Order theory, Priestley duality and semantic interpretation of logic

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Bachelorthesis

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\[ \langle P(\{1, 2, 3\}), \subseteq \rangle \]
\[ \langle \mathbb{N}; \leq \rangle \]
\[ Q \subseteq \langle \mathbb{N}_0; \preceq \rangle \]
Abstract
After an introduction to order theory we will establish a way of associating any poset with a lattice endowed with extremely good properties. And we can also establish the reverse correspondence. But we are interested in a correspondence from posets to distributive lattices and back, which gets us to the same poset we started with. This we can do but only for special distributive lattices.
From the finite case we will arrive at the general case, were we will see that Boolean algebras are dual to Boolean spaces, and bounded distributive lattices are dual to Priestley spaces.
Classical propositional logic is closely related to this. We will introduce the Lindenbaum-Tarski algebra, which can be treated in the same way, obtaining very nice results about maximal consistent theories.

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Chapter 1

Introduction

This thesis starts with an introduction to order-theory. Sets can be ordered in different ways. Different ordering will give us different properties, which will be showed in examples in the first chapter, Chapter 2.

In Chapter 3 we will introduce lattices, which are partial ordered sets with very nice properties. Then we will establish a way of associating any poset with a lattice endowed with extremely good properties. And we can also establish the reverse correspondence, in a trivial way, because every distributive lattice is also a poset. But we are interested in a correspondence from posets to distributive lattices and back, which gets us to the same poset we started with. This we can do but only for special distributive lattices.

After having introduced join irreducibles in Chapter 5 we will be able to prove that every finite Boolean algebra is isomorphic to $\mathcal{P}(X)$ for some finite set $X$. And after that even that every Boolean algebra is isomorphic to a subalgebra of $\mathcal{P}(X)$. But it would be nice to have an isomorphism not only for the finite case, but for Boolean algebras in general. Therefore we will make in the last chapter the set of prime ideals of any Boolean algebra into a topology space in such a way that it is dual to a Boolean algebra.

At the end we will see that Boolean algebras are dual to Boolean spaces, and bounded distributive lattices are dual to Priestley spaces. For the Boolean algebras we will use the Stone representation theorem, and for the bounded distributive lattices there is Priestley’s representation theorem.

Also for the classical propositional logic these are all very nice results. Almost directly the logic will be implemented in this thesis, starting with the consequence relation that can be seen as an order relation. The Lindenbaum algebra can then be interpreted as a lattice and it is in fact a lattice. We will show a same kind of duality between the Lindenbaum algebra (which we will first manipulate to make it into a Boolean algebra) and a canonical model whose points are maximal consistent theories.
Chapter 2

Order

He is older than that girl. Flour is an ingredient of bread and of pasta. Amsterdam is the city with the greatest number of inhabitants of The Netherlands. In our daily lives we compare a lot of things. We form groups and order their elements. A group of people can be ordered by length, by age or by weight. But we could also order some cities by surface or by number of inhabitants. We see that there are many different things to order, and also the way of ordering can differ. We could order dishes by ingredients or by calories, and clothes by size or by color, and there are also various ways to order families. And further we have linear orders, for example time and lists.

The following chapter outlines the mathematical formalization of this idea, and how we can apply this in logic, where order for example models the consequence relation.

2.1 Order

2.1.1 Pre-orders and Partial orders

Definition 2.1. Let $P$ be a set. An order (or partial order) on $P$ is a binary relation $\leq$ on $P$ such that, for all $x, y, z \in P$,
1. $x \leq x$ (reflexivity)
2. $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)

A set $P$ equipped with an order relation $\leq$ is said to be an ordered set or a partially ordered set (poset). If we want to specify the order relation we write $\langle P, \leq \rangle$.

On any set $=$ is an order, the discrete order. A quasi-order is a relation $\leq$ on a set $P$ that is reflexive and transitive but not antisymmetric. This relation is also called a pre-order.
2.1.2 Examples

- We can order sets by inclusion. The inclusion relation is typically used to order the powerset \( \mathcal{P}(X) \), the set of all subsets of \( X \). For \( A, B \in \mathcal{P}(X) \) we say \( A \leq B \) if and only if \( A \subseteq B \), and we write \( \langle \mathcal{P}(X), \subseteq \rangle \).

- The set \( \mathbb{N}_0 \) can be ordered with its usual order, i.e. \( m \leq n \iff m = n \) or \( m < n \). This gives us the poset \( \langle \mathbb{N}_0; \leq \rangle \), where for all \( m, n \in \mathbb{N}_0 \) either \( m \leq n \) or \( n \leq m \). In general, an ordered set \( P \) is called a totally ordered set or a chain if for all \( p, q \in P \) either \( p \leq q \) or \( q \leq p \) (if any two elements of \( P \) are comparable). The poset \( \langle \mathbb{N}_0; \leq \rangle \) is a chain. A set \( X \) in which \( x \nmid y \) (\( x \not\leq y \) and \( y \not\leq x \)) for every \( x, y \in X \) is called an anti-chain.

- The set \( \mathbb{N}_0 \) can also be ordered in a different way. Define the order \( \preccurlyeq \) as \( x \preccurlyeq y \) if and only if \( x \) divides \( y \) (that is \( \exists k \in \mathbb{N}_0 \) such that \( kx = y \)). The obtained poset is \( \langle \mathbb{N}_0; \preccurlyeq \rangle \).

2.1.3 Partial orders and strict orders

**Example 2.2.** Let \( P \) be a set on which a binary relation \(<\) is defined such that, for all \( x, y, z \in P \),

a) \( x < x \) is false (antireflexive)

b) \( x < y \) and \( y < z \) imply \( x < z \)

We can prove that if \( \leq \) is defined by \( x \leq y \iff (x < y \) or \( x = y) \), then \( \leq \) is an order on \( P \). We have to check the three conditions for an order:

1. The relation \( \leq \) is reflexive, because \( \forall x \in P \) holds \( x = x \), hence \( x \leq x \)

2. Then \( x \leq y \) and \( y \leq x \) imply \( x = y \), because if not both are equalities, we have \( x < x \) which is false. The relation is antisymmetric.

3. Transitivity can be shown as follows: Let \( x \leq y \) and \( y \leq z \). Then we obtain one of the four following (in-)equalities: \( x < y < z \), \( x < y = z \), \( x = y < z \) or \( x = y = z \). This implies \( x < z \) or \( x = z \) and hence \( x \leq z \).

2.1.4 Order in propositional logic

**Definition 2.3.** Let \( Fm \) be any logical language. Then a consequence relation is a relation \( \vdash \subseteq \mathcal{P}(Fm) \times Fm \) which satisfies the following conditions

1. if \( \phi \in \Gamma \), then \( \Gamma \vdash \phi \).

2. if \( \Gamma \vdash \delta \) for every \( \delta \in \Delta \subseteq Fm \) and \( \Delta \vdash \phi \), then \( \Gamma \vdash \phi \).
3. if $\Gamma \vdash \phi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \phi$.

Note that 1. and 2. imply 3. Fix therefor $\Gamma \vdash \phi$ and $\Gamma \subseteq \Gamma'$. Then by 1., $\Gamma' \vdash \delta$ for all $\delta \in \Gamma$. Then because $\Gamma \vdash \phi$ and by 2. we obtain $\Gamma' \vdash \phi$.

If we now take the restriction of $\vdash$ to $Fm \times Fm$, where the first $Fm$ is identified with the subset $\{\phi\} | \phi \in Fm \subseteq \mathcal{P}(Fm)$, then this is a pre-order.

Reflexivity is certainly the case, by 1. and $\phi \in \{\phi\}$. The consequence relation is not antisymmetric, because it can be the case that $\phi \vdash \psi$ and $\psi \vdash \phi$ but $\phi \neq \psi$. Take for example the two formulas $\neg \phi$ and $\phi \rightarrow \bot$. We know $\phi \rightarrow \bot \vdash \neg \phi \lor \bot$, and $\neg \phi \lor \bot \vdash \neg \phi$, hence (by 2.) $\phi \rightarrow \bot \vdash \neg \phi$, and vice versa $\neg \phi \vdash \phi \rightarrow \bot$. But these two formulas are not the same. They are logically equivalent, i.e. they have the same truth values under any assignment, but they are not identical because they are different strings of symbols. Hence the consequence relation is not antisymmetric. But the relation is transitive, which follows directly from 2. If we take $\Gamma = \{\psi\}$ and $\Delta = \{\delta\}$, then if $\{\psi\} \vdash \delta$ and $\{\delta\} \vdash \phi$, we obtain $\{\psi\} \vdash \phi$.

The relation $\vdash$ is reflexive and transitive but not antisymmetric and hence a pre-order.

### 2.1.5 Pre-orders and equivalence relations

Let $\langle X, \leq \rangle$ be a pre-order. Define the relation $\equiv \subseteq X \times X$ as follows:

$$x \equiv y \iff x \leq y \text{ and } y \leq x.$$ 

The relation $\equiv$ is an equivalence relation. Then consider the quotient set $X/\equiv := \{[x] | x \in X\}$, where $[x] = \{y \in X | x \equiv y\}$. Now define the following relation over $X/\equiv$

$$[x] \leq_{\equiv} [y] \iff \exists x' \in [x], \exists y' \in [y]: x' \leq y'.$$

The relation $\leq_{\equiv}$ is a partial order.

We can apply this to logic. Take therefor $\langle Fm, \vdash \rangle$ as a pre-order, where $Fm$ is any logical language. The equivalence relation will be defined as

$$\phi \equiv \psi \iff \phi \vdash \psi \text{ and } \psi \vdash \phi \text{ (notation } \phi \vdash_{\equiv} \psi).$$

This is the interderivability relation, a proof theoretic relation, which tells us that $\phi$ logically implies $\psi$, and vice versa. The quotient set then becomes $Fm/\equiv := \{[\phi] | \phi \in Fm\}$, where $[\phi] = \{\psi \in Fm | \phi \equiv \psi\}$. We define also the following relation over $Fm/\equiv$

$$[\phi] \leq_{\equiv} [\psi] \iff \exists \phi' \in [\phi], \exists \psi' \in [\psi]: \phi' \vdash \psi'.$$

This relation is (as in the general case) a partial order.

We obtained a set the of equivalence classes equipped with the order relation $\leq_{\equiv}$. This is the partial ordered set $\langle Fm_{\equiv}, \leq_{\equiv} \rangle$. The pre-ordered set $\langle Fm, \vdash \rangle$ we started with, has now turned into a partial ordered set, introducing equivalence relations.

As we promised in the introduction of this chapter, a first connection already emerges between order and the consequence relation. In the next chapter the poset $\langle Fm_{\equiv}, \vdash_{\equiv} \rangle$ will become the Lindenbaum-Tarski algebra under the condition that $\equiv$ is a congruence.
2.1.6 Lexicographic order

Let $P$ and $Q$ be two posets. We can define different orders on the Cartesian product $P \times Q$. The canonical product order is defined component-wise, $(p_1, q_1) \leq_{\pi} (p_2, q_2) \iff p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$.

**Proposition 2.4.** The product order is the greatest order relation on $P \times Q$ which makes both projections order preserving.

*Proof.* Let $\leq' \subseteq P \times Q$ be an order relation such that both $\pi_1 : P \times Q \rightarrow P$ and $\pi_2 : P \times Q \rightarrow Q$ are order preserving. Let us show that if $(p, q) \leq' (p', q')$ then $(p, q) \leq_{\pi}(p', q')$. We need to show that $p \leq_P p'$ and $q \leq_Q q'$. Since $\pi_1$ is order preserving, $(p, q) \leq' (p', q')$ implies $\pi_1(p, q) \leq_{\pi} \pi_1(p', q')$ which is equal to $p \leq_P p'$. Likewise we can show that $q \leq_Q q'$. This is enough to show that $\leq' \subseteq \leq_{\pi}$. ∎

Another order on $P \times Q$ is the lexicographic order, defined by $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 < p_2$ or $(p_1 = p_2$ and $q_1 \leq q_2$).

