The Strong Largeur d’Arborescence

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Abstract

An important graph parameter is tree-width. The tree-width of a graph $G$, which we denote by $\text{tw}(G)$, is defined as the smallest integer $k$ such that $G$ is contained in a clique-sum of copies of $K_{k+1}$. Two related graph parameters are the largeur d’arborescence and the strong largeur d’arborescence, which we denote by $\text{la}_{\square}(G)$ and $\text{la}_{\bowtie}(G)$, respectively. The parameter $\text{la}_{\square}(G)$, introduced by Colin de Verdière [6], is defined as the smallest integer $r$ such that $G$ is a minor of the Cartesian product $T \square K_r$ for some tree $T$. The parameter $\text{la}_{\bowtie}(G)$, recently introduced by Marianna E.-Nagy, Monique Laurent, and Antonios Varvitsiotis [8], is based on the strong product instead of the Cartesian product. It is defined as the smallest integer $r$ such that $G$ is a minor of the strong product $T \bowtie K_r$ for some tree $T$.

We define a new family of graph parameters $\text{la}_t(G)$ in terms of tree-decompositions, with $t$ a non-negative integer (or $t = \infty$). Given $t$, we call the parameter $\text{la}_t(G)$ the largeur d’arborescence of order $t$, and we show that this parameter is minor-monotone, i.e., $\text{la}_t(H) \leq \text{la}_t(G)$ whenever $H$ is obtained from $G$ by a series of edge deletions, edge contractions, and vertex deletions. We also show that this family contains the tree-width (for $t = 0$), the largeur d’arborescence (for $t = 1$), and the strong largeur d’arborescence (for $t = \infty$).

To investigate the parameter $\text{la}_t(G)$ for $t \geq 1$, we introduce the notion of partially eared graphs. This allows us to deal with vertices of degree 2, as these vertices play a special role in our study of $\text{la}_t(G)$. Based on this notion we give a characterization of $\text{la}_t(G)$ in terms of vertex elimination orderings. We also show that it is NP-hard to compute $\text{la}_t(G)$.

Additionally, for $t \geq 1$, we look at minimal forbidden minors for the class of graphs $G$ with $\text{la}_t(G) \leq r$. We show that these graphs are 3-connected after suppressing all vertices of degree 2. We use this property to characterize the class of graphs $G$ with $\text{la}_t(G) \leq 2$, thus recovering earlier results known for the cases $t = 1$ and $t = \infty$. We also give two families of minimal forbidden minors for $\text{la}_t(G) \leq r$, and construct minimal forbidden minors for $\text{la}_\infty(G) \leq r$ from minimal forbidden minors for $\text{la}_\infty(G) \leq r - 1$. Finally, we present an algorithm for finding minimal forbidden minors for $\text{la}_t(G) \leq r$. We use this algorithm to reconstruct the complete set of minimal forbidden minors for $\text{la}_1(G) \leq 3$, to give a (possibly complete) set of 194 minimal forbidden minors for $\text{la}_\infty(G) \leq 3$, and a set of 1303 minimal forbidden minors for $\text{la}_1(G) \leq 4$. 
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Introduction

A fundamental notion in graph theory is tree-width. Informally, the tree-width of a graph $G$, which we denote by $\text{tw}(G)$, is a measure of the tree-likeness of $G$ in some sense. It has a number of different (but equivalent) characterizations, one of which is the following. The tree-width of a graph $G$ is the smallest integer $k$ such that $G$ is contained in a clique-sum of copies of $K_{k+1}$, where $K_{k+1}$ denotes the complete graph on $k+1$ vertices. For example, if $G$ is a forest, then $\text{tw}(G) = 1$, since a forest is a clique-sum of edges.

The notion of tree-width is important both theoretically and in practice. It underlies the results about graph minors by Robertson and Seymour, perhaps one of the deepest results in graph theory. Also, on a practical view, it has been shown that many NP-hard problems are linear time solvable on graphs with some constant upper bound on the tree-width. Examples are the Hamiltonian circuit problem and the maximum independent set problem (see [3] for an overview).

An important property of tree-width is minor-monotonicity. A graph parameter $f(G)$ is called minor-monotone if $f(H) \leq f(G)$ for any graph $G$ and any minor $H$ of $G$, where a minor of $G$ is a graph obtained from $G$ by a series of edge deletions, edge contractions, and vertex deletions. From the celebrated results of Robertson and Seymour it follows that for any minor-monotone graph parameter $f(G)$ and any number $k$, the class of graphs $G$ with $f(G) \leq k$ is characterized by a finite set $\mathcal{F}$ of minimal forbidden minors. That is, $\mathcal{F}$ is the minimal set of graphs with the property that for any graph $G$, $f(G) \leq k$ if and only if no graph in $\mathcal{F}$ is a minor of $G$. For $f(G) = \text{tw}(G)$ and $k = 1, 2, 3$, the sets of minimal forbidden minors are given in Table [1].
A parameter related to tree-width is the *largeur d’arborescence*, which we denote by $\mathrm{la}(G)$. This topological graph parameter is defined as the smallest integer $r$ such that $G$ is a minor of the Cartesian product $T \Box K_r$ for some tree $T$. It satisfies

$$\mathrm{tw}(G) \leq \mathrm{la}(G) \leq \mathrm{tw}(G) + 1$$

(see [14] for the lower bound and [6] for the upper bound). From the definition it directly follows that $\mathrm{la}(G)$ is minor-monotone. The motivation of this parameter is that it is an upper bound on the spectral graph invariant $\nu(G)$, introduced by Colin de Verdière in [6] (where it is called $\nu^R_1(G)$).

For a graph $G = (V, E)$, the parameter $\nu(G)$ is defined as the maximum corank (dimension of the kernel) of a positive semidefinite $|V| \times |V|$ matrix $A$ such that

(i) for $i \neq j$, $A_{ij} = 0$ if and only if $ij \notin E$,

(ii) any symmetric $|V| \times |V|$ matrix $X$ satisfying $AX = 0$ and $X_{ij} = 0$ for $i = j$ or $ij \in E$ is equal to the zero matrix.

This parameter is also minor-monotone [6]. We refer to [6, Theorem 2] for a proof of the inequality $\nu(G) \leq \mathrm{la}(G)$. The main idea is to show that $\nu(T \Box K_r) \leq r$ for any tree $T$, and to then use the minor-monotonicity of $\nu(G)$.

A related spectral graph invariant is $\nu^= (G)$, introduced by Hein van der Holst in [16]. It has the same definition as $\nu(G)$, except that (i) is replaced by

(i') if $i \neq j$ and $ij \notin E$, then $A_{ij} = 0$.

Clearly, $\nu(G) \leq \nu^= (G)$ for any graph $G$. In [16] it is also shown that

$$\nu^= (G) \leq \mathrm{tw}(G) + 1.$$  

A second topological graph parameter related to tree-width is the *strong largeur d’arborescence*, which we denote by $\mathrm{la}_\Box(G)$. It is based on the strong
product instead of the Cartesian product. Recall that the strong product $G \boxtimes H$ of two graphs $G = (V, E)$ and $H = (W, F)$ is the graph on $V \times W$ with two distinct vertices $(u, u'), (v, v')$ connected if and only if $(u = v$ or $uv \in E)$ and $(u' = v'$ or $u'v' \in F)$. Given a graph $G$, the parameter $la_0(G)$ is defined as the smallest integer $r$ such that $G$ is a minor of $T \boxtimes K_r$ for some tree $T$. This parameter is introduced by Marianna E.-Nagy, Monique Laurent, and Antonios Varvitsiotis in [8] and serves as an upper bound for the extreme Gram dimension, a linear algebra type graph parameter related to the matrix completion problem, also introduced in [3].

The (extreme) Gram dimension is defined as follows. A positive semidefinite matrix with an all-ones diagonal is called a correlation matrix. We denote the set of all $n \times n$ correlation matrices by $E_n$. Given a graph $G = (V, E)$, $E(G)$ is defined as the projection of $E_n$ onto the subspace $\mathbb{R}^E$ indexed by $E$. In other words, $E(G)$ is the set of all partial symmetric matrices whose entries are specified at positions corresponding to the edge set of $G$ and which can be completed to a correlation matrix. The Gram dimension $gd(G, a)$ of an element $a \in E(G)$ is defined as the smallest rank of a positive semidefinite completion of $a$. Now recall that an extreme point of a convex set $K$ is an element $x \in K$ such that $x = ty + (1 - t)z$ with $y, z \in K$ and $t \in [0,1]$ implies that $x = y$ or $x = z$. We denote the set of all extreme points of $K$ by $\text{Ext}(K)$. The Gram dimension $gd(G)$ and the extreme Gram dimension $egd(G)$ of a graph $G$ are defined as:

$$gd(G) = \max_{a \in E(G)} gd(G, a),$$

$$egd(G) = \max_{a \in \text{Ext}(E(G))} gd(G, a).$$

It is clear that $egd(G) \leq gd(G)$. In [10] it is shown that

$$\nu^-(G) \leq gd(G) \leq tw(G) + 1.$$  

Also, both parameters $gd(G)$ and $egd(G)$ are minor-monotone [10, 8].

We define a new family of graph parameters $la_t(G)$, with $t$ a non-negative integer (or $t = \infty$). Given $t$, we call the parameter $la_t(G)$ the largeur d’arborescence of order $t$. Its definition is in terms of tree-decompositions, which are used in one of the characterizations of tree-width. We show that $la_t(G)$ is minor-monotone, and that for any graph $G$ it holds that

$$la_0(G) = tw(G) + 1,$$

$$la_1(G) = la_\boxtimes(G),$$

$$la_\infty(G) = la_\boxtimes(G).$$
In the rest of the thesis we focus on the case $t \geq 1$. We give a different representation of a graph, called a partially eared graph. Using this representation we prove additional properties of $\text{la}_t(G)$. Moreover, we use this representation to characterize $\text{la}_t(G)$ in terms of ordered partitions of the vertex set of $G$, which is similar to the characterization of tree-width in terms of elimination orderings. We also look at minimal forbidden minors for the class of graphs $G$ with $\text{la}_t(G) \leq r$. We investigate their structure, identify two families of minimal forbidden minors, and give the complete sets of minimal forbidden minors for $\text{la}_t(G) \leq 2$ for $t = 1$ and $t \geq 2$. We also present an algorithm for finding minimal forbidden minors for $\text{la}_t(G) \leq r$.

**Outline of the thesis**

In Chapter 1 we introduce the new family of graph parameters $\text{la}_t(G)$. We first define the tree-width of a graph in terms of tree-decompositions (Section 1.1), and then introduce the notion of an open tree-decomposition (Section 1.2). This notion gives rise to the family of parameters $\text{la}_t(G)$, which we define in Section 1.3. In that section we also show that the parameter $\text{la}_t(G)$ is minor-monotone, that $\text{la}_0(G) = \text{tw}(G) + 1$, and that $\text{la}_\infty(G) = \text{la}_{\infty}(G)$.

In Chapter 2 we introduce the notion of partially eared graphs. Section 2.1 contains several definitions. In Section 2.2 we redefine $\text{la}_t(G)$ using partially eared graphs, and in Section 2.3 we prove equality of both definitions when $t \geq 1$. In Section 2.4 we give an alternative characterization of $\text{la}_t(G)$ in terms of ordered partitions of the vertex set of $G$. In Chapter 3 we use the notion of partially eared graphs to prove NP-hardness of computing $\text{la}_t(G)$ (Section 3.1), to investigate the behaviour of $\text{la}_t(G)$ under clique-sums (Section 3.2), and to study the case $t = 1$ (Section 3.3). In Chapter 4 we look at minimal forbidden minors for the class of graphs $G$ with $\text{la}_t(G) \leq r$. In Section 4.1 we show that these graphs are 3-connected after suppressing (i.e., contracting one of the two incident edges of a vertex of degree 2) all vertices of degree 2. In Section 4.2 we give two classes of minimal forbidden minors for $\text{la}_t(G) \leq r$, and in Section 4.3 we give the complete sets of minimal forbidden minors for $\text{la}_t(G) \leq 2$ for $t = 1$ and $t \geq 2$. In Section 4.4 we give an operation on a graph $G$ which increases $\text{la}_\infty(G)$ by one, and show that under certain conditions this operation transforms a minimal forbidden minor for $\text{la}_\infty(G) \leq r$ into a minimal forbidden minor for $\text{la}_\infty(G) \leq r + 1$. Finally, in Section 4.5 we present an algorithm for finding minimal forbidden minors for $\text{la}_t(G) \leq r$, and explain the results obtained from running this algorithm.
Preliminaries

Let $G = (V, E)$ be a graph. For the vertex set and edge set of $G$ we also use the notation $V(G)$ and $E(G)$, respectively. We denote an edge $\{u, v\} \in E$ by $uv$. We will always assume that $G$ is finite and simple (i.e., no loops or parallel edges). For $u, v \in V$, we write $u \simeq v$ if $u = v$ or $uv \in E$. For $U \subseteq V$, we define $N(U) = \{v \in V \setminus U \mid uv \in E$ for some $u \in U\}$ as the set of neighbours of $U$. By $G[U]$ we denote the subgraph of $G$ induced by $U$, and by $G - U$ we denote the graph obtained from $G$ by removing the vertices in $U$ and their incident edges. Given $F \subseteq \[V\]_2$ (with $\[V\]_2$ the set of all 2-element subsets of $V$), we define the graph $G + F = (V, E \cup F)$. If $U = \{u\}$ or $F = \{e\}$, then we also write $N(u)$, $G - u$ or $G + e$ instead of $N(\{u\})$, $G - \{u\}$ and $G + \{e\}$, respectively. If $w \in V$ is a vertex of degree two, then we write $G * w$ for the graph obtained from $G$ by suppressing the vertex $w$, i.e., $G * w = (G - w) + uv$, where $u, v$ are the two neighbours of $w$.

A pair $\{U, W\}$ of two sets is called a separation of $G$ if $U \cup W = V$ and $G$ has no edge between $U \setminus W$ and $W \setminus U$. The number $|U \cap W|$ is the order of the separation. If both $U \setminus W$ and $W \setminus U$ are non-empty, then the separation is called proper. In this case, the subset $U \cap W \subseteq V$ is called a cutset or a $k$-vertex-cut of $G$, where $k$ is the order of $\{U, W\}$. If $k = 1$, then the vertex $u \in V$ with $U \cap W = \{u\}$ is called a cutvertex of $G$. The graph $G$ is $k$-connected if $|V| > k$ and $G$ does not contain any $(k - 1)$-vertex-cut.

Let $e = uv \in E$. For $w \notin V$, the graph $(V \cup \{w\}, (E \setminus \{e\}) \cup \{uw, wv\})$ is the graph obtained from $G$ by subdividing the edge $e$. The graph $G \setminus e = (V, E \setminus \{e\})$ is the graph obtained from $G$ by deleting the edge $e$, and $G/e$ is the graph obtained from $G$ by contracting the edge $e$, i.e., by replacing both $u, v$ by a new vertex $u'$ such that $N(u') = N(\{u, v\})$. A minor of $G$ is a graph $H$ (written as $H \preceq G$) obtained from $G$ by a series of edge deletions, edge contractions, and vertex deletions. If in addition $H \neq G$, then we say that $H$ is a proper minor of $G$ and write $H \prec G$. If $G$ is connected, then $H$ is a minor of $G$ if and only if there is a partition of $V$ into nonempty subsets.
\{W_u \mid u \in V(H)\}$, such that each $G[W_u]$ is connected and, for each edge $uv \in E(H)$, there exists at least one edge between $W_u$ and $W_v$ in $G$. In this case the collection $\{W_u \mid u \in V(H)\}$ is called an $H$-partition of $G$. A graph parameter $f(\cdot)$ is minor-monotone if $f(H) \leq f(G)$ for any $H \preceq G$.

A subgraph $P \subseteq G$ is called an induced path in $G$ if $P$ is both a path and an induced subgraph of $G$. The length of $P$ is the number of edges in $P$. A subset $U \subseteq V$ is called an independent set in $G$ if every two vertices in $U$ are non-adjacent. A subset $C \subseteq V$ is called a clique in $G$ if every two vertices in $C$ are adjacent. Let $G_1, G_2$ be two graphs. We write $G_1 \cup G_2$ for the union of $G_1$ and $G_2$, and we write $G_1 \subseteq G_2$ if $G_1$ is a subgraph of $G_2$. If $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2)$ is a clique in both $G_1$ and $G_2$, then we say that $G$ is the $k$-clique-sum of $G_1$ and $G_2$, where $k = |V(G_1) \cap V(G_2)|$. 
Chapter 1

The parameter $\lambda_t(G)$

In this chapter we introduce the new family of graph parameters $\lambda_t(G)$. We first define tree-width in terms of tree-decompositions, and then give the definition of an open tree-decomposition in Section 1.2. Based on this definition we define $\lambda_t(G)$ in Section 1.3.

1.1 Tree-decomposition

There are a number of ways to characterize the tree-width of a graph. One of them is to use the tree-decomposition of a graph (see for example [7]). We first define a more general notion.

Definition 1.1.1. A tree-model is a pair $\mathcal{T} = (T, \mathcal{X})$, where $T$ is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of finite sets indexed by the nodes of $T$, such that for every $u \in \bigcup \mathcal{X}$, the subset $\{i \in V(T) \mid u \in X_i\}$ induces a subtree $T_u$ of $T$.

Given a tree-model $\mathcal{T} = (T, \mathcal{X})$ and two elements $u, v \in \bigcup \mathcal{X}$, we distinguish the following ways in which $T_u$ and $T_v$ can be related.

Definition 1.1.2. Let $\mathcal{T} = (T, \mathcal{X})$ be a tree-model, and let $u, v \in \bigcup \mathcal{X}$. If $T_u$ and $T_v$ intersect, i.e., if there is some $i \in V(T)$ such that $u, v \in X_i$, then we say that $u, v$ meet in $\mathcal{T}$. If $V(T_u) \cup V(T_v)$ induces a subtree of $T$, i.e., if there exist $i \simeq j \in V(T)$ such that $u, v \in X_i \cup X_j$, then we say that $u, v$ touch in $\mathcal{T}$.

