Quark/gluon jets discrimination using thrust at NNLL

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Abstract

There is much to learn from studying jets and their substructure. In particular, we can make a distinction between quark jets and gluon jets, which is relevant for searches for New Physics, where quark jets might indicate New Physics and gluon jets are just the QCD background. We study the event shape variable thrust and use it as a quark/gluon jet discriminant. The thrust cross section suffers from logarithmic terms which become large and diverge in the small thrust limit. We use the Renormalization Group Equations and formalism of Soft Collinear Effective Theory to resum these large logarithms. Analytical calculations of the thrust distribution up to next-to-next-to-leading-logarithmic order are compared with predictions of two Monte Carlo event generators: HERWIG and PYTHIA. We find that PYTHIA and HERWIG are at opposite sides of our analytical predictions, but still within the uncertainties.
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Chapter 1.

Introduction

Collisions of particles in particle colliders often produce jets: narrow cones of hadrons and other particles. This can be pictured as in Fig. 1.1 where a hard collision has produced two back-to-back jets. There is much to learn from studying these jets and their substructure. In particular, we can assign a quark or gluon tag to a jet and differentiate between them. Being able to distinguish between quark and gluon jets is relevant for example for searches for New Physics, where New Physics are often accompanied by quark jets (e.g. squarks in SUSY), while the QCD background gluon jets. Studies to discriminate between quark/gluon jets using several jet observables have been done before using analytical calculations and parton showers. It was found that there were large differences in the predicted discrimination power between several parton showers [1]. In this thesis we will look into this issue and study quark/gluon discrimination by using an event shape observable called “thrust.” Analytical calculations will be compared with the predictions the Monte Carlo event generators PYTHIA [2] and HERWIG [3].

![Diagram showing dijet production from a hard collision, accompanied by soft radiation.](image)

Figure 1.1.: Dijet production from a hard collision, accompanied by soft radiation.

There are some ambiguities and subtleties in what exactly a quark/gluon jet is, but in this study we simply look at quark/gluon initiated jets from the processes $\gamma/Z \rightarrow q\bar{q}$ and $H \rightarrow gg$ (Fig. 1.2). These can occur from electron-positron collisions, which we use in this thesis. Although the process for the gluon jets is not very interesting, since the coupling of the Higgs to electrons is very small, it provides us a well-defined
way to produce gluon jets. Furthermore, electron-positron collisions provide us a clean environment to do analytical calculations. We will calculate the cross section differential in thrust for the above mentioned processes. As we will later see, we encounter the problem of large logarithms when we do this. This problem will be solved by using the Renormalization Group Equations (RGE) which will resum the large logarithms.

The outline of this thesis is as follows: in chapter 2 we give a brief overview of the background used in this thesis. We discuss QCD, SCET, Monte Carlo event generators and thrust. The calculation of the thrust cross section is separated into two parts: the singular part, which we go through in chapter 3, and the nonsingular part, which we go through in chapter 4. Chapter 5 deals with how the results from chapter 3 and chapter 4 are combined to produce the final results with uncertainties and how we account for the hadronization. These final results are shown in chapter 6, where they are compared with the predictions of the Monte Carlo event generators. Finally, we conclude in chapter 7.
Chapter 2.
Monte Carlos, SCET and thrust

2.1. Quark/gluon jets tagging

There have been many quark/gluon discriminants proposed, for example the two-parameter family of generalized angularities \[ \lambda_\beta^\kappa = \sum_{i \in \text{jet}} z_i^\kappa \theta_i^\beta, \] where \( i \) runs over the jet constituents, \( z_i \) is the momentum fraction, \( \theta_i \) a normalized angle to the jet axis and \( \kappa, \beta \) are the two parameters that determine the weighting of the momentum and angle. Five of these generalized angularities \((\kappa, \beta) = \{(0, 0), (2, 0), (1, 0.5), (1, 1), (1, 2)\}\) have been studied as quark/gluon discriminants in \[ \text{(2.2)} \]. Their results are shown in Fig. 2.1.

In this thesis we will be using the event shape thrust as our classifier. One may ask why we use this variable thrust, which looks at the whole event, to discriminate between quark and gluon jets, rather than a more suitable jet observable. In general, it is true that observables which look exclusively to jets, like jet thrust where only particles within a certain jet radius are considered, are better to study jets. Thrust however, has an advantage over these other observables in that it can be analytically calculated to very high order compared to these other observables. As we expect that quark/gluon jet discrimination performance is very sensitive to effects beyond leading order calculations, analytical higher order calculations are important to study quark/gluon jets discrimination. The observable thrust allows us to do this, which is the reason we use it in this thesis.

Earlier studies \[ \text{[1]} \] have studied quark/gluon jet discrimination and the classifier separation as predicted by several different parton-shower generators. There was good agreement between the different parton-showers on the quark samples, whereas on the gluon samples there were larger variations between the different parton-showers. The reason that there is good agreement on the quarks sample is because these Monte Carlo programs have been tuned to match LEP data on the quarks. Since for the gluon samples there has not been done such a tuning in the Monte Carlos (we will see one exception later), it is not surprising that the variations are much larger here. In \[ \text{[1]} \] it was generally
found that the Monte Carlo program Pythia was most optimistic about quark/gluon jet discrimination and Herwig the least. For this reason we will restrict our studies to only these two Monte Carlos and compare the findings with our analytical results.

2.2. Monte Carlo event generators

Monte Carlo event generators are programs which simulate processes happening at colliders using Monte Carlo techniques. At the hard scale of the collision, the hard process is generated by using matrix element calculations (usually at LO or NLO). To give a picture about the substructure of a jet and the distributions of the various produced particles, higher order effects should be included. These higher order effects are simulated by a parton shower algorithm. The parton shower evolution starts at the hard scale and evolves to a low scale of order 1 GeV. At this scale the Monte Carlo program switches from the parton shower to some hadronization model to describe the confinement of partons into hadrons. We use two different Monte Carlo event generators in our study: Herwig and Pythia. Since we are studying the thrust distribution of quark and gluon jets obtained from $e^+e^- \to (\gamma/Z)^* \to q\bar{q}$ and $e^+e^- \to h^* \to gg$, we simulate an electron-positron collider and choose to look only to these two processes (for Herwig we actually used a $\tau^+\tau^-$ collider to produce the gluon jets. Because we will eventually only use the normalized thrust distribution of these jets, this does not matter for the results). We run the simulations at a centre of mass energy of 125 GeV and turn initial state radiation off. The Monte Carlo programs each use different parton shower algorithms and different hadronization models, which have multiple settings (tunes). In this study we use the default tunes. For Herwig we consider two different parton showers:
Chapter 2. Monte Carlos, SCET and thrust

2.2. Figure 2.2.: Graphical representation of the process generation and the parton shower in HERWIG. An electron and positron collide and produce $q\bar{q}$ through an intermediate $Z_0$-boson. Thereafter, the partons radiate off gluons.

2.3. Figure 2.3.: Graphical representation of the hadronization which occurs after the parton shower in HERWIG. Coloured partons are turned into colourless hadrons by the Cluster hadronization model of HERWIG.

2.3. Herwig 7.0.4 vs Herwig 7.1.0

In this thesis we used the latest HERWIG versions available at the time for the Monte Carlo predictions. These are HERWIG 7.0.4 and HERWIG 7.1.0. There have been a few major changes in switching from these versions for our study, so we briefly discuss this here.

Changes have been made in what to preserve in the parton shower after multiple emissions, and for the first time data with gluon-initiated jets has been included in...
the tuning \cite{[6]}. The resulting normalized thrust distributions for quarks and gluons are shown in Fig. 2.4 and Fig. 2.5. For the quarks, there are some small differences in the peak region at parton level. At hadron level they become more or less the same again. For the gluons however, we immediately see large differences. At parton level the new thrust distribution of HERWIG 7.1.0 is less peaked with the peak shifted more towards a higher value of $\tau$, which also results in a broader thrust distribution at hadron level. This is due to the tuning to gluon-initiated jets data and the changes in the parton shower. As we will see later, these changes significantly improve the agreement of the resulting thrust distribution with our analytical results. In the following, we then use HERWIG 7.1.0 as our main version for HERWIG for our comparisons.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{quarks_parton_level}
\includegraphics[width=0.45\textwidth]{quarks_hadron_level}
\caption{Normalized thrust distributions of HERWIG 7.0.4 (blue) vs HERWIG 7.1.0 (red) for the quarks at parton level (left) and hadron level (right) at $Q = 125$ GeV. The parton shower used in these plots is the (default) angular shower.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{gluons_parton_level}
\includegraphics[width=0.45\textwidth]{gluons_hadron_level}
\caption{Same as Fig. 2.4 but for gluons.}
\end{figure}

2.4. Quantum Chromodynamics

Like QED, Quantum Chromodynamics (QCD) is a quantum field theory, but instead of describing electrodynamics, it describes the strong interaction. QCD has some simi-
ties with QED, but also many differences, which make the theory behave very differently from QED. In this section we will give a short overview of some properties of QCD.

QCD describes interactions between quarks and gluons. There are 6 types of quarks: up (u), down (d), charm (c), strange (s), top (t) and bottom (b). The masses of these quarks vary greatly: the up-quark is the lightest with a mass of \( \sim \frac{2}{3} \text{ MeV} \) and the top-quark the heaviest with a mass of \( \sim 175 \text{ GeV} \). The top-quark mass is much greater than the masses of all the other quarks (the second heaviest quark is the bottom quark with a mass of only \( \sim 4 \text{ GeV} \)). Because of this we often do calculations with the \( m_t \rightarrow \infty \) limit and \( n_f = 5 \). The other five quarks are also treated as massless.

The QCD Lagrangian is given by:

\[
\mathcal{L} = \sum_q \bar{\psi}_q^i \gamma^\mu (D_\mu)_{ij} \psi_q^j - m_q \bar{\psi}_q^i \psi_q^i - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu},
\]

(2.3)

where \( \psi_q^i \) is a quark field of flavour \( q \) (u, d, s, c, b, t) and colour \( i \) (red, green blue), \( \gamma^\mu \) the Dirac matrix, since the quarks are fermions, \( D_\mu \) the covariant derivative in QCD, \( m_q \) the mass of the quark and \( F_{\mu\nu}^a \) the gluon field strength tensor. The covariant derivative is

\[
(D_\mu)_{ij} = \delta_{ij} \partial_\mu - ig_s t^a_{ij} A_\mu^a,
\]

(2.4)

where \( g_s \) is the strong coupling, \( A_\mu^a \) the gluon field with colour index \( a \) (running from 1 to 8) and \( t^a_{ij} \) the Gell-Mann matrices \( \lambda^a_{ij} \) of \( SU(3) \) with some normalization: \( t^a_{ij} = \frac{1}{2} \lambda^a_{ij} \).

The gluon field strength tensor is given by

\[
F_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + g_s f_{abc} A_{\mu}^b A_{\nu}^c
\]

(2.5)

From this extra third term which is absent in QED, we see that there are three gluon and four gluon interactions in the Lagrangian of QCD. This is a new feature in QCD which QED does not have. This new third term and the resulting self-interactions of the gluons arise in QCD, because the gauge group of QCD is a non-abelian gauge group: \( SU(3) \) (see App. E for more details).

Another key feature of QCD is that it is asymptotic free. The running of the strong coupling is governed by the QCD \( \beta \)-function (see App. A).

\[
\frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s),
\]

(2.6)

where the \( \beta \)-function looks like

\[
\beta(\alpha_s) = -2\alpha_s \left( \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \beta_2 \left( \frac{\alpha_s}{4\pi} \right)^3 + \ldots \right)
\]

(2.7)

The fact that there is an overall minus sign in the \( \beta \)-function (together with that \( \beta_0 \) is positive) leads to the result that the strong coupling decreases as the scale increases. By solving Eq. (2.6), the numeric value of the strong coupling at any scale \( \mu \) can be related
to the value of the strong coupling at some initial scale $\mu_0$. To one-loop, the solution is:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)\beta_0}{2\pi} \log \left( \frac{\mu}{\mu_0} \right)} \quad (2.8)$$

From here, it can be seen that the strong coupling grows as we go to lower energies and that at a certain low scale

$$\mu = \Lambda_{QCD} = \mu_0 e^{-\frac{2\pi}{\alpha_s(\mu_0)\beta_0}} \quad (2.9)$$

the coupling even diverges. This is the reason why perturbative techniques start to break down at low scales in QCD and nonperturbative methods are needed in this region. In this thesis we also encounter this problem and have to make use of nonperturbative hadronization models.

### 2.5. SCET

In this section we briefly look into SCET [7–11] and some of its properties needed for our calculations. More detailed information on SCET can be found in [12, 13].

Soft-Collinear Effective Theory (SCET) is a top-down effective field theory, which can be derived from QCD. SCET is very suitable to describe jet physics with interactions of soft and collinear particles in the presence of a hard interaction. This is exactly the situation in this thesis where we study the processes $e^+e^- \rightarrow (\gamma/Z)^* \rightarrow q\bar{q}$ and $e^+e^- \rightarrow h \rightarrow gg$. In SCET, particles are not necessarily integrated out, but modes are separated according to how their momentum components scale. In collider processes we call the the momentum scale corresponding to the hard interaction the hard scale, $Q$.

Collinear degrees of freedom are energetic particles which move in or near a preferred direction (the jet axes). Soft degrees of freedom are particles which have no preferred direction and have momenta much lower than $Q$. We will have different momentum regions and separate particles accordingly. In SCET we can thus have different fields which would represent one field in the full QCD theory. For example, we have collinear quarks, soft quarks, collinear gluons and soft gluons.

#### 2.5.1. Lightcone coordinates

We use coordinates which make the different scalings in momentum components more transparent. These will be the lightcone coordinates, defined by the vectors $n^\mu = (1, \vec{n})$ and $\bar{n}^\mu = (1, -\vec{n})$, where $\vec{n}$ is the direction of the jet and which satisfy $n^2 = 0 = \bar{n}^2$ and $n \cdot \bar{n} = 2$. For example, the vectors $n^\mu = (1, 0, 0, 1)$ and $\bar{n}^\mu = (1, 0, 0, -1)$ satisfy these relations. Any vector $p^\mu$ can then be decomposed in this lightcone basis:

$$p^\mu = \frac{n^\mu}{2} \bar{n} \cdot p + \frac{\bar{n}^\mu}{2} n \cdot p + p_\perp^\mu \equiv (p^+, p^-, p_\perp^\mu), \quad (2.10)$$

where $p^+ = n \cdot p$, $p^- = \bar{n} \cdot p$ and $p_\perp^\mu$ are the remaining components orthogonal to both $n$ and $\bar{n}$. The four-momentum squared of $p^\mu$ in this notation is: $p^2 = p^+ p^- - p_\perp^2$. 

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In this thesis we look at jet physics. The simplest process of jet production is the process $e^+e^- \rightarrow \text{dijets}$. At lowest order this is $e^+e^- \rightarrow \gamma^* \rightarrow q\bar{q}$, where the $q$ and $\bar{q}$ quarks will each form a jet. In the CM frame, the momentum of the photon is $q^\mu = (Q, 0, 0, 0)$, which determines the hard scale. By momentum conservation the two jets will be back-to-back and we can easily find the two vectors $n^\mu = (1, \vec{n})$ and $\bar{n}^\mu = (1, -\vec{n})$ to set up our lightcone coordinate system. The plane orthogonal to $\vec{n}$ divides the space in two hemispheres, with one jet in each of them. The particles in the jet collinear to $n^\mu$ have their minus momentum much bigger than their $\perp$-momentum:

$$p^- \sim Q \gg p_\perp \sim Q\lambda,$$

where $\lambda \ll 1$ is a small dimensionless parameter. By using the fact that we are considering fluctuations about $p^2 = 0$, we find from $p^+p^- - p_\perp^2 \sim p^+Q - Q^2\lambda^2 \sim 0$ the scaling of the plus momentum: $p^+ \sim Q\lambda^2$. We thus find the scaling of $n^\mu$-collinear momenta as $p^\mu \sim Q(\lambda^2, 1, \lambda)$. The same way we find for the $\bar{n}^\mu$-collinear momenta scaling in the other hemisphere: $p^\mu \sim Q(1, \lambda^2, \lambda)$.

In the lightcone coordinates, the scalings of the different modes are now clearly visible. To summarize: the hard modes are momenta modes which scale as

$$p^\mu_h \sim Q(1, 1, 1).$$

(2.11)

The collinear and anti-collinear modes have momenta scaling like

$$p^\mu_c \sim Q(\lambda^2, 1, \lambda)$$

$$p^\mu_{c\bar{c}} \sim Q(1, \lambda^2, \lambda),$$

(2.12)

and soft momenta scale like

$$p^\mu_s \sim Q(\lambda^2, \lambda^2, \lambda^2),$$

(2.13)

where $\lambda$ is as a small dimensionless parameter with $\lambda \ll 1$. In the literature these are often called “ultrasoft” modes, whereas “real” soft modes scale like

$$p^\mu_s \sim Q(\lambda, \lambda, \lambda).$$

(2.14)

This is the difference between SCET I, where the homogeneous modes have ultrasoft scaling such that $p_{us}^2 \sim Q^2\lambda^4$ is parametrically different than for the collinear modes $p_c^2 \sim Q^2\lambda^2$, and SCET II theories where the homogeneous modes have soft scaling such that $p_s^2 \sim Q^2\lambda^2 \sim p_c^2$. We use a SCET I theory, so we do not have soft modes scaling like in Eq. (2.14), and can then drop the “ultra” in “ultrasoft” and call our ultrasoft modes just soft without any risk of confusion.

