Spontaneous Unitarity Breaking in Macroscopic Quantum Systems

by

Jonas Veenstra

5942802


60 ECTS

Supervisor: Prof. dr. J. van Wezel
Examiner: Prof. dr. J.S. Caux

Institute for Theoretical Physics
Abstract

A system’s Hilbert space grows exponentially with the number of particles it contains due to the linearity of quantum mechanics. Classical state space, however, only scales linearly with system size. This would lead one to conclude that classical states become an increasingly rare phenomenon in the thermodynamic limit. The opposite is obviously true, indicating that a symmetry is irreversibly broken as systems grow in size.

In the first part of this thesis, it is shown that the decoherence approach [1] to solving this paradox succeeds partially but only describes ensemble-averaged quantities. A well known dynamical approach to wavefunction collapse by the name of CSL is treated [2,3], which modifies the Schrödinger equation and it is argued that despite its phenomenological accuracy, it merely represents an ad hoc solution introducing undesirable and unexplained nonlinear dynamics into quantum theory.

In the second part, the suppression of non-classical states in equilibrium as a result of the spontaneous symmetry breaking (SSB) is studied. By analyzing two well-known examples of SSB, its vital ingredients are identified and subsequently adapted to enable a dynamical description of the breakdown of time-translation and -reversal symmetry. Since this symmetry is generated by the unitarity of the Hamiltonian, a non-unitary symmetry breaking field is a natural choice but it is shown that a previous proposal [4] is irreconcilable with Born’s rule. Finally, an alternative field in the form of a random matrix is proposed which does yield the right statistics but does not break time-reversal symmetry in a satisfying way. To overcome these problems, a tentative mechanism is proposed that ensures permanent localization when the right order of limits is taken.
Acknowledgements

I would like to thank Jasper for his relentless optimism in times of despair and his inspiring insights when mine had run out. I also wish to thank my fellow students for their pleasant company.
## Contents

1 Introduction  

2 The Quantum-to-Classical Transition  
   2.1 The von Neumann Measurement Scheme  
   2.2 Collapse of the wavefunction  
   2.3 Many Worlds Interpretation  
   2.4 Decoherence  
   2.4.1 Einselection  
   2.4.2 Basis superselection  
   2.4.3 Born’s rule from Envariance  
   2.5 Spontaneous Collapse Theories  
   2.5.1 Ghirardi–Rimini–Weber Theory (GRW)  
   2.5.2 Continuous Spontaneous Localization (CSL)  

3 Spontaneous Symmetry Breaking  
   3.1 The Lieb-Mattis Model  
   3.2 Breaking SU(2) symmetry  
   3.3 Breaking time translation symmetry  

4 Simulation of Unitarity Breaking  
   4.1 The Harmonic Crystal  
   4.2 Unitarity Breaking in the Harmonic crystal  
   4.3 Symmetry breaking with a random matrix field  
   4.3.1 Generating Born’s rule  
   4.3.2 Avoiding delocalization  

5 Conclusion and Discussion
1. Introduction

Symmetry breaking is the process by which a certain asymmetrical situation is selected from a larger set of situations that together is invariant under a symmetry group. Such a selection can be effectuated by the application of some asymmetric force. This force then completely determines which asymmetric state is selected. However, even without an explicit force, it turns out that an asymmetric situation can emerge spontaneously under the right conditions. A consequence of this spontaneity is that the selection of which asymmetric outcome is realized, is not controlled and thus cannot be known beforehand.

As an example of classical symmetry breaking, we can break the (approximately) rotational symmetry of a pine tree by running it over with a bulldozer. The initial situation exhibits a rotational symmetry around the axis which is aligned with the earth’s gravitational field. In other words, the tree is invariant under the rotational transformations that make up the tree’s symmetry group. Then, by having the bulldozer exert a force on the tree, its original symmetry is lost as the tree is tumbles down to the ground. Indeed, performing the same symmetry transformation on the felled tree yields a differently oriented tree. Of all possible directions that a knocked over tree can assume, a single one was selected by the driver of our bulldozer. We could also have waited for our pine tree to grow to such a height that a gust of wind, sweeping the treetop, would achieve the same goal as the bulldozer. It then becomes less predictable what direction the final state of the tree will end up in but it is clear that this is determined by the direction of the wind. Furthermore, it is obvious that the gust of wind requires considerably less force than the bulldozer to achieve the same goal, since its centre of force lies much higher up the tree. Extrapolating, it would seem that even an infinitesimal force is able to make an infinitely tall tree come crashing down.

Obviously, trees never become infinitely tall, partially due to their increasing sensitivity to perturbations as they grow in length. For very large trees, the perturbation threshold becomes so small that no reasonable experiment could possibly measure the symmetry breaking force and we say that the symmetry of the tree is spontaneously broken.

Spontaneous symmetry breaking (SSB) is more subtle when applied to quantum systems. Hamiltonians governing such systems may exhibit a symmetry which is then automatically respected by the system’s groundstate. By the linearity of quantum theory, this groundstate may be a superposition of states. In fact, quantum systems governed by symmetric Hamiltonians exhibit groundstates that are superpositions of multiple states connected by a corresponding symmetry transformation. However, in the thermodynamic limit, this symmetry is obviously lost, judging by the absence of superposed states in the classical world. Instead, a different groundstate is found which is not even an eigenstate of the system’s Hamiltonian, with the ferromagnet as notable exception.

Spontaneous symmetry breaking is one of the prime examples of emergence in physics, the process by which collections of entities exhibit fundamentally different properties than their constituent parts separately, as is well summarized by Anderson’s ‘More is Different’ [5]. SSB has successfully been applied to explain the behaviour of macroscopic quantum systems such as superconductors, antiferromagnets and crystals.

However, SSB does not provide a description of dynamical quantum systems. It only tells us that such systems are infinitely susceptible to perturbation in the thermodynamic limit.
which causes these systems to assume an asymmetric equilibrium state. In fact, the dynamical description provided by the Schrödinger equation does not even allow for such evolutions to occur, since conventional quantum theory demands that time evolution operators be norm-preserving or non-unitary. Yet, any system that transitions from a non-classical state to a classical state necessarily undergoes non-unitary evolution, thereby breaking the time-translation symmetry which is generated by the unitarity of time evolution. This issue becomes particularly evident when considering measurements of quantum systems which generally involve interactions between macroscopic and microscopic systems, such that quantum superpositions are amplified to macroscopic proportions. The ensuing contradiction between the mathematical description of quantum mechanics through Schrödinger’s equation and our classical reality has pestered physicists since the theory’s inception.

It has been proposed by van Wezel [4,6] that the loss of time-translation symmetry may be yet another instance of SSB, and it has also been shown [7] that despite its traditionally static character, SSB can be modelled dynamically. The fact that the symmetry breaking field necessary to invoke the breaking of time-translation symmetry is of a non-unitary form means an appeal to physics that current quantum theory is unable to describe. Such phenomena may for example find their origin in the ill-definedness of time-translation in general relativity when superpositions are included in its description. The main aim of this work will be to investigate the dynamics of quantum systems under the influence of non-unitary symmetry breaking fields and to see whether the predicted breakdown of time-translation symmetry yields result that correspond to our classical expectations.

This thesis is structured as follows. In order to fully appreciate the fundamental problem of the quantum-to-classical transition, the measurement paradox will be analyzed in chapter 2. After a bit of history, two important proposed resolutions will be analyzed. The decoherence programme [1], which tries to explain the emergence of classical states as a consequence of unavoidable and ubiquitous environmental perturbation, is shown to only provide a solution to the preferred basis problem, accounting for only half of the measurement problem. Secondly, a theory of spontaneous dynamical collapse [8] is explained and its reliance on nonlinear fields is criticized. Furthermore, the line of reasoning behind the origin and justification of non-unitary randomly fluctuating fields is reviewed, as proposed by Penrose [9].

In chapter 3 of this thesis, the Lieb-Mattis antiferromagnet, an important example of SSB, is studied. The breaking of unitarity symmetry in this system as proposed by [7] is analyzed and it is found that the corresponding symmetry is only broken in special cases. It is shown that a randomly fluctuating symmetry breaking field, necessary to simulate Born’s law, does not break unitarity symmetry and instead produces very non-classical dynamics. In chapter 4, a simpler example of SSB is studied in order to verify whether the dynamics proposed in [6] break unitarity and adhere to Born’s rule. Although it appears that unitarity is indeed broken, it is shown that Born’s rule cannot emerge from the proposed symmetry breaking field. To overcome these issues, a more general field is proposed, which fluctuates randomly both in time and space. The ensuing dynamics are shown to yield Born probabilities but time-translation symmetry is not immediately broken. To resolve this, a tentative solution making use of the continuity of space is proposed and shown to break unitarity in the right order of limits. Finally, conclusions are drawn regarding the viability of the proposed mechanism and some recommendations for future research are stated.
Perhaps the most fundamental difference between classical and quantum mechanics consists in the latter formalism’s inclusion of the superposition principle in their description of states. Whereas classical propositions can be either false or true, quantum logic allows linear combinations of true and false. Similarly, we can superpose classically allowed states to form quantum states, making the resulting Hilbert space linear, in contrast to the classical state space. A corollary of this linearity is that the size of a system’s Hilbert space becomes exponentially larger than its classical state space as we increase the degrees of freedom of a system. Consequently, quantum mechanics in the thermodynamic limit does not immediately seem to coincide with our perception of the classical world since the classical portion of the Hilbert space rapidly diminishes in that limit. In spite of this mismatch, quantum mechanics is one of the most precisely tested theories to have been developed in modern times [10]. This state of affairs indicates that quantum mechanics either admits a limited range of application or requires a mechanism that effectively prohibits, renders improbable the appearance of non-classical states in the thermodynamic limit. The former option provides a quick and easy explanation for the absence of quantum effects in macroscopic systems. However, divorcing the quantum realm from the classical realm and formulating distinct laws neither explains the origin of such an arbitrary separation nor works towards the much cherished principle of unification. Even worse, it requires us to formulate a transition point from one theory to the other and to explain why the breakdown occurs at exactly that scale. The Copenhagen interpretation, as this approach is loosely known, nonetheless gained and has retained an overwhelming following among physicists to the present day. It is briefly discussed in 2.2.

A rather more elegant picture is to think of classicality as a limiting case of quantum mechanics. In line with Anderson’s ’More is different’ [5], this would mean that the collective behaviour of a quantum mechanical system fundamentally differs from that of its individual subparts. Due to the complexity of quantum systems in general and the large numbers of constituent particles present in typical macroscopic systems, straightforward calculation of such effects quickly becomes intractable. Instead, we can only hope to approximate the effects by averaging over all possibilities, as is customary in statistical physics, and hope that the outcome is representative of reality. This approach is discussed in section 2.4.

In the following, we will consider the quantum measurement process as an example of what happens when the realms of quantum and classical physics meet. By magnifying microscopic quantum states to macroscopic proportions with the use of entanglement, it will be made obvious that the quantum mechanical time evolution generated by the Schrödinger equation alone does not suffice to describe the macroscopic world. A resolution relying on additional interpretational structures will be discussed in section 2.3 and attempts to include collapse into the dynamics of quantum theory are the subject of section 2.5.
2.1 The von Neumann Measurement Scheme

The measurement problem is best understood by investigating the so-called von Neumann ideal measurement scheme. In this setup, an experimenter wishes to measure some observable $\mathcal{O}_S$ of a microscopic system $S$ by probing the system with a macroscopic measurement device $A$ which measures $\mathcal{O}_S$, thus entangling both systems. The measurement is assumed to be a unitary process because it can be described by the action of an evolution operator $\hat{U}$ on the ensemble of system and apparatus $S \otimes A$, as prescribed by the Schrödinger equation.

Before we perform the experiment, we must first prepare the apparatus in a state $|A_0\rangle$ ready for measurement, for example by having the pointer indicate zero. Then, in order to be able to read off the results of our experiment we allow the two systems to interact for a while:

$$\hat{U}(|S_i\rangle \otimes |A_0\rangle) = |S_i\rangle \otimes |A_i\rangle$$

(2.1)

Here, the time evolution operator $\hat{U}$ is generated by the interaction Hamiltonian $H_{SA}$, inducing entanglement between the apparatus and the measured quantum system. Moreover, the self-interaction Hamiltonians are taken to be zero since the measurement only takes a small amount of time. The interaction causes the resulting apparatus state to correspond exactly to the initial system state, i.e. an information transfer takes place such that the apparatus now contains a record of the system state. The von Neumann scheme thus assumes an ideal measurement in the sense that the interaction does not affect the state of the system before it is recorded. In a good measuring device, all possible outcomes can be represented by a set of states $\{|A_i\rangle\}$ that is mutually orthogonal such that each microscopic state $|S_i\rangle$ corresponds to a distinct pointer state $|A_i\rangle$, which can then be read off. Moreover, it is assumed that the correlation between $S_i$ and $A_i$ is perfect, i.e. the measurement of a state $|S_i\rangle$ will always force the apparatus to assume a corresponding state $|A_i\rangle$.

The situation becomes problematic when we want to measure a system state that is in a linear combination of states $|S_i\rangle = \sum_n a_n |s_n\rangle$. Since the time evolution operator is linear, the measurement scheme then predicts a superposition of microscopic system plus apparatus states:

$$\hat{U}(|S_i\rangle \otimes |A_0\rangle) = \sum_n a_n |s_n\rangle \otimes |A_n\rangle$$

(2.2)

What it means for the combined system to be in this entangled state has been the subject of debate for almost a century. The most literal interpretation does not agree with experiment in an obvious way since we always find the pointer of our measurement apparatus to point only at a single definite outcome instead of a variety of superimposed values. Furthermore, the decomposition of the total state into a separable system and apparatus state as found on the right hand side of (2.2) is not necessarily unique. This is true even with the restriction that both the apparatus states and system states be mutually orthogonal. Indeed, if at least two of the coefficients squared $|a_n|^2$ have equal values, as is the case with many often considered quantum states such as the Bell state, the biorthogonal decomposition theorem states that there must exist a distinct decomposition of the final total state:

$$\sum_n a_n |s_n\rangle \otimes |A_n\rangle = \sum_n a'_n |s'_n\rangle \otimes |A'_n\rangle$$

(2.3)

Whereas the left hand side of (2.3) tacitly implies that the observable $\hat{O} = \sum_n a_n |s_n\rangle \langle s_n|$ has been measured, a different observable $\hat{O}' = \sum_n a'_n |s'_n\rangle \langle s'_n|$ seems to have been measured on the right side. The fact that we cannot derive with certainty from the final state
of a measurement which observable has been measured appears to conflict with the assumption that we can decide which observable we want to measure by picking the appropriate apparatus. Furthermore, the pairs $O$ and $O'$ do not in general commute and thus cannot be measured simultaneously without a loss in precision. However, in the absence of a mechanism to decide which basis was preferred, the state on the left hand side of (2.2) must contain all information regarding both possibilities, in seeming contradiction with the non-commutativity of the observables [11]. Thus, even if we were to explain the disappearance of all but one of the superimposed states after measurement, the question remains why that particular basis is preferred over another.