Let $\mathcal{T} = (T, \mathcal{X})$ be a tree-model. Note that if $u, v \in \bigcup \mathcal{X}$ meet in $\mathcal{T}$, then $u, v$ also touch in $\mathcal{T}$. We now define a tree-decomposition of a graph as follows (this is the same definition as in [7]).
Definition 1.1.3. A tree-decomposition of a graph \( G = (V,E) \) is a tree-model \( T = (T,X) \) with \( \bigcup X = V \) such that for every edge \( uv \in E, u,v \) meet in \( T \).

An important property of a tree-decomposition is its width. Let \( T = (T,X) \) be a tree-model. We call the number \( \max_{i \in V(T)} |X_i| - 1 \) the width of \( T \). The tree-width of \( G \) is defined as the smallest possible width of a tree-decomposition of \( G \), and is denoted by \( tw(G) \). The “-1” in the definition of the width of a tree-model has the purpose of giving a forest a tree-width equal to 1.

1.2 Open tree-decomposition

In this section we define a slightly weaker version of a tree-decomposition of a graph, one for which it is sufficient that any two adjacent vertices of the graph touch in the decomposition.

1.2.1 Definition

We start with the definition.

Definition 1.2.1. An open tree-decomposition of a graph \( G = (V,E) \) is a tree-model \( T = (T,X) \) with \( \bigcup X = V \) such that for every edge \( uv \in E, u,v \) touch in \( T \).

Notice that a tree-decomposition of a graph \( G \) is also an open tree-decomposition of \( G \). The other direction is not true in general. However, as we will see in Lemma 1.2.4, there is a transformation which transforms an open tree-decomposition of \( G \) into a tree-decomposition of \( G \). The main ingredient of this transformation is the following operation of subdividing an edge in a tree-model.

Lemma 1.2.2 (subdividing an edge in a tree-model). Let \( T = (T,X) \) be a tree-model. Suppose the tree \( T' \) is obtained from \( T \) by subdividing an edge \( ij \in E(T) \). Denote the new vertex of \( T' \) by \( k \). Let \( X_k \) be a finite set satisfying \( X_i \cap X_j \subseteq X_k \cap (\bigcup X) \subseteq X_i \cup X_j \). Then the pair \( (T',X \cup \{X_k\}) \) is a tree-model.

Proof. For \( u \in \bigcup X' \), let \( T'_u \) denote the subgraph of \( T' \) induced by \( \{i' \in V(T') \mid u \in X_{i'}\} \). We have to show that for every \( u \in \bigcup X' \), \( T'_u \) is a tree. Let \( u \in \bigcup X' \). If \( u \not\in \bigcup X \), then \( T'_u \) is a tree on the single node \( k \). Now suppose \( u \in \bigcup X \). If \( ij \in E(T_u) \), with \( T_u \) as in Definition 1.1.1 then \( u \in X_i \cap X_j \subseteq \)
$X_k$, and thus $T'_u$ is the tree obtained from $T_u$ by subdividing the edge $ij$. Now suppose $ij \not\in E(T_u)$. If $u \in X_k$, then $u \in X_k \cap \bigcup X \subseteq X_i \cup X_j$, say $u \in X_i$, and $T'_u$ is the tree obtained from $T_u$ by adding the vertex $k$ and edge $ik$. Otherwise, $u \not\in X_k$, and $T'_u$ is identical to $T_u$. We conclude that the pair $(T', X')$ is a tree-model.

### 1.2.2 Largeur and spread

We now introduce two parameters of a tree-model that refine the width of a tree-model: its largeur and its spread.

**Definition 1.2.3.** The **largeur** of a tree-model $T = (T, \mathcal{X})$ is defined as

$$r = \max_{i \in V(T)} |X_i|,$$

and its **spread** is defined as

$$s = \max_{i \approx j \in V(T)} |X_i \cup X_j|.$$

Note that the largeur $r$ of a tree-model is exactly one larger than its width, and that its spread $s$ satisfies $r \leq s \leq 2r$. An example of a graph together with three different decompositions is given in Figure 2.3.

![Graph and Tree Decompositions](image)

**Figure 1.1:** A graph $G$ together with a tree-decomposition $T$ of $G$ of width 2, an open tree-decomposition $T'$ of $G$ of largeur 3 and spread 6, and an open tree-decomposition $T''$ of $G$ of largeur 2 and spread 3.
We now show how to transform an open tree-decomposition of a graph $G$ into a tree-decomposition of $G$.

**Lemma 1.2.4.** If a graph $G = (V, E)$ has an open tree-decomposition of spread $s$, then $G$ has a tree-decomposition of largeur $s$.

**Proof.** Suppose $T = (T, \mathcal{X})$ is an open tree-decomposition of $G$ of spread $s$. If $|V(T)| = 1$, then the largeur of $T$ is $s$, and every pair of vertices $u, v \in E$ meet in $T$. Hence, in this case, $T$ is a tree-decomposition of largeur $s$.

Now suppose $|V(T)| > 1$. We transform $T$ into a tree-decomposition $T'$ of $G$ of largeur $s$ as follows. First we obtain the tree $T'$ from $T$ by subdividing every edge of $T$, creating a new vertex $k_{ij}$ for every $ij \in E(T)$. Then for each new vertex $k_{ij} \in V(T')$, we define $X_{k_{ij}} = X_i \cup X_j$. Finally, we set $T' = (T', \mathcal{X}')$ with $\mathcal{X}' = \{X_{k_{ij}} | ij \in E(T)\}$. The pair $T'$ is a tree-model by Lemma 1.2.2.

The largeur $r$ of $T'$ satisfies $r = \max_{k \in V(T')} |X_k| = \max_{ij \in E(T')} |X_{k_{ij}}| = \max_{ij \in E(T')} |X_i \cup X_j| = s$, where we use that $|V(T)| > 1$. We now show that $T'$ is a tree-decomposition of $G$. If $uv \in E$, then $u, v$ touch in $T$, i.e., $u, v \in X_i \cup X_j$ for some $i \equiv j \in V(T)$. As $|V(T)| > 1$, we may assume $i \neq j$. Hence, $u, v \in X_{k_{ij}}$, and thus $u, v$ meet in $T'$. We conclude that $T'$ is a tree-decomposition of $G$ of largeur $s$. \qed

Instead of taking the union $X_i \cup X_j$ for every $ij \in E$ in the above transformation, we can also take the intersection $X_i \cap X_j$. Then we get the following.

**Lemma 1.2.5.** If a graph $G$ has a tree-decomposition of largeur $r$, then $G$ has a tree-decomposition of largeur $r$ and spread $r$.

**Proof.** Suppose $T = (T, \mathcal{X})$ is a tree-decomposition of $G$ of largeur $r$. If $|V(T)| = 1$, then the spread of $T$ is also $r$. Now suppose $|V(T)| > 1$. We transform $T$ into a tree-decomposition $T' = (T', \mathcal{X}')$ of $G$ of largeur $r$ and spread $r$ as follows. First we obtain the tree $T'$ from $T$ by subdividing every edge of $T$, and for each newly created vertex $k_{ij}$, we define $X_{k_{ij}} = X_i \cap X_j$. Then we set $\mathcal{X}' = \mathcal{X} \cup \{X_{k_{ij}} | ij \in E(T)\}$ and we define $T' = (T', \mathcal{X}')$. The pair $T'$ is a tree-model by Lemma 1.2.2. Its largeur is clearly $r$. Furthermore, since every edge of $T'$ is of the form $\{i, k_{ij}\}$ for some $ij \in E(T)$, the spread $s$ of $T'$ is equal to $s = \max_{ij, j' \in E(T')} |X_{ij} \cup X_{ij'}| = \max_{ij \in E(T')} |X_i \cup X_{k_{ij}}| = \max_{ij \in E(T')} |X_i| = r$, where we use that $|V(T)| > 1$ and $|V(T')| > 1$. Furthermore, as $\mathcal{X} \subseteq \mathcal{X}'$, every pair of vertices which meet in $T$ also meet in $T'$. We conclude that $T'$ is a tree-decomposition of $G$ of largeur $r$ and spread $r$. \qed
1.2.3 Smoothness

In some cases it is helpful if a tree-model has a specific form. In [4], a tree-decomposition \((T, \mathcal{X})\) of largeur \(r\) is called smooth if \(|X_i| = r\) for all \(i \in V(T)\), and \(|X_i \cap X_j| = r - 1\) for all \(ij \in E(T)\). We need a slightly more general notion, and define smoothness of a tree-model as follows.

Definition 1.2.6. A tree-model \(T = (T, \mathcal{X})\) of largeur \(r\) is called smooth if \(|X_i| = r\) for all \(i \in V(T)\), and \(X_i \neq X_j\) for all \(i \neq j \in V(T)\).

Lemma 1.2.7. Let \(G\) be a graph. Any open tree-decomposition \(T = (T, \mathcal{X})\) of \(G\) of largeur \(r\) and spread \(s\) can be transformed into a smooth open tree-decomposition of \(G\) of largeur \(r\) and spread at most \(\max\{r + 1, s\}\).

Proof. Let \(T = (T, \mathcal{X})\) be an open tree-decomposition of \(G\) of largeur \(r\) and spread \(s\). Define \(s' = \max\{r + 1, s\}\) and consider the following transformations.

1. Let \(ij \in E(T)\) with \(|X_i| < r\) and \(|X_i \cup X_j| = s'\). Subdivide \(ij\), and for the new node \(k\) choose \(X_k\) such that \(X_i \subseteq X_k \subseteq X_i \cup X_j\) and \(|X_k| = r\).

2. Let \(ij \in E(T)\) with \(|X_i| < |X_j|\). Add some \(u \in X_j \setminus X_i\) to \(X_i\).

3. Let \(ij \in E(T)\) with \(X_i = X_j\). Contract the edge \(ij\), and for the new node \(k\) set \(X_k = X_i\).

Note that \(T\) remains a tree-model of the same largeur under each of these transformations, where we use Lemma 1.2.2 for the first transformation. Also, if transformation 1 is applied to an edge \(ij \in E(T)\), creating a node \(k\), then \(X_i \cup X_j \subseteq X_k \cup X_j\). This shows that if a pair of vertices touch in \(T\) before this transformation, then they also do after. Hence, \(T\) remains an open tree-decomposition of \(G\) under the first transformation. For the other two transformations this is clear.

We now prove by induction on \(\sum_{i \in V(T)} r - |X_i|\) that after modifying \(T\) using the first two transformations, it is possible to satisfy \(|X_i| = r\) for all \(i \in V(T)\), while the spread of \(T\) is at most \(s'\). If \(\sum_{i \in V(T)} r - |X_i| = 0\), we do not have to do anything. Now suppose \(\sum_{i \in V(T)} r - |X_i| > 0\). As the largeur of \(T\) is \(r\), there exists an edge \(ij \in E(T)\) with \(|X_i| < |X_j|\). For each \(i' \neq j\) adjacent to \(i\) such that \(|X_i \cup X_{i'}| = s'\), we apply transformation 1 to the edge \(ii'\), creating a new node \(k_{i'}\). Note that this does not change the value of \(\sum_{k \in V(T)} r - |X_k|\), as each newly created node \(k_{i'}\) satisfies \(|X_{k_{i'}}| = r\). Also note that this does not change the spread of \(T\). Now, each \(k \neq j\) adjacent to \(i\) satisfies \(|X_i \cup X_k| < s'\), as for each newly created node \(k_{i'}\) we have \(X_i \subseteq X_{k_{i'}}\).
and thus $|X_i \cup X_{k'}| = |X_{k'}| = r < s'$. Next, we apply transformation 2 to the edge $ij$. By the above, the spread of $T$ remains at most $s'$. Also, the value of $\sum_{k \in V(T)} r - |X_k|$ decreases by one. Therefore, by the induction hypothesis, after modifying $T$ further, it is possible to satisfy $|X_k| = r$ for all $k \in V(T)$, while keeping the spread of $T$ at most $s'$.

Thus, we may now assume that $T$ is an open tree-decomposition of $G$ of largeur $r$ and spread at most $s'$, satisfying $|X_i| = r$ for all $i \in V(T)$. Finally, we repeatedly apply transformation 3 until it is not possible anymore, and we have

$$X_i \neq X_j \text{ for each } ij \in E(T).$$

(1)

Now, if there exist $i, j \in V(T)$ with $X_i = X_j$, then from $|X_k| = r$ for all $k \in V(T)$ it follows that $X_i = X_k$ for each $k \in V(T)$ on the (unique) path from $i$ to $j$. This contradicts (1), and therefore we can conclude that $T$ is smooth.

Notice that the procedure from the above proof transforms a tree-model $T$ into a tree-model $T'$ such that every pair of vertices which touch in $T$, also touch in $T'$. Moreover, we see that any pair of vertices which meet in $T$, also meet in $T'$, as this is the case after each of the three transformations used in the procedure. We will use this observation in the proof of Lemma 2.2.5.

### 1.3 Definition and basic properties of $l_a(G)$

It becomes more interesting when we look at both the largeur and the spread of an open tree-decomposition. We now define a new sequence of parameters $l_a(G)$ for any integer $t \geq 0$ (or $t = \infty$) that refines the notion of tree-width. Given $t$, the definition of $l_a(G)$ is similar to the definition of tree-width, but instead of depending on the width of a tree-decomposition, it depends on both the largeur and the spread of an open tree-decomposition of $G$.

**Definition 1.3.1.** For an integer $t \geq 0$, the largeur d’arborescence of order $t$ of a graph $G$, denoted by $l_a(G)$, is defined as the smallest integer $r$ for which $G$ has an open tree-decomposition of largeur $r$ and spread at most $r + t$. The value $t = \infty$ is also allowed, and $l_a(G)$ is defined as the smallest integer $r$ for which $G$ has an open tree-decomposition of largeur $r$.

We will always assume that $r$ denotes an integer and that $t$ denotes an integer or infinity. Given $r, t \geq 0$, we call an open tree-decomposition of $G$ of largeur at most $r$ and spread at most $r + t$ a certificate for $l_a(G) \leq r$. 

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If in addition $\mathcal{T}$ is smooth, then we say that $\mathcal{T}$ is a *smooth certificate* for $\text{la}_t(G) \leq r$. This definition is motivated by the following lemma.

**Lemma 1.3.2.** Let $r, t \geq 0$ and let $G$ be a graph. Then $\text{la}_t(G) \leq r$ if and only if there exists a certificate for $\text{la}_t(G) \leq r$. Moreover, if $t \geq 1$, then $\text{la}_t(G) \leq r$ implies that there exists a smooth certificate for $\text{la}_t(G) \leq r$.

**Proof.** From the definition of $\text{la}_t(G)$ it follows from Lemma 1.2.5 that $\text{la}_t(G) \leq r$ if and only if there exists a certificate $\mathcal{T}$ for $\text{la}_t(G) \leq r$. Now suppose $\mathcal{T} = (T, \mathcal{X})$ is a certificate for $\text{la}_t(G) \leq r$ of largeur $r'$ and spread $s \leq r + t$. If $s \leq r' + t$, then $\text{la}_t(G) \leq r' \leq r$ and we are done. Otherwise, $s > r' + t$. Let $ij \in E(T)$ with $|X_i \cup X_j| = s$. Obtain $T'$ from $T$ by subdividing the edge $ij$, and for the new node $k$ choose $X_k$ such that $X_i \cap X_j \subseteq X_k \subseteq X_i \cup X_j$ and $|X_k| = s - t$. This is possible as $|X_i \cap X_j| \leq r' < s - t \leq s = |X_i \cup X_j|$. Then $(T', \mathcal{X} \cup \{X_k\})$ is an open tree-decomposition of $G$ of largeur $s - t$ and spread at most $s$. It follows that $\text{la}_t(G) \leq s - t \leq r$, which was to be shown.

Note that in general there does not exist a smooth certificate for $\text{la}_0(G) \leq r$. For example, if $G$ is a graph without any edges on at least two nodes, then the spread of any smooth open tree-decomposition of $G$ is at least 2, and therefore there does not exist a smooth certificate for $\text{la}_0(G) \leq 1$.

**Lemma 1.3.3.** Let $G$ be a graph and let $t \geq 0$. Then

$$\text{tw}(G) + 1 - t \leq \text{la}_t(G) \leq \text{tw}(G) + 1 \leq 2 \text{la}_t(G).$$

In particular, $\text{la}_0(G) = \text{tw}(G) + 1$.

**Proof.** Write $r = \text{tw}(G) + 1$. As $G$ has a tree-decomposition of largeur $r$, it follows from Lemma 1.2.5 that $G$ has a tree-decomposition $\mathcal{T}$ of largeur $r$ and spread $r$. Since $\mathcal{T}$ is also an open tree-decomposition of $G$, we have $\text{la}_t(G) \leq r = \text{tw}(G) + 1$.

For the other two inequalities, write $r = \text{la}_t(G)$. Let $\mathcal{T}$ be an open tree-decomposition of $G$ of largeur $r$ and spread $s \leq r + t$. By Lemma 1.2.4, $G$ has a tree-decomposition of largeur $s$. Therefore, $\text{tw}(G) + 1 \leq s$, and thus $\text{tw}(G) + 1 - t \leq \text{la}_t(G)$. Finally, since the spread of any tree-model is at most twice its largeur, we have $s \leq 2r$, and therefore $\text{tw}(G) + 1 \leq 2 \text{la}_t(G)$.

Apart from $t = 0$, there are two other special values for $t$, namely $t = 1$. 