### 2.5.2. Collinear fields

The collinear SCET Lagrangian for a quark is given by [12]

$$\mathcal{L} = \bar{\xi}_n \left( i\vec{n} \cdot \vec{D} + i\vec{D}_\perp \frac{1}{i\vec{n} \cdot \vec{D}} i\vec{D}_\perp \right) \frac{\not{D}}{2} \xi_n$$

(2.15)
A few of the resulting Feynman rules for a collinear quark are given below.

\begin{equation}
\frac{i \tilde{n}}{2} \frac{\tilde{n} \cdot p}{(n \cdot p)(\bar{n} \cdot p) + p_\perp^2 + i0}
\end{equation}

\begin{equation}
igT n_{\mu} \frac{\beta}{2}
\end{equation}

\begin{equation}
igT \left[ n_{\mu} + \frac{\gamma_{\mu}}{\bar{n} \cdot p} + \frac{\gamma_{\mu}}{\bar{n} \cdot p'} - \frac{\gamma_{\mu}}{\bar{n} \cdot p \bar{n} \cdot p'} \right] \frac{\beta}{2}
\end{equation}

The Feynman rule for a gluon propagator is just (in the Feynman gauge):

\begin{equation}
-ig \delta_{ab} q^{\mu \nu} \frac{1}{q^2 + i0}
\end{equation}

2.5.3. Wilson lines

In SCET we encounter Wilson lines, and for our study we encounter two kinds of them: collinear Wilson lines and ultrasoft Wilson lines. The power counting of collinear gluon fields leads to collinear Wilson lines. The components of the collinear gluon field scale the same as the components of collinear momentum:

\begin{equation}
A_\mu^n \sim k_\mu^C \sim (\lambda^2, \lambda^0, \lambda)
\end{equation}
We see that $\bar{n} \cdot A \sim \lambda^0$, which means that there is no suppression (no extra powers of $\lambda$) for adding $\bar{n} \cdot A_n$ fields to operators in SCET. We could add multiple of these $\bar{n} \cdot A_n$ and it would still be as important as having only added one in the power counting. Summing them, this effectively results in the replacement of $\bar{n} \cdot A_n$ with the Wilson line

$$W[\bar{n} \cdot A_n] = \sum \exp \left( -g \frac{\bar{n} \cdot A_n}{\bar{n} \cdot P} \right), \quad (2.21)$$

where the $P$ is the label momentum operator (for more details on this see [12]). $W[\bar{n} \cdot A_n]$ Fourier transformed gives the following expression in position space:

$$W(x) = P \exp \left[ ig \int_{-\infty}^{0} ds \bar{n} \cdot A(x + s\bar{n}) \right]. \quad (2.22)$$

The structure of these collinear Wilson lines is dictated by gauge invariance. Under a gauge transformation $U(x) = \exp (i\alpha A(x) T^A)$ the collinear field $\xi_n$ transforms as

$$\xi_n(x) \to U_n(x)\xi_n(x), \quad (2.23)$$

while the Wilson line transforms as

$$W_n(x) \to U_n(x)W_n(x) \quad (2.24)$$

Combining both, we obtain gauge invariant operators $\chi \equiv W_n^\dagger \xi_n$ and $\bar{\chi} \equiv \bar{\xi}_n W_n$.

Similarly, the ultrasoft Wilson lines

$$Y_n(x) = P \exp \left[ ig \int_{-\infty}^{0} ds n \cdot A_{us}(x + s\bar{n}) \right], \quad (2.25)$$

are constructed, which are used to decouple the collinear and ultrasoft modes [12].

### 2.6. Thrust

In this thesis we look into the process $e^+e^- \to \text{jets}$. Kinematically dominant is the production of two jets, but it is also possible that more jets are produced. The event shape variable thrust can be used to distinguish dijet events from events with more than two jets. Thrust is defined as [12, 14]:

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i \cdot \vec{n}_T|}{\sum_i |\vec{p}_i|}, \quad (2.26)$$

where $i$ runs over all the final state particles and $\vec{n}_T$ is the unit thrust axis. The thrust vector $\vec{n}_T$ is chosen such that it maximizes $T$. Collinear (or anti-collinear) particles have a large projection onto the thrust axis, giving $T$ near 1. Events with $T$ near 1 are thus 2-jet like, while lower values of $T$ indicate broader jets or multiple jets. It is more
convenient to use the variable $\tau = 1 - T$. In this thesis we use $\tau$ as our variable and refer to $\tau$ when talking about thrust. Some calculations are however done in $T$, but the different notation should be enough to distinguish between the two thrust definitions. This now means that in the limit $\tau \to 0$ we are in the situation of two thin pencil-like jets and moving away from this limit we go to the cases of broader dijets or multiple jets. At the thrust endpoint $\tau = \frac{1}{2}$ we are in the situation of a totally spherically symmetric distribution of final state particles. The observable thrust thus gives us information about how an event looks like and how jet-like it is.

From the definition of thrust in Eq. (2.26), we see that it is very similar to the generalized angularity $(1, 2)$ in Eq. (2.1), since thrust can be written as

$$\tau = 1 - T = 1 - \sum_i z_i \cos \theta_i \approx 1 - \sum_i z_i \left(1 - \frac{\theta_i^2}{2}\right) = \frac{1}{2} \sum_i z_i \theta_i^2. \quad (2.27)$$

We then expect to find similar results for the predictions of the Monte Carlo event generators for thrust.

So we study the collider observable thrust and use the framework provided by SCET for this. This means that the thrust $\tau$ is our small parameter in Eq. (2.12): $\lambda^2 = \tau$ and that the two lightcone vectors $n^\mu$ and $\bar{n}^\mu$ take the form

$$n^\mu = (1, \vec{n}_T), \quad \bar{n}^\mu = (1, -\vec{n}_T). \quad (2.28)$$
Chapter 3.
Singular cross section

3.1. Factorization of the singular cross section

We will use the factorization theorem for thrust to calculate the singular cross section. SCET allows us to factorize the cross section for thrust into several parts: a hard function, jet functions and a soft function. Symbolically we will have \( \frac{d\sigma}{d\tau} \sim H \cdot J \otimes \bar{J} \otimes S \), where the \( \otimes \) denotes a convolution. Here \( H \) is called the hard function which arises from integrating out the hard modes of QCD and matching to SCET. The hard function is defined as the absolute value squared of the Wilson coefficient. \( J \) and \( \bar{J} \) are the jet functions which describe the collinear jets in the \( n \)- and \( \bar{n} \)-directions respectively. Finally, \( S \) is the soft function, which describes the soft modes, allowing interactions between the jets. Each of these functions will be discussed in more detail in the next subsections. To be more precise, the factorization theorem tells us that the thrust cross section factorizes in the following way \[15, 16\]:

\[
\frac{1}{\sigma_{i,0}} \frac{d\sigma_i}{d\tau} = |C_i(Q, \mu)|^2 \int ds_1 J_i(s_1, \mu) \int ds_2 \bar{J}_i(s_2, \mu) \int dk S_i(k, \mu) \frac{1}{\delta(\tau - s_1 + s_2 - \frac{k}{Q})} - \frac{1}{Q^2} - \frac{k}{Q} \tag{3.1}
\]

where \( \sigma_{i,0} \) denotes the Born cross section, \( C_i \) are the Wilson coefficients with \( Q \) the hard scale and \( i = q, g \) corresponding to the quark jets and gluon jets processes respectively, \( J_i, \bar{J}_i \) the jet functions which are functions of \( s_1, s_2 \) the invariant masses of the collinear radiation in the jets, and \( S_i \) the soft function which depends on the soft momentum \( k \) of the soft radiation.

We call the first term in Eq. (3.1), which is factorized, the singular part of the cross section. Singular terms scaling like \( 1/\tau \) are contained in this part. The remainder is called the nonsingular cross section. As the name suggests, this part contains the nonsingular terms (the nonsingular part of the cross section actually does contain singularities, but they are integrable singularities and less singular than the ones in the singular part of the cross section). The nonsingular cross section is suppressed by \( O(\tau) \) and not as important as the singular part at low values of the thrust \( \tau \). At higher values of \( \tau \) these power suppressed terms do become important, but for now we will focus on the singular part of the thrust cross section.
Chapter 3. Singular cross section

The main advantage of the factorization theorem is that the hard, jet and soft functions all depend on their own scale, which are widely separated. Instead of one multiscale function we now have multiple single scale functions. For each of these functions, we thus have a single scale where we can calculate it. There are also other observables and processes for which a factorization theorem holds, and there could be some universality for these several different functions, meaning that the same function could be used for different processes.

When we do a perturbative expansion of the thrust cross section, we encounter logarithms of \( \tau \), which become large for \( \tau \ll 1 \). These large logarithms spoil the convergence of the perturbative expansion and make our thrust cross section diverge. Luckily, as we will see later, we can use the renormalization group equations to resum these large logarithms and solve the divergence at \( \tau \ll 1 \) problem. The cross section for thrust will then not diverge in the \( \tau \to 0 \) limit anymore and instead go to 0 (Fig. 3.1).

We first look at the perturbative parts of the factorized singular cross section.

3.2. Hard function

The hard function is defined as the absolute square of the Wilson coefficient. As SCET is an effective field theory, we naturally have Wilson coefficients \( C \) encoding the high energy theory. We will be needing the Wilson coefficients \( C_q \) and \( C_g \) for our quark and gluon jets processes. These are obtained by matching calculations done in QCD to SCET for our two processes. This is shown for the quarks up to one-loop order in Fig. 3.2 and Fig. 3.3.
Figure 3.2.: Matching from QCD to SCET for the quarks at tree level. The dashed arrowed line represents a (anti-)collinear quark and the circle with the cross denotes $\bar{\chi}_n \Gamma \chi_n$.

Figure 3.3.: Matching from QCD to SCET for the quarks at one-loop level. The dashed arrowed line represents a (anti-)collinear quark. The last two SCET diagrams have a Wilson line attached to the (anti-)collinear quark.

To one-loop order this gives for the quark Wilson coefficient [17]:

$$C_q(Q, \mu) = 1 + \frac{\alpha_s(\mu)C_F}{4\pi} \left[-\log^2 \left(-\frac{Q^2 - i0}{\mu^2}\right) + 3\log \left(-\frac{Q^2 - i0}{\mu^2}\right) - 8 + \frac{\pi^2}{6}\right], \quad (3.2)$$

from which the quark hard function is then easily obtained:

$$H_q(Q, \mu) = |C(Q, \mu)|^2 = 1 + \frac{\alpha_sC_F}{2\pi} \left(-4\log^2 \frac{Q}{\mu} + 6\log \frac{Q}{\mu} - 8 + \frac{7\pi^2}{6}\right). \quad (3.3)$$

The matching for the gluons can be done immediately in one step from QCD to SCET, or in two steps: first integrating out the topquark from QCD and then from QCD with five flavours to SCET (Fig. 3.4). The Wilson coefficient from the one-step matching is then the product of the two Wilson coefficients from the two-step matchings [18]. The gluon hard function up to one-loop is given by [18]

$$C_g(Q, \mu) = \alpha_s \left\{ 1 + \frac{\alpha_s}{4\pi} \left[-C_A \log^2 \left(-\frac{Q^2 - i0}{\mu^2}\right) + \left(5 - \frac{\pi^2}{6}\right)C_A - 3C_F\right] \right\}. \quad (3.4)$$

It has an overall extra factor of $\alpha_s$ relative to the quarks Wilson coefficient, since $H \rightarrow gg$.
starts at order $\alpha_s^2$. The gluon hard function is then given by

$$H_g(Q, \mu) = |C_g(Q, \mu)|^2 = \alpha_s^2 \left\{ 1 + \frac{\alpha_s}{2\pi} \left[ -4C_A \log^2 \left( \frac{Q}{\mu} \right) + \left( 5 + \frac{7\pi^2}{6} \right) C_A - 3C_F \right] \right\}.$$  

Besides the extra $\alpha_s$, it has the same form as the quark hard function. Symbolically they both look like: $H \sim 1 + \alpha_s [c_2 L^2 + c_1 L + c_0]$, where $L = \log(Q/\mu)$ and the $c_i$ are (different) numbers multiplying the different power of logarithms. This trend continues in the perturbative expansion at higher orders, with two extra higher powers logarithms coming at each order. This is also the case for the jet and soft functions (although there we do not have simple logarithms but plus distributions instead).

### 3.3. Jet function

In this section we will calculate the jet function for the quarks to one-loop order. We use the optical theorem to relate the jet function to the imaginary part of a forward scattering amplitude. The jet function can be defined as the vacuum matrix element of a two-point collinear function \footnote{[12]}

$$J(s) = \text{Im}[\mathcal{J}(s)], \quad \mathcal{J}(s) = \frac{-i}{\pi \omega} \int d^4x \, e^{ik \cdot x} \langle 0 | T \bar{n}_{\omega,0} n (0) \frac{\gamma^\mu}{4N_c} \chi_n(x) | 0 \rangle,$$

where $s = p^2 = k^+ \omega$ and $p^\mu = k^+ \bar{n}^\mu / 2 + \omega n^\mu / 2$. At tree level we basically just have a simple collinear quark propagator. The spin and colour indices are contracted giving the following traces

$$\mathcal{J}^{\text{tree}} = \frac{-i}{\pi \omega} (-1) \frac{i\omega}{\omega k^+ + i0} \text{Tr} \left[ \frac{\gamma^\mu}{2} \frac{\gamma^\mu}{4} \right] \text{Tr} \left[ \frac{1}{N_c} \right] = \frac{-1}{\pi} \frac{1}{\omega k^+ + i0} \left( \frac{4n \cdot \bar{n}}{8} \right) \left( \frac{N_c}{N_c} \right) = \frac{-1}{\pi(s + i0)}$$

(3.7)
Chapter 3. Singular cross section

Taking the imaginary part of this now gives the tree level jet function, which is just a delta function:

\[ J^{\text{tree}}(s) = \Im \left[ \frac{-1}{\pi(s + i0)} \right] = \delta(s). \]  

(3.8)

At one-loop the following diagrams besides the tree level diagram in Fig. 3.5 have to be calculated. Using the SCET Feynman rules (in Feynman gauge) gives for diagram a:

![Diagram](image)

\[ \int \frac{d^4k}{(2\pi)^4} \left( igT^A \left[ n_\mu + \frac{\gamma_\mu (\not{n} + \not{k}_\perp)}{\not{n} \cdot (p + k)} + \frac{\gamma_\mu \not{p}_\perp}{\not{n} \cdot p} \right] - \frac{\not{p}_\perp (\not{n} + \not{k}_\perp)}{\not{n} \cdot (p + k) \not{n} \cdot p} \right) \frac{\not{n}}{2} \left( \frac{1}{2} \frac{\not{n} \cdot (p + k)}{2(p + k)^2 + i0} \right) (\not{g}T^A \frac{\not{n}_\nu}{\not{n} \cdot k - i0}) \left( \frac{1}{2} \frac{\not{n} \cdot p}{p^2 + i0} \right) \right) \]

\[ = g^2 T^A T^A \frac{\not{n} \cdot (p + k)}{2} \frac{\not{n}}{2} \frac{\not{n} \cdot p}{2p^2 + i0} \int \frac{d^4k}{(2\pi)^4 (k^2 + i0)((p + k)^2 + i0)(\not{n} \cdot k - i0)} \]

(3.9)

We will forget about the prefactors for the moment and calculate the integral

\[ I = \int \frac{d^4k}{(2\pi)^4} \frac{(n \cdot \bar{n})(\bar{n} \cdot (p + k))}{(k^2 + i0)((p + k)^2 + i0)(\bar{n} \cdot k - i0)} \].