Together, the problem of definite outcomes (why do we never measure superposed values) and the preferred basis problem (why do measurements return outcomes expressed in one specific basis rather than any other) constitute the measurement problem. Instead of trying to interpret the macroscopic entangled state found in (2.2), one might ask whether the assumptions made about the measuring process are valid. In analyzing an overly idealized situation, it could be argued that the physics responsible for the collapse of the wavefunction do not come into play. If the assumed perfect correlation between quantum state and apparatus state or the orthogonality of apparatus states turns out to hide certain physics, the measurement problem might be off the table. In addition, the measurement scheme assumes the possibility of preparing the apparatus into a ready-to-measure state $|A_0\rangle$ and does not take into account possible interactions between the environment and the apparatus.

The decoherence program, as will be discussed in section 2.4, builds exactly on the idea that in opposition to classical systems, quantum systems are inherently fragile with respect to interactions with the environment. However, it can be shown [12] that even under severely weakened assumptions, the contradiction between experimental results and theoretical description persists.
2.2 Collapse of the wavefunction

For a long time, the contradiction was considered provisorically solved by postulating a non-unitary process which transforms the sum on the right side of (2.2) to one of its constituent terms with a probability proportional to its norm upon measurement:

\[
\sum_{n} a_n |s_n\rangle \otimes |A_n\rangle \xrightarrow{\text{Non-unitary}} |s_m\rangle \otimes |A_m\rangle,
\]

with probability \(|a_m|^2\) (2.4)

The addition of these postulates to the mathematical backbone of the theory then represents the starting point of an ill-defined collection of interpretations collectively known as the Copenhagen interpretation. There are many ways to formulate the ensemble of postulates and even their number may vary but for the present purposes we shall use an insightful version consisting of five axioms based on refs. [13–15]:

(i) The quantum state of a system \(S\) is represented by a vector \(|\psi\rangle\) in the system’s Hilbert space \(\mathcal{H}_S\)

(ii) Quantum evolutions are unitary (e.g., generated by the Schrödinger equation)

(iii) Immediate repetition of a measurement yields the same outcome

(iv) Measurement outcomes are restricted to an orthonormal set \(\{|s_k\rangle\}\) of eigenstates of the measured observable

(v) The probability of finding a given outcome is \(p_k = |\langle s_k | \psi \rangle|^2\), where \(|\psi\rangle\) denotes the preexisting state of the system,

The first two axioms contain the mathematical structure of quantum theory. (i) implies linearity and thus the superposition principle while (ii) incorporates time into the framework. Postulate (iii) derives from the classical common sense that immediately after measurement the state of a system will not have changed and connects the mathematics of the previous postulates to a statement pertaining to reality. Note that immediate repetitions are not necessarily experimentally feasible and that most realistic measurements indeed cause demolition of the measured states, as will be discussed later on. However, the postulate is necessary to introduce the fundamental notion of predictability into the theory without which the concept state loses its meaning.

The last two postulates are the source of the continued debate surrounding the foundations of quantum mechanics. Postulate (iv) implicitly breaks a preexisting symmetry of states by allowing only a very specific selection of states to be realized after measurement. It is this symmetry breaking that should be explained instead of postulated in the eyes of most physicists. The last postulate is immediately recognized as Born’s rule, which is equally undeserving of its status as postulate. Rather, we would like these two postulates to emerge from the first three postulates.

The consequence of the addition of these postulates is not that macroscopic superpositions are forbidden, but that they can never be observed by definition, as they are destroyed by measurement. The exact mechanism that underlies this sudden collapse of the wavefunction is left unspecified so that it remains unclear what exactly constitutes measurement and whether it is an instantaneous or continuous process.
The Copenhagen approach thus relies on two distinct descriptions of time evolution, without a prescription as to when each one is to apply. Quantum states evolve unitarily and deterministically according to the Schrödinger equation while unobserved, whereas the act of measurement enforces a non-unitary and probabilistic process that projects the quantum state vector onto the eigenspace associated with the eigenvalue of the quantity measured. von Neumann, who first formulated the projection postulate described above, believed the human consciousness to play a part in the shift of time evolution mechanism. However, opening the door to subjective influence prompts a host of philosophical issues and paradoxical gedanken experiments because it allows us to conscientiously divorce pre-measurement (i.e. the state of the system before reading off the results (2.2)) from actual measurement (the state of the system after reading off). As a result, we would be able to construct macroscopical superpositions that only cease to exist when human consciousness becomes involved, so that one can argue that the moon only exists when we perceive it. This solipsistic state of affairs is not a desirable feature for a science that strives to describe an observer-independent reality and so for many decades physicists have looked on in search for (more) objective explanations for wavefunction collapse. Pragmatically, the orthodox interpretation predicts the quantum phenomena that we are able to measure extremely well but advances in the experimental wing of quantum physics already allow for the creation of superpositions of increasingly large systems. Without an exact division between the classical and quantum regimes, the Copenhagen interpretation is destined to be replaced by a more fundamental mechanism elucidating the disappearance of quantum superpositions.
2.3 Many Worlds Interpretation

Another approach to interpreting the macroscopic superposition is to insist on the validity of the outcome of the measurement process in (2.2) and place those state vectors we fail to measure outside of our personally perceived universe. With every measurement, the universe then branches up into a multitude of universes, in each of which a possible outcome is realized. No secondary mechanism to force the superposition to become classical is needed since from an external perspective the superposition is still intact and remains that way. Where the orthodox interpretation tries to account for the fact that superpositions are never observed, MWI proposes that the observer becomes part of the superposition. Interpreted as such, the Schrödinger equation does not tell us we should observe an indeterminate outcome as in (2.2), but instead represents $n$ observers measuring $n$ distinct outcomes. The projection postulate can then be removed since there is no external observer to set the process of collapse in motion in the first place but more importantly because all constituent states of the total wavefunction $|\Psi\rangle$ are realized. This total wavefunction neatly evolves according to the Schrödinger equation, at first sight minimizing the amount of postulates necessary to make the theory complete.

Whereas the coefficients corresponding to state vectors are interpreted as the probability of measuring a certain outcome within the orthodox framework (the Born rule), their role in a Many Worlds scenario is not immediately clear. It can be argued that they represent the relative frequencies with which a ‘world’ containing a corresponding state branches off but such ad hoc solutions are ontologically undesirable and say nothing about the origin of the probabilistic behaviour described by the Born rule.

In addition to the general issues that stem from the preferred basis problem as discussed in section 2.1, the validity of the MWI scenario hinges even more on its resolution. While the orthodox interpretation simply avoids the measurement problem by postulating an unspecified mechanism that exactly fills the gap between our understanding of quantum mechanics and everyday observation, MWI has to face the music and requires a formal solution to the preferred basis problem. Fortunately, the first physicist to separate the preferred basis problem and the problem of definite outcomes, Wojciech Zurek [1], also came up with a solution in observing that the symmetry between different bases is effectively broken by environmental interaction. This will be discussed in the next section.
2.4 Decoherence

Although some still insist on the special role measurement has acquired within the framework of the orthodox interpretation, the ontological and epistemological ramifications of ascribing the occurrence of objective phenomena to subjective measurement produce a hard pill to swallow for the majority of physicists [16]. Instead, it makes more sense think of the act of measurement as nothing but an ordinary quantum interaction, the likes of which have been extensively studied. Then, all interactions between the quantum system, the measurement apparatus and everything else potentially become equally important. This implies that we must be very careful with the idealization of the measurement process and reevaluate the potential effects of ‘noise’ generated by the environment.

Ever since the scientific revolution, scientists have had to devise methods in order to cope with the inevitable noise that plagued their experiments. By considering idealized situations, while minimizing noise and neglecting negligible effects, classical theory and experiment were often successfully reconciled. The situation is very different in the quantum case, as it is notoriously hard to shield any quantum system from its immediate environment, inevitably leading to unwanted correlations in the form of quantum entanglement. This entanglement causes the ensemble of system and environment to become inseperable such that the system by itself can no longer be exactly described without considering its environment. In a sense, the information contained within the quantum system leaks into the environment upon interaction, and although the state vector describing the collection of subsystems evolves unitarily, the same is not necessarily true of the subsystems individually.

2.4.1 Einselection

In order to take a shot at quantifying these effects, we will have to be more concrete in defining the environment. The environment of a quantum state is anything but static and includes at most all degrees of freedom contained in its lightcone. Once these degrees of freedom start to interact with the quantum system, all hopes of observing pure quantum effects vanish.

The environment’s inherently uncontrollable and chaotic character can obviously not be exactly simulated but it could prove worthwhile to approximate its effect on a quantum measurement by averaging over all possible interactions an environment $|E\rangle$ can induce. By defining the probabilities $p_n$ that some subset of the environment $|\epsilon_n\rangle$ interacts, we may treat the average effective environment as a quantum mechanical object. However, the state vector formalism is unable to describe statistical ensembles of pure quantum states such as a particular configuration of the environment. Fortunately, we may use the density matrix $\rho_E = \sum_n p_n |\epsilon_n\rangle \langle \epsilon_n|$ to represent all environmental state vectors and their statistical weights at once. In this representation, a particular environment state $|\epsilon_n\rangle$ really consists of a number of individual microstates $|e_i\rangle$ describing the states of all constituent particles. For the sake of simplicity, we will assume that no interaction between these environmental microstates occurs, or equivalently that the interaction Hamiltonian $H_{EE}$ is zero and $|\epsilon_n\rangle$ is a product state of microstates $|e_i\rangle$.

We are now able to revisit the von Neumann measurement scheme with the inclusion of the environment. At the start of the experiment, both $|S\rangle$ and $|A\rangle$ are assumed to be prepared
as pure states expressed in matrix form, and since no interaction has occurred yet, the total system is still separable. Furthermore, in order to keep things tractable, we set all self-interaction Hamiltonians to zero and we assume that the environment does not come into play before the apparatus has finished recording the quantum state through interaction.

\[
\hat{U}_{AE} [\hat{U}_{SA} (\rho_S \otimes \rho_A) \hat{U}_{SA}^* \otimes \rho_E] \hat{U}_{AE}^* = \rho_{SAE} = \sum_{m,n} a_m^* a_n \langle s_m | A_m \rangle \langle s_n | A_n | \epsilon_n \rangle (2.5)
\]

Here, \(\rho_{SAE}\) denotes the state of the entire system after subsystems have been allowed to interact for a while. However, since we are interested in finding expectation values of the quantum system, we do not care about those degrees of freedom of the environment \(|E\rangle\) that have not interacted with \(|SA\rangle\). By including into \(|SA\rangle\) those degrees of freedom of \(|E\rangle\) that have interacted, the separation between the systems of interest remains clear. However, we would like to find an expression for \(\rho_{SA}\), denoting the post-measurement system-apparatus state, since the relevant expectation values should not depend on the idle environmental degrees of freedom of the environment. To get rid of this portion of the environment, we may use the partial trace operation. This can be seen by considering a measurement of an observable \(\hat{O}_{SA}\) on \(|SA\rangle\). By definition, its expectation value is found by taking the full trace over \(|SA\rangle\):

\[
\langle \hat{O}_{SA} \rangle = \text{Tr}_{SA} (\rho_{SA} \hat{O}_{SA}) = \text{Tr}_{SA} (\rho_{SA} \hat{O}_{SA} \otimes I_E) = \text{Tr}_{SAE} (\rho_{SAE} (\hat{O}_{SA} \otimes \hat{I}_E)) (2.6)
\]

From this it is easily seen that \(\rho_{SA} = \text{Tr}_E (\rho_{SAE})\). However, since we have defined the environment to be a statistical mixture while both the system and apparatus state were prepared as pure states, \(\rho_{SAE}\) also represents a mixed state. Then, if we are to define \(\rho_{SA}\) as above, this will have profound interpretational consequences. In effect, the system \(|SA\rangle\) has evolved from a pure state into a statistical mixture, meaning that any conclusion drawn about the final state \(\rho_{SA}\), only applies to ensemble-averaged quantities. Furthermore, the justification of the partial trace operation as derived above tacitly presupposes the Born rule by identifying the expectation value of an observable as the trace over that observable multiplied by the appropriate density matrix. Keeping this in mind, performing the partial trace then yields:

\[
\rho_{SA} = \sum_{m,n} a_m^* a_n \langle s_m | A_m \rangle \langle s_n | A_n | \epsilon_n \rangle (2.7)
\]

So far, we have not assumed anything about the orthogonality of the different configurations of the environment, but clearly it constitutes a measure for the off-diagonal terms of the density matrix, representing non-classical states. Obviously, the exact value of \(\langle \epsilon_n | \epsilon_m \rangle\) depends on the distribution of the environment as well as the form of the interaction Hamiltonians, but we can gain some intuition by tuning the size of the environment.

If, on top of the environment we have already considered, we add the exact same environment \(N\) times and let these interact only with \(|SA\rangle\), while preventing entanglement between them, our new environment can be written as a separable state \(|\epsilon'_n\rangle = \bigotimes_N |\epsilon_n\rangle\).

The orthogonality of these new environments can then be expressed as follows:

\[
\langle \epsilon_n | \epsilon_n' \rangle = (\cos \theta_{nm})^N = (\cos \theta_{nm})^N (2.8)
\]
Here cos(θ_{nm}) cannot be equal to one, since that would break the one-to-one correspondence between system and apparatus states, meaning that the apparatus is not functioning properly, counter to what we have assumed. Eq. (2.8) then suggests that for an increasing volume of environmental states, the non-classical states vanish, provided that the environment does not contain entangled states. Since the environment grows exponentially with time, we see that the superposed states of which |SA⟩ consists, quickly lose their coherence resulting in a density matrix with only classical states. Thus, the environment induces decoherence of the initial quantum state such that only classical states are selected to survive. This phenomenon was named ‘einselection’ after Environment-Induced Selection by Zurek [1]. An indication of the extreme speed with which this decoherence process occurs can be deduced from Table (1). Although we have assumed a toy environment consisting of identical particles that are invulnerable to entanglement, explicit calculations of more realistic environments show a similar tendency of exponential suppression of coherence [1,17,18]. It must be stressed that coherence is only lost locally since the reduced density matrix only bears information about the fate of the system plus apparatus states |SA⟩. Decoherence effects cannot be observed in the global density matrix ρ_{SAE}.

Table 1: The localization rate in the position basis with units of cm^{-2}s^{-1} is shown for mesoscopic objects of differing length a as induced by different environments. This rate is inversely proportional to the decoherence time, denoting the time it takes the particle to localize to up to one wavelength. Taken from [18].