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and $t = \infty$. In Section 3.3 we show that $la_t(G) = la_\infty(G)$ \footnote{However, an open tree-decomposition of $G$ is not the same as a two-sided tree-decomposition of $G$ (a notion introduced in \cite{15} to characterize $la_\infty(G)$), although it is possible to construct a two-sided tree-decomposition of width $r$ from an open tree-decomposition of largeur $r$ and spread at most $r + 1$, and vice versa.} and at the end of this section we show that $la_\infty(G) = la_{\hat{\nabla}}(G)$. Notice that Lemma 1.3.3 then gives the following known bounds for the parameters $la_\square(G)$ and $la_{\hat{\nabla}}(G)$ (shown respectively in \cite{15} and \cite{8}):

$$tw(G) \leq la_\square(G) \leq tw(G) + 1,$$

$$\frac{tw(G) + 1}{2} \leq la_{\hat{\nabla}}(G) \leq tw(G) + 1.$$ 

As $tw(K_r) = r - 1 = la_\square(K_r)$ and $la_{\hat{\nabla}}(K_{2r}) = r$, the two left-most lower bounds are tight. In Section 4.2 we show that the two upper bounds are also tight. Note that from the definition of the parameter $la_t(G)$ it follows that $t_1 \leq t_2$ implies $la_{t_2}(G) \leq la_{t_1}(G)$. We therefore have the following chain of inequalities:

$$la_{\hat{\nabla}}(G) = la_\infty(G) \leq \cdots \leq la_2(G) \leq la_1(G) \leq la_0(G) = tw(G) + 1.$$

An important property of the parameters $tw(G)$, $la_\square(G)$ and $la_{\hat{\nabla}}(G)$ is minor-monotonicity. We now show that for any $t \geq 0$, $la_t(G)$ is minor-monotone.

**Proposition 1.3.4.** Let $t \geq 0$, let $G = (V, E)$ be a graph, and let $G' \preceq G$ be a minor of $G$. Then $la_t(G') \leq la_t(G)$.

**Proof.** Given $e \in E$, we have $la_t(G \setminus e) \leq la_t(G)$, since any open tree-decomposition of $G$ is also an open tree-decomposition of $G \setminus e$. Now suppose $G' = G/e$ for some $e = uv \in E$, and let $u'$ denote the new vertex of $G'$ created by contraction of the edge $e$. Write $r = la_t(G)$ and let $T = (T, \mathcal{X})$ be a certificate for $la_t(G) \leq r$. We construct a certificate $T'$ for $la_t(G') \leq r$ as follows.

For $i \in V(T)$, let $X'_i$ be the set obtained from $X_i$ by replacing every occurrence of $u$ or $v$ by $u'$. Define $\mathcal{X}' = \{X'_i \mid i \in V(T)\}$ and set $T' = (T, \mathcal{X}')$. Notice that $\bigcup \mathcal{X}' = V(G')$. We first show that $T'$ is a tree-model. For $w \in V(G')$, let $T'_w$ denote the subgraph of $T$ induced by $\{i \in V(T) \mid w \in X'_i\}$. We have to show that for all $w \in V(G')$, $T'_w$ is a tree. Let $w \in V(G')$.

If $w \neq u'$, then $T'_w$ is the tree identical to $T_w$. Otherwise $w = u'$, and $V(T'_w) = V(T_u) \cup V(T_v)$. Since $uv \in E$, $u, v$ touch in $T$, i.e., $V(T_u) \cup V(T_v)$
induces a subtree of \( T \). Therefore, \( \mathcal{T}' \) is a tree-model. Clearly, the largeur of \( \mathcal{T}' \) is at most \( r \) and its spread is at most \( r + t \).

We now show that \( \mathcal{T}' \) is an open tree-decomposition of \( G' \). Let \( xy \in E(G') \) be an edge of \( G' \). We have to show that \( x, y \) touch in \( \mathcal{T}' \), i.e., that \( V(T_x') \cup V(T_y') \) induces a subtree of \( T \). If \( x \neq u' \) and \( y \neq u' \), then \( T_x' = T_x, T_y' = T_y \), and \( xy \in E \). It follows that \( x, y \) touch in \( \mathcal{T} \), i.e., \( V(T_x) \cup V(T_y) = V(T_x') \cup V(T_y') \) induces a subtree of \( T \). Otherwise, say \( x = u' \). Then \( V(T_x') = V(T_u) \cup V(T_v) \) and \( T_y' = T_y \). Also, from \( x'y \in E(G') \) it follows that \( u'y \in E \) or \( vy \in E \), say \( u'y \in E \). Then \( V(T_u) \cup V(T_y) \) induces a subtree of \( T \), and hence, \( V(T_x') \cup V(T_y') = V(T_u) \cup V(T_v) \cup V(T_y) \) induces a subtree of \( T \). Therefore, \( \mathcal{T}' \) is an open tree-decomposition of \( G' \). Hence, \( \mathcal{T}' \) certifies \( \mathrm{la}_\ell(G') \leq r \). We conclude that for any \( G' \preceq G \), \( \mathrm{la}_\ell(G') \leq \mathrm{la}_\ell(G) \).

We conclude this section by proving the equality \( \mathrm{la}_\infty(G) = \mathrm{la}_\ell^G(G) \) for any \( G \).

**Theorem 1.3.5.** For any graph \( G = (V, E) \), \( \mathrm{la}_\infty(G) = \mathrm{la}_\ell^G(G) \).

**Proof.** First we show that \( \mathrm{la}_\ell^G(G) \leq \mathrm{la}_\infty(G) \). Write \( \mathrm{la}_\infty(G) = r \) and let \( \mathcal{T} = (T, X) \) be a smooth certificate for \( \mathrm{la}_\infty(G) \leq r \). Construct a graph \( H \) with

\[
V(H) = \{w_{u,i} \mid u \in X_i, i \in V(T)\},
\]

\[
E(H) = \{w_{u,i}w_{v,j} \mid w_{u,i} \neq w_{v,j} \in V(H), i \sim j\}.
\]

Since \( |X_i| = r \) for all \( i \in V(T) \), \( H = T \boxtimes K_r \). We now show that \( G \preceq H \) by constructing a \( G \)-partition of \( H \). For \( u \in V \), we define \( W_u = \{w_{u,i} \mid i \in T_u\} \). As \( T_u \) is a subtree of \( T \), \( W_u \) induces a connected subgraph of \( H \). Now suppose \( uv \in E \). As \( \mathcal{T} \) is an open tree-decomposition of \( G \), there exist \( i \sim j \in V(T) \) such that \( u, v \in X_i \cup X_j \), say \( u \in X_i \) and \( v \in X_j \). Then \( w_{u,i}w_{v,j} \in E(H) \) is an edge between \( W_u \) and \( W_v \). Hence, the collection \( \{W_u \mid u \in V \} \) is a \( G \)-partition of \( H \). Therefore, \( G \preceq H = T \boxtimes K_r \). This shows that \( \mathrm{la}_\ell^G(G) \leq r = \mathrm{la}_\infty(G) \).

Now suppose \( G \preceq T \boxtimes K_r \) for some tree \( T \). As the parameter \( \mathrm{la}_\infty(G) \) is minor-monotone, it is enough to show that \( \mathrm{la}_\infty(T \boxtimes K_r) \leq r \). We construct a certificate for \( \mathrm{la}_\infty(T \boxtimes K_r) \leq r \) as follows. Write \( V(T \boxtimes K_r) = \bigcup_{i \in V(T)} X_i \), where the \( X_i \)'s are pairwise disjoint sets of size \( r \) (i.e., \( X_i \) is the vertex set of the copy of \( K_r \) corresponding to \( i \in V(T) \)). Then the pair \( (T, \{X_i \mid i \in V(T)\}) \) is a certificate for \( \mathrm{la}_\infty(T \boxtimes K_r) \leq r \). This shows that \( \mathrm{la}_\infty(G) \leq \mathrm{la}_\ell^G(G) \), and therefore, \( \mathrm{la}_\infty(G) = \mathrm{la}_\ell^G(G) \). 

\( \square \)
Chapter 2

Partially eared graphs

In an open tree-decomposition $\mathcal{T} = (T, \mathcal{X})$ of a graph $G = (V, E)$, every vertex $u \in V$ has a subtree $T_u$ of $T$ assigned to it. In this chapter we show that in the definition of $\text{la}_t(G)$, the subtree of a vertex of degree 2 can be replaced by an additional condition on the subtrees of the two neighbours of the vertex. This allows for more compact open tree-decompositions which share some of the characteristics of tree-decompositions. As a result, several theorems about $\text{tw}(G)$ can be generalized to the parameters $\text{la}_t(G)$ for $t \geq 1$. In particular, we define a characterization of $\text{la}_t(G)$ in terms of an elimination ordering.

2.1 Definitions

Let $G = (V, E)$ be a graph. If $w \in V$ has exactly two neighbours $u, v \in V$, then we say that $u, v$ are connected by an ear with top $w$. Now consider the following definition.

Definition 2.1.1. A partially eared graph is a triple $\mathcal{G} = (V, E, \mathcal{E})$, where $E, \mathcal{E}$ are two sets of edges (with possibly a non-empty intersection). The graph $\psi(\mathcal{G}) = (V, E \cup \mathcal{E})$ is called the base graph of $\mathcal{G}$.

Let $\mathcal{G} = (V, E, \mathcal{E})$ be a partially eared graph. We call the sets $V, E, \mathcal{E}$ (also written as $V(\mathcal{G}), E(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$) the vertex set, the first edge set and the second edge set of $\mathcal{G}$, respectively. We call the elements of $E \cup \mathcal{E}$ the edges of $\mathcal{G}$. We say that an edge $e$ of $\mathcal{G}$ is incident with a vertex $u \in V$ if $e = uv$ for some $v \in V$. We think of an edge $e \in \mathcal{E}$ as a contracted ear connecting $u, v$. More precisely, we define the following transformation on a partially eared graph. We say that a partially eared graph $\mathcal{G}'$ is obtained from $\mathcal{G}$ by
subdividing the edge \( e = uv \in E \), if \( G' = (V \cup \{w_e\}, E \cup \{uw_e, w_ev\}, E \setminus \{e\}) \) for some \( w_e \notin V \). In other words, this operation removes the edge \( e \) from \( E \) and connects \( u, v \) by an ear with top \( w_e \). If we subdivide all edges in \( E \), then we are left with a partially eared graph \( G' = (W, F, E) \) with \( F = \emptyset \). We call the graph \((W, F)\) the underlying graph of \( G \), which we define formally as follows.

**Definition 2.1.2.** Let \( G = (V, E, E_0) \) be a partially eared graph. The *underlying graph* of \( G \) is the graph \( \phi(G) = (W, F) \) with

\[
\begin{align*}
W &= V \cup \{w_e | e \in E\}, \\
F &= E \cup \{uw_e | u \in V, e \in E, u \text{ is incident with } e\},
\end{align*}
\]

where we assume that \( w_e \notin V \) for each \( e \in E \).

The following figure gives an example of a partially eared graph \( G \) together with its underlying graph \( \varphi(G) \) and its base graph \( \psi(G) \).

![Diagram of a partially eared graph and its underlying and base graphs](image)

Figure 2.1: A partially eared graph \( G = (V, E, E_0) \) with \( V = \{1, 2, 3\} \), \( E = \{13, 23\} \), and \( E = \{12, 13, 23\} \). A wiggly line represents an edge in \( E \setminus E_0 \), and a thick line represents an edge in \( E \cap E_0 \). The underlying graph \( \varphi(G) \) and the base graph \( \psi(G) \) of \( G \) are also shown.

Let \( G_1, G_2 \) be two partially eared graphs. We define \( G_1 \cup G_2 \) as the partially eared graph \((V(G_1) \cup V(G_2), E(G_1) \cup E(G_2), E(G_1) \cup E(G_2))\). Also, we write \( G_1 \subseteq G_2 \) if \( V(G_1) \subseteq V(G_2) \), \( E(G_1) \subseteq E(G_2) \), and \( E(G_1) \subseteq E(G_2) \).

Let \( G = (V, E, E_0) \) be a partially eared graph. We call a subset \( U \subseteq V \) an *eared clique* in \( G \) if \( U \) is a clique in the graph \((V, E \cap E_0)\). If \( G = G_1 \cup G_2 \), where \( G_1, G_2 \) are two partially eared graphs such that \( V(G_1) \cap V(G_2) \) is an eared clique in both \( G_1 \) and \( G_2 \), then we say that \( G \) is an *eared clique k-sum* of \( G_1 \) and \( G_2 \), where \( k = |V(G_1) \cap V(G_2)| \).
For $U \subseteq V$, we define

$$N(U) = \{v \in V \setminus U \mid uv \in E \text{ for some } u \in U\},$$

$$\overline{N}(U) = \{v \in V \setminus U \mid uv \in \overline{E} \text{ for some } u \in U\}.$$  

Also, we define $\mathcal{G} - U$ as the partially eared graph obtained from $\mathcal{G}$ by removing the vertices in $U$ from $V$ and their incident edges from $E$ and $\overline{E}$. Furthermore, we define $\mathcal{G} \div U$ as the partially eared graph obtained from $\mathcal{G} - U$ by turning $N(U) \cup \overline{N}(U)$ into an eared clique. If $U = \{u\}$, then we also write $N(u), \overline{N}(u), \mathcal{G} - u$ and $\mathcal{G} \div u$ instead of $N(\{u\}), \overline{N}(\{u\}), \mathcal{G} - \{u\}$ and $\mathcal{G} \div \{u\}$, respectively.

For $e \in E$, let $\mathcal{G} \setminus e = (V, E \setminus \{e\}, \overline{E})$ denote the partially eared graph obtained from $\mathcal{G}$ by removing the edge $e$ from $E$. Also, for $e = uv \in E$, we define $\mathcal{G} / e$ as the partially eared graph obtained from $\mathcal{G}$ by contracting the edge $e$, i.e., replacing both $u, v$ by a new vertex $u'$ such that $N(u') = N(\{u, v\})$ and $N(u') = \overline{N}(\{u, v\})$. Finally, for $e \in \overline{E}$, we define $\mathcal{G} \star e = (V, E \cup \{e\}, \overline{E} \setminus \{e\})$ as the partially eared graph obtained from $\mathcal{G}$ by suppressing the edge $e$. Note that $\varphi(\mathcal{G} \star e) = \varphi(\mathcal{G}) \star w_e$. The following figure shows some of these operations.

![Figure 2.2: Some operations on a partially eared graph $\mathcal{G}$.](image)

### 2.2 Open tree-decomposition of a partially eared graph

We extend the notion of an open tree-decomposition of a graph to an open tree-decomposition of a partially eared graph. We first define a more restricted form of touching of two vertices in a tree-model.

**Definition 2.2.1.** Let $T = (T, \mathcal{X})$ be a tree-model. We say that two distinct elements $u, v \in \bigcup \mathcal{X}$ sharply touch in $T$ if there exists an edge $ij \in E(T)$
such that \( u \in X_i, v \in X_j \) and \( X_i \{u\} = X_j \{v\} \). In this case we also say that \( u, v \) sharply touch at the edge \( ij \).

Let \( T = (T, \mathcal{X}) \) be a tree-model. Note that for any \( ij \in E(T) \), there are at most two elements which sharply touch at the edge \( ij \). Moreover, if \( u, v \in \bigcup \mathcal{X} \) sharply touch at \( ij \in E(T) \), then \( u, v \) touch in \( T \) (as \( u, v \in X_i \cup X_j \)), but \( u, v \) do not meet in \( T \) (as \( u \in X_i \setminus X_j \) and \( v \in X_j \setminus X_i \), and therefore the subtrees \( T_u \) and \( T_v \) do not intersect). We now define an open tree-decomposition of a partially eared graph as follows.

**Definition 2.2.2.** An open tree-decomposition of a partially eared graph \( \mathcal{G} = (V, E, \overline{E}) \) is a tree-model \( T = (T, \mathcal{X}) \) with \( \bigcup \mathcal{X} = V \) such that

- \( T1 \) for every edge \( uv \in E \setminus \overline{E} \), \( u, v \) touch in \( T \),
- \( T2 \) for every edge \( uv \in E \setminus \overline{E} \), \( u, v \) meet or sharply touch in \( T \),
- \( T3 \) for every edge \( uv \in E \cap \overline{E} \), \( u, v \) meet in \( T \).

Figure 2.3 contains an example of an open tree-decomposition of a partially eared graph. Now let \( \mathcal{G} = (V, E, \overline{E}) \) be a partially eared graph. If \( \overline{E} = \emptyset \), then the underlying graph \( \varphi(\mathcal{G}) \) of \( \mathcal{G} \) satisfies \( \varphi(\mathcal{G}) = (V, E) \). In this case, a tree-model is an open tree-decomposition of \( \mathcal{G} \) if and only if it is an open tree-decomposition of \( \varphi(\mathcal{G}) \). This is not true when \( \overline{E} \neq \emptyset \), since then \( V(\mathcal{G}) \neq V(\varphi(\mathcal{G})) \). However, in the next section we show that we can transform an open tree-decomposition of \( \mathcal{G} \) into an open tree-decomposition of \( \varphi(\mathcal{G}) \), and vice versa.

![Figure 2.3: A partially eared graph \( \mathcal{G} \) and an open tree-decomposition \( T \) of \( \mathcal{G} \) of largeur 3 and spread 4.](image-url)

One easy consequence of Definition 2.2.2 is the following. Suppose \( T = (T, \mathcal{X}) \) is a tree-model. If \( u, v \in \bigcup \mathcal{X} \) meet in \( T \), then trivially \( u, v \) meet or
sharply touch in $T$. Furthermore, if $u, v$ meet or sharply touch in $T$, then $u, v$ touch in $T$. Hence, we have the following.

**Lemma 2.2.3.** An open tree-decomposition $T$ of a partially eared graph $G = (V, E, E)$ is also an open tree-decomposition of the base graph $\psi(G)$, of the graph $G \setminus e$ (for $e \in E$), and of $G \star e$ (for $e \in E$).

A basic fact about a tree-decomposition $T = (T, X)$ of a graph $G = (V, E)$ is the following. If $U \subseteq V$ is a clique in $G$, then $U \subseteq X_i$ for some $i \in V(T)$. A similar statement for an eared clique in a partially eared graph is the following.

**Lemma 2.2.4.** Let $T = (T, X)$ be an open tree-decomposition of a partially eared graph $G = (V, E, E)$, and let $U \subseteq V$ be an eared clique. Then $U \subseteq X_i$ for some $i \in V(T)$.