The lexicographic order is different from the canonical order, and it is normally not so nicely behaved. The only aspect in which the lexicographic order scores better than the canonical product order is that if $P$ and $Q$ be chains, then also $P \times Q$, ordered lexicographically, is a chain. To show this take $(p_1, q_1)$ and $(p_2, q_2)$ arbitrarily. Sure is that either $p_1 \leq p_2$ or $p_2 \leq p_1$. If $p_1 < p_2$ we have $(p_1, q_1) \leq (p_2, q_2)$, and also if $p_2 < p_1$ we are done. Assume now that $p_1 = p_2$. Then again we have $p_2 \leq q_2$ or $q_2 \leq p_2$ and the lexicographic order gives us $(p_1, q_1) \leq (p_2, q_2)$ or $(p_2, q_2) \leq (p_1, q_1)$ respectively. This means that any two elements of $P \times Q$ are comparable, hence $P \times Q$ is a chain in the lexicographic order.

2.2 Diagrams

We can represent an ordered set in a diagram. Therefore we need to introduce the term covering.

2.2.1 Covering

**Definition 2.5.** Let $P$ be an ordered set, and let $x, y, z \in P$. We say $y$ covers $x$ if $x < y$ and $x \leq z < y$ implies $z = x$. We use the notation $x \prec y$.

This covering relation is irreflective and intransitive: Let $P$ be an ordered set. Because $x < x$ is false for all $x \in P$, and thus $x \not\prec x$ for all $x \in P$, the covering relation $\prec$ is an irreflective relation. A relation $\circ$ is intransitive if $x \circ y$ and $y \circ z$ imply that $x \circ z$ is false, for all $x, y, z \in P$. We see that $\prec$ is intransitive because for all $x, y, z \in P$, if $x \prec y$ and $y \prec z$ then $x \prec y \prec z$, which implies that $x \leq y < z$ where $y \neq x$ and hence $x \not\prec z$.

In case $P$ is a finite ordered set, the following holds for every $x, y \in P$ such that $x \not= y$:

$$x < y \text{ iff } x < x_1 < \cdots < x_n < y \text{ for some } x_1, \ldots, x_n \in P.$$
Assume $x < y$. If there is no element $x_1 \in P$ such that $x \leq x_1 < y$ and $x_1 \neq x$ we are finished and conclude $x < y$. Otherwise we continue and ask if there is an elements $x_2$ such that $x \leq x_2 < x_1$ and $x_2 \neq x$, and if there is an element $x_3$ such that $x_1 \leq x_3 < y$ and $x_3 \neq x_1$. In the case there is such an element, continue the same procedure. Otherwise we have for example $x \prec x_1$. Once finished (this will always be the case, because $P$ is finite) we have found the elements that will form the covering chain.

Suppose for the other direction that $x \prec x_1 \prec \cdots \prec x_n \prec y$ for some $x_1, \ldots, x_n \in P$. This means $x < x_1 < \cdots < x_n < y$ and we can conclude that $x < y$.

### 2.2.2 Construction of a diagram

To explain how to draw a diagram we use the following example. Let $P = \{a, b, c, d, e\}$ be a set ordered by $a < b$, $a < c$, $a < e$, $b < e$, $c < e$, $d < c$ and $d < e$. We construct a diagram for $P$ linking every element of $P$ to a point in the Euclidean plane $\mathbb{R}^2$. And every covering pair $x \prec y$ will be drawn as a line segment between the two points. Line segments may not accidentally cross points. Further, every element $x$ has to be lower than $y$ (that is, the second coordinate has to be smaller) whenever $y$ covers $x$. A few possible diagrams of $P$ will now be:

![Diagram example]

$P = \{a, b, c, d, e\}$

The elements of $P$ are points in the diagram. And only the covering pairs are connected by a line segment. For example $a$ is connected with $b$ because $a$ is covered by $b$. But there is no line segment from $d$ to $e$ because $d < c < e$ and hence $e$ does not cover $d$. Further we see that $b$ is lower than $e$ because $b < e$.

### 2.2.3 Examples

In the figure on the next page you find examples of diagrams of different posets. The poset $\langle P(\{1, 2, 3\}), \subseteq \rangle$ and a chain of 4 elements (notation $4 = \{1, 2, 3, 4\}$) and the chain $\mathbb{N}_0$ both ordered by $\leq$. The last diagram is a subset of the poset $\langle \mathbb{N}_0, \leq \rangle$, namely $Q := \langle\{1, 2, 4, 5, 6, 12, 20, 30, 60\}; \preceq \rangle$. 

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2.3 Duality Principle

Definition 2.6. Let $P = \langle X, \leq \rangle$ be a poset. The dual of $P$, notation $P^\partial$, is the poset $\langle X, \geq \rangle$ where $x \geq y$ in $P^\partial$ iff $x \leq y$ in $P$.

2.4 Maps

Definition 2.7. Let $P$ and $Q$ be ordered sets and $\phi$ a map from $P$ to $Q$.

1. We call $\phi$ an order-preserving map if $p \leq_P q$ implies $\phi(p) \leq_Q \phi(q)$, for all $p, q \in P$.

2. We call $\phi$ an order-embedding if $p \leq_P q$ if and only if $\phi(p) \leq_Q \phi(q)$, for all $p, q \in P$.

3. We call $\phi$ an (order-)isomorphism if $\phi$ is onto and an order-embedding.

Example 2.8. An order-embedding is by definition always an injective order-preserving map. It is now an interesting question which map is an order-preserving map but not an order-embedding. Define $P = \{a, b, c\}$ such that $a < c$ and $b < c$. Define the map $\phi$ as in the figure below.

This is obviously a bijection. The map is order-preserving, because $a < c$ implies $\phi(a) < \phi(c)$ and $b < c$ implies $\phi(b) < \phi(c)$. But $\phi$ is not an order-embedding, because $\phi(b) < \phi(a)$ while $a \neq b$. This example shows that the notion of an injective order-preserving map
is not enough to characterize substructures in the context of posets and order-preserving maps, as is the case with injective homomorphisms in algebras, injective continuous maps in topological spaces and injective functions in sets.

2.5 Down-sets

2.5.1 Subposets and down-sets

In an ordered set we can indicate subsets, which themselves are again ordered sets, because they are naturally endowed with the inherited order.

**Definition 2.9.** Given a poset $P = \langle X, \leq \rangle$. We call $Q = \langle Y, \leq \rangle$ a **subposet** of $P$ if $Y \subseteq X$ and if for every $x, y \in Y$, $x \leq y$ in $Q$ iff $x \leq y$ in $P$.

Take for example the poset $P = \langle \mathbb{N}, \leq \rangle$. The set of all even numbers $\{2, 4, 6, \cdots \}$ equipped with the same order-relation is a subposet of $P$. An other example of a subposet is a down-set.

**Definition 2.10.** A subset $S$ of a poset $P$ is a **down-set** iff $s \in S$, $x \in P$ and $x \leq s$ imply $x \in S$. For arbitrary $S \subseteq P$ and $y \in P$ we define:

$$\downarrow S = \{x \in P \mid (\exists s \in S)x \leq x\} \quad \text{and} \quad \downarrow y = \{x \in P \mid x \leq y\}$$

These downsets are generated by $S$ and $y$ respectively. A down-set of the form $\downarrow y$ (generated by one element of $P$) is called a principal down-set. The set of all down-sets of a poset $X$ is denoted by $\mathcal{O}(X)$. Dually we can define up-sets, notation $\uparrow S$ and $\uparrow y$.

2.5.2 Examples

- In the figure below we see the diagram of a poset $P$ the diagram for the set of all down-sets of $P$. Not all the down-sets in $\mathcal{O}(P)$ are principal downsets. The downset $\{a, c\}$ is a principal downset, because it is equal to $\downarrow c$. But the down-set $\{a, c, d\}$ is generated by the set $\{c, d\}$ and hence not a principal down-set.
For an antichain $X$ the following equality holds: $\mathcal{O}(X) = \mathcal{P}(X)$. The principal downsets all contain only one element. Because $\mathcal{O}(X) \subseteq \mathcal{P}(X)$, and because now every element of $X$ is a down-set, and also every union of elements of $X$ is a down-set, we have $\mathcal{P}(X) \subseteq \mathcal{O}(X)$, and hence $\mathcal{O}(X) = \mathcal{P}(X)$

### 2.5.3 Properties of down-sets

**Lemma 2.11.** Let $P$ be an ordered set, and $x, y \in P$. Then the following are equivalent:

1. $x \leq y$
2. $\down{x} \subseteq \down{y}$
3. $(\forall Q \in \mathcal{O}(P)) y \in Q \Rightarrow x \in Q$

**Proof.** 

1. $\Rightarrow$ 2. Assume $x \leq y$. Then if $x \in \down{x}$, thus $x \in \{ a \in P \mid a \leq x \}$. Because $x \leq y$ then also $x \in \{ a \in P \mid a \leq y \} = \down{y}$. Hence for all $x \in \down{x}$ we have $x \in \down{y}$, which implies $\down{x} \subseteq \down{y}$.

2. $\Rightarrow$ 3. Assume $\down{x} \subseteq \down{y}$. Take a $Q \in \mathcal{O}(P)$. Then if $y \in Q$ we have $\down{x} \subseteq \down{y} \subseteq Q$, which implies $x \in Q$.

3. $\Rightarrow$ 1. Assume $(\forall Q \in \mathcal{O}(P)) y \in Q \Rightarrow x \in Q$. Suppose $x > y$. Take $Q = \down{y}$, thus $y \in Q$. But because $x > y$, $x \notin \down{y}$ hence $x \notin Q$, which is a contradiction. We can conclude that $x \leq y$.

**Lemma 2.12.** Let $P$ be an ordered set and $\mathcal{O}(P)$ the set of all downsets of $P$. The map $\phi : P \rightarrow \mathcal{O}(P)$ is an order-embedding.

**Proof.** Define the map $\phi$ by sending an element of $P$ to its downset: $x \rightarrow \down{x}$. Further we take the inclusion order on $\mathcal{O}(P)$. To check if $\phi$ is an order-embedding we first take $x, y \in P$ such that $x \leq y$. Then $x \down{} = \{ z \in P \mid z \leq x \}$ and $\down{y} = \{ z \in P \mid z \leq y \}$. While $x \leq y$ we know that $z \leq x \leq y$ holds for all $z \in \down{x}$ and hence for all $z \in \down{x}$ we have $z \in \down{y}$. So $\down{x} \subseteq \down{y}$ and hence $\phi$ is an order-preserving map.

For the other direction, assume $\down{x} \subseteq \down{y}$. Then we know that $p \in \down{y}$ for all $p \in \down{x}$. Hence for all $p \leq x$ holds $p \leq y$. Since $x \in \down{x}$ because $x \leq x$ holds $x \in \down{y}$ and hence $x \leq y$.