<table>
<thead>
<tr>
<th>Environment</th>
<th>a = 10^{-3} cm dust particle</th>
<th>a = 10^{-5} cm dust particle</th>
<th>a = 10^{-6} cm large molecule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cosmic background radiation</td>
<td>10^6</td>
<td>10^{-6}</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>300 K photons</td>
<td>10^{19}</td>
<td>10^{12}</td>
<td>10^{6}</td>
</tr>
<tr>
<td>Sunlight (on earth)</td>
<td>10^{21}</td>
<td>10^{17}</td>
<td>10^{13}</td>
</tr>
<tr>
<td>Air molecules</td>
<td>10^{36}</td>
<td>10^{32}</td>
<td>10^{30}</td>
</tr>
<tr>
<td>Laboratory vacuum</td>
<td>10^{23}</td>
<td>10^{19}</td>
<td>10^{17}</td>
</tr>
</tbody>
</table>

It will prove useful for the following to note that the off-diagonal entries of the density matrix never quite reach zero, signifying a depart from the orthodox way of thinking about classical states. Under the Copenhagen interpretation, a measurement can only ever yield a definite value if the system under observation is in an eigenstate of the measured observable [13]. This eigenstate-eigenvalue link (E-E link) is not a necessary element of quantum mechanics and must be weakened if decoherence (or any other dynamical model) is to explain the emergence of the seemingly definite states we experience in the macroscopic world [19,20]. It has been argued [21] that the establishment of an approximate or fuzzy link induces an anomaly since the wavefunction describing a collection of separable fuzzy objects would then suffer from even more fuzziness by amplification. The anomaly can be resolved but has some consequences for the interpretation of probability. For a discussion, see [22]. More importantly, for measurements of observables with continuous spectra such as position and momentum, the E-E link requires that the system ends up in an eigenstate |x_i⟩ of ˆX post-measurement, such that ⟨x|x_i⟩ = 1. However, such inner products are only well-defined if the eigenvalue spectrum is discrete. Thus, in the position basis, it is clear that such states are not allowed.
mathematically and should thus be considered as ‘improper eigenstates’ [23]. Instead, it makes more sense to define a localized state in a continuous parameter basis as having a probability distribution, the bulk of which is contained in some finite region Σ. The issue of defining localization more precisely will become important in section 4.3.

Summing up, we have seen that the inclusion of the environment into the measurement scheme leads, under certain assumptions about that environment and with the use of the partial trace operation, to a mixed state representing exactly those states we would expect to find after an ensemble of measurements. This is true regardless of whether the initial system is in a pure or in a mixed state. Additionally, statistical mixtures are not uniquely described in the density matrix formalism. Consider for example two $N$-particle statistical mixtures of a two state system. Mixture $A$ consists of equal numbers of $|0\rangle$ and $|1\rangle$ states, while mixture $B$ consists of equal numbers of $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ states. Evidently, the density matrix describing the mixture of classical states $A$ is indistinguishable from the quantum mixture $B$. The absence of a one-to-one correspondence between statistical ensembles and their density matrix representation means that einselection cannot solve the measurement problem for singular events, unless combined with an interpretational structure, as we will see below.

The fact that taking the partial trace makes it impossible to distinguish between both situations, signals that we have lost a valuable piece of information somewhere along the way. More specifically, the reduced density matrix represents an improper mixture [24], meaning that the statistics included in it relate only to the subpart $|SA\rangle$ of the non-separable state $|SAE\rangle$. In other words, we have first allowed the environment to monitor $|SA\rangle$ through decoherence, but in the end we are forced to pretend that no entanglement between these systems exists, as improper mixtures do not convey this information.

The reason this rather subtle difference concerns us is has to do with interpretations. Decoherence by itself does not constitute an interpretation, nor is it an adaptation of quantum mechanics. Instead, it is the consequence of taking in account the environment and recognizing that closed quantum systems do not exist. If we had been able to derive a pure state density matrix $\rho_{SA}$ with vanishing off-diagonal such as the one of eq. (2.9), we would now have a convincing solution to the problem of definite outcomes and this thesis would end here.

Alternatively, a proper mixture as a final state would still be reconcilable with reality if we were to adapt our interpretation accordingly. A proper mixture indicates our ignorance with respect to a certain state, as a result of some statistical process in preparing it. Ending up with a proper mixture despite an initial pure state seems problematic unless we deny that quantum mechanics is able to describe pure states and instead is a theory of ensembles. This statistical interpretation [25, 26] finds its roots in the observation that quantum mechanics has essentially only been verified by considering the outcomes of many identical experiments and noticing the strikingly accurate match with Born’s rule.

However, the reality is that environmentally induced decoherence provides us with an improper mixture. It has been argued [27, 28] that this type of mixed state cannot simply be interpreted as a state which we are forced to admit ignorance about. Specifically,

To identify improper mixtures with proper ones is definitely illegitimate, at least whenever the difference between the two [...] has consequences that are in principle observable. [29]

Thus, a mechanism similar to the collapse postulate of section (2.2) still needs to be invoked in order to convert the mixed state density matrix into a pure state density matrix. For a
two-state system, this is to say that the following transformation cannot be derived from
the effects of environmental decoherence alone:
\[ \rho_{\text{red}} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad p = |\alpha|^2 \]
Furthermore, the interpretation of the entries of the reduced density matrix as measurement
outcome probabilities is at this point still presupposed [30], since the transformation from
density matrix to expectation value relies on Born’s rule. Thus, even if all the aforementioned
objections were to be refuted, Born’s rule remains a postulate and definite outcomes do not
emerge through the inclusion of the environment. However, as we shall see in section 2.4.3,
environmental decoherence can be shown to induce Born’s rule. Before that, we will examine
the effect of decoherence on the preferred basis problem.

2.4.2 Basis superselection

As we have seen, the many worlds interpretation of section (2.3) was conceived especially
to avoid the problem of definite outcomes, but instead faces considerable difficulty with the
origin of Born’s rule and the more generally applying preferred basis problem. Let us see
whether the involvement of environmental interaction is able to mitigate these issues.
At first glance, the situation is considerably altered by the addition of |\mathcal{E}\rangle in equation (2.3).
The biorthogonal decomposition theory no longer applies, and instead the triorthogonal
uniqueness theorem conveniently predicts that \textit{any} expansion of a total state vector into
a sum over a product of three separable states, each living in their own Hilbert space, is
unique. This theorem does not immediately solve the problem, since the theorem does not
assure the existence of such a decomposition for every total state [31]. Additionally, it is
not obvious why an uncontrolled and seemingly random environment consistently produces
the same preferred basis of pointer states seen in the macroscopic world.

Essentially, the idea is that a specific set of pointer states \{|a_n\rangle\} is robust under inter-
action with the environment while all other possible bases of states are rapidly suppressed.
The deciding factor in this superselection cannot be the exact configuration of the envi-
ronment, since that would imply the emergence of a different preferred basis after each
experiment. Moreover, even if the selection were dependent on environmental specifics, we
would run into issues similar to those of the previous section, because we would be forced
to use a mixed state to describe the environment. Instead, we must look to the form of
the different interaction Hamiltonians and their relative strength. These Hamiltonians are a
representation of the potential generated by forces arising from the presence of particles [11].
So far, we have considered the combined apparatus and system state |S,A\rangle to be vulnerable
to environmental interaction without specifying which of the two couples most to the envi-
ronment, i.e. the relation between \(H_{SE}\) and \(H_{AE}\). However, as was seen in Table (1), the
localization rate is proportional to the size of the object under environmental influence, so
we can assume that \(H_{SE} \ll H_{AE}\).
For the sake of simplicity, we will assume that we can measure (i.e. copy) the state of |S|
with the $|\mathcal{A}\rangle$ before the environment starts interacting with the apparatus. Then, after the apparatus measurement is performed by bringing $|\mathcal{S}\rangle$ and $|\mathcal{A}\rangle$ into interaction, the environment continually measures the apparatus, which is now in a macroscopic superposition and without a preferred basis.

Now, one can distinguish between measurements that alter or destroy the current state of a quantum system before it can be recorded by an apparatus and measurements that leave the system intact. This implies that even if two measurements of the former type are performed within an arbitrarily small time interval, the two outcomes will very likely differ. It was shown by Braginsky [32] and others that the distinction is captured by the commutativity between the interaction Hamiltonian $H_{S\mathcal{A}}$ and the observable $O_S$ that the apparatus is set up to measure. If these two operators commute, we have performed a non-demolition experiment. Otherwise, the state of the system is appreciably altered and correlations between the outcome recorded by the apparatus and the initial quantum state are lost.

In the same vein, the environment will destroy those correlations that do not commute with the interaction Hamiltonian $H_{AE}$, while performing a non-demolition measurement in the basis for which both $H_{AE}$ and the observed quantity $O_A$ are diagonal. Thus, under the assumption that the apparatus faithfully copies the state $|S\rangle$, or equivalently that $[O_S, H_{S\mathcal{A}}] = 0$, the environment is subsequently able to select a certain preferred basis by destroying all correlations between system and apparatus states that cannot be expressed as eigenstates of $O_A$.

The resulting Russian doll situation in which each subsystem is measured by an even larger subsystem then prevents us from having to postulate a specific universally preferred basis. Instead, the form of the interaction Hamiltonian between subsystem and system determines which correlations between subsystem and subsystem are retained. We are now only left with the task of characterizing the general form of interaction Hamiltonians. A first guess from everyday experience is that environmental interaction Hamiltonians must be a function of position operators $X_n = |x_n\rangle \langle x_n|$. Indeed, it seems a natural choice since the forces involved in the interaction between the chaotic mess of particles that makes up the environment and a quantum system are typically functions of position. More careful and explicit analysis has shown that in cases where the self-interaction Hamiltonian represents a kinetic term which commutes with the momentum basis couples more strongly then the interaction Hamiltonian, a type of 'self-measurement' is carried out. This situation typically occurs in the microscopic realm, so that the emergence of a preferred basis in momentum space is allowed [33].

This solution to the preferred basis problem is actually quite elegant in that it neither admits to an ad hoc solution specifically construed to fit the data, nor constitutes an interpretation that is impossible to corroborate. The only possible bases that we could possibly encounter this way are the ones in terms of quantities that are reflected by the laws of nature, and not exotic combinations of them. On the other hand, it is not straightforward to find the exact form of interaction Hamiltonians for realistic situations. Given that the decoherence program is a relatively recent development, there is ample room for further investigation.
2.4.3  Born’s rule from Envariance

There is only one strong objection left standing against MWI besides the ontological unease it causes and the general problem of lacking falsifiability that all interpretations suffer from. Namely, how do the Born probabilities emerge, and what is their significance in the MWI framework? Additionally, one of the reasons that the partial trace operation of section (2.4) cannot be trusted is due to its presupposition of what it tries prove: Born’s rule. By once again considering environmental interaction as an inherent part of any time evolution, recent work by Zurek [34] has shown that under a number of assumptions, Born’s rule emerges naturally from quantum mechanics.

As said, any non-circular derivation of the Born rule must avoid utilizing reduced density matrices. Instead, this time we will not be considering unknown and uncontrolled distributions of environmental states, but consider a class of unitary evolutions that alter the state of subsystems while preserving the total state. These ‘envariant’ transformations $\hat{u}_X$ are defined as follows:

$$\hat{U}_A \hat{U}_B |\psi_{AB}\rangle = (\hat{u}_A \otimes \hat{I}_B)(\hat{I}_A \otimes \hat{u}_B)|\psi_{AB}\rangle = \hat{I}_{AB} |\psi_{AB}\rangle = |\psi_{AB}\rangle \quad (2.10)$$

It is here that the counterintuitive nature of quantum entanglement becomes particularly evident. Subsystems $A$ and $B$ are both transformed in such a way as to cancel their combined effect on the total system $|\psi_{AB}\rangle$. Classically, this effect can only be reconstructed if $B = \bar{A}$, i.e. if the two systems combined cover all state space. For example, after applying a transformation that swaps the positions of Mars and Venus, the only subsequent transformation that does not affect Mars and Venus again but still restores the initial state, must affect the complement of Mars and Venus. However, in the quantum case, this symmetry exists locally by virtue of entanglement.

To see how Born’s rule emerges from this symmetry, we now let both subsystems be two-level systems with equal coefficients and let the unitaries $\hat{u}_X$ enforce a swap between states of subsystem $X$. Full knowledge of the pure state $|\psi_{AB}\rangle$ is assumed, from which we are able to derive the state of the subsystems:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|a_1\rangle |b_1\rangle + |a_2\rangle |b_2\rangle) \quad (2.11)$$

$$\hat{u}_X = |x_1\rangle \langle x_2| + |x_2\rangle \langle x_1| \quad (2.12)$$

Clearly, any swap on system $A$ is countered by a subsequent swap on system $B$, so that $\hat{u}_X$ is indeed an envariant transformation. In contrast to the classical situation, we could add extra degrees of freedom into the state of eq. (2.11) without affecting the state’s envariance. If we were now to perform measurements of some observable of subsystem $A$ and $B$ consecutively while forbidding both external interaction and unitary evolution in between measurements, we would find that the outcomes would correspond with probability 1. This perfect correlation is a defining property of entanglement and introduces probability without referring to Born’s law. If we now assume that probabilities are only determined by the coefficients of the initial wavefunction and independent of the state vector itself or hidden variables, it must be that a measurement on subsystem $A$ yields equal probabilities to a measurement on subsystem $B$. We can write this in the following way:
\[
p(a_1|\hat{O}_A|\psi_{AB}) = p(b_1|\hat{O}_B|\psi_{AB}) \tag{2.13}
\]

That is, the chance of finding \(|a_1\rangle\) after a measurement \(\hat{O}_A\) on the combined system is equal to finding \(|b_1\rangle\) after a measurement \(\hat{O}_B\). Now, a swap on subsystem \(A\) should not change the observable properties of subsystem \(B\) and vice versa, since only the identity operator is applied to the latter. On the other hand, the correlations between subsystems are swapped. Since we have assumed that probabilities are only dependent on the wavefunction’s prefactors, this means that the probabilities associated to the swapped states also swap, such that the following identities hold:

\[
p(a_1|\hat{O}_A(\hat{I}_A \otimes \hat{u}_B)|\psi_{AB}) = p(a_1|\hat{O}_A|\psi_{AB}) \tag{2.14a}
\]

\[
= p(b_2|\hat{O}_B|\psi_{AB}) \tag{2.14b}
\]

Furthermore, we know from equation (2.10) that a subsequent counterswap restores the initial state, so that the probabilities are necessarily preserved:

\[
p(a_1|\hat{O}_A(\hat{u}_A \otimes \hat{I}_B)(\hat{I}_A \otimes \hat{u}_B)|\psi_{AB}) = p(a_1|\hat{O}_A|\psi_{AB}) \tag{2.15}
\]

It then follows that we can remove the rightmost operation on \(|\psi_{AB}\rangle\) on the LHS of this expression, since it only applies on subsystem \(B\), whereas the probabilities pertain only to subsystem \(A\):

\[
p(a_1|\hat{O}_A(\hat{u}_A \otimes \hat{I}_B)|\psi_{AB}) = p(a_1|\hat{O}_A|\psi_{AB}) \tag{2.16}
\]

Together, these identities can be massaged into the following expression, revealing that the probabilities of finding either of the eigenvalues upon a measurement are equal:

\[
p(a_1|\hat{O}_A|\psi_{AB}) = p(a_2|\hat{O}_A|\psi_{AB}) \tag{2.17}
\]

Without invoking Born’s rule, we have nonetheless obtained a prediction for the probabilities of finding the eigenvalues of \(A\) upon measurement. The result seems rather trivial since we have considered two state system with equal coefficients, but the result can be reproduced for an \(N\)-state system with equal coefficients. Then, recalling that the subdivision into systems is merely an arbitrary choice, one can construct unequal coefficients. Since the central premise of decoherence is that quantum systems are open, an environment is always available to entangle with, enforcing the emergence of probabilities as described above.