(3.10)

We use the Georgi parameter trick

\[ a^{-1}b^{-1}c^{-1} = 2 \int_0^\infty dx \int_0^\infty dy (c + bx + ay)^{-3}, \]

(3.11)

and choose \( a = k^2 + i0, b = (p + k)^2 + i0 \) and \( c = \bar{n} \cdot k - i0 \). We then get for our integral:

\[ I = 2 \int \frac{d^4k}{(2\pi)^4} \bar{n} \cdot (p + k) \int_0^\infty dx \int_0^\infty dy \frac{2}{[k^2 + i0 + ((p + k)^2 + i0)x + (\bar{n} \cdot k - i0)y]^3}, \]

(3.12)
which after some arranging gives
\[ I = 2 \int \frac{d^4k}{(2\pi)^4} \bar{n} \cdot (p + k) \int_0^\infty dx \int_0^\infty dy \frac{2}{[(1 + x)(k^2 + 2k \cdot q - \Delta)]^2}, \] (3.13)
where we defined
\[ q^\mu = \frac{xp^\mu + y\bar{n}^\mu}{1 + x}, \quad \Delta = -xp^2 + i0 \] (3.14)
We now have our integral \( I \) in the following nice form
\[ 4\tilde{\mu}^2 \int_0^\infty dx \int_0^\infty dy \frac{1}{(1 + x)^3} \int \frac{d^4k}{(2\pi)^d} \frac{\bar{n} \cdot p + \bar{n} \cdot k}{(k^2 + 2k \cdot q - \Delta)^3}, \] (3.15)
where we have gone to \( d = 4 - 2\epsilon \) dimensions and \( \tilde{\mu}^2 = \mu^2 e^{\gamma_E} \) to do this integral with dimensional regularization in the \( \overline{\text{MS}} \)-scheme. We use the master integral
\[ \int \frac{d^4k}{(2\pi)^d} \frac{1}{(k^2 + 2k \cdot q - \Delta)^\alpha} = \frac{(-1)^\alpha}{(2\pi)^d} \frac{i\pi^{d/2}}{(\Delta + q^2)^{-\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \] (3.16)
for the first term in \( I \), and an other master integral
\[ \int \frac{d^4k}{(2\pi)^d} \frac{k^\mu}{(k^2 + 2k \cdot q - \Delta)^\alpha} = \frac{(-1)^{\alpha-1}}{(2\pi)^d} \frac{i\pi^{d/2}}{(\Delta + q^2)^{-\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} q^\mu, \] (3.17)
which is obtained from the first master integral by taking a derivative with respect to \( q^\mu \), for the second term in \( I \). This gives us:
\[ 4\tilde{\mu}^2 \int_0^\infty dx \int_0^\infty dy \frac{1}{(1 + x)^3} \left[ \frac{\bar{n} \cdot p}{(2\pi)^{4-2\epsilon}} \frac{-i\pi^{2-\epsilon}}{(\Delta + q^2)^{1+\epsilon}} \frac{\Gamma(1 + \epsilon)}{2} + \frac{\bar{n} \cdot q}{(2\pi)^{4-2\epsilon}} \frac{i\pi^{2-\epsilon}}{(\Delta + q^2)^{1+\epsilon}} \frac{\Gamma(1 + \epsilon)}{2} \right]. \] (3.18)
We will write \( I = I_1 + I_2 \), where \( I_1 \) is the first term and \( I_2 \) the second term in \( I \). We will calculate \( I_1 \) first:
\[ I_1 = 4\tilde{\mu}^2 \frac{\bar{n} \cdot p}{(2\pi)^{4-2\epsilon}} (-i\pi^{2-\epsilon}) \frac{\Gamma(1 + \epsilon)}{2} \int_0^\infty dx \frac{1}{(1 + x)^3} \int_0^\infty dy \left[ -p^2 x + 2(\bar{n} \cdot p)xy + i0 \right]^{-1-\epsilon} \] (3.19)
\[ = 4\tilde{\mu}^2 \frac{\bar{n} \cdot p}{(2\pi)^{4-2\epsilon}} (-i\pi^{2-\epsilon}) \frac{\Gamma(1 + \epsilon)}{2} \int_0^\infty dx \frac{x^{-1-\epsilon}}{(1 + x)^{1-2\epsilon}} \int_0^\infty dy \left[ -p^2 + 2(\bar{n} \cdot p)y + i0 \right]^{-1-\epsilon}. \]
The \( x \) and \( y \) integrals are now factorized. The \( x \)-integral gives
\[ \int_0^\infty dx \frac{x^{-1-\epsilon}}{(1 + x)^{1-2\epsilon}} = \frac{\Gamma(1 - \epsilon)\Gamma(-\epsilon)}{\Gamma(1 - 2\epsilon)}, \] (3.20)
and the \( y \) integral
\[ \int_0^\infty dy \left[ -p^2 + 2(\bar{n} \cdot p)y + i0 \right]^{-1-\epsilon} = \frac{-p^2}{2\bar{n} \cdot p \epsilon}, \] (3.21)
We then obtain for $I_1$
\[
I_1 = \left( \frac{\hat{\mu}^2}{-p^2 - i0} \right) \epsilon^{(-i\pi^2 - \epsilon)} \frac{\Gamma(1 + \epsilon)\Gamma(1 - \epsilon)}{\epsilon \Gamma(1 - 2\epsilon)} \tag{3.22}
\]
\[
= \frac{i}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \left( \frac{\mu^2}{-p^2 - i0} \right) + \frac{1}{2} \log^2 \left( \frac{\mu^2}{-p^2 - i0} \right) - \frac{\pi^2}{12} \right].
\]
Inspecting the second part $I_2$, we see that it is actually the same as $I_1$ with the replacement
\[
\bar{n} \cdot p \rightarrow \bar{n} \cdot q = \bar{n} \cdot \frac{px + \bar{y}y}{1 + x} = \bar{n} \cdot p x.
\tag{3.23}
\]
and an additional minus sign. This results in a different $x$-integral giving
\[
\int_0^\infty dx \frac{x^{-1 - \epsilon}}{(1 + x)^{2-2\epsilon}} \left( \frac{x}{1 + x} \right) = \frac{\Gamma(1 - \epsilon)^2}{\Gamma(2 - 2\epsilon)}.
\tag{3.24}
\]
We then find for $I_2$:
\[
I_2 = \left( \frac{\hat{\mu}^2}{-p^2 - i0} \right) \epsilon^{(-i\pi^2 - \epsilon)} \frac{\Gamma(1 + \epsilon)\Gamma(1 - \epsilon)}{\epsilon \Gamma(2 - 2\epsilon)} = \frac{i}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \log \left( \frac{\mu^2}{-p^2 - i0} \right) + 2 \right].
\tag{3.25}
\]
Adding $I_1$ and $I_2$ gives us the final result for our integral $I$:
\[
I = \frac{i}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \left( \frac{\mu^2}{-p^2 - i0} \right) + \frac{1}{2} \log^2 \left( \frac{\mu^2}{-p^2 - i0} \right) + \log \left( \frac{\mu^2}{-p^2 - i0} \right) + 2 - \frac{\pi^2}{12} \right].
\tag{3.26}
\]
Putting the prefactors back in front and doing the traces we end up with the following expression for diagram a:
\[
\text{Tr} \left[ \frac{\hat{\mu}^2}{2 \pi^2} \right] \frac{1}{N_c} \text{Tr}[T^A T^A] g^2 \left( \frac{-i}{\pi\omega} \right) (-1) \frac{\bar{n} \cdot p}{p^2 + i0} \times I = \frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log \left( \frac{-s - i0}{\mu^2} \right) + \log^2 \left( \frac{-s - i0}{\mu^2} \right) - 2 \log \left( \frac{-s - i0}{\mu^2} \right) + 4 - \frac{\pi^2}{6} \right].
\tag{3.27}
\]
We now move on to the other diagrams in Fig. 3.5. Diagram b gives the same as diagram a. Diagram c is 0, because it is proportional to $\bar{n} \mu \bar{n} \sigma^\mu = 0$ from the contraction of the two Wilson line vertices by the gluon propagator. For diagram d we get the following expression after using the SCET Feynman rules:
\[
\left( \frac{n \cdot \bar{n}}{2p^2} \right)^2 (-1) \frac{1}{N_c} T^A T^A g^2 \left( \frac{-i}{\pi\omega} \right) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \cdot \frac{\bar{n} \cdot (p + k)}{(p + k)^2} \cdot \frac{(\bar{\phi}_\perp + \bar{\kappa}_\perp)}{\bar{n} \cdot (p + k)} \bar{n} \cdot p \bar{n} \cdot (p + k) \tag{3.28}
\]
\[
\left[ \frac{(n \cdot \bar{n})}{\bar{n} \cdot (p + k)} \bar{n} \cdot p \bar{n} \cdot (p + k) - \frac{(\phi_\perp + \kappa_\perp)}{\bar{n} \cdot (p + k)} \bar{n} \cdot (p + k) \right] \cdot \frac{\phi_\perp}{\bar{n} \cdot (p + k)} \frac{(\phi_\perp + \kappa_\perp)}{\bar{n} \cdot (p + k)}
\]
Diagram d eventually evaluates to
\[
\frac{1}{\pi(-s-i0)} \frac{\alpha_s C_F}{4\pi} \left[ -\frac{1}{\epsilon} + \log\left( \frac{-s-i0}{\mu^2} \right) - 1 \right].
\] (3.29)

Lastly, diagram e is 0, because it gives a scaleless integral which is zero in dimensional regularization. Summing all the one-loop jet diagrams we find
\[
J^{(1)}_{\text{bare}}(s) = \frac{1}{\pi(-s-i0)} \frac{\alpha_s C_F}{4\pi} \left[ \frac{4}{\epsilon^2} + \frac{3}{\epsilon} - \frac{4}{\epsilon} \log\left( \frac{-s-i0}{\mu^2} \right) + 2 \log^2\left( \frac{-s-i0}{\mu^2} \right) - 3 \log\left( \frac{-s-i0}{\mu^2} \right) + 7 - \frac{\pi^2}{3} \right]
\] (3.30)

where we used the identities in Eq. (B.2) and have dropped the i0’s to end up with the plus distributions \( L_n \), defined in App. B. Now we renormalize the jet function by adding a counterterm which cancels all the \( 1/\epsilon \) poles:
\[
J^{(1)}_{\text{ren}}(s) = J^{(1)}_{\text{bare}}(s) + Z^{(1)}(s),
\] (3.32)

where the counterterm is given by
\[
Z^{(1)}(s) = -\frac{\alpha_s C_F}{4\pi} \left[ \delta(s) \left( \frac{4}{\epsilon^2} + \frac{3}{\epsilon} - \frac{4}{\epsilon} \log\left( \frac{s}{\mu^2} \right) - 3 - 2 \frac{\pi^2}{3} \delta(s) + 4 \frac{1}{\mu^2} L_1\left( \frac{s}{\mu^2} \right) \right) \right].
\] (3.33)

We are then finally left with the renormalized one-loop jet function:
\[
J^{(1)}_{\text{ren}}(s) = \frac{\alpha_s C_F}{4\pi} \left[ (7 - \pi^2)\delta(s) - 3 \frac{1}{\mu^2} L_0\left( \frac{s}{\mu^2} \right) + 4 \frac{1}{\mu^2} L_1\left( \frac{s}{\mu^2} \right) \right],
\] (3.34)

which agrees with [20].

### 3.4. Soft function

Finally, we look at the soft function. We will be needing the thrust soft function \( S(l, \mu) \), which can be obtained from the hemisphere soft function \( S_{\text{hemi}}(l^+, l^-, \mu) \). At one-loop the diagrams in Fig. 3.6 have to be calculated for the hemisphere soft function. We first look at diagram a. Using the Feynman rules we get the following expression for \( S_a \):
\[
S_a = \int \frac{d^4q}{(2\pi)^4} g_\mu \frac{\bar{n}^\mu}{q^- + i0} T^{a}_{\mu} - i\delta_{ab} q^\mu g_\nu g_\sigma \frac{n'^\nu}{q^+ - i0} T^{b}_{\mu} \delta(l^+) \delta(l^-)
\] (3.35)

\[
= -2ig_\sigma^2 C_F \int \frac{d^4q}{(2\pi)^4} (q^- + i0)(q^+ - i0)(q^2 + i0)
\]

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The deltafunctions $\delta(l^+)\delta(l^-)$ arise, because there is no gluon final state in this diagram, so $l^+, l^-$ are set to 0. This integral is scaleless and will vanish in dimensional regularization. The same happens for diagram b. Next are the real diagrams c and d:

$$2g^2 C_F \hat{\mu}^{2\epsilon} \int \frac{d^d q}{(2\pi)^d (2\pi)^2} \delta(q^2) \theta(q^+ - q^-) \delta(l^- - q^-) \delta(l^+) + \theta(q^- - q^+) \delta(l^+ - q^+) \delta(l^-) \theta(q^+) \theta(q^-)$$

We can calculate this diagram by rewriting the measure in its light-cone coordinate parts:

$$d^d q = \frac{1}{2} dq^+ dq^- d^{d-2} \vec{q}_\perp,$$

and then doing the integral over $\vec{q}_\perp$ first, because there is no dependence on $\vec{q}_\perp$ in the integrand, except in $\delta(q^2) = \delta(q^+ q^- - \vec{q}_\perp^2)$:

$$\int d^{d-2} \vec{q}_\perp \delta(q^+ q^- - \vec{q}_\perp^2) = \int dr \theta(r) r^{d-3} A_{d-3} \delta(q^+ q^- - r^2),$$

where we have gone to spherical coordinates $r^2 = \vec{q}_\perp^2$ and $A_n$ is the surface area of the $n$-sphere with radius 1. Then by using the delta-function and

$$A_n = 2\pi V_n = 2\pi \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

we arrive at:

$$\int d^{d-2} \vec{q}_\perp \delta(q^+ q^- - \vec{q}_\perp^2) = 2\pi \frac{\pi^{-\epsilon}}{\Gamma(1 - \epsilon)} \frac{(q^+ q^-)^{-\epsilon}}{2} \theta(q^+) \theta(q^-) = \frac{\pi^{1-\epsilon}}{\Gamma(1 - \epsilon)} (q^+ q^-)^{-\epsilon} \theta(q^+) \theta(q^-)$$
Inserting this result in Eq. (3.36) and doing the remaining integrals over $q^+$ and $q^-$ then gives:

$$g^2 C_F \hat{\mu}^{2\epsilon} 2^{-3+2\epsilon} \pi^{-2+\epsilon} \frac{\delta(l^+)\theta(l^-)}{(l^-)^{1+2\epsilon}} + \frac{\delta(l^-)\theta(l^+)}{(l^+)^{1+2\epsilon}}. \tag{3.41}$$

Diagrams e, f and g, are all individually 0. This can easily be seen by looking at the attachment of a gluon from a $n$- to $n$-direction (or $\bar{n}$- to $\bar{n}$-direction) Wilson line, resulting in $n \cdot n = 0$ (or $\bar{n} \cdot \bar{n} = 0$).

Summing all the diagrams and switching from the bare coupling to the renormalized coupling, we then arrive at:

$$\frac{\alpha_s(\mu) C_F}{\pi} \frac{\mu^{2\epsilon} e^{\epsilon\gamma}}{\epsilon \Gamma(1-\epsilon)} \left( \frac{\delta(l^-)\theta(l^+)}{(l^+)^{1+2\epsilon}} + \frac{\delta(l^+)\theta(l^-)}{(l^-)^{1+2\epsilon}} \right). \tag{3.42}$$

The one-loop bare hemisphere soft function is thus:

$$S^{\text{bare}}(t^+, l^-) = \delta(t^+)\delta(l^-) + \frac{\alpha_s(\mu) C_F}{\pi} \frac{e^{\epsilon\gamma}}{\epsilon \Gamma(1-\epsilon)} \left( \frac{\mu}{\xi} \right)^{2\epsilon} \left( \frac{\delta(l^-)\theta(l^+)}{\xi} \left( \frac{\xi}{l^+} \right)^{1+2\epsilon} + \frac{\delta(l^+)\theta(l^-)}{\xi} \left( \frac{\xi}{l^-} \right)^{1+2\epsilon} \right), \tag{3.43}$$

where $\xi$, a dummy variable with mass dimension 1, is introduced so that we can use the following relation for dimensionless variables:

$$\frac{\theta(x)}{x^{1+2\epsilon}} = -\frac{\delta(x)}{2\epsilon} + \left[ \frac{\theta(x)}{x} \right]_+ - 2\epsilon \left[ \frac{\theta(x) \log x}{x} \right]_+ + O(\epsilon^2). \tag{3.44}$$