Because for all $p, q \in P$, $p \leq q$ if and only if $\phi(x) \subseteq \phi(q)$, the map $\phi$ is an order-embedding.
Chapter 3

Lattices

3.1 Upper bounds and Lattices

3.1.1 Definitions

Definition 3.1. Let $P$ be an ordered set and let $S \subseteq P$. An element $x \in P$ is an upper bound of $S$ if $s \leq x$ for all $s \in S$. A lower bound is defined dually. For the set of all upper bounds of $S$ and for the set of all lower bounds of $S$ we write

$$S^u = \{x \in P \mid (\forall s \in S) s \leq x\} \quad \text{and} \quad S^l = \{x \in P \mid (\forall s \in S) x \leq s\}.$$ 

Since $\leq$ is transitive, $S^u$ is always an up-set, and $S^l$ a down-set. If $S^u$ has a least element, then it is called the least upperbound of $S$, also known as the supremum of $S$. Dually the greatest element of $S^l$, if it exists, is called the greatest lowerbound, or the infimum of $S$. Instead of writing $\sup S$ (and $\inf S$) we introduce the terms ‘join’ and ‘meet’, and we write $\bigvee S$ and $\bigwedge S$. This way we get, in case $S$ contains two elements, $\sup \{x, y\} = x \lor y$ and $\inf \{x, y\} = x \land y$.

Definition 3.2. The ordered set $P$ is called a lattice if $x \lor y$ and $x \land y$ exist for any two elements $x, y \in P$. An ordered set is called a complete lattice if for every subset $S \subseteq P$ both $\bigvee S$ and $\bigwedge S$ exist. A bounded lattice is a lattice that has a top and a bottom, also called 0 and 1.

If the join of two elements does not exist, it means that they have no least common upper bound, or that there is no common upper bound at all.

3.1.2 Examples

- The poset $(\mathcal{P}(X); \subseteq)$ is a lattice for every set $X$. For every $A, B \in \mathcal{P}(X)$ both $A \lor B$ and $A \land B$ exist. In this case they correspond to $A \cup B$ and $A \cap B$.

- Every chain is a lattice. Let $X$ be a chain. For every two elements $x, y \in X$ either $x \leq y$ or $y \leq x$. Then, assuming that $x \leq y$, $x \land y = x$ and $x \lor y = y$. Hence for every two elements the join and meet exist, which makes the chain a lattice.
• There are also posets that are not lattices. Consider the set \( Q = \{1, 2, 4, 5, 6, 12, 20, 30, 60\} \), equipped with the order \( \preceq \) that is defined as \( m \preceq n \) if \( m \) divides \( n \), as we have seen in the previous chapter. When we draw the diagram of \( Q \), we can easily see \( Q \) is not a lattice.

![Diagram of Q](image)

\[ \{2, 5\}^u = \{20, 30, 60\} \]

The join of 2 and 5 doesn’t exist, because there is no least common upper bound, and hence the poset is not a lattice.

### 3.1.3 A canonical embedding

**Lemma 3.3.** The set \( \mathcal{O}(P) \) of all downsets of \( P \) is a complete lattice.

**Proof.** To prove this it suffices to show that for every collection \( D \) of downsets also \( \bigcup D \) and \( \bigcap D \) are downsets. Because then for every collection of downsets the join and the meet exist.

Take a collection of downsets \( D \subseteq \mathcal{O}(P) \). We need to proof that \( \bigcap D \) is a downset. Let \( y \in \bigcap D \) and assume \( x \leq y \in \bigcap D \) for an arbitrary \( x \) in \( P \). While \( y \in \bigcap D \) we know that \( y \in C \) for all downsets \( C \) in the collection \( D \). But while \( C \) being a downset \( x \leq y \in C \) implies that \( x \in C \) for all \( C \), which gives us \( x \in \bigcap D \). Hence \( \bigcap D \) is a downset. We can conclude that the meet of every collection of downsets is a downset.

For the join the proof is almost the same. Let \( y \in \bigcup D \) and assume \( x \leq y \in \bigcup D \) for an arbitrary \( x \) in \( P \). While \( y \in \bigcup D \) we know that \( y \in C \) for at least one downset \( C_1 \) in the collection \( D \). But then \( x \leq y \in C_1 \) implies that \( x \in C_1 \), which gives us \( x \in \bigcup D \). Hence \( \bigcap D \) is a downset. We can conclude that the join of every collection of downsets is a downset.

Now for all \( D \subseteq \mathcal{O}(P) \) the join and meet exist, and hence \( \mathcal{O}(P) \) is a complete lattice. \( \square \)

We established a way of associating any poset with a lattice endowed with extremely good properties. But can we also establish the reverse correspondence? Yes, in a trivial way, every distributive lattice is also a poset. But we are interested in a correspondence from posets to distributive lattices and back, which gets us to the same poset we started with. Can we do this? Yes we can but only for special distributive lattices. Therefore we need more theory, so we will get back to this after having introduced join irreducibles.
3.2 Properties of Lattices

Every chain is a lattice, hence \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Q} \) and \( \mathbb{R} \) are lattices under their usual order. But they are not complete lattices, because the top is missing.

**Theorem 3.4.** Let \( L \) be a lattice. Then \( \lor \) and \( \land \) satisfy, for all \( a,b,c \in L \),

- \((L1)\) \( (a \lor b) \lor c = a \lor (b \lor c) \) \text{ (associativity)}
- \((L1)^\theta\) \( a \land b \land c = a \land (b \land c) \)
- \((L2)\) \( a \lor b = b \lor a \) \text{ (commutativity)}
- \((L2)^\theta\) \( a \land b = b \land a \)
- \((L3)\) \( a \lor a = a \) \text{ (idempotency)}
- \((L3)^\theta\) \( a \land a = a \)
- \((L4)\) \( a \lor (a \land b) = a \) \text{ (absorption)}
- \((L4)^\theta\) \( a \land (a \lor b) = a \)

**Proof.** The duality enables us to prove only \((L1)-(L4)\).

For \((L3)\) we know
\[
\{a, a\}^u = \{x \in L \mid a \leq x \text{ and } a \leq x\}
= \{x \in L \mid a \leq x\}
\]

Because \(\leq\) is reflexive \((a \leq a)\) the least upperbound is \(a\), hence \(a \lor a = a\).

Then for \((L2)\) the following equality holds:
\[
\{a, b\}^u = \{x \in L \mid a \leq x \text{ and } b \leq x\}
= \{x \in L \mid b \leq x \text{ and } a \leq x\}
= \{b, a\}^u
\]

Since the set of upperbounds of \(a\) and \(b\) is equal to the set of upperbounds of \(b\) and \(a\), also the least upperbound will be one and the same, hence \(a \lor b = b \lor a\).

Then \((L4)\) is easily done, when we see that \(a \land b \leq a\). Knowing this, we can conclude directly that \(a \lor (a \land b) = a\).

For \((L1)\) we need to show that \((a \lor b) \lor c = \sup\{a, b, c\}\). Because then it follows with \((L2)\) that \((a \lor b) \lor c = a \lor (b \lor c)\). We want to show that \(\{a \lor b, c\}^u = \{a, b, c\}^u\). We see the following
\[
d \in \{a \lor b, c\}^u \iff a \lor b \leq d \text{ and } c \leq d
\iff d \in \{a, b\}^u \text{ and } c \leq d
\iff a \leq d \text{ and } b \leq d \text{ and } c \leq d
\iff d \in \{a, b, c\}^u.
\]

Thus \(\sup\{a \lor b, c\} = \sup\{a, b, c\} = \sup\{a, b \lor c\}\), and hence \((a \lor b) \lor c = a \lor (b \lor c)\). \(\square\)
Theorem 3.5. Let \( (L; \vee; \wedge) \) be a non-empty set equipped with two binary operations which satisfy (L1)-(L4) and (L1\(^0\))-(L4\(^0\)). Then

1. For all \( a, b \in L \) we have \( a \vee b = b \) if and only if \( a \wedge b = a \).
2. Define \( \leq \) on \( L \) by \( a \leq b \) if \( a \vee b = b \). Then \( \leq \) is an order-relation.
3. With \( \leq \) defined as in (2) \( (L; \leq) \) is a lattice in which the original operations agree with the induces operations, that is, \( \forall a, b \in L : a \vee b = \sup a, b \) and \( a \wedge b = \inf a, b \).

3.3 Filters and Ideals and their logical significance

3.3.1 Filters and Ideals

Definition 3.6. Let \( L \) be a lattice. Then a non-empty set \( J \subseteq L \) is an ideal if

1. \( a \vee b \in J \) for all \( a, b \in J \) and
2. \( a \in L, b \in J \) and \( a \leq b \) imply \( a \in J \).

In words we can describe an ideal as a non-empty down-set closed under join. Further we call the dual of an ideal, that is a non-empty up-set closed under meet a filter. If the ideal (filter) does not coincide with \( L \) we call it a proper ideal (filter). A principal filter is a filter generated by an element \( x \in L \), i.e. it is the collection \( \{ y \in L \mid x \leq y \} \) for some \( x \in L \). In the same way \( \{ y \in L \mid y \leq x \} \) is a principal ideal generated by \( x \). A principal ideal (filter) generated by an element \( a \in L \), is the down-set \( \downarrow a \) (up-set \( \uparrow a \)).

If a lattice \( L \) has a maximum element 1, then every filter in \( L \) contains it. Dually if \( L \) has a minimum element 0, then every ideal in \( L \) contains it.

Example 3.7. Let \( L \) and \( K \) be bounded lattices, and \( f : L \to K \) a \( \{0, 1\}\)-homomorphism. We can easily show that \( f^{-1}(0) \) is an ideal of \( L \) and \( f^{-1}(1) \) is an filter of \( L \).

We know that \( f(0) = 0 \) and \( f(1) = 1 \) because \( f \) is bounded, and because of the homomorphism we have \( f(a \vee b) = f(a) \vee f(b) \) and \( f(a \wedge b) = f(a) \wedge f(b) \) for all \( a, b \in L \).

Now we wonder if \( a \vee b \in f^{-1}(0) \) for all \( a, b \in f^{-1}(0) \). There holds

\[
a, b \in f^{-1}(0) \iff f(a) = 0 \text{ and } f(b) = 0 \iff f(a \vee b) = f(a) \vee f(b) = 0 \vee 0 = 0
\]

and hence \( a \vee b \in f^{-1}(0) \). Now it is left to show that \( f^{-1}(0) \) is a downset, thus we take \( a \in L \) and \( b \in f^{-1}(0) \) such that \( a \leq b \). There holds

\[
f(b) = 0 \text{ and } a \vee b = b \iff f(a \vee b) = f(b) = 0 \iff f(a) \vee f(b) = f(a) \vee 0 = 0 \iff f(a) = 0
\]

and hence \( a \in f^{-1}(0) \), which shows that \( f^{-1}(0) \) is in fact an ideal. The proof of \( f^{-1}(1) \) being a filter is dual to this proof.
3.3.2 The logical significance of filters and ideals

Given a logic \( \langle Fm, \vdash \rangle \). Suppose that the language has \( \lor \) and \( \land \) with the properties above. A theory for the logic is a subset \( \Sigma \subseteq Fm \) such that \( \forall \phi \in Fm (\Sigma \vdash \phi \implies \phi \in \Sigma) \). We say that \( \Sigma \) is closed under the consequence relation. Now for all \( \Sigma \subseteq Fm \) if \( \Sigma \) is a theory, then \( \{[\phi] \mid \phi \in \Sigma\} \) is a filter of the lattice \( \langle Fm/\models, \subseteq \rangle \), i.e. the Lindenbaum-Tarski algebra.