**Proof.** For every pair of vertices $u, v \in U$, we have $uv \in E \cap \overline{E}$, and thus $u, v$ meet in $T$, i.e., the subtrees $T_u$ and $T_v$ intersect. Hence, as any collection $\mathcal{F}$ of pairwise intersecting subtrees of some tree satisfies $\bigcap \mathcal{F} \neq \emptyset$ (see for example [12, Corollary 6a]), there exists an $i \in V(T)$ such that $i \in V(T_u)$ for all $u \in U$. Therefore, $U \subseteq X_i$, as was to be shown. $\square$

We now show that it is possible to transform an open tree-decomposition of a partially eared graph $G$ into a smooth open tree-decomposition of $G$, in analogy to Lemma [1.2.7] for graphs.

**Lemma 2.2.5.** Let $G = (V, E, E)$ be a partially eared graph. Any open tree-decomposition $T = (T, X)$ of $G$ of largeur $r$ and spread $s$ can be transformed into a smooth open tree-decomposition of $G$ of largeur $r$ and spread at most $\max\{r + 1, s\}$.

**Proof.** Let $T = (T, X)$ be an open tree-decomposition of $G$ of largeur $r$ and spread $s$. We will first modify $T$ such that

$$\forall u, v \in V \text{ which sharply touch at some } i j \in E(T), \ |X_i| = |X_j| = r. \quad (1)$$

Suppose two vertices $u, v \in V$ sharply touch in $T$, i.e., there exists an edge $ij \in E(T)$ such that $u \in X_i, v \in X_j$, and $X_i \setminus \{u\} = X_j \setminus \{v\}$. If $|X_i| < r$, then we subdivide the edge $ij$, and for the new node $k$ we set $X_k = X_i \cup X_j$. As any $w, w' \in X_i \cup X_j$ now meet at $k$, $T$ remains an open tree-decomposition of $G$. Also, the spread does not change. Moreover, as $|X_k| = |X_i \cup X_j| = |X_i| + 1 \leq r$, the largeur of $T$ also remains the same. After repeating this for all $u, v \in V$ which sharply touch in $T$, (1) is satisfied.

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Next, we apply the three-step procedure from the proof of Lemma 1.2.7 to \( T \) to obtain a smooth tree-model \( T' \) of largeur \( r \) and spread at most \( \max\{r + 1, s\} \). By the observation directly below this procedure, any pair of vertices which touch (respectively meet) in \( T \), also touch (respectively meet) in \( T' \). Now suppose \( u, v \in V \) sharply touch in \( T \), say at the edge \( ij \in E(T) \). By \([\text{1}]\), we have \( |X_i| = |X_j| = r \), and thus the edge \( ij \) will not be subdivided during the procedure, nor will any new vertices be added to \( X_i \) or \( X_j \). Hence, \( u, v \) still sharply touch at the edge \( ij \) in \( T' \). This shows that \( T' \) is an open tree-decomposition of \( G \).

We conclude that \( T' \) is a smooth open tree-decomposition of \( G \) of largeur \( r \) and spread at most \( \max\{r + 1, s\} \).

Finally, in analogy to \( \text{lat}(G) \) for a graph \( G \), we define \( \text{lat}(\mathcal{G}) \) for a partially eared graph \( \mathcal{G} \) as follows.

**Definition 2.2.6.** Let \( \mathcal{G} \) be a partially eared graph and \( t \geq 0 \). Then \( \text{lat}(\mathcal{G}) \) is defined as the smallest integer \( r \) such that \( \mathcal{G} \) has an open tree-decomposition of largeur \( r \) and spread at most \( r + t \). A certificate for \( \text{lat}(\mathcal{G}) \leq r \) is an open tree-decomposition of \( \mathcal{G} \) of largeur at most \( r \) and spread at most \( r + t \).

Let \( r, t \geq 0 \) and let \( \mathcal{G} \) be a partially eared graph. With the same argument as in the proof of Lemma 1.3.2 one can show that \( \text{lat}(\mathcal{G}) \leq r \) if and only if there exists a certificate for \( \text{lat}(\mathcal{G}) \leq r \), and moreover, that \( t \geq 1 \) and \( \text{lat}(\mathcal{G}) \leq r \) imply that there exists a smooth certificate for \( \text{lat}(\mathcal{G}) \leq r \), i.e., a smooth open tree-decomposition of \( \mathcal{G} \) of largeur at most \( r \) and spread at most \( r + t \) (for the second statement we rely on Lemma 2.2.5 instead of Lemma 1.2.7).

### 2.3 The equality \( \text{lat}(\mathcal{G}) = \text{lat}(\varphi(\mathcal{G})) \)

The goal of this section is to show that \( \text{lat}(\mathcal{G}) = \text{lat}(\varphi(\mathcal{G})) \) for any \( t \geq 1 \) and any partially eared graph \( \mathcal{G} \). For \( t = 0 \), this equality does not hold for every \( \mathcal{G} \), as \( \mathcal{G} = (\{1, 2\}, \{12\}, \{12\}) \) has \( \text{la}_0(\mathcal{G}) = 2 \), while \( \varphi(\mathcal{G}) = K_3 \) and \( \text{la}_0(K_3) = \text{tw}(K_3) + 1 = 3 > 2 \). We prove the equality \( \text{la}_t(\mathcal{G}) = \text{la}_t(\varphi(\mathcal{G})) \) for \( t \geq 1 \) by showing that for any \( \mathcal{G}' \) obtained from a partially eared graph \( \mathcal{G} \) by subdividing an edge \( e \in \mathcal{E} \) (see Section 2.1), we have \( \text{la}_t(\mathcal{G}') = \text{la}_t(\mathcal{G}) \). We first show a slightly stronger version of the inequality \( \text{la}_t(\mathcal{G}') \leq \text{la}_t(\mathcal{G}) \).

**Lemma 2.3.1.** Let \( \mathcal{G} = (V, E, \mathcal{E}) \) be a partially eared graph and \( e = uv \in \mathcal{E} \). Let \( \mathcal{G}' \) be obtained from \( \mathcal{G} \) by removing \( e \) from \( \mathcal{E} \), adding a new vertex \( w \) to \( V \), and two new edges \( uw, wv \) to \( \mathcal{E} \). Then \( \text{la}_t(\mathcal{G}') \leq \text{la}_t(\mathcal{G}) \) for any \( t \geq 1 \).
Therefore, touch sharply touch or meet in $T$.

Finally, we have to check whether the conditions on the spread of $T$ and a certificate for $T$.

Let $G$ be a partially eared graph and let $T'$ be a tree-model. We show that $T'$ is a tree-model. We show that $T'$ is a certificate for $la_t(G') \leq r$. Since $|X_i \cup X_j| = |X_i \cup \{w\}| \leq r + 1 \leq r + t$, the spread of $T'$ is at most $r + t$. Clearly, the largeur of $T'$ is equal to $r$. Finally, we have to check whether the conditions on $T'$ imposed by the two edges incident with $w$ are satisfied, i.e., whether the vertices $u, w$ and $w, v$ sharply touch or meet in $T'$. This is the case, as the vertices $u, w$ sharply touch in $T'$ at the edge $ij$, and the vertices $w, v$ meet in $T'$ at the node $j$. Therefore, $T'$ certifies $la_t(G') \leq r$.

Otherwise, if $u, v$ do not meet in $T$, then $uv \in E \setminus E$, and $u, v$ sharply touch in $T$. Let $ij \in E(T)$ be such that $u \in X_i, v \in X_j$ and $X_i \setminus \{u\} = X_j \setminus \{v\}$. Obtain $T'$ from $T$ by subdividing the edge $ij$, and for the new node $k$ set $X_k = (X_i \setminus \{u\}) \cup \{w\}$. Define $T' = (T', X \cup \{X_k\})$. By Lemma 1.2.2, $T'$ is a tree-model. We show that $T'$ is a certificate for $la_t(G') \leq r$. It is clear that the largeur of $T'$ is equal to $r$ and that the spread of $T'$ is equal to the spread of $T$. Also, since $u, w$ sharply touch at the edge $ik$, and $w, v$ sharply touch at the edge $kj$, the conditions on $T'$ imposed by the two edges incident with $w$ are satisfied. However, as $T'$ is obtained from $T$ by subdividing the edge $ij$, we have to check the conditions on $T'$ imposed by the edges $xy$ of $G'$ with $x \in X_i \setminus X_j$ and $y \in X_j \setminus X_i$. These are satisfied, as only the vertices $u, v$ satisfy $u \in X_i \setminus X_j$ and $v \in X_j \setminus X_i$, and $uv$ is not an edge of $G'$. Therefore, $T'$ certifies $la_t(G') \leq r$. We conclude that $la_t(G') \leq la_t(G)$.

Lemma 2.3.2. Let $G = (V, E, E)$ be a partially eared graph and $e = uv \in E$ an edge. Let $G'$ be obtained from $G$ by subdividing the edge $e$, i.e., by removing $e$ from $E$, adding a new vertex $w$ to $V$, and two new edges $uw, vw$ to $E$. Then $la_t(G') = la_t(G)$ for any $t \geq 1$.

Proof. Let $t \geq 1$. By Lemma 2.3.1 and Lemma 2.2.3, $la_t(G') \leq la_t(G)$. We now show the other inequality. Write $la_t(G') = r$ and let $T' = (T', X')$ be a certificate for $la_t(G') \leq r$, where $X' = \{X_i \mid i \in V(T')\}$. Since $uw, vw \in E$, it follows from Definition 2.2.2 that the vertices $u, w$ touch in $T'$ as well as the vertices $w, v$. Our goal is to construct a certificate $T$ for $la_t(G) \leq r$. In particular, the vertices $u, v$ should meet or sharply touch in $T$. We first define an intermediate tree-model $T''$. If the vertices $u, v$ or $w, v$ meet or sharply touch in $T'$, then we set $T'' = T'$ and $T'' = T'$. Otherwise, both pairs of vertices $u, v$ and $w, v$ do not
meet or sharply touch in $T'$, and we construct a certificate $T''$ for $l_a(\mathcal{G}') \leq r$ in which $w, v$ sharply touch as follows. As $w, v$ touch but do not meet in $T'$, there exists an edge $ij \in E(T')$ such that $w \in X_i \setminus X_j$ and $v \in X_j \setminus X_i$. As $w, v$ do not sharply touch in $T'$, no vertices sharply touch at $ij$. We obtain the tree $T''$ from $T'$ by subdividing the edge $ij$, and denote the new node of $T''$ by $k$. We define $X_k = (X_i \setminus \{w\}) \cup \{v\}$ and $T'' = (T'', X'' \cup \{X_k\})$.

By Lemma 1.2.2 $T''$ is a tree-model. Note that $w, v$ sharply touch in $T''$ at the edge $ik$. We show that $T''$ certifies $l_a(\mathcal{G'}) \leq r$. Clearly, the largeur of $T''$ is equal to $r$ and its spread is at most $r + t$. Now, as $T''$ is obtained from $T'$ by subdividing the edge $ij$, and $T'$ is an open tree-decomposition of $\mathcal{G}'$, it is sufficient to check whether the conditions imposed by the edges $xy$ of $\mathcal{G}'$ with $x \in X_i \setminus X_j$ and $y \in X_j \setminus X_i$ are preserved. As no vertices sharply touch at $ij$, and $(X_i \cup X_j \setminus \{w\}) = X_k \cup X_i$, we only have to check the conditions imposed by the edges $wy$ with $y \in X_j \setminus X_i$. As there are two edges incident with $w$ in $\mathcal{G}'$, we have $y \in \{u, v\}$. Then $y = v$, as $u \not\in X_j$, since by assumption $u, v$ do not meet in $T'$. The condition imposed by the edge $uv$ is satisfied, as $w, v \in X_i \cup X_k$. Therefore, $T''$ certifies $l_a(\mathcal{G'}) \leq r$.

We now construct a certificate $T$ for $l_a(\mathcal{G}) \leq r$. For $i \in V(T'')$, let $Y_i = (X_i \setminus \{w\}) \cup \{u\}$ if $w \in X_i$, and $Y_i = X_i$ otherwise. Define $T = T''$ and $T = (T, \{Y_i \mid i \in V(T')\})$. The pair $T$ is a tree-model, as $V(T_u) = V(T''_u) \cup V(T''_w)$ induces a subtree of $T$, since $u, w$ touch in $T''$. Moreover, since $u, v$ or $w, v$ meet or sharply touch in $T''$, the vertices $u, v$ meet or sharply touch in $T$. Therefore, $T$ certifies $l_a(\mathcal{G}) \leq r$. We conclude that $l_a(\mathcal{G}) \leq l_a(\mathcal{G'})$, and thus $l_a(\mathcal{G}) = l_a(\mathcal{G'})$.

By repeatedly applying Lemma 2.3.2 we now prove the equality $l_a(\mathcal{G}) = l_a(\varphi(\mathcal{G}))$ for $t \geq 1$.

**Proposition 2.3.3.** Let $\mathcal{G} = (V, E, E)$ be a partially eared graph and $t \geq 1$. Then $l_a(\varphi(\mathcal{G})) = l_a(\mathcal{G})$.

**Proof.** Let $\mathcal{G}'$ be obtained from $\mathcal{G}$ by subdividing every edge in $E$. By repeatedly applying Lemma 2.3.2 $l_a(\mathcal{G}') = l_a(\mathcal{G})$. As $E(\mathcal{G}') = \emptyset$, a tree-model is an open tree-decomposition of $\mathcal{G}'$ if and only if it is an open tree-decomposition of $\varphi(\mathcal{G}')$. Therefore, $l_a(\mathcal{G}') = l_a(\varphi(\mathcal{G}'))$. Hence, as $\varphi(\mathcal{G}') = \varphi(\mathcal{G})$, we have $l_a(\mathcal{G}) = l_a(\varphi(\mathcal{G}))$. \qed

### 2.4 $(r, t)$-elimination ordering

An alternative characterization of tree-width is the following. First, an elimination ordering of a graph $G = (V, E)$ is a permutation $\pi$ of $V$. Given
an elimination ordering \( \pi \) of \( G \), the fill-in graph of \( G \) with respect to \( \pi \) is constructed as follows: for \( i = 1 \ldots n \), we add an edge between each pair of non-adjacent higher numbered neighbours of the \( i \)th vertex in \( \pi \). Now, \( \text{tw}(G) \) is the smallest integer \( k \) such that there exists an elimination ordering \( \pi \) of \( G \) where every vertex has at most \( k \) higher numbered neighbours in the fill-in graph of \( G \) with respect to \( \pi \) [5]. In this section we give a similar characterization of \( \text{la}_t(G) \). We will make use of the following proposition.

**Proposition 2.4.1.** Let \( r, t \geq 1 \) and let \( \mathcal{G} = (V, E, \mathcal{L}) \) be a partially eared graph with \( |V| > r \). Then \( \text{la}_t(\mathcal{G}) \leq r \) if and only if there exists a subset \( U \subseteq V \) with \( 1 \leq |U| \leq t \) such that

\[
(i) \quad |U \cup \overline{N(U)}| \leq r, \text{ or both } |U| = 1 \text{ and } |N(U) \cap \overline{N(U)}| < r,
(ii) \quad \text{la}_t(\mathcal{G} \setminus U) \leq r.
\]

**Proof.** We first prove the “only if” part. Note that it is sufficient to consider the case when \( \text{la}_t(\mathcal{G}) = r \). Let \( \mathcal{T} = (T, \mathcal{X}) \) be a smooth certificate for \( \text{la}_t(\mathcal{G}) \leq r \). Since \( |V| > r \), we have \( |V(T)| > 1 \). Let \( i \in V(T) \) be a leaf of \( T \) and denote its neighbour by \( j \). Set \( U = X_i \setminus X_j \). As \( T \) is smooth of largeur \( r \) and spread at most \( r + t \), we have \( 1 \leq |U| \leq t \). Notice that for each \( u \in U \), \( V(T_u) = \{ i \} \). Therefore, given two vertices \( u \in U \) and \( v \in V \), if \( u, v \) sharply touch in \( T \), then they sharply touch at the edge \( ij \), and if \( u, v \) meet in \( T \), then they meet at the node \( i \). Now, if \( |U| > 1 \), then no vertices sharply touch at \( ij \), and thus, if \( v \in N(U) \), then \( u, v \) must meet in \( T \) for some \( u \in U \), i.e., \( v \in X_i \). Hence, in this case, \( U \cup \overline{N(U)} \subseteq X_i \), and therefore \( |U \cup \overline{N(U)}| \leq r \). Otherwise, \( |U| = 1 \), say \( U = \{ u \} \) with \( u \in V \). If \( v \in N(u) \cap \overline{N(u)} \), then \( u,v \in E \cap \overline{E} \), and thus \( u,v \) meet in \( T \), i.e., \( v \in X_i \). Therefore, \( N(u) \cap \overline{N(u)} \subseteq X_i \setminus \{ u \} \), and thus \( |N(u) \cap \overline{N(u)}| < r \). We are left to show (ii). If \( v \in N(U) \cup \overline{N(U)} \), then \( u, v \) touch in \( T \) for some \( u \in U \), and thus \( v \in X_j \). Therefore, \( N(U) \cup \overline{N(U)} \subseteq X_j \). We define \( T' = (T - i, \mathcal{X} \setminus \{ X_i \}) \). As \( N(U) \cup \overline{N(U)} \subseteq X_j \), \( T' \) is a certificate for \( \text{la}_t(\mathcal{G} \setminus U) \leq r \), and (ii) is satisfied.

We now show the “if” part. Let \( U \subseteq V \) be a subset with \( 1 \leq |U| \leq t \) satisfying the conditions (i) and (ii). Let \( T' = (T', \mathcal{X}) \) be a certificate for \( \text{la}_t(\mathcal{G} \setminus U) \leq r \). As \( N(U) \cup \overline{N(U)} \) is an eared clique in \( \mathcal{G} \setminus U \), there exists by Lemma 2.2.4 a node \( j \in V(T') \) such that \( N(U) \cup \overline{N(U)} \subseteq X_j \). Let \( T \) be obtained from \( T' \) by adding a leaf \( i \) adjacent to \( j \). We distinguish two cases.

