Pricing derivatives under CVA, DVA and funding costs

Student: Manuela Facchini (10085564)  

Supervisors:  
Dr. Peter Spreij (UvA)  
Dr. Jaroslav Krystul (Double Effect)  
Dr. Bert-Jan Nauta (Double Effect)

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Abstract:

In this master’s thesis we are going to discuss different approaches to the pricing problem of derivatives when counterparty credit risk and funding costs are taken into account. We first focus on approaches that consider fixed funding costs and compare their methods. In particular, we will analyze more accurately one of the proposed pricing equations: a discrete time backward induction equation. However, we note that also other methods yield a similar structure: the inclusion of fixed funding costs leads to a recursive pricing problem. Subsequently we address an approach that assumes funding costs which reflect the asset composition of the party asking for funding. We present a balance sheet model for this party and derive a pricing equation. From this pricing equation it is then possible to conclude that the risky price of a derivative does not depend on the funding rate.

Keywords:

Counterparty credit risk (CCR), credit valuation adjustment (CVA), debt valuation adjustment (DVA), funding costs, elastic case, default time, defaultable claims.
Chapter 1

Introduction

The purpose of this thesis is to study the pricing of over-the-counter (OTC) derivatives when counterparty credit risk (possibly of both the counterparty and the investor) as well as funding costs are taken into account. The main approach to this pricing problem is already settled. It consists in calculating first the price of the derivative considering both parties default-free and subsequently account for bilateral counterparty credit risk and funding costs. This procedure leads to an adjusted price of the derivative. How to account for bilateral counterparty credit risk and funding costs in a consistent framework is still a topic of ongoing discussion and different approaches have been proposed. In particular, we will focus on two different approaches: the first assumes fixed funding costs, whereas the second assumes funding costs which reflect the asset composition of the counterparty asking for funding. This second approach can be referred to as \textit{Elastic Case}.

1.1 Counterparty Credit Risk

In this section we will briefly explain what unilateral and bilateral counterparty credit risk is and what the motivations for studying bilateral counterparty credit risk are. Further, we want to highlight the relationship between these two concepts and funding costs, so that a better understanding of the formulas in the following chapters can be achieved.

According to [Gregory, 2009], counterparty credit risk (CCR) is the risk that a counterparty in a financial contract will default prior to the expiration of the contract and fail to make future payments. CCR became more and more important over the last few years, especially for OTC derivatives. Scenarios in which (large) counterparties may default are more realistic nowadays and this led to the necessity of including CCR in the fair valuation of contracts. Taking counterparty credit risk into consideration when valuing a derivative is intuitively motivated in [Brigo, 2011b, p. 15-16]: “Clearly, all things being equal, we would always prefer entering a trade with a default-free counterparty than with a default risky one. Therefore we charge the default risky one a supplementary amount besides the default-free cost of the contract.” First, the pricing counterparty would consider only the risk of default of the other counterparty by introducing an adjustment term, referred to as credit valuation adjustment (CVA), without considering its own risk of default. The CVA term should be subtracted from the value of the derivative computed by assuming that both counterparties are default-free. We will refer to this value as \textit{clean value} or \textit{risk-free value}. This way of
integrating CCR yields in general different prices for the two counterparties. Therefore, the pricing counterparty has to consider its own risk of default as well and adjust the risk-free value of a derivative by introducing a second term that accounts for the credit risk of the investor and is referred to as debt valuation adjustment (DVA). When valuation of a contract is performed, it is important to specify from which point of view the valuation occurs. In the reminder of this thesis, unless specified differently, we will consider two entities: an investor ($I$) that wants to price a deal with a counterparty ($C$). Even though counterparty credit risk applies to a more general class of contracts between financial counterparties, we will refer to them as derivatives. In what follows we will introduce in detail the factors we consider in the valuation of derivatives. We try to stay as general and abstract as possible, since we do not want to consider any special cases and different approaches to this problem yet.

1.1.1 Unilateral Counterparty Credit Risk

Taking into account the possible loss due to the default of a counterparty leads to an adjusted value $\bar{V}$ of the risk-free value $V$ of a derivative. Unilateral CVA (UCVA) assumes that the valuating counterparty is default-free. Hence, from a point of view of the investor the adjusted price will be $\bar{V}_I = V_I - UCVA_I$, where with the subscript $I$ we indicate that the value is calculated by the investor. On the other hand, for the counterparty the adjusted value would be $\bar{V}_C = V_C - UCVA_C$, where $V_I = -V_C$ [Brigo et al., 2011]. Since in general $UCVA_I \neq UCVA_C$, because the two counterparties have a different credit risk, the adjusted value calculated by $I$ is not the opposite of the adjusted value calculated by $C$. Therefore, the two parties do not agree on the risky price of the derivative.

1.1.2 Bilateral Counterparty Credit Risk

In order to achieve symmetry both counterparties have to integrate a DVA term, which accounts for their own risk of default. Clearly, the credit risk of the counterparty calculated by the investor should coincide with the DVA calculated by the counterparty. Thus, $CVA_I = DVA_C$ has to hold by consistency. The same reasoning holds from the point of view of the counterparty and hence we also must have $CVA_C = DVA_I$. Introducing this DVA term leads to a bilateral credit valuation adjustment. However, in [Brigo et al., 2011] it is shown that a mere introduction of a unilateral DVA (UDVA) term is not enough to consider bilateral credit valuation adjustment consistently. Below we explain briefly what the problem with this approach is.

Simplified formula: One possible way to consider bilateral CVA, is to include both unilateral adjustments (UCVA and UDVA) when calculating the adjusted value of a derivative. By doing so, we can derive the following relations

$$\bar{V}_I = V_I - UCVA_I + UDVA_I$$

$$= -V_C - UDVA_C + UCVA_C$$

$$= -(V_C - UCVA_C + UDVA_C)$$

$$= -V_C.$$

This chain of equalities shows that price symmetry is attained when introducing the
UDVA term. However, there is a problem in the reasoning leading to (1.1): no first to
default time is considered.

First to default event: The approach above accounts for the default risk of both parties.
Nevertheless, each of the adjustment terms is calculated as if only one party could
default. As a consequence, in (1.1) there is no term taking into account who is the first
counterparty to default. Therefore, the terms UCVA and UDVA need to be adjusted in
order to incorporate the first to default time. In the following chapters we will always
consider the bilateral adjustment that takes into account the first to default time and
call the adjustments simply CVA and DVA.

1.1.3 Funding Costs

With funding costs we refer to the interest rate paid by financial institutions for the funds that
they deploy in their business. At the moment there is no standard procedure to include both
CCR and funding costs in the fair valuation of a contract. First, in [Pallavicini et al., 2011,
p. 5] it is explained that funding costs can not be taken into account by simply introducing a
funding cost adjustment (FCA) term into (1.1). Second, funding costs are (possibly) different
for each party and thus, including them in the valuation leads to an asymmetric risky price.
As a consequence, the counterparties would not agree on the derivative price anymore, see
[Burgard and Kjaer, 2012]. Third, there are different opinions on whether and when funding
costs should be taken into account, see for example [Hull and White, 2012], [Nauta, 2012] or
[Laughton and Vaisbrot, 2012].

When considering funding costs it needs to be decided whether borrowing and lending rates
are assumed to be the same. If different rates are assumed, then funding and investing
rates are different as well and it is possible to consider deal specific funding costs. This
case can be found in [Pallavicini et al., 2011]. What is more, it is important to distinguish
between fixed funding spreads\footnote{With funding spread we mean the difference between the funding rate and the risk-free rate.} and funding spreads that adjust after each new transaction,
see [Nauta, 2012]. Assuming that the investor and his counterparty have a fixed funding
spread, taking funding costs into account results in a recursive relation between them and
future cash flows originating from the derivative contract. On the other hand, if funding
spreads are supposed to be adjusted after each transaction, [Nauta, 2012] shows that for
specific cases funding costs do not influence the risky price. This result would lead again to
a symmetric price.

1.2 Problem Definition

As we mentioned at the beginning of this chapter, the purpose is to compute the risky value
of a derivative, in particular we will price an option considering counterparty risk and funding
costs. Other factors influencing the risky price, such as collateral\footnote{Collateral is an asset pledged by a borrower to a lender, usually in return for a loan. However, collateral is also used for OTC derivatives to manage credit exposure among financial market participants.}, close-out netting rules\footnote{Close-out netting rules reduce all cash flows happening at default between the counterparties to one single net cash flow before recovery is applied.} or
re-hypothecation\footnote{Re-hypothecation allows the use of collateral for purposes outside the deal for which it has been posted.} are, for simplicity, neglected in our assumptions. For the pricing we decided
to take over the approach presented in [Pallavicini et al., 2011] for fixed funding costs\(^5\) and the approach proposed in [Nauta, 2012], which corresponds to the Elastic Case. Moreover, we want to extend the approach in [Nauta, 2012, p. 3-5] to multiple funding intervals and considering a general derivative. Since in [Nauta, 2012] it is showed that on one funding interval the risky value of a bond does not depend on funding costs, the expected result for multiple funding intervals is that the risky value of a derivative does not depend on funding costs either.

The remainder of this thesis is organized as follows. In chapter 2 we compare different approaches to the pricing problem including bilateral counterparty risk and funding costs in a consistent framework. Chapter 3 delves deeper into the chosen approach and states the pricing formulas derived to compute the risky value of a derivative. In chapter 4 we show the existence of a replicating strategy for defaultable claim and discuss briefly the hedging strategy chosen in [Pallavicini et al., 2011]. Finally, in the last chapter, we discuss the Elastic Case and extend it to multiple funding intervals.

\(^5\)This decision will be motivated in the following chapter.
Chapter 2

Literature Review

After introducing the main topic, we continue with some notation before comparing different approaches to the pricing problem mentioned in the introduction.

2.1 Notation

The deterministic parameter $R^X \in [0,1]$ denotes the recovery rate of the market value of the transaction that the counterparty of $X$ gets when party $X$ defaults. This notation leads automatically to the definition of loss given default, $\text{LGD}^X = 1 - R^X$, which is the loss incurred by the counterparty upon default of party $X$, [Pallavicini et al., 2011, p. 8]

N.B. If no recovery rate is specified, we assume $R^X = 0$.

Throughout this thesis we denote the maximum of a general random variable $Y$ by $Y^+ = Y(\omega)^+ := \max(Y(\omega), 0)$ and the the minimum by $Y^- = Y(\omega)^- := \min(Y(\omega), 0)$.

Furthermore, let $T \in \mathbb{R}_+$ denote the maturity of the derivative and let $t \in [0, T]$ be a time index. Consider now a measurable space $(\Omega, \mathcal{G}_T)$, with $\mathcal{G}_T$ defined below. All random times we will consider are $[0, T] \cup \{\infty\}$-valued.

Let $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ be the filtration containing the information of the default-free market, i.e. $\mathcal{F}$ represents all the observable market quantities but the default events, and let $\mathcal{H} = \{\mathcal{H}_t\}_{t \in [0, T]}$ be a filtration such that $\tau_I$, the default time of the investor, and $\tau_C$, the default time of the counterparty, are $\mathcal{H}$-stopping times. Then, the first to default time $\tau$, defined by $\tau := \tau_I \wedge \tau_C$, is an $\mathcal{H}$-stopping time as well.

Successively we consider the filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0, T]}$, with $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$. Clearly $\tau_I$ and $\tau_C$ are both stopping times w.r.t. $\mathcal{G}$ since $\mathcal{H}_t \subset \mathcal{G}_t$ for all $t$. In the remainder of this thesis we will assume all the processes to be adapted to $\mathcal{G}$. Consider now the stopped filtration

$$\mathcal{G}_{\tau_X} = \{G \in \mathcal{G}_\infty : G \cap \{\tau_X \leq t\} \in \mathcal{G}_t, \ \forall \ t \in [0, T]\},$$

where $\mathcal{G}_\infty := \sigma(\mathcal{G}_t, t \in [0, T])$ and $X \in \{I, C\}$. We endow the filtered measurable space $(\Omega, \mathcal{G}, \mathcal{G}_T)$ with a probability measure $\mathbb{P}$. In the following $\mathbb{P}$ will always be referred to as the real world probability measure. Next to $\mathbb{P}$ we also consider a risk neutral probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ (see section 3.1 for the definition) and denote the conditional expectation under $\mathbb{Q}$ given $\mathcal{G}_t$ or $\mathcal{G}_{\tau_X}$ by $\mathbb{E}_t$ or $\mathbb{E}_{\tau_X}$, respectively. Since we work with a finite time horizon, we assume from now onwards that $t \in [0, T]$. Note also that we assume all filtrations to satisfy the usual conditions, i.e. for a general filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ it holds that the filtration is
right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ null-sets of $\mathcal{F}_T$. Note that since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $(\Omega, \mathcal{G}_T)$ the $\mathbb{P}$ null-sets of $\mathcal{G}_T$ coincide with the $\mathbb{Q}$ null sets of $\mathcal{G}_T$. We define $\text{CVA}_T$ as the expected cost to the investor caused by the default of the counterparty, prior to the default of the investor self, and $\text{DVA}_T$ as the expected cost to the counterparty caused by the default of the investor, prior to the default of the counterparty. The following definition is adapted from [Brigo, 2011a].

**Definition 2.1.1.** Let $D(t, \tau)$ denote the discount factor associated to the money market account and let $\text{NPV}(\tau)$ denote the net present value\(^1\) of the derivative at time $\tau$, for any stopping time $\tau$. A subscript $\mathcal{I}$ or $\mathcal{C}$ indicates that the net present value is calculated from a point of view of the investor or the counterparty, respectively. Then, we define the CVA term from a point of view of the investor by

$$
\text{CVA}_{\mathcal{I},t} := \mathbb{E}_t \left[ 1_{\{t < \tau \leq T\}} \text{LGD}^{\mathcal{C}} D(t, \tau) \left| \text{NPV}_{\mathcal{I}}(\tau) \right|^+ \right]
$$

(2.1)
and the DVA term from a point of view of the investor by

$$
\text{DVA}_{\mathcal{I},t} := \mathbb{E}_t \left[ 1_{\{t < \tau \leq T\}} \text{LGD}^{\mathcal{I}} D(t, \tau) \left| -\text{NPV}_{\mathcal{I}}(\tau) \right|^+ \right].
$$

(2.2)

Note that by definition it holds $\text{NPV}_{\mathcal{I}}(\tau) = -\text{NPV}_{\mathcal{C}}(\tau)$. In [Brigo, 2011b, p. 16] Expression (2.1) is interpreted as the price of “an option on the residual value of the portfolio, with a random maturity given by the default time of the counterparty.”

In the following section we look at different approaches on how to include funding costs. It is important to notice that when considering funding costs, we substitute the term $\text{NPV}$ with a general close-out amount.

### 2.2 Summaries

In order to facilitate the comparison we introduce partly the same notation when reviewing the following papers, mainly taking over the notation from [Pallavicini et al., 2011]. There are two aspects which we would like to point out before considering the papers separately. The first is that in all the papers the assumption of fixed funding spreads is made. Second, in [Burgard and Kjaer, 2009] and [Crépey, 2011] an additive decomposition of the form $\bar{V} = V - A$ is assumed, where $A$ includes the adjustment terms stemming from the bilateral credit risk and the funding costs.