There are a number of assumptions implicit in the derivation of Born’s rule, the validity of some of which has been the subject of debate \[35\]. Specifically, the assertion that probabilities associated to one subsystem are invariant under envariant transformations, implicit in equation (2.14b), has been claimed to be unjustified. To see this, recall that envariant transformations \(\hat{u}_X\) cannot be observed locally, as can be readily seen from equation (2.16). Only the combined system \(AB\) is affected. Although the probability associated to state \(|a_1\rangle\) is not dependent on whether it is entangled with \(|b_1\rangle\) or \(|b_2\rangle\), it does not follow directly from envariance that an envariant transformation \(\hat{u}_B\), besides swapping entangle-ments, leaves probabilities intact. After all, an envariant transformation does change the global state, which contains ‘all [information] that is needed (and all that is available) to determine the state of the subsystem \([\mathcal{A}]\)’ \[36\]. In addition, besides assuming that coefficients completely determine the probability distribution, the more fundamental assumption has
been made that probability exists to begin with. Both these assumptions have encountered resistance [37], but we will not pursue these issues here. Finally, it is assumed that there exists a preferred basis, but the considerations of the last section have convinced us of that. Despite these objections and others relating to the generality of the approach, the theory of envariance is unique in its claimed ability to derive Born’s rule from quantum physics in its original form [38].

In this section, we have seen that the decoherence program succeeds quite satisfyingly in providing an mechanism for the emergence of a preferred basis. On the other hand, a subsequent mechanism enforcing the effective ‘collapse’ into classically allowed states, has been shown to suffer from certain interpretational issues provoked by the misplaced use of the partial trace operation. Furthermore, einselection seems to admit a circularity in its derivation by appealing to Born’s law. This circularity is arguably redeemed by the law’s derivation from envariance, but as of yet it is unclear whether this recent development will permit alternative derivations of einselection that do not necessitate the reduced density matrix formalism.

The incorporation of einselection into the MWI interpretation removes the strongest objection it faces, and with envariance providing the origin of probabilities, it would seem we do not need to look further. Furthermore, since no collapse is required to take place, the issues surrounding the derivation of environment-induced collapse do not affect its credibility. However, the acceptance of einselection also introduces new issues. The prediction that the position basis will predominantly be preferred, given the ubiquity of position-dependent interactions in macroscopic systems, entails that all state vectors included in a wavefunction are realized in different branches of the universe. However, it is not at all certain that space is quantized, such that branching is no longer well defined since the amount of projection operators is then no longer denumerable. Thus, pending the successful quantization of space, the consequences of a position measurement on a Gaussian wavepacket in position space are not clear in a no-collapse scenario.

Besides all objections that can be raised against the validity of the various environment-induced phenomena and the technical difficulties that persist when incorporated into the MWI framework, it is perhaps the ontological leap of faith required that is most unsatisfactory. In combination with the seeming impossibility of verification, implicit in any ‘interpretation’ of quantum mechanics, it would only seem reasonable to look on for a more fundamental theory. However, several decades of intense research have not proven enough to conceive of a more fundamental framework in which to solve the measurement problem. Therefore, it has been proposed to look for effective theories that assume the existence of some very small non-unitary field, either generated by the gravitational quantum effects or ‘new physics’ to induce perturbations to the Schrödinger equation. One of such proposals is discussed in the next section.
2.5 Spontaneous Collapse Theories

Given the accuracy of conventional quantum mechanics, we have to be very careful with modifications to the theory, since they may cause flagrant violations of experimental observation. Besides enforcing definite outcomes, perturbations added to the Schrödinger equation may only produce deviations that lie beyond the experimentally accessible regime. Secondly, a mechanism must be introduced in order to explain the emergence of Born’s law. Evidently, any modification to quantum theory should not conflict with fundamental principles such as conservation of energy.

Several authors have suggested the incompatibility between gravitation and quantum mechanics to be related to the macro-objectification problem [39, 40]. One specific argument by Penrose [9] consists in the realization that a massive superposition in position space entails the superposition of different spacetime metrics, assuming that gravitational effects can be implemented straightforwardly into a description of quantum mechanics. Although there will exist a mapping between sections of these different spacetimes, distinct points in each spacetime cannot be said to correspond exactly. As a consequence, the notion of time translation is no longer well-defined, leading to an incompatibility between the quantum-mechanical time evolution of each spacetime. More specifically, in a quantumgravitational picture, a stationary state $|\phi\rangle$ will always entangle with a gravitational field $|G\phi\rangle$ such that the total state is an eigenstate of the time-translation operator $\hat{T}$, with the eigenstates representing the energy of the total state. For superposition of two stationary states

$$|\Psi\rangle = \alpha |\phi\rangle |G\phi\rangle + \beta |\chi\rangle |G\chi\rangle$$

(2.18)

such a time-translation operator cannot be straightforwardly obtained, leading to an uncertainty $E_{\Delta}$ in the energy eigenvalue of the system, which is shown to be a function of its mass density. Without a well-defined time-translation operator, the notion of stationarity loses its meaning. It is not known what exactly this implies for the stability of superpositions, but Penrose has argued that they become unstable after a finite time $\tau \propto E_{\Delta}^{-1}$, as a result.

Other proposals to clarify this ‘problem of time’ have in general focused on quantizing general relativity with the aim of constructing a unified theory of quantum gravity. The general approach of spontaneous collapse theories is instead to include gravitational effects into quantum theory with the specific aim to shape the time evolution operator in such a way as to enforce effective wavefunction collapse under the right circumstances. These approaches generally have a phenomenological character and do not attempt to provide a fundamental description of nature. Furthermore, in order to describe single events rather than the statistical mixtures we encountered in section (2.4), the quantum state reduction will be expressed in the state vector formalism.

2.5.1 Ghirardi—Rimini—Weber Theory (GRW)

The first attempts to formulate a modified quantum theory including a collapse mechanism were put forward by Ghirardi et al. [2], based on a previous work on the exponential decay in quantum systems and work by Pearle [3] on the addition of nonlinear terms to the Schrödinger equation. In essence, the state vector $|\psi\rangle$ undergoes a non-unitary process that occurs at random times, such that $|\psi\rangle$ is transformed in the following way:
\[ |\psi\rangle \rightarrow \frac{\hat{P}_{x,p} |\psi\rangle}{\|\hat{P}_{x,p} |\psi\rangle\|} \]

where \( \hat{P}_{x,p} = \exp \left( -\frac{(x - x')^2}{\sigma} \right) \) (2.19)

This transformation multiplies \(|\psi\rangle\) by a Gaussian function of width \(\sigma\), as can be seen in Figure 2.1. GRW postulates the position basis to be universally preferred which, as we have seen, is partially confirmed by einselection. This preference is originally a postulate of GRW, but can be generalized to any type of basis, as we will see in the next section. Since the einselection of preferred basis, as discussed in section 2.4.2, occurs on extremely short timescales, a combination of the two theories is conceivable. Then, decoherence first selects an eigenbasis as determined by the environment, after which spontaneous collapse occurs as described by GRW. For the sake of simplicity, we will here treat collapse in position basis.

\[ < \psi(x) | \psi(x) > \]
\[ < \psi(x) | P_c P_c | \psi(x) > \]
\[ ||P_c | \psi(x) > ||^2 \]

Figure 2.1: Example of GRW collapse.

Then, \(x\) denotes the centre of this Gaussian which is chosen according to the probability distribution \(p = |\psi|^2\). It is this probability rule that makes the theory nonlinear and at the same time assures Born’s rule. No explanation is provided regarding the origin of this probability distribution nor the physical nature of the transformations or ‘hits’. Hits occur according to a Poisson distribution (i.e. at random times) but with a mean frequency \(\nu\) which is dependent on the number of particles \(N\) that are described by \(|\psi\rangle\). This new physical constant is then tuned as to conform to experiment, ensuring that microscopic systems only very rarely localize. On the other hand, macroscopic systems are continually bombarded by hits, such that they localization occurs on a timescale impossible to probe. Furthermore, once the wavefunction localized, it becomes invulnerable to normal unitary evolution which has a delocalizing effect. The exact value of \(\nu\) is chosen such that the theory’s predictions exactly match the current data. However, GRW does provide a falsifiable
description for what happens at the interface between the two realms and thus goes beyond the Copenhagen interpretation in this respect. Furthermore, perfect localization is never realized, as the width of the Gaussian $\sigma$, which is also introduced as a new physical constant, cannot be zero due to energy constraints. Since localization in position space means delocalization in momentum space, Gaussian hits imply an energy gain inversely proportional to their width. This delimits the possible regime of $\sigma$, as sufficiently small values would have been experimentally observed. Conversely, for large values of $\sigma$, the wavefunction no longer localizes in an empirically adequate manner. The parameterspace spanned by these two new constants of nature can be represented diagrammatically, see Figure 2.3. GRW thus requires a weakening of the eigenstate-eigenvalue link, akin to the decoherence based approach, since the infinite tails of the Gaussian imply that any state vector with a nonzero value at $t = 0$ will also be nonzero for $t > 0$. GRW can be seen as an attempt to widen the range of the Copenhagen collapse to cover both the classical and the quantum realm. In addition, the hitting mechanism no longer depends on the subjective property of measurement but instead depends on the objective property that is the particle number $N$. The approach has been successfully extended to include description of identical particles, a diffusion mechanism in order to prevent warming up as a result of sharp localizations, and a relativistic scenario has also been developed [41]. Arguably, the most interesting improvement to the original GRW theory turns the instantaneous localizations into a dynamical process.

2.5.2 Continuous Spontaneous Localization (CSL)

In order to make collapse a continuous phenomenon, we will need to adjust the probability rule. Simply chopping up the Gaussian hits into infinitesimal hits while preserving the GRW probability conflicts drastically with Born’s rule. Additionally, the hitting mechanism must be turned into a continuous function while still exhibiting random behaviour. This is achieved by introducing a field $w(x,t)$ fluctuating randomly in both time and space, which indicates the centre of the Gaussian hit. Mathematically, the most natural choice is to use white noise, which includes all possible wavelengths and thus has a flat frequency spectrum. On the other hand, the inclusion of infinitely high frequencies is not very physical so recent work has focused on the introduction of a cut-off in the spectrum [42].

In any case, the field $w(x,t)$ will induce a continuous random walk in the Hilbert space $\mathcal{H}$ of the system under consideration. In order to avoid the possibility of delocalization, the ‘edges’ of $\mathcal{H}$, representing the classical states for which all but one weights go to zero, must be absorbant. This is to say that the probability for the wavefunction to delocalize must decrease as it localizes. The probability rule of GRW accomplishes this but does not suffice as it breaks Born’s rule. This can be seen by considering a time evolution with time steps of $dt \to 0$, equivalent to a continuous random walk. In this scenario, the component of the initial wavefunction with the highest weight will always win, since the infinitesimal steps taken will statistically favour the direction of that component. Instead, CSL introduces a rule expressing the probability that nature chooses a certain function $w(x,t)$ with $t \in [0,T]$ is dependent on the total norm of the resulting wavefunction $\psi(x,T)$:
Figure 2.2: Example of CSL collapse for $dt > 0$. Observe that the wavepackets have a noticeable tendency to approach each other due to the fact that $a \sim \sigma$.

$$\frac{T}{dt} \text{Prob}\{w(x, t)\} = C \prod_{i=0}^{T/dt} dw(x, t_i) \langle \psi(x, T) | \psi(x, T) \rangle$$  \hspace{1cm} (2.20)

At first sight, it would seem that this implies equal probability for all possible configurations of $w(x, t)$. However, since the continuous hits correspond to a non-unitary transformation, the norm of the wavefunction $\langle \psi, t | \psi, t \rangle$ is not conserved. Equation (2.20) then states that fields $w(x, t)$ resulting in the largest norms, are the most likely to occur. The normalization factor $C$ is then a function of the timeslice $dt$, which is taken to 0, and the width of the Gaussian $\sigma$. For the case $C = 1$, it is straightforward to see that when the centre of the Gaussian hit corresponds to a component $\psi(x)$ with almost no weight, the tails will annihilate the bulk of the weight, such that the norm decreases severely. Conversely, a hit centered around the heaviest weighing component, will best preserve the total norm. Implicit in this model is that the space between components $a$ cannot be smaller than the width of the Gaussian $\sigma$, because this leads to severe deviations from the Born rule. In fact, if we do allow $a < \sigma$, localization probabilities will start to depend on the weights of surrounding components. However, even for $a > \sigma$ the effect persists as can be seen in Figure 2.2, although it vanishes exponentially fast for increasing $a$.

The modified Schrödinger equation can then be written in the following form:

$$\frac{d |\psi(x, t)\rangle}{dt} = [-iH_S - \sigma^{-1}(w(x, t) - A(x))^2] |\psi(x, t)\rangle$$ \hspace{1cm} (2.21)

Here, $H_S$ denotes the normal unitary Hamiltonian of the system and $A(x)$ is a generalization of the position operator so that collapse also functions in other bases. Specifically,
in order to establish a connection to gravitation as the culprit responsible for state reduction, the majority of research has focused on an operator proportional to the mass density of a sphere of radius \( \sigma \) around a point \( x \) [8]. In analogy with the GRW approach, CSL also invokes the existence of two constants. In the end, these constants of nature must be established experimentally but suggestions for possible values have been proposed [43, 44]. So far, experiments have not been able to probe the scales at which predictions of collapse theories and standard quantum theory diverge. Figure 2.3 shows the state of affairs as of 2012.

![Figure 2.3: Parameter space spanned by the two constants of nature required in (a) GRW and (b) CSL models. ‘ERR’ marks the experimentally refuted region, and the region ‘NCR’ covers those values of (\( \lambda, \sigma \)) for which no collapse takes place on timescales that correspond with the classical world. The unmarked region encompasses values that produce collapse but are as of yet inaccessible to experiment. The two dots indicate the original GRW proposal, and a later one by Adler. Adapted from [45]](image)

Given the Gaussian nature of the non-unitary hits, it is not surprising that on average the suppression of all but the ‘winning’ component of \( |\psi\rangle \) occurs exponentially. In fact, it turns out that the master equation, governing the time evolution of density operators, of CSL and environment-induced decoherence show a strong resemblance when certain forms of the environment are considered [46]. This could indicate that both theories, at least ensemble averaged, effectively describe the same phenomenon. It would also mean that the parameters (\( \lambda, \sigma \)) do not represent physical constants at all, but merely serve as indicators
for the makeup of the interacting environment. A crucial difference of course lies in the fact that CSL can be shown to predict collapse behaviour when expressed in the state vector formalism [47], whereas decoherence only demonstrates an exponential suppression of the off-diagonal terms of the density matrix. On the other hand, the similarity implies that the only way to distinguish and thus potentially disprove either theory, would be to successfully shield experiments from decoherence effects and to use mesoscopic measurement instruments to probe whether superposed states persist or not. Examples of such experiments are described in [48,49] and typically focus on suppressing the effect of decoherence in order to distinguish between theories.

It will be useful for the following to note that strictly speaking, equation (2.21) does not contain any nonlinear terms. Instead, all nonlinearity is contained within the probability rule by the suppression of those ‘trajectories’ $w(x,t)$ that lead to the smallest final norm. The fact that CSL requires nature to only produce noise of a very specific type – i.e. norm-preserving noise – without specifying its source and the exact mechanism leading to this selection, is rather problematic. The situation can be mitigated by arguing that nature makes no such selection but that instead the wavefunction couples more strongly to norm-preserving noise than to norm-destructive noise. Even then, a deeper theory remains necessary to elucidate such behaviour.

