u_0(x,t|s) \) and \( \delta u(x,t|s) \). Using this notation we write the approximation we found in the previous chapter in the following form:

\[
\delta U(x|s) = \omega^2 \int G_0(x,x')m(x')U_0(x'|s)dx',
\]

(5.1)

where \( m(x) = 2s_0(x)\delta s(x) \) is the reflection distribution [1]. We now regard \( U_0(x|s) \) as a ”downgoing wave” caused by the source and \( \delta U(x|s) \) as the reflected ”upgoing wave”. We therefore expect to measure \( \delta U(x|s) \) at the location of the geophones. Thus if we call \( D(g|s) \) the data measured at geophone \( g \) due to a signal sent out by source \( s \) we can re-interpret the equation as:

\[
D(g|s) = \omega^2 \int G_0(x,x')m(x')U_0(x'|s)dx'.
\]

(5.2)

Since the right side is linear in the function \( m \) we may write this as a linear equation:

\[
d = Lm.
\]

(5.3)
Where $d$ stands for the data $D(g|s)$, $m$ for the reflection distribution function and $L$ for the linear operator as in equation (5.2). When actually doing calculations the problem will always be discretized and we may approximate equation (5.3) by a matrix equation:

$$d_i = \sum_{j=1}^{N} L_{ij} m_j, \quad (5.4)$$

where we divided the region under interest into a regular grid with $N$ cells, $d_i$ is the data vector for source-geophone pair $i$ ($i \in \{1, \ldots, M\}$), $m_j = m(x_j)$ with $x_j$ the center of the $j$th cell, and $L_{ij} = \omega^2 G_0(g_i, x_j) U_0(x_j|s)$ where $U_0(x_j|s)$ is the background wave caused by a source at location $s$.

In general this system of equations is overdetermined and should therefore be interpreted in a least squares sense, that is we try to find $m$ such that $\|d - Lm\|_2 \leq \|d - \tilde{L}m\|_2$ for all $\tilde{m} \in \mathbb{R}^N$. Note that $\{Lm | m \in \mathbb{R}^N\} = \text{Col}(L)$. So we are trying to find $y = Lm \in \text{Col}(L)$ that minimizes $\|y - d\|$. But this is just finding the best approximation of $d$ in $\text{Col}(L)$ and we know this is the orthogonal projection of $d$ on $\text{Col}(Y)$, which is characterized by $y - d \perp \text{Col}(Y)$. Being orthogonal to $\text{Col}(L)$ means being orthogonal to the columns of $L$. Thus $y$ is the best approximation of $d$ in $\text{Col}(L)$ if $L^*(y - d) = 0$. If we plug in $y = Lm$ we obtain the so-called normal equations:

$$L^*Lm = L^*d. \quad (5.5)$$

If $L^*L$ is invertible the solution is:

$$m = (L^*L)^{-1} L^*d. \quad (5.6)$$

This can be approximated by:

$$m_i = \frac{(L^*d)_i}{(L^*L)_{ii}}, \quad (5.7)$$

when $L^*L$ is "diagonally dominant" [1]. We can then also normalize $L^*L$ so the diagonal elements are normalized to unity. We then obtain the compact formula

$$m \approx L^*d, \quad (5.8)$$

which says that the model can be obtained by applying the adjoint of the forward modelling operator on the data. If we write this out in a matrix equation it reads

$$m_j = \sum_{i=1}^{M} L_{ji} d_i = \sum_{i=1}^{M} \omega^2 G_0(g(i)|x_j) d_i U_0(x_j|s(i)). \quad (5.9)$$

If we remember where this equation came from we can see that this is an approximate form of:

$$m(x) \approx \int_{-\infty}^{\infty} \int_{g} \int_{s} \omega^2 G_0(g, x) D(g|s) U_0(x|s) dgdsd\omega. \quad (5.10)$$
where we integrate over all frequencies to stack together the images for all frequencies. Note that if we choose suitable function spaces for \( m(x) \) and \( D(g|s) \) with inner products \( \langle f(x), g(x) \rangle_1 = \int f(x)g(x)dx \) and \( \langle A(g|s), B(g|s) \rangle_2 = \int_{-\infty}^{\infty} \int A(g|s)B(g|s)dgdsd\omega \) it would also be the adjoint in that sense.