The first paper we consider is [Burgard and Kjaer, 2009]. The main result is an extended Black-Scholes partial differential equation (PDE) representation for the risky value $\bar{V}$ of a derivative that takes into account bilateral counterparty credit risk and funding costs. It is assumed that the underlying asset has no default risk and that its price process $S(t)$ follows a Markov process with generator $A_t$. In particular $S(t)$ is assumed to follow a time in-homogenous geometric Brownian motion, which under $\mathbb{P}$ is specified by

$$
\frac{dS}{S} = \mu(t) dt + \sigma(t) dW,
$$

where $W(t)$ is a Brownian motion and $\mu(t)$ and $\sigma(t)$ are deterministic functions of $t$. The risky value of the derivative on $S$ is denoted by $\bar{V}(t, S)$ and in order to determine it, a replication

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\(^1\)The net present value is by definition the sum of discounted cash flows of an investment.
strategy consisting of a hedging portfolio and a cash account, whose value is denoted by $F(t)$, are considered.

To define the hedging portfolio two additional traded assets are introduced: $P_I$ and $P_C$. These are default risky, zero-recovery, zero-coupon bonds of the investor and its counterparty, respectively, with notional 1 at maturity. The portfolio value that replicates the risky value of the derivative at time $t$ is then given by

$$-\bar{V}(t) = \delta(t)S(t) + \alpha_I(t)P_I(t, T) + \alpha_C(t)P_C(t, T) + F(t),$$

where $\delta$ denotes the units of $S$ to be bought, $\alpha_I$ units of $P_I$ and $\alpha_C$ units of $P_C$. The hedging strategy includes the (re-)purchase of the investor’s bonds in order to lower its own credit risk. In [Burgard and Kjaer, 2009] it is shown that the purchase is completely funded by the cash account $F$ and hence there is no need to issue more debt, since issuing more debt via bonds to buy back bonds would not change a party’s credit risk. An important assumption characterising the approach described in this paper is that cash accounts accrue at different deterministic rates: $r(t)$ denotes the risk-free rate, $r_I(t)$ the yield on the recovery-less bond of the investor, $r_C(t)$ the yield on the recovery-less bond of the counterparty and $r_F(t)$ the funding rate at which the borrowed cash for the replication strategy accrues. These rates, together with the spreads originating from these rates, e.g. the spread on bond $P_I$ defined by $\lambda_I := r_I - r$, the spread on bond $P_C$ defined by $\lambda_C := r_C - r$ and the funding spread defined by $s_F := r_F - r$, determine the dynamics of the cash account. Once the dynamics are defined and the self-financing condition on the replicating portfolio is imposed, applying Itô’s Lemma leads to a PDE whose solution is $\bar{V}$ (note that the $t$ and $S$ variables are omitted in the first line)

$$\begin{cases}
\partial_t \bar{V} + A_t \bar{V} - r \bar{V} & = (\lambda_I + \lambda_C) \bar{V} + s_F M^+ - \lambda_I (R^2 M^- + M^+) - \lambda_C (R^C M^+ + M^-) \\
\bar{V}(T, S) & = H(S)
\end{cases}$$

(2.4)

where $A_t$ is defined by

$$A_t V(t, S) := \frac{1}{2} \sigma(t)^2 S(t)^2 \partial^2_s V(t, S) + (q_S(t) - \gamma_S(t)) S(t) \partial^1_S V(t, S).$$

Here $q_S$ and $\gamma_S$ are rates related to financing costs and dividends, respectively, $M$ the mark-to-market\(^3\) value of $V$ to the investor\(^4\) and $H(S)$ the payoff of the derivative. In equation (2.4) funding costs are taken into account through the second term on the right hand side. All the other terms on the right hand side are related to counterparty risk and one can see that in absence of default risk, equation (2.4) reduces to the standard Black and Scholes equation. Subsequently two different cases regarding the mark-to-market value are distinguished. The first assumes that the mark-to-market value is equal to the risky value of the derivative and

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\(^2\)This strategy has been proposed also in [Piterbarg, 2009]. However, it is only possible in specific cases, see [Burgard and Kjaer, 2012, p. 2].

\(^3\)Mark-to-market is a measure of the fair value of accounts that can change over time, such as assets and liabilities, [Investopedia, 2013].

\(^4\)By International Swaps and Derivatives Association (ISDA) specifications “the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the view of the surviving party) just prior to default, whereas the full mark-to-market value has to be paid to the defaulting party if the mark-to-market value is negative (from the view of the surviving party)”, [Burgard and Kjaer, 2009, p. 5]. However, it is unclear whether funding costs should be included in the mark-to-market value of the derivative and thus a general mark-to-market value $M(t, S)$ is considered in [Burgard and Kjaer, 2009].
the second case assumes that it is equal to the risk-free value of the derivative. Which one of the two cases is applied for pricing a derivative is contract specific. Furthermore, for both mark-to-market values, two different funding rates are assumed: \( r_f = r \) and \( r_f = r + s_F \). Finally, after focusing on the special case of a call option, the paper concludes by proposing possible extensions to a more general framework than the one imposed.

Subsequently we consider [Pallavicini et al., 2011], which will be discussed in more detail in Chapter 3. In this paper credit and debt valuation adjustments along with funding costs are taken into account in a coherent way by means of a recursive formula. The fundamental contrast to the previous approach is the requirement to model default times\(^5\). In [Pallavicini et al., 2011] it is assumed that the risky price \( \tilde{V}_t(C; F) \) of a derivative, with \( C \) the collateral account and \( F \) the cash account needed for trading, is determined by all the cash flows happening when the trading position is entered. The following cash accounts are considered: the sum \( \Pi(t, T) \) of all discounted payoff terms in the interval \( (t, T] \), the collateral margining costs \( \gamma(t, T; C) \) within the interval \( (t, T] \), the funding and investing costs \( \varphi(t, T; F) \) in the interval \( (t, T] \) and the on-default cash flow \( \theta_\tau(C, \varepsilon) \). \( \varepsilon \) is the amount of losses or costs the surviving party would incur upon a default event (in the following we refer to it as close-out amount\(^6\)) and \( \tau \) is the first default time between the two parties, where \( Q(\tau = \tau_C) = 0 \) is assumed. This approach leads to the following expression for the Bilateral Collateralized Credit and Funding Valuation Adjusted (BCCFVA) price of a derivative

\[
\tilde{V}_t(C; F) = \mathbb{E}_t [\Pi(t, T \wedge \tau) + \gamma(t, T \wedge \tau; C) + \varphi(t, T \wedge \tau; F)] + \mathbb{E}_t [1_{\{\tau < T\}}D(t, \tau)\theta_\tau(C, \varepsilon)]. \tag{2.5}
\]

The FCA term mentioned in section 1.1.3, enters the formula through \( \varphi(t, T \wedge \tau; F) \), whereas CVA and DVA enter the formula through \( \Pi(t, T \wedge \tau) \) and \( \theta_\tau(C, \varepsilon) \). We would like to point out that the recursion in the formula is introduced through the funding costs\(^7\), since “the derivative price at time \( t \) depends on the funding strategy after \( t \), and in turn the funding strategy after \( t \) will depend on the derivative price at following times”, see [Pallavicini et al., 2011, p. 13]. By including funding costs we thus consider the positions entered at time \( t \) to obtain the cash amount needed to establish the hedging strategy. As in the previous approach the hedging strategy, perfectly replicating the derivative to be priced, is formed by a cash account \( F_t \) and (the value of) a portfolio of hedging instruments \( H_t \). Hence, we have \( \tilde{V}_t(C; F) = F_t + H_t \).

The approach differs from the other two methods through the introduction of the stopping time \( \tau \). Since in [Pallavicini et al., 2011] distinction is made between funding and investing costs, \( \varphi(t, T; F) \) depends on the liquidity policy of the investor and thus the funding strategy needs to be modeled explicitly. Two examples are given, the first considering funding via the bank’s treasury and the second funding directly on the market\(^8\). Moreover, this approach distinguishes between funding and investing costs. These costs are denoted with \( P^{f+}_t(T) \) and

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\(^5\)Note that this paper is not concerned with the modeling of default times. To further investigate this aspect the authors refer to [Brigo et al., 2009].

\(^6\)As in [Burgard and Kjaer, 2009], since the mark-to-market value is not specified by regulations, it is possible to consider different close-out amounts.

\(^7\)This is made explicit in section 3.3.

\(^8\)This second case considers also a recovery rate for the funder upon default of the investor, which is not considered in [Burgard and Kjaer, 2012].
2.2. SUMMARIES

$P^f_t(T)$, respectively. The (forward) funding/investing rates defined by

$$f_t(T)^\pm := \frac{1}{T-t} \left( \frac{1}{P^f_t(T)} - 1 \right)$$

are assumed to be fixed with respect to a change in the asset composition of a given party and hence the derivatives value depends on the funding rate.

Another difference to [Burgard and Kjaer, 2009] and [Crépey, 2011] is that all cash flows are discounted by using the risk-free discount factor $D(t,T)$. The authors use a risk-free discount factor because all the costs and risks concerning the deal are taken into account by considering the respective cash flows and hence no additional spreads to the risk-free rate need to be considered when discounting these cash flows.9

In order to consider funding costs a partition $t = t_1 < \cdots < t_m = T$, $m \geq 2$, is chosen and at the beginning of each interval $(t_j, t_{j+1})$, $j = 1, \ldots, m-1$, the cash amount obtained in the previous time interval, including funding costs, has to be reimbursed and a new funding (depending on the derivative’s risky value at time $t_j$) is asked. In [Pallavicini et al., 2011] it is then shown that under specific assumptions10 and substituting the corresponding expressions for funding and investing costs leads to

$$\tilde{V}_t(0; F) = \mathbb{E}_t \left[ \sum_{j=1}^{m-1} \mathbb{1}_{\{\tau = \tau_C > t_j\}} \left( \prod_{i=1}^{j} \frac{P^f_{t_i}(t_{i+1})}{P^f_{t_i}(t_{i+1})} \right) \Pi(t_j, t_{j+1}) D(t_j, t_j) \right],$$

with $t_1 = t$, $P^f_{t_i}(t_{i+1})$ the zero-coupon bond price, $D(t_i, t_{i+1})$ the discount factor and by considering a close-out amount equal to11

$$\varepsilon_{\sigma, \tau} = \mathbb{E}_\tau \left[ \Pi(\tau, T) + \varphi(\tau, T; (\bar{V})) \right].$$

A more intuitive expression for the risky price is obtained by letting the mesh size of the partition introduced above go to zero (we state the result without proof):

$$\mathbb{E}_t \left[ \int_t^T \Pi(u, u + du) e^{-\int_u^t dv (f^+_u + \lambda^{<\mathcal{I}})} \right],$$

where $\lambda^{<\mathcal{I}}$ is the counterparty’s default intensity conditional on the investor not having defaulted earlier.

To conclude, [Pallavicini et al., 2011] derives recursive solutions of (2.5) for implementation.

Furthermore, we mention the backward stochastic differential equation (BSDE) approach studied in [Crépey, 2011]. Here the point of view is the one of the counterparty and not the investor. The considered framework is very general and allows to analyze a wide range of different cases, such as simultaneous default, including all cases from [Burgard and Kjaer, 2009] and [Pallavicini et al., 2011]. The main results are concrete strategies to hedge counterparty

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9 This approach is also shared by [Morini and Prampolini, 2010].

10 The case with zero collateral and $H_t = 0$ is considered. Furthermore zero recovery, positive payoff and negligible mismatch between close-out amounts calculated by the investor and the counterparty are assumed.

11 Note that this close-out amount corresponds to a mark-to-market value $M(t, S)$ equal to the risk-free value of a derivative in [Burgard and Kjaer, 2009].
risk (including also funding costs) and minimize the variance of the hedge error in form of BSDE’s and, in specific cases, in form of PDE’s.

Finally we refer to [Fries, 2010]. This approach does not aim to calculate the risky price through a credit and funding adjustment, but includes bilateral counterparty risk and funding costs by discounting the net amount needed over a period by an appropriately defined forward bond. This forward bond contains the time-value of a cash flow (taking thus funding costs into account) as well as the credit risk of a considered party. The main result in [Fries, 2010] is a recursive formula to valuate the net cash position required at the present time $t$, accounting for funding costs, credit risk and collateral posting.

2.3 Chosen Approach

We decided to elaborate the approach proposed in [Pallavicini et al., 2011]. The motivations for this decision are multiple. First, pricing a derivative by the cash flows originating from the deal is an optimal basis to compare the approach with the balance sheet approach proposed in [Nauta, 2012]. Second, the possibility to distinguish between funding and investing costs and to analyze different funding policies includes desirable features for a wide scenario consideration. Third, we note that the approach in [Pallavicini et al., 2011] allows for general derivatives, contrary to [Burgard and Kjaer, 2009] and [Crépey, 2011] where only derivatives with simple contingent claims are considered$^{12}$. Finally, we point out that like [Pallavicini et al., 2011], both [Burgard and Kjaer, 2009] and [Crépey, 2011] conclude that funding costs are not a simply additive term but need to be taken into consideration through a recursive formula. However, the BSDE approach proposed in [Crépey, 2011] is very technical and the PDE approach proposed in [Burgard and Kjaer, 2009] is not appropriate for high dimensional problems.

$^{12}$Although it is mentioned in [Burgard and Kjaer, 2009] that an extension to derivatives with more general payments, such as interest rate swaps, is possible, we prefer to focus on the main goal of this thesis, avoiding additional derivations.
Chapter 3

Derivative Pricing under CCR and Funding Costs

In this chapter we are going to describe in detail the chosen approach. We will introduce the mathematical framework assumed in [Pallavicini et al., 2011], give additional definitions and adapt the results to the case with zero collateral. Moreover, we will integrate the chosen approach with more recent results from [Pallavicini et al., 2012].

3.1 Preliminaries

Most of the definitions and results in this section are derived from [Filipović, 2009] and the set of lecture notes [van der Vaart, 2005]. Let $P(t,T)$ denote the value of a risk-free zero-coupon bond with maturity $T$ (also called $T$-bond). A $T$-bond makes a sure payment of $x$ units of cash at maturity, i.e. $P(T,T) = x$. Without loss of generality we can assume $x = 1$. Hence, we can see $P(t,T)$ as the time-$t$ value of 1 unit of cash at time $T$. We assume the following conditions on $P(t,T)$ and the market we are going to consider:

- there exists a frictionless market for $T$-bonds for all $T > 0$;
- $P(T,T) = 1$ for all $T$;
- for a fixed $t$, $P(t,T)$ is $\mathbb{P}$-a.s. continuously differentiable in the $T$-variable.

The function $T \mapsto P(t,T)$ is called discount curve and by the third point of the assumptions stated above it is a smooth curve. This reflects the fact that for a given $t$, $P(t,T)$ should smoothly vary with $T$. On the other hand, bond prices are highly sensitive to changes on the market and unknown before time $t$. This is modeled by letting $\{P(t,T); 0 \leq t \leq T\}$ be an adapted stochastic process. In particular we will assume that it is a nonnegative, càdlàg semimartingale on the stochastic basis $(\Omega, \mathcal{G}_T, \mathcal{G}, \mathbb{P})$. Recall that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ contains the information available in the market at time $t$ and $\mathcal{H}_t$ the information about the default times at time $t$. All other asset price processes are assumed to be semimartingales on $(\Omega, \mathcal{G}_T, \mathcal{G}, \mathbb{P})$.