Using this relation and expanding in $\epsilon$ results in:

$$S^{\text{bare}}(t^+, l^-) = \delta(t^+)\delta(l^-) + \frac{\alpha_s(\mu) C_F}{\pi} \left[ S^{\text{bare}}_{\text{div}}(t^+, l^-) + S^{\text{bare}}_{\text{fin}}(t^+, l^-) \right], \tag{3.45}$$

where $S^{\text{bare}}_{\text{div}}(t^+, l^-)$ contains the divergent terms:

$$S^{\text{bare}}_{\text{div}}(t^+, l^-) = -\frac{\delta(t^+)\delta(l^-)}{\epsilon^2} + \frac{\delta(l^+)}{\epsilon \xi} \left[ \frac{\xi \theta(l^-)}{l^-} \right]_+ + \frac{\delta(l^-)}{\epsilon \xi} \left[ \frac{\xi \theta(l^+)}{l^+} \right]_+ - \frac{\delta(l^+)\delta(l^-)}{\epsilon} \log \frac{\mu^2}{\xi^2}, \tag{3.46}$$

and $S^{\text{bare}}_{\text{fin}}(t^+, l^-)$ the finite terms:

$$S^{\text{bare}}_{\text{fin}}(t^+, l^-) = \frac{\delta(t^+)\delta(l^-)}{\xi} \left( \frac{\pi^2}{12} - \log^2 \frac{\mu^2}{\xi^2} \right) + \frac{\delta(l^+)}{\xi} \log \frac{\mu^2}{\xi^2} \left[ \frac{\xi \theta(l^-)}{l^-} \right]_+ - \frac{\delta(l^-)}{\xi} \left[ \frac{\theta(l^-) \log (l^-/\xi)}{l^-/\xi} \right]_+ - \frac{2\delta(l^-)}{\xi} \left[ \frac{\theta(l^-) \log (l^-/\xi)}{l^-/\xi} \right]_+$$

$$+ \frac{\delta(l^-)}{\xi} \log \frac{\mu^2}{\xi^2} \left[ \frac{\xi \theta(l^+)}{l^+} \right]_+ \tag{3.47}$$
Renormalizing the hemisphere soft function then gives the renormalized hemisphere soft function:

\[ S_{\text{ren}}^{\text{bare}}(l^+, l^-, \mu) = \delta(l^+) \delta(l^-) + \frac{\alpha_s(\mu) C_F}{\pi} S_{\text{lim}}^{\text{bare}}(l^+, l^-). \] (3.48)

Since thrust is symmetrical under the exchange of the two hemispheres, we might actually be able to factorize the soft functions in two independent parts. At the level of one-loop, there is only one single gluon in the real radiation graph, which can only go in one of the two hemispheres. So at order \( \alpha_s \) we can write:

\[ S_{\text{ren}}(l^+, l^-, \mu) = S_{\text{ren}}(l^+, \mu) S_{\text{ren}}(l^-, \mu). \] (3.49)

where the soft functions depending only on one of the two variables \( l^\pm \) are given by:

\[
S_{\text{ren}}(l^\pm, \mu) = \frac{\alpha_s(\mu)}{\pi} C_F \left[ \delta(l^\pm) \left( \frac{\pi^2}{24} - \frac{1}{4} \log^2 \frac{\mu^2}{\xi^2} \right) + \frac{1}{\xi} \log \frac{\mu^2}{\xi^2} \left[ \xi \theta(l^\pm) \frac{l^\pm}{\log (l^\pm/\xi)} \right] - \frac{2}{\xi} \left[ \theta(l^\pm) \log (l^\pm/\xi) \right] \right].
\] (3.50)

At order \( \alpha_s^2 \) there are graphs with more than one real parton, which means that this factorized form of the soft function does not hold. For the calculation of thrust we use the thrust soft function, which can be easily obtained from the hemisphere soft function by:

\[ S(l, \mu) = \int \frac{dl^+ dl^-}{2} S_{\text{ren}}^{\text{bare}}(l - l^+, l^-). \] (3.51)

Using the definitions of the plus distributions \( L_n(x) \) in Eq. (B.1), the thrust soft function to one-loop order finally reads:

\[
S(l, \mu) = \delta(l) + \frac{\alpha_s(\mu) C_F}{\pi} \left[ \left( \frac{\pi^2}{12} - 2 \log^2 \frac{\xi}{\mu} \right) \delta(l) - 4 \log \left( \frac{\xi}{\mu} \right) \frac{1}{\xi} L_0 \left( \frac{l}{\xi} \right) - 4 \frac{1}{\xi} L_1 \left( \frac{l}{\xi} \right) \right]
\] (3.52)

This is the one-loop soft function for the quark case. The one-loop soft function for the gluons will be the same, except for the replacement of \( C_F \rightarrow C_A \). This comes from the fact that the gluons are in the adjoint representation instead of the fundamental representation of the quarks, so that instead of \( T^a \) we get \( F^a \) from the gluon off the Wilson line emission vertex, and hence \( F^a F^a = C_A \) (see App. E).

### 3.5. Renormalization Group Equations

As we have seen in the previous sections, the hard, jet and soft functions all contained logarithms which grow large in the \( \tau \rightarrow 0 \) limit. In this section we show how we use the Renormalization Group Equations to resum the logarithms to all orders in \( \alpha_s \).
Each of these functions depends only (besides the renormalization scale) on their own scale. What we want to do is to evaluate each function at its own scale

\[ \mu_H \sim Q, \quad \mu_J \sim s \sim Q\sqrt{\tau}, \quad \mu_S \sim k \sim Q\tau, \quad (3.53) \]

so that the large logarithms in the functions are no longer large. Then we use the Renormalization Group Equations to evolve each function to a common scale \( \mu \). This evolution will be done on each function by its evolution kernel \( U_F \). The RGE improved version of the cross section Eq. (3.1) then reads:

\[
\frac{1}{\sigma_{i,0}} \frac{d\sigma_i}{d\tau} = H_i(Q, \mu_H) U_{H}(\mu, \mu_H) \int ds_1 ds'_1 J_i(s'_1, \mu_J) U_{J_i}(s_1 - s'_1, \mu, \mu_J) \times \int ds_2 ds'_2 J_i(s'_2, \mu_J) U_{J_i}(s_2 - s'_2, \mu, \mu_J) \int dk dk' S_i(k', \mu_S) U_{S_i}(k - k', \mu, \mu_S) \times \delta \left( \tau - s_1 + s_2 \frac{k}{Q^2} \right) + \frac{d\sigma^\text{ons}}{d\tau},
\]

where now each function is replaced with a product/convolution of the function evaluated at their own scale and their evolution factor evolving it from their own scale to a common scale \( \mu \). We use the convention that the preceding scale is the ending scale and the succeeding scale is the initial scale for the evolution factors.

We first look at the RGE for the hard function (or almost equivalently, the Wilson coefficient).

### 3.5.1. RGE hard function

We demand the usual renormalization scale independence equation; the bare coefficient should not depend on the renormalization scale:

\[
0 = \mu \frac{d}{d\mu} C_{\text{bare}} = \mu \frac{d}{d\mu} [Z_C(\mu)C(\mu)] = C \mu \frac{d}{d\mu} Z_C + Z_C \mu \frac{d}{d\mu} C \quad (3.55)
\]

from which we define the anomalous dimension of the Wilson coefficient:

\[
\mu \frac{d}{d\mu} C = - \frac{1}{Z_C} C \mu \frac{dZ_C}{d\mu} \equiv \gamma_C C. \quad (3.56)
\]

For the quarks this anomalous dimension to order \( \alpha_s \) is:

\[
\gamma_C = \frac{\alpha_s}{4\pi} \left[ 4C_F \log \left( \frac{-Q^2 - i0}{\mu^2} \right) - 6C_F \right]. \quad (3.57)
\]

In this process the anomalous dimension can always be written in the following form [12]:

\[
\gamma_C(\mu, \omega) = \Gamma_{\text{cusp}}[\alpha_s(\mu)] \log \left( \frac{-Q^2 - i0}{\mu} \right) + \gamma_C[\alpha_s(\mu)], \quad (3.58)
\]
where the anomalous dimension is now separated in a cusp anomalous dimension $\Gamma_{\text{cusp}}$ and non-cusp anomalous dimension $\gamma_C$. This formula holds to all orders in $\alpha_s$ ($\Gamma_{\text{cusp}}$ and $\gamma_C$ have expansions in $\alpha_s$), which allows us to solve the RGE order by order. The general solution of this type of RGE is found in App. C. The solution for the hard function then reads:

$$H(Q, \mu) = U_H(\mu, \mu_0)H(Q, \mu_0),$$

(3.59)

where $U_H(\mu, \mu_0)$ is the evolution kernel for the hard function running it from an initial scale $\mu_0$ to $\mu$ and is formally given by:

$$U_H(Q, \mu, \mu_0) = \left|e^{-2K_T(\mu, \mu_0) + K_\gamma(\mu, \mu_0)} \frac{-Q^2 - i0}{\mu_0^2} \right|^2.$$  

(3.60)

Using the anomalous dimensions for the quark Wilson coefficient given in App. A and the results from App. C, we then find for the quark hard function at LL order:

$$K_T(\mu, \mu_0) = -\frac{2C_F}{\beta_0^2} \frac{4\pi}{\alpha_s(\mu_0)} \left(1 - \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} - \log \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)$$

(3.61)

$$K_\gamma(\mu, \mu_0) = 0$$

$$\omega_H(\mu, \mu_0) = \frac{4C_F}{\beta_0} \log \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)},$$

which gives us the following expression for the quark hard function evolution kernel:

$$U_H(Q, \mu, \mu_0) = \operatorname{Exp} \left[\frac{4C_F}{\beta_0^2} \frac{4\pi}{\alpha_s(\mu_0)} \left(1 - \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} - \log \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right) \left(\frac{\mu_0^2}{Q^2}\right)\right].$$

(3.62)

### 3.5.2. RGE jet and soft function

For the jet functions and soft function the RGE’s are similar as the one for the hard function, but more complicated, since the jet and soft function are no longer simple functions but distributions. Instead of a multiplicative RGE we now have one with a convolution [12]:

$$\frac{d}{d\mu} F(t, \mu) = \int dt' \gamma_F(t - t') F(t', \mu),$$

(3.63)

where $F = J, \bar{J}, S$. There are several methods to solve an equation like this. It can be solved by going to Fourier space or Laplace space, where the RGE turns into a multiplicative RGE like the one of the hard function. In App. C the general method to solve the RGE for the jet and soft function by going to Laplace space is shown. This results in the solution

$$F(t, \mu) = \int dt' F(t - t', \mu_F) U_F(t', \mu, \mu_F),$$

(3.64)
where \( U_F(t', \mu, \mu_F) \) is the evolution kernel evolving the jet or soft function from \( \mu_F \) to \( \mu \). The evolution kernel of the jet function looks like \([12]\):

\[
U_J(s, \mu, \mu_0) = e^{\gamma_E \omega(\mu, \mu_0)} e^{2 K_T(\mu, \mu_0) + K_s(\mu, \mu_0)} \frac{1}{\Gamma(-\omega(\mu, \mu_0))} \left[ \frac{1}{\mu_0^2} \mathcal{L}^{-\omega(\mu, \mu_0)} \left( \frac{s}{\mu_0^2} \right) - \frac{1}{\omega(\mu, \mu_0)} \delta(s) \right],
\]

(3.65)

where \( \mathcal{L}^a \) is the plus distribution defined in Eq. (B.4). The evolution kernel of the soft function looks very similar to the jet’s one, as the structure of their anomalous dimension is the same.

### 3.6. Implementation

Having seen the factorization of the singular thrust cross section, the hard function, jet functions and soft function, their RGE’s and the resummation of the logarithms, we now look into the implementation of all the calculations to produce the thrust distribution.

We calculate the thrust distribution from LL to NNLL’ order. The necessary ingredients for this at each order are displayed in Tab. 3.1. At the highest order we considered, NNLL’, we need for both quarks and gluons the hard function \([17, 18]\), jet function \([20, 21]\) and soft function \([22, 23]\) to two-loops, the three-loop cusp anomalous dimensions \([24]\), two-loop non-cusp anomalous dimensions \([25]\) and three-loop QCD \(\beta\)-function \([26]\). These ingredients have been calculated before and can be found in the literature. Their expressions are collected in App. A. Using all the ingredients and keeping terms up to certain order in \(\alpha_s\) we calculate the thrust spectrum for the quarks and gluons. We use the following method for this: at LL and NLL only the 0-loop order hard, jet and soft functions are used. These are just 1 or delta-functions. The thrust spectrum is then basically just the evolution kernels at their appropriate order:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = U_H(\mu, \mu_h) \cdot U_J(\mu, \mu_j) \otimes U_J(\mu, \mu_j) \otimes U_S(\mu, \mu_s).
\]

(3.66)

At NLL’ and NNLL order, the one-loop fixed order ingredients are used. These are multiplied with each other and only terms up to order \(\alpha_s\) are then kept. Schematically (suppressing the evolution kernels for readability):

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} \sim H^{(1)} J^{(0)} \bar{J}^{(0)} S^{(0)} + H^{(0)} J^{(1)} \bar{J}^{(0)} S^{(0)} + H^{(0)} J^{(0)} \bar{J}^{(1)} S^{(0)} + H^{(0)} J^{(0)} \bar{J}^{(0)} S^{(1)},
\]

(3.67)
where $F^{(n)}$ means the function $F$ up to $n$-th order. The same method goes for NNLL$'$ with terms being kept one order higher in $\alpha_s$.

Since the evolution factors obey $U_F(\tau, \mu, \mu) = \delta(\tau)$, we can make the calculation easier by choosing the end scale $\mu$ as $\mu_J$ or $\mu_S$ to effectively remove one convolution. Here we have chosen $\mu = \mu_S$. We can do another simplification by noting that

\[
\mu \frac{d}{d\mu} J \otimes \bar{J} = J \mu \frac{d}{d\mu} \bar{J} + \bar{J} \mu \frac{d}{d\mu} J = J \bar{\gamma} + \bar{J} \gamma.
\]

Since the jet functions $J$ and $\bar{J}$ have the same form and $\bar{\gamma} = \gamma$, we can do the convolution of the two evolution factors $U_J$ and $U_{\bar{J}}$ simply by having only one evolution factor, where then $\gamma_J$ is replaced by $2\gamma_J$.

The resulting thrust distributions using the canonical scales in Eq. (3.53) are shown in Fig. 3.7.

![Figure 3.7](image1.png)

**Figure 3.7.:** Singular part of the differential thrust cross section for the quarks (left) and gluons (right) at orders LL, NLL, NLL$'$, NNLL and NNLL$'$ for $Q = 125$ GeV. The thrust distributions are normalized to their Born cross section. Canonical scales have been used: $\mu_H = Q, \mu_J = Q\tau^{1/2}$ and $\mu_S = Q\tau$.

![Figure 3.8](image2.png)

**Figure 3.8.:** The same as Fig. 3.7 but now with $\pi^2$-resummation on. Improvement on the convergence is most noticeable for the gluons.
3.6.1. $\pi^2$-resummation

The hard function may have large $\pi^2$-terms, making the corrections not so small anymore. This is the case for the gluons at one loop, which can be seen by looking at the difference between NLL and NLL′ gluons thrust distribution in Fig. 3.7. These $\pi^2$-terms come from the analytic continuation of $L = \log \frac{-Q^2 - \mu^2}{\mu^2} \rightarrow \log \frac{Q^2}{\mu^2} - i\pi$. The convergence of the hard function may be improved by making the scale complex: $\mu_H = iQ$ (or $\mu_H = -iQ$), so that $\mu^2 < 0$. Using this “$\pi^2$-resummation” scheme [27–29], we show the following numerical results for the quark and gluon hard functions in Tab. 3.2. The results for the thrust distributions with $\pi^2$-resummation are also shown in Fig. 3.8. We see that for the gluons case, the first and second order loop corrections get significantly smaller. At one loop it changes from a 82% correction to 27% correction on the leading order and at two loop the correction on the one loop correction changes from 20% to 3%. For the quarks case, the first order correction actually gets larger by the $\pi^2$-resummation from 8% to 15%, but at second order it improves from 2.7% to 0.2%. The worsening of the perturbative convergence at the first order correction for quarks is due to the $-8$ term already present in the hard function, which partially cancels the $\pi^2$ term.

For the goal of discriminating quark from gluon jets, we will eventually only need the normalized thrust distributions. Whether we do or do not use $\pi^2$-resummation is then not relevant anymore. In all the following results we do not use $\pi^2$-resummation.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_s^0$</th>
<th>$\alpha_s^1$</th>
<th>$\alpha_s^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quark (no $\pi^2$-resummation)</td>
<td>1</td>
<td>+ 0.08418</td>
<td>+ 0.02900</td>
</tr>
<tr>
<td>Quark with $\pi^2$-resummation</td>
<td>1</td>
<td>- 0.14542</td>
<td>- 0.00129</td>
</tr>
<tr>
<td>Gluon (no $\pi^2$-resummation)</td>
<td>1</td>
<td>+ 0.81817</td>
<td>+ 0.36001</td>
</tr>
<tr>
<td>Gluon with $\pi^2$-resummation</td>
<td>1</td>
<td>+ 0.27346</td>
<td>+ 0.04050</td>
</tr>
</tbody>
</table>

Table 3.2.: Corrections on the hard function with and without $\pi^2$-resummation at $Q = 125$ GeV.
Chapter 4.