If \( |U \cup \overline{N(U)}| \leq r \), then set \( X_i = U \cup \overline{N(U)} \) and define the tree-model \( \mathcal{T} = (T, \mathcal{X} \cup \{ X_i \}) \). Clearly, the largeur of \( \mathcal{T} \) is at most \( r \). Also, as \( |U| \leq t \) and \( X_i \cup X_j \subseteq U \cup X_j \), the spread of \( \mathcal{T} \) is at most \( r + t \). To show that \( \mathcal{T} \) is an open tree-decomposition of \( \mathcal{G} \), it is enough to show that the conditions
imposed on $T$ by all edges of $G$ of the form $uv$ with $u \in U$ and $v \in V$ are satisfied. Let $u \in U$ and $v \in V$. If $wv \in E$, then $v \in N(U) \cup U$, and since $N(U) \cup U \subseteq X_i \cup X_j$, this shows that $u, v$ touch in $T$. If $uv \in E$, then from $N(U) \cup U \subseteq X_i$ it follows that $u, v$ meet in $T$. Hence, $T$ certifies $\lambda_t(G) \leq r$.

Otherwise, $|U \cup N(U)| > r$, and by (i), $U = \{u\}$ for some $u \in V$ and $|N(u) \cap N(u)| < r$. In particular, $|N(u)| \geq r$. Then from $N(u) \subseteq X_j$ it follows that $|X_j| \geq |N(u)| \geq r > |N(u) \cap N(u)|$. Let $v \in V \setminus (N(u) \cap N(u))$. We set $X_i = (X_j \setminus \{v\}) \cup \{u\}$ and define the tree-model $T = (T, X \cup \{X_i\})$. Note that the largeur of $T$ is equal to $r$ and that its spread is at most $r + t$. To show that $T$ is an open tree-decomposition of $G$, it is enough to show that the conditions imposed on $T$ by all edges $uv$ of $G$ with $w \in V$ are satisfied. If $w \neq v$, then this is clear, as then $w \in (N(u) \cup N(u)) \setminus \{v\} \subseteq X_i$, and thus $u, w$ meet at node $i$ in $T$. If $w = v$, then it is also clear, since $w \notin E \cap E$, and $u, v$ sharply touch in $T$ at the edge $ij$. Therefore, $T$ certifies $\lambda_t(G) \leq r$.

We conclude that in both cases, $\lambda_t(G) \leq r$.

We now define an $(r, t)$-elimination ordering as follows.

**Definition 2.4.2.** Let $G = (V, E, E)$ be a partially eared graph. Given $r, t \geq 1$, an $(r, t)$-elimination ordering of $G$ is a sequence $(V_1, V_2, \ldots, V_m)$ of pairwise disjoint subsets of $V$ with $\bigcup_{k=1}^{m} V_k = V$, $1 \leq |V_k| \leq t$ for $1 \leq k < m$, $|V_m| \leq r$, and such that the following holds. Define $G_1 = G$, and for $1 \leq k < m$, define $G_{k+1} = G_k - V_k$. Now, for each $1 \leq k < m$, it should hold that $|V_k \cup N(V_k)| \leq r$, or both $|V_k| = 1$ and $|N(V_k) \cap N(V_k)| < r$, where $N(V_k)$ and $N(V_k)$ are taken in $G_k$.

For a graph $G = (V, E)$, we say that a sequence $(V_1, V_2, \ldots, V_m)$ is an $(r, t)$-elimination ordering of $G$ if $(V_1, V_2, \ldots, V_m)$ is an $(r, t)$-elimination ordering of the partially eared graph $(V, E, \emptyset)$.

Let $(V_1, V_2, \ldots, V_m)$ be an $(r, t)$-elimination ordering of a partially eared graph $G$. Note that $|V_k| \leq r$ for any $1 \leq k \leq m$, and that $(V_2, V_3, \ldots, V_m)$ is an $(r, t)$-elimination ordering of $G - V_1$. Using Proposition 2.4.1 we can now prove the following alternative characterization of $\lambda_t(G)$ for $t \geq 1$.

**Theorem 2.4.3.** Let $r, t \geq 1$ and let $G = (V, E, E)$ be a partially eared graph. Then $\lambda_t(G) \leq r$ if and only if $G$ has an $(r, t)$-elimination ordering.

**Proof.** First suppose $\lambda_t(G) \leq r$. We prove by induction on $|V|$ that $G$ has an $(r, t)$-elimination ordering. If $|V| \leq r$ it is clear. Now let $|V| > r$. By Proposition 2.4.1 there exists a subset $U \subseteq V$ with $1 \leq |U| \leq t$ such that $\lambda_t(G - U) \leq r$, and, $|U \cup N(U)| \leq r$ or both $|U| = 1$ and $|N(U) \cap N(U)| < r$. Then by the induction hypothesis, $G - U$ has an $(r, t)$-elimination ordering
(V_1, V_2, \ldots, V_m). It follows that (U, V_1, V_2, \ldots, V_m) is an (r, t)-elimination ordering of \mathcal{G}.

Now suppose \mathcal{G} has an (r, t)-elimination ordering (V_1, V_2, \ldots, V_m). We prove by induction on m that \text{la}_t(\mathcal{G}) \leq r. If m = 1 it is clear. Now let m > 1. As (V_2, \ldots, V_m) is an (r, t)-elimination ordering of \mathcal{G} - V_1, \text{la}_t(\mathcal{G} - V_1) \leq r by the induction hypothesis. If |V(\mathcal{G})| \leq r, then \text{la}_t(\mathcal{G}) \leq r. Otherwise, Proposition 2.4.1 shows that \text{la}_t(\mathcal{G}) \leq r (take U = V_1). \qed

Notice that for an (r, t)-elimination ordering (V_1, V_2, \ldots, V_m) of a partially eared graph \mathcal{G}, we have |N(V_1) \cup \overline{N}(V_1)| \leq r. Indeed, as \text{la}_t(\mathcal{G} - V_1) \leq r and N(V_1) \cup \overline{N}(V_1) is an eared clique in \mathcal{G} - V_1, this follows from Lemma 2.2.4.
Chapter 3

A closer look at the parameter $\text{la}_t(G)$

In this chapter we use the results from the previous chapter to investigate the parameter $\text{la}_t(G)$ further. We use the close connection with $\text{tw}(G)$ to show that computing $\text{la}_t(G)$ is hard, and to characterize the behaviour of $\text{la}_t(G)$ under (eared) clique-sums. Finally, in Section 3.3 we consider the special case $t = 1$.

3.1 Complexity

It is well-known that computing the tree-width is hard (see for example [1]). We extend this hardness result to $\text{la}_t$ for any $t \geq 1$. For this we use a reduction based on the following theorem. In this theorem we use the notation $\hat{G}$ to denote the graph obtained from $G = (V, E)$ by connecting the endpoints of each edge in $G$ by an ear, see Figure 3.1. Note that $\hat{G}$ is the underlying graph (see Section 2.1) of the partially eared graph $(V, E, E)$.

Theorem 3.1.1. Let $G = (V, E)$ be a graph and $t \geq 1$. Then $\text{la}_t(\hat{G}) = \text{tw}(G) + 1$.

Proof. Construct the partially eared graph $\mathcal{G} = (V, E, E)$ and note that $\hat{G} = \varphi(\mathcal{G})$. Since $E(\mathcal{G}) = E(G)$, it follows from Definition 2.2.2 that any open tree-decomposition $\mathcal{T}$ of $\mathcal{G}$ has the property that $u, v$ meet in $\mathcal{T}$ for each $uv \in E$. Therefore, a tree-model $\mathcal{T}$ is an open tree-decomposition of $\mathcal{G}$ if and only if it is a tree-decomposition of $G$. Moreover, Lemma 1.2.5 states that if $\mathcal{T}$ is a tree-decomposition of $G$ of largeur $r$, then $G$ has a tree-decomposition of
largeur $r$ and spread $r$. Hence, $\lambda_t(G) = \tau(G) + 1$. Therefore, by Proposition \ref{prop:tw}, $\lambda_t(\widehat{G}) = \lambda_t(\phi(\mathscr{G})) = \lambda_t(\mathscr{G}) = \tau(G) + 1$.

Hence, $\lambda_t(G) = \tau(G) + 1$. Therefore, by Proposition \ref{prop:tw}, $\lambda_t(\widehat{G}) = \lambda_t(\phi(\mathscr{G})) = \lambda_t(\mathscr{G}) = \tau(G) + 1$.

**Theorem 3.1.1.** Let $t \geq 0$. For a given graph $G$ and a given integer $r$, it is NP-complete to determine whether $\lambda_t(G) \leq r$.

**Proof.** As it is NP-complete to determine whether $\tau(G) \leq r$, it is clear for $t = 0$. Now let $t \geq 1$. The problem is in NP as an $(r,t)$-elimination ordering of $G$ is a short certificate for $\lambda_t(G) \leq r$. Furthermore, Theorem \ref{thm:lat} states that $\tau(G) + 1 = \lambda_t(\widehat{G})$ for any graph $G$. Since $\widehat{G}$ can be obtained from $G$ in polynomial time, this shows that the problem of deciding whether $\tau(G) \leq r$ can be reduced in polynomial time to the problem of deciding whether $\lambda_t(G) \leq r$. Therefore, the statement of the theorem follows.

**3.2 Clique-sums**

Using Lemma \ref{lemma:tw}, we can prove the following statement, which is an analogue of the fact that if $G$ is the clique-sum of two graphs $G_1, G_2$, then $\tau(G) = \max\{\tau(G_1), \tau(G_2)\}$.

**Proposition 3.2.1.** Let $\mathscr{G} = (V, E, \mathcal{E})$ be the eared clique-sum of two partially eared graphs $\mathscr{G}_1$ and $\mathscr{G}_2$, i.e., $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ and $V(\mathscr{G}_1) \cap V(\mathscr{G}_2) = \emptyset$. Then $\lambda_t(\mathscr{G}) = \max\{\lambda_t(\mathscr{G}_1), \lambda_t(\mathscr{G}_2)\}$ for any $t \geq 0$.

**Proof.** Let $t \geq 0$ and write $r = \max\{\lambda_t(\mathscr{G}_1), \lambda_t(\mathscr{G}_2)\}$. By Lemma \ref{lemma:tw}, $r \leq \lambda_t(\mathscr{G})$. We now show that $\lambda_t(\mathscr{G}) \leq r$. For $k = 1, 2$, let $T_k = (T_k, \mathcal{E}_k)$ be a certificate for $\lambda_t(\mathscr{G}_k) \leq r$. We may assume that the vertex sets of $T_1$ and $T_2$
are disjoint. Write \( U = V(\mathcal{G}_1) \cap V(\mathcal{G}_2) \) and note that \((\bigcup \mathcal{X}_1) \cap (\bigcup \mathcal{X}_2) = U\). For \( k = 1, 2 \), \( U \) is an eared clique in \( \mathcal{G}_k \), so by Lemma 2.2.4 there exists a node \( i_k \in V(T_k) \) such that \( U \subseteq X_{i_k} \). Let \( T \) be the tree obtained from \( T_1 \cup T_2 \) by adding a new node \( j \) adjacent to both \( i_1 \) and \( i_2 \). Define \( X_j = U \) and \( T = (T, \mathcal{X}_1 \cup \mathcal{X}_2 \cup \{ X_j \}) \). The pair \( T \) is a tree-model, as \((\bigcup \mathcal{X}_1) \cap (\bigcup \mathcal{X}_2) = X_j \). We now show that \( T \) certifies \( \text{la}_t(\mathcal{G}) \leq r \). It is clear that the largeur of \( T \) is at most \( r \) and that its spread is at most \( r + t \). Next, suppose \( e = uv \in E \setminus E \). As \( \mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \), we have \( e \in E(\mathcal{G}_1) \setminus E(\mathcal{G}_1) \) or \( e \in E(\mathcal{G}_2) \setminus E(\mathcal{G}_2) \), say \( e \in E(\mathcal{G}_1) \setminus E(\mathcal{G}_1) \). Then \( u, v \) touch in \( T_1 \), so by the construction of \( T \), \( u, v \) touch in \( T \). Likewise, the condition imposed on \( T \) by an edge \( e \in E \setminus E \) is satisfied. Now suppose \( e = uv \in E \cap E \). If \( u, v \in U \), then \( u, v \) meet in \( T \) at \( j \). Otherwise, either \( u, v \in V(\mathcal{G}_1) \), or \( u, v \in V(\mathcal{G}_2) \), but not both. Say \( u, v \in V(\mathcal{G}_1) \). Then \( e \in E(\mathcal{G}_1) \cap E(\mathcal{G}_1) \), and thus \( u, v \) meet in \( T_1 \), and hence \( u, v \) also meet in \( T \). Therefore, \( T \) certifies \( \text{la}_t(\mathcal{G}) \leq r \). We conclude that \( \text{la}_t(\mathcal{G}) = \max\{ \text{la}_t(\mathcal{G}_1), \text{la}_t(\mathcal{G}_2) \} \).

If \( G \) is the clique-sum of two graphs \( G_1, G_2 \), then it does not hold in general that \( \text{la}_t(G) = \max\{ \text{la}_t(G_1), \text{la}_t(G_2) \} \) for \( t \geq 1 \). For example, \( K_3 \) is a clique-sum of copies of \( K_3 \), while \( \text{la}_t(K_3) = 3 > 2 = \text{la}_t(K_3) \) for \( t \geq 1 \), see Proposition 4.2.2. However, when we only consider \((\leq 1)\)-clique-sums, then the equation does hold, as the following corollary shows.

**Corollary 3.2.2.** Let \( G \) be the \((\leq 1)\)-clique-sum of two graphs \( G_1, G_2 \). Then \( \text{la}_t(G) = \max\{ \text{la}_t(G_1), \text{la}_t(G_2) \} \) for any \( t \geq 1 \).

**Proof.** Let \( t \geq 1 \). For \( k = 1, 2 \), define the partially eared graph \( \mathcal{G}_k = (V(G_k), E(G_k), \emptyset) \). Let \( \mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \). Note that \( \text{la}_t(\mathcal{G}_k) = \text{la}_t(G_k) \) for \( k = 1, 2 \), and that \( \text{la}_t(\mathcal{G}) = \text{la}_t(G) \). Now, as \( |V(\mathcal{G}_1) \cap V(\mathcal{G}_2)| \leq 1 \), the intersection \( V(\mathcal{G}_1) \cap V(\mathcal{G}_2) \) is trivially an eared clique in both \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \). Hence, by Proposition 3.2.1, \( \text{la}_t(\mathcal{G}) = \max\{ \text{la}_t(\mathcal{G}_1), \text{la}_t(\mathcal{G}_2) \} \). Therefore, \( \text{la}_t(G) = \max\{ \text{la}_t(G_1), \text{la}_t(G_2) \} \).

### 3.3 The case \( t = 1 \)

Let us first prove that \( \text{la}_1(G) = \text{la}_{\Box}(G) \) for any graph \( G \).

**Theorem 3.3.1.** For any graph \( G = (V, E) \), \( \text{la}_1(G) = \text{la}_{\Box}(G) \).

**Proof.** First we show that \( \text{la}_{\Box}(G) \leq \text{la}_1(G) \). Write \( r = \text{la}_1(G) \) and let \( T = \)
$(T, \mathcal{X})$ be a smooth certificate for $\lambda_1(G) \leq r$. Construct a graph $H$ with

$$V(H) = \{ w_{u,i} \mid u \in X_i, \ i \in V(T) \},$$
$$E(H) = \{ w_{u,i}w_{u,j} \mid u \neq v \in X_i, \ i \in V(T) \} \cup
\{ w_{u,i}w_{u,j} \mid u \in X_i \cap X_j, \ ij \in E(T) \} \cup
\{ w_{u,i}w_{v,j} \mid u \in X_i \setminus X_j, \ v \in X_j \setminus X_i, \ ij \in E(T) \}.$$  

Since $\mathcal{T}$ is smooth with largeur $r$ and spread at most $r + 1$, $H = T \Box K_r$ (here we use that $|X_i \setminus X_j| = 1 = |X_j \setminus X_i|$ for each $ij \in E(T)$). We now show that $G \leq H$ by constructing a $G$-partition of $H$. For $u \in V$, we define $W_u = \{ w_{u,i} \mid i \in T_u \}$. As $T_u$ is a subtree of $T$, $W_u$ induces a connected subgraph of $H$. Now suppose $uv \in E$. As $\mathcal{T}$ is an open tree-decomposition of $G$, there exist $i \simeq j \in V(T)$ such that $u, v \in X_i \cup X_j$, say $u \in X_i$ and $v \in X_j$. Then $w_{u,i}w_{v,j} \in E(H)$ is an edge between $W_u$ and $W_v$. Hence, the collection $\{ W_u \mid u \in V \}$ is a $G$-partition of $H$. Therefore, $G \leq H = T \Box K_r$. This shows that $\lambda_1(G) \leq r = \lambda_1(G)$.

Next, for the inequality $\lambda_1(G) \leq \lambda_{r\Box}(G)$, it is enough to show that $\lambda_1(T \Box K_r) \leq r$ for any tree $T$ (by the minor monotonicity of $\lambda_1(G)$). Obtain the partially eared graph $\mathcal{G}$ from $K_2 \Box K_r$ by turning the two maximal cliques $V_1, V_2$ of $K_2 \Box K_r$ into eared cliques. Write $V_1 = \{ u_1, u_2, \ldots, u_r \}$. Then $\lambda_1(\mathcal{G}) \leq r$, as $(\{ u_1 \}, \{ u_2 \}, \ldots, \{ u_r \}, V_2)$ is an $(r, 1)$-elimination ordering of $\mathcal{G}$. Therefore, by Proposition 3.2.1, $\lambda_1(\mathcal{G}') \leq r$ for any eared clique $r$-sum $\mathcal{G}'$ of copies of $\mathcal{G}$. As $T \Box K_r$ is of the form $\psi(\mathcal{G}')$ for any tree $T$, we have $\lambda_1(T \Box K_r) \leq r$, which was to be shown. \qed

We can also show (Proposition 3.3.3) that $\lambda_1(G)$ does not change under subdivision of edges (which is also shown in 15, Theorem 3.4). We first make an observation.

**Lemma 3.3.2.** Let $\mathcal{T} = (T, \mathcal{X})$ be a smooth tree-model of largeur $r$ and spread $s \leq r + 1$. Suppose $u, v \in \bigcup \mathcal{X}$ touch in $\mathcal{T}$. Then $u, v$ meet or sharply touch in $\mathcal{T}$.