We now introduce different rates used throughout this thesis. Let $t$ be the current time, and let $T$ and $S$ be two time points such that $t \leq T \leq S$ hold.
Definition 3.1.1 (Simply compounded forward rate). The simply compounded forward rate for \([T, S]\) contracted at \(t\) is given by
\[
F(t; T, S) := \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right)
\]
and the time-\(t\) forward price for the zero-coupon bond spanning \([T, S]\) is
\[
P(t; T, S) := \frac{P(t, S)}{P(t, T)}.
\]

Definition 3.1.2 (Simply compounded spot rate). The simply compounded spot rate for \([t, T]\) is given by
\[
F(t, T) := F(t; t, T) = \frac{1}{T - t} \left( \frac{1}{P(t, T)} - 1 \right).
\]

Definition 3.1.3 (Continuously compounded forward rate). The continuously compounded forward rate for \([T, S]\) contracted at \(t\) is given by
\[
R(t; T, S) := - \log P(t, S) - \log P(t, T).
\]

Definition 3.1.4 (Continuously compounded spot rate). The continuously compounded spot rate for \([t, T]\) (or simply yield) is given by
\[
R(t, T) := R(t; t, T) = - \frac{\log P(t, T)}{T - t}.
\]

Definition 3.1.5 (Instantaneous forward rate). The instantaneous forward rate (or simply forward rate) with maturity \(T\) contracted at \(t\) is given by the limit, if it exists,
\[
f(t, T) := \lim_{S \downarrow T} R(t; T, S) = - \frac{\partial \log P(t, T)}{\partial T}.
\]

The forward curve at time \(t\) is then given by the map \(T \mapsto f(t, T)\). Note that if the map \(T \mapsto - \log P(t, T)\) is \(\mathbb{P}\)-a.s. continuously differentiable, then it follows from Definition 3.1.5 and from \(P(T, T) = 1\) that
\[
P(t, T) = e^{- \int_t^T f(u, T) \, du}.
\]

In this case we can give the following definition.

Definition 3.1.6 (Instantaneous short rate). The instantaneous short rate (or simply short rate) at time \(t\) is given by
\[
r(t) = f(t, t) := \lim_{T \downarrow t} f(t, T).
\]

Subsequently we define the continuously compounded money market account \(\beta(t)\), which can be identified with an asset growing at time \(t\) instantaneously at short rate \(r(t)\).

Definition 3.1.7 (Continuously compounded money market account). The continuously compounded money market account \(\beta(t)\) is defined by the ODE
\[
d\beta(t) = r(t)\beta(t) \, dt, \quad \beta(0) = 1.
\]
3.2. CASH FLOWS

Solving the equation above yields
\[ \beta(t) = e^{\int_0^t r(u) \, du}. \]
Recall now that a numeraire is any asset with price process \( N(t) \), s.t. \( N(t) > 0 \) for all \( t \). Then, we note that \( \beta(t) \) is a numeraire and we will assume its existence in the market.

**Definition 3.1.8 (Numéraire pair).** Let \( A \) be a vector of asset price processes of the considered market. A numéraire pair \((N, N)\) consists of a probability measure \( N \) on a measurable space that is equivalent to \( \mathbb{P} \) and a numéraire \( N \), s.t. \( A/N \) is a \( N \)-martingale.

Together with \( \beta(t) \) we also assume the existence of a measure \( \mathbb{Q} \) such that \((\mathbb{Q}, \beta(t))\) is a numéraire pair. By the First Fundamental Theorem of Asset Pricing it follows then that the market is arbitrage free. Let \( V(t) \) be the time-\( t \) value of a counterparty risk-free derivative. The \( \mathcal{G}_T \)-measurable random variable \( V(T) \) denotes the payment of a derivative and is assumed to have finite variance. Then Definition 3.1.8 implies the following pricing formula
\[
V(t) = \beta(t) \mathbb{E}_t^\mathbb{Q} \left( \frac{V(T)}{\beta(T)} \right). \tag{3.1}
\]
Hence, for a \( T \)-bond we have
\[
P(t, T) = \mathbb{E}_t^\mathbb{Q} \left[ e^{-\int_t^T r(u) \, du} \right]. \tag{3.2}
\]
We refer to \( \mathbb{Q} \) as the risk neutral measure. In the following we will drop the superscript \( \mathbb{Q} \) and denote the \( \mathcal{G}_t \) conditional expectation w.r.t. the risk neutral measure by \( \mathbb{E}_t \).

Another important measure in the pricing of derivatives is the forward measure, defined in the following way.

**Definition 3.1.9 (T-forward measure).** The \( T \)-forward measure \( \mathbb{Q}^T \) uses \( P(t, T) \) as a numéraire and is defined as the measure which makes \( V(t)/P(t, T) \) a martingale. Hence, it must hold
\[
V(t) = P(t, T) \mathbb{E}_t^{\mathbb{Q}^T} \left[ \frac{V(T)}{P(T, T)} \right] = P(t, T) \mathbb{E}_t^{\mathbb{Q}^T} [V(T)],
\]
where \( \mathbb{E}_t^{\mathbb{Q}^T} \) is the \( \mathcal{G}_t \)-conditional expectation with respect to \( \mathbb{Q}^T \).

From this definition it is immediate that forward prices are martingales under the forward measure (take for example \( V(t) = P(t, S) \), with \( S \geq T \)). An explicit connection between the risk neutral measure and the \( T \)-forward measure is given by the density process
\[
\mathbb{E}_t \left[ \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \right] = \frac{P(t, T)/P(0, T)}{\beta(t)/\beta(0)},
\]
where \( d\mathbb{Q}^T/d\mathbb{Q} \) denotes the Radon-Nikodym derivative of \( \mathbb{Q}^T \) w.r.t. \( \mathbb{Q} \).

3.2 Cash flows

We recall that the basic assumption made in [Pallavicini et al., 2011] is that to price a derivative we have to consider all the cash flows occurring when the trading position is entered. For the trading position that we are going to consider, these cash flows include the derivative cash flows (including the ones coming from the hedging instruments), cash flows required by the collateral margining procedure, cash flows required by the funding and investing procedures and cash flows occurring on default events.
Assumption 3.2.1. In the remainder of the thesis we assume that the trading position does not involve posting of collateral, i.e. $C = 0$. Consequently all the cash flows originating from the collateral margining procedure are zero\(^1\).

**Derivative cash flows:** The term $\Pi(t, T)$ denotes the sum of all discounted payoff terms in the interval $(t, T]$. In particular, in equation (2.5) the cash flows up to the first default time are considered. Note that on the interval $(t, T \land \tau]$ the cash flows are given by

$$\Pi(t, T \land \tau) = \mathbb{1}_{\{\tau \leq T\}}\Pi(t, \tau) + \mathbb{1}_{\{\tau > T\}}\Pi(t, T).$$

In general we assume that the considered derivatives have integrable payoffs, i.e.

$$\Pi(t, T) \in L^1(\Omega, \mathcal{G}_T, Q).$$

**Cash flows at default:** The cash flows at default, denoted by $\theta_t(\varepsilon)$, depend on which is the first counterparty to default and the conventions used to calculate the residual value of the derivative, called close-out amount and denoted with $\varepsilon$. In [Pallavicini et al., 2011] we find the following definition for the close-out amount priced at time $\tau = T \land \tau$:

$$\varepsilon_t := \mathbb{1}_{\{\tau = \tau_c < T\}}\varepsilon_{I, \tau} + \mathbb{1}_{\{\tau = \tau_c < T\}}\varepsilon_{C, \tau},$$

where $\varepsilon_{I, \tau}$ is the close-out amount priced at time $\tau$ by the investor on counterparty’s default and $\varepsilon_{C, \tau}$ the close-out amount priced by the counterparty on investor’s default\(^2\). It is also important to notice that in general one can assume the close-out amount to be risky. This means that after the first default happens, the surviving party is not assumed to be risk-free. As a matter of consistency also funding costs are included in the calculation of the residual value. This type of close-out amount is often referred to as ‘substitution close-out’. If we assume a risk-free close-out amount and neglect funding costs in its calculation, we have $\varepsilon_{I, \tau} = \varepsilon_{C, \tau} = \mathbb{E}_\tau[\Pi(\tau, T)]$. Including the funding costs in the close-out amount on the other hand leads to

$$\varepsilon_{I, \tau} = \varepsilon_{C, \tau} = \mathbb{E}_\tau[\Pi(\tau, T) + \varphi(\tau, T; F)],$$

where $\varphi(t, T; F)$ is defined in equation 3.5. More about the impact of different conventions in the calculation of the close-out amount can be found in [Brigo and Morini, 2010].

**Funding costs:** Even though [Pallavicini et al., 2011] distinguishes between funding and investing costs, for the time being we consider only funding costs in order to simplify the notation. When we take into account funding costs, a partition $t = t_1 < \cdots < t_m$, $m \geq 2$, is chosen and at the beginning of each interval $(t_j, t_{j+1}]$, $j = 1, \ldots, m - 1$, the cash amount obtained in the previous time interval, including funding costs, has to be reimbursed and a new funding (depending on the derivative’s risky value at time $t = t_j$) is asked. As mentioned in section 2.2, funding cash flows depend on the liquidity policy of the investor. In the following we consider an investor funding via bank’s treasury.

---

\(^1\)Since we look at this special case, we will redefine some of the objects introduced in [Pallavicini et al., 2011].

\(^2\)Recall that all quantities are seen from a point of view of the investor and hence $\varepsilon_{C, \tau} > 0$ means that the investor is a creditor of the counterparty.
This means that in the case of default of the investor no more cash flows occur between the funder and the investor. Hence, the cash flows occurring when entering a funding or investing position at time $t_j$ are
\[
\Phi(t_j, t_{j+1}) = \mathbb{1}_{\{\tau > t_j\}} \left( F_{t_j} - N_{t_j} D(t_j, t_{j+1}) \right),
\]
where
\[
N_{t_j} := \frac{F_{t_j}}{P_{t_j}^-(t_{j+1})} + \frac{F_{t_j}^+}{P_{t_j}^+(t_{j+1})},
\]
$P_{t_j}^\pm(T)$ representing the price of a funding contract and $P_{t_j}^\pm(T)$ the price of an investing contract, both worth one unit of cash at maturity $T$. Hence, $N_{t_j}$ is the amount the investor has to pay at time $t_{j+1}$ if $F_{t_j} > 0$ or receives if $F_{t_j} < 0$. In [Pallavicini et al., 2011] the processes $(P_{t_j}^\pm(T))_t$ are assumed to be adapted to $G$.

We are now able to determine the funding costs on the interval $(t, T]$ without taking explicitly default events into account, that is dropping the indicator function $\mathbb{1}_{\{\tau > t_j\}}$ in the cash flows above. We will see that default events are included automatically when considering funding costs on the intervals $(t, T \wedge \tau]$ and $(\tau, T]$. Hence, we define the funding costs on $(t, T]$ by summing the prices of all cash flows required by the funding and investing procedures
\[
\varphi(t, T; F) := \sum_{j=1}^{m-1} \mathbb{1}_{\{t \leq t_j < T\}} D(t, t_j) \mathbb{E}_{t_j} \left[ F_{t_j} - N_{t_j} D(t_j, t_{j+1}) \right]
\]
\[
= \sum_{j=1}^{m-1} \mathbb{1}_{\{t \leq t_j < T\}} D(t, t_j) \left( F_{t_j} - F_{t_j} - \frac{P_{t_j}^-(t_{j+1})}{P_{t_j}^+(t_{j+1})} - F_{t_j}^+ \frac{P_{t_j}^+(t_{j+1})}{P_{t_j}^-(t_{j+1})} \right), \quad (3.5)
\]
We recall that the funding and investing rates in [Pallavicini et al., 2011] are defined by
\[
f_t(T)^\pm := \frac{1}{T - t} \left( \frac{1}{P_t^\pm(T)} - 1 \right)
\]
and hence simple compounding is assumed.

The term $F_t$ represents the amount to be funded/invested at time $t$ and according to [Pallavicini et al., 2011] we have
\[
F_t = \bar{V}_t(C; F) - H_t, \quad (3.6)
\]
where $H_t$ is (the value of) a portfolio of hedging instruments. In particular, according to [Pallavicini et al., 2011, p. 17], one can “assume that the hedging strategy is constituted by a set of rolling par-swaps, so that at each time the hedge portfolio’s value process $H_t$ is zero”. This approach will be further discussed in section 4.2.

3.3 Pricing Equations

Adapting formula (2.5) to the case with zero collateral leads to
\[
\bar{V}_t(0; F) = \mathbb{E} \left[ \Pi(t, T \wedge \tau) + \varphi(t, T \wedge \tau; F) + \mathbb{1}_{\{\tau < T\}} D(t, \tau) \theta_\epsilon(0, \epsilon) \big| \mathcal{G}_t \right]. \quad (3.7)
\]
Further, we define the discounted cash flows coming from the derivative and the collateral margining procedure (excluding funding costs) on the interval \( (t_j, t_{j+1}] \) by

\[
\bar{\Pi}_T(t_j, t_{j+1}; C) := \Pi(t_j, t_{j+1} \wedge \tau) + \gamma(t_j, t_{j+1} \wedge \tau; C) + \mathbb{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) \theta_\tau(C, \varepsilon),
\]

where the parameter \( T \) shows explicitly that the close-out amount \( \varepsilon \) refers to a derivative with terminal maturity \( T \). Adapting this definition to the case \( C = 0 \) yields

\[
\bar{\Pi}_T(t_j, t_{j+1}; 0) := \Pi(t_j, t_{j+1} \wedge \tau) + \mathbb{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) \theta_\tau(0, \varepsilon).
\]

To simplify the notation we put

\[
\bar{V}_t(F) := \bar{V}_t(0; F),
\]

\[
\bar{\Pi}_T(t_j, t_{j+1}) := \bar{\Pi}_T(t_j, t_{j+1}; 0)
\]

and\(^3\)

\[
\theta_\tau(\varepsilon) := \theta_\tau(0, \varepsilon)
\]

\[
= \mathbb{1}_{\{\tau = \tau_C < \tau_F\}} \left( \varepsilon_{\bar{\tau}_\tau} - LGD^2 \varepsilon_{\bar{\tau}_\tau} \right)
\]

\[
+ \mathbb{1}_{\{\tau = \tau_F < \tau_H\}} \left( \varepsilon_{\bar{\tau}_\tau} - LGD^2 \varepsilon_{\bar{\tau}_\tau} \right).
\]

As in [Pallavicini et al., 2011], we would like to express the risky price at time \( t_j \) in terms of \( \bar{V}_{t_{j+1}} \). Using equation (3.7) as a starting point, we can recover the following expression\(^4\)

\[
\bar{V}_t_j(F) = \mathbb{E}_{t_j} \left[ \bar{V}_{t_{j+1}}(F) D(t_j, t_{j+1}) + \bar{\Pi}_T(t_j, t_{j+1}) \right]
\]

\[
- \mathbb{1}_{\{\tau > t_j\}} \left( F_{t_j}^- \frac{P_{t_j}(t_{j+1})}{P_{t_j}^-(t_{j+1})} + F_{t_j}^+ \frac{P_{t_j}(t_{j+1})}{P_{t_j}^+(t_{j+1})} - F_{t_j} \right). \tag{3.8}
\]

Assuming now \( H_t = 0 \) we get \( F_t = \bar{V}_t(F) \). By substitution into equation (3.8) and changing to the \( t_{j+1}\)-forward measure (see Section A.1 in the Appendix) we get the following set of recursive equations:

\[
\mathbb{1}_{\{\tau < t_j\}} \bar{V}_{t_j}(F) = \mathbb{1}_{\{\tau < t_j\}} \mathbb{E}_{t_j} \left[ \bar{V}_{t_{j+1}}(F) D(t_j, t_{j+1}) + \bar{\Pi}_T(t_j, t_{j+1}) \right]
\]

\[
\mathbb{1}_{\{\tau > t_j\}} \bar{V}_{t_j}^-(F) = \mathbb{1}_{\{\tau > t_j\}} P_{t_j}^- (t_{j+1}) \left( \mathbb{E}_{t_j}^{+1} \left[ \bar{V}_{t_{j+1}}(F) + \frac{\bar{\Pi}_T(t_j, t_{j+1})}{D(t_j, t_{j+1})} \right] \right)^-
\]

\[
\mathbb{1}_{\{\tau > t_j\}} \bar{V}_{t_j}^+(F) = \mathbb{1}_{\{\tau > t_j\}} P_{t_j}^+(t_{j+1}) \left( \mathbb{E}_{t_j}^{+1} \left[ \bar{V}_{t_{j+1}}(F) + \frac{\bar{\Pi}_T(t_j, t_{j+1})}{D(t_j, t_{j+1})} \right] \right)^+.
\]

\(^3\)See [Pallavicini et al., 2011, p. 11] for the general definition of \( \theta_\tau(C, \varepsilon) \).