Nonsingular cross section

Up to now, we have only considered the singular part of the thrust cross section. The factorization theorem only holds for the singular part and the $O(\tau)$ terms we have missed are the nonsingular part of the cross section. These nonsingular parts of the thrust distribution become important at the higher values of $\tau$, and should be included if we also want to describe the thrust distribution accurately at the higher values of $\tau$. We write the total cross section as

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = \frac{1}{\sigma_0} \frac{d\sigma^{\text{sing}}}{d\tau} + \frac{1}{\sigma_0} \frac{d\sigma^{\text{ns}}}{d\tau},$$

where $d\sigma^{\text{sing}}/d\tau$ is the singular part of the cross-section obtained from the resummation calculations and $d\sigma^{\text{ns}}/d\tau$ are the $O(\tau)$ corrections we will add to it. We use the following expansion for the nonsingular terms:

$$\frac{1}{\sigma_0} \frac{d\sigma^{\text{ns}}}{d\tau} = \frac{\alpha_s}{2\pi} f_1(\tau) + \left(\frac{\alpha_s}{2\pi}\right)^2 f_2(\tau) + \ldots,$$

where the first term corresponds to the NLO calculation, the second to the NNLO calculation and so on. In this section we will calculate the first term of this expansion for the quarks and gluons. These nonsingular thrust differential cross sections can be obtained by calculating the whole NLO cross sections and subtracting from this the singular parts from the resummed calculations to avoid double counting. Since the nonsingular cross section vanishes for $\tau = 0$, we only need to calculate the real emission diagrams, since the virtual diagrams correspond to $\tau = 0$.

In Fig. 4.1 we already show the results of the calculations in this chapter. The absolute size of the nonsingular part is plotted against the singular part cross section. For both the quarks and gluons we see that in the small $\tau$-region the singular contribution dominates over the nonsingular contribution and that at $\tau = \frac{1}{3}$ they are exactly the same size. The nonsingular contribution is always negative in our case, which makes the cross section zero for $\tau \geq \frac{1}{3}$.

4.1. Born cross section

The matrix elements needed for the NLO calculations are obtained by using the amplitudes calculated in [30], where they used the spinor helicity formalism to calculate helicity
amplitudes, in contrast to calculating the amplitude for arbitrary spin and then summing over them. Since the helicity amplitudes correspond to different external states, they can each be squared and summed over without interference to obtain the full squared matrix element (App. F). We will first use the helicity amplitudes for the Born cross section of the quarks. This is just the \( e^+e^0 \rightarrow \gamma \rightarrow q\bar{q} \) process, for which the cross section is the well known expression

\[
\sigma_{q,0} = \frac{4\pi\alpha^2_{EM}N_c}{3s}. \tag{4.3}
\]

We have for the amplitude:

\[
A_q(+-+-) \equiv A_q^{(0)}(1^+, 2^-, 3^+, 4^-) = -2i\frac{[13\langle 24\rangle]}{s_{12}}, \tag{4.4}
\]

which squares to

\[
|A_q(+-+-)|^2 = 4\frac{s_{13}s_{24}}{s_{12}^2}. \tag{4.5}
\]

The other amplitudes can be obtained by charge conjugation: \( A_q(+--+|_{3\leftrightarrow 4}, A_q(-++- = A_q(+--)|_{1\leftrightarrow 2} \) and \( A_q(-+-+ = A_q(+--)|_{1\leftrightarrow 2, 3\leftrightarrow 4}. \)

Squaring them and summing all four results in:

\[
|A_q|^2 = 8\frac{s_{13}s_{24} + s_{14}s_{23}}{s_{12}^2}. \tag{4.6}
\]

Here \( s_{12} = s \). Choosing \( p_3 = E(1, 0, 0, 1), p_4 = E(1, 0, 0, -1) \) and \( p_1 = E(1, \sin \theta, 0, \cos \theta) \), where \( E = \sqrt{s}/2 \), we get:

\[
|A_q|^2 = 8\frac{4E^4(1 - \cos \theta)^2 + 4E^4(1 + \cos \theta)^2}{s^2} = 4(1 + \cos^2 \theta). \tag{4.7}
\]
Here $|A_q|^2$ is again the naked amplitude, where some factors have been left out. Including these factors, we have for the full matrix element squared:

$$|M|^2 = 16\pi^2 \alpha_{EM}^2 |A_q|^2. \quad (4.8)$$

Summing over the color and averaging over the spins we get:

$$\langle |M|^2 \rangle = \frac{1}{4} N_c |M|^2, \quad (4.9)$$

and including the phase space integral gives:

$$\sigma_{q,0} = \frac{1}{2s} \int \frac{d^3p_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3p_4}{(2\pi)^3} \frac{1}{2E_4} (2\pi)^4 \delta^4(q - p_3 - p_4) \langle |M|^2 \rangle = \frac{1}{32\pi s} \int d(\cos \theta) \langle |M|^2 \rangle. \quad (4.10)$$

This indeed results in

$$\sigma_{q,0} = \frac{4\pi \alpha_{EM}^2 N_c}{3s}, \quad (4.11)$$

the expression for the tree level $e^+e^- \rightarrow q\bar{q}$ cross section as in Eq. (4.3).

### 4.2. NLO calculation quarks

We now start with the NLO calculation for the quarks. At NLO order we thus have to calculate the real radiation diagram in Fig. 4.2.

Using the short notation $A_q(+-++-+) \equiv A_q^{(0)}(1^+, 2^+, 3^-, 4^+, 5^-)$ for the notation used in [30], where the + and − in the argument denote the helicities of $ggqe^+e^-$, we have [31]:

$$A_q(+-++-) = -2\sqrt{2} \frac{(35)^2}{(12)(13)(45)}, \quad (4.12)$$

and squaring this gives:

$$|A_q(+-++-)|^2 = 8s_{35}^2 s_{12}s_{13}s_{45}, \quad (4.13)$$

Figure 4.2.: Real radiation diagram $e^+e^- \rightarrow q\bar{q}g$. 


The other helicity amplitudes can be obtained from this one by using parity and charge conjugation transformations. Doing a charge conjugation on the quark line changes \( q \) into \( \bar{q} \) and vice versa, which corresponds to exchanging 2 and 3 with each other in the expression for \( |A_q(\pm\pm\pm\pm)|^2 \):

\[
|A_q(\pm\pm\pm\pm)|^2 = 8 \frac{s_{25}^2}{s_{12}s_{13}s_{45}},
\]

(4.14)

The same can be done on the lepton line to obtain \( A_q(\pm\pm\pm\pm) = A_q(\pm\pm\pm\pm)|_{4\leftrightarrow5} \), and on both fermion lines for \( A_q(\pm\pm\pm\pm) = A_q(\pm\pm\pm\pm)|_{2\leftrightarrow3, 4\leftrightarrow5} \). Then by using parity we can also obtain the \( A_q(-\pm\pm\pm\pm), A_q(-\pm\pm\pm\pm), A_q(-\pm\pm\pm\pm) \) and \( A_q(-\pm\pm\pm\pm) \) amplitudes, which squared are the same as the \( A_q(\pm\pm\pm\pm) \) amplitudes. Summing these all gives:

\[
|A_q|^2 = 16 \frac{s_{35}^2 + s_{34}^2 + s_{25}^2 + s_{24}^2}{s_{12}s_{13}s_{45}}
\]

(4.15)

Looking ahead to the three particle phase space integration we have to do to calculate the cross section from the amplitudes, it is convenient to switch to the following variables:

\[
x_i = \frac{2p_i \cdot q}{q^2}, \quad q^2 = s
\]

(4.16)

for the final state particles \( i = 1, 2, 3 \). In this parametrization we have \( 0 \leq x_i \leq 1, x_1 + x_2 + x_3 = 2 \), and in the centre of mass frame: \( x_i = 2E_i/\sqrt{s} \). From the identity:

\[
x_i = \frac{2p_i \cdot q}{q^2} = \frac{2p_i \cdot (p_1 + p_2 + p_3)}{q^2} = \frac{s_{12} + s_{13} + s_{14}}{q^2},
\]

(4.17)

we find the following identities:

\[
s_{12} = \frac{q^2}{2}(x_1 + x_2 - x_3)
\]

\[
s_{13} = \frac{q^2}{2}(x_1 + x_3 - x_2)
\]

\[
s_{23} = \frac{q^2}{2}(x_2 + x_3 - x_1)
\]

(4.18)

Along with identities such as \( s_{12} = (p_1 + p_2)^2 = (q - p_3)^2 = q^2(1 - x_3), s_{45} = q^2 = s, s_{24}^2 + s_{25}^2 = (s_{24} + s_{25})^2 - 2s_{24}s_{25} = s^2 x_2^2 - 2s_{24}s_{25} \), and eliminating \( x_1 \) by using \( x_1 = 2 - x_2 - x_3 \), we rewrite \( |A_q|^2 \) as:

\[
|A_q|^2 = \frac{16 x_2^2 + x_3^2 - 2s_{24}s_{25} - 2s_{34}s_{35}}{s} \frac{1}{(1 - x_2)(1 - x_3)}
\]

(4.19)

These helicity amplitudes are the “naked” amplitudes, where the colour structure, electromagnetic and weak strengths have been extracted:

\[
M_{q\bar{q}g} = e^2 \left( Q^I Q^I + v_{L,R}^I v_{L,R}^I P_z(s_{II}) \right) i T_{a234}^{a_1} g_A A_q
\]

(4.20)

The total squared matrix element including all these is then:

\[
|M_{q\bar{q}g}|^2 = e^4 C_F g_A^2 |A_q|^2 = 64\pi^3 \alpha_s^2 g_A^2 C_F |A_q|^2
\]

(4.21)
4.2.1. Three particle phase space

To calculate the cross section we now have to integrate the squared matrix element over the three particle phase space. The general form of the cross section for three final particles is:

\[
\sigma = \frac{1}{2s} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{2E_1} \frac{d^3p_3}{2E_2} \frac{d^3p_3}{2E_3} (2\pi)^4 \delta^4(q - p_1 - p_2 - p_3)|M|^2. \tag{4.22}
\]

Rewriting \(d^3p_3/(2E_3) = d^4p_3\delta^+(p_3^2)\), we do the \(d^4p_3\) integral using the \(\delta^4(q - p_1 - p_2 - p_3)\) delta-function:

\[
\sigma = \frac{1}{2s} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{2E_1} \frac{d^3p_2}{2E_2} \frac{1}{(2\pi)^3} (2\pi)^4 \delta^+((q - p_1 - p_2)^2)|M|^2. \tag{4.23}
\]

Writing out the argument of the delta-function and rewriting \(p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 = x_1 x_2 (s/4)(1 - \cos \theta_{12})\), where \(\theta_{12}\) is the angle between \(\vec{p}_1\) and \(\vec{p}_2\), we get:

\[
\sigma = \frac{1}{2s} \frac{1}{(2\pi)^5} \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \delta^+ \left( s - s x_1 - s x_2 + \frac{s x_1 x_2}{4} (1 - \cos \theta_{12}) \right) |M|^2. \tag{4.24}
\]

Using spherical coordinates \(d^3p_i = E_i^2 dE_i d\phi_i d\cos \theta_i\) for both \(i = 1, 2\), and using the fact that the integrand is only dependant on \(\cos \theta_{12}\), we can do the two angle integrals of one particle, say particle 1, giving a factor \(4\pi\), and only the \(d\phi_2\) integral, giving a factor \(2\pi\). We then have:

\[
\sigma = \frac{1}{4s} \frac{1}{(2\pi)^3} \int dE_1 dE_2 E_1 E_2 d(\cos \theta_{12}) \delta^+ \left( s - s x_1 - s x_2 + \frac{s x_1 x_2}{4} (1 - \cos \theta_{12}) \right) |M|^2. \tag{4.25}
\]

Using the delta-function to do the \(d(\cos \theta_{12})\) integral and changing variables from \(E_i\) to \(x_i\) by using \(E_i = \frac{x_i \sqrt{s}}{2}\):

\[
\sigma = \frac{1}{4s} \frac{1}{(2\pi)^3} \int dx_1 dx_2 \frac{x_1 x_2 s^2}{16} \frac{2}{x_1 x_2} |M|^2 = \frac{1}{256\pi^3} \int dx_1 dx_2 |M|^2. \tag{4.26}
\]

For the quarks the squared matrix element can be rewritten as

\[
|A_q|^2 = \frac{16 x_2^2 + x_3^2 - 2 s_2 s_3 - 2 s_2 s_{25} - 2 s_3 s_{35}}{s (1 - x_2)(1 - x_3)} \quad \tag{4.27}
\]

\[
= \frac{16}{s} \left( 1 - \frac{1}{2} (1 - \cos \theta_2) (1 + \cos \theta_3) x_2^2 + \frac{1}{2} (1 - \cos \theta_3) (1 + \cos \theta_2) x_3^2 \right),
\]

which after the phase space integral then gives

\[
\sigma = \frac{4\pi \alpha^2_{EM} N_c \alpha_s C_F}{3s} \frac{1}{2\pi} \int dx_2 dx_3 x_2^2 + x_3^2 \frac{x_2^2}{(1 - x_2)(1 - x_3)}. \tag{4.28}
\]
4.2.2. Thrust cross section

If we would want to calculate the total cross section, we would find that the real emission diagrams we have just calculated contain infrared divergences. These would be cancelled by the virtual diagrams. But since we are interested in the differential cross section in thrust, we do not have to worry about this and we can restrict the final states to those who have a thrust $T \leq T_c$. Here, $T_c$ is a thrust cut which regulates the infrared divergence. In the case of three final state particles we have that the thrust $T = 1 - \tau = \max\{x_1, x_2, x_3\}$. Together with the fact that $x_1 + x_2 + x_3 = 2$, we have that $2/3 \leq T < 1$, so $2/3 < T_c < 1$. From these we can infer the boundaries of the $x_1, x_2$ integrals, which become functions of $T_c$. The upper boundaries are set to $T_c$ by $T \leq T_c$. The lower boundary of $x_1$ is set to $2 - 2T_c$, which is the lowest value allowed, corresponding to the case $x_2 = x_3 = T_c$. The lower boundary of $x_2$ is then fixed to $2 - T_c - x_1$. We thus arrive at the expression for the cross section with a thrust cut $T_c$:

$$\sigma_q(T_c) = \sigma_{q,0} \frac{\alpha_s C_F}{2\pi} \int_{2-2T_c}^{T_c} dx_1 \int_{2-T_c-x_1}^{T_c} dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}.$$  (4.29)

where $\sigma_{q,0}$ is the Born cross section of $e^+e^- \rightarrow q\bar{q}$ given in Eq. (4.3). Doing the $x_1$ and $x_2$ integrals gives:

$$\sigma_q(T_c) = \sigma_{q,0} \frac{\alpha_s C_F}{2\pi} \left[ -8 - \frac{\pi^2}{3} + 15T_c - \frac{9T_c^2}{2} + (6 - 12T_c) \arctanh \left( 3 - \frac{2}{T_c} \right) \right.\left. + 2 \log^2 \left( -1 + \frac{1}{T_c} \right) + 4\text{Li}_2 \left( -1 + \frac{1}{T_c} \right) \right].$$  (4.30)

Switching to thrust $\tau = 1 - T$ and differentiating this with respect to $\tau_c$ we obtain the cross section differential in $\tau$ with a thrust cut:

$$\frac{d\sigma_q}{d\tau} = \sigma_{q,0} \frac{\alpha_s C_F}{2\pi} \frac{1}{\tau(\tau - 1)} \left[ 3 - 9\tau - 3\tau^2 + 9\tau^3 - (4 - 6\tau + 6\tau^2) \log \left( \frac{1}{\tau} - 2 \right) \right].$$  (4.31)

Expanding this for $\tau$ near zero, gives:

$$\sigma_{q,0} \frac{\alpha_s C_F}{2\pi} \left. -3 - 4 \log \frac{\tau}{\tau_c} \right|_{\tau_c},$$  (4.32)

which is exactly the singular part as obtained from the resummed calculation. Subtracting the singular part from the full NLO calculation then leaves us with the nonsingular parts. We have thus calculated the first term in the expansion of Eq. (4.2) for the quarks: $f_{\tau}^q(\tau)$, which encodes the nonsingular part for the quarks at NLO level:

$$f_{\tau}^q(\tau) = \frac{C_F}{\tau(\tau - 1)} \left[ (-6\tau^2 + 6\tau - 4) \log \left( \frac{1}{\tau} - 2 \right) + 9\tau^3 - 3\tau^2 - 9\tau + 3 \right] \theta \left( \frac{1}{3} - \tau \right) + C_F \frac{3 + 4 \log \tau}{\tau}.$$  (4.33)
4.3. NLO calculation gluons

Using the same method, we do the same for the gluons. Now we have to calculate the diagrams in Fig. 4.3. Here we also have a three final state particles, and the phase space integration will be much the same. The helicity amplitudes for these diagrams are also taken from [30]. We calculate the “gggH,” where one gluon splits into two gluons and “gq\bar{q}H,” where one gluon splits into a q\bar{q} separately. The case where there are three final state gluons will give a term proportional to C_A, while the second case where there is one gluon and two quarks in the final state will give a term proportional to n_fT_F.