Instead, we may ask to what extent a linear addition to the Schrödinger equation is able to achieve collapse dynamics and what the phenomenological sacrifice of such an approach will be. At first sight, the removal of the nonlinear part of the CSL dynamics, as contained in the probability rule, will result in a simple random walk in the system Hilbert space $\mathcal{H}$. Clearly, the absorbant property necessary to ensure the robustness of classical states under a stochastic field then no longer occurs. In the next sections, we will try to restore that robustness while using the theory of spontaneous breaking of time-translation symmetry to argue for a different interpretation of the white noise field $w(x,t)$. 

26
3. Spontaneous Symmetry Breaking

We are fortunate enough to live in a world that is brimming with symmetries, since it allows physicists to formulate symmetrical laws that find general application. It is also fortunate that our universe does not admit to all possible symmetries. In such a world it would not be possible to recognize structure, because everything would be similar in every respect, all the time. It is by virtue of the fact that some symmetries are broken, that we can perceive the underlying (approximate) subsymmetries that we try to describe.

Symmetries of a system may be broken explicitly due to the inclusion of symmetry breaking terms in the Hamiltonian describing it. Such terms may find their origin in the presence of certain forces or objects and cause the system’s groundstate to become asymmetric. For example, drawing a dot on one of the wings of a butterfly breaks the mirror symmetry that was present before.

A special case occurs when a symmetric Hamiltonian gives rise to asymmetric groundstates that are not even eigenstates of the Hamiltonian. Instead of occupying the symmetric groundstate that is an eigenstate of the Hamiltonian, the system may then spontaneously break this symmetry, forcing the system into an asymmetric groundstate. This happens when there exists a finite symmetry breaking field and the energy spectrum becomes degenerate in the thermodynamic limit. If the broken symmetry is a continuous one, the asymmetric groundstate will be infinitely degenerate, and its selection will effectively appear to be guided by a probabilistic process if the symmetry breaking field is effectively isotropic.

Due to the linearity of quantum theory that is implicit in the superposition principle, we can create new symmetric states by adding up all states that are related by a symmetry transformation. The resulting singlet state is generically found to be the exact groundstate for a host of important models, as is proven by Marshall’s theorem [50]. Obviously, such a quantum state is never realized in the thermodynamic limit, indicating that a symmetry is spontaneously broken. Given the probabilistic process that occurs in spontaneous symmetry breaking (SSB) and the subsequent superselection of states, it may thus be sensible to think about the emergence of classicality as an instance of SSB.

We can find out about the occurrence of SSB by looking at the behaviour of the order parameter of the system under observation. If a system is ordered with respect to the order parameter in the thermodynamic limit, while such order is absent when the system contains only small numbers of particles, it is clear that a phase transition has taken place, caused by SSB. Another indication of SSB is found by analyzing behaviour of the energy spectrum in the thermodynamic limit. If the system exhibits a spectrum that becomes degenerate in the thermodynamic limit, it will become infinitely susceptible to symmetry breaking in that limit.

In the following, we will take a look at a particular model exhibiting spontaneous rotational symmetry breaking in order to see if we can extend the methods and conclusions of SSB to a model in which unitarity symmetry is broken, as is the case with wavefunction collapse.
3.1 The Lieb-Mattis Model

In order to appreciate the power of spontaneous breaking, we will first take a look at the Lieb-Mattis model, which is essentially the infinite wavelength limit of any short-ranged Heisenberg model and accurately describes its gapless excitations in the thermodynamic limit \[51\]. The energy levels of states within this thin spectrum scale inversely with the number of particles and thus cannot be directly observed for large systems. It is this property that is essential to spontaneous symmetry breaking and will effectively demarcate the quantum from the classical realm.

We will treat the bipartite case but the result can be extended to a range of lattice types \[7\]. The antiferromagnetic Lieb-Mattis model is characterized by infinite range interactions between spins on sublattice A and spins on sublattice B. Spins located on the same sublattice do not interact, so that for \( N = 2N_A \) spins there will be \( N^2/4 \) interactions, all with equal weight \( J \). Although this model may seem unphysical, it effectively describes the low energy limit of the antiferromagnetic nearest neighbor Heisenberg model, the Hamiltonian of which reads:

\[
\hat{H} = J \sum_{i,\delta} \hat{S}_i \cdot \hat{S}_{i+\delta} = J \sum_{k,\delta} e^{ik\delta} \hat{S}_{-k} \cdot \hat{S}_k \tag{3.1}
\]

where \( \delta = \{+\hat{x}_1, +\hat{x}_2, ..., +\hat{x}_d \} \) is a positive unit vector that connects nearest neighbor spins to \( \hat{S}_i \) and \( J > 0 \), such that the energy is minimized when neighboring spins do not align. This seems to imply that the groundstate can be constructed by having all the spins on one sublattice point opposite to all the spins on the other sublattice. Classically, this so-called Néel state is indeed the ground state but it is not an eigenstate of the Hamiltonian. This is easy to see if we rewrite the Hamiltonian in terms of spin ladder operators:

\[
\hat{H} = J \sum_{i,\delta} \frac{1}{2} (\hat{S}_i^+ \hat{S}_{i+\delta}^- + \hat{S}_i^- \hat{S}_{i+\delta}^+ + \hat{S}_i^z \hat{S}_{i+\delta}^z) \tag{3.2}
\]

Whereas in the ferromagnetic case, the ladder operators kill the groundstate of maximum (or minimum) spin, the Néel state is projected onto a state that is not proportional to it. In order to find out more about the quantum antiferromagnetic groundstate it is useful to write the spin operators as bosonic operators by using the Holstein-Primakoff transformation. Furthermore, we may rotate the spins of one sublattice by \( \pi \) around the \( x \)-axis to make further calculation more convenient. This rotation is tantamount to the transformations \( \hat{S}_i^+ \leftrightarrow \hat{S}_i^- \) and \( \hat{S}_i^z \leftrightarrow -\hat{S}_i^z \) and is allowed because we are free to choose coordinate systems.

Lastly, we can approximate the HP-transformation by expanding around the small parameter \( S^{-1} \) and neglecting \( O(S^0) \) and lower terms, since we are interested in the behaviour of the antiferromagnet in the thermodynamic limit. The transformation then amounts to the following:

\[
\begin{align*}
\hat{S}_i^+_{i \in A} &\rightarrow \sqrt{2\sigma} a_i \\
\hat{S}_i^-_{i \in A} &\rightarrow \sqrt{2\sigma} a_i^\dagger \\
\hat{S}_i^z_{i \in A} &\rightarrow \sigma - a_i^\dagger a_i \\
\hat{S}_i^+_{i \in B} &\rightarrow \sqrt{2\sigma} b_i \\
\hat{S}_i^-_{i \in B} &\rightarrow \sqrt{2\sigma} b_i^\dagger \\
\hat{S}_i^z_{i \in B} &\rightarrow b_i^\dagger b_i - \sigma
\end{align*}
\]

for spin operators with size \( \sigma \) on sublattices \( A \) and \( B \) respectively. After a Fourier transformation we find that the resulting expression contains non-diagonal terms:
\[ H = -J N \frac{z \sigma^2}{2} + J z \sigma \sum_k ((a_k^\dagger a_k + b_k^\dagger b_k) + \gamma_k (a_k^\dagger b_{-k}^\dagger + a_k b_{-k})) \] (3.4)

where \( z = 2d \) is the coordination number of the bipartite lattice in \( d \) dimensions and \( \gamma_k = z^{-1} \sum_{\delta} e^{ik\delta} \). To diagonalize the entire Hamiltonian we use a Bogoliubov transformation which once again preserves the canonical commutation relations:

\[ a_k = \cosh(\theta_k) a_k + \sinh(\theta_k) b_k^\dagger \]
\[ b_k = \cosh(\theta_k) b_k + \sinh(\theta_k) a_k^\dagger \] (3.5)

The non-diagonal terms vanish for \( \gamma_k = -\tanh 2\theta_k \) so that we are left with a well-behaving Hamiltonian, the groundstate of which is just the classical groundstate with negative quantum corrections. The condition that takes care of the off-diagonal terms contains singularities for the modes \( k = 0 \) and \( k = \pi \). The former mode corresponds to the infinite wavelength interactions that describe the dynamics of the entire antiferromagnet while the latter accounts for global rotations of one of the sublattices. Isolating these parts yields the Lieb-Mattis Hamiltonian:

\[ \hat{H}_{LM} = J \sum_{k=0,\pi} \gamma_k \hat{S}_{-k} \cdot \hat{S}_k = \frac{2J}{N} \hat{S}_A \cdot \hat{S}_B = \frac{J}{N} (\hat{S}_T^2 - \hat{S}_A^2 - \hat{S}_B^2) \] (3.6)

where \( \hat{S}_T = \hat{S}_A + \hat{S}_B \). The groundstate is simply the spin singlet state with maximal values for \( S_A \) and \( S_B \):

\[ E_{LM} = \frac{J}{N} |S_T(S_T+1) - S_A(S_A+1) - S_B(S_B+1)| \] (3.7)

\[ |\psi_0\rangle_{LM} = \left| S_T = 0, S_A = S_B = \frac{N\sigma}{2} \right> \quad E_0 = -\frac{J}{8} (N+4) \] (3.8)

Contrary to the classical Neél groundstate, the quantum antiferromagnetic groundstate found above is an eigenstate of the Lieb-Mattis Hamiltonian. This hints at the possibility that somewhere during the transition from quantum to classical, a symmetry is broken and the classical Neél groundstate is recovered. There are two ways of exciting this ground state: we can either increase the total spin, or decrease one of the sublattice spins. The cost of the first type excitation is inversely proportional to the system size and constitutes the thin spectrum mentioned before. In the thermodynamic limit these excitations have a vanishing energy relative to the groundstate and are not directly observable as a consequence. More precisely, since the amount of possible excitation only grows linearly with \( N \) while the amount of total states grows exponentially with \( N \), the contribution of these excitations to the partition function vanishes. The second type of excitation is proportional to the coupling constant \( J \) only and is commonly known as the magnon. Magnons give rise to collective excitations in the form of spin waves and remain gapped in the Lieb-Mattis model due to its infinite interactions. Consequently, these excitations remain observable in the thermodynamic limit as can be seen in Figure 3.1. For our present purposes we will only be interested in the properties of the thin spectrum under the influence of a symmetry breaking term in the Hamiltonian.
Figure 3.1: The ratio between the groundstate energy and thin spectrum energies as a function of the number of spins $N$ (left). In the thermodynamic limit, these thin states become degenerate while the magnon excitations (right) remain distinctly non-degenerate.
3.2 Breaking SU(2) symmetry

Both the Heisenberg model and the Lieb-Mattis model possess spin rotation symmetry SU(2). This is to say that the Hamiltonian is unchanged by a global and continuous rotation of the spin operators in the system it describes. The form of the Hamiltonian explicitly hints at this: $\hat{S}_i \cdot \hat{S}_j$ represents the angle between both operators which is clearly conserved under a global rotation. A continuous symmetry by Noether’s theorem implies the existence of a conserved quantity, which in this case is the total spin $S = \sum_i s_i$ since it commutes with the Hamiltonian.

The classical Néel state is not invariant under a global rotation of spin operators. Instead, a nontrivial SU(2) transformation maps the classical state onto a different classical state orthogonal to the initial state. The fact that the classical Néel state does not abide by the same symmetries as the Hamiltonian indicates that in occupying one of many degenerate groundstates, the system’s symmetry is broken. A consequence and a good indicator of this loss of symmetry in the quantum to classical transition is that the system’s order parameter changes values. The order parameter for an antiferromagnet is the staggered magnetization $M = \hat{S}_A - \hat{S}_B$ and it clearly attains a maximal value in the completely ordered Néel state.

However, the operator we use to measure the staggered magnetization $\hat{M} = \hat{S}_A - \hat{S}_B$ does not commute with the Hamiltonian which means that the order parameter is not a constant of motion. This is where the antiferromagnetic case diverges from the ferromagnetic case, the order parameter of which is a conserved quantity.

To further investigate the situation it is useful to consider a situation in which we slightly perturb the symmetric Hamiltonian. By introducing a symmetry breaking field $b(S_A^z - S_B^z)$ which effectively forces the order parameter to obtain a non-zero value, we can explicitly break the symmetry. This seems a rather ad hoc and unphysical way of forcing the system into its classical groundstate, but taking the limit $b \to 0$ in the end removes this suspicion and approximately restores the original Lieb-Mattis Hamiltonian. In fact, we could use any type of field, as long as it contains a staggered component. A completely uniform field is the only field for which symmetry breaking of the quantum antiferromagnetic groundstate does not occur. We thus arrange our new symmetry breaking Hamiltonian in such a way that the energy is minimized when sublattice $A$ is magnetized in the positive $z$-direction and sublattice $B$ is magnetized in the opposite direction:

$$\hat{H} = \hat{H}_{LM} - b(S_A^z - S_B^z)$$

(Equation 3.9)

Evaluating the eigenstates directly is problematic because the operators in the Hamiltonian do not have a common basis of eigenstates so in general we would have to calculate Clebsch-Gordan coefficients for large values of the involved quantum numbers. Fortunately, the situation is somewhat simplified by the fact that since we are interested in the groundstate and its thin spectrum excitations only, we will set $S_A = S_B = N/4$ from now on. Also, since the spectrum of the Lieb-Mattis Hamiltonian found in (3.6) is degenerate for different values of the total spin in the $z$-direction $M_T$, we might as well set it to zero. Furthermore, by picking the right basis of quantum numbers we can avoid calculating CG-coefficients. Even though the derivation of the Lieb-Mattis spectrum without symmetry breaking field was done in the basis $|S_A, S_B, S_T, M_T\rangle$, we could equivalently pick the basis $|S_A, S_B, M_A, M_B\rangle$, which is an eigenbasis of the symmetry breaking term. This only requires writing the total spin operator, found in $H_{LM}$, in that same basis:
Here, $\phi$ has eigenfunctions the eigenvalues of the spin operators occurring in Eq. (3.10). Assuming this Hamiltonian for the matrix element $A$ a perturbed quantum harmonic oscillator:

we can rewrite the recursion relation as a well-known partial differential equation describing the zero magnon state:

After adding two resolutions of the identity and applying operators, the full Hamiltonian can be written as follows, where $|M_A| = |S_A = S_B = N/4, M_A, M_B = -M_A\rangle$ denotes the solution for the groundstate $M_A$ can take on values $\{-N/4, -N/4 + 1, ..., N/4\}$, the function $A(M_A, i)$ can be considered smooth and differentiable in the thermodynamic limit. For large values of $M_A$, the two coefficients on the right side of (3.13) approach each other, allowing us to straightforwardly calculate the eigenfunction of $\hat{H}$. In addition, we will assume that $N >> M_A$. If the solution for the groundstate $A(M_A, 0)$ turns out to have vanishing contributions in both these limits, our assumption will have been valid. Then, using the definition of the double derivative:

we can rewrite the recursion relation as a well-known partial differential equation describing a perturbed quantum harmonic oscillator:

$$
\mathbf{S}_T^2 = \frac{1}{2}(\mathbf{S}_A^+ + \mathbf{S}_B^+)(\mathbf{S}_A^- + \mathbf{S}_B^-) + \frac{1}{2}(\mathbf{S}_A^- + \mathbf{S}_B^-)(\mathbf{S}_A^+ + \mathbf{S}_B^+) + (\mathbf{S}_A^z + \mathbf{S}_B^z)^2
$$