Equation (5.10) is called the migration formula. We will now give an interpretation. Let us write equation (5.10) in a more suggestive form:

\[
m(x) \approx \int_{s} \int_{-\infty}^{\infty} \left[ \int_{g} G_0(g, x) D(g|s) dg \right] \left[ \omega^2 \overline{U_0(x|s)} \right] d\omega ds.
\]

(5.11)

If we remember that \( G_0(x, x_s) \) is the acausal Green’s function we may interpret \( \int_{g} G_0(g, x) D(g|s) dg \) as a backpropagated version of the wavefield measured by the geophones. If we now apply formula (2.14) from section 2.3 to equation (5.10) we see that it can be written in the following form

\[
m(x) \approx -\frac{1}{2\pi} \int_{s} \int_{-\infty}^{\infty} F^{-1} \left[ \int_{g} G_0(g, x) D(g|s) dg \right] \frac{\partial^2}{\partial t^2} u_0(x, t|s) dt ds,
\]

(5.12)

which means we are multiplying the backpropagated upward going reflected wave with the second time derivative of the forward propogated wave at time \( t \) and location \( x \) over all time to give us our reflection distribution at location \( x \). Intuitively this makes sense because when the upgoing wave and downgoing wave are at the same location at the same time we expect there to be a reflector at that location. This method of migration is called Reverse Time Migration or RTM for short.

We could also directly use an adjoint for equation (4.8). This is called Kirchhoff migration and is described in [11].

5.2 MATLAB

Here we will discuss a series of MATLAB files from [6], which illustrate the born modeling and migration which we have discussed in the previous sections. This is done for a two dimensional situation. The code can be applied to different velocity models. We will use a three layer system to describe the different steps. We divide the area in a grid of \( dx \times dx \) sized cells. A velocity model is defined in the form of a matrix whose \((j, k)\)-th entry represents the wave velocity in the \((j, k)\)-th cell, where \( j \) represents the \( x \)-coordinate and \( k \) the \( z \)-coordinate. The slowness field is then calculated as the reciprocal of the velocity field. Then it is smoothed to obtain a background slowness field. The difference between the original slowness field and the smoothed slowness field then becomes \( \delta s \).

The smoothing is done by taking for every entry of the matrix the average of a small box of the matrix centred at that entry. If we do this for our three layer system we obtain the fields as in figure 5.1.
Figure 5.1: Velocity model used for Born forward modeling and RTM. On the left the "true" velocity field is shown. In the middle the smoothed velocity field, which is used as the background velocity. On the right the reflection distribution derived from the background and true velocity field is shown.

A source and geophone geography is defined. Then born modeling is done numerically to generate data. Afterwards this data is used to perform RTM and produce an image. Then a so-called adjoint test is done to see in how far RTM is the adjoint process of born modeling and if the code functions properly.

5.2.1 Finite difference method

To approximate the wave equation the MATLAB-programmes use the so-called Finite Difference method. In this method derivatives are replaced by finite differences. Because of this we can express the field at a next time step using the current field and the one from the previous time step. More specifically we use a second-order approximation for the second order time derivative:

\[
\left( \frac{\partial^2 u}{\partial t^2} \right)_{j,k}^n \approx \frac{u_{j,k}^{n+1} - 2u_{j,k}^n + u_{j,k}^{n-1}}{dt^2},
\]

and an eighth order approximation for the second spatial derivative in the x-direction:

\[
\left( \frac{\partial^2 u}{\partial x^2} \right)_{j,k}^n \approx \frac{1}{dx^2} \left( c_1 u_{j,k}^n + \sum_{i=2}^{5} c_i \left[ u_{j+i-1,k}^n + u_{j-i+1,k}^n \right] \right).
\]