4.3.1. gggH contribution

We do the gggH first. Using the short notation \( A(+++) \equiv A^{(0)}(1^+, 2^+, 3^+, 4H) \), we have [32]:

\[
A(+++) = \frac{1}{\sqrt{2}} \frac{m_H^4}{(12)(23)(31)}, \quad |A(+++)|^2 = \frac{m_H^8}{2s_{12}s_{23}s_{13}}
\]  

(4.34)

By parity we also get \( A(−−−) \), which squared is the same as \( |A(+++)|^2 \). The other helicity amplitudes are:

\[
A(+-+) = \frac{1}{\sqrt{2}} \frac{[12]^3}{[13][23]}, \quad |A(+-+)|^2 = \frac{s_{12}^4}{2s_{12}s_{23}s_{13}}
\]  

(4.35)

From this one we get \( A(+−−) = A(++)|_{2 \leftrightarrow 3} \) and \( A(−++ ) = A(++)|_{1 \leftrightarrow 3} \). Parity then also provides us \( A(−−−), A(−−+) \) and \( A(+−−) \), which by parity invariance are again the same as their \( A(++) \) counterparts. Summing them all gives:

\[
|A_{ggg}|^2 = \frac{m_H^8 + s_{12}^4 + s_{13}^4 + s_{23}^4}{s_{12}s_{13}s_{23}}
\]  

(4.36)

Using the same parametrization as in the quark case (Eq. (4.16)), we find:

\[
|A_{ggg}|^2 = \frac{m_H^8 + s^4(x_1 - 1)^4 + s^4(x_3 - 1)^4 + s^4(x_1 + x_3 - 1)^4}{s^3(1 - x_1)(1 - x_3)^2}
\]  

(4.37)
The helicity amplitude is related to the full amplitude by:

\[ M_{ggg} = \frac{2T_F \alpha_s g_s}{3 \pi v} i (i f \alpha_1 \alpha_2 \alpha_3) A_{ggg}, \]  

(4.38)

so that:

\[ |M_{ggg}|^2 = \frac{16 T_F^2 \alpha_s^3 C_A}{9 \pi v^2} |A_{ggg}|^2. \]  

(4.39)

After doing the similar phase space integral we find for the differential thrust cross section in \( \tau \) for the gggH contribution:

\[
\left( \frac{d\sigma}{d\tau} \right)_{gggH} = \sigma_{g,0} \frac{\alpha_s}{2 \pi} \frac{C_A}{3 \tau (\tau - 1)} \left[ 11 - 68 \tau + 144 \tau^2 - 132 \tau^3 + 45 \tau^4 \right. \\
- (12 - 24 \tau + 36 \tau^2 - 24 \tau^3 + 12 \tau^4) \log \left( \frac{1 - 2 \tau}{\tau} \right). 
\]  

(4.40)

Expanding this expression for small \( \tau \) gives \( \sigma_{g,0} \frac{C_A}{3 \tau} \frac{-11 - 12 \log \tau}{3 \tau} \), which is the same as the \( C_A \) part of the singular cross section.

4.3.2. gqqH contribution

Now we consider the gqqH contribution. The notation is again \( A(+-+) \equiv A^{(0)}(1^+, 2_q^+, 3\bar{q}, 4H) \), and we have [32]:

\[ A(+-+) = \frac{1}{\sqrt{2}} \left[ \frac{[12]^2}{[23]} \right], \quad |A(+-+)|^2 = \frac{s_{12}^2}{s_{23}} \]  

(4.41)

Charge conjugation on the quark line gives \( A(-++) = A(+-+)|_{2\leftrightarrow3} \), and parity again gives the same for \( |A(-+-)|^2 \) and \( |A(--+)|^2 \). Summing all gqq squared amplitudes, we find:

\[ |A_{gqq}|^2 = \frac{s_{12}^2 + s_{13}^2}{s_{23}}. \]  

(4.42)

The full amplitude is related to this helicity amplitude as:

\[ M_{gqq} = \frac{2 T_F \alpha_s g_s}{3 \pi v} i T_{\alpha_2 \alpha_3} A_{gqq}, \]  

(4.43)

which gives us the full squared amplitude:

\[ |M_{gqq}|^2 = \frac{16 T_F^2 \alpha_s^3 T_F}{9 \pi v^2} |A_{gqq}|^2. \]  

(4.44)

Using the same parametrization again, we have:

\[ |A_{gqq}|^2 = -s \frac{x_1^2 + (x_2 - x_3)^2}{x_1 - x_2 - x_3}. \]  

(4.45)
Using $x_1 = 2 - x_2 - x_3$, this becomes:

$$|A_{gqq}|^2 = \frac{s}{x_2 + x_3 - 1} \left( 2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 \right) .$$

Doing the phase space integral over this, calculating the differential cross section in the thrust cut $T_c$ and going over to $\tau_c$, results in:

$$\int dx_2 dx_3 |A_{gqq}|^2 = s \int_{2-T_c}^{T_c} dx_2 \int_{2-T_c-x_2}^{T_c} dx_3 \frac{2 - 2x_2 + x_2^2 - 2x_3 + x_3^2}{x_2 + x_3 - 1} .$$

Adding all prefactors in front, this gives us for the $gq\bar{q}H$ part of the cross section

$$\frac{d\sigma}{d\tau}_{gq\bar{q}H} = \sigma_{g,0} \frac{2T_Fn_f}{3\tau} \left[ 2 - 21\tau + 60\tau^2 - 45\tau^3 + 6 \log \left( \frac{1}{\tau} - 2 \right) \tau_c (1 - 2\tau_c + 2\tau_c^2) \right] ,$$

which expanded near small $\tau$ gives $\sigma_{g,0} \frac{3n_fT_F}{3\tau_c}$, which is indeed the same as the $n_fT_F$ part of the singular differential cross section from our resummation calculation.

### 4.3.3. Full NLO gluons differential cross section

Adding the $C_A$ and $n_fT_F$ contributions together, we find for the full NLO gluons differential cross section:

$$\frac{d\sigma}{d\tau} = \sigma_{g,0} \frac{\alpha_s}{2\pi} \left\{ C_A \frac{1}{3\tau(\tau - 1)} \left[ 11 - 68\tau + 144\tau^2 - 132\tau^3 + 45\tau^4 
\right.ight.$$

$$\left. - (12 - 24\tau + 36\tau^2 - 24\tau^3 + 12\tau^4) \log \left( \frac{1 - 2\tau}{\tau} \right) \right\}$$

$$+ T_Fn_f \frac{2}{3\tau} \left[ 2 - 21\tau + 60\tau^2 - 45\tau^3 + (6\tau - 12\tau^2 + 12\tau^3) \log \left( \frac{1 - 2\tau}{\tau} \right) \right]$$

$$\right\} ,$$

which expanded near small $\tau$ gives $\sigma_{g,0} \frac{2n_fT_F}{3\tau_c}$, which is indeed the same as the $n_fT_F$ part of the singular differential cross section from our resummation calculation.
Subtracting the singular part from this, we have obtained the nonsingular part for the gluons at NLO, which we include in our thrust calculation as in Eq. (4.1):

\[
f_{1}^{g}(\tau) = \begin{cases} 
\frac{C_{A}}{3\tau(\tau - 1)} \left[ 11 - 68\tau + 144\tau^{2} - 132\tau^{3} + 45\tau^{4} \right] \\
- (12 - 24\tau + 36\tau^{2} - 24\tau^{3} + 12\tau^{4}) \log\left( \frac{1 - 2\tau}{\tau} \right) \\
+ \frac{2T_{F}n_{f}}{3\tau} \left[ 2 - 21\tau + 60\tau^{2} - 45\tau^{3} + (6\tau - 12\tau^{2} + 12\tau^{3}) \log\left( \frac{1 - 2\tau}{\tau} \right) \right] \theta \left( \frac{1}{3} - \tau \right) \\
+ \frac{(11 + 12\log \tau)C_{A} - 4n_{f}T_{F}}{3\tau} \end{cases} \tag{4.50}
\]
Chapter 5.
Uncertainties and hadronization

5.1. Thrust regions

We have seen the general shape of the resummed thrust distributions of quark and gluon jets in Fig. 3.7. In this section we discuss the shape and the various thrust regions in more detail. In Fig. 5.1 a typical plot of the thrust distribution is shown. The distribution starts at zero at \( \tau = 0 \), quickly grows to a peak at a certain value of \( \tau \) close to 0, and then dies off as \( \tau \) increases. Based on the general shape of the thrust distribution and the different physics that are important at different values of the thrust, we can distinguish several different regions for the thrust distribution.

![Figure 5.1.: Typical shape of the thrust distribution. Several regions are distinguished.](image)

The first region is the nonperturbative region where \( \tau \lesssim \Lambda_{\text{QCD}}/Q \). In this region perturbative techniques break down since in this region the strong coupling is not small anymore. The second region is the resummation region. This region is also called the peak region, because this is often where the peak of the distribution is found. In the resummation region, where the thrust is very small: \( \tau \ll 1 \), the singular contributions
dominate over the nonsingular contributions. The nonsingular contributions are power suppressed while the singular contributions scaling like $\frac{1}{\tau}, \frac{\log \tau}{\tau}$ are very large. The singular contributions need to be resummed in this region. The next region is the tail region. This is at higher values of $\tau$, where the logarithms are not large anymore and the nonsingular terms become relevant. In the tail region the singular and nonsingular contributions become equally important and should both be included at the same order in $\alpha_s$. The singular and nonsingular parts become the same in absolute value size, with opposite sign, and large cancellations take place between the two. These cancellations will be spoiled if the resummation is kept on in this region. Between the resummation region and the tail region we also speak of a transition region, connecting the two regions where the logarithms are still large, but are decreasing in size. At the $\tau$ values near the very endpoint of the thrust, $\tau \sim 0.5$, we sometimes also speak of a far-tail region. This is the region where we have multijet events, where there is certainly no distinction between the different scales, and where fixed-order perturbation calculations are sufficient to make accurate predictions. The boundaries between these different regions is not exact and we have to find a proper description that smoothly connects the different regions.

### 5.2. Profile functions

The hard, jet and soft functions each depend on their own scales: $\mu_H, \mu_J$ and $\mu_S$. The proper resummation of the logarithms is achieved by making these scales functions of $\tau$. We call these renormalization scale functions profile functions. There is no unambiguous procedure for choosing these profile functions, but we are guided by several things, like the relative sizes of the singular and the nonsingular contributions and the known behaviour of the scales in certain regions. As discussed before, the resummation cannot be continued all the way until the thrust endpoint. We know that in the far-tail region the resummation must be turned off already, since this is where cancellations of the same order between the singular and nonsingular contributions take place and we have to avoid the non-physical consequences if these cancellations do not exactly take place. To avoid these non-physical consequences, like the cross-section becoming negative, we have to turn the resummation off, which is achieved by letting all the different scales go to a common scale, the fixed-order scale. In our case the fixed-order scale is set to the hard scale: $\mu_{\text{FO}} = Q$. Also in the region where $\tau$ is very small, we are near $\Lambda_{\text{QCD}}$ and find ourself in the nonperturbative region, where it is also not obvious how to set the profile scales.

In this section we discuss how we choose the profile functions to resum the logarithms properly. A first guess would be to just use the canonical scales for the profile functions: $\mu_H(\tau) = Q, \mu_J(\tau) = Q\sqrt{\tau}, \mu_S(\tau) = Q\tau$. These profile functions set the hard, jet and soft function at their canonical renormalization scale where the logarithms are minimized. These functions are suitable for the resummation region where $\tau$ is very small (but not so small where it corresponds to the nonperturbative region), but not for the region where $\tau$ is higher valued (tail/far-tail regions). In the far-tail region where the cross-section is
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\[ \mu_H = Q, \quad \mu_J = Q \sqrt{\tau}, \quad \mu_S = Q \tau \]

Figure 5.2.: In the left panel are shown the canonical profile scales: \( \mu_H = Q, \quad \mu_J = Q \sqrt{\tau}, \quad \mu_S = Q \tau \).

In the right panel are shown the profile functions used to address the different points in different thrust regions. Both are at \( Q = 125 \text{ GeV} \).

described by fixed-order perturbation theory, we want that all the scales approach \( Q \), and certainly that at the thrust end-point \( \tau = 0.5 \) they are all exactly the same. As can be seen in the left panel of Fig. 5.2, the canonical scales do not satisfy this requirement and better profile functions are needed. We use the following profile scales (right panel of Fig. 5.2) from [33]:

\[ \begin{align*}
\mu_H &= Q \\
\mu_J &= Q \sqrt{f_{\text{run}}(\tau)} \\
\mu_S &= Q f_{\text{run}}(\tau),
\end{align*} \]

where

\[ f_{\text{run}}(x) = \begin{cases}
  x_0 \left(1 + \left(\frac{x}{2x_0}\right)^2\right) & x \leq 2x_0 \\
  x & 2x_0 \leq x \leq x_1 \\
  x + \frac{(2x_2-x_2)(x-x_1)^2}{2(x_2-x_1)(x_3-x_1)} & x_1 \leq x \leq x_2 \\
  1 - \frac{(2x_1-x_2)(x-x_3)^2}{2(x_1-x_2)(x_3-x_2)} & x_2 \leq x \leq x_3 \\
  1 & x_3 \leq x
\end{cases} \]  

\[ (5.2) \]

\( f_{\text{run}} \) (and its derivative), is a continuous function which takes care of the resummation (and turning off the resummation) of the scales in the different regions. The first line of Eq. (5.2) is the nonperturbative region, where we let \( \mu_J \) approach \( Q \sqrt{x_0} \) and \( \mu_S \) approach \( Q x_0 \) as \( x \) goes to 0. \( x_0 \) will be chosen such that \( Q \sqrt{x_0} \) and \( Q x_0 \) are still bigger than \( \Lambda_{\text{QCD}} \). The second line corresponds to the resummation region, where \( \mu_J \) and \( \mu_S \) follow the canonical scalings. The last line is the fixed-order region where the resummation is turned off by setting \( f_{\text{run}} \) to 1. The third and fourth lines are the transition region, where a quadratic scaling is used for a smooth transition from the resummation region to the fixed-order region. The function \( f_{\text{run}} \) has four parameters, of which one is dependent on the others: the midpoint of the transition region is fixed as \( x_2 = (x_3 + x_1)/2 \). We will choose central values for the other three and vary around these central values to
get uncertainties. $x_3$ is the point where the resummation is turned off completely, and we choose $x_3 = 1/3$, as this is where the singular and nonsingular contributions start to cancel each other exactly (at NLL’+NLO level). $x_0$ determines the boundary of the nonperturbative region, and we set $x_0 = 3/Q$, corresponding to $3 \text{ GeV} > \Lambda_{\text{QCD}}$. Lastly, $x_1$ should be chosen at a value such that the three different scales are still well separated. We choose $x_1 = 0.1$. These $x_i$-values are still chosen with some arbitrariness, so we will vary them around their central values as part of our uncertainty estimate. How we exactly do this is explained in the next section.

### 5.3. Uncertainties

Without some uncertainties, we cannot make a comparison between the thrust distributions at different orders and say something about the convergence. In this section we discuss how we define the uncertainties to make a uncertainty band around the central plots.

Traditionally, the fixed-order uncertainty is obtained by scaling all the scales with a factor 2 up and down. We will also use this method to obtain a fixed-order uncertainty. Since we always want to keep $\mu_H \geq \mu_J \geq \mu_S$, we cannot vary them independently, so we should vary them all simultaneously. In Fig. 5.3 the factor 2 up and down variation of the scales used for the fixed-order uncertainty are plotted. This variation however leaves the ratio between the three scales the same. The evolution factors containing logarithms of these ratios then remain the same, which means that this variation is not suitable to assess the uncertainty corresponding to the resummation. In order to estimate the size of the higher order terms in the resummed logarithm series, the ratios of the scales are

![Fixed-order scale variations](image)

Figure 5.3.: FO scale variations with a factor 2 up and down on our profile scales at $Q = 125 \text{ GeV}$. 