**Proof.** If $u, v$ do not meet in $\mathcal{T}$, then there exists an edge $ij \in E(T)$ such that $u \in X_i \setminus X_j$, and $v \in X_j \setminus X_i$. Since $s \leq r + 1$, $|X_i \cup X_j| \leq r + 1$, and since $\mathcal{T}$ is smooth, $|X_i| = |X_j| = r$. Hence, $X_i \setminus \{ u \} = X_j \setminus \{ v \}$. Therefore, $u, v$ sharply touch at the edge $ij$. Hence, $u, v$ meet or sharply touch in $\mathcal{T}$. \qed

Using this observation we now show the following.

**Proposition 3.3.3.** Let $G = (V, E)$ be a graph. Let $G'$ be obtained from $G$ by subdividing an edge $e \in E$. Then $\lambda_1(G) = \lambda_1(G')$.  

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Proof. As $G \preceq G'$, $la_1(G) \leq la_1(G')$. For the other inequality we define the partially eared graph $\mathcal{G} = (V, E \setminus \{e\}, \{\{e\}\})$. Note that $G'' = \varphi(\mathcal{G})$. By Lemma 3.3.2 any smooth open tree-decomposition of $G$ of largeur $r$ and spread at most $r + 1$ is an open tree-decomposition of $\mathcal{G}$. Hence, $la_1(\mathcal{G}) \leq la_1(G)$, and therefore $la_1(G') = la_1(\varphi(\mathcal{G})) = la_1(\mathcal{G}) \leq la_1(G)$, where we use Proposition 2.3.3 for the second equality.

$\square$
Chapter 4

Minimal forbidden minors

Since the parameter \( l_t(G) \) is minor-monotone for any \( t \geq 0 \), it follows from the graph minor theorem of Robertson and Seymour [11] that for any \( r, t \geq 0 \), the class of graphs \( G \) with \( l_t(G) \leq r \) can be characterized by a finite collection \( F_{r,t} \) of minimal forbidden minors. That is, \( F_{r,t} \) is the minimal collection of graphs such that for any graph \( G \), \( l_t(G) \leq r \) if and only if no graph in \( F_{r,t} \) is a minor of \( G \). Note that \( G \in F_{r,t} \) if and only if \( l_t(G) > r \) and \( l_t(G') \leq r \) for any \( G' \prec G \). Also note that if \( G \in F_{r,t} \), then \( l_t(G) = r + 1 \). Indeed, given \( G \in F_{r,t} \), we have \( l_t(G - v) \leq r \) for any \( v \in V(G) \).

In this chapter we look at the sets \( F_{r,t} \) for \( t \geq 1 \). Since the spread of a tree-model is at most twice its largeur, we have for \( t \geq r \) that \( l_t(G) \leq r \) if and only if \( l_r(G) \leq r \). Therefore, \( F_{r,r} = F_{r,r+1} = \cdots = F_{r,\infty} \). Moreover, given three integers \( r, t_1, t_2 \) satisfying \( 1 \leq t_1 < t_2 \leq r \), we have \( F_{r,t_1} \neq F_{r,t_2} \), since \( l_{t_2}(K_{r+t_2}) = r \), while \( l_{t_1}(K_{r+t_2}) > r \). In other words, for fixed \( r \), there are exactly \( r \) different sets \( F_{r,t} \) (excluding the set \( F_{r,0} \)).

In Section 4.1 we show that \( F_{r,t} \) consists of graphs which are 3-connected after suppressing all vertices of degree 2. In Section 4.2 we give two classes of minimal forbidden minors for \( l_t(G) \leq r \), and in Section 4.3 we determine the sets \( F_{2,1} \) and \( F_{2,2} \). In Section 4.4 we give an operation on a graph \( G \) which increases \( l_t(G) \) by one, and show that under certain conditions this operation transforms a minimal forbidden minor for \( l_t(G) \leq r \) into a minimal forbidden minor for \( l_t(G) \leq r + 1 \). Finally, in Section 4.5 we present an algorithm for finding minimal forbidden minors for \( l_t(G) \leq r \), and explain the results obtained from running this algorithm.
Let $G$ be a minimal forbidden minor for $\lambda_t(G) \leq r$ for $t \geq 1$, i.e., $G \in \mathcal{F}_{r,t}$. If $\{U,W\}$ is a proper separation of $G$ of order 1, then $\lambda_t(G) = \max\{\lambda_t(G[U]),\lambda_t(G[W])\}$ by Corollary 3.2.2. Let $\lambda_t(G) = \lambda_t(G[U])$. As $G[U] \prec G$, this contradicts the minimality of $G$. Hence, $G$ is 2-connected when $|V| > 2$. The goal of this section is to show (Proposition 4.1.4) that there exists a partially eared graph $\mathcal{G}$ with $\varphi(\mathcal{G}) = G$ such that $\psi(\mathcal{G})$ is $t$-connected when $|\mathcal{V}(\mathcal{G})| > 3$. We first prove three lemmas which tell us more about the structure of a minimal forbidden minor for $\lambda_t(G) \leq r$. As in the rest of this chapter we exclude the case $t = 0$. The statements do hold for the parameter $\lambda_0(G)$, but our proofs rely on partially eared graphs, and the equality $\lambda_0(\mathcal{G}) = \lambda_0(\varphi(\mathcal{G}))$ does not hold for every partially eared graph $\mathcal{G}$.

Lemma 4.1.1. Suppose a graph $G$ has an induced path $P$ of length at least 3. Then $G \notin \mathcal{F}_{r,t}$ for any $t \geq 1$ and $r \geq 0$.

Proof. Let $t \geq 1$ and $r \geq 0$. Let $u, w_1, w_2, v$ be four consecutive vertices of $P$. Obtain $G'$ from $G$ by contracting the edge $w_1w_2$, and denote the new vertex of $G'$ by $w$. We show that $\lambda_t(G) \leq \lambda_t(G')$. Note that $w$ is the top of an ear connecting $u, v$ in $G'$. Define

$$\mathcal{G} = (V(G'), E(G') \setminus \{uw, wv\}, \{uw, wv\}),$$

$$\mathcal{G}' = (V(G') \setminus \{w\}, E(G') \setminus \{uw, wv\}, \{w\}).$$

Note that $G \preceq \varphi(\mathcal{G})$, and thus $\lambda_t(G) \leq \lambda_t(\varphi(\mathcal{G})) = \lambda_t(\mathcal{G})$. Furthermore, by Lemma 2.3.1, $\lambda_t(\mathcal{G}) \leq \lambda_t(\mathcal{G}')$. Hence, $\lambda_t(G) \leq \lambda_t(\mathcal{G}') = \lambda_t(\varphi(\mathcal{G}')) = \lambda_t(G')$, where we use that $\varphi(\mathcal{G}') = G'$. As $G'$ is a proper minor of $G$, $G \notin \mathcal{F}_{r,t}$. □

Lemma 4.1.2. Suppose $G = (V, E)$ is a graph with two vertices $u, v \in V$ which are connected by more than one ear. Then $\lambda_t(G \star w) = \lambda_t(G)$ for any $t \geq 1$, where $w$ is the top of an ear connecting $u, v$.

Proof. Let $t \geq 1$. As $G \star w \preceq G$, we have $\lambda_t(G \star w) \leq \lambda_t(G)$. Now let $w' \neq w \in V$ be the top of another ear connecting $u, v$. Define the partially eared graph $\mathcal{G}$ by removing both $w, w'$ from $G$ and turning $\{u, v\}$ into an eared clique, i.e.,

$$\mathcal{G} = (V \setminus \{w, w'\}, E \setminus \{uw, wv, uw', w'v\}) \cup \{uv\}, \{uw\}.$$
Note that $\varphi(\mathcal{G}) = G \ast w$. We show that $la_t(G) \leq la_t(\mathcal{G})$. Write $r = la_t(\mathcal{G})$ and let $T = (T, \mathcal{X})$ be a certificate for $la_t(\mathcal{G}) \leq r$. By Lemma 2.2.4, there exists a $j \in V(T)$ with $u, v \in X_j$. Obtain the tree $T'$ from $T$ by adding two leaves $i, i'$ adjacent to $j$. Set $X_i = \{w\}$, $X_{i'} = \{w'\}$, and define $T' = (T', \mathcal{X} \cup \{X_i, X_{i'}\})$. The only edges of $G$ not in $\mathcal{G}$ are incident with $w$ or $w'$. Since $u, v, w \in X_j \cup X_i$, and $u, v, w' \in X_j \cup X_{i'}$, $T'$ certifies $la_t(G) \leq r$. Hence, $la_t(G) \leq la_t(\mathcal{G}) = la_t(\varphi(\mathcal{G})) = la_t(G \ast w)$. Therefore, $la_t(G \ast w) = la_t(G)$. \qed

Note that Lemma 4.1.2 shows that if $G \in \mathcal{F}_{r,t}$, then every pair of vertices of $G$ is connected by at most one ear.

**Lemma 4.1.3.** Let $t \geq 1$ and $r \geq 0$. Let $G = (V, E)$ be a graph and let $\{U_1, U_2\}$ be a separation of $G$ of order 2 with $|U_1 \setminus U_2| > 1$ and $|U_2 \setminus U_1| > 1$. Then $G \notin \mathcal{F}_{r,t}$.

**Proof.** Write $G_1 = G[U_1]$, $G_2 = G[U_2]$, and $U_1 \cap U_2 = \{u, v\}$. We may assume that $G$ is 2-connected, otherwise the statement is clear (see the beginning of this section). For $k = 1, 2$, let $\mathcal{G}_k$ be the partially eared graph obtained from $G_k$ by turning $\{u, v\} \subseteq V(G_k)$ into an eared clique. We first show that $\varphi(\mathcal{G}_1) < G$. We do this by showing that $G_2$ has a proper minor $H$ with $H = K_3$ and $u, v \in V(H)$.

Since $G$ is 2-connected, there exists a $u - v$ path $P \subseteq G_2$. If $P = G_2$, then $P$ is an induced path in $G$ of length at least 3 (as $|U_2| > 3$), and the statement follows from Lemma 4.1.1. Otherwise, $P \subseteq G_2$. If we choose $P$ with minimum possible length, then there exists a vertex $w \in U_2 \setminus V(P)$. Let $Q \subseteq G_2$ be a $u - v$ path with $w \in V(Q)$. It is easy to see that $P \cup Q$ has a minor $H$ with $H = K_3$ and $u, v \in V(H)$. Since $|V(H)| = 3 < |U_2|$, $H$ is a proper minor of $G_2$. Therefore, $\varphi(\mathcal{G}_1) < G$. Likewise, $\varphi(\mathcal{G}_2) < G$.

Next, write $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. Since $G$ is a subgraph of $\varphi(\mathcal{G})$, $la_t(G) \leq la_t(\varphi(\mathcal{G})) = la_t(\mathcal{G})$. By Proposition 3.2.1, $la_t(\mathcal{G}) = \max\{la_t(\mathcal{G}_1), la_t(\mathcal{G}_2)\}$, say $la_t(\mathcal{G}_2) = la_t(\mathcal{G}_1)$. Then the graph $\varphi(\mathcal{G}_1)$ is a proper minor of $G$ satisfying $la_t(\varphi(\mathcal{G}_1)) = la_t(\mathcal{G}_1) \geq la_t(G)$, which shows that $G \notin \mathcal{F}_{r,t}$. \qed

By combining the previous three lemmas we can now show the main result of this section.

**Proposition 4.1.4.** Let $G = (V, E)$ be a minimal forbidden minor for $la_t(G) \leq r$ for $t \geq 1$ and $r \geq 0$. Then $G$ is 2-connected when $|V| > 2$, and there exists a partially eared graph $\mathcal{G}$ with $\varphi(\mathcal{G}) = G$ such that $\psi(\mathcal{G})$ is 3-connected when $|V(\mathcal{G})| > 3$. 

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Proof. At the start of this section we have already shown that $G$ is 2-connected when $|V| > 2$. Next, if $|V| \leq 3$, then we can take $\mathcal{G} = (V, E, \emptyset)$. Otherwise, $|V| > 3$. Let $W = \{w \in V \mid \deg(w) = 2\}$. This is an independent set, which can be seen as follows. Suppose $w_1, w_2 \in W$ are two adjacent vertices, say $N(w_1) = \{u_1, u_2\}$ and $N(w_2) = \{u_1, u_2\}$ for $u_1, u_2 \in V$. Then $u_1 \neq u_2$, since otherwise $u_1$ is a cutvertex of $G$ (as $|V| > 3$). Hence, $\{u_1, u_2, w_1, w_2\}$ induces a path of length 3 in $G$, contradicting the minimality of $G$ by Lemma 4.1.1. Therefore, $W$ is an independent set. Then the set $F = \{uw \mid N(w) = \{u, v\} \text{ for some } w \in W\}$ satisfies $F \subseteq E(G - W)$. Therefore, $\mathcal{G} = (V \setminus W, E(G - W), F)$ is a partially eared graph. By Lemma 4.1.2 any pair of vertices in $G$ is connected by at most one ear, and thus $\varphi(\mathcal{G}) = G$.

Now suppose $\{u, v\} \subseteq V$ is a 2-vertex-cut of $G$. Let $\{U_1, U_2\}$ be a proper separation of $G$ with $U_1 \cap U_2 = \{u, v\}$. By Lemma 4.1.3 we may assume $|U_1| = 3$, say $U_1 \setminus U_2 = \{w\}$. Then $\deg(w) = 2$, i.e., $w \in W$, and therefore $\{u, v\}$ is not a 2-vertex-cut of $\psi(\mathcal{G}) = (G - W) + F$. Moreover, as any cutset of $\psi(\mathcal{G})$ is a cutset of $G$, $\psi(\mathcal{G})$ contains no $(\leq 2)$-vertex-cut, i.e., $\psi(\mathcal{G})$ is 3-connected. □

4.2 Two classes of minimal forbidden minors

Let $t \geq 1$, $r \geq 0$, and let $\mathcal{G}$ be a partially eared graph with $\varphi(\mathcal{G}) \in \mathcal{F}_{r,t}$. Then $\la_t(\mathcal{G}) = \la_t(\varphi(\mathcal{G})) > r$, and if $\mathcal{G}'$ is obtained from $\mathcal{G}$ by deleting/contracting/suppressing an edge of $\mathcal{G}$, then $\varphi(\mathcal{G}') \prec \varphi(\mathcal{G})$, and therefore $\la_t(\mathcal{G}') = \la_t(\varphi(\mathcal{G}')) \leq r$. We now show a slightly stronger version of the converse, which we will use to prove that certain graphs are minimal forbidden minors for $\la_t(G) \leq r$.

Lemma 4.2.1. Let $t \geq 1$, $r \geq 0$, and let $\mathcal{G} = (V, E, E)$ be a partially eared graph such that $\la_t(\mathcal{G}) > r$, $\la_t(\mathcal{G} \setminus e) \leq r$ for each $e \in E$, $\la_t(\mathcal{G} \ast e) \leq r$ for each $e \in E \setminus E$, and $\la_t(\mathcal{G} / e) \leq r$ for each $e \in E \setminus E$. Then $\varphi(\mathcal{G}) \in \mathcal{F}_{r,t}$.

Proof. As $\la_t(\varphi(\mathcal{G})) = \la_t(\mathcal{G}) > r$, we are left to show minimality of $\varphi(\mathcal{G})$. We will use the following: $\la_t(\mathcal{G} \ast e) \leq r$ for any $e \in E$. Indeed, if $e \in E \setminus E$, then this holds by assumption, and if $e \in E \cap E$, then $\mathcal{G} \ast e = (\mathcal{G} \setminus e) \ast e$, and thus $\la_t(\mathcal{G} \ast e) = \la_t((\mathcal{G} \setminus e) \ast e) \leq \la_t((\mathcal{G} \setminus e) \setminus e) \setminus e \leq r$ by Lemma 2.2.3.

The edge set of $\varphi(\mathcal{G})$ is equal to

$$E(\varphi(\mathcal{G})) = E \cup \{uw_e \mid u \in V, e \in E, u \text{ is incident with } e\}.$$ 

We consider the two types of edges separately. Suppose $f \in E(\varphi(\mathcal{G}))$ is an edge of the form $f = uw_e$ for $u \in V$ and $e \in E$. Then $\varphi(\mathcal{G}) / f = \varphi(\mathcal{G} \ast e)$,
and thus $\text{la}_t(\varphi(\mathcal{G})/f) = \text{la}_t(\varphi(\mathcal{G} \ast e)) = \text{la}_t(\mathcal{G} \ast e) \leq r$. Also, the graph $\varphi(\mathcal{G}) \setminus f$ is (a subgraph of) a 1-clique-sum of $\varphi(\mathcal{G} \ast e)$ and $K_2$, and thus $\text{la}_t(\varphi(\mathcal{G}) \setminus f) \leq \max\{\text{la}_t(\varphi(\mathcal{G} \ast e)), \text{la}_t(K_2)\} \leq \text{la}_t(\varphi(\mathcal{G} \ast e)) \leq r$, where we use Corollary 3.2.2 and the fact that $K_2 \preceq \varphi(\mathcal{G} \ast e)$.