Where the constants \( c_i \) are respectively -205/7, 8/5, -1/5, 8/315 and -1/560. Where these come from is discussed in [7]. The second spatial derivative in the y-direction is approximated in the same manner as the one in the x-direction. If we substitute this into the wave equation (2.1) we obtain:

\[
\frac{1}{v_{j,k}^2} \frac{u_{j,k}^{n+1} - 2u_{j,k}^n + u_{j,k}^{n-1}}{dt^2} = \frac{1}{dx^2} \left( 2c_1 u_{j,k}^n + \sum_{i=2}^{5} c_i \left[ u_{j+i-1,k}^n + u_{j-i+1,k}^n \right] \right) + f_{j,k}^n.
\]
where $u_{j,k}^n$ is the wavefield at the $(j,k)$-th $x,z$ location and at the $n$th time step, $v_{j,k}$ is the velocity field at the $(j,k)$-th position and $f_{j,k}^n$ represents the forcing term. We now define $\alpha_{j,k} = v_{j,k}^2 dt^2/dx^2$. We can rewrite the equation to calculate the wavefield at timestep $n + 1$ from the wavefield at the previous two timesteps:

$$u_{j,k}^{n+1} = (2 + 2c_1)u_{j,k}^n - u_{j,k}^{n-1} + \alpha_{j,k} \left(2c_1u_{j,k}^n + \sum_{i=2}^{n} c_i \left[u_{j+i-1,k}^n + u_{j-i+1,k}^n + u_{j,k+i-1}^n + u_{j,k-i+1}^n\right]\right) + \beta_{j,k}f_{j,k}^n. \quad (5.16)$$

An important thing to note is that we use a finite matrix to describe our system when computing. If we used normal boundaries for our region of interest we would get reflections that we do not want. A solution to this would be to use a much larger matrix than our region of interest, so that the waves do not have the time to reflect and come back to our region of interest, but this is computationally very expensive. To deal with this so called absorbing boundary conditions are used: a layer is created around our region of interest in which the solutions decay. A detailed description of this falls outside the scope of this thesis, but for more information see [12].

### 5.2.2 Born modeling

In the programme doing the Born-modeling a background wavefield $P$ and a perturbation wavefield $Q$ are defined. The background wavefield is updated at each time step by equation (5.16) with $f_{j,k}^n$ a predetermined source function $s$ which only acts at the source location. For the three layer system a so-called Ricker wavelet was used. See figure 5.2.

![Ricker wavelet](image)

**Figure 5.2:** The ricker wavelet which is used as input signal.

To update the perturbed field we first need a Finite Difference version of equation (4.6). For this we need an approximation of $\frac{\partial^2 u}{\partial t^2}$. Note that if there is no source at a location then the background wavefield satisfies $\frac{\partial^2 u}{\partial t^2} = v^2 \Delta u$. If we use a second order
Finite Difference scheme for this we obtain:

\[
(m \frac{\partial^2 u}{\partial t^2})^n_{j,k} \approx m_{j,k}v_{j,k}(u_{j+1,k}^n + u_{j-1,k}^n + u_{j,k+1}^n + u_{j,k-1}^n - 4u_{j,k}^n).
\]  

(5.17)

So this term is used as the forcing term \(f^n_{j,k}\) for the perturbation field. In figure 5.3 we see a screenshot of the background and perturbation wavefield obtained with the born modeling code.

![Background wave field P, t=0.35 s](image1)

![Perturbation wave field Q, t=0.35 s](image2)

Figure 5.3: Snapshot of the Born modeling done in [6]. On the left the background wavefield is shown and on the right the perturbation wavefield.

Something worth noting is that the background field is not purely "downgoing", but also contains(weaker) reflections.

5.2.3 RTM

The RTM-function takes a velocity model and seismogram as input. It first uses forward modeling as in equation (5.16) to calculate the background wavefield at the final two timesteps. Notice now that equation (5.15) is symmetric in time. That is, we can also use it to calculate \(u_{j,k}^{n-1}\) from the next two timesteps:

\[
u_{j,k}^{n-1} = (2 + 2c_1)u_{j,k}^n - u_{j,k}^{n+1}
+ \alpha_{j,k} \left(2c_1u_{j,k}^n + \sum_{i=2}^{n} c_i [u_{j+i,k}^n + u_{j-i+1,k}^n + u_{j,k+i-1}^n + u_{j,k-i+1}^n] + \beta_{j,k}f_{j,k}^n\right).\]  

(5.18)

Using this the forward propogated downgoing wave is backpropogated in time again. At the same time the upward going wave is approximated by using the seismogramdata as sources. That is, each geophone acts a source for the backward propogation of the upgoing wave, in accordance with the Huygens’ principle. The image is created by multiplying the velocity squared times the laplacian of downgoing wavefield with the

28
upgoing wavefield at each timestep and summing, in accordance with equation (5.12). For our three layer system data was generated by born modeling and then used as input for the RTM function. The resulting image is shown in figure 5.4.

Figure 5.4: RTM image made using code from [6], based on the velocity model in 5.1 and a synthetic seismogram made with Born modeling.

With some imagination, we can see the two lines of reflection we started out with. There is also quite some noise. A part of this can be explained by the fact that the "downgoing" background wavefield also has upgoing components which will be multiplied with the backpropagated perturbation field to add to the image. Also we should keep in mind that the RTM is only an adjoint process and not an inverse process. We are also only using a finite amount of geophones. The lines are also deformed at the ends, causing them to be "smiles". This can be explained by the absence of measured data from the reflections at the end of the lines, because there are no geophones more to the right or to the left. Another important cause of artefacts relates to aperture: we cannot measure all directions and therefore miss k-vectors in our spectrum. Also the ricker wavelet misses the lower frequencies. For more information on this, see [11].

In figure 5.5 we see an image created by using a constant background velocity and a "single-point" reflectivity function. The same process of using Born modeling to obtain a seismogram and then using RTM to make an image was used. We see that instead of just getting back the point, we get a "cross" around the point. This can be explained by a lack of frequencies in the input signal. Again, for more information, see [11].
5.2.4 Adjoint test

In section 5.1 we described RTM to be the adjoint of Born modeling. Remember that if $T : X \rightarrow Y$ is a linear operator between two Hilbert spaces we call $S : Y \rightarrow X$ its adjoint if:

$$\langle Tx, y \rangle_Y = \langle x, Sy \rangle_X,$$

for all $x \in X$ and $y \in Y$. Applying that to our situation we check whether the RTM function indeed acts as the adjoint of Born modeling. In the space where the reflectivity distribution lies we use:

$$\sum_{j,k} m_{j,k}^2$$

as inner product and for our data(seismogram) we use the following inner product:

$$\sum_{g,n} \left( u_{g,k}^n \right)^2.$$

This corresponds with the integral inner products in section 5.1 up to scaling(but the same scaling for both inner products). To test whether or not two operators are each other’s adjoints one can take some elements $x \in X$ and $y \in Y$ see if $\langle Tx, y \rangle = \langle x, Sy \rangle$. In the case of the three layer model this was done with $m$ and $Lm$ as elements and $L$ and $R$ as operators, where $m$ is the reflectivity distribution, $L$ is born forward modeling and
$R$ is RTM. The results were $\langle Lm, Lm \rangle = 957.0849$, $\langle m, RLm \rangle = 957.0667$. By using the obtained image from RTM as reflectivity for born modeling it was also possible to calculate $\langle Lm, LRLm \rangle = 13430328.0751$ and compare it with $\langle RLm, RLm \rangle = 13412715.6765$. These values support the idea that Born modeling and RTM are adjoint operations. The adjoint test also serves as a test for the functioning of the code.

5.3 Comments on practicality

We’d now like to make a short comment on the matrix mentioned in section 5.1. For our three layer system we have $201 \times 81 = 16281$ locations and 100 geophones. So, with our one source, that would mean that the matrix from section 5.1 would have 1628100 entries. And that is for just one time step. We also have 2001 timesteps. This is just for our simple small 2D system. In real situations these numbers grow quickly as there are many more geophones, more sources and a larger area under interest, which is also 3D. In practice the matrix is therefore not calculated explicitly. But even then huge amounts of data are necessary for the imaging process.

Also we need a background velocity to do RTM in a real situation. Finding a suitable background velocity field is called seismic velocity estimation. To go into detail about this falls outside the scope of this thesis, but for more information how it is done and can be described mathematically, see [5].
6 Conclusion

We have discussed the wave equation with variable constants and it was shown that it has unique solutions in an appropriate Sobolev space if the wave speed $c$ was measurable and bounded away from zero. It was also shown that the solution then depends continuously on $c$. This is important because if we do not have this continuity we cannot hope to find good approximations if we do not precisely know $c$. After this we linearized the wave equation and obtained the Born forward modeling formula. This was done in both the time and frequency domain and it was shown both results agree. This Born formula then lead to a migration formula, which consisted of applying the adjoint of the Born operator to the data to obtain an image. The procedures of Born forward modeling and migration were then illustrated using MATLAB code from [6].

Further research could be done to look at microlocal analysis and the results from Beylkin [9], which mathematically justifies all migration algorithms by proving that using the adjoint of the modeling operator gives back the right discontinuities of the parameterfunctions describing the subsurface. It would also be worthwhile to take a closer look at the concept of aperture and how this effects the images obtained by using migration methods. One could for example look at [11] for this. Another point of interest could be to take a better look at the numerical methods used to execute migration algorithms, such as Finite Difference methods and Absorbing boundary conditions.
Populaire samenvatting

Vaak wilt men in de seismologie een “afbeelding” van de ondergrond maken, bijvoorbeeld voor olie- of gasbedrijven zodat ze de locatie van olie en gas in de ondergrond kunnen vaststellen, maar het kan ook voor wetenschappelijke redenen interessant zijn. De manier waarop zo’n afbeelding wordt gemaakt is met golven. In het algemeen worden veel afbeeldingen gemaakt met behulp van golven. Denk hierbij aan optische experimenten, waarbij we objecten bekijken met behulp van lichtgolven, maar ook aan ultrasound echoscopie voor medische toepassingen of radar. Bij die laatste twee zijn het geluidsgolven die gebruikt worden om een afbeelding te maken. Dat is in ons geval ook zo. Er wordt dus een experiment met geluidsgolven uitgevoerd om informatie over de ondergrond te vinden. Als het experiment op land wordt uitgevoerd worden grote vibrerende trucks of explosieven gebruikt om die geluidsgolven te creëren. Die geluidsgolven weerkaatsen dan in de ondergrond en worden op het oppervlak gemeten met een soort microfoons, zie figuur 6.1. Zo’n experiment kan ook op het water worden uitgevoerd. Dan worden zogenaamde ”airguns” gebruikt om geluidsgolven te creëren en lijnen vol met microfoons op het wateroppervlak om de gereflecteerde golven te meten.

Als de golven gemeten zijn aan het oppervlak, moet aan de hand van die data een afbeelding gemaakt worden van de ondergrond. Daar gaat deze scriptie over. Wiskundig gezien worden golven beschreven door de golfvergelijking. De oplossing van de golfvergelijking.

Figure 6.1: Illustration of the seismic imaginging experiment on land from the website (http://www.innoseis.com/seismic-surveying/)
jking is een functie die voor elk punt en voor elk tijdstip zegt hoe de golf eruitziet. De
golfsnelheid speelt een rol in de golfvergelijking. Maar de golfsnelheid verschilt in ver-
schillende materialen en moet zelf dus ook beschreven worden door een functie van de
plaats. Het "exact" oplossen van de golfvergelijking met variabele golfsnelheid blijkt erg
lastig te zijn. Maar door wat versimpelingen te maken kunnen we wel een benaderende
oplossing vinden van de golfvergelijking met variabele golfsnelheid. Dit geeft ons dan een
"operator" die de golfsnelheid vertaalt naar de golffunctie. Eigenlijk zijn we natuurlijk
géïnteresseerd in het omgekeerde proces: gegeven de golffunctie op bepaalde plaatsen(die
we gemeten hebben), willen we juist de golfsnelheidsfunctie kunnen reconstrueren, want
die vertelt ons waar wat voor materialen in de ondergrond zitten. We willen wiskundig
gezegd eigenlijk een "inverse" van de net genoemde operator. Dit blijkt ook heel moeilijk
te zijn, maar we kunnen wel iets doen wat er op lijkt. Deze "benaderende inverse" geeft
ons dan de afbeelding waar we naar op zoek zijn.
Bibliography