\(^4\)The calculations are reported in the Appendix.
Chapter 4

Hedging of defaultable claims

In this chapter we reproduce the existence proof of a replicating portfolio for a defaultable claim. The proof is based on [Bielecki and Rutkowski, 2001], considering in particular a non-dividend paying defaultable claim with zero recovery, paying a promised payoff at maturity. To conclude the chapter we will comment on the hedging strategy proposed in [Pallavicini et al., 2011].

4.1 Proof of existence

The idea of this proof is to first show that if the default-free market is arbitrage-free and if a suitable definition of an additional security is chosen, then the market containing this additional security is arbitrage-free as well. This allows us to express the price process of a defaultable zero-coupon bond. Subsequently, a suitable martingale representation theorem yields a replicating strategy for a defaultable claim based on continuous trading in default-free securities and defaultable bonds. We use the same notation and setting introduced in section 2.1, with the difference that here \( \tau \) is a general random time on \((\Omega, \mathcal{G}, \mathcal{G}_T^*, Q)\), with \(0 < T^* < \infty\). Recall that \( \beta \) denotes the money market account.

We now state the assumptions holding for this chapter and section A.3 in the Appendix. First, note that all stochastic processes considered in this proof are assumed to have right-continuous and left-limited sample paths. Suppose that the default-free market is arbitrage-free and complete. Further, we assume that the martingale invariance property holds under \(Q\), that is, every \(\mathcal{F}\)-martingale under \(Q\) is also a \(\mathcal{G}\)-martingale under \(Q\) and that the filtration \(\mathcal{F}\) is such that any \(\mathcal{F}\)-martingale is continuous\(^2\). Furthermore, whenever we consider self-financing strategies, we will consider them to be admissible (on the risk-free market). This means that the wealth process (defined hereafter) associated to a given strategy has to be non negative at any point in time. Also, we consider only non-dividend paying claims with zero recovery\(^3\).

\(^1\)From the literature we know that the martingale representation theorem we are looking for has been established for a restricted class of martingales, see [Bielecki and Rutkowski, 2001, p156]. In particular, the case where the market filtration is generated by some Brownian motion is considered.

\(^2\)An example of a filtration supporting continuous martingales is a filtration generated by a number of Brownian motions.

\(^3\)The general proof considering dividends, recovery claim and recovery process can be found in [Bielecki and Rutkowski, 2001].
Let $X$ be an $\mathcal{F}_T$-measurable random variable (on the probability space above) denoting the promised payoff of a defaultable contingent claim with maturity $T < T^*$ and let $X^d(t, T)$ denote the time $t$-value of this contingent claim.

**Definition 4.1.1.** The price process $X^d(\cdot, T)$ of a defaultable claim with maturity $T$ is given as

$$X^d(t, T) = \beta_t \mathbb{E}^Q \left[ \beta_T^{-1} X 1_{\{\tau > T\}} | \mathcal{G}_t \right] \quad \forall t \in [0, T].$$

Equation (4.1) will be referred to as risk-neutral valuation formula. It is important to emphasize that we are considering a defaultable claim and hence equation (4.1) cannot be derived from the pricing equation for risk-free contingent claims. The arguments presented below will justify Definition 4.1.1.

Consider $n$ price processes $S^1, \ldots, S^n$ of non-dividend paying primary assets of the default-free market. Assume further that $S^1, \ldots, S^n$ are semimartingales and put $S^n_t = \beta_t$ for all $t \in [0, T]$. Let $S^n_T$ be the price process of an additional security. We assume that it represents the current value of all future cash flows associated to that security, so that $S^n_T = 0$. Let us also introduce the discounted price processes $\tilde{S}^i_t$, $i = 0, \ldots, n$, by setting $\tilde{S}^i_t = \beta_t^{-1} S_i$. We will show that the following Proposition holds.

**Proposition 4.1.2.** $S^0$ satisfies, for $t \in [0, T]$,

$$S^0_t = \beta_t \mathbb{E}^Q \left[ \beta_T^{-1} X 1_{\{\tau > T\}} | \mathcal{G}_t \right].$$

**Proof** Buy one unit of $S^0$ and consider the $\mathcal{G}$-predictable trading strategy $\psi = (1, 0, \ldots, 0, \psi^n)$. The wealth process associated to $\psi$ is then given by $U_t(\psi) = S^0_t + \psi^n \beta_t$. Further, we consider the gains process

$$G_t(\psi) = X 1_{\{\tau > T\}} 1_{\{t \geq T\}} + \int_{(0, t]} dS^n_u + \int_{(0, t]} \psi^n_u d\beta_u.$$ 

Hence, imposing $U_t(\psi) - U_0(\psi) = G_t(\psi)$ we have that $\psi$ is a self-financing strategy. Applying Lemma A.3.1 we may rewrite the discounted wealth process $\tilde{U}_t(\psi) = \beta_t^{-1} U_t(\psi)$ of the considered strategy $\psi$ as

$$\tilde{U}_t(\psi) = \tilde{U}_0(\psi) + \tilde{S}^0_t - \tilde{S}^0_0 + \beta_t^{-1} X 1_{\{\tau > T\}} 1_{\{t \geq T\}}.$$ 

Recalling now that we assumed the risk-free market with primary assets $S^1, \ldots, S^n$ to be arbitrage-free, the discounted wealth process of any self-financing strategy follows a $Q$-martingale. Moreover, since we took $\psi$ self-financing (and admissible), $\tilde{U}(\psi)$ also follows a $Q$-martingale with respect to $\mathcal{G}$. This implies that for any $t \in [0, T]$ it holds

$$\mathbb{E}^Q \left[ \tilde{U}_T(\psi) - \tilde{U}_t(\psi) | \mathcal{G}_t \right] = 0.$$ 

---

4We define the wealth process of a trading strategy $\phi$ in $\mathbb{R}^{n+1}$ by the formula $U_t(\phi) = \sum_{i=0}^n \phi_i S^i_t$, for all $t \in [0, T]$.

5The gains process of a trading strategy $\phi$ in $\mathbb{R}^{n+1}$ is defined by

$$G_t(\phi) = \phi^n X 1_{\{\tau > T\}} 1_{\{t \geq T\}} + \sum_{i=0}^n \int_{(0, t]} \phi_i u dS^i_u \text{ for all } t \in [0, T].$$

6Here we also used the fact that we consider only admissible trading strategies belonging to a suitable class of strategies. See for example [Spreij, 2012, p. 59] for a characterization of the wealth process in a discrete time setting.
4.1. PROOF OF EXISTENCE

Using equation (4.3) and recalling that we put \( \tilde{S}_0^T = 0 \), we obtain
\[
S_t^0 = \beta_t \mathbb{E}^Q \left[ \beta_T^{-1} X 1_{\{\tau > T\}} \right] |G_t] \quad \forall \ t \in [0, T).
\]

\( \square \)

The following Proposition states a desirable characteristic of the extended market, i.e. the absence of arbitrage opportunities if the default-free market is considered to be arbitrage-free. The proof of this proposition can be found in [Bielecki and Rutkowski, 2001, p. 39]. However, we want to emphasize that equation (4.2) is used in the proof and hence motivates Definition 4.1.1.

**Proposition 4.1.3.** For any self-financing trading strategy \( \phi = (\phi_0, \ldots, \phi^n) \) the discounted wealth process \( \tilde{U}_t(\phi), t \in [0, T] \), follows a local martingale under \( Q \) with respect to the filtration \( G \).

This shows that assuming an arbitrage-free market of non-defaultable securities and adding to this market a defaultable claim, whose price process is defined by Definition 4.1.1, leads to an extended market which is arbitrage-free as well. Hence, Proposition 4.1.3 justifies Definition 4.1.1, which in turn allows us to express the price process of a defaultable zero-coupon bond using the risk-neutral valuation formula (4.1):
\[
D^0(t, T) = \beta_t \mathbb{E}^Q \left[ \beta_T^{-1} X 1_{\{\tau > T\}} \right] |G_t].
\]

We now want to show the existence of a replicating strategy for an arbitrary defaultable claim based on continuous trading in default-free securities and defaultable bonds. Let us first define the \( \mathbb{F} \)-hazard process of \( \tau \).

**Definition 4.1.4.** The \( \mathbb{F} \)-hazard process of \( \tau \) under \( Q \), denoted by \( \Gamma \), is defined through
\[
\Gamma_t := -\ln G_t = -\ln (1 - F_t) \quad \forall \ t \in [0, T],
\]
where \( F_t = Q(\tau \leq t | \mathcal{F}_t) \).

Note that in Section A.4 we define the \( \mathbb{F} \)-hazard process of \( \tau \) in terms of the process \( \lambda \). In this section we will not use this particular characterization. However, we will assume that \( \Gamma \) follows a continuous and increasing process (hence we can apply the results from section A.4). Let us further introduce the process \( L_t := 1_{\{\tau > t\}} \exp(\Gamma_t) \) and the (strictly positive) \( \mathbb{F} \)-martingale, for \( t \in [0, T] \),
\[
m_t = \mathbb{E}^Q \left[ \beta_t^{-1} e^{-\Gamma_T} 1_{\{\tau > T\}} | \mathcal{F}_t \right] .
\]

Assume that the defaultable bond with price process \( D^0(t, T) \) and discounted price process \( Z^0(t, T) = \beta_t^{-1} D^0(t, T) \) is a traded security. Consider now the defaultable claim introduced at the beginning of this section, with promised payoff \( X \) at maturity \( T \). Assume its price process is given by Definition 4.1.1. Let \( S^0_t \) denote the discounted value of this defaultable claim, i.e. \( \tilde{S}_0^T = 1_{\{\tau > T\}} \mathbb{E}^Q \left[ \beta_T^{-1} X 1_{\{\tau > T\}} \right] |G_t] \). In particular, recalling that \( X \) is \( \mathcal{F}_T \)-measurable, we can apply Lemma A.4.1 to rewrite\(^7\)
\[
\tilde{S}_t^0 = 1_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^Q \left[ \beta_T^{-1} e^{-\Gamma_T} X | \mathcal{F}_t \right] .
\]

\(^7\)See the first part of the proof of Proposition A.3.4 for remarks.
Hence, by definition of the processes $L$ and $m^X$, where $m^X_t = \mathbb{E}^Q \left[ \beta^{-1} e^{-\Gamma_T} X | \mathcal{F}_t \right]$, it holds $\tilde{S}^0_t = L_t m^X_t$. Lemma A.3.2 tells us that $L_t$ follows a $\mathbb{G}$-martingale and since $m^X$ is a bounded martingale (for the filtration $\mathbb{F}$ and hence also for $\mathbb{G}$ by the martingale invariance property), Lemma A.3.2 also implies that the product $L_t m^X_t$, and hence $\tilde{S}^0_t$, is a $\mathbb{G}$-martingale. The following Lemma is stated without proof:\footnote{For the proof see [Bielecki and Rutkowski, 2001, p. 249].}

**Lemma 4.1.5.** The $\mathbb{G}$-martingale $\tilde{S}^0_t$ admits the integral representation

$$\tilde{S}^0_t = \tilde{S}^0_0 + \int_{0}^{t \land \tau} e^{\Gamma_u} (Z^0_t)^X dm^X_u - \int_{0}^{t \land \tau} e^{\Gamma_u} m^X_u d\hat{M}_u.$$  

(4.4)

We are now able to construct a replicating strategy for the considered defaultable claim. Let us introduce the two processes $Y_1 = \exp(-\Gamma_T)$ and $Y_2 = X \exp(-\Gamma_T)$. By completeness of the default-free market $Y_1$ and $Y_2$ admit replicating strategies. Hence, we may consider them as primary securities.

**Proposition 4.1.6.** Let us denote $\zeta^X_t = m^X_t m_t^{-1}$. On the set $\{ t \leq \tau \}$, the replicating strategy for the discounted price process $\tilde{S}^0$ equals

$$\phi^0_t = \zeta^X_t \quad \phi^1_t = e^{\Gamma_t} \zeta^X_t \quad \phi^2_t = e^{\Gamma_t},$$

where the hedging instruments are: the discounted price process $Z^0(t, T)$ of the $T$-maturity zero-coupon bond with zero recovery and the discounted price process of the default-free claims $Y_1$ and $Y_2$. On the set $\{ t > \tau \}$, the replicating strategy is identically equal to zero.

**Proof** First note that we will consider all price processes on the set $\{ t \leq \tau \}$. If default happened before time $t$, the price process of the considered claim is equal to zero and so is the replicating strategy. By definition, $m$ is the discounted price process of $Y_1$ and $m^X$ is the discounted price process of $Y_2$. From equation (A.4), replacing $L_t m_t$ with $Z^0(t, T)$, it follows that

$$dZ^0(t, T) - L_t^- dm_t = -L_t^- m_t^- d\hat{M}_t = -e^{\Gamma_t} m_t^- d\hat{M}_t,$$

where the standard notation $L_t^-$ indicates the left-hand limit of the process. Rewriting the equality above for the price process $m^X$ yields

$$m^X_t m_t^{-1} dZ^0(t, T) - m_t^- m_t^- L_t^- dm_t = -e^{\Gamma_t} m_t^- d\hat{M}_t.$$

Combining equation (4.4) with the equality above, we obtain

$$\tilde{S}^0_t = \tilde{S}^0_0 + \int_{0}^{t \land \tau} \zeta^X_u dZ^0(u, T) - \int_{0}^{t \land \tau} e^{\Gamma_u} \zeta^X_u dm_u + \int_{0}^{t \land \tau} e^{\Gamma_u} m^X_u d\hat{M}_u.$$  

□

This concludes the existence proof of a replicating strategy for a non-dividend paying defaultable claim with zero recovery.
4.2 Hedging with par-swaps

In [Pallavicini et al., 2011] the replicating strategy of the risky derivative is given by \( F_t + H_t \), with \( F_t \) denoting the value of a cash account and \( H_t \) the value of a portfolio of hedging instruments. The proposed hedging strategy is based on a set of rolling par-swaps. Recall that a par swap has zero net present value at initiation date and that the notional principal is not exchanged. This implies that at each time point \( H_t = 0 \) and hence, the amount to be funded/invested at time \( t \) is equal to the time \( t \) value of the risky derivative. In section 3.3 we saw that this leads to a simplification in the pricing equations. The chosen hedging strategy may be motivated, when considering interest rate swaps, by the following argument based on [Miron and Swannell, 1995]. Suppose a hedging portfolio is constituted by a set of swaps. This portfolio will have a certain exposure to the change of interest rates, which should offset the exposure of the derivative to be hedged. In [Miron and Swannell, 1995, p. 133] we read that a set of par-swaps can be found, having the same exposure to interest rates as the original set of swaps. Such a par-swap portfolio is then called an equivalent position. This equivalence is determined by looking at the delta vector of the two portfolios, which contains the change of the value of a swap with respect to the interest rates. However, this way of defining equivalent positions considers only market risk. In order to include counterparty risk, probably default related rates have to be included when calculating the delta vector.
Chapter 5

The elastic case for multiple funding intervals

In this chapter we want to extend the elastic case proposed in [Nauta, 2012, p. 3-5] to multiple funding intervals and general derivatives. In order to do so, we first give a brief summary of the most important aspects and assumptions of the elastic case for one funding interval. Subsequently we generalize the mathematical and economic framework to allow for multiple funding intervals. In section 5.2 we present the balance sheet model assumed for the investor and derive then a pricing equation for the value of a derivative, traded with a defaultable counterparty. We will refer to this value as risky value. Finally, we add a second derivative, traded with a different counterparty, to the balance sheet and observe that this does not change the pricing equation, yet it influences the fair funding rate.

For the extension we consider the following assumptions:

- the investor, hereafter referred to as bank, is completely funded by equity and hence can not default;
- all recovery rates are zero;
- the risk-free rate is deterministic;
- the growth rate of the assets representing cash is equal to the risk-free rate;
- the funding dates and the cash flow dates of the derivative coincide;
- all values are seen from the point of view of the bank.

The first assumption has important implications on the economic and mathematical outline of the elastic case. The bank, being completely funded by equity, does not have any liabilities on its balance sheet. This means that under this assumption, by definition of equity\footnote{From an economic point of view the equity of a business is defined as the difference between assets and liabilities.}, the total assets should equal the total equity. Another consequence of this assumption is that the bank is default-free. In [Nauta, 2013] the restriction imposed by the first assumption is relaxed in order to allow for a debt account in the balance sheet. The second assumption is made for simplifying the notation. An extension to deterministic,
non-zero recovery rates is straightforward and will be briefly discussed in the next section.

Throughout this chapter we will use the following notations and definitions, partly extending the setup of Chapter 3:

- \( C \) denotes the counterparty, \( B \) the bank and \( F \) the funder;
- the considered derivatives are defined on the time interval \([0, T] \), \( T < \infty \);
- \( r \) denotes the risk-free rate and \( D(t, T) = e^{-r(T-t)} \), \( 0 \leq t \leq T \), the discount factor;
- let \( \{G_t\}_{t \geq 0} \) be a filtration satisfying the usual conditions on a given probability space, then \( \mathbb{E}_t \) denotes the conditional expectation w.r.t. \( G_t \) under the risk-neutral measure;
- the random time \( \tau \) denotes the default time of \( C \) (if more than one counterparty is considered, we denote the default time corresponding to the respective counterparty by \( \tau_C \)) and by the definition of \( H_t \) and \( G_t \), \( \tau \) is an \( (H_t) \)- and \( (G_t) \)-stopping time (also in this case, if more than one counterparty is considered, we denote the filtration corresponding to the default time \( \tau_C \) by \( (H^C_t) \));
- \( A^{\text{tot}} \) denotes the value of the total assets (including the assets \( A \) related to the deal with the counterparty\(^2 \) and the cash \( C \)) and \( E^{\text{tot}} \) the total equity;
- \( H_t \) denotes the value of the replicating portfolio (consists of hedging instruments and cash) for the counterparty risk-free derivative, i.e. \( H_t \) hedges the market risk only (not the counterparty risk);
- with \( H_{t,i}^+ \) we indicate the right-hand limit

\[
\lim_{s \downarrow t} H_s, \quad s \in D
\]

where \( D \) is a countably dense subset of \( \mathbb{R}_+ \) (the almost sure existence of the right-hand limit is proved for continuous time submartingales in [Karatzas and Shreve, 1988, p. 16-17]);
- \( P = \{0 = t_1, \ldots, t_k = T \} , k \geq 0 \), denotes a partition of \([0, T] \) and \( \bar{V}_T \) is the pay-off of the derivative;
- \( \bar{V}_{t,i}^+ \) denotes the risky value of the derivative at time \( t_i \in P \), after the cash flow of the derivative has been paid; hence we define

\[
\bar{V}_{t_1}^+ := \bar{V}_{t_1},
\]

\[
\bar{V}_{t_k}^+ := 0
\]

\(^2\)If more than one counterparty is considered, we denote the asset corresponding to the respective counterparty by \( A^C \).
and
\[
\bar{V}_{t_i} := \bar{V}_{t_i} - \mathbb{1}_{\{\tau > t_i\}} \Pi(t_{i-1}, t_i) D(t_{i-1}, t_i)^{-1}
\]
for \(i = 1, \ldots, k - 1\), with \(\Pi(t_{i-1}, t_i)\) denoting the sum of the discounted cash flows\(^3\) of the derivative on the interval \((t_{i-1}, t_i)\);

We conclude this section with a few additional notes. First, the processes \((\bar{V}_t)\) and \((H_t)\) are both assumed to be integrable. In order to avoid confusion between the notation above and the one used in Chapter 3, we emphasize that in this chapter the process \(H_t\) refers to the value of the replicating portfolio of the counterparty risk-free derivative, whereas in the preceding chapter \(H_t\) refers to the value of the hedging instruments of the derivative with counterparty risk - see equation (3.6).

### 5.1 Economic principles

Before extending the framework to allow for multiple funding intervals, we want to recall from [Nauta, 2012] what the elastic funding assumption states and which two economic principles are used to determine the fair funding rate and the fair value of a derivative in presence of counterparty credit risk. The elastic funding assumption states that “the funding costs of the bank are adjusted immediately after each new transaction and fully reflect the new asset composition”. Further, denote the equity of a business by \(E\) and the assets by \(A\). Then, the first economic principle is given by
\[
\mathbb{E}[E_T] = E_0
\]
and the second economic principle is given by
\[
\mathbb{E}[A_T - A_0] = \mathbb{E}[E_T - E_0].
\]

Note that in [Nauta, 2012] the risk-free rate is assumed to be zero and hence equations (5.1) and (5.2) do not include the discount factor. Subsequently, for the extension of the two economic principles, we assume a non-zero risk-free rate.

The first economic principle is based on the assumption that it should be equally attractive for investors to invest in the equity of the bank as in any other financial asset and is used to determine the fair funding rate. The second economic principle is based on the assumption that the expected return on the equity equals the expected return on the assets. This simplifies to \(\mathbb{E}[A_T] = \mathbb{E}[E_T]\), considering that in [Nauta, 2012] \(A_0\) and \(E_0\) are constants and coincide. Subsequently we will adapt these two principles to multiple funding intervals. In order to achieve this we need to extend the framework given in [Nauta, 2012].

First we consider the actions taken by the bank. The bank \(B\) wants to enter a deal with counterparty \(C\) and needs funding for this purpose. The funding is asked repetitively on an interval \([0, T^F]\), with \(T^F \leq T\). We will refer to this interval as \textit{funding period}. Furthermore, we consider a partition \(P^F = \{0 = t_1, t_2, \ldots, t_m = T^F\}\) of \([0, T^F]\). The elements of this partition represent the dates at which funding is asked. The subintervals \((t_{i-1}, t_i), i = 1, \ldots, m - 1\), are referred to as \textit{funding intervals}. At the end of every funding interval we consider the following actions:

\(^3\)We consider the final pay-off of the derivative separately from the cash flows occurring on the interval \((t_{k-1}, t_k)\).
5.1. ECONOMIC PRINCIPLES

(a) Assets and equity values are observed (we assume they are $G$-adapted).

(b) Market positions related to the deal are unwound and the funding obtained at the previous funding time is returned, according to specified rules (see Definition 5.1.1 below), to the funder.

(c) New funding is asked and the trade and hedging positions related to the deal are entered again.

Since we want to consider the case with multiple funding intervals, we distinguish between equity funder $E^F$ (the stockholders’ equity) and equity bank $E^B$, with $E^{\text{tot}} = E^F + E^B$. Moreover, to distinguish between the different values of $E^F$ and $E^B$ after each of the actions mentioned above, we introduce for $t \in P^F$ two vector-valued stochastic processes, taking values in $\mathbb{R}^3$ and defined on $\Omega$:

\[
E^F_t := \begin{bmatrix} E^F_{\alpha,t} \\ E^F_{\beta,t} \\ E^F_{\gamma,t} \end{bmatrix} \quad (5.3)
\]

\[
E^B_t := \begin{bmatrix} E^B_{\alpha,t} \\ E^B_{\beta,t} \\ E^B_{\gamma,t} \end{bmatrix} \quad (5.4)
\]

The indices $\alpha, \beta$ and $\gamma$ refer to the values taken after the corresponding actions mentioned above. It is important to notice that the fraction $E^B$ of the total equity refers to the equity owned by the bank and hence no funding costs have to be paid on it to the funder. Nevertheless, $B$ expects a return on the amount invested in the deal with $C$. This expected return on equity will be specified in the following. For now we simply give an interpretation for the elements of the vectors in (5.3) and (5.4). $E^F_{\alpha,t}$ is the amount $B$ has to pay back to $F$ at time $t$. This amount will depend on the funding rate, the funding amount obtained at the previous funding date and whether or not there has been a default of the counterparty. $E^F_{\beta,t}$ is the value of the equity of the funder once the repayment occurred and $E^F_{\gamma,t}$ is the funding amount $B$ asks from $F$ at time $t$ for the next funding interval.

Similarly we can describe the elements of the vector $E^B_t$. The first element, $E^B_{\alpha,t}$, represents the amount the bank gets back from its investment, $E^B_{\beta,t}$ denotes the value of the equity of the bank left after action $\beta$ and can be positive or negative and $E^B_{\gamma,t}$ denotes the equity owned by the bank after entering again the deal with $C$.

We now state the first economic principle used in this work. It is a generalization of (5.1) and we assume it holds for any equity investor, i.e. (5.5) holds for the corresponding components of $E^F$ as well as $E^B$.

**Principle 1** (First economic principle - EP I). For an investment at time $t_i \in P^F$,

\[
D(t_1, t_i)^{-1} \mathbb{E}_{t_i} \left[ D(t_1, t_{i+1}) E^F_{\alpha,t_{i+1}} \right] = E^F_{\gamma,t_i} \quad (5.5)
\]

must hold for all $i = 1, \ldots, k - 1$.

We now introduce the stochastic vector $C_t$, which represents the cash account of $B$ after the respective actions. For $t \in P^F$ define

\[
C_t := \begin{bmatrix} C^F_{\alpha,t} \\ C^F_{\beta,t} \\ C^F_{\gamma,t} \end{bmatrix}, \quad (5.6)
\]
where \( \gamma_{\alpha,t} \) can be interpreted as the accrued cash over the funding interval with endpoint \( t \), \( \gamma_{\beta,t} \geq 0 \) as the cash left in the bank account after action \( \beta \), \( \gamma_{\beta,t} < 0 \) as the temporary accumulated loss over all funding intervals preceding \( t \) and \( \gamma_{\gamma,t} \) the cash not needed for reinvesting in the deal with \( C \).

Let us now consider how the funding occurs. The idea is that the bank asks for a certain funding amount and pays it back at predetermined time points. The funding rate, in contrast to the papers summarized in Chapter 2, is not assumed to be fixed, but will be determined through the first economic principle. What is more, the funding will be returned only if the counterparty of \( B \) did not default before that time.

**Definition 5.1.1.** The funding contract between \( B \) and \( F \) is defined by the following steps:

1. Determine a partition \( P^F = \{t_1, \ldots, t_m\} \) of \([0, T^F]\), with \( t_m = T^F < \infty \), where at every date \( t_i \), \( i = 2, \ldots, m \), \( B \) has to repay the funding obtained for the period \((t_{i-1}, t_i]\) and at dates \( t_i \), for \( i = 1, \ldots, m-1 \), (possibly) asks new funding for the period \((t_i, t_{i+1}]\).

2. \( E^F_{\gamma,t_i} \) is the amount \( B \) asks from \( F \) at time \( t_i \), \( i = 1, \ldots, m-1 \), and is determined by
   \[
   E^F_{\gamma,t_i} := \left[ \bar{V}_{t_i} - \mathbb{1}_{\{\tau > t_i\}} H_{t_i} - C_{\beta,t_i} \right]^+;
   \]

3. \( E^F_{\alpha,t_{i+1}} \) is the amount \( B \) returns to \( F \) at time \( t_{i+1} \), \( i = 1, \ldots, m-1 \), and is defined by
   \[
   E^F_{\alpha,t_{i+1}} = f_{s_F}(\tau, E^F_{\gamma,t_i}) := \mathbb{1}_{\{\tau > t_{i+1}\}} E^F_{\gamma,t_i} e^{s_F(t_{i+1} - t_i)}, \tag{5.7}
   \]
   where at every funding date \( t_i \) the funding rate \( s_{F,i} \) for the time interval \((t_i, t_{i+1}]\) is determined by EP I (if \( f_{s_F}(\tau, 0) > 0 \));

4. If \( C \) defaults and \( B \) has a negative cash position, the contract between \( B \) and \( F \) ends and \( B \) resorts to internal resources to achieve a positive balance.

Note that no recovery is assumed in case of default. By redefining \( f_{s_F} \) in (5.7) it is easy to generalize to deterministic recovery rates different from zero:

\[
f_{s_F}(\tau, E^F_{\gamma,t_i}) := \mathbb{1}_{\{\tau > t_{i+1}\}} E^F_{\gamma,t_i} e^{s_F(t_{i+1} - t_i)} + \mathbb{1}_{\{\tau \leq t_{i+1}\}} R^F E^F_{\gamma,t_i} e^{s_F(t_{i+1} - t_i)}.
\]

Furthermore, we note that by definition \( E^F_{\gamma,t_i} \) is \( \mathcal{G}_{t_i} \)-measurable and \( E^F_{\alpha,t_{i+1}} \) is \( \mathcal{G}_{t_{i+1}} \)-measurable. Also, the funding rate \( s_{F,i} \) consists of the risk-free rate \( r \) and a funding spread. Since we want to consider a fixed funding rate \( s_{F,i} \) over the interval \((t_i, t_{i+1}]\), we will assume that the risk-free rate and the funding spread are constant as well on each of the intervals \((t_i, t_{i+1}]\). The funding rate can be determined by substituting \( R^F_{\gamma,t_i} \) defined in (5.7) into (5.5):

\[
E^F_{\gamma,t_i} = D(t_{i+1}, t_i)^{-1} \mathbb{E}_{t_i} [D(t_{i+1}, t_i+1) E^F_{\alpha,t_{i+1}}]
= D(t_{i+1}, t_i+1) \mathbb{E}_{t_i} \left[ \mathbb{1}_{\{\tau > t_{i+1}\}} E^F_{\gamma,t_i} e^{s_F(t_{i+1} - t_i)} \right]
= D(t_{i+1}, t_i+1) \mathbb{P} (\tau > t_{i+1}|\mathcal{G}_{t_i}) E^F_{\gamma,t_i} e^{s_F(t_{i+1} - t_i)};
\]

\[4\] We assume that in order to cover this loss \( B \) will ask additional funding for the next funding interval.
where the last equality holds by right-continuity of \((\mathcal{G}_t)\). This yields on the event \(\{\tau > t_i\}\)
\[
 e^{s_F, (t_{i+1} - t_i)} = \frac{1}{D(t_i, t_{i+1}) \mathbb{P} (\tau > t_i + 1 | \mathcal{G}_t)}.
\] (5.8)

In case of a positive recovery rate the above formula for the funding rate changes slightly. However, there are no additional complications from a computational point of view.

Given the first economic principle, we also note that the value at time \(t_j \in \mathcal{P}^F\) of the difference between the time \(t_j\) funding amount \(E^F_{\alpha, t_{i+1}}\) and the time \(t_{i+1}\) payment \(E^F_{\alpha, t_{i+1}}\) is 0 for all \(i \geq j + 1\). This means that the expected future loss from a point of view of \(\mathcal{B}\) due to the contract with \(F\) is 0.

**Proposition 5.1.2.** Let \(E^F_{\gamma, t_i}\) and \(E^F_{\alpha, t_{i+1}}\) be as in Definition 5.1.1. Then, given EP I
\[
 D(t_1, t_j)^{-1} \mathbb{E} \left[ D(t_1, t_{i+1}) E^F_{\alpha, t_{i+1}} - D(t_1, t_i) E^F_{\gamma, t_i} | \mathcal{G}_{t_j} \right] = 0
\] holds for all \(i \geq j + 1\).

**Proof** Using the tower property and the \(\mathcal{G}_t\)-measurability of \(E^F_{\gamma, t_i}\) we have
\[
 D(t_1, t_j)^{-1} \mathbb{E} \left[ D(t_1, t_{i+1}) E^F_{\alpha, t_{i+1}} - D(t_1, t_i) E^F_{\gamma, t_i} | \mathcal{G}_{t_j} \right]
 = D(t_1, t_j)^{-1} \mathbb{E} \left[ \mathbb{E} \left[ D(t_1, t_{i+1}) E^F_{\alpha, t_{i+1}} - D(t_1, t_i) E^F_{\gamma, t_i} | \mathcal{G}_{t_j} \right] | \mathcal{G}_{t_j} \right]
 = D(t_1, t_j)^{-1} \mathbb{E} \left[ \mathbb{E} \left[ D(t_1, t_{i+1}) E^F_{\alpha, t_{i+1}} | \mathcal{G}_{t_j} \right] - D(t_1, t_i) E^F_{\gamma, t_i} | \mathcal{G}_{t_j} \right].
\]

Then, by EP I the argument of the conditional expectation w.r.t. \(\mathcal{G}_{t_j}\) is 0. \(\square\)

Let us now consider the vector
\[
 A_t := \begin{bmatrix} A_{\alpha, t} \\ A_{\beta, t} \\ A_{\gamma, t} \end{bmatrix}, \tag{5.9}
\]
representing the value of the bank’s assets concerning the deal with \(\mathcal{C}\). In particular, \(A_{\alpha, t}\) corresponds to the value of the assets at the end of the funding interval, \(A_{\alpha, t_i}\) to the value of the assets at the beginning of the funding interval and \(A_{\gamma, t}\) stands for the value of the assets once the deal with \(\mathcal{C}\) has been unwound.

If we consider a time \(t_i \in \mathcal{P}^F\), then \(A_{\alpha, t_i} - A_{\alpha, t_{i+1}}\) determines the return on the assets of the bank on the interval \((t_i, t_{i+1})\). Imposing that the expected discounted return on the total assets equals the expected discounted return on the total equity, we can postulate the generalization of the second economic principle:

**Principle 2** (Second economic principle - EP II). At time \(t_i \in \mathcal{P}^F\) the following equality must hold for \(i = 1, \ldots, m - 1\)
\[
 D(t_1, t_i)^{-1} \mathbb{E}_{t_i} \left[ D(t_1, t_{i+1}) \left( E^\text{tot}_{\alpha, t_{i+1}} - E^\text{tot}_{\gamma, t_i} \right) \right] = D(t_1, t_i)^{-1} \mathbb{E}_{t_i} \left[ D(t_1, t_{i+1}) \left( A^\text{tot}_{\alpha, t_{i+1}} - A^\text{tot}_{\gamma, t_i} \right) \right]. \tag{5.10}
\]

The fair value of the trade with \(\mathcal{C}\) can be determined through this principle (see Section 5.3). Note that under the assumptions holding in [Nauta, 2012] in section 2, (5.10) reduces to
\[
 \mathbb{E}[A_T] = \mathbb{E}[E_T].
\]
5.2 Balance sheet model

Below we give an overview of a simplified balance sheet of the bank for the funding dates \( t_1, t_2 \) and \( t_3 \), distinguishing between the actions \( \alpha, \beta \) and \( \gamma \). In each column the random variables on the left-hand side represent the assets, cash and equity the bank has in its balance sheet at that given time and after the given action. It is important to note that after action \( \alpha \) the total assets and total equity of the bank are not equal, since \( A, C, E^F \) and \( E^B \) grow all at (possibly) different rates over a funding interval. In particular, we assume that \( C \) grows at the risk-free rate, whereas to determine the rate at which \( E^F \) grows (funding rate) and the rate at which \( E^B \) grows we impose EP I.

Suppose now that at time \( t_1 \) bank \( B \) wants to make a deal with counterparty \( C \) and that this deal is not in default, i.e. \( \tau > t_1 \). In order to make this deal the bank enters a contract with the funder \( F \). This contract satisfies Definition 5.1.1, where we assume that \( T^F = T \) and that the funding dates coincide with the time points at which the cash flows of the deal with \( C \) occur. Suppose also that the equity owned by the bank at time \( t_1 \) is given by a constant \( K \), where \( 0 \leq K < \infty \). Under these assumptions the funding amount asked by \( B \) is given by the risky value of the derivative and the value of the replicating portfolio, decreased by the equity owned by \( B \).

The balance sheet of the bank at time \( t_1 \), after action \( \gamma \), is given below. The dynamics of the assets and the equity are motivated after Table 5.3, when it will be possible to consider a general situation of the deal between \( B \) and \( C \).

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{\gamma,t_1} = \bar{V}<em>{t_1+} - 1</em>{{\tau &gt; t_1}}H_{t_1+} )</td>
<td>( E^F_{\gamma,t_1} = [\bar{V}<em>{t_1+} - 1</em>{{\tau &gt; t_1}}H_{t_1+} - K]^+ )</td>
</tr>
<tr>
<td>( C_{\gamma,t_1} = [K - (\bar{V}<em>{t_1+} - 1</em>{{\tau &gt; t_1}}H_{t_1+})]^+ )</td>
<td>( E^B_{\gamma,t_1} = K )</td>
</tr>
</tbody>
</table>

We assume that on the interval \((t_1, t_2)\) no transactions regarding the deal with \( C \) and the contract with \( F \) take place. At time \( t_2 \) the bank observes on the market the values that are observable. All the other values are \( G_{t_2} \)-measurable by assumption or by definition. Over this interval the cash position, if positive, grows at the risk-free rate. Hence, we have

\[
C_{\alpha,t_2} = D^{-1}(t_1, t_2)C_{\gamma,t_1}.
\]  

The value of the remaining assets depends on the derivative and the replicating portfolio. Since there is a default possibility for counterparty \( C \), this has to be taken into account. Indeed, the risky value \( \bar{V}_{t_2} \) of the derivative will be zero if default happened before time \( t_2 \) and so the value of the assets will be equal to the value of the replicating portfolio. Moreover, the equity owned by the bank and the equity obtained through the funder both grow according to the function defined in (5.7), substituting the respective variables.
5.2. BALANCE SHEET MODEL

Table 5.2: Balance sheet at time $t_2$ after action $\alpha$, $A^{\text{tot}} \neq E^{\text{tot}}$

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\alpha,t_2} = V_{t_2} - \mathbb{1}<em>{{\tau &gt; t_1}}H</em>{t_2}$</td>
<td>$E^{F}<em>{\alpha,t_2} = f</em>{sp}(\tau, E^{F}_{\gamma,t_1})$</td>
</tr>
<tr>
<td>$C_{\alpha,t_2} = D^{-1}(t_1, t_2)C_{\gamma,t_1}$</td>
<td>$E^{B}<em>{\alpha,t_2} = f</em>{sb}(\tau, (E^{B}_{\gamma,t_1})^+)$</td>
</tr>
</tbody>
</table>

After observing the changes in the different accounts, $B$ unwinds the market positions related to the deal with $C$ and returns the funding as it is specified in point 3 of Definition 5.1.1. Since we look at all the values from a point of view of the bank, $E^{F}_{\beta,t_2}$ will be zero. In order to establish a balance between assets and equity, we temporarily close the derivatives’ account, transferring the value to the cash account $C_{\beta,t_2}$. Moreover, we see that $A^{\text{tot}}_{\beta,t_2} = E^{\text{tot}}_{\beta,t_2}$ holds only if $E^{B}_{\beta,t_2} = C_{\beta,t_2}$.

Table 5.3: Balance sheet at time $t_2$ after action $\beta$, $A^{\text{tot}} = E^{\text{tot}}$

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\beta,t_2} = 0$</td>
<td>$E^{F}_{\beta,t_2} = 0$</td>
</tr>
<tr>
<td>$C_{\beta,t_2} = C_{\alpha,t_2} + A_{\alpha,t_2} - E^{F}_{\alpha,t_2}$</td>
<td>$E^{B}<em>{\beta,t_2} = C</em>{\beta,t_2}$</td>
</tr>
</tbody>
</table>

At this point, if no default happened on the interval $(t_1, t_2)$, $B$ wants to enter again the deal with $C$. To motivate the equalities in Table 5.1 and Table 5.4 we look at different scenarios at a general funding time $t_i \in F^F$

- First of all suppose no default happened up to time $t_i$. If $E^{B}_{\beta,t_i} \geq 0$, then $B$ has cash left after repaying the funding. If $E^{B}_{\beta,t_i} < 0$, then $B$ has a temporary loss. On the other hand also $V_{t_i+} - H_{t_i+}$ can assume positive and negative values. If $V_{t_i+} - H_{t_i+} \geq 0$, $B$ sees a cash outflow, but the risky value of the derivative and the replicating portfolio have a positive value on the balance sheet. Viceversa, if $V_{t_i+} - H_{t_i+} < 0$, there is a positive cash flow. However, the registered value of the deal is negative for $B$. We analyze four cases separately:

1. Suppose $V_{t_i+} - H_{t_i+} \geq 0$ and $E^{B}_{\beta,t_i} \geq 0$, then the amount $B$ asked for funding is determined by $E^{F}_{\gamma,t_i} = V_{t_i+} - H_{t_i+} - E^{B}_{\beta,t_i}$. Here we can again distinguish between two cases:
   - $V_{t_i+} - H_{t_i+} > E^{B}_{\beta,t_i}$
   - $V_{t_i+} - H_{t_i+} \leq E^{B}_{\beta,t_i}$

The first case reflects the situation in which $B$ will ask less funding than needed for the deal with $C$. Once the funding has been obtained, the total amount of $E^{F}_{\gamma,t_i}$ and $E^{B}_{\beta,t_i}$ are used to buy the derivative. Hence, $A_{\gamma,t_i} = V_{t_i+} - H_{t_i+}$ and $C_{\gamma,t_i} = 0$.

The second case reflects the situation in which $B$ does not need to ask for funding. Hence, $A_{\gamma,t_i} = V_{t_i+} - H_{t_i+}$ and $C_{\gamma,t_i} = E^{B}_{\beta,t_i} - (V_{t_i+} - H_{t_i+})$. 

2. Suppose now \( \bar{V}_{t_i^+} - H_{t_i^+} < 0 \) and \( E_{\beta, t_i}^B \geq 0 \), then \( B \) does not need to ask for funding at time \( t_i \) and hence \( E_{\gamma, t_i}^F = 0 \). The amount held in the cash account is given by the positive cash flow from the deal with \( C \) and by the cash position after action \( \beta \): \[ C_{\gamma, t_i} = C_{\beta, t_2} - (\bar{V}_{t_2^+} - \mathbb{1}_{\{t > t_2\}} H_{t_2^+}). \]

3. Then we suppose \( \bar{V}_{t_i^+} - H_{t_i^+} \geq 0 \) and \( E_{\beta, t_i}^B < 0 \). In this case \( B \) needs to ask funding for both the deal with \( C \) and the negative cash position. Thus, \( E_{\gamma, t_i}^F = \bar{V}_{t_i^+} - H_{t_i^+} - E_{\beta, t_i}^B \) and the value of the cash account (after asking for the new funding amount and (re)buying the derivative) is 0.

4. The last case to consider is \( \bar{V}_{t_i^+} - H_{t_i^+} < 0 \) and \( E_{\beta, t_i}^B < 0 \). This case can be analyzed with similar arguments to the ones used in the first case.

- Suppose now default happened before time \( t_i \). In that case \( A_{\gamma, t_i} = 0 \), because the risky value of the derivative is zero and hence there is no need to hedge. Moreover, if \( B \) had a positive cash position at time \( t_i \), this will stay on the balance sheet, i.e. \( C_{\gamma, t_i} = C_{\beta, t_i}^+ \).

Summarizing the different cases, one can see that the balance sheet dynamics after action \( \gamma \) can be described by the following equations:

\[
E_{\gamma, t_i}^F = [\bar{V}_{t_i^+} - \mathbb{1}_{\{t > t_i\}} H_{t_i^+} - C_{\beta, t_i}]^+;
A_{\gamma, t_i} = \bar{V}_{t_i^+} - H_{t_i^+},
C_{\gamma, t_i} = [C_{\beta, t_i} - (\bar{V}_{t_i^+} - \mathbb{1}_{\{t > t_i\}} H_{t_i^+})]^+.
\]

Moreover, since the total assets and the total equity of the bank after action \( \gamma \) need to be in balance, \( E_{\gamma, t_i}^B \) must be equal to \( C_{\beta, t_i} \).

Table 5.4: Balance sheet at time \( t_2 \) after action \( \gamma \), \( A^{tot} = E^{tot} \)

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{\gamma, t_2} = \bar{V}<em>{t_2^+} - \mathbb{1}</em>{{t &gt; t_2}} H_{t_2^+} )</td>
<td>( E_{\gamma, t_2}^F = [\bar{V}<em>{t_2^+} - \mathbb{1}</em>{{t &gt; t_2}} H_{t_2^+} - C_{\beta, t_2}]^+ )</td>
</tr>
<tr>
<td>( C_{\gamma, t_2} = [C_{\beta, t_2} - (\bar{V}<em>{t_2^+} - \mathbb{1}</em>{{t &gt; t_2}} H_{t_2^+})]^+ )</td>
<td>( E_{\gamma, t_2}^B = C_{\beta, t_2} )</td>
</tr>
</tbody>
</table>

The balance sheets at time \( t_3 \) are derived by following the dynamics of time \( t_2 \). It is important to note that at time \( t_3 \), if \( \tau \leq t_2 \), then the only account which may have non-zero value is the cash account. Note also that after default of the counterparty, if \( C_{\beta, t_2} > 0 \), then the cash amount accrues at the risk-free rate till the end of the funding period.

Balance sheet at time \( t_3 \) after action \( \alpha \), \( A^{tot} \neq E^{tot} \)

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{\alpha, t_3} = \bar{V}<em>{t_3} - \mathbb{1}</em>{{t &gt; t_3}} H_{t_3} )</td>
<td>( E_{\alpha, t_3}^F = f_{\gamma} (\tau, E_{\gamma, t_2}) )</td>
</tr>
<tr>
<td>( C_{\alpha, t_3} = D^{-1}(t_2, t_3) C_{\gamma, t_2} )</td>
<td>( E_{\alpha, t_3}^B = f_{\beta} (\tau, (E_{\gamma, t_2})^+) )</td>
</tr>
</tbody>
</table>
Balance sheet at time $t_3$ after action $\beta$, $A^{\text{tot}} = E^{\text{tot}}$

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\beta,t_3} = 0$</td>
<td>$E_{\beta,t_3}^{F} = 0$</td>
</tr>
<tr>
<td>$C_{\beta,t_3} = C_{\alpha,t_3} + A_{\alpha,t_3} - E_{\alpha,t_3}^{F}$</td>
<td>$E_{\beta,t_3}^{B} = C_{\beta,t_3}$</td>
</tr>
</tbody>
</table>

Balance sheet at time $t_3$ after action $\gamma$, $A^{\text{tot}} = E^{\text{tot}}$

<table>
<thead>
<tr>
<th>Total Assets</th>
<th>Total Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\gamma,t_3} = \bar{V}<em>{t_3} + 1</em>{{\tau&gt;t_3}} H_{t_3}$</td>
<td>$E_{\gamma,t_3}^{F} = [\bar{V}<em>{t_3} + 1</em>{{\tau&gt;t_3}} H_{t_3} - C_{\beta,t_3}]^{+}$</td>
</tr>
<tr>
<td>$C_{\gamma,t_3} = [C_{\beta,t_3} - (\bar{V}<em>{t_3} + 1</em>{{\tau&gt;t_3}} H_{t_3})]^{+}$</td>
<td>$E_{\gamma,t_3}^{B} = C_{\beta,t_3}$</td>
</tr>
</tbody>
</table>

### 5.3 Derivation of the fair price

Once we specified the balance sheet model, we determine the risky value of the derivative in the bank’s asset composition. We assume that the risky value satisfies the second economic principle, i.e. equality (5.10) must hold. In particular, since we assume a deterministic risk-free rate and $A_{\gamma,t_i}^{\text{tot}} = E_{\gamma,t_i}^{\text{tot}}$, we can rewrite (5.10) as

$$\mathbb{E}_t \left[ E_{\alpha,t_{i+1}}^{\text{tot}} \right] = \mathbb{E}_t \left[ A_{\alpha,t_{i+1}}^{\text{tot}} \right].$$

(5.12)

**Proposition 5.3.1.** Given EP I and EP II, it holds for $i = 1, \ldots, m - 1$

$$\bar{V}_{t_i} + 1_{\{\tau>t_i\}} H_{t_i} = D(t_i, t_{i+1}) \mathbb{E}_t \left[ \bar{V}_{t_{i+1}} \right] - 1_{\{\tau>t_i\}} D(t_i, t_{i+1}) \mathbb{E}_t \left[ H_{t_{i+1}} \right].$$

**Proof**

$$\bar{V}_{t_i} + 1_{\{\tau>t_i\}} H_{t_i} = A_{\gamma,t_i},$$

(5.13)

$$= E_{\gamma,t_i}^{\text{tot}} - C_{\gamma,t_i},$$

$$= E_{\gamma,t_i}^{F} + E_{\gamma,t_i}^{B} - C_{\gamma,t_i},$$

$$= D(t_i, t_{i+1}) \mathbb{E}_t A_{\alpha,t_{i+1}}^{\text{tot}} + D(t_i, t_{i+1}) \mathbb{E}_t E_{\alpha,t_{i+1}}^{B} - C_{\gamma,t_i}$$

(5.14)

$$= D(t_i, t_{i+1}) \mathbb{E}_t A_{\alpha,t_{i+1}}^{\text{tot}} - C_{\gamma,t_i}$$

(5.15)

$$= D(t_i, t_{i+1}) \mathbb{E}_t A_{\alpha,t_{i+1}} + D(t_i, t_{i+1}) \mathbb{E}_t C_{\alpha,t_{i+1}} - C_{\gamma,t_i}$$

$$= D(t_i, t_{i+1}) \mathbb{E}_t A_{\alpha,t_{i+1}} + D(t_i, t_{i+1}) \mathbb{E}_t \left[ D^{-1}(t_i, t_{i+1}) C_{\gamma,t_i} \right] - C_{\gamma,t_i}$$

(5.16)

$$= D(t_i, t_{i+1}) \mathbb{E}_t A_{\alpha,t_{i+1}}$$

where (5.13) holds because $A_{\gamma,t_i}^{\text{tot}} = E_{\gamma,t_i}^{\text{tot}}$, (5.14) holds by EP I, (5.15) by EP II and (5.16) holds by the relation in (5.11). \qed

**Proposition 5.3.2.** Assuming EP I and EP II, the risky value of a derivative at time $t_i$ is given by

$$\bar{V}_{t_i}^{+} = D(t_i, t_{i+1}) \mathbb{E}_t \bar{V}_{t_{i+1}}.$$
CHAPTER 5. THE ELASTIC CASE FOR MULTIPLE FUNDING INTERVALS

Proof We want to show that

\[ H_{t_i^+] = D(t_i, t_{i+1}) E_{t_i} \left[ H_{t_{i+1}} \right]. \]  

(5.17)

By definition of replicating portfolio we have that \( H_t = V_t \) for all \( t \in [0, T] \), where \( V_t \) denotes the counterparty risk-free price of the derivative. Considering derivatives with possible cash flows at times \( t_i \in P^F \) we can not apply the pricing formula (3.1). However, on the funding interval \((t_i, t_{i+1}]\) the integrability and measurability properties of \( V_t/\beta(t) \) are preserved and the martingale property holds since possible cash flows do not affect the derivative’s value on these intervals:

\[ D(s, t) E [V_t | F_s] = V_s \quad \text{for all} \quad t_i < s < t \leq t_{i+1}. \]

The discussions in Section A.4 (in particular the implication (a) \( \Rightarrow \) (c) of Lemma A.4.2) imply that the martingale property above holds also w.r.t. to \( (G_i)_{i \in (t_i, t_{i+1}]} \). Since \( V_t/\beta(t) \) is a martingale on \((t_i, t_{i+1}]\), the right-hand limit \( \lim_{s \uparrow t_i} V_s/\beta(s) \) exists almost surely. Moreover, by continuity of \( \beta(s) \), \( \lim_{s \uparrow t_i} V_s \) exists almost surely as well and we can define \( V_{t_i^+} := \lim_{s \uparrow t_i} V_s \).

Then, consider a sequence \( (t_n) \) s.t. \( t_n \downarrow t_i \) as \( n \to \infty \). By definition of filtration, \( G_{t_n} := G_{t_n+1/n} \) is a decreasing sequence of sub-sigma-algebras of \( G \) and since \( V_{t_{i+1}} \) is integrable, we can apply Levy’s downward Theorem to obtain

\[
\lim_{n \to \infty} E \left[ V_{t_{i+1}} | G_{t_n} \right] = E \left[ V_{t_{i+1}} \left| \bigcap_n G_{t_n} \right. \right].
\]

From the arguments above and assuming that \( t_i + 1/n \in D \), with \( D \) countably dense subset of \( \mathbb{R}_+ \), we see that the following equalities hold true for \( n \) s.t. \( t_i + 1/n \leq t_i+1 \):

\[
\begin{align*}
H_{t_i^+} &= V_{t_i^+} \\
&= \lim_{t_n \uparrow t_i} V_{t_{i+n}} \\
&= \lim_{n \to \infty} D(t_{i+n}, t_{i+1}) E \left[ V_{t_{i+1}} | G_{t_{i+n}} \right] \\
&= \lim_{n \to \infty} D(t_i, t_{i+1}) \lim_{n \to \infty} E \left[ V_{t_{i+1}} | G_{t_{i+n}} \right] \\
&= D(t_i, t_{i+1}) \left[ V_{t_{i+1}} \left| \bigcap_n G_{t_{i+n}} \right. \right] \\
&= D(t_i, t_{i+1}) \left[ V_{t_{i+1}} \left| G_{t_{i+n}} \right. \right] \\
&= D(t_i, t_{i+1}) \left[ H_{t_{i+1}} | G_{t_{i+n}} \right].
\end{align*}
\]

By right-continuity of \( G \), we conclude that (5.17) holds. This implies that

\[
\bar{V}_{t_i^+} = 1_{\{\tau > t_i\}} H_{t_i^+} + D(t_i, t_{i+1}) E_{t_i} \bar{V}_{t_{i+1}} - D(t_i, t_{i+1}) E_{t_i} \left[ 1_{\{\tau > t_i\}} H_{t_{i+1}} \right]
\]

\[
= 1_{\{\tau > t_i\}} H_{t_i^+} + D(t_i, t_{i+1}) E_{t_i} \bar{V}_{t_{i+1}} - 1_{\{\tau > t_i\}} D(t_i, t_{i+1}) E_{t_i} \left[ H_{t_{i+1}} \right]
\]

\[
= D(t_i, t_{i+1}) E_{t_i} \bar{V}_{t_{i+1}}.
\]

\( \square \)
Using the relation in Proposition 5.3.2 repeatedly over the whole funding period and recalling that we consider the pay-off of the derivative separately from the cash flows occurring on \([t_{m-1}, t_m]\), we can write

\[
\tilde{V}_{t_1} = D(t_1, t_m)E_{t_1}\tilde{V}_{t_m} + \sum_{i=2}^{m} D(t_1, t_{i-1})E_{t_1}\left[1_{\{\tau > t_i\}}\Pi (t_{i-1}, t_i)\right].
\] (5.18)

**Conclusion:** We conclude that the risky price \(\tilde{V}_{t_1}\) of a derivative does not depend on the funding rate under the assumption of the two economic principles (5.5) and (5.10). In particular it is interesting to notice that this result is independent from the definition of assets and equity. The definition of \(E^F\) for example does not influence the fair price of the derivative, but is needed to determine the fair funding rate for the bank.

### 5.3.1 Example - European call option

Suppose now that \(B\) wants to buy a European call option on a non-dividend paying stock, with maturity \(T = T^F\) and strike price \(M\), and that counterparty \(C\) is the seller. The value \(\tilde{V}_t\) is positive for the bank and equal to zero if the counterparty defaults. At time \(t_1\) \(C\) is not in default by assumption and so \(B\) can value the option by formula (5.18):

\[
\tilde{V}_{t_1} = D(t_1, T)E_{t_1}\left[1_{\{\tau > T\}} (S_T - M)^+\right],
\]

Note that the sum in formula (5.18) does not appear since a European call option produces cash flows only at maturity. Below we consider the balance sheet equations. We assume that because of this deal the bank needs to ask funding at times \(t_1\) and \(t_2\).

First, note that there are no cash flows originating from the option and thus \(\tilde{V}_{t_1+} = \tilde{V}_{t_1}\). Second, recall that \(H_t\) denotes the replicating portfolio for the traded derivative neglecting counterparty risk. Hence, assuming a Black-Scholes model to price the same option with a risk-free counterparty and a dynamic hedging strategy, we have \(H_t = \tilde{V}_t\) for all \(t \in [0, T]\). Then the balance sheet equations at time \(t_1\) after action \(\gamma\) look as follows:

\[
E^F_{\gamma,t_1} = [\tilde{V}_{t_1} - 1_{\{\tau > t_1\}}V_{t_1+} - K]^+,
\]

\[
A_{\gamma,t_1} = \tilde{V}_{t_1} - 1_{\{\tau > t_1\}}V_{t_1+},
\]

\[
C_{\gamma,t_1} = [K - (\tilde{V}_{t_1} - 1_{\{\tau > t_1\}}V_{t_1+})]^+.
\]

On the event \(\{\tau > t_1\}\) the funding rate for the interval \((t_1, t_2]\) can be determined by (5.8):

\[
e^{\gamma F_{t_1}(t_2-t_1)} = \frac{e^{\gamma(t_2-t_1)}{Q (\tau > t_2|\mathcal{G}_{t_1})}}{Q (\tau > t_1|\mathcal{G}_{t_1})}.
\]

Applying Lemma A.4.1 in section A.4 we can compute

\[
Q (\tau > t_2|\mathcal{G}_{t_1}) = E^Q [1_{\{\tau > t_1\}}1_{\{\tau > t_2\}}]|\mathcal{G}_{t_1}]
\]

\[
= 1_{\{\tau > t_1\}}e^{\int_{t_1}^{t_2} \lambda Q(s) \, ds}E^Q [1_{\{\tau > t_2\}}]|\mathcal{F}_{t_1}]
\]

\[
= 1_{\{\tau > t_1\}}e^{\int_{t_1}^{t_2} \lambda \beta_{t_1}}Q (\tau > t_2|\mathcal{F}_{t_1})
\]

\[
= 1_{\{\tau > t_1\}}e^{\int_{t_1}^{t_2} \lambda \beta_{t_1}}Q (\tau > t_2|\mathcal{F}_{t_1}).
\]
Hence, on \( \{ \tau > t_1 \} \) it holds
\[
e^{s_{F,1}(t_2-t_1)} = e^{\alpha t_1} Q(\tau > t_2 | F_{t_1}).
\]

Subsequently we suppose that at time \( t_2 \) the seller of the option is not in default. In this case \( \mathbb{1}_{\{\tau > t_2\}} = 1 \) and \( B \) returns the funding obtained at time \( t_1 \), accrued at the funding rate \( s_{F_1}^{t_1,t_2} \), and determines again the funding amount needed for the deal with \( C \).
\[
E^F_{\gamma,t_2} = \left[ \bar{V}_2 - \mathbb{1}_{\{\tau > t_2\}} V_{t_2} - C_{\beta,t_2} \right] +
= \left[ \bar{V}_2 - \mathbb{1}_{\{\tau > t_2\}} V_{t_2} + \mathbb{1}_{\{\tau > t_1\}} V_{t_2} + E^F_{\alpha,t_2} \right].
\]

Since the value of a European call option is continuous in time on \([0,T]\), the right-hand limit \( V_{t_2}^\prime \) coincides with \( V_{t_2} \). We see that in this particular example the bank asks the same amount it had to return. However, the new funding rate is determined through
\[
e^{s_{F,2}(t_3-t_2)} = e^{\alpha t_2} Q(\tau > t_3 | F_{t_2}).
\]

Consider two different counterparties \( C \) and \( C' \) with \( \tau_C \) and \( \tau_C' \) random variables, denoting the default times of \( C \) and \( C' \), respectively. Suppose that \( B \) already determined the price \( V_{t_1}^C \) through the pricing formula (5.18). If at time \( t_1 \) \( B \) also wants to make a deal with counterparty \( C' \), the bank has to ask for additional funding. Therefore, \( B \) will ask for an amount \( E^F_{\gamma,t_1} \) equal to \( A_{\gamma,t_1}^C := A_{\gamma,t_1}^C + A_{\gamma,t_1}^{C'} \). As in the case of a single counterparty, \( E^F_{\gamma,t_1} \) will grow according to a funding rate \( s_{F_{\text{new},1}} \) over the interval \( (t_1,t_2] \). This new funding rate will be determined by EP I, where in the case of two counterparties we may extend (5.7) by defining\(^5\)
\[
E^{\alpha}_{\gamma,t_1} := \mathbb{1}_{\{\tau_C > t_{i+1}\}} \omega^C_{t_i} E^F_{\gamma,t_i} e^{s_{F_{\text{new},i}}(t_{i+1}-t_i)} + \mathbb{1}_{\{\tau_C' > t_{i+1}\}} \omega^{C'}_{t_i} E^{\alpha}_{\gamma,t_i} e^{s_{F_{\text{new},i}}(t_{i+1}-t_i)},
\]
where \( \omega^C_{t_i} = A^C_{\gamma,t_i}/A_{\gamma,t_i} \) is the percentage of \( E^F_{\gamma,t_i} \) and \( E^B_{\gamma,t_i} \) the bank invests to conclude the deal with \( C \) and \( \omega^{C'}_{t_i} = A^{C'}_{\gamma,t_i}/A_{\gamma,t_i} \) the percentage of \( E^F_{\gamma,t_i} \) and \( E^B_{\gamma,t_i} \) the bank invests to

\(^5\)Note that given this definition of \( E^{\alpha}_{\gamma,t_i} \), both counterparties have to be in default in order to proceed with point 4 of Definition 5.1.1.
conclude the deal with $C'$. The choice of taking the same percentage for $F$ and $B$ implies that they both invest proportionally the same amount in the deals with $C$ and $C'$ (if both $E_{\gamma,t_i}^S$ and $E_{\gamma,t_i}^F$ are strictly positive). Consequently, the counterparty risk borne by $C$ and $C'$ is distributed in a fair way among the bank and the funder. If we proceed as we did in section 5.3, we find that by additivity of the value of the assets and the equity value the following relation holds for $i = 1, \ldots, m - 1$

\[
\bar{V}_{t_i}^C + \bar{V}_{t_i}^{C'} - \mathbb{1}_{\{\tau_C > t_i\}} H_{t_i}^C = A_{\gamma,t_i}^C + A_{\gamma,t_i}^{C'}
\]

\[
= A_{\gamma,t_i}
\]

\[
= D(t_i, t_{i+1}) \mathbb{E}_{t_i} A_{\alpha,t_{i+1}}
\]

\[
= D(t_i, t_{i+1}) \mathbb{E}_{t_i} \left[ A_{\alpha,t_{i+1}}^C + A_{\alpha,t_{i+1}}^{C'} \right]
\]

\[
= D(t_i, t_{i+1}) \mathbb{E}_{t_i} \left[ \bar{V}_{t_{i+1}}^{C'} - \mathbb{1}_{\{\tau_C > t_i\}} H_{t_{i+1}}^C \right]
\]

\[
+ D(t_i, t_{i+1}) \mathbb{E}_{t_i} \left[ \bar{V}_{t_{i+1}}^{C'} - \mathbb{1}_{\{\tau_C > t_i\}} H_{t_{i+1}}^C \right] .
\]

and as before we obtain

\[
\bar{V}_{t_i}^C + \bar{V}_{t_i}^{C'} = D(t_1, t_m) \mathbb{E}_{t_1} V_{t_m}^C + \sum_{i=2}^{m} D(t_1, t_{i-1}) \mathbb{E}_{t_1} \left[ \mathbb{1}_{\{\tau > t_i\}} \Pi^C (t_{i-1}, t_i) \right]
\]

\[
+ D(t_1, t_m) \mathbb{E}_{t_1} V_{t_m}^{C'} + \sum_{i=2}^{m} D(t_1, t_{i-1}) \mathbb{E}_{t_1} \left[ \mathbb{1}_{\{\tau' > t_i\}} \Pi^{C'} (t_{i-1}, t_i) \right] .
\]

Finally, substituting (5.18) yields

\[
\bar{V}_{t_i}^{C'} = D(t_1, t_m) \mathbb{E}_{t_1} \bar{V}_{t_m}^{C'} + \sum_{i=2}^{m} D(t_1, t_{i-1}) \mathbb{E}_{t_1} \left[ \mathbb{1}_{\{\tau_C > t_i\}} \Pi^{C'} (t_{i-1}, t_i) \right] .
\]

**Conclusion:** We see that by imposing EP I for the total new amount $E_{\gamma,t_i}$ we need at the beginning of each funding interval and then applying EP II, the price of the derivative traded with counterparty $C'$ is independent of $s_{F, i}$ as well as $s_{F,i}$ for all $i = 1, \ldots, m - 1$. Under the assumptions stated at the beginning of this chapter, calculating the price of a deal with a new counterparty is a merely additive problem.
Appendix A

Results from the literature

In the following we state some important results from the literature.

A.1 Radon-Nikodym Theorem

For completeness we state the Radon-Nikodym Theorem. However, since in this thesis only probability measures are considered, we give a simplified version.

**Theorem A.1.1** (Radon-Nikodym Theorem, [Andersen and Piterbarg, 2010]). Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be equivalent probability measures on the common measure space $(\Omega, \mathcal{F})$. There exists a unique (a.s.\(^1\)) non-negative random variable $R$ with $\mathbb{E}^\mathbb{P}(R) = 1$, such that

$$\tilde{\mathbb{P}}(A) = \mathbb{E}^\mathbb{P}(R\mathbb{1}_A), \text{ for all } A \in \mathcal{F}.$$  

Considering two measures induced by numéraires, the next theorem shows that the density of the Radon-Nikodym derivative can be expressed in terms of the numéraires.

**Theorem A.1.2** (Change of Numéraire, [Andersen and Piterbarg, 2010]). Consider two numéraires $N(t)$ and $M(t)$, inducing equivalent martingale measures $Q^N$ and $Q^M$, respectively. If the market is complete, then the density of the Radon-Nikodym derivative relating the two measures is uniquely given by

$$\mathbb{E}_t^{Q^N} \left( \frac{dQ^M}{dQ^N} \right) = \frac{M(t)/M(0)}{N(t)/N(0)}.$$  

The change from the risk-neutral measure to the $T$-forward measure at the end of Section 3.3 is determined through the Radon-Nikodym derivative process $\mathbb{E}_t[dQ^T/dQ]$. The following Lemma shows how:

**Lemma A.1.3.** Let $0 \leq t \leq T$ be given and let $Y$ be a $\mathcal{G}_T$-measurable random variable. Then

$$\mathbb{E}_t \left[ \frac{dQ^T}{dQ} \right] \mathbb{E}_t^{Q^T}[Y] = \mathbb{E}_t \left[ \frac{dQ^T}{dQ} Y \right]. \tag{A.1}$$

\(^{1}\)This means that if $R$ and $\tilde{R}$ are two such random variables, then the event \{\(R \neq \tilde{R}\)\} has zero probability under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$. 

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A.2. **Calculations Leading to the Pricing Equation in Chapter ??**

Given the results above and since \( \mathbb{E}_t[dQ^T/dQ] \) is strictly positive, we can switch to the \( t_{j+1} \)-forward measure in equation (3.8) by

\[
\mathbb{E}_{t_j} \left[ \frac{Y_{t_{j+1}}}{\beta_{t_{j+1}}} \right] = \frac{P_{t_j}(t_{j+1})}{\beta(t_j)} \mathbb{E}_{t_{j+1}}^{\tau}[Y_{t_{j+1}}],
\]

where \( Y_{t_{j+1}} \) is a general \( G_{t_{j+1}} \)-measurable random variable.

### A.2 Calculations Leading to the Pricing Equation in Chapter 3

We want to show that equation (3.8) holds. From formula (3.7) we get for \( t = t_{j+1} \)

\[
\tilde{V}_{t_{j+1}} = \mathbb{E}_{t_{j+1}} \left[ \Pi(t_{j+1}, T \wedge \tau) + \varphi(t_{j+1}, T \wedge \tau; F) + \mathbbm{1}_{\{t_{j+1} < \tau < T\}} D(t_{j+1}, \tau) \theta_{\tau}(\varepsilon) \right]
\]

and for \( t = t_j \)

\[
\tilde{V}_j = \mathbb{E}_{t_j} \left[ \Pi(t_j, T \wedge \tau) + \varphi(t_j, T \wedge \tau; F) + \mathbbm{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) \theta_{\tau}(\varepsilon) \right].
\]

These equalities hold on \( \{ \tau > t_{j+1} \} \) and \( \{ \tau > t_j \} \), respectively. All the remaining equalities in this section are to be interpreted conditional on the event \( \{ \tau > t_j \} \).

First, note that

\[
\Pi(t_j, T \wedge \tau) = \Pi(t_j, t_{j+1} \wedge \tau) + D(t_j, t_{j+1}) \Pi(t_{j+1}, T \wedge \tau),
\]

\[
\varphi(t_j, T \wedge \tau; F) = \varphi(t_j, t_{j+1} \wedge \tau; F) + D(t_j, t_{j+1}) \varphi(t_{j+1}, T \wedge \tau; F),
\]

\[
\mathbbm{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) = \mathbbm{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) + \mathbbm{1}_{\{t_{j+1} < \tau < T\}} D(t_{j+1}, \tau) \theta_{\tau}(\varepsilon).
\]

Substituting the equalities above yields

\[
\tilde{V}_j = \mathbb{E}_{t_j} \left[ \Pi(t_j, t_{j+1} \wedge \tau) + \varphi(t_j, t_{j+1} \wedge \tau; F) + \mathbbm{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau) \theta_{\tau}(\varepsilon) \right]
\]

\[
+ \mathbb{E}_{t_j} \left[ \mathbb{E}_{t_{j+1}} \left[ D(t_j, t_{j+1}) \left( \Pi(t_{j+1}, T \wedge \tau) + \varphi(t_{j+1}, T \wedge \tau; F) + \mathbbm{1}_{\{t_{j+1} < \tau < T\}} D(t_{j+1}, \tau) \theta_{\tau}(\varepsilon) \right) \right] \right]
\]

\[
= \mathbb{E}_{t_j} \left[ D(t_j, t_{j+1}) \tilde{V}_{t_{j+1}} \right] + \mathbb{E}_{t_j} \left[ \Pi(t_j, t_{j+1}) + \mathbb{E}_{t_j} [\varphi(t_j, t_{j+1} \wedge \tau; F)] \right],
\]

where in the first equality we used the tower property and in the second equality we used the fact that \( D(t_j, t_{j+1}) \) is \( F_{t_{j+1}} \)-measurable. Subsequently we notice that

\[
\varphi(t_j, t_{j+1} \wedge \tau; F) = \mathbbm{1}_{\{t_{j+1} < \tau\}} \varphi(t_j, t_{j+1}; F) + \mathbbm{1}_{\{t_{j+1} \geq \tau\}} \varphi(t_j, \tau; F)
\]

and using equality (3.5) yields

\[
\varphi(t_j, t_{j+1} \wedge \tau; F) = \mathbbm{1}_{\{t_{j+1} < \tau\}} \left( F_{t_j} - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) \right)
\]

\[
+ \mathbbm{1}_{\{t_{j+1} \geq \tau\}} \mathbbm{1}_{\{t_j < \tau\}} \left( F_{t_j} - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) \right)
\]

\[
= \left( \mathbbm{1}_{\{t_{j+1} < \tau\}} + \mathbbm{1}_{\{t_{j+1} \geq \tau\}} \mathbbm{1}_{\{t_j < \tau\}} \right) \left( F_{t_j} - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) \right)
\]

\[
= \mathbbm{1}_{\{t_j < \tau\}} \left( F_{t_j} - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) - F_{t_j}^\tau P_{t_j}(t_{j+1}) P_{t_j}^\tau(t_{j+1}) \right),
\]
where in the last equality we used the fact that $\mathbb{1}_{\{t_{j+1} < \tau\}} = \mathbb{1}_{\{t_j < \tau\}} \mathbb{1}_{\{t_{j+1} < \tau\}}$.

### A.3 Necessary results for the proof of existence

In this section we summarize part of the results from [Bielecki and Rutkowski, 2001] needed to show the existence of a replicating strategy for a defaultable claim. For the definitions of the processes appearing below see section 4.1.

**Lemma A.3.1.** The discounted wealth $\tilde{U}_t(\psi) = \beta - 1_t U_t(\psi)$ of a self-financing trading strategy $\psi$ satisfies, for every $t \in [0, T]$

$$\tilde{U}_t(\psi) = \tilde{U}_0(\psi) + \tilde{S}_0^0 - \tilde{S}_0^0 + \beta - 1_t X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{t \geq T\}}.$$ 

**Lemma A.3.2.** The process $L$ given by the formula $L_t := \mathbb{1}_{\{\tau > t\}} \exp \Gamma_t$, follows a $\mathbb{G}$-martingale under $Q$. Furthermore, for any bounded $\mathbb{F}$-martingale $m$, the product $L m$ is a $\mathbb{G}$-martingale.

A necessary assumption for the Lemma above to hold is that the inclusion $\mathcal{F}_t \subset \mathcal{G}_t$ is strict. This is however already implied by the definition of hazard rate, since $\mathcal{F}_t < 1$ for all $t \in [0, T]$.

**Proposition A.3.3.** Assume that the $\mathbb{F}$-hazard process $\Gamma$ of $\tau$ follows a continuous, increasing process. Then the process $\hat{M}_t = H_t - \Gamma_{t \wedge \tau}$ follows a $\mathbb{G}$-martingale, specifically,

$$\hat{M}_t = \int_{(0,t]} e^{-\Gamma_u} dL_u.$$ 

Furthermore, $L$ solves the linear integral equation

$$L_t = 1 - \int_{(0,t]} L_{u-} \, d\hat{M}_u. \tag{A.2}$$

Note that from equation (A.2) it follows

$$dL_t = -L_{t-} \, d\hat{M}_t. \tag{A.3}$$

We now state Itô’s product rule for two right-continuous processes $V$ and $W$, both with finite variation:

$$V_t W_t = V_0 W_0 + \int_0^t V_{s-} \, dW_s + \int_0^t W_{s-} \, dV_s + [V, W]_t,$$

where $[V, W]_t = \sum_{0 \leq s \leq t} (V_s - V_{s-}) (W_s - W_{s-})$. We will use this formula in the proof of the following Proposition.

**Proposition A.3.4.** Let the price process $D^0(t, T)$ be given by $\beta \mathbb{E}^Q \left[ \beta^{-1} \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t \right]$. Then

$$dD^0(t, T) = D^0(t-, T) \left( r_t \, dt - \, d\hat{M}_t \right) + \beta_t L_{t-} \, dm_t.$$
A.4. INTENSITY PROCESS

Proof Let \( Z^0(t, T) = \beta_t^{-1} D^0(t, T) \). This is clearly a \( \mathcal{G} \)-martingale under \( \mathbb{Q} \). We now want to apply Lemma A.4.1 to derive an expression of \( Z^0(t, T) \) in terms of the hazard process. Noting that \( 1_{\{\tau > T\}} \) is a continuous and increasing process, the process \( m \) is continuous and \( L \) is a process of finite variation. Hence, we can apply Itô’s product rule stated above to get

\[
d(L_t m_t) = L_t \, dm_t + m_t \, dL_t,
\]

where the covariation term is zero by continuity of \( m \). Equation (A.3) yields then

\[
d(L_t m_t) = L_t \, dm_t + m_t \left( -L_t \, d\tilde{M}_t \right) = L_t \, dm_t - Z^0(t-, T) \, d\tilde{M}_t.
\]

Finally, we can apply Itô’s formula to \( \beta_t Z^0(t, T) \):

\[
dD^0(t, T) = d\left( \beta_t Z^0(t, T) \right) = Z^0(t-, T) \, d\beta_t + \beta_t \, dZ^0(t, T) = Z^0(t-, T) \beta_t \, r_t \, dt + \beta_t L_t \, dm_t - \beta_t Z^0(t-, T) \, d\tilde{M}_t = D^0(t-, T) \left( r_t \, dt - d\tilde{M}_t \right) + \beta_t L_t \, dm_t.
\]

\( \Box \)

A.4 Intensity process

We briefly state the most important assumptions and results of the intensity-based approach in [Filipović, 2009]. Consider the mathematical framework from Chapter 5, with \( \mathbb{E} \) denoting expectations under the real-world probability measure \( \mathbb{P} \). The first assumption that is made allows to express conditional probabilities of default in terms of a process \( \lambda \).

(D1) There exists a nonnegative \( (\mathcal{F}_t) \)-progressive process \( \lambda \) such that

\[
\mathbb{P} (\tau > t | \mathcal{F}_t) = e^{-\int_0^t \lambda(s) \, ds}.
\]

Lemma A.4.1. Assume (D1), and let \( Y \) be a nonnegative random variable. Then

\[
\mathbb{E} \left[ 1_{\{\tau > t\}} Y | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\int_0^t \lambda(s) \, ds} \mathbb{E} \left[ 1_{\{\tau > t\}} Y | \mathcal{F}_t \right]
\]

for all \( t \).
The second assumption is

(D2) \( P(\tau > t | \mathcal{F}_\infty) = P(\tau > t | \mathcal{F}_t), \quad t > 0. \)

Lemma A.4.2. The following properties are equivalent:

(a) (D2) holds.

(b) Every bounded \( \mathcal{F}_\infty \)-measurable \( X \) satisfies \( \mathbb{E}[X|\mathcal{G}_t] = \mathbb{E}[X|\mathcal{F}_t] \).

(c) Every \( (\mathcal{F}_t) \)-martingale is a \( (\mathcal{G}_t) \)-martingale.

It is then possible to find an equivalent probability measure \( Q \sim P \) such that (D1) and (D2) hold under \( Q \) for a process \( \lambda_Q = \mu \lambda \), with \( \mu \) a positive \( (\mathcal{G}_t) \)-predictable process satisfying

\[
\int_0^t \lambda_Q(s) \, ds < \infty \quad \text{for all } t \in \mathbb{R}_+.
\]
Bibliography