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varied by introducing the following ‘varying function’:

\[ f_{\text{vary}}(x) = \begin{cases} 
 2 \left( 1 - \frac{x^2}{x_3^2} \right) & 0 \leq x \leq \frac{x_3}{2} \\
 1 + 2 \left( 1 - \frac{x}{x_3} \right) & \frac{x_3}{2} \leq x \leq x_3 \\
 1 & x_3 \leq x.
\] \tag{5.3}

This function goes to 2 as \( \tau \to 0 \) and approaches 1 as \( \tau \to x_3 \). This function, to the power \( \alpha \), is then used as a multiplicative factor to the soft function, and the up and down variation of the soft function is then achieved by varying the parameter \( \alpha \). By the canonical relation between the three scales \( \mu_J = \sqrt{\mu_H \mu_S} \), the jet function is then modified by the factor \( f_{\text{vary}}^{\alpha/2}(\tau) \). To also allow a variation of \( \mu_J \) independent of \( \mu_S \), we introduce another parameter \( \beta \), so that finally:

\[ \begin{align*}
\mu_H &= Q \\
\mu_J &= Q \left( f_{\text{vary}}^\alpha(\tau) f_{\text{run}}(\tau) \right)^{1/2} \\
\mu_S &= Q f_{\text{vary}}^\alpha(\tau) f_{\text{run}}(\tau).
\end{align*} \tag{5.4}
\]

The parameter \( \beta \) changes the canonical relation between the scales to \( \mu_J^2 = \mu_H^{1+2\beta} \mu_S^{1-2\beta} \). The central values of \( \alpha \) and \( \beta \) are 0 and we will vary these around 0 to obtain the resummation uncertainty. We vary \( \alpha \) from 1 to \( -1 \), which then results in a typical factor of 2 variation for the soft scale and a \( \sqrt{2} \) for the jet scale, and \( \beta \) from \( 1/6 \) to \( -1/6 \). All in all we have five free parameters for our profile functions, for which we use the following central values and variations:

\[ \{x_0, x_1, x_3, \alpha, \beta\} = \{3/Q \pm 25\%, 0.1 \pm 25\%, 1/3, 0 \pm 1, 0 \pm 1/6\} \tag{5.5} \]

The variation of these parameters are done one by one, keeping the others fixed. Ideally, we would like to vary the parameters continuously over their range, but since this is computationally intense we can take discrete values between the ranges. In practice, this discretization works well enough and an uncertainty band is obtained of which the edges correspond to the endpoints of the parameter ranges, so that only the boundary points of the variations need to be done.

Although we vary the parameters symmetrically around their central value, the resulting changed thrust distributions corresponding to the variations are in general not symmetrically above and below the central thrust distribution. There are several ways how to define the uncertainty bands. The method we take is that for each value of \( \tau \) we take the maximal deviation from the central value obtained from all the variations and use this as the uncertainty. We do this separately for the fixed-order uncertainty obtained from the fixed-order variations

\[ \delta_{\text{FO}}(\tau) = \max_{\mu = \{2Q, Q/2\}} \left| \frac{d\sigma}{d\tau} - \frac{d\sigma_{\text{central}}}{d\tau} \right|. \tag{5.6} \]
as well for the resummation uncertainty obtained from varying the resummation parameters
\[ \delta_{\text{resum}}(\tau) = \max_{(\alpha,\beta,x_0,x_1)} \left| \frac{d\sigma}{d\tau} - \frac{d\sigma_{\text{central}}}{d\tau} \right|. \] (5.7)

The total perturbative uncertainty is then defined by adding the fixed-order and resummation uncertainty in quadrature
\[ \delta_{\text{pert}}(\tau) = \sqrt{\delta_{\text{FO}}(\tau)^2 + \delta_{\text{resum}}(\tau)^2}. \] (5.8)

We add and subtract this perturbative uncertainty to the central thrust distribution, which defines the boundaries of the uncertainty band. The central thrust distribution thus lies in the middle of the band. The jet and soft profile scales and their variations are plotted in Fig. 5.4.

\[ \mu_J(\text{central value}) \]
\[ \mu_S(\text{central value}) \]

Figure 5.4.: Jet and soft profile functions from [33] at \( Q = 125 \text{ GeV} \). The blue and orange solid lines are the jet and soft scales at the central value for the parameters. The variations of the parameters fill up the bands. The values and variations of the parameters are as in Eq. (5.5).

### 5.4. Hadronization

We model hadronization effects by replacing the perturbative soft function with a convolution of the perturbative soft function with a nonperturbative shape function \( F(k) \):
\[ S_i(k,\mu) = \int dk' S_i^{\text{pert}}(k-k') F_i(k'), \] (5.9)

where \( k' \sim \Lambda_{\text{QCD}} \). We choose a simple form for the nonperturbative shape function which has the following desired properties:
\[ \int dk' F_i(k') = 1, \quad \int dk' k' F_i(k') = 2\Omega_i. \] (5.10)
The first property makes sure that the total cross section is not changed by the shape function and that only the shape of the distribution is changed. The second property is the first moment, which is interpreted as the average shift of the perturbative thrust distribution due to the nonperturbative effects. This can be shown in the tail region where \( k' \sim \Lambda_{\text{QCD}} \ll k \), so that we can make the expansion:

\[
S_i = \int dk' S_i^{\text{pert}}(k - k') F_i(k') \sim \int dk' \left( S_i^{\text{pert}}(k) - \frac{dS_i^{\text{pert}}(k)}{dk} k' \right) F_i(k') \quad (5.11)
\]

\[
= S_i^{\text{pert}}(k) \int dk' F_i(k') - \frac{dS_i^{\text{pert}}(k)}{dk} \int dk' k' F_i(k') = S_i^{\text{pert}}(k) - \frac{dS_i^{\text{pert}}(k)}{dk} 2\Omega_i \sim S_i^{\text{pert}}(k - 2\Omega_i).
\]

The second point is important since in some cases there is some data on this \( \Omega_i \) in the literature, which encodes the shift of the distribution in the tail region. We choose the following form for the shape function \[35\]:

\[
F_i(k') = \frac{k'}{\Omega_i^2} e^{k'/\Omega_i}. \quad (5.12)
\]

We thus have one nonperturbative parameter \( \Omega_i \). For the value of this parameter for the quarks we choose \( \Omega_q = 0.4 \), due to reasons from extraction the first moment from the shift of the distribution in the tail region \[36\]. For the gluons we assume Casimir scaling \( \Omega_g = (C_A/C_F)\Omega_q \).

### 5.4.0.1. Nonperturbative uncertainty

With these additional parameters \( \Omega_i \) we define a nonperturbative uncertainty by separately varying \( \Omega_q \) and \( \Omega_g \) with \( \pm 50\% \) and simultaneously varying both by \( 75\% \). As with the perturbative uncertainty, we again take the maximum deviation from the central value as the uncertainty \( \delta_{\text{nonp}} \). The nonperturbative uncertainty is uncorrelated to the perturbative uncertainty, so the total uncertainty is defined as

\[
\delta_{\text{total}}(\tau) = \sqrt{\delta_{\text{pert}}(\tau)^2 + \delta_{\text{nonp}}(\tau)^2}. \quad (5.13)
\]
Chapter 6.

Results

6.1. Convergence plots

Now that we have explained how we make the thrust distributions and their uncertainty bands, we show the results in this section. We plot the singular part of the thrust distributions at the orders NLL, NLL' and NNLL' against each other.

In Fig. 6.1 and the left panel of Fig. 6.3 we see that the bands overlap each other and become smaller as we go to higher orders, which demonstrate the convergence of the resummed predictions. Note that the far tails of the distribution do not overlap (best seen for the gluons in the left panel of Fig. 6.3), which is due to the missing nonsingular corrections. In Fig. 6.2 and the right panel of Fig. 6.3 we show the differences when we add the nonsingular contribution at NLL' order. At low values of $\tau$ there is almost no difference, but in the tail region the difference is noticeable, and the thrust distribution now correctly goes to 0 at $\tau = \frac{1}{3}$.

6.2. Comparisons with the Monte Carlos

We compare the analytical calculations with the predictions of the Monte Carlos. We do this at parton level, where we have not included the hadronization modelling yet, and at hadron level, where the hadronization modelling is included. In Fig. 6.4 and Fig. 6.5 the comparisons of NNLL' with PYTHIA and HERWIG for the quarks are shown. For the quarks we have chosen to show the singular NNLL', since we are zooming in on the peak region, and there is a significant shift in the peak position going from NLL' to NNLL'. The tail region is now a bit off, which is due to the missing NNLO nonsingular corrections. We see that the parton showers differ quite a lot at parton level. HERWIG still has a lot of events at $\tau = 0$, which is probably due to a higher shower cutoff [1]. This gets compensated by the hadronization model, and we see that at hadron level all the Monte Carlos become very similar. This is expected due to the tuning of the hadronization models to LEP data for the quarks.

In Fig. 6.6 the comparisons are shown for the gluons. Here we see large differences for the several parton showers. These occur at parton level and persist at hadron level. At parton level for small values of $\tau$, PYTHIA is at the lower boundary and HERWIG at the upper boundary of the uncertainty band of NLL'+NLO. In the tail region our analytical
Chapter 6. Results

Figure 6.1.: The normalized thrust spectrum for the quarks at NLL (green), NLL’ (blue) and NNLL’ (red) in the peak region (left) and the tail region (right) at $Q = 125$ GeV. The bands indicate the perturbative uncertainty, obtained using Eq. (5.8). These plots clearly demonstrate the convergence of our resummed predictions.

Figure 6.2.: Comparison of the singular part only (NLL’) and singular plus nonsingular part (NLL’+NLO) quark thrust distribution. Shown left is the peak region where the nonsingular part is negligible in comparison with the singular part, resulting in almost no discernible difference. Shown right is the far tail region where the difference becomes noticeable. Having included the nonsingular part, the thrust distribution now correctly goes to 0 at $\tau = \frac{1}{3}$.

Figure 6.3.: Same as Fig. 6.1 and Fig. 6.2 but now for the gluons: left are the convergence plots and right NLL’ vs NLL’+NLO.
predictions are 0 at $\tau = 1/3$, since there are only three partons at NLO, whereas PYTHIA still has events with $\tau > 3$. PYTHIA however does not do this with any formal accuracy, i.e. it does not contain NNLO corrections. At hadron level our analytical calculation agrees most with PYTHIA. In these figures for the gluons we have also included the prediction of HERWIG 7.0.4 to highlight the changes for the latest version of HERWIG, which greatly improve the agreement of HERWIG with the calculations.

### 6.3. Classifier separation

To quantify the separation power of quark and gluon jets we use the classifier separation as in \[1\]. The classifier separation $\Delta$ is defined as

$$
\Delta = \frac{1}{2} \int d\lambda \frac{[p_q(\lambda) - p_g(\lambda)]^2}{p_q(\lambda) + p_g(\lambda)},
$$

(6.1)
Figure 6.6.: The normalized thrust spectrum for gluons at NLL’+NLO (blue) compared to PYTHIA (purple) and HERWIG (green, yellow) for the angular and dipole shower respectively. The angular shower of the older version HERWIG 7.0.4 is also included in grey. Shown are results at parton level (left) and hadron level (right) for $Q = 125$ GeV.

Figure 6.7.: The integrand of the classifier separation as in Eq. (6.1) at NLL’+ NLO order (left panel) and of PYTHIA’s prediction (right panel) at parton level (blue) and hadron level (orange).

where $p_q(\lambda), p_g(\lambda)$ are the quark and gluon jets probability distributions as functions of the classifier $\lambda$. In our case $\lambda = \tau$ and $p_q(\tau), p_g(\tau)$ are the normalized thrust distributions. The classifier separation is a number between 0 and 1 rating the discrimination power of quark/gluon jets, with $\Delta = 0$ meaning no discrimination power at all and $\Delta = 1$ corresponding to perfect discrimination power. We will calculate this classifier separation for the predictions of the Monte Carlo programs HERWIG, PYTHIA and for the analytical calculations, and see how they compare.

The integrand of Eq. (6.1) is basically the fractional difference between the quarks and gluons at a given value of thrust, and this can be used to see in what regions of thrust most of the separation power comes from. These are shown in Fig. 6.7 and Fig. 6.8. We see that most of the separation power comes from the small $\tau$-region. This was expected since this is the region where the peaks of the distributions are and where the quark/gluon thrust distributions differ most. After the peaks however, there is still some significant contribution of the separation power from the middle regions. For the analytical calculations we only have the complete thrust distribution at NLL’+ NLO.
level, and so we can only calculate the classifier separation at this order. However, since most of the separation power comes from the low $\tau$-regions, we can still calculate the classifier separation at NNLL' order if we restrict to small values of $\tau$. We integrate the classifier separation integrand from 0 to $\tau_{\text{cut}}$, and choose $\tau_{\text{cut}}$ such that we still include most of the separation power. We cannot pick $\tau_{\text{cut}}$ to be too large, because we have to stay in the region where the nonsingular part (the NNLO which we lack for NNLL') is still much smaller than the singular part. Guided by these two constraints we pick $\tau_{\text{cut}} = 0.15$.

The numerical values of the classifier separation are shown in Tab. 6.1 and Tab. 6.2. The numerical values for the classifier separation with a thrust cut are shown in Tab. 6.3 and Tab. 6.4. To make the easier comparisons, the values are also shown in Fig. 6.9. As expected, we see that the perturbative uncertainties on the classifier get smaller as we go to higher orders.

![Figure 6.8: The integrand of the classifier separation as in Eq. (6.1) of HERWIG’s prediction. Left is the (default) angular shower and right the dipole shower. Parton level is in blue, hadron level in orange.](image)

![Figure 6.9: Classifier separation for the analytical calculations and Monte Carlos. In the left panel are shown the values for PYTHIA, HERWIG and NLL’ + NLO. In the right panel are shown the values of the classifier separation with $\tau_{\text{cut}} = 0.15$ for PYTHIA, HERWIG, NLL’ and NNLL’. The left errorbar centred at the parton level value represents the perturbative uncertainty. The right outer errorbar represents the total uncertainty due to the perturbative and the nonperturbative variations, while the right inner errorbar is only the perturbative uncertainty.](image)
### ∆ at parton level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>FO uncertainty</th>
<th>Resummation uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLL’+ NLO</td>
<td>0.230</td>
<td>± 0.043</td>
<td>± 0.069</td>
</tr>
<tr>
<td>PYTHIA</td>
<td>0.266</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG angular</td>
<td>0.156</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG dipole</td>
<td>0.138</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.1.: Values of the classifier separation for the analytical calculations at NLL’+ NLO level and the Monte Carlos at parton level with $Q = 125$ GeV. The perturbative fixed-order and resummation uncertainties for the analytical calculations are shown in the second and third column.

### ∆ at hadron level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>Perturbative uncertainty</th>
<th>Nonperturbative uncertainty</th>
</tr>
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<tbody>
<tr>
<td>NLL’+ NLO</td>
<td>0.279</td>
<td>± 0.081</td>
<td>± 0.083</td>
</tr>
<tr>
<td>PYTHIA</td>
<td>0.303</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG angular</td>
<td>0.221</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG dipole</td>
<td>0.135</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.2.: Same as Tab. 6.1 but at hadron level.

### ∆($\tau < \tau_{\text{cut}}$) at parton level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>FO uncertainty</th>
<th>Resummation uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLL</td>
<td>0.164</td>
<td>± 0.043</td>
<td>± 0.084</td>
</tr>
<tr>
<td>NLL’</td>
<td>0.220</td>
<td>± 0.043</td>
<td>± 0.070</td>
</tr>
<tr>
<td>NNLL’</td>
<td>0.235</td>
<td>± 0.031</td>
<td>± 0.043</td>
</tr>
<tr>
<td>NLL’+ NLO</td>
<td>0.228</td>
<td>± 0.045</td>
<td>± 0.072</td>
</tr>
<tr>
<td>PYTHIA</td>
<td>0.195</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG angular</td>
<td>0.133</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG dipole</td>
<td>0.108</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.3.: Same as Tab. 6.1 but now for the classifier with a $\tau_{\text{cut}} = 0.15$.

### ∆($\tau < \tau_{\text{cut}}$) at hadron level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>Perturbative uncertainty</th>
<th>Nonperturbative uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NNLL’</td>
<td>0.284</td>
<td>± 0.053</td>
<td>± 0.087</td>
</tr>
<tr>
<td>NLL’+ NLO</td>
<td>0.277</td>
<td>± 0.085</td>
<td>± 0.086</td>
</tr>
<tr>
<td>PYTHIA</td>
<td>0.230</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG angular</td>
<td>0.192</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HERWIG dipole</td>
<td>0.103</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.4.: Same as Tab. 6.2 but now for the classifier with a $\tau_{\text{cut}} = 0.15$. 
Chapter 6. Results

Figure 6.10.: The normalized thrust spectrum for the quarks (left) and gluons (right) at NLL (green), NLL' (blue) and NNLL' (red) at $Q = 1$ TeV.

Figure 6.11.: The normalized thrust spectrum for quarks at NNLL' (red) compared to Pythia (purple) and Herwig (green, yellow) for the angular and dipole shower respectively. Shown are results at parton level for $Q = 1$ TeV in the peak region (left) and tail region (right). The band shows the perturbative uncertainty obtained using Eq. (5.8).

6.4. 1 TeV

All the analyses have been done at the hard scale $Q = 125$ GeV. We repeat the analyses at a higher scale of $Q = 1$ TeV to see what happens when we change the scale. In general we find similar results as at $Q = 125$ GeV, see Fig. 6.11 and Fig. 6.12 for the quarks, and Fig. 6.13 for the gluons. For the resummed calculations, the ratio's between the hard, jet and soft scales are relevant as they appear in the evolution kernels, and changing the overall scale does not affect these. The strong coupling constant $\alpha_s$ is however lower now at $Q = 1$ TeV. This means that for our dimensionless variable thrust $\tau$ we expect “less” radiation, so that the peak of the thrust distributions will be shifted towards lower values of $\tau$. We confirm this from the results in Fig. 6.10. This effect is most visible in the thrust distribution for the gluons, which are now more similar to the quark ones. This results in overall lower values of the classifier separation, see Tab. 6.5, Tab. 6.6, Tab. 6.7 and Tab. 6.8.
Figure 6.12.: Same as Fig. 6.11 but at hadron level.