Next, suppose $e \in E(\varphi(\mathcal{G}))$ is an edge with $e \in E$. Since $\varphi(\mathcal{G}) \setminus e = \varphi(\mathcal{G} \setminus e)$, we have $\text{la}_t(\varphi(\mathcal{G} \setminus e)) = \text{la}_t(\varphi(\mathcal{G} \setminus e)) = \text{la}_t(\varphi(\mathcal{G} \setminus e) \leq r$. We are left to show that $\text{la}_t(\varphi(\mathcal{G})/e) \leq r$. This is clear when $e \in E \cap E$, since then $\varphi(\mathcal{G})/e$ is a 1-clique-sum of $\varphi(\mathcal{G} \ast e)/e$ and $K_2$. Now suppose $e = uv \in E \setminus E$. If there is no $x \in V$ with $xu, xv \in E$, then $\varphi(\mathcal{G})/e = \varphi(\mathcal{G} / e)$, and $\text{la}_t(\varphi(\mathcal{G})/e) = \text{la}_t(\varphi(\mathcal{G} / e)) = \text{la}_t(\varphi(\mathcal{G} / e) \leq r$ by assumption. Otherwise, let $x \in V$ be a vertex with $xu, xv \in E$. Then $w_{xu}, w_{xv}$ are the tops of two ears connecting $x$ and $u'$ in $\varphi(\mathcal{G})/e$, where $u'$ is the vertex of $\varphi(\mathcal{G})/e$ created by contracting the edge $e$. From Lemma 4.1.2 it follows that $\text{la}_t(\varphi(\mathcal{G})/e) = \text{la}_t((\varphi(\mathcal{G})/e) \ast w_{xu})$. As $(\varphi(\mathcal{G})/e) \ast w_{xu} = \varphi(\mathcal{G} \ast xu)/e$, this shows that $\text{la}_t(\varphi(\mathcal{G})/e) = \text{la}_t((\varphi(\mathcal{G})/e) \ast w_{xu}) = \text{la}_t(\varphi(\mathcal{G} \ast xu)/e) \leq \text{la}_t(\varphi(\mathcal{G} \ast xu)) \leq r$. Hence, $\varphi(\mathcal{G}) \in F_{r,t}$.

4.2.1 The class $\widetilde{K}_{r+1}$

We now give the first class of minimal forbidden minors for $\text{la}_t(G) \leq r$, which are the graphs of the form $\widetilde{K}_{r+1}$, i.e., graphs obtained from a complete graph $K_{r+1}$ by connecting the endpoints of each edge by an ear (see Section 3.1 in [8] these graphs are denoted by $F_r$).

**Proposition 4.2.2.** Let $r, t \geq 1$. Then $\widetilde{K}_{r+1} \in F_{r,t}$.

**Proof.** Write $G = (V, E) = K_{r+1}$, and define the partially eared graph $\mathcal{G} = (V, E, E)$. Let $T = (T, X)$ be an open tree-decomposition of $\mathcal{G}$. As $V$ is an eared clique in $\mathcal{G}$, $V \subseteq X_i$ for some $i \in V(T)$. Hence, the largeur of $T$ is at least $r+1$, and thus $\text{la}_t(\mathcal{G}) > r$. As $E(\mathcal{G}) = E(\mathcal{G})$, we are left to show by Lemma 4.2.1 that $\text{la}_t(\mathcal{G} \setminus e) \leq r$ for every $e \in E$. This in turn follows from the fact that $(\{u\}, V \setminus \{u\})$ is an $(r, t)$-elimination ordering of $\mathcal{G} \setminus e$, where $u$ is one of the endpoints of $e$.

Proposition 4.2.2 also shows the following. Let $\mathcal{G}$ be a partially eared graph with $|V(\mathcal{G})| \leq r$ and write $G = \varphi(\mathcal{G})$. Then $G \preceq \widetilde{K}_r$, and therefore, by Proposition 4.2.2 $\text{la}_t(G) = r$ if and only if $G = \widetilde{K}_r$ for $t \geq 1$. Also note that as $\text{tw}(\widetilde{K}_r) = r - 1$ for $r \neq 2$, we have $\text{la}_t(\widetilde{K}_r) = \text{tw}(\widetilde{K}_r) + 1$ for any $t \geq 0$ and $r \neq 2$. Therefore, the bound $\text{la}_t(G) \leq \text{tw}(G) + 1$ from Section 1.3 is tight.
4.2.2 The class $K_{r+2}^*$

We now consider partially eared graphs $\mathcal{G}$ with base graph $\psi(\mathcal{G}) = K_{r+2}$. We first show that $K_{r+2}$ is a minimal forbidden minor for $\text{la}_1(G) \leq r$.

**Proposition 4.2.3.** Given $r \geq 1$, the graph $K_{r+2}$ is a minimal forbidden minor for $\text{la}_1(G) \leq r$.

**Proof.** Let $G = (V, E) = K_{r+2}$. Since $|N(u)| > r$ for each $u \in V$, $G$ has no $(r, 1)$-elimination ordering, i.e., $\text{la}_1(G) > r$. Now let $e = uv \in E$. Then $(\{u\}, \{v\}, V \setminus \{u, v\})$ is an $(r, 1)$-elimination ordering of $G \setminus e$, which shows that $\text{la}_1(G \setminus e) \leq r$. Also, from $G/e = K_{r+1}$ it follows that $\text{la}_1(G/e) \leq r$. Hence, $G$ is a minimal forbidden minor for $\text{la}_1(G) \leq r$.

Next, we consider the case $t \geq 2$. We start with an observation.

**Lemma 4.2.4.** Let $r, t \geq 2$ and let $\mathcal{G} = (V, E, E)$ be a partially eared graph with $\psi(\mathcal{G}) = K_{r+2}$. Then $\text{la}_t(\mathcal{G}) \leq r$ if and only if $\mathcal{G}$ satisfies

$$\exists U \subseteq V \text{ with } |U| = 2 \text{ such that } |N(U)| \leq r - 2. \quad (1)$$

**Proof.** First suppose $\text{la}_t(\mathcal{G}) \leq r$. Let $V_1$ be the first set in an $(r, t)$-elimination ordering of $\mathcal{G}$. Then $|V_1| \geq 2$, since $|N(u) \cup N(u)| = r + 1 > r$ for any $u \in V$. Now take any subset $U \subseteq V_1$ with $|U| = 2$. Since $|V_1 \cup N(V_1)| \leq r$, we have $|U \cup N(U)| \leq r$, i.e., $|N(U)| \leq r - 2$. Hence, $\mathcal{G}$ satisfies $[1]$. For the converse, suppose $\mathcal{G}$ satisfies $[1]$ for some $U \subseteq V$. Then $|U| = 2 \leq t$, $|V \setminus U| = r + 2 - 2 = r$, and $|U \cup N(U)| = |U| + |N(U)| \leq r$. This shows that $(U, V \setminus U)$ is an $(r, t)$-elimination ordering of $\mathcal{G}$, and hence, $\text{la}_t(\mathcal{G}) \leq r$.

Finally, we characterize the partially eared graphs $\mathcal{G}$ with $\psi(\mathcal{G}) = K_{r+2}$ and $\varphi(\mathcal{G}) \in \mathcal{F}_{r,t}$.

**Proposition 4.2.5.** Let $r, t \geq 2$. Let $\mathcal{G} = (V, E, E)$ be a partially eared graph with $\psi(\mathcal{G}) = K_{r+2}$. Then $\varphi(\mathcal{G}) \in \mathcal{F}_{r,t}$ if and only if $E \cap E = \emptyset$, $\mathcal{G}$ does not satisfy $[1]$, and for each $e \in E$, $\mathcal{G} \ast e$ satisfies $[1]$.

**Proof.** First suppose $\varphi(\mathcal{G}) \in \mathcal{F}_{r,t}$. Then $\text{la}_t(\mathcal{G}) > r$ and $\text{la}_t(\mathcal{G} \ast e) \leq r$ for each $e \in E$, and thus, by Lemma 4.2.4, we only have to show that $E \cap E = \emptyset$ (here we use that $\psi(\mathcal{G} \ast e) = K_{r+2}$). Assume for the sake of a contradiction that there exists an edge $e \in E \cap E$. Then $\psi(\mathcal{G} \setminus e) = K_{r+2}$, and since $\text{la}_t(\mathcal{G} \setminus e) \leq r$, the graph $\mathcal{G} \setminus e$ satisfies $[1]$ for some $U \subseteq V$. Then $\mathcal{G}$ also satisfies $[1]$ (for the same $U$), a contradiction.
Now suppose $E \cap E = \emptyset$, $G$ does not satisfy (1), and for each $e \in E$, $G * e$ satisfies (1). By Lemma 4.2.4, we only have to show that $\lambda_t(G \setminus e) \leq r$ and $\lambda_t(G/e) \leq r$ for each $e \in E$. The first inequality holds since $(\{u\}, \{v\}, V \setminus \{u, v\})$ is an $(r, t)$-elimination ordering of $G \setminus uv$ for $uv \in E$, and the second inequality holds since $(\{w\}, V(G/e) \setminus \{w\})$ is an $(r, t)$-elimination ordering of $G/e$, where $w$ is any vertex of $G/e$ not equal to the new vertex created by the contraction.

For $r = 2$, Proposition 4.2.5 gives exactly one minimal forbidden minor for $\lambda_t(G) \leq r$, which we denote by $K_{\sim}3$, see Figure 4.1 (this graph can also be found in [8], where it is called $H_3$). For $r \geq 3$, we define the $K_{\sim}r+1$-family as the set of minimal forbidden minors for $\lambda_t(G) \leq r$ obtained from Proposition 4.2.5. The three graphs of the $K_{\sim}5$-family are also shown in Figure 4.1.

![Figure 4.1: The graph $K_{\sim}4$ and the $K_{\sim}5$-family (depicted as partially eared graphs).](image)

### 4.3 Forbidden minor characterization for $\lambda_t(G) \leq 2$

We first give the complete set of minimal forbidden minors for $\lambda_t(G) \leq 1$ for $t \geq 1$.

**Theorem 4.3.1.** Let $t \geq 1$. The complete set $\mathcal{F}_{1,t}$ of minimal forbidden minors for $\lambda_t(G) \leq 1$ is equal to $\mathcal{F}_{1,t} = \{K_3\}$.

**Proof.** By Proposition 4.2.2, $K_3 = \overline{K}_2 \in \mathcal{F}_{1,t}$. Now suppose $G \in \mathcal{F}_{1,t}$. Clearly, $|V(G)| > 2$. Then by Proposition 4.1.4, $G$ is 2-connected. Hence, $K_3 \preceq G$, and therefore $G = K_3$, which was to be shown.

We now give the complete sets of minimal forbidden minors for $\lambda_t(G) \leq 2$ for $t = 1$ and $t \geq 2$ (these sets can also be found in [9] and [8], respectively).
Theorem 4.3.2. The complete set $\mathcal{F}_{2,1}$ of minimal forbidden minors for $la_1(G) \leq 2$ is equal to $\mathcal{F}_{2,1} = \{\hat{K}_3, K_4\}$.

Proof. By Proposition 4.2.2 and Proposition 4.2.3, $\{\hat{K}_3, K_4\} \subseteq \mathcal{F}_{2,1}$. Now suppose $G \in \mathcal{F}_{2,1}$. By Proposition 4.1.4, there exists a partially eared graph $\mathcal{G}$ with $\varphi(\mathcal{G}) = G$ such that $\psi(\mathcal{G})$ is 3-connected when $|V(\mathcal{G})| > 3$. If $|V(\mathcal{G})| \leq 3$, then $G \preceq \hat{K}_3 \in \mathcal{F}_{2,1}$, and thus $G = \hat{K}_3$. Otherwise, $\psi(\mathcal{G})$ is 3-connected. Then $K_4 \preceq \psi(\mathcal{G})$, and as $\psi(\mathcal{G}) \preceq G$, we have $G = K_4$. It follows that $\mathcal{F}_{2,1} = \{\hat{K}_3, K_4\}$. $\square$

Theorem 4.3.3. Let $t \geq 2$. The complete set $\mathcal{F}_{2,t}$ of minimal forbidden minors for $la_t(G) \leq 2$ is equal to $\mathcal{F}_{2,t} = \{\hat{K}_3, K_4^\sim, W_5\}$, where $W_5$ is the wheel graph on 5 vertices.

Proof. By Proposition 4.2.2 and Proposition 4.2.5, $\{\hat{K}_3, K_4^\sim\} \subseteq \mathcal{F}_{2,t}$. Now suppose $G \in \mathcal{F}_{2,t}$. By Proposition 4.1.4, there exists a partially eared graph $\mathcal{G}$ with $\varphi(\mathcal{G}) = G$ such that $\psi(\mathcal{G})$ is 3-connected when $|V(\mathcal{G})| > 3$. If $|V(\mathcal{G})| \leq 3$, then $G \preceq \hat{K}_3 \in \mathcal{F}_{2,t}$, and thus $G = \hat{K}_3$. If $|V(\mathcal{G})| = 4$, then $\psi(\mathcal{G}) = K_4$. In this case, $G = K_4^\sim$ by Proposition 4.2.5. Otherwise, $|V(\mathcal{G})| \geq 5$. It is well-known that any 3-connected graph on at least 5 vertices has a $W_5$ minor [13, Tutte’s Wheel Theorem]. Therefore, $W_5 \preceq \psi(\mathcal{G}) \preceq G$. We have $la_t(W_5) > 2$, since $|N(U)| > 2$ for each $U \subseteq V(W_5)$ with $1 \leq |U| \leq 2$ (see the observation directly below Theorem 2.4.3). Then from $\hat{K}_3 \not\preceq W_5$ and $K_4^\sim \not\preceq W_5$ it follows that $\mathcal{F}_{2,t} = \{\hat{K}_3, K_4^\sim, W_5\}$. $\square$

4.4 Constructing minimal forbidden minors

Let $t \geq r \geq 1$. In this section we give an operation on a graph $G$ which increases $la_t(G)$ by one. We show that under certain conditions this operation transforms a minimal forbidden minor for $la_t(G) \leq r$ into a minimal forbidden minor for $la_t(G) \leq r + 1$. We first need a definition.

Definition 4.4.1. Let $r \geq 1$ and let $\mathcal{G} = (V, E, \overline{E})$ be a partially eared graph. We say that a subset $W \subseteq V$ is an $r$-forcing-set in $\mathcal{G}$ if, by a series of the following two operations, we can obtain $W = V$.

1. Add a vertex $u \in V \setminus W$ to $W$ satisfying $|(N(u) \cup \overline{N}(u)) \cap W| \geq r$.
2. Suppose $U$ is the vertex set of a component of $\psi(\mathcal{G}) - W$ with $\overline{N}(u) = \emptyset$ for each $u \in U$, and moreover, $|U| < r$ or both $|U| = r$ and $|N(U)| \geq r$, where $\overline{N}(u)$ and $N(U)$ are taken in $\mathcal{G}$. Add the vertices in $U$ to $W$. 

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For a graph $G = (V, E)$, we say that $W \subseteq V$ is an $r$-forcing-set in $G$ if $W$ is an $r$-forcing-set in the partially eared graph $(V, E, \emptyset)$.

**Lemma 4.4.2.** Let $r \geq 1$ and $t \geq r - 1$. Let $G = (V, E, E)$ be a partially eared graph and $x \in V$ a vertex satisfying $N(x) = N(x)$, where $N(x)$ is an $r$-forcing-set in $G - x$. If $l_{a_t}(G - x) \geq r$, then $l_{a_t}(G) > r$.

**Proof.** Suppose for the sake of a contradiction that $l_{a_t}(G) \leq r$, say with certificate $T = (T, A')$. We give a transformation of $T$ resulting in $V(T_u) \cap V(T_x) \neq \emptyset$ for each $u \in V$. Then $x \in X_i$ for each $i \in V(T)$ with $|X_i| = r$. Therefore, if we remove $x$ from $X_i$ for each $i \in V(T_x)$, then we get a certificate for $l_{a_t}(G - x) \leq r - 1$, contradicting the assumption that $l_{a_t}(G - x) \geq r$.

Set $W = N(x) \cup \{x\}$. As $N(x) = N(x)$, we have for $w \in N(x)$ that $wx \in E \cap E$, and thus $w$, $x$ meet in $T$. This shows that

$$V(T_u) \cap V(T_x) \neq \emptyset \forall w \in W. \quad (2)$$

Since $W \setminus \{x\}$ is an $r$-forcing-set in $G - x$, and $N(x) \subseteq W$, $W$ is an $r$-forcing-set in $G$, i.e., we can obtain $W = V$ using the operations from Definition 4.4.1.

We now show how to update $T$ for each of these operations such that (2) remains true.

Suppose $u \in V \setminus W$ is a vertex with $|(N(u) \cup N(u)) \cap W| \geq r$. We show that $V(T_u) \cap V(T_x) \neq \emptyset$. Suppose to the contrary that $V(T_u) \cap V(T_x) = \emptyset$. Let $i \in V(T_x)$ be the node of $T_x$ closest to $T_u$. If $w \in W$ and $w \notin X_i$, then $V(T_w) \cap V(T_x) = \emptyset$ by (2), and thus $i$ separates $T_w$ and $T_u$. It follows that $u$, $w$ do not touch, and therefore $w \notin N(u) \cup N(u)$. Hence, $(N(u) \cup N(u)) \cap W \subseteq X_i$. Moreover, this inclusion is proper, as $x \in X_i$ and $x \notin N(u) \cup N(u)$ (since $N(x) \cup N(x) \subseteq W$ and $u \notin W$). Hence, $|X_i| > |(N(u) \cup N(u)) \cap W| \geq r$, contradicting the fact that the largeur of $T$ is at most $r$. Therefore, $V(T_u) \cap V(T_x) \neq \emptyset$, and thus we can add $u$ to $W$ while keeping (2) true.

Now let $U$ be the vertex set of a component of $\psi(G) - W$ with $N(u) = \emptyset$ for each $u \in U$, and moreover, $|U| < r$ or both $|U| = r$ and $|N(U)| \geq r$. We first consider the case in which

$$V(T_u) \cap V(T_x) = \emptyset \forall u \in U. \quad (3)$$

Define $T^\prime = T[V(T_u)]$ since $U$ induces a connected subgraph of $\psi(G)$, $T^\prime$ is a tree. By (3), $T^\prime$ satisfies $V(T_x) \cap V(T_x) = \emptyset$. Let $i \in V(T_x)$ be the node of $T_x$ closest to $T^\prime$. If $w \in N(U)$, then $u, w$ touch in $T$ for some $u \in U$, and $V(T_w) \cap V(T_x) \neq \emptyset$ (as $w \in N(U) \subseteq W$), and thus $u \in X_i$ as before. Therefore, $N(U) \subseteq X_i$. Moreover, since $N(x) \subseteq W$ and $U \cap W = \emptyset$, we have
$N(U) \subseteq X_i \setminus \{x\}$. If $|U| \geq r$, then by assumption $|N(U)| \geq r$, and thus $|X_i| > |(N(U)) \cap W| \geq r$, a contradiction. Therefore, $|U| < r$. We modify $\mathcal{T}$ as follows. We first remove all vertices in $U$ from $X_i$ for each $i \in V(T)$. Next, we add a leaf $j$ adjacent to $i$ to $T$, and we add $X_j = U \cup \{x\}$ to $\mathcal{T}$.