(3.10)
Then, by applying some changes of variable, this expression can be rewritten as an unperturbed oscillator with frequency and energy:

\[ \omega = \frac{N}{2} \]  

\[ \nu_i = \frac{N}{2J} \left( \frac{b^2 N}{2J} - \frac{J N}{8} - E_{\phi_i} \right) = \left( i + \frac{1}{2} \right) \omega \]  

The groundstate indeed only has relevant contributions for \( N >> M_A >> 1 \) if we take the limit \( N \to \infty \) before \( b \to 0 \), so that our assumptions are validated, and the eigenfunctions and associated energy eigenvalues of \( |\phi_i\rangle \) are:

\[ A(M_A, 0) = \left( \frac{\omega}{\pi} \right)^{\frac{1}{2}} e^{-\frac{1}{4} \omega (M_A - \frac{b^2 N}{2J})^2} \]  

\[ E_{\phi_0} = \frac{b^2 N}{2J} - \frac{J}{8} (N + 4) \]  

Now, \( E_{\phi_0} \) is an approximation but the physically relevant terms agree with previous results, which approximate the groundstate and spectrum in terms of the basis \( |S_A, S_B, S_T, M_T\rangle \) [52]. Furthermore, if we set \( b = 0 \), we indeed retrieve the unbroken groundstate energy of Eq. (3.8). We would now like to know how the groundstate and the order parameter \( M_S \) behave for different regimes of \( N \) and \( b \). It turns out that the appropriate limits \( N \to \infty \) and \( b \to 0 \) do not commute, such that we find the classical and quantum groundstate:

\[ \lim_{b \to 0 \pm} \lim_{N \to \infty} A(M_A, 0) = \delta_{M_A, \pm \frac{N}{2}} \]  

\[ \lim_{N \to \infty} \lim_{b \to 0 \pm} A(M_A, 0) = \text{constant} \]  

and similarly for the system’s order parameter:

\[ \lim_{b \to 0 \pm} \lim_{N \to \infty} \langle \hat{S}_A^z - \hat{S}_B^z \rangle = \pm 1 \]  

\[ \lim_{N \to \infty} \lim_{b \to 0 \pm} \langle \hat{S}_A^z - \hat{S}_B^z \rangle = 0 \]

Apparently, taking the thermodynamic limit before letting the symmetry breaking field vanish leads to the classical groundstate we would expect, whereas a reverse order of limits yields the quantum antiferromagnetic groundstate. This phenomenon is commonly known as spontaneous symmetry breaking and it implies that the quantum groundstate of a system becomes infinitely vulnerable to perturbations as its size increases. Only if we succeed at perfectly shielding the system from perturbation, can we construct a macroscopic superposition.

In the thermodynamic limit, the thin spectrum becomes completely degenerate, so that even the slightest external influence is capable of changing the state of the system. Of course, no system is infinitely-sized and thus there exists a threshold for the symmetry breaking influence for each system beyond which the system is robust. Furthermore, in the derivation of spontaneous symmetry breaking we have invoked a perfectly staggered magnetic field, but it is sufficiently clear that for large enough systems, even a single symmetry breaking component \( b S_i^z \) is able to set the breaking of symmetry in motion. Given the exponentially small field needed for truly macroscopic systems, a child playing with refrigerator magnets in Japan would suffice.
3.3 Breaking time translation symmetry

In the last section, we have explicitly broken the SU(2) rotation symmetry of the antiferromagnet and we have argued that for sufficiently large systems, the need for an explicit field disappears, making the process a spontaneous one. However, due to the fact that the symmetry breaking field is unitary, nothing withholds the system from returning to its initial state. This can be made visible by considering the action of unitary evolution on a two-state system represented by the Bloch sphere. If we pick its poles to be the stationary states of the generator of the time evolution $G$, then the trajectories generated will leave the norms $\langle i| i \rangle$ invariant (see fig. 3.2(a)). Of course, we could pick the poles to represent non-stationary states, which results in a Rabi cycle (see fig. 3.2(b)). However, in both these cases the evolution causes the wavefunction to cycle around all available states on a trajectory indefinitely. Furthermore, since the amount of available states scales with $N^2$, the turning time goes to infinity in the thermodynamic limit, such that the effect is not observable.

![Figure 3.2: Wavefunction trajectories generated by (a) a unitary Hamiltonian with eigenstates $\{|0\rangle, |1\rangle\}$, (b) a unitary Hamiltonian for which $\{|0\rangle, |1\rangle\}$ are not stationary states and (c) a non-unitary Hamiltonian $H_\pm$, which sends all possible states towards one of its stationary states depending on its sign. Note that the state vector is continually renormalized during non-unitary evolution. Adapted from [53].](image)

Instead, wavefunction collapse is a non-unitary process (see fig 3.2(c)) and thus in order to accomplish a successful collapse, we will need to invoke a non-unitary influence. Within the decoherence approach of section (2.4), this effect is achieved by dividing up subsystems in such a way as to effectively achieve a local non-unitary evolution, although only for ensemble averaged quantities. The dynamical collapse theories of section (2.5) made use of a nonlinear non-unitary field, the origin of which was left unspecified, to ensure collapse at the state vector level.

It has been proposed by van Wezel and colleagues [4, 7, 53] to use the mechanism described above to break the time-translation symmetry that is implicit in unitary evolution by adding a non-unitary field to the Hamiltonian. Then, just by multiplying the symmetry breaking
field by $i$ in eq. (3.9) and taking the limits in the right order, the systems is forced into a classical state from which it cannot escape. In terms of the Bloch sphere, we would be adding the vectors of fig 3.2(a) to those of 3.2(c), resulting in a trajectory that spirals towards one of the poles, depending on the sign of $b$. Now, non-unitary time evolution is not norm preserving, and in general adds energy to the system, in apparent contradiction to energy conservation. However, these issues vanish as we take the thermodynamic limit and the limit of the non-unitary field $b$ to zero respectively, which we are allowed to do since we have assumed from the outset that the symmetry breaking field to couple to the thin states. On the other hand, this limit cannot strictly be taken to zero, just as the thermodynamic limit only constitutes an approximation for the size of macroscopic objects. Instead, if the theory is to provide experimental predictions, a fundamental constant $\lambda$, denoting the magnitude of the order parameter field, will be necessary in order to ensure that collapse starts to happen at a certain mass scale, as is the case with CSL. Unfortunately, the Lieb-Mattis Hamiltonian with a non-unitary symmetry breaking field can no longer be massaged into harmonic oscillator form, so that we will have to solve the Schrödinger equation numerically. For now, we will be most interested in the localization time $t_{loc}$ in the two limits of eq. (3.20), so we will have to simulate the evolution for different values and extrapolate the dependencies of $t_{loc}$ (see fig. 3.3). The time evolution operator is given by:

$$U(t) = \exp\left(\frac{i t}{\hbar} \left[ \frac{J}{N} (\hat{S}_T^2 - \hat{S}_A^2 - \hat{S}_B^2) - i b (\hat{S}_A^z - \hat{S}_B^z) \right] \right)$$

(3.22)

In the $M_A$ basis, a symmetry breaking field pointing in the $z$-direction, effectively multiplies all components $|M_A\rangle$ by the exponential function $e^{ibM_A t}$, implying that exact localization will only occur for either $N \to \infty$ or $t \to \infty$. Even though macroscopic systems consist of extremely large numbers of particles, taking the thermodynamic limit can only serve as an approximation of the effects that occur in finite-sized systems. Thus, $t_{loc}$ must be considered as the time it takes to reach approximate localization, as was also the case with both the decoherence and dynamical collapse approaches. Once again, since we are allowed to weaken the eigenstate-eigenvalue link, this does not form a problem in principle. On the other hand, the asymmetric groundstate, represented by a delta-function in the $M_A$-basis, is completely robust under non-unitary influence. This is to say that the exactly localized state $|M_A = -N/4\rangle$ is completely unfazed by a non-unitary field that points in any other direction, since only the norm of the wavefunction is altered and not its distribution. Thus, the reasoning of refs. [4,7] goes that unitarity has been broken since once localization occurs, robustness is ensured and the situation has become irreversible, due to the fact that delocalization only occurs through unitary evolution, the strength of which vanishes in the thermodynamic limit (see bottom right of fig. 3.3).

However, the subtle difference between exact and approximate localization starts to play an important role at this point. The approximately localized state that is attained in a finite-sized system after a finite time evolution, has nonzero values for all components of the wavefunction that started out with nonzero values (see fig. 3.4). Since only zero-valued components are resilient to the non-unitary field, a realistic measurement on a symmetric groundstate with a macroscopic device can never yield a robust asymmetric state as outcome. For example, after an initial state has approximately localized under the influence of a non-unitary field in the positive $z$-direction, the effect is easily reversed by introducing a non-unitary in the negative $z$-direction. This is to say that regardless of system size and
Figure 3.3: Time evolution of the order parameter $\langle \hat{M}_S \rangle$, governed by the Lieb-Mattis Hamiltonian with a symmetry breaking non-unitary field. All graphs but the last one use the symmetric groundstate as initial function. **Top**: Doubling the values of $N$ (*left*) or $b$ (*right*) results in a halving of $t_{\text{loc}}$, if localization is interpreted in an approximative way, indicating that $t_{\text{loc}} \propto (Nb)^{-1}$ **Bottom**: The interplay between the unitary and non-unitary components has an effect on $t_{\text{loc}}$ when measured intermediately (*left*), but all trajectories collapse onto each other in the thermodynamic limit, as the unitary component vanishes in that limit. This becomes visible when considering an asymmetric groundstate under unitary evolution for different values of $N$ (*right*).

Of course, the choice of a time-independent staggered field is not a physically realistic choice and more importantly does not produce the right probabilities. Instead, in order to generate Born’s rule, it is a natural choice to invoke a white noise field $b(t)$ to represent the sum of all infinitesimal influences that affect the system. For the 1D unitarity breaking evolution time, the wavefunction always remains unstable with respect to the non-unitary field. Summarizing, it is true that in the relevant limits an approximately localized state is the end result of non-unitary dynamics. It is also the case that an exactly localized state is stable under those same dynamics. However, the contention that these two aspects of non-unitary dynamics imply that unitarity has been broken is unfounded, since it presupposes that we can switch between exact and approximate localization at will. This problem is not easily solved and we will try to find different mechanisms to ensure stability without exact localization in the next section.
Figure 3.4: **Left:** Starting out from the symmetric state, the wavefunction in the $|M_A\rangle$-basis approaches the ordered state $|M_A = N/4\rangle$ but only becomes exactly localized for either $N \to \infty$ or $t \to \infty$. **Right:** The exactly localized state only becomes vulnerable to the unitarity breaking field after the wavefunction spreads out unitarily. This recurrence time goes to infinity in the thermodynamic limit. The inset shows the time evolution of the order parameter.

Field we have considered, this brings about a random walk in order parameter space with a diminishing average stepsize as the system approaches its ordered state. If we pretend for a moment that an ordered state which is stable under further non-unitary influence can indeed be reached within finite time, it is immediately clear that the system with a 1D non-unitary field abides by Born’s rule. After all, an antiferromagnet with a 1D field effectively represents a two state system with only two possible ordered states, both of which are equally likely to emerge. However, we cannot be completely sure that the rule also holds for more complex situations. For a 2D unitarity breaking field, there are infinitely many ordered states, since the ordered state can emerge in any direction on the $y - z$ plane:

$$H_{SB} = ib_1(t)(\hat{S}_A^\lambda - \hat{S}_B^\lambda) + ib_2(t)(\hat{S}_A^y - \hat{S}_B^y)$$  \hspace{1cm} (3.23)

The extra staggered field in the $y$-direction is not an eigenoperator of the rest of the Hamiltonian which means that a simulation of time evolution generated by eq. (3.23) will require a change of basis after every timestep. This quickly becomes computationally expensive for larger values of $N$ since it involves calculating large numbers of Clebsch-Gordan coefficients. Alternatively, Wigner functions could be used to visualize the time evolution of the conjugate variable pair of spins. However, as Wigner functions are defined in terms of density matrices and our ultimate goal will be to describe singular events, they will be of no use. Therefore, we will instead consider a different example of spontaneous symmetry breaking in the next section: the harmonic crystal. Crystals also exhibit a thin spectrum but they are most easily described in the position basis which allows for ordered states at every point in space. This will enable us to see whether a white noise non-unitary field as proposed in [4] will indeed yield the statistics dictated by Born’s rule.
4. Simulation of Unitarity Breaking

In this section another example of spontaneous symmetry breaking will be discussed which is better suited to test the implications of unitarity breaking. As we saw in the last sections, spontaneous symmetry breaking requires certain ingredients. Similar to the Lieb-Mattis antiferromagnet of the last section, the harmonic crystal also exhibits a global symmetry in its quantum groundstate and reveals a thin spectrum in the thermodynamic limit. However, whereas the antiferromagnet’s rotational symmetry can only give rise to two asymmetric groundstates under a 1D order parameter field, the crystal asymmetric groundstate is infinitely degenerate in 1D. This will allow us to verify Born’s rule for generalized wavefunctions under the action of a unitarity breaking field.

The treatment of the translational symmetry breaking of crystals in the next section is based on [54] and the steps involved are fairly analogous to those in the Lieb-Mattis model. Subsequently, it will be shown that the proposed unitarity breaking field of [4] invokes issues similar to those discussed in section 3.3, such that robustness under non-unitary influence is never achieved. Moreover, it is found that for asymmetric but non-classical initial conditions, the probability rule does not apply, to such an extent that it is in direct conflict with conventional quantum mechanics.

To resolve these issues, a more general symmetry breaking field is proposed in the form of a random matrix in section 4.3. This choice of unitarity breaking field adheres to Born’s law and arguably represents a more natural choice with regard to its purported gravitational origin. The instability of classical states is not restored with this new choice of field, but it is shown that under certain assumptions and in taking the right order of limits, a robust but approximate localization may be ensured.

4.1 The Harmonic Crystal

We typically experience crystals as objects with a fixed position which is to say that they are translationally asymmetric. However, a quick glance at the Hamiltonian for a 1D crystal reveals that this translational asymmetry is not expected from a quantum mechanical viewpoint. It is not immediately obvious what part of the Hamiltonian gives rise to the preferred position that crystals seem to adopt:

\[ H = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2} \sum_j (x_j - x_{j+1})^2 \]  \hspace{1cm} (4.1)

Here, \( j \) labels the individual particles with mass \( m \) and momentum \( p \) that make up the crystal. The coupling between neighboring particles is described by \( \kappa \). Although the Hamiltonian describes the energies associated to the positions of the crystal’s constituent particles, a symmetry with regard to the entire crystal is present.