Conversely if \( F \) is a filter of the Lindenbaum-Tarski algebra, then \( \Sigma = \cup\{[\phi] \mid [\phi] \in F\} \) is a theory of \( \langle Fm, \vdash \rangle \). Proof: Let \( \phi \) such that \( \Sigma \vdash \phi \), so by finitarity there is a \( \Sigma' = \{\gamma_1, \ldots, \gamma_n\} \) such that \( \Sigma \vdash \phi \). Now take \( \gamma = \land \gamma_i \). Then by the 'filter condition' \( [\gamma] \in F \iff [\gamma_1 \land \ldots \land \gamma_n] = [\gamma_1] \land \ldots \land [\gamma_n] \in F \) and \( \gamma \vdash \phi \) imply \( F \ni [\gamma] \leq [\phi] \). Hence \( [\phi] \in F \), i.e. \( \phi \in \Sigma \).

Filters and ideals are important for logic. Filters for example model logical theories.

In propositional logic the meet-symbol \( \land \) is interpreted as 'and'. In proves it is used as follows:

- \( \phi \land \psi \vdash \phi \) and \( \phi \land \psi \vdash \psi \).
- \( \{\phi, \psi\} \vdash \phi \land \psi \), we would like to refer to it as the 'filter condition'.

Then for the join-symbol \( \lor \) we have the interpretation 'or' which is used in propositional logic as follows:

- \( \phi \vdash \phi \lor \psi \) and \( \psi \vdash \phi \lor \psi \).
- If \( P \vdash \phi \lor \psi \), then either \( P \vdash \phi \) or \( P \vdash \psi \), for a theory \( P \). One of the two has to be chosen, and we could refer to this property as the 'prime ideal condition'.

If \( T \) is a theory (a set of formulas closed under logical consequence) and \( T \vdash a \), then there is a finite set of theorems \( b_1, \ldots, b_n \in T \) such that \( \land_{i=1}^n b_i \vdash a \). But this means that \( a \in T \), because e theory is closed under logical consequences. In general if \( \Gamma \) is a theory such that \( \Gamma \vdash \phi \), then \( \Gamma' \vdash \phi \) for all \( \Gamma' \supseteq \Gamma \).

We can see theories as filters, because it has the two properties for being a filter:

- If \( a, b \in T \), thus \( T \vdash a \land b \) (see filter condition above), then also \( a \land b \in T \) because \( T \) is closed under logical consequences.
- If \( a \) is a theorem, \( b \in T \) and \( b \vdash a \), then follows, again by logical consequence, that \( a \in T \).

If filters are theories, then we can think of ideals as cotheories.

3.3.3 Examples

- The lattice \( L \) itself is both a filter and an ideal.
• If 0 is the minimum and 1 is the maximum of $L$, then $\{0\}$ is an ideal and $\{1\}$ a filter of $L$.

• All the filters and ideals of a chain are principal.

• Let us search for a non-principal filter. In the finite-cofinite algebra the subset $F := \{Y \in FC(X) \mid Y \text{ is cofinite} \}$ of all cofinite subsets of $X$ forms a non-principal filter. To check if it is a filter let $Y_1, Y_2 \in F$. Then $X \setminus Y_1$ and $X \setminus Y_2$ are finite and thus $X \setminus Y_1 \cup X \setminus Y_2 = X \setminus (Y_1 \cap Y_2)$ is finite. Hence $Y_1 \cap Y_2$ is cofinite and $(Y_1 \cap Y_2) \in F$. Now assume $Y_1 \in FC(X), Y_2 \in F$ and $Y_2 \subseteq Y_1$. It is clear $Y_1$ contains more elements than (or the same amount of elements as) $Y_2$, because $Y_2$ is included in $Y_1$. Then the complement of $Y_1$ contains less elements than the complement of $Y_2$ which is finite. We may conclude $Y_1$ is cofinite, and hence $Y_1 \in F$.

We showed $F$ is a filter. It is a non-principal filter because there is not one element in $FC(X)$ that generates $F$. Assume there is such an element $A \in FC(X)$. Then $F = \{V \in FC(X) \mid A \subseteq V\}$. This element $A$ has to be cofinite, because $A \in F$. We write $A := X \setminus \{a_1, \ldots, a_n\}$, where $n \in \mathbb{N}$. But then the cofinite set $B := X \setminus \{a_1, \ldots, a_n, b\}$ cannot be a member of $F$, because $A \not\subseteq B$. This is a contradiction, whence $F$ contains all cofinite subsets of $X$.

We have found a non-principal filter. And this set of cofinite subsets is in fact a maximal proper filter: an ultrafilter. Because if we add another element, then it will be a finite subset (all the cofinite subsets are already taken). But this finite subset is the complement of some element in $F$. Then the empty set will thus be in $F$. This implies that all subsets of $FC$ belong to $F$, because the empty set is a subset of every other subset. We may conclude that $F$ is a maximal proper filter, because adding one element will give us the whole set.
Chapter 4

Boolean Algebra

4.1 Definitions and examples

Definition 4.1. Let $L$ be a lattice. We call $L$

1. distributive if it satisfies the distributive law:
\[
(\forall a, b, c \in L) \ a \land (b \lor c) = (a \land b) \lor (a \land c).
\]

2. modular if it satisfies the modular law:
\[
(\forall a, b, c \in L) \ a \geq c \rightarrow a \land (b \lor c) = (a \land b) \lor c.
\]

Example 4.2. 1. Every powerset lattice $\mathcal{P}(X)$ is distributive, because in this case the join and meet are the union and intersection.

Definition 4.3. Let $L$ be a lattice with 0 and 1. For $a \in L$, we say $b \in L$ is a complement of $a$ if $a \land b = 0$ and $a \lor b = 1$. If $a$ has a unique complement, we denote this complement by $a'$.

Definition 4.4. A lattice $L$ is called a Boolean lattice if

1. $L$ is distributive,
2. $L$ has 0 and 1,
3. each $a \in L$ has a (necessarily unique) complement $a' \in L$.

We see that a Boolean lattice is a special kind of distributive lattice. The necessity of the uniqueness of the complement of every element $a \in L$ can be shown as follows. Let $S$ be a distributive lattice in which every element has at least one complement. Assume $x \in L$, and let $y$ and $z$ both be complements of $x$. Then $x \land y = 0$, $x \land z = 0$, $x \lor y = 1$ and $x \lor z = 1$. This means that $y = y \lor 0 = y \lor (x \land z) = (y \lor x) \land (y \lor z) = 1 \land (y \lor z) = y \lor z$, which implies $z \leq y$. In the same way $z = y \lor z$, which implies $y \leq z$. Hence $y = z$. Each $a \in L$ has a necessarily unique complement.
Definition 4.5. A Boolean algebra is a structure $\langle B; \lor, \land, ', 0, 1 \rangle$ such that

1. $\langle B; \lor, \land \rangle$ is a distributive lattice,
2. $a \lor 0 = a$ and $a \land 1 = a$ for all $a \in B$,
3. $a \lor a' = 1$ and $a \land a' = 0$ for all $a \in B$.

In the following chapters we will prove that every finite Boolean algebra is isomorphic to $\mathcal{P}(X)$ for some finite set $X$. And after that even that every Boolean algebra is isomorphic to a subalgebra of $\mathcal{P}(X)$.

4.1.1 Examples of Boolean algebras

- The finite cofinite subsets of $\mathbb{N}$ is a Boolean algebra, but not isomorphic to any powerset algebra, because of cardinality reason. The collection of all the cofinite subsets is a non principal ultrafilter.

- $\mathcal{P}(X)$ is a Boolean algebra for every $X$. This algebra is join-generated by its atoms (see next chapter).

- The algebra of formula’s (as strings of symbols) is not a Boolean algebra, because the consequence relation restricted to formulas is not antisymmetric as we have seen in section 2.1.4.

- The Lindenbaum algebra (see next section) of classical propositional logic is a Boolean algebra. As we will see in the next chapter it is atomless.

4.1.2 The Lindenbaum algebra as Boolean algebra

To be a Boolean lattice $LA_{\models}$ needs to have a top and bottom and also join and meet have to be defined as also the complement for every element in $LA$.

Define top and bottom as follows:

$$1 = [\phi] \text{ where } \phi \text{ is a tautology} \quad 0 = [\neg \phi] \text{ where } \phi \text{ is a tautology}$$

Further define

$$[\phi] \lor [\psi] = [\phi \lor \psi]$$

$$[\phi] \land [\psi] = [\phi \land \psi]$$

$$[\phi]' = [\neg \phi]$$

$LA$ has become a Boolean lattice, but we will go on and make a Boolean algebra of it. It is easy to see that $\langle LA_{\models}; \leq \rangle$ is a lattice with join and meet given by $\lor$ and $\land$. Then $\langle LA_{\models}; \lor, \land \rangle$ is also distributive, and finally there holds $[\phi] \lor [\phi]' = 1$ and $[\phi] \land [\phi]' = 0$. The Boolean algebra $\langle LA_{\models}; \lor, \land, ', 0, 1 \rangle$ we obtained is also known as the Lindenbaum algebra.
4.2 The finite intersection property

A subset \( A \) of a Boolean algebra has the finite intersection property (from now on called ‘fip’) if the infimum of any finite subset of \( A \) is not equal to 0. In this section we will discover that we can extend precisely these subsets to proper filters, and further how to extend a proper filter to an ultrafilter. All these steps can also be done in propositional logic almost in the exact same way.

4.2.1 The fip and the interpretation in propositional logic

Let \( B \) be a Boolean algebra and let \( A \) be a set of elements of \( B \). If \( A \) has the fip, then for any element \( x \in B \) either \( A \cup \{x\} \) or \( A \cup \{x'\} \) has the fip.

To show this assume neither one of them has the fip. Then \( \bigwedge (A \cup \{x\}) = 0 \) and \( \bigwedge (A \cup \{x'\}) = 0 \). We can write this in a different way, having in mind \( A \) does have the fip:

\[
0 = (a \land x) \lor (a \land x') = a \land (x \lor x') = a \land 1 = a,
\]

which is a contradiction. It cannot be the case that neither one of them has the fip.

The proof of the fact that, if one of the two has the fip, then the other cannot have the fip, will not be showed here.

The following lemma shows that any subset that has the fip can be extended to a filter.

Let therefor \( B \) be a Boolean algebra, and define

\[
A^0 = \{ x \in B \mid \text{for some } a \in A, a \leq x \} \quad \text{and} \quad A^c = \{ \inf(X) \mid X \text{ is a finite subset of } A \}
\]

**Lemma 4.6.** For any subset \( A \) of a Boolean algebra, the set \( (A^c)^0 \) is a filter. Any filter containing \( A \) contains \( (A^c)^0 \). \( (A^c)^0 \) is proper if and only if \( A \) has the fip.

The interpretation in propositional logic is very interesting. We saw in the previous chapter that theories can be seen as filters. The lemma above shows that a set of theorems can be extended to a consistent theory if and only if the set of theorems does not cause a contradiction. The next question is then if we could extend a set of theorems to a maximal consistent theory. For the answer we need to prove the ‘Ultrafilter Theorem’, but first we consider the following lemma.

**Lemma 4.7.** If \( F \) is a filter in a Boolean algebra \( B \), \( F \) is an ultrafilter if and only if for each \( x \in B \) either \( x \in F \) of \( x' \in F \), but not both.

In propositional logic we can interpretate this lemma now in terms of maximal consistent theories. An ultrafilter is a maximal proper filter, it has no proper extension. Every extension of an ultrafilter would contain the element 0, which makes the extension equal to the Boolean algebra itself.

A maximal consistent theory is an ultrafilter (in the appropriate context, see 3.3.2, because extending the maximal consistent theory would give us an inconsistent theory. A maximal
consistent theory always contains either $\phi$ or $\neg\phi$, for all $\phi$ in the set of theorems. In fact, extending this theory by adding a theorem leads to a contradiction: $\phi \land \neg\phi$, and thus to an inconsistent theory.