Clearly, $\mathcal{T}$ remains a tree-model of largeur at most $r$ and spread at most $r + t$. Moreover, since $U \cup N(U) \subseteq X_i \cup X_j$ and $N(U) = \emptyset$, $\mathcal{T}$ remains a certificate for $\la_t(\mathcal{G}) \leq r$. Now note that $V(T_u) \cap V(T_x) = \{j\}$ for each $u \in U$, and therefore (2) remains true if we add all vertices in $U$ to $W$.

We are left to show the case that $U' = \{u \in U \mid V(T_u) \cap V(T_x) \neq \emptyset\}$ is non-empty. In this case, we first add the vertices in $U'$ to $W$ (2) remains true. Now note that $U \setminus U'$ has a partition into vertex sets of components of $\psi(\mathcal{G}) - W$. Each of these vertex sets satisfies (3), and thus we can apply our previous result on each, resulting in the addition of all vertices in $U$ to $W$. Hence, it is possible to update $\mathcal{T}$ for both operations from Definition 4.4.1 such that (2) remains true. This gives the desired transformation of $\mathcal{T}$ resulting in $V(T_u) \cap V(T_x) = \emptyset$ for each $u \in V$.

Let $t \geq r - 1$ and suppose $\mathcal{G}'$ is a partially eared graph with $\la_t(\mathcal{G}') = r$. Let $W \subseteq V$ be an $r$-forcing-set in $\mathcal{G}'$. Obtain $\mathcal{G}$ from $\mathcal{G}'$ by adding a vertex $x$ and edges such that $N(x) = N(x) = W$. Lemma 4.4.2 shows that $\la_t(\mathcal{G}) > r$. We now prove that if in addition $\varphi(\mathcal{G}) \in F_{r-1,t}$, $|W| = r$, and $\la_t(\mathcal{G} - x) \leq r$, then $\varphi(\mathcal{G}) \in F_{r,t}$.

**Proposition 4.4.3.** Let $r \geq 1$ and $t \geq r - 1$. Let $\mathcal{G} = (V,E,E')$ be a partially eared graph and $x \in V$ a vertex satisfying $N(x) = N(x)$, where $|N(x)| = r$ and $N(x)$ is an $r$-forcing-set in $\mathcal{G} - x$. Suppose $\varphi(\mathcal{G} - x) \in F_{r-1,t}$ and $\la_t(\mathcal{G} - x) \leq r$. Then $\varphi(\mathcal{G}) \in F_{r,t}$.

**Proof.** Since $N(x)$ is an $r$-forcing-set in $\mathcal{G} - x$ and $\la_t(\mathcal{G} - x) \geq r$, we have $\la_t(\mathcal{G}) > r$ by Lemma 4.4.2. We now show minimality. First suppose $\mathcal{G}'$ is obtained from $\mathcal{G}$ by deleting/contracting/suppressing an edge of $\mathcal{G}$ not incident with $x$. Then $\varphi(\mathcal{G}' - x) < \varphi(\mathcal{G} - x)$, and thus $\la_t(\mathcal{G}' - x) \leq r - 1$ (as $\varphi(\mathcal{G} - x) \in F_{r-1,t}$), and therefore $\la_t(\mathcal{G}') \leq r$. Next, suppose $\mathcal{G}' = \mathcal{G} \setminus e$, where $e$ is an edge of $\mathcal{G}$ incident with $x$. Then $|N(x) \cap N(x)| = r - 1 < r$ in $\mathcal{G}'$. Moreover, $\la_t(\mathcal{G}' - x) \leq r$, since $\mathcal{G}' - x = \mathcal{G} - x$ and $\la_t(\mathcal{G} - x) \leq r$ by assumption. Hence, by Proposition 2.4.1 $\la_t(\mathcal{G}') \leq r$. Therefore, $\varphi(\mathcal{G}) \in F_{r,t}$.

If we take $r = 3$ and $t \geq 3$ then we get the following.

**Corollary 4.4.4.** Let $G = (V,E) \in \{K_3, K_4^-, W_5\}$ and let $U \subseteq V$ be a subset of size 3. Suppose $U$ is not of the form $U = \{u,v,w\}$, with $w$ an
ear connecting $u, v$. Obtain $G'$ from $G$ by adding a vertex $x$ adjacent to all vertices in $U$, and connecting $u, x$ by an ear for each $u \in U$. For $t \geq 3$, $G'$ is a minimal forbidden minor for $\varphi(G) \leq 3$.

**Proof.** We first show that $U$ is a 3-forcing-set in $G$. If $G - U$ has more than one component, then each component has at most 2 vertices, since $|V| \leq 6$, and therefore $U$ is a 3-forcing-set in $G$. Otherwise, $C = V \setminus U$ induces a connected subgraph of $G - U$. If $N(C) \subseteq U$, then $U$ contains a vertex $w$ with $N(w) \subseteq U$. As $G$ is 2-connected, $w$ is the top of an ear connecting the other two vertices in $U$, contradicting our assumption. Hence, $N(C) = U$, and thus $|N(C)| = 3$. Therefore, $U$ is a 3-forcing-set in $G$. Obtain the partially eared graph $\mathcal{G}$ from $G$ by adding a vertex $x$ and edges such that $N(x) = N(x) = U$. Then $\varphi(\mathcal{G} - x) \leq 3$, since $(U, V \setminus U)$ is an $(3, t)$-elimination ordering of $\mathcal{G} \setminus x$. Hence, by Proposition 4.4.3, $\varphi(\mathcal{G}) = G'$ is a minimal forbidden minor for $\varphi(G) \leq 3$.

This corollary gives 19 minimal forbidden minors for $\varphi(G) \leq 3$ for $t \geq 3$ (including $K_4$), see Figure 1 in the appendix. Another application of Proposition 4.4.3 is the following, in which we apply the proposition to the class $K_{r+1}^\sim$.

**Corollary 4.4.5.** Let $r \geq 1$ and $t \geq r - 1$. Let $G = (V, E)$ be a graph from the $K_{r+1}^\sim$-family. Let $U \subseteq V$ be a subset with $|U| = r$ and $\deg(u) \geq 3$ for each $u \in U$. Obtain $G'$ from $G$ by adding a vertex $x$ adjacent to all vertices in $U$, and connecting $u, x$ by an ear for each $u \in U$. Then $G' \in \mathcal{F}_{r,t}$.

**Proof.** Let $\mathcal{G}$ be the partially eared graph with $\varphi(\mathcal{G}) = G$ and $\psi(\mathcal{G}) = K_{r+1}^\sim$. As $\deg(u) \geq 3$ for each $u \in U$, $U \subseteq V(\mathcal{G})$. Write $V(\mathcal{G}) \setminus U = \{v\}$ for some $v \in V$. As $|N(v) \cup N(v)| = r$, $U$ is an $r$-forcing-set in $\mathcal{G}$. Obtain the partially eared graph $\mathcal{G}'$ from $\mathcal{G}$ by adding a vertex $x$ and edges such that $N(x) = N(x) = U$. Then $\varphi(\mathcal{G}' - x) \leq r$, since $(\{v\}, U)$ is an $(r, t)$-elimination ordering of $\mathcal{G}' \setminus x$. Hence, by Proposition 4.4.3, $\varphi(\mathcal{G}') = G' \in \mathcal{F}_{r,t}$.

### 4.5 Searching for minimal forbidden minors

A next step is to find the forbidden minor characterization of the class of graphs $G$ with $\varphi(G) \leq 3$. For $t = 1$ this has been done in [15] in the setting of multigraphs, where a collection $\mathcal{F}$ of 13 minimal forbidden minors is given and proven to be complete. In this section we present an algorithm for finding minimal forbidden minors for $\varphi(G) \leq r$. We use this algorithm to reconstruct the collection $\mathcal{F}$, and to find minimal forbidden minors for
$l_{a_2}(G) \leq 3$, $l_{a_3}(G) \leq 3$, and $l_{a_1}(G) \leq 4$. However, we do not prove by hand that any of the found graphs are in fact minimal forbidden minors (except for the ones already given in Section 4.2 and Section 4.4).

### 4.5.1 Overview

Proposition 4.1.4 states that any minimal forbidden minor for $l_{a_t}(G) \leq r$ is of the form $\varphi(G)$ for some partially eared graph $G$ with $\psi(G)$ 3-connected when $|V(G)| > 3$. Therefore, we work as follows. Suppose two integers $n,m$ are given. We first generate all 3-connected graphs $H$ on at most $n$ vertices (and all graphs on at most 3 vertices), and for each $H$, we generate all partially eared graphs $G$ with $\psi(G) = H$ and $\varphi(G)$ of at most $m$ vertices. Then, for each $G$, we check whether $\varphi(G)$ is a minimal forbidden minor for $l_{a_t}(G) \leq r$ using Lemma 4.2.1. For $n$ and $m$ large enough, we find all minimal forbidden minors for $l_{a_t}(G) \leq r$ in this way.

### 4.5.2 Algorithm

For the generation of the 3-connected graphs we use Tutte’s Wheel Theorem [13]. That is, we start with the set of all wheels on at most $n$ vertices. Then we obtain all 3-connected graphs on at most $n$ vertices by recursively applying the following two operations in all possible ways.

- Add an edge between two non-adjacent vertices.
- Split a vertex $u$ with $\deg(u) \geq 4$ into two adjacent vertices $v$ and $v'$ such that $N(u) = N\{v, v'\}$, $|N(v)| \geq 3$, $|N(v')| \geq 3$, and $N(v) \cap N(v') = \emptyset$.

We also check for graph isomorphisms to avoid generating the same 3-connected graph many times. Then, for each 3-connected graph $H = (W, E)$, we generate partially eared graphs $G = (W, E, F)$ with $\psi(G) = H$ by generating sets $E$ and $E$ with $E \cup E = F$.

Finally, to check whether $\varphi(G) \in F_{r,t}$, we check if $l_{a_t}(G) > r$, $l_{a_t}(G \setminus e) \leq r$ for each $e \in E$, $l_{a_t}(G \ast e) \leq r$ for each $e \in E \setminus E$, and $l_{a_t}(G / e) \leq r$ for each $e \in E \setminus E$ (see Lemma 4.2.1). Thus we need an algorithm to decide whether $l_{a_t}(G) \leq r$. Here we use the characterization of $l_{a_t}(G)$ in terms of $(r,t)$-elimination orderings. To determine whether $l_{a_t}(G) \leq r$, we try to find an $(r,t)$-elimination ordering of $G$. If we succeed, then $l_{a_t}(G) \leq r$. If we fail after having exhausted all possibilities, then $l_{a_t}(G) > r$. A recursive algorithm which returns an $(r,t)$-elimination ordering if one exists (and which
returns false otherwise) can be obtained from the definition of an \((r, t)\) elimination ordering, and is given here as Algorithm 1. An implementation of the complete algorithm can be found at [http://ewps.nl/thesis](http://ewps.nl/thesis).

**Algorithm 1:** A recursive algorithm for finding an \((r, t)\)-elimination ordering of a partially eared graph \(\mathcal{G}\).

```plaintext
Function FindEliminationOrdering(\(\mathcal{G}, r, t\)):
    if |V(\(\mathcal{G}\))| ≤ r then
        return (V(\(\mathcal{G}\)))
    foreach U ⊆ V(\(\mathcal{G}\)) with 1 ≤ |U| ≤ \(\min\{r, t\}\) do
        if |U ∪ N(U)| ≤ r or both |U| = 1 and |N(U) ∩ N(U)| < r then
            \(\pi = \text{FindEliminationOrdering}(\mathcal{G} - U, r, t)\)
            if \(\pi \neq \text{false}\) then
                return (U, \(\pi_1, \pi_2, \ldots, \pi_{\text{length}(\pi)}\))
    return false
```

4.5.3 Results

We are mostly interested in minimal forbidden minors for \(l_{a_2}(G) \leq 3\). We therefore set \(r = 3\) and \(t = 3\). The following table gives an overview of the numbers of minimal forbidden minors for \(l_{a_3}(G) \leq 3\) we find for different values of \(n\) and \(m\).
Table 4.1: On position \((n,m)\) the number of minimal forbidden minors \(G\) for \(\text{la}_3(G) \leq 3\) with \(|V(G)| = m\) and such that there exists a partially eared graph \(\mathcal{G}\) with \(\varphi(\mathcal{G}) = G\) and \(\psi(\mathcal{G})\) 3-connected on \(n\) vertices. *) only considering 3-connected graphs with at most 19 edges.

<table>
<thead>
<tr>
<th>(n \setminus m)</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>(\geq 15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>24</td>
<td>20</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>10</td>
<td>35</td>
<td>36</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>9</td>
<td>12</td>
<td>18</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>9</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>11</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0*</td>
<td>0*</td>
<td>0*</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>(\geq 12)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

In total we find 194 minimal forbidden minors for \(\text{la}_3(G) \leq 3\). The gaps (question marks) in the table indicate that for these values of \(n\) and \(m\) the algorithm did not finish within a couple of hours. Looking at the shape of the non-zero entries in the table, we have some reason to believe that we found most (or even all) minimal forbidden minors for \(\text{la}_3(G) \leq 3\).

We also ran the program for different values of \(r\) and \(t\). We found 20 minimal forbidden minors for \(\text{la}_1(G) \leq 3\), 74 for \(\text{la}_2(G) \leq 3\), and 1303 for \(\text{la}_1(G) \leq 4\). This last value is definitely only a lower bound on the total number of minimal forbidden minors. Moreover, if we modify the contraction operation on a partially eared graph as to mimic the contraction on multigraphs, we find a set of 13 minimal forbidden minors for \(\text{la}_1(G) \leq 3\).

This set corresponds to the complete set of 13 multigraph minimal forbidden minors for \(\text{la}_0(G) \leq 3\) given in [15]. We refer to [http://ewps.nl/thesis](http://ewps.nl/thesis) for a list of all the graphs we have found.
Discussion

The family of parameters $l_4(G)$ offers a new approach for studying the topological graph parameters $l_{□}(G)$ and $l_{□}(G)$. It gives an alternative characterization of these parameters in terms of (open) tree-decompositions, and exposes the close connection between $\text{tw}(G)$, $l_{□}(G)$ and $l_{□}(G)$. In particular, it shows that there exists a range of minor-monotone graph parameters between $\text{tw}(G)$ and $l_{□}(G)$.

The parameter $l_{□}(G)$ was introduced in [8] as an upper bound on the parameter $\text{egd}(G)$ related to the matrix completion problem. It was used to characterize the class of graphs $G$ with $\text{egd}(G) \leq 2$. We can reproduce this result using Theorem 4.3.3.

**Theorem 1.** The complete set $\mathcal{F}$ of minimal forbidden minors for $\text{egd}(G) \leq 2$ is equal to $\mathcal{F} = \{\hat{K}_3, K_{\sim 4}\}$.

*Proof.* We will need the following values of $\text{egd}(G)$ shown in [8]: $\text{egd}(K_{3,3}) \leq 2$, $\text{egd}(\hat{K}_3) > 2$, and $\text{egd}(K_{\sim 4}) > 2$. First suppose $G \in \mathcal{F}$ with $G \neq \hat{K}_3$ and $G \neq K_{\sim 4}$. Then $G$ is 2-connected, since if $G$ is the $(\leq 1)$-clique-sum of two graphs $G_1$ and $G_2$, then $\text{egd}(G) = \max\{\text{egd}(G_1), \text{egd}(G_2)\}$ [8]. From $\text{egd}(\hat{K}_3) > 2$, $\text{egd}(K_{\sim 4}) > 2$ and the minor-monotonicity of $\text{egd}(G)$ it follows that $G$ has no minor equal to $\hat{K}_3$ or $K_{\sim 4}$. Therefore, if $W_5 \not\cong G$, then by Theorem [4.3.3] $\text{egd}(G) \leq l_{□}(G) \leq 2$, a contradiction. Hence, $W_5 \cong G$.

We have $|V(G)| \geq 6$, as $\text{egd}(G') \leq 2$ for any graph $G'$ on at most 5 nodes [8, Lemma 2.6]. It is an easy exercise to show that the only 2-connected graph on at least 6 nodes with a $W_5$-minor and without a $K_{\sim 4}$-minor is the graph $K_{3,3}$. Since $\text{egd}(K_{3,3}) \leq 2$, we reach a contradiction. Therefore, $\mathcal{F} \subseteq \{\hat{K}_3, K_{\sim 4}\}$. Moreover, since $\text{egd}(\hat{K}_3) > 2$, $\text{egd}(K_{\sim 4}) > 2$ and both $\hat{K}_3$ and $K_{\sim 4}$ are minimal forbidden minors for $l_{□}(G) \leq 2$ (Theorem [4.3.3]), we have $\mathcal{F} = \{\hat{K}_3, K_{\sim 4}\}$. \hfill □

The proof of Theorem 1 shows that for any 2-connected graph $G \neq K_{3,3}$ on at least 6 nodes, $\text{egd}(G) \leq 2$ if and only if $l_{□}(G) \leq 2$. It would
be interesting to find such a statement for the class of graphs $G$ with $\text{egd}(G) \leq 3$. As $\text{egd}(G) \leq r$ for any graph $G$ on strictly less than $\binom{r+2}{2}$ vertices [8, Lemma 2.6], any minimal forbidden minor $G$ for $\text{lag}_2(G) \leq 3$ with $|V(G)| < \binom{3+2}{2} = 10$ satisfies both $\text{egd}(G) \leq 3$ and $\text{lag}_2(G) > 3$. Our computer search shows that there are 99 such graphs. This may indicate that such a statement for the graphs with $\text{egd}(G) \leq 3$ needs many exceptions.
Bibliography


Appendix

Figure 1: Some minimal forbidden minors for $\lambda_{\mathbb{G}}(G) \leq 3$, constructed as in Corollary 4.4.4 (depicted as partially eared graphs).