In order to bring out the thin spectrum containing the energies of collective modes we first
rewrite the position and momentum operators in terms of bosonic creation and annihilation operators:

\[
p_j = \frac{i}{2}(2m\kappa)^{1/4}(b_j^\dagger - b_j) \quad x_j = \frac{1}{2}(2m\kappa)^{-1/4}(b_j^\dagger + b_j) \quad (4.2)
\]

Then, similar to the procedure followed in section 3.1, a Fourier transform is applied and the result is diagonalized by a Bogoliubov transformation. Again, the parameters used in the Bogoliubov transform contain singularities for \( k = 0 \), which describes the collective behaviour of the system. Since spontaneous breaking is a global phenomenon, it is only the \( k = 0 \) part of the Hamiltonian exhibiting the thin spectrum that is of interest, while the remaining part governs the internal degrees of freedom due to the coupling of neighboring particles (phonons). Turning back the Bogoliubov transformation and setting \( k = 0 \) then yields:

\[
H_{k=0} = \sqrt{\frac{\kappa}{2m}}(1 - \frac{1}{2}(b_0^\dagger - b_0)^2) = \frac{p_{k=0}^2}{2m} + \sqrt{\frac{\kappa}{2m}} \quad (4.3)
\]

The total momentum \( p_{\text{tot}} \) of the system is then obtained by multiplying the momentum operator of the \( k = 0 \) mode by the number of particles \( N \). The resulting expression can then be interpreted as the Hamiltonian for a free particle of mass \( Nm \) in a constant potential which we may neglect:

\[
H_{\text{collective}} = \frac{p_{\text{tot}}^2}{2Nm} \quad (4.4)
\]

It is clear that the separation between energy levels of the eigenstates of this Hamiltonian, describing the collective behaviour of the crystal, vanishes in the thermodynamic limit, signalling the presence of a thin spectrum. Since the absolute groundstate is identified as the state for which \( p_{\text{tot}} = 0 \), the uncertainty in the position of the crystal must be maximal. Differently said, since the Hamiltonian is still translationally symmetric, the wavefunction must be constant over all space. This is obviously in conflict with everyday experience, so we may try to break this symmetry by adding a term dependent on the centre of mass position of the crystal \( x_{\text{tot}} \):

\[
H_{\text{SB}} = \frac{p_{\text{tot}}^2}{2Nm} + \frac{1}{2}Nm\omega^2 x_{\text{tot}}^2 \quad (4.5)
\]

As was discussed in section 3, the choice of order parameter field is to a certain extent arbitrary. The only constraints are that the field be asymmetric with respect to the symmetry-broken quantity and that it vanishes in the right order of limits. Our particular choice of order parameter field then becomes clear as the Hamiltonian of eq. (4.5) is trivially identified as governing the quantum harmonic oscillator with groundstate solution:

\[
\psi_0(x_{\text{tot}}) = \left(\frac{m\omega N}{\pi}\right)^{-1/4}e^{-\frac{m\omega N/2}{2}x_{\text{tot}}^2} \quad (4.6)
\]
Analyzing the two relevant limits then verifies that we are dealing with a case of spontaneous symmetry breaking. If the strength of the order parameter field $\omega$ is taken to zero, the wavefunction becomes constant in space, whereas a deltafunction at $x_{tot} = 0$ is found in the thermodynamic limit. The fact that these two limits commute then verifies the occurrence of spontaneous symmetry breaking:

$$\lim_{\omega \to 0} \lim_{N \to \infty} \psi_0 = \delta_{x,x_{tot}}$$

$$\lim_{N \to \infty} \lim_{\omega \to 0} \psi_0 = \text{constant}$$

(4.7)

### 4.2 Unitarity Breaking in the Harmonic crystal

We would once again like to break the time-translation and time-reversal symmetry that is contained in the unitary character of conventional quantum mechanics by adding a non-unitary term to the collective Hamiltonian. This term will just be the imaginary version of the translational symmetry breaking field of the previous section, such that the time evolution operator is given by the following expression:

$$U(t) = \exp \left( - \frac{it}{\hbar} \left[ \frac{p_{tot}^2}{2Nm} + \frac{1}{2} Nm \omega^2 x_{tot}^2 \right] \right)$$

(4.8)

In the position base, the wavefunction is spread out by the unitary part of the Hamiltonian, while it is multiplied by a Gaussian function centered around $x_{tot}$. The situation is analogous to the antiferromagnetic case, and the unitarity breaking field is identical to the extra term added to the CSL Schrödinger equation of section 2.5.2. The principle of energy conservation is not immediately violated because the non-unitary field only couples to the thin spectrum states, which have a vanishing energy in the thermodynamic limit.

As before, an exactly localized state cannot be altered by the non-unitary field until the unitary component has caused the wavefunction to spread out sufficiently. But due to the extensivity of the unitary coupling, this spreading out no longer occurs in the thermodynamic limit. Starting the evolution from the symmetric quantum groundstate, it is clear that under the non-unitary field causes an instability and pushes the system into an approximately localized state if $\omega$ is kept constant. However, once components obtain a nonzero weight, neither unitary nor non-unitary evolution can remove this weight completely unless the system is of infinite size or the evolution is allowed to run for an infinite time. In other words, an exactly localized state can become approximately localized but not vice versa. In effect, this means that a collapse mechanism is still needed to completely break unitarity, if we insist that classical states are represented by exactly localized states. Conversely, if we allow classical states to be represented by approximately localized states, then we lose the robustness that exact localization provides, since all nonzero wavefunction components are equally susceptible to non-unitary influence.

In order to simulate the chaotic nature that the non-unitary field is supposed to represent, we now make the field time-dependent to see whether Born’s rule applies. This is to say that the centre of the Gaussian hit fluctuates in time, in analogy to the CSL dynamics:
Figure 4.1: Example of the time evolution of two Gaussian wavepacket under a fluctuating non-unitary field. Even though it may appear that the wavefunction localizes at some point (left), a log plot (right) clearly shows that this is not the case. Under the proposed symmetry breaking field, wavefunction components that have nonzero weight at $t = t_0$, will have nonzero weight for all $t > t_0$. Note furthermore that the non-unitary Gaussian hits benefit the centre of the considered Hilbert space, implying that fluctuations will come to a halt when the wavefunction localizes around the centre.

$$\frac{d}{dt}\langle\psi(x,t)\rangle = \left[-i\frac{p^2_{\text{tot}}}{2Nm} + \frac{1}{2}Nm\omega^2(x - w(t))^2\right]\langle\psi(x,t)\rangle$$

(4.9)

However, without the nonlinear probability rule that is postulated in the CSL framework, localization will never be retained throughout time evolution governed by eq.(4.9). Instead, the dynamics result in a random walk that only comes to a halt when the wavefunction has localized around the centre of the considered space (see fig. 4.1). To see whether Born’s rule does survive in the absence of the probability rule, we will introduce a cut-off for the sharpness of the localization above which we will pretend that localization has occurred and is further retained.

Whereas the probability rule in CSL dictates that Gaussian hits are most likely to occur within regions that contain most weight, no such prescription is given in [4, 7]. As a consequence, Born’s rule breaks down for almost every initial wavefunction, as can be seen in fig. 4.2. The tails of the Gaussian always cause the wavefunction to collapse on the centre of mass of the wavefunction, completely ignoring Born’s rule. This effect can only be undone by making position space periodical, but is not immediately clear what the physical meaning of such a configuration would be. Another implication of the Gaussian tails is the tendency of wavepackets to move towards a particular direction, a phenomenon we already encountered in the discussion of CSL.

The proposed dynamics breaks unitarity in the sense that once the wavefunction localizes around the wavefunction centre of mass, there is no way back. Yet, the behaviour that we find is not at all in accordance with Born’s law. Although the choice of a Gaussian symmetry breaking field is a natural one, it has only been shown to work in conjunction with some nonlinear probability rule.

Now, abiding by the conditions that we already imposed on the order parameter field,
Figure 4.2: Examples of the time evolution of various initial wavefunctions. **Top left:** Starting from the symmetric initial state, the wavefunction invariably localizes at the centre of the considered space. Initial states consisting of three (**Top right**) and five (**Bottom left**) exhibit a similar behaviour, again in contradiction with Born’s rule. **Bottom right:** An initial superposition with zero weight at the centre of the sample will nonetheless evolve into a localized state at the centre after the wavefunction has spread out under unitary influence. However, this effect vanishes in the thermodynamic limit.

namely that it be asymmetric and of vanishing size, we will try to find a more general field that does generate Born’s rule while modelling the effect of the previously mentioned fluctuations induced by gravity. Such a field will be considered in the next section.

### 4.3 Symmetry breaking with a random matrix field

We have seen that Gaussian hits demonstrate a tendency to move the weight towards the centre of mass of the wavefunction. This is also the case in a CSL scenario, but the probability rule makes sure that the effect is minimized by making it unlikely that hits land in areas with a small probability distribution, speeding up localization. Without such a probability rule, hits land on random coordinates and a random walk ensues which invariably ends in a localized state at the centre of mass of the initial probability distribution.

To avoid this centralizing behaviour, we will have to get rid of the Gaussian tails. As a matter of fact, all symmetrical functions that have tails are to be excluded. Instead, we propose a random field with entries both uncorrelated in time and space, such that each
wavefunction component is multiplied by a different number at every timestep. With the right constraints, this will mean that we can approximately prevent the violation of norm and energy conservation. Recall that the limit $\omega \to 0$ does not exactly preserve the norm or energy. In fact, $\omega$ must be a finite albeit extremely small quantity for collapse to take place and thus the conservation laws are violated by an equally small amount. A random field could preempt the emergence of such small deviations from the conservation laws by demanding that the norm is preserved after each timestep. This is not possible with randomly occurring Gaussian hit since the evolution of the norm is extremely dependent on the specifics of the existing wavefunction and the placement of the centre of the Gaussian. A matrix containing random variables multiplying each spatial component will have a mean multiplication that can be set to unity. Regardless of this possibility, we will renormalize the wavefunction after every step in the following.

The new Schrödinger equation including the random field is then given by:

$$\frac{d}{dt} |\psi(x,t)\rangle = \left[ -i \frac{p_{\text{tot}}^2}{2Nm} + NmB(x,t) \right] |\psi(x,t)\rangle$$

(4.10)

The entries $b$ of the random field $B(x,t)$ are allowed to be both positive and negative and may in principle include values in the range $[-\infty, \infty]$, in analogy to typical white noise. However, as before, we will consider the limits $\bar{b} \to 0$ and $N \to \infty$, where $\bar{b}$ denotes the average matrix entry value. The ensuing time evolution can effectively be seen as a high-dimensional random walk, where every dimension represents a component of the wavefunction. Given the difficulties associated with quantizing space, we will take for granted that space is continuous and only discretize the wavefunction in components for the sake of simulation.

The wavefunction thus undergoes a random walk in Hilbert space, but for our purposes it will be more interesting to look at the absolute length of the individual wavefunction components (see 4.3). Since the (renormalized) norm of a single component can only attain a maximal value of 1, the space in which the random walk occurs is real and spanned by axes of length 1. Due to the boundary condition that $\sum |c_i|^2 = 1$, an $N$-component wavefunction random walk is executed in a $(N - 1)$-simplex with orthogonal corners, where its vertices are identified with the classical states. Furthermore, the stepsize is random but constrained to a certain average proportional to $\bar{b}$. More importantly, as the random walker approaches one of the vertices of the simplex, the stepsize diminishes, such that the exact position of the vertex is never reached. This is indicative of the fact that exact localization only occurs in the limits $t \to \infty$ or $N \to \infty$. Since robustness under non-unitary time influence only occurs for exactly localized states, it is clear that the random field offers no solution to the delocalization problem. This is to say that, in analogy to the Gaussian case, approximately localized states are never protected from non-unitary influence, such that delocalization can and must occur after finite time. However, the completely random nature of the field combined with the absence of correlation between entries may generate Born statistics, unlike the linear Gaussian field.

4.3.1 Generating Born’s rule

First, we want to make sure that the dynamics of eq.(4.10) yield the right statistics. To do so, we will once again have to introduce a norm cut-off $\gamma$ above which we will consider a
Figure 4.3: Some examples of random walks of a three state system starting from an equiprobable state. Since the sum of the norms is constrained to unity, only two dimensions are necessary to visualize the random walk. Colours (blue to red) denote the passage of time and the vertices of the graph denote localized states.

Given component of the wavefunction to have localized. This is to say that if $|c_i|^2 > \gamma$, the wavefunction $|\psi\rangle$ will be considered to have collapsed to $|\psi_i\rangle$. For now, we will neglect the unitary part of the Hamiltonian, as for large systems its contribution goes to zero anyway. It is immediately clear that if the steps of the random walk are too large in proportion to $\gamma$, Born’s rule will surely fail. Instead, we need to ensure that the random walk is able to make sufficiently many steps before localization occurs. In fact, as fig. 4.4 shows, it turns out that Born statistics are approached as $b \to 0$, for some fixed value of $\gamma$. On the other hand, if we vary $\gamma$ for a given stepsize, it is clear that the statistics will approximate Born’s rule in the limit $\gamma \to 1$. In other words, if the threshold $\gamma$ is lowered, the average amount of steps necessary for a single component norm to reach $\gamma$ will also decrease, and thus the accuracy of the ensuing Born statistics.

Besides the circumstantial evidence provided by numerics, we can convince ourselves of...
the legitimacy of these considerations by taking a closer look at a simplified version of the random walk described above. It is rather inconvenient to have to deal with a random walk in an infinite-dimensional bounded space with steps that are both random and have an average size depending on the walker’s current position. The walk is simplified by considering the logarithm of the wavefunction norms while allowing the entries of the random field $B(x,t)$ to be $\pm b$:

$$\log|\psi(x,t)|^2 = L(x,t)$$

The simplification to fixed stepsizes and a diagonalized random matrix has no consequences for the time-averaged properties of the random walk and taking the logarithm of the norm implies that stepsizes becomes position-independent. Then, we can write the non-unitary evolution and subsequent renormalization of $L$ as a nonlinear operation:

$$L(x,t_1) = \log\left(\frac{|e^{B(x,t_0)}\psi(x,t_0)|^2}{\sum_x|\psi(x,t_1)|^2}\right) = \frac{B(x,t_0) + L(x,t_0)}{\log\sum_x|\psi(x,t_1)|^2}$$

The resulting dynamics are reminiscent of a symmetric gambler’s ruin game. In the original version of this game, two gamblers play heads or tails with the loser having to pay a dollar to the winner. The game ends when one of the players loses all their money. It can easily
be shown that the probability of winning this game is proportional to the distribution of the total amount of money between players. This is to say that if player $A$ has 90 dollars and player $B$ has 10 dollars, their respective probabilities of winning will be 0.9 and 0.1. The game can be generalized to $n$ players while preserving the probability rule which is essentially Born’s rule [55].

It is obvious that a game of gambler’s ruin presupposes a finite amount of money in order to effectuate a finite duration. Since the wavefunction norms never quite reach zero, the cut-off $\gamma$ will determine the end of the game. Furthermore, the fact that we are considering the unnormalized wavefunction means that the amount of money is not fixed throughout the duration of the game. This implies that when one of the wavefunction components ‘wins’, the other components are not necessarily on the brink of ‘losing’. Nonetheless, the winning behaviour of components is an indication for the extent to which Born’s rule is adhered, since the probability for more than one component to reach $\gamma$ vanishes as the limits $b \to 0$ and $\gamma \to 1$ are taken. Lastly, due to the renormalization after every timestep, the ‘stakes’ of the game fluctuate as a function of the resulting normalization. Thus, if the norm of the wavefunction increases, the effective stepsize will be smaller than $b$ and vice versa.