But let us come back to our question. The following theorem, the 'Ultrafilter Theorem', is the last piece of the puzzle. Adding this theorem, we will be able to construct a maximal consistent theory out of a set of (consistent) theorems.

**Theorem 4.8. (The Ultrafilter Theorem)** Every filter in a Boolean algebra can be extended to an ultrafilter.

We will get to this theorem in Chapter 6.
Chapter 5

Finite representation

5.1 Join-irreducible elements

Definition 5.1. Let $L$ be a lattice. An element $x \in L$ is **join-irreducible** if

1. $x \neq 0$
2. $x = a \lor b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

We use the notation $J(L)$ to indicate the set of all join-irreducible elements of $L$. In the same way we write $M(L)$ for the set of meet-irreducible elements of $L$, which are defined dually.

Let $P$ be an ordered set, and let $Q \subseteq P$. Then $Q$ is called **join-dense** if for every element $a \in P$ there is a subset $A$ of $Q$ such that $a = \bigvee_{p} A$.

Example 5.2. For every two elements $x$ and $y$ in a chain we have either $x \leq y$ or $y \leq x$. This implies that either $x \lor y = x$ or $x \lor y = y$ and hence every element in a chain is join-irreducible.

In a diagram it is easy to distinguish the join-irreducible elements, these are namely the points with only one lower cover. Assume for example that point $x$ has two lower covers $a$ and $b$. This means $x = a \lor b$ while $a < x$ and $b < x$. A point with two lower covers can therefore never be a join-irreducible element.

At last we take the lattice $\mathcal{P}(X)$. The join-irreducible elements of $\mathcal{P}(X)$ are the singletons $\{x\}$ for $x \in X$, because those elements have only one lower cover, which is 0.

Lemma 5.3. Let $L$ be a lattice with least element 0. Then

1. $0 \prec x$ implies $x \in J(L)$
2. if $B$ is a Boolean lattice, $x \in J(L)$ implies $0 \prec x$. 


Proof. For the first part suppose by way of contradiction that $0 < x$ and $x \notin \mathcal{J}(L)$. Thus $x = a \lor b$ and $a < x$ and $b < x$. Now $0 < x$ implies $a = b = 0$ and hence $x = 0$ which is a contradiction (because $0 < x$).

For the other part of the proof assume $L$ is a Boolean lattice and $x \in \mathcal{J}(L)$. Then $x = a \lor b$ implies $x = a$ or $x = b$. Suppose that $0 \leq y < x$. To prove that $0 < x$ we need to show that $y = 0$. The fact that $y \lor y' = 1$ and the distributivity of the Boolean lattice gives us

$$x = x \lor y = (x \lor y) \land (y \lor y') = y \lor (x \land y').$$

Now $x = y \lor (x \land y')$ and $x < y$ imply $x = x \land y'$. Because $y < x \leq y'$ we obtain $y = x \land y \leq y' \land y = 0$. Hence $y = 0$ and we proved that $0 < x$. □

5.2 Atoms

In the previous example of $\mathcal{P}(L)$ we have seen that the join-irreducible elements were exactly the elements that cover 0.

Definition 5.4. Let $L$ be a lattice with least element 0. Then $a \in L$ is called an atom if $0 < a$.

Lemma 5.5. This will be lemma 5.4 of the book. (We need it for the next proof.)

The set atoms of $L$ is denoted by $\mathcal{A}(L)$. The lattice $L$ is called atomic if for every $x \in L$ (where $x \neq 0$) there is an $a \in \mathcal{A}(L)$ such that $a \leq x$.

Theorem 5.6. Let $B$ be a finite Boolean algebra. Then the map

$$\eta : a \mapsto \{ x \in \mathcal{A}(B) \mid x \leq a \}$$

is an isomorphism of $B$ onto $\mathcal{P}(X)$ where $X = \mathcal{A}(B)$, with the inverse of $\eta$ given by $\eta^{-1}(S) = \bigvee S$ for $S \in \mathcal{P}(X)$.

Proof. We will first show that $\eta$ maps $B$ onto $\mathcal{P}(X)$. Clearly $\emptyset = \eta(0)$. Let $S = \{ a_1, \ldots, a_k \}$ be a set of atoms, thus $S \in \mathcal{P}(X)$, and define $a = \bigvee S$.

We claim $S = \eta(a)$. Because for all $i$, $a_i$ is an atom and $a_i \leq a = a_1 \lor \ldots \lor a_k$, there holds $S \subseteq \eta(a)$. For the other inclusion let $x$ be an atom such that $x \leq a = a_1 \lor \ldots \lor a_k$. For each $i$ we have $0 \leq x \land a_i \leq x$. Assume that $x \land a_i = 0$ for all $i$. Then because $x \leq a$ and because of distributivity we have $x = x \lor a = x \lor (a_1 \lor \ldots \lor a_k) = (x \lor a_1) \lor \ldots \lor (x \lor a_k) = 0 \lor \ldots \lor 0 = 0$. But $x$ is an atom, and covers 0, hence $x > 0$. Contradiction! There must be at least one $j$ such that $x \land a_j = x$. But both $x$ and $a_j$ are atoms, whence $x = a_j$. For our arbitrarily chosen $x \in \eta(a)$ we know that $x \in S$, and hence $\eta(a) \subseteq S$. We can conclude that $S = \eta(a)$, and thus that $\eta$ maps $B$ onto $\mathcal{P}(X)$.

To show that $\eta$ is an isomorphism let $a, b \in B$. If $a \leq b$ we easily see that $\eta(a) \subseteq \eta(b)$ because $\{ x \in \mathcal{A}(B) \mid x \leq a \} \subseteq \{ x \in \mathcal{A}(B) \mid x \leq b \}$. Assume $\eta(a) \subseteq \eta(b)$. By lemma 5.4 we have $a = \bigvee \{ x \in \mathcal{A}(B) \mid x \leq a \} = \bigvee \eta(a)$, and in the same way $b = \bigvee \eta(b)$. Because $\eta(a) \subseteq \eta(b)$ we obtain $a = \bigvee \eta(a) \leq \bigvee \eta(b) = b$. The map $\eta$ is an isomorphism of $B$ onto $\mathcal{P}(X)$. □
Chapter 6

Maximality Principles

6.1 Prime ideals and filters

**Definition 6.1.** Let $L$ be a lattice. A non-empty subset $J$ of $L$ is called an **ideal** if

1. $a, b \in J$ implies $a \lor b \in J$ and
2. $a \in L$, $b \in J$, and $a \leq b$ imply $a \in J$.

We call $J$ a **proper ideal** if $J \neq L$. A proper ideal $J$ of $L$ is said to be **prime** if $a, b \in L$ and $a \land b \in J$ imply $a \in J$ or $b \in J$. The set of prime ideals of $L$ is denoted by $\mathcal{I}_p(L)$.

Dually defined we have a filter, a proper filter and a prime filter. We denote the set of prime filters of $L$ by $\mathcal{F}_p(L)$.

**Lemma 6.2.** A subset $J$ of a lattice $L$ is a prime ideal iff $L \setminus J$ is a prime filter.

**Proof.** Let $L$ be a lattice and $J$ a prime ideal. Then the following three properties hold:

- $a, b \in J$ implies $a \lor b \in J$,
- $a \in L$, $b \in J$ and $a \leq b$ imply $a \in J$,
- $a, b \in L$ and $a \land b \in J$ imply $a \in J$ or $b \in J$.

To prove that $L \setminus J$ is a prime filter we need to prove the three dual properties. For the first assume $a, b \in L \setminus J$. If $a \land b \in J$ then $a \in J$ or $b \in J$ because of the third property of $J$, which is a contradiction. Hence $a, b \in L \setminus J$ implies $a \land b \in L \setminus J$.

Then assume $a \in L$, $b \in L \setminus J$ and $a \geq b$. If $a \in J$, the second property of $J$ tells us that $a \in J$, $b \in L$ and $b \leq a$ imply $b \in J$, which is again a contradiction. Hence $a \in L$, $b \in L \setminus J$ and $a \geq b$ imply $a \in L \setminus J$.

At last assume that $a, b \in L$ and $a \lor b \in L \setminus J$. If both $a$ and $b$ are elements of $J$ we have a contradiction, because the first property of $J$ claims that if $a, b \in J$ also $a \lor b \in J$, which is not the case. Hence $a, b \in L$ and $a \lor b \in L \setminus J$ imply $a \in L \setminus J$ or $b \in L \setminus J$.

The three dual properties hold for $L \setminus J$ and thus $L \setminus J$ is a prime filter. The proof of the converse is obtained dually. \[\square\]
Lemma 6.3. Let L be a distributive lattice and let \( x \in L \) with \( x \neq 0 \) in case L has a zero. Then the following are equivalent:

1. \( x \) is join-irreducible
2. if \( a, b \in L \) and \( a \leq b \) then \( x \leq a \) or \( x \leq b \).

Proof. \( 1. \Rightarrow 2. \) Assume \( x \in J(L) \). Thus for \( x = a \vee b \) we have \( x = a \) or \( x = b \). Suppose \( x \leq a \vee b \). Then by distributivity of L and because \( x \leq a \vee b \), \( x = x \land (a \vee b) = (x \land a) \vee (x \land b) \). And the join-irreducibility of \( x \) implies \( x = a \land b \) or \( x = x \land b \), hence \( x \leq a \) or \( x \leq b \). We conclude that if \( a, b \in L \) and \( a \leq b \) then \( x \leq a \) or \( x \leq b \).

2. \( \Rightarrow 1. \) Assume now that if \( a, b \in L \) and \( a \leq b \) then \( x \leq a \) or \( x \leq b \). Suppose \( x = c \lor d \). We want to show that \( x = c \) or \( x = d \). Because \( x = c \lor d \) we can write also \( x \leq c \lor d \). This implies that \( x \leq c \) or \( x \leq d \). But \( x = c \lor d \) implies \( c \leq x \) and \( d \leq x \). We obtained \( x \leq c \leq x \) or \( x \leq d \leq x \), hence either \( x = c \) or \( x = d \). Which means that \( x \) is join-irreducible.

\[ \square \]

Lemma 6.4. Let L be a lattice satisfying (DCC). Suppose \( a, b \in L \) and \( a \not\leq b \). Then there exists \( x \in J(L) \) such that \( x \leq a \) and \( x \not\geq b \).

Proof. Suppose \( a, b \in L \) and \( a \not\leq b \). Let \( S = \{ x \in L \mid x \leq a \) and \( x \not\leq b \} \). Because \( a \in S \), S is not-empty. By lemma 2.39 (dual) the subset \( S \subseteq L \) has a minimal element. We need to proof that this minimal element, say \( x \), is in \( J(L) \).

Assume \( x \not\in J(L) \). Then \( x = c \lor d \) and \( c \leq x \) and \( d \leq x \). Because \( x \) is a minimal element of \( S \), \( c \) and \( d \) are not in \( S \). Thus \( c \leq b \) and \( d \leq b \). Hence \( x = c \lor d \leq b \), which implies \( x \leq b \). But \( x \in S \), so we have reached a contradiction. Hence \( x \in J(L) \) and \( x \leq a \) and \( x \not\geq b \).

\[ \square \]

Lemma 6.5. Let L be a finite distributive lattice and let \( a \in L \). Then the map \( x \mapsto L \setminus \uparrow x \) is an order-isomorphism of \( J(L) \) onto \( \mathcal{I}_p(L) \), that maps \( x \in J(L) \mid x \leq a \) onto \( \{ I \in \mathcal{I}_p(L) \mid a \not\in I \} \).

Proof. We first prove that \( \uparrow x \) is a prime filter. For \( a, b \in \uparrow x \) holds \( a \geq x \) and \( b \geq x \) which implies \( a \land b \geq x \). Hence \( a \land b \in \uparrow x \).

Then suppose \( a \in L \), \( b \in \uparrow x \) and \( a \geq b \). This means that \( b \geq x \) and \( a \geq b \) and thus \( a \uparrow x \).

At last suppose \( a, b \in L \) and \( a \lor b \in \uparrow x \). Because \( a \lor b \geq x \), lemma 5.1 gives us \( a \in \uparrow x \) or \( b \in \uparrow x \).

We conclude that \( \uparrow x \) is a prime filter, because it has the three properties it needs to have. In a finite lattice every ideal/filter is principal, i.e. of the form \( \uparrow x \) or \( \downarrow x \) which is the up- or downset generated by \( a \). As we have seen in lemma 6.1 above, \( L \setminus \uparrow x \) is a prime ideal. Onto because, take \( I \in \{ I \in \mathcal{I}_p(L) \mid a \not\in I \} = \{ L \setminus \downarrow x \mid a \not\in L \setminus \downarrow x \) and \( x \in J(L) \}. For every ideal \( L \setminus \uparrow x \) there is a \( x \in J(L) \), because we take the complement of the filter generated by
Lemma 6.6. Let $L$ be a finite distributive lattice and let $a \nleq b$ in $L$. Then there exists $I \in \mathcal{I}_p(L)$ such that $a \notin I$ and $b \in I$.

Proof. Let $L$ be a finite distributive lattice. Because $L$ is finite the (DDC) holds. Suppose now that $a \nleq b$. Then by lemma 2.45 there is a $x \in \mathcal{J}(L)$ such that $x \leq a$ and $x \nleq b$. By lemma 10.8 the map $x \mapsto L \setminus \uparrow x$ is an order-isomorphism of $\mathcal{J}(L)$ onto $\mathcal{I}_p(L)$, that maps $\{x \in \mathcal{J}(L) \mid x \leq a\}$ onto $\{I \in \mathcal{I}_p(L) \mid a \notin I\}$. The $x \in \mathcal{J}(L)$ is mapped onto $L \setminus \uparrow x$, where $a \notin L \setminus \uparrow x$. We want to know now whether $b \in L \setminus \uparrow x$ or not. Suppose $b \notin L \setminus \uparrow x$, then $b \in \uparrow x$, which means that $x \leq b$. Contradiction! Hence $b \in L \setminus \uparrow x$. We have found the $I \in \mathcal{I}_p(L)$ such that $a \notin I$ and $b \in I$. 

6.2 Maximal ideals and ultrafilters

Definition 6.7. Let $L$ be a lattice and let $I \subseteq L$ be an ideal. We call $I$ a maximal ideal if, for every ideal $J \subseteq L$ which contains $I$ properly (thus $I \subset J$), we have $J = L$. A maximal filter, also known as an ultrafilter, is dually defined.

Theorem 6.8. Let $L$ be a distributive lattice with $1$. Then every maximal ideal in $L$ is prime. (Dually in a distributive lattice with $0$, every ultrafilter is a prime filter.)

Proof. Let $L$ be a distributive lattice with $1$. Let $I$ be a maximal ideal in $L$. An ideal $I$ is prime if $a, b \in L$ and $a \land b \in I$ imply $a \in I$ or $b \in I$.

Suppose $a \land b \in I$ and without loss of generality suppose that $a \notin I$. We need to prove that $b \in I$. Define $I_a = \downarrow \{a \lor c \mid c \in I\}$. This is an ideal because

- let $a \lor x_1$ and $a \lor x_2$ be elements of $I_a$, thus $x_1, x_2 \in I$. Then, because $x_1 \lor x_2 \in I$, we have $(a \lor x_1) \lor (a \lor x_2) = a \lor (x_1 \lor x_2) \in I_a$.

- if $a \lor x_1 \in L$, $a \lor x_2 \in I_a$ and $a \lor x_1 \leq a \lor x_2$, then $x_1 \leq x_2$ hence $x_1 \in I$. We conclude that $a \lor x_1 \in I_a$.

Let now $c_1 \in I$. Then $a \lor c_1 \geq c_1$ implies $c_1 \in I_a$, because $I_a$ is a downset. Hence is $I \subseteq I_a$. We started with $I$ being a maximal ideal, which implies that $I_a = L$. Because $1 \in I_a$, there is a $d \in I$ such that $1 = a \lor d$. We get $d \lor (a \land b) \in I$, and by distributivity $(d \lor a) \land (d \lor b) \in I \iff 1 \land (d \lor b) \in I \iff d \lor b \in I$. Because $d \lor b \geq b$ we obtain $b \in I$, which we wanted to prove. Every maximal ideal in $L$ is prime. 

Theorem 6.9. Let $B$ be a Boolean lattice and let $I$ be a proper ideal in $B$. Then the following are equivalent:

1. $I$ is a maximal ideal

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2. \( I \) is a prime ideal

3. for all \( a \in B \) holds: \( a \in I \iff a' \notin I \).

**Proof.** We will prove 1. \( \Rightarrow \) 2. \( \Rightarrow \) 3. \( \Rightarrow \) 1. to obtain the equivalence.

1. \( \Rightarrow \) 2. Assume \( I \) is a maximal ideal. To prove that \( I \) is also a prime ideal we need to show that for \( a \land b = x \in I \) either \( a \in I \) or \( b \in I \), or both. Suppose \( x = a \land b \in I \) and assume \( a \notin I \). Required is now that \( b \in I \).

Define \( I_a = \{ c \in \mathcal{I} \mid c \notin I \} \). For all \( c \in I \) we have \( c \leq a \lor c \) and hence \( c \in I_a \), which implies \( I \subseteq I_a \). But because \( I \) is a maximal ideal, and \( 0 \in I \) we obtain \( a \lor 0 = a \in I_a \Rightarrow I \subsetneq I_a \Rightarrow I_a = B \). Now \( 1 \in I_a \), thus there is a \( d \in B \) such that \( a \lor d = 1 \). But then

\[
(a \land b) \lor d = (a \land d) \lor (b \lor d) \in I \implies b \lor d \in I \implies b \leq b \lor d \in I \implies b \in I.
\]

The maximal ideal is also a prime ideal.

2. \( \Rightarrow \) 3. Assume \( I \) is a prime ideal. Because \( a \land a' = 0 \in I \) and \( I \) is a prime ideal either \( a \in I \) or \( a' \in I \). Assume without loss of generality that \( a \in I \). Suppose \( a' \in I \) aswell. Then also \( a \lor a' = 1 \in I \) which is a contradiction because \( I \) is proper. Hence \( a \in I \iff a' \notin I \).

3. \( \Rightarrow \) 1. Assume for all \( a \in B \) it is the case that \( a \in I \iff a' \notin I \). Let \( J \) be an ideal such that \( I \subsetneq J \) and fix \( a \in J \setminus I \). Because \( a \notin I \), we have \( a' \in I \). But then \( 1 = a \lor a' \in J \), which implies \( J = B \), and hence \( I \) is maximal.

\( \square \)

Let \( \mathcal{M} = (X, \ldots, V) \) be a model and \( T_w = \{ \phi \mid w \models \phi \} \) be a theory. The valuation \( V \) is a function that maps a proposition letter to the subset of \( X \) where this letter is true. The theory is thus a maximal consistent theory, because for every theorem \( \phi \) either \( \phi \in T_w \) or \( \neg \phi \notin T_w \).

Every consistent theory is the intersection of all the maximal consistent theories extending it. If \( \Gamma \models \phi \), then there is a maximal consistent theory such that \( \Gamma \subseteq \Gamma' \) and \( \phi \notin \Gamma' \). Hence \( \Gamma = \cap \{ \Gamma' \mid \Gamma' \text{ is a maximal consistent theory and } \Gamma \subseteq \Gamma' \} \).

**Theorem 6.10.** (The ultrafilter theorem) Every filter in a Boolean algebra can be extended to an ultrafilter.

**Proof.** Let \( F \) be a filter in a Boolean algebra \( B \). Define \( \mathcal{F} := \{ F_i \subseteq B \mid F_i \text{ is a filter and } F \subseteq F_i \} \). Clearly \( \mathcal{F} \) is not-empty, because \( F \in \mathcal{F} \). Order \( \mathcal{F} \) by inclusion, and we obtain the poset \( (\mathcal{F}; \subseteq) \).

We claim now that every chain in \( (\mathcal{F}; \subseteq) \) has an upper bound. To show this let \( S = \{ S_i \in \mathcal{F} \mid i \in I \} \) be a chain in \( \mathcal{F} \). Define \( S := \bigcup_{i \in I} S_i \). If \( x, y \in S \), then there are \( i, j \in I \) such that \( x \in S_i \) and \( y \in S_j \). Without loss of generality we may assume \( S_i \subseteq S_j \), because \( S \) is
a chain. Thus \( x \in S_i \subseteq S_j \), imply \( x \in S_j \). Because \( S_j \) is a filter also \( x \wedge y \in S_j \subseteq S \). Now suppose \( z \in B \) and \( x \leq z \). Then \( z \in S_i \subseteq S_j \subseteq S \), hence \( z \in S \). Finally, for all \( i \in I \), we have \( 0 \notin S_i \), and hence \( 0 \notin S \). We can conclude now that \( S \) is a proper filter containing \( F \), which means \( S \in S \). Hence \( S \) is an upper bound for \( S \) in \( \mathcal{F} \).

Every chain in \( \langle \mathcal{F}; \subseteq \rangle \) has an upper bound. By Zorn’s Lemma we can conclude that \( \mathcal{F} \) has a maximal element. This maximal element is a filter \( G \) that contains \( F \) and for every filter \( F_i \) such that if \( F_i \supseteq G \), holds \( F_i \subseteq G \) (because \( F \subseteq F_i \), hence \( F_i = G \) and \( G \) is a ultrafilter. \)

If \( Fm \) is a language over a denumerable set of variables, then \( |Fm| = \aleph_0 \) and we have an alternative proof of the ultrafilter theorem for the Boolean algebra \( \langle Fm/\equiv, \leq \equiv \langle \) which does not need Zorn’s Lemma (or any other property equivalent to AC):

This is done in the so called Henkin construction. Since \( |Fm| = \aleph_0 \), we can enumerate the formulas of \( Fm \), \( Fm = \{ \gamma_0, \gamma_1, \ldots, \gamma_i, \ldots | i \in \mathbb{N} \} \). Let \( T \) be our consistent theory. Define a chain \( \Gamma_0 \subseteq \Gamma_1 \ldots \subseteq \Gamma_n \subseteq \ldots \) for \( n \in \mathbb{N} \) by induction as follows:

\[
\Gamma_0 := T.
\]

Let \( n \geq 0 \). Suppose we have defined \( \Gamma_n \). In order to define \( \Gamma_{n+1} \) consider \( \gamma_n \) and

\[
\Gamma_{n+1} := \begin{cases} 
\Gamma_n \cup \{ \gamma_n \} & \text{if } \Gamma_n \cup \{ \gamma_n \} \not\vdash \bot \\
\Gamma_n \cup \{ \neg \gamma_n \} & \text{if } \Gamma_n \cup \{ \neg \gamma_n \} \vdash \bot 
\end{cases}
\]

Let \( \Gamma := \bigcup_{n \in \mathbb{N}} \). It now rests us to show that \( \Gamma \) is a maximal consistent theory. This is obvious, because for every formula \( \gamma \) in our language is decided whether it is contained in \( \Gamma \) or not, and if not then \( \neg \gamma \) is contained in \( \Gamma \). Adding a formula will lead to absurdity, and hence \( \Gamma \) is a maximal consistent theory.
Chapter 7

Representation

We have seen that a finite Boolean algebra is isomorphic to a powerset algebra. But it would be nice to have an isomorphism not only for the finite case, but for Boolean algebras in general. We will make therefore the set of the prime ideals of any Boolean algebra into a topology space in such a way that it is homeomorphic to a Boolean algebra. At the end we will see that Boolean algebras are dual to Boolean spaces, and bounded distributive lattices are dual to Priestley spaces. For the Boolean algebras we will use the Stone representation theorem, and for the bounded distributive lattices there is Priestley’s representation theorem. Also for the logic this is a very nice result. We will give a same order-homeomorphism between the Lindenbaum algebra (which we will first manipulate to make it into a Boolean algebra) and a canonical model whose points are maximal consistent theories.

7.1 Topology

For the Priestley space and the Boolean space we need a bit of topology. This section describes some basics of topology. There are also two lemmas that are needed for some proofs in this chapter.

A topology on \( X \) is a family of subsets of \( X \) closed under finite intersections, arbitrary unions and which contains \( \emptyset \) and \( X \). A topological space is a set \( X \) together with a topology on \( X \), notation \((X; \mathcal{T})\).

Let \( \mathcal{B} \) be a family of subsets of \( X \), containing \( \emptyset \) and \( X \), and which is closed under finite intersections. Then \( \mathcal{B} \) is a basis for the topology \( \mathcal{T} \) which contains all arbitrary unions of sets in \( \mathcal{B} \).

Let \( \mathcal{S} \) be a family of subsets of \( X \), containing \( \emptyset \) and \( X \). We can form \( \mathcal{B} \) by taking all the finite intersections of sets in \( \mathcal{S} \). Then \( \mathcal{B} \) is again a basis for the topology containing all arbitrary unions of sets in \( \mathcal{B} \). We call \( \mathcal{S} \) a subbasis for \( \mathcal{T} \).

Let \((X; \mathcal{T})\) and \((X'; \mathcal{T}')\) be topological spaces and \( f : X \to X' \) a map. Then \( f \) is continuous iff \( U \subseteq X' \) open implies \( f^{-1}(U) \subseteq X \) open. And \( f \) is said to be a homeomorphism
if $f$ is bijective and if both $f$ and $f^{-1}$ are continuous.

A topological space $(X; T)$ is said to be **Hausdorff** if for all $x, y \in X$ such that $x \neq y$ there are $U, V \subseteq X$ open such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

A topological space $(X; T)$ is compact if for every open cover $U$ of $X$ there is a finite subcover $U \subseteq U$ of $X$.

**Lemma 7.1.** Let $(X; T)$ be a compact Hausdorff space. A subset $Y$ of $X$ is compact iff $Y$ is closed.

**Proof.** Assume $Y \subseteq X$ is a compact subset of $X$. Fix a $z \in X \setminus Y$. Then for all $y \in Y$ we have $y \neq x$ and because $X$ is Hausdorff there are open $U_y$ and $V_y$ such that $z \in U_y$ and $y \in V_y$. Define $Y := \{V_y \mid y \in Y, z \notin V_y\}$. This is an open cover of $Y$. We assumed $Y$ is compact, thus there are $y_1, \ldots, y_n \in Y$ such that $Y \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Define then $U_z := \bigcap_{i=1}^{n} U_{y_i}$ (note this is an open set). Then $z \in U_z$ and $U_z \cap Y = \emptyset$. For all $z \in X \setminus Y$ we can find a $U_z$ such that $U_z \cap Y = \emptyset$. We conclude $X \setminus Y = \bigcup_{z \in X \setminus Y} U_z$ is open, and hence $Y$ is closed.

Assume then that $Y$ is a closed subset of $X$. Let $U = \{U_i\}_{i \in I}$ be an open cover of $Y$. Because $Y$ is closed, $X \setminus Y$ is open, and hence $U \cup X \setminus Y$ is an open cover of $X$. We know that $X$ is compact, thus there is a finite subcover, $U_1 \cup \ldots \cup U_n \cup X \setminus Y$, of $X$. But then $Y \subseteq U_1 \cup \ldots \cup U_n$. We found a finite subcover for an arbitrary open cover of $Y$, hence $Y$ is compact.

**Lemma 7.2. Alexander’s Subbasis Lemma**

Let $(X; T)$ be a topological space and $S$ be a subbasis for $T$. Then $X$ is compact if every open cover of $X$ by members of $S$ has a finite subcover.

7.2 Stone’s representation

7.2.1 Boolean space

The Stone’s representation theorem will give us a order-homeomorphism between Boolean algebras and Boolean spaces. First we will define these terms:

**Definition 7.3.** A topological space $(X; T)$ is called a totally disconnected space if for all $x, y \in X$ such that $x \neq y$ there exists a clopen (open and closed) subset $U \subseteq X$ such that $x \in U$ and $y \notin U$.

**Definition 7.4.** A topological space is called a Boolean space if it is both compact and totally disconnected.

Now we have to ask ourselves which set $X$ to choose to achieve the right representation. It turns out to be the prime ideal space of the Boolean algebra, $(I_p(B); T)$, that gives us
exact the right properties for the order-homeomorphism. The topology on \( \mathcal{I}_p(B) \) will be defined as follows:

\[
\mathcal{T} := \{ U \subseteq \mathcal{I}_p(B) \mid U \text{ is a union of } X_a \text{ for a collection of } a \in B \}
\]

where \( X_a := \{ I \in \mathcal{I}_p(B) \mid a \notin I \} \). These sets \( X_a \) of prime ideals of \( B \) are clopen sets (we will show in this section that the clopen subsets of \( X \) are exactly the sets \( X_a \) for \( a \in B \)). This is a nice choice, because in a topological space the family of clopen subsets form a Boolean algebra. (have I showed this yet?)

To show that \( (\mathcal{I}_p(B); T) \) is in fact a Boolean space we will first show that the space is compact.

**Proposition 7.5.** Let \( B \) be a Boolean algebra. Then the prime ideal space \( (\mathcal{I}_p(B); T) \) is compact.

**Proof.** We have to show that for every open cover of \( X := \mathcal{I}_p(B) \) there exist finitely many members of this open cover whose union is \( X \). Let \( \mathcal{U} \) be an open cover of \( X \). The collection \( \mathcal{B} := \{ X_a \mid a \in B \} \) (where \( X_a = \{ I \in \mathcal{I}_p(B) \mid a \notin I \} \)) forms a basis for the topology \( \mathcal{T} \). All unions of \( X_a \) are open sets, and without loss of generality we may assume that \( \mathcal{U} \subseteq \mathcal{B} \). We write \( \mathcal{U} = \{ X_a \mid a \in A \} \) where \( A \subseteq B \). Let \( J \) be the smallest ideal containing \( A \), that is \( J = \{ b \in B \mid b \leq a_1 \lor \ldots \lor a_n \text{ for some } a_1, \ldots, a_n \in A \} \). Suppose \( J \) is not proper. Then \( 1 \in J \), and \( 1 = a_1 \lor \ldots \lor a_n \) for some \( a_1, \ldots, a_n \in A \). We obtain

\[
\mathcal{I}_p(B) = X = X_1 = X_{a_1 \lor \ldots \lor a_n} = \{ I \in \mathcal{I}_p(B) \mid a_1 \lor \ldots \lor a_n \notin I \} = \{ I \in \mathcal{I}_p(B) \mid a_1 \notin I \lor \ldots \lor a_n \notin I \} = \{ I \in \mathcal{I}_p(B) \mid a_1 \notin I \} \cup \ldots \cup \{ I \in \mathcal{I}_p(B) \mid a_n \notin I \} = \bigcup_{i=1}^{n} X_{a_i}
\]

Hence we found a finite subcover \( \{ X_{a_1}, \ldots, X_{a_n} \} \) of \( \mathcal{U} \) that covers \( X \).

Suppose that \( J \) is proper. Then (BPI) confirms us the existence of a prime ideal \( I \) such that \( J \subseteq I \). But then \( A \subseteq I \in \mathcal{I}_p(B) = X \). Also \( a \in I \) implies \( I \notin \{ I \in \mathcal{I}_p(B) \mid a \notin I \} \) for all \( a \in A \). Hence \( I \notin X_a \) for all \( a \in A \). This means that \( \mathcal{U} \) doesn’t cover \( X \). Contradiction. We conclude that \( J \) has to be not proper which implies that for every open cover of \( X \) there exists a finite subcover. Hence the prime ideal space \( (\mathcal{I}_p(B); T) \) is compact. \( \square \)

Now we have proved the compactness, we have to prove that \( (\mathcal{I}_p(B); T) \) is totally disconnected. We therefore need the following proposition, which also shows that the clopen subsets of \( \mathcal{I}_p(B) \) are exactly the sets \( X_a \) for \( a \in B \), as we promised above.

**Proposition 7.6.** Let \( X := \mathcal{I}_p(B) \) and let \( (X; T) \) be the prime ideal space of the Boolean algebra \( B \). Then the clopen subsets of \( X \) are exactly the sets \( X_a \) for \( a \in B \). Further, given distinct points \( x, y \in X \), there exists a clopen subset \( V \) of \( X \) such that \( x \in V \) and \( y \notin V \).

**Proof.** The sets \( X_a \) form the basis for the topology and hence are open sets. Then \( X \setminus X_a = \mathcal{I}(B) \setminus \{ I \in \mathcal{I}_p(B) \mid a \notin I \} \). Theorem 10.12 tells us that \( a \notin I \iff a' \notin I \). Thus \( X \setminus X_a = \{ I \in \mathcal{I}_p(B) \mid a' \notin I \} = X_{a'} \subseteq B \). Because \( X_{a'} \) is open, all sets \( X_a \) are closed and open:
closed, and by A7 compact. Hence there is a finite
A
Let now
x
B
prime ideal in
X
Proof.
Define
Lemma 7.8.
Let
y
X
such that
y
∈
A
we have
I
clopen.
Lemma 7.7.
Let
I
∈
A
for some
A
⊆
B
. But
U
is also
closed, and by A7 compact. Hence there is a finite
A
1
⊆
A
such that
U
= \bigcup a \in A_1 = X_{a_1 \vee \ldots \vee a_n} = X_{V A_1}
. The clopen subsets of
X
are exactly the sets
X_a
for
a \in B
.

Lemma 7.7. Let \((X; T)\) be a Boolean space.

1. Let \(Y\) be a closed subset of \(X\) and \(x \notin Y\). Then there exists a clopen set \(V\) such that \(Y \subseteq V\) and \(x \notin V\).

2. Let \(Y\) and \(Z\) be closed disjoint subsets of \(X\). Then there exists a clopen set \(U\) such that \(Y \subseteq U\) and \(Z \cap U = \emptyset\).

Proof. 1. Let \(Y\) be a closed subset of \(X\) and \(x \notin Y\). Then proposition ... (11.3) tells us that for every \(y \in Y\) there exists a clopen subset \(V_y\) of \(X\) such that \(y \in V_y\) and \(x \notin V_y\). Thus \(\bigcup_{y \in Y} V_y \supseteq Y\) is an open cover of \(Y\) and \(x \notin \bigcup_{y \in Y} V_y\). Because \(Y\) is closed, lemma ... implies the compactness of \(Y\), and thus there is a finite subcover of \(Y\), namely \(V' := \bigcup_{i=1}^n V_i \subseteq Y\).

A finite set of clopen sets is clopen, and hence \(V'\) is a clopen set such that \(Y \subseteq V'\) and \(x \notin V'\).

2. Let \(Y\) and \(Z\) be closed disjoint subsets of \(X\). The Boolean space \((X; T)\) is totally disconnected which means that for \(x, y \in X\) such that \(x \neq y\) there is a clopen subset \(V\) of \(X\) such that \(x \in V\) and \(y \notin V\). We define \(U_1^{y,z}\) as a clopen subset of \(X\) such that \(y \in U_1^{y,z}\) and \(z \notin U_1^{y,z}\). Define then \(U_1 := \{U_1^{y,z} \mid y \in Y\}\). This is an open cover of \(Y\). Because \(Y\) is closed and the Boolean space is Hausdorff, we know by the lemma that \(Y\) is compact. We can now find a finite subcover \(U_1 := \bigcup_{i=1}^n U_1^{y_i,z_i}\) of \(Y\) such that \(z \notin U_1^{y_i}\).

After this we define \(U_2 := \{X \setminus U_1^z \mid z \in Z\}\) where \(X \setminus U_1^z \not\supseteq Y\) and \(z \in X \setminus U_1^z\) holds for all \(z \in Z\). This \(U_2\) is again an open cover, this time of \(Z\) and \(Y \notin U_2\). For the same reasons also \(Z\) is compact, and hence we can find a finite subcover \(U_2 := \bigcup_{i=1}^m X \setminus U_1^{z_i}\) of \(Z\) such that \(Y \not\subseteq U_2\).

Define finally \(U := X \setminus U_2 = X \setminus \bigcup_{1 \leq i \leq m} X \setminus U_1^{z_i} = \bigcap_{1 \leq i \leq m} U_1^{z_i}\). Then \(Y \subseteq U\) and \(Z \cap U = \emptyset\).

Lemma 7.8. Let \(Y\) be a Boolean space, let \(B\) be the algebra \(\mathcal{P}^T(Y)\) of clopen subsets of \(Y\) and let \(X\) be the dual space of \(B\). Then \(Y\) and \(X\) are homeomorphic.

Proof. Define \(\epsilon : Y \to X\) by \(\epsilon(y) := \{a \in B \mid y \notin a\}\). Then as we see below \(\epsilon(y)\) is a prime ideal in \(B\):

- Assume \(c, d \in \epsilon(y)\). Then \(y \notin c \cup d\) (= \(c \lor d\)), whence \(y \notin c\) and \(y \notin d\).
- If \(c \in B\), \(d \in \epsilon(y)\) and \(c \subseteq d\), then \(y \notin d\) and thus \(y \notin c\). This implies \(c \in \epsilon(y)\).

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• Let $I$ be an ideal such that $\epsilon(y) \subseteq I$. Then there is an element $x \in I \setminus \epsilon(y)$, thus $y \notin x$. Both $x$ and $x'$ are clopen sets and $y \notin x'$, hence $x' \in \epsilon(y) \subseteq I$. Because $x, x' \in I$ and because $I$ is an ideal, $x \lor x' \in I \Rightarrow I = X$.

We now want to prove that $\epsilon$ is a continuous bijection, because then by the lemma of section Topology it follows that $\epsilon$ is a homeomorphism.

Let us start with the injectivity. Let $y \neq z$ in $Y$. Because $Y$ is totally disconnected there is a clopen subset $a$ of $Y$ such that $y \in a$ and $z \notin a$. But then $\epsilon(y) \neq \epsilon(z)$. Hence $\epsilon$ is an injection.

To show that $\epsilon$ is continuous we need to show that every $\epsilon^{-1}(X_a)$ is clopen in $Y$ for every $a \in B$. Because $X_a = \{ I \in X \mid a \notin I \}$, we obtain

$$\epsilon^{-1}(X_a) = \{ y \in Y \mid \epsilon(y) \in X_a \} = \{ y \in Y \mid a \notin \epsilon(y) \} = \{ y \in Y \mid a \notin \{ b \in B \mid y \notin b \} \} = \{ y \in Y \mid a \in \{ b \in B \mid y \notin b \} \} = \{ y \in Y \mid y \in a \} = a.$$

This is obviously a clopen subset of $Y$, and hence $\epsilon$ is continuous.

Finally, for surjectivity, assume there is a $x \in X \setminus \epsilon(Y)$. By Topology lemma ... we know, because $\epsilon$ is continuous, that $\epsilon(Y)$ is compact, and hence $\epsilon(Y)$ is a closed subset of $X$.

By proposition ... and lemma ... above there is a clopen subset $X_a$ of $X$ such that $\epsilon(Y) \subseteq X_a$ and $x \notin X_a$. But then $x \in X \setminus X_a = X_{a'}$ and $X_{a'} \cap \epsilon(y) = \emptyset$, which implies $\emptyset = \epsilon^{-1}(X_{a'}) = a'$. This is a contradiction and hence $\epsilon$ is a surjection.

By lemma ... of the section Topology it follows now that $\epsilon$ is a homeomorphism.

\[ \Box \]

### 7.3 Priestley

**Theorem 7.9.** Let $L$ be a bounded distributive lattice. Then the prime ideal space $(\mathcal{I}_p(L); T)$ is compact.

**Proof.** We want to prove that for every open cover $U$ of $X := \mathcal{I}_p(L)$ there is a finite subcover of $X$. Alexander’s Subbasis Lemma helps us to prove only that for every open cover of $X$ that consists of sets of a subbasis $\mathcal{S}$ there is a finite subcover of $X$.

Let $U := \{ X_b \mid b \in A_0 \} \cup \{ X \setminus X_c \mid c \in A_1 \}$, where $A_0, A_1 \subseteq L$. Let $J$ be the ideal generated by $A_0$ (which is $\{ 0 \}$ if $A_0 = \emptyset$) and let $G$ be the filter generated by $A_1$ (which is $\{ 1 \}$ if $A_1 = \emptyset$). Suppose $J \cap G = \emptyset$. With (DPI) we can find a $I \in \mathcal{I}_p(L)$ and $F = L \setminus I \in \mathcal{F}_p(L)$ such that $J \subseteq I$ and $G \subseteq F$. Thus $G \cap I = \emptyset$. Now $b \in I$ implies $I \notin X_b$ for all $b \in A_0$. Further $c \notin I$ implies $I \notin X_c$ for all $c \in A_1$ and hence $I \notin X \setminus X_c$ for all $c \in A_1$. But then $U$ doesn’t cover $\mathcal{I}_p(L)$, which is a contradiction.

We may conclude now that $J \cap G \neq \emptyset$. Take an $a \in J \cap G$, and assume $A_0$ and $A_1$ are both non-empty. Then because $a \in J$ and $a \in G$ there are $a_1, \ldots, a_n \in A_0$ such that $a \leq a_1 \lor \ldots \lor a_n$ and there are $b_1, \ldots, b_m \in A_1$ such that $a \leq b_1 \land \ldots \land b_m$. Hence

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We obtain $X = X_1 = X_{a_1 \lor \ldots \lor a_n} \cup (X \setminus X_{b_1 \land \ldots \land b_m}) = \bigcup_{i=1}^{n} X_{a_i} \cup (X \setminus \bigcap_{i=1}^{m} X_{b_i}) = X_{a_1} \cup \ldots \cup X_{a_n} \cup (X \setminus X_{b_1} \cup \ldots \cup X \setminus X_{b_m})$. We have found the finite subcover. \qed

**Definition 7.10.** A Priestley space is a totally order-disconnected space that is compact.

**Lemma 7.11.** Let $\langle X; \leq, T \rangle$ be a Priestley space.

1. $x \leq y$ in $X \iff y \in U$ implies $x \in U$ for every $U \in O^{T}(X)$.

2. Let $Y$ be a closed downset in $X$ and let $x \notin Y$. Then there exists a clopen downset $U$ such that $Y \subseteq U$ and $x \notin U$.

   • Let $Y$ and $Z$ be disjoint closed subsets of $X$ such that $Y$ is a down-set and $Z$ is an up-set. Then there exists a clopen down-set $U$ such that $Y \subseteq U$ and $Z \cap U = \emptyset$.

**Proof.** 1. In the Priestley space $\langle X; \leq, T \rangle$ there is a clopen downset $U \in O^{T}(X)$ such that $x \in U$ and $y \notin U$ for all $x, y \in X$ with $x \not\leq y$.

Let $x, y \in X$ and $x \leq y$. Then for the case $x = y$ we have for all $U \in O^{T}(X)$ that if $y \in U$ then $x \in U$. For the case $x < y$ (thus $x \not\leq y$) we have property of a Priestley space namely that there is a $U \in O^{T}(X)$ such that $x \in U$ and $y \notin U$. But if $x < y$ and $y \in U$ then because $U$ is a down-set also $x \in U$. Hence if $y \in U$ then also $x \in U$, but the other way is not always true.

2. Let $Y$ be a closed down-set in $X$ and let $x \notin Y$. For each $y \in Y$ holds $x \not\leq y$ otherwise $x \in Y$. Because of the Priestley space there is a clopen down-set $U \in O^{T}(X)$ such that $y \in U_y$ and $x \notin U_y$. Thus $\{U_y \mid y \in Y\}$ forms an open cover of $Y$. Because $Y$ is closed, the lemma ... sais $Y$ is compact. Hence there is a finite subcover $\{U_{y_i} \mid y_1, \ldots, y_n \in Y\}$ of $Y$.

By construction $x \notin \bigcup_{i=1}^{n} U_{y_i}$ and $Y \subseteq \bigcup_{i=1}^{n} U_{y_i}$. \qed
Chapter 8

Populaire samenvatting

Stel je voor dat iemand een prijs wint en op reis mag met een paar familieleden. Maar wel met familieleden die op een speciale manier geselecteerd worden, namelijk als volgt: We beginnen met de winnaar. Alle kinderen van de winnaar mogen mee. En vervolgens ook de kinderen van deze personen, enzovoort. Daarnaast mogen ook die mensen mee, die een ouder zijn van twee personen uit de al geselecteerde mensen.

Concreter (zie ook het plaatje): stel je voor dat Piet de prijs wint. Clara en Joost, de kinderen van Piet, mogen dan mee op reis. Vervolgens mag ook de moeder van Clara en Joost mee op reis, omdat zij de ouder is van deze twee. Als Piet geen broers en zussen zou hebben, dan mogen de vader en moeder van Piet helaas niet mee, omdat zij allebei geen ouder zijn van twee personen. We hebben nu met twee simpele eisen een manier gevonden om een aantal mensen uit te kiezen uit bijvoorbeeld een familie.

Dit kunnen we nu heel algemeen maken en in willekeurige geordende verzamelingen toepassen. Een geordende verzameling is bijvoorbeeld de verzameling natuurlijke getallen, die geordend is met de natuurlijk ordening (zo geldt bijvoorbeeld $5 < 7$). Vervolgens kunnen we overstappen naar de propositielogica en op een bepaalde manier de proposities ordenen, bijvoorbeeld door te bekijken welke premissen nodig zijn om bepaalde conclusies te kunnen trekken. Uit de zin ”Het is maandag en het regent.” kunnen we natuurlijk concluderen dat het regent. Op deze manier brengen we een ordening aan, door te zeggen welke proposities afleidbaar zijn uit andere propositie. En nu kunnen we net zoals bij de familie ook binnen de propositielogica een specifieke
verzameling proposities aanwijzen die consistent is (niet tot een absurditeit leidt).

We kunnen dit steeds verder doorvoeren. Er zijn speciale geordende verzamelingen, die heel mooie eigenschappen hebben. Daardoor kunnen we deze verzamelingen gebruiken om ingewikkelde wiskundige structuren duidelijker te maken.
Bibliography