Figure 6.13.: The normalized thrust spectrum for gluons at NLL’+NLO (blue) compared to \textsc{Pythia} (purple) and \textsc{Herwig} (green, yellow) for the angular and dipole shower respectively. Shown are results at parton level (left) and hadron level (right) for $Q = 1$ TeV.

### at parton level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>FO uncertainty</th>
<th>Resummation uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLL’+ NLO</td>
<td>0.202</td>
<td>$\pm 0.016$</td>
<td>$\pm 0.070$</td>
</tr>
<tr>
<td>\textsc{Pythia}</td>
<td>0.225</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>\textsc{Herwig} angular</td>
<td>0.144</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>\textsc{Herwig} dipole</td>
<td>0.125</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.5.: Values of the classifier separation for the analytical calculations at NLL’+ NLO level and the Monte Carlos at parton level with $Q = 1$ TeV. The perturbative fixed-order and resummation uncertainties for the analytical calculations are shown next to the central value.

### at hadron level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>Perturbative uncertainty</th>
<th>Nonperturbative uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLL’+ NLO</td>
<td>0.214</td>
<td>$\pm 0.072$</td>
<td>$\pm 0.018$</td>
</tr>
<tr>
<td>\textsc{Pythia}</td>
<td>0.228</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>\textsc{Herwig} angular</td>
<td>0.162</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>\textsc{Herwig} dipole</td>
<td>0.126</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.6.: Same as Tab. 6.5 but at hadron level.
### $\Delta(\tau < \tau_{\text{cut}})$ at parton level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>FO uncertainty</th>
<th>Resummation uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLL</td>
<td>0.166</td>
<td>± 0.017</td>
<td>± 0.069</td>
</tr>
<tr>
<td>NLL’</td>
<td>0.196</td>
<td>± 0.016</td>
<td>± 0.038</td>
</tr>
<tr>
<td>NNLL’</td>
<td>0.202</td>
<td>± 0.010</td>
<td>± 0.019</td>
</tr>
<tr>
<td>NLL’+ NLO</td>
<td>0.202</td>
<td>± 0.016</td>
<td>± 0.040</td>
</tr>
<tr>
<td>Pythia</td>
<td>0.183</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Herwig angular</td>
<td>0.131</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Herwig dipole</td>
<td>0.106</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.7.: Same as Tab. 6.5, but now for the classifier with a $\tau_{\text{cut}} = 0.15$.

### $\Delta(\tau < \tau_{\text{cut}})$ at hadron level

<table>
<thead>
<tr>
<th></th>
<th>Central value</th>
<th>Perturbative uncertainty</th>
<th>Nonperturbative uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>NNLL’</td>
<td>0.214</td>
<td>± 0.021</td>
<td>± 0.020</td>
</tr>
<tr>
<td>NLL’+ NLO</td>
<td>0.214</td>
<td>± 0.043</td>
<td>± 0.020</td>
</tr>
<tr>
<td>Pythia</td>
<td>0.187</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Herwig angular</td>
<td>0.148</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Herwig dipole</td>
<td>0.107</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.8.: Same as Tab. 6.6, but now for the classifier with a $\tau_{\text{cut}} = 0.15$. 

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Chapter 6. Results
Chapter 7.

Conclusion

In this thesis we have studied the thrust distributions of quark and gluon jets and used these as a discriminant for quark/gluon jets. We have done analytical calculations up to NNLL' and compared the results against the predictions of two Monte Carlo event generators: PYTHIA and HERWIG (for HERWIG we used two different parton showers and two versions, HERWIG 7.0.4 and HERWIG 7.1.0). Although we find similar results as earlier studies [1], as in that PYTHIA is most optimistic about quark/gluon discrimination, HERWIG least optimistic, and the (central) analytical calculation somewhere in between these two, we find that our analytical results are consistent with both Monte Carlos. This is due to the new changes (for gluon jets) in the latest Herwig version, since our earlier results with HERWIG 7.0.4 were not consistent with our analytical predictions. Even though this significant improvement in HERWIG, our predictions still seem to favour PYTHIA a bit more. Our analyses have been done at $Q = 125$ GeV. Having repeated them at $Q = 1$ TeV, we draw similar conclusions there.

We have shown the perturbative convergence from going to NLL order to NNLL' order. The perturbative uncertainty can be reduced by going to even higher orders, but at NNLL' the largest contribution of the total uncertainty already comes from the non-perturbative uncertainty. Improvements on the nonperturbative modelling thus require the most attention.
Appendix A.

Beta-function coefficients, anomalous dimensions and fixed-order coefficients

All the used coefficients for the $\beta$-function, anomalous dimensions and fixed-order coefficients needed up to NNLL' order used in this thesis are given in this section [12, 18].

The running of the strong coupling constant is

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s), \quad (A.1)$$

where $\beta(\alpha_s)$ is the QCD $\beta-$function. Expanding the beta function in powers of $\alpha_s$ we have

$$\beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \quad (A.2)$$

The explicit expression of $\alpha_s$ up to three loops is then given by [37]:

$$\frac{1}{\alpha_s(\mu)} = \frac{X}{\alpha_s(M_Z)} + \frac{\beta_1}{4\pi\beta_0} \log X + \frac{\alpha_s(M_Z)}{16\pi^2 X} \left( \frac{\beta^2_1}{\beta^2_0} - \frac{\beta_2}{\beta_0} \right) (1 - X) + \frac{\beta^2_1}{\beta^2_0} \log X, \quad (A.3)$$

where $X = 1 + \alpha_s(M_Z) \log \left(\frac{\mu}{M_Z}\right)^{\frac{3\beta_0}{\pi}}$ and $\alpha_s(M_Z)\beta_0$ is the coupling at the scale of the mass of the Z-boson. In the calculations of this thesis we use the value $\alpha_s(M_Z) = 0.118$. The used coefficients for the $\beta$-function are [26]:

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f,$$

$$\beta_1 = \frac{34}{3} C_A^2 - \left( \frac{20}{3} C_A + 4C_F \right) T_F n_f,$$

$$\beta_2 = \frac{2857}{54} C_A^3 + \left( C_F^2 - \frac{205}{18} C_F C_A - \frac{1415}{54} C_A^2 \right) 2T_F n_f + \left( \frac{11}{9} C_F + \frac{79}{54} C_A \right) 4T_F^2 n_f^2, \quad (A.4)$$

The cusp anomalous dimension expanded in $\alpha_s$ is

$$\Gamma^i_{\text{cusp}}(\alpha_s) = \sum_{n=0}^{\infty} \Gamma^i_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \quad (A.5)$$
Appendix A. Beta-function coefficients, anomalous dimensions and fixed-order coefficients

The cusp anomalous dimensions for the quarks are [24]

\[
\Gamma_q^0 = 4C_F \\
\Gamma_q^1 = 4C_F \left[ \left( \frac{67}{9} - \frac{\pi^2}{3} \right) C_A - \frac{20}{9} T_F n_f \right] \\
\Gamma_q^2 = 4C_F \left[ \left( \frac{245}{6} - \frac{134\pi^2}{27} + \frac{11\pi^4}{45} + \frac{22\zeta_3}{3} \right) C_A^2 \\
- \left( \frac{418}{27} - \frac{40\pi^2}{27} + \frac{56\zeta_3}{3} \right) C_A T_F n_f - \left( \frac{55}{3} - 16\zeta_3 \right) C_F T_F n_f - \frac{16}{27} T_F^2 n_f^2 \right] 
\]

The cusp anomalous dimensions for the gluons can be obtained from the quark ones by Casimir scaling

\[
\Gamma_g^0 = C_F \Gamma_q^0 \\
\Gamma_g^1 = C_F \Gamma_q^1 \\
\Gamma_g^2 = C_F \Gamma_q^2
\]

The non-cusp anomalous dimension has a same expansion in powers of \(\alpha_s\)

\[
\gamma^i(\alpha_s) = \sum_{n=0}^{\infty} \gamma^i_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1} 
\]

The non-cusp anomalous dimensions for the quarks are [25]

\[
\gamma_{q0}^q = -6C_F, \\
\gamma_{q1}^q = -C_F \left[ \left( \frac{82}{9} - 52\zeta_3 \right) C_A + \left( 3 - 4\pi^2 + 48\zeta_3 \right) C_F + \left( \frac{65}{9} + \pi^2 \right) \beta_0 \right], \\\n\gamma_{q2}^q = 6C_F, \\
\gamma_{j0}^q = -6C_F, \\
\gamma_{j1}^q = C_F \left[ \left( \frac{146}{9} - 80\zeta_3 \right) C_A + \left( 3 - 4\pi^2 + 48\zeta_3 \right) C_F + \left( \frac{121}{9} + \frac{2\pi^2}{3} \right) \beta_0 \right], \\\n\gamma_{s0}^q = 0, \\
\gamma_{s1}^q = C_F \left[ \left( -\frac{128}{9} + 56\zeta_3 \right) C_A + \left( -\frac{112}{9} + \frac{2\pi^2}{3} \right) \beta_0 \right]. 
\]

And for the gluons [25]

\[
\gamma_{c0}^g = -2\beta_0, \\
\gamma_{c1}^g = \left( -\frac{118}{9} + 4\zeta_3 \right) C_A^2 + \left( \frac{38}{9} + \frac{\pi^2}{3} \right) C_A \beta_0 - 2\beta_1, \\\n\gamma_{j0}^g = 2\beta_0, \\
\gamma_{j1}^g = \left( \frac{182}{9} - 32\zeta_3 \right) C_A^2 + \left( \frac{94}{9} - \frac{2\pi^2}{3} \right) C_A \beta_0 - 2\beta_1, \\\n\gamma_{s0}^g = 0, \\\n\gamma_{s1}^g = C_A \left[ \left( -\frac{128}{9} + 56\zeta_3 \right) C_A + \left( -\frac{112}{9} + \frac{2\pi^2}{3} \right) \beta_0 \right]. 
\]
Note that the soft non-cusp anomalous dimensions of the quarks and gluons have Casimir scaling and that indeed $2\gamma_C^q(\alpha_s) + 2\gamma_j^q(\alpha_s) + \gamma_S^q(\alpha_s) = 0$ as required by the $\mu$-independence of the cross-section.

Explicit expressions of the quark and gluon Wilson coefficients, jet and soft functions up to two-loop order, expressed in their anomalous dimensions are given below:

\[
C_q(Q, \mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} \left[ - \frac{\Gamma_0^q}{4} L^2 - \frac{\gamma_C^q}{2} L + \gamma_0^q \right] + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left[ \frac{(\Gamma_0^q)^2}{32} L^4 + \frac{\Gamma_0^q(3\gamma_C^q + 2\beta_0)}{24} L^3 ight. \\
+ \left. \frac{(\gamma_C^q)^2}{8} + 2\beta_0\gamma_C^q - 2\Gamma_1^q - 2\Gamma_0^q\gamma_C^q L^2 - \gamma_C^2 + \gamma_C^q\gamma_0^q + 2\beta_0\gamma_0^q L + c_i^q \right] \quad (A.11a)
\]

\[
C_g(Q, \mu) = \alpha_s \left[ 1 + \frac{\alpha_s(\mu)}{4\pi} \left[ - \frac{\Gamma_0^g}{4} L^2 - \frac{\gamma_C^g}{2} L + \gamma_0^g \right] + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left( \frac{(\Gamma_0^g)^2}{32} L^4 ight. \\
+ \left. \frac{(\gamma_C^g)^2}{8} + 6\beta_0\gamma_C^g + 8\beta_1^g - 2\Gamma_1^g - 2\Gamma_0^g\gamma_C^g L^2 \right) \right] \quad (A.11b)
\]

\[
J_i(s, \mu) = \delta(s) + \frac{\alpha_s(\mu)}{4\pi} \left[ \frac{\Gamma_0^g}{\mu^2} L_1 \left( \frac{s}{\mu^2} \right) - \frac{\gamma_j^i}{2\mu^2} L_0 \left( \frac{s}{\mu^2} \right) + j_i^j(\delta(s)) \right] + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left( \frac{(\Gamma_0^g)^2}{6} + \Gamma_1^g \right) \frac{1}{\mu^2} L_1 \left( \frac{s}{\mu^2} \right) \\
- \left( - j_0^i \left( \beta_0 + \frac{\gamma_j^i}{2} \right) + \frac{\pi^2}{12} + (\Gamma_0^g)^2 \frac{1}{\mu^2} L_0 \left( \frac{s}{\mu^2} \right) \right) + j_i^j(\delta(s)) \quad (A.11c)
\]

\[
S_i(k, \mu) = \delta(k) + \frac{\alpha_s(\mu)}{4\pi} \left[ \frac{\Gamma_0^g}{\mu^2} L_1 \left( \frac{k}{\mu} \right) - \frac{\gamma_S^i}{2} L_0 \left( \frac{k}{\mu} \right) + s_i^0 \delta(k) \right] + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left( \Gamma_0^g L_1 \left( \frac{k}{\mu} \right) + \Gamma_1^g \frac{1}{\mu} \right) \frac{1}{\mu^2} L_1 \left( \frac{k}{\mu} \right) \\
- \left( -2\beta_0\gamma_S^i + 2\beta_0\gamma_S^i + (\gamma_S^i)^2 - \frac{8\pi^2}{3} + 4\Gamma_1^g \right) \frac{1}{\mu^2} L_0 \left( \frac{k}{\mu} \right) + s_i^0 \delta(k) \quad (A.11d)
\]

The constants appearing in the hard, jet and soft function, which are not obtained by the anomalous dimensions from the RGE-equations are given here [12, 22]:

\[
c_i^q = \left( -8 + \frac{\pi^2}{6} \right) C_F \quad (A.12a)
\]

\[
c_i^g = \left( \frac{255}{8} + \frac{7\pi^2}{2} - 30\zeta_3 - \frac{83\pi^4}{360} \right) C_F^2 + \left( \frac{251157}{648} - \frac{337\pi^2}{108} + \frac{313\zeta_3}{9} + \frac{11\pi^4}{45} \right) C_F C_A \\
+ \left( \frac{4085}{162} + \frac{23\pi^2}{27} + \frac{4\zeta_3}{9} \right) C_F T_F n_f \quad (A.12b)
\]
Appendix A. Beta-function coefficients, anomalous dimensions and fixed-order coefficients

\( c_0^g = \left(5 + \frac{\pi^2}{6}\right)C_A - 3C_F \)  \hspace{1cm} (A.12b)

\[
c_1^g = (7C_A^2 + 11C_A C_F - 6C_F \beta_0) \ln \left(\frac{-q^2 - i0}{m_t^2}\right) + \left(-\frac{419}{27} + \frac{7\pi^2}{6} + \frac{\pi^4}{72} - 44\zeta_3\right)C_A^2 \\
+ \left(-\frac{217}{2} - \frac{\pi^2}{2} + 44\zeta_3\right)C_A C_F + \left(\frac{2255}{108} + \frac{5\pi^2}{12} + \frac{23\zeta_3}{3}\right)C_A \beta_0 - \frac{5}{6}C_A T_F + \frac{27}{2}C_F^2 \\
+ \left(\frac{41}{2} - 12\zeta_3\right)C_F \beta_0 - \frac{4}{3}C_F T_F + \mathcal{O}\left(\frac{q^2}{4m_t^2}\right)
\]

\( j_0^g = (7 - \pi^2)C_F \)  \hspace{1cm} (A.12c)

\[
j_1^g = \left(\frac{205}{8} - \frac{67\pi^2}{6} + \frac{14\pi^4}{15} - 18\zeta_3\right)C_F^2 + \left(\frac{53129}{648} - \frac{208\pi^2}{27} - \frac{17\pi^4}{180} - \frac{206\zeta_3}{9}\right)C_F C_A \\
+ \left(\frac{4057}{162} + \frac{68\pi^2}{27} + \frac{16\zeta_3}{9}\right)C_F T_F n_f
\]

\( j_0^g = \left(\frac{4}{3} - \pi^2\right)C_A + \frac{5}{3}\beta_0 \)  \hspace{1cm} (A.12d)

\[
j_1^g = \left(\frac{4255}{108} - \frac{26\pi^2}{9} + \frac{151\pi^4}{180} - 72\zeta_3\right)C_A^2 - \left(\frac{115}{108} + \frac{65\pi^2}{18} - \frac{56\zeta_3}{3}\right)C_A \beta_0 \\
- \left(\frac{25}{9} - \frac{\pi^2}{3}\right)\beta_0^2 + \left(\frac{55}{12} - 4\zeta_3\right)\beta_1
\]

\( s_0^g = \frac{\pi^2}{3}C_F \)  \hspace{1cm} (A.12e)

\[
s_1^g = \left(-\frac{3\pi^4}{10} + \