Now, consider a wavefunction consisting of a large number of components, the weights of which are all equal. It is clear that the norm of this wavefunction will only fluctuate minimally under the action of $\hat{B}$ since it is likely that the amount of components multiplied by $e^b$ is approximately equal to the those multiplied by $e^{-b}$. As a consequence, the stepsize remains approximately equal as long as the wavefunction is distributed symmetrically over a large amount of components. Conversely, more asymmetrically distributed wavefunctions will experience larger norm fluctuations, leading to variations in the stepsize of the random walk. In the next section we will try to use this property to enforce stable localization.

### 4.3.2 Avoiding delocalization

In the previous subsection, we used the cut-off $\gamma$ to define localization and assumed that the dynamics stopped at that point. However, in the absence of a nonlinear probability rule, there is no mechanism that ensures that localized states are robust under non-unitary influence. Although it is true that higher-dimensional random walks have a vanishing probability of returning to their initial position, this does not mean that delocalization will simply not occur. This obviously represents a serious problem, as classical objects do not exhibit randomly delocalizing behaviour.

So far, we have not included the effect of the unitary component on the random walk. In the position basis, the operator $\hat{p}$ causes the wavefunction to spread out, resulting in a perturbation of the random walk. In the following, we will consider a model in which localization occurs as before, but delocalization becomes unlikely when the limits $N \to \infty$, $\bar{b} \to 0$ and $\gamma \to 1$ are taken.

Let us consider an initial wavefunction consisting of $N$ delta-functions that are spatially separated. This is the typical starting point of a quantum measurement performed by a macroscopic instrument. A quantum superposition of sharply localized states is amplified to macroscopical proportions. Then, the action of the non-unitary field will increase the weight of some delta functions, while diminishing the weight of others. Meanwhile, the unitary component of the Hamiltonian will slowly spread out these delta-functions, turning them into Gaussians. As a result, the non-unitary component will affect a growing number
of wavefunction components. However, we have assumed the delta-functions to be separated in space and since the unitary component becomes extremely small in the thermodynamic limit, the tails of these Gaussians will not meet before localization occurs.

Consequently, the net action of the non-unitary time evolution on each Gaussian changes as it spreads out. Whereas the evolution of the weight of a delta-function is determined by a single entry of $B(x, t)$, a spread out Gaussian will come under the influence of a growing number of entries. Since the entries of $B(x, t)$ are allowed to be positive as well as negative, the net effect of $B(x, t)$ on the ensemble of components that comprise the Gaussian will diminish.

Before, we assigned a different random walk dimension to every single wavefunction component, but since space has not been quantized successfully as of yet, we must assume that space is a continuous parameter. This means that axes representing the different compo-

Figure 4.5: Time evolution of 100 two-state wavefunctions with an equiprobable initial superposition of two deltafunctions for different initial values of the order parameter field. Here, a 'state' does not correspond to a specific eigenstate of the Hamiltonian since space is taken to be continuous. Instead, we interpret a state in the approximate sense, and include all the weight that 'escapes' from the initial delta function through unitary time evolution. The plots show the gradual slowing down of the random walk as a function of time such that time evolution freezes into place. As the limit $N \to \infty$ is taken, the likelihood of a wavefunction ending up outside the small region $\gamma$ vanishes.
ponents of the wavefunction become ill-defined. Now, if we choose the axes to represent the weight of the Gaussian wavepackets, the random walk will be perturbed by the unitary spreading of the wavefunction causing a diminishing stepsize until the random walker is stopped in his tracks. In essence, we coarse-grain the measurement outcome and let the pointer states denote physically distinguishable states (Gaussian wavepackets separated by some distance that can be experimentally probed) instead of mathematically distinguishable states (delta-functions at each infinitesimally differing value).

We can model this effect by assuming that the amount of components that represent a single dimension in the random walk is given by \( n = rt + 1 \), where \( r \) is a number representing the rate at which the wavefunction spreads out. Then, \( n \) is also the amount of entries of \( B(x,t) \) that will affect the wavefunction, meaning that the net effect of the weight \( |c_i|^2 \) on a Gaussian wavepacket is described by:

\[
|c_i|^2_{t+1} = \frac{e^{\pm 2b}}{n^2} |c_i|^2_t
\]

The logarithm of the weight of the wavefunction components again undergoes a random walk but this time with stepsize \( 2b/n^2 \). It can be proven [56] that if the sum of all steps of a random walk does not diverge, the random walk has a final position. Since this sum converges in our case, the random walk has a final position:

\[
\sum_{t=1}^{\infty} \frac{2b}{n^2_t} \propto \frac{2b\pi^2}{6}
\]

Now, for localization to occur, we must be sure that the random walker has ample time to arrive in the region bounded by \( \gamma \) denoting approximate localization before the stepsize vanishes and the system becomes static. In fact, the probability for the wavefunction to freeze into a non-classical state must go to zero. This emphasizes the need for a better definition of what it means for a macroscopic object to behave classically. As we have seen, the eigenstate-eigenvalue link does not provide a satisfying definition of localization, especially when the parameters considered form a continuous spectrum as is the case with position and momentum.

Both the decoherence approach and dynamical collapse theories tell us that there exists an effective force, generated by the environment and gravitation respectively, which continuallylocalizes a particular state until a balance with the unitary delocalizing time-evolution is reached. This equilibrium position then quantifies the region \( \Sigma \) bounded by \( \gamma \) around the axes of the Hilbert space that will represent what we call a localized state. In the present case, the driving force represented by the random field does not continually push the wavefunction towards localization. Instead, the field randomly localizes and delocalizes the wavefunction until the interplay with the unitary component causes its net effect to vanish. As can be seen in fig. (4.5), higher values for \( b \) increase the probability for the wavefunction to reach a state contained in some region \( \Sigma \). Thus, in the thermodynamic limit, the unitary evolution causes a stabilization of the non-unitary time evolution but is too small to cause localized wavefunction to spread out to measurable non-classical states. We are thus dealing with four separate limits, two of which certainly do not commute. There is no algebraic expression that we can rely on, since the stochastic character of the time evolution can only be simulated numerically. It is clear, however, that the limit \( b \rightarrow 0 \) implies that the frequency of hits must be infinite in order to achieve localization, i.e. \( dt \rightarrow 0 \). Furthermore, the threshold delimiting localized from delocalized state \( \gamma \) must be
close enough to 1 to avoid measurement ambiguities from occurring. Then, it would seem that in analogy to eq. (4.15), the wavefunction behaves in the following way:

\[
\lim_{\gamma \to 1} \lim_{b \to 0} \lim_{N \to \infty} \lim_{dt \to 0} \left( \frac{T}{dt} \prod_{n} \hat{U}(t = ndt) \right) \psi_0 \approx \psi_i
\]

Here, \(\psi_i\) represents an approximately localized function around some position \(i\) and the time evolution operator \(\hat{U}\) is discretized in timeslices \(dt\) and generated according to eq. (4.10), where each timeslice generates a different non-unitary field \(B(x,t)\). Naturally, the limit \(t \to 0\) is prioritized in order to reflect the continuity of time. Subsequently, the original non-commuting limits are taken and finally the limit \(\gamma \to 1\) is taken to define classicality.
5. Conclusion and Discussion

The main aim of this thesis has been to simulate and analyze the dynamics of spontaneous breaking of time-translation symmetry in macroscopic systems in order to explain the emergence of classical states in the thermodynamic limit. To motivate the necessity of a description in terms of SSB, it has been made obvious that macroscopical amplification of quantum states, particularly evident during measurement procedures, is described by conventional quantum mechanics in a manner irreconcilable with classical reality. The axioms neither tell us why superpositions are suppressed in the classical limit nor motivate nature’s preference to appear classically only in the position basis.

The inclusion of a ubiquitous and chaotic environment in the quantum mechanical description of measurement provides a promising explanation to the latter issue. The ensuing unavoidable entanglement between the environment and systems of interest effectively determines the preferred basis by disturbing pre-existing correlations. Only correlations expressed in a basis for which the associated observable commutes with the interaction Hamiltonian remain robust. All other correlations are destroyed before they can be recorded, thereby precluding them from our experience. The fact that interaction Hamiltonians, representing the forces between systems, are generally functions of position then matches well with the evident preference of classical systems to decohere into eigenstates of the position operator. Environmental decoherence has not proven to constitute an explanation for the appearance of definite outcomes in single measurements, as encountered for example in single shot double slit experiments. The necessary tracing out of unobserved environmental degrees of freedom from the density matrix always leads to an improper mixture of states which cannot be interpreted as an indication for the actual outcomes of a certain measurement.

The CSL framework was considered in order to serve as a comparison to unitarity symmetry breaking. This approach modifies unitary time-evolution by including a non-unitary, nonlinear and stochastic field. The non-unitarity of this term may for example be explained by considering the conflict between the general covariance of general relativity and quantum mechanical unitary time evolution, and its stochastic nature stems from a poor man’s attempt to effectively model the omnipresent perturbation that is caused by this conflict. Furthermore, in order to generate probabilities and to relieve Born’s rule from its status as postulate, such random fluctuations are indispensible for any dynamical collapse theory. However, in return, the CSL framework postulates that the gravitational fluctuations couple to individual particles and affect the wavefunction in a nonlinear way as to maximize its norm. This tendency remains unexplained and so the theory can at most function as a phenomenological account of a deeper-lying theory.

Spontaneous symmetry breaking may be the key to formulating a more fundamental mechanism that causes classicality to emerge in the thermodynamic limit. It was shown that the collective Hamiltonian of certain quantum systems exhibits a thin spectrum. The energy levels contained in this spectrum scale inversely with the particle number, and thus become degenerate in the large $N$ limit. This behaviour signals the possibility of SSB, as the ground state becomes ill-defined and infinitely fragile under perturbations in the thermodynamic limit. Such perturbations are taken to originate from environmental noise in ‘conventional’ spontaneous symmetry breaking. However, breaking time-translation sym-
metry in this manner requires a different approach because the symmetry is generated by unitary time evolution. Thus, we have attempted to break this symmetry by including a symmetry breaking field that generates non-unitary time evolution, analogous to the CSL approach. The crucial difference between both formulations consists in the fact that CSL requires a nonlinear coupling to individual particles, whereas SSB relies on groundstate degeneracy in macroscopic systems, which renders non-classical states extremely susceptible even the slightest of perturbations. To avoid the need of a more fundamental theory, these perturbations are constrained to be linear to first order.

It has been shown that for very specific initial states, the originally proposed symmetry breaking field generates a process that cannot be reversed, thus breaking unitarity. For most asymmetric initial configurations however, the wavefunction will exhibit a random walk, such that localization may occur but only for finite times. The conclusion that unitarity is spontaneously broken under non-unitary influence because (1) non-classical states tend to localize and (2) localized states are robust is incorrect due to a conflation of the notions of approximate and exact localization. Statement (1) assumes the former type, in which the E-E link is broken, and statement (2) is only correct for exactly localized states. Even if approximately localized states were to be found robust under further non-unitary evolution, it has been made clear that Born’s rule is blatantly violated. The Gaussian tails of the symmetry breaking field always benefit the centre of a finite region of Hilbert space, regardless of the initial wavefunction.

Since the non-unitary order parameter field is supposed to represent the energy fluctuations induced by instability of superposed spacetimes, there is some choice as to which field to use. A field in the form of a random field was shown to yield Born statistics, while at the same time meeting the conditions of asymmetry and vanishing size in the relevant limit. However, the dynamics generated by the random field still represent a random walk, meaning that permanent localization is never ensured. It would seem that such behaviour cannot be generated by linear stochastic dynamics unless some slowdown is introduced such that the random walk has a final position. This can be achieved by considering the unitary evolution which effectively spreads out the wavefunction, altering the susceptibility of the wavefunction to non-unitary hits. Whereas a delta-function will evolve under non-unitary time evolution as a function of only a single entry, more spread out Gaussians are hit by a larger number of entries. Since single entry hits can both increase and decrease components, the net effect of a random field hit will diminish with the number of components in a spread out delta-function. Thus, in the right limits and with a number of implicit assumptions that will be further discussed in the next section, it was shown that the probability of a measurement ending up in a non-classical state vanishes due to SSB. This constitutes a solution to the problem of definite outcomes reached merely by the inclusion of a linear and vanishingly small non-unitary order parameter field to the Schrödinger equation. Taken together with the emergence of Born’s rule through the action of a randomly fluctuating field and incorporating the solution to the preferred basis problem in terms of decoherence, the proposed dynamics successfully simulate wavefunction collapse without the introduction of nonlinear physics or an appeal to ensemble averaged quantities.

Given the success of SSB in wide range of fields, it may be argued to be a fundamental mechanism that we can rely on when redefining the postulates of quantum mechanics. In that sense, whereas CSL lacks a fundamental clarification regarding the origin of the non-linear behaviour of gravitationally induced fields, a dynamical collapse theory in terms of SSB has the advantage of relying on a well-tried concept, adding credence to the approach.
However, giving up nonlinearity comes at a cost. It has turned out that a linear stochastic modification to the Schrödinger equation does not straightforwardly yield Born’s rule while at the same time guaranteeing the irreversibility of localization. Born’s rule has been shown to emerge for a symmetry breaking field which fluctuates both in space and time and we have attempted to show that the interplay between unitary and non-unitary time evolution may generate Born’s rule and force the macroscopic wavefunction to localize permanently at the same time. It is clear that without including nonlinear physics into quantum theory, the traditional notion of wavefunction localization, in which the E-E link is upheld and localized states are represented by delta functions, must be let go of in a collapse scenario. After all, any linear evolution of an initial superposition will inevitably produce a superposition as outcome. Thus, in order for a linearly modified Schrödinger to correctly produce wavefunction collapse behaviour, the localized wavefunction must be represented as a ‘packet’, consisting of vector components of nonzero length distributed over a distance that is at most equal to the best resolution a measuring device could possibly attain. Due to the continuous nature of the position basis, this is not hard to imagine since it is always conceivable that a perceived delta function really represents a Gaussian function that is too sharp to measure.

Then, taking the sharpness of the Gaussian $\gamma$ as a measure for the extent to which a wavefunction has localized, it was shown that in the right limits, the probability for a delocalized macroscopic object to appear vanishes. First, the limit $\gamma \to 0$ was taken in order to ensure continuity of time. Secondly, the two noncommuting limits of $b$ and $N$ were taken. If the limit $N \to \infty$ is taken first, the non-unitary component becomes infinitely strong, immediately localizing the wavefunction with probability that approaches certainty. Unitary evolution subsequently spreads out the localized wavefunction, such that the its susceptibility to non-unitary influence diminishes gradually, eventually leading to frozen localized wavefunction. Conversely, if the limit $b \to 0$ is taken first, the non-unitary component vanishes, the wavefunction will simply evolve according to the conventional Schrödinger equation. Since the model that was considered features an extensive quantity as unitary component, macroscopic systems are static in the latter limit.

It must be emphasized that although the results of section 4.3.2 were derived by using the continuity of space in a smart and sound way, we have not been able to produce the same results by numerically simulating the modified Hamiltonian. Specifically, the decreasing effect of the non-unitary field as a result of the interplay between unitary and non-unitary time evolution has not proven to be easily obtainable. It would be interesting to quantify this effect more thoroughly.
References


[23] D. Wallace. What is orthodox quantum mechanics?, April 2016. (p. 15)


[31] M. Schlosshauer. Decoherence, the measurement problem, and interpretations of quantum mechanics. 76, 01 2004. (p. 16)


[56] K.E. Morrison. Random Walks with Decreasing Steps. (p. 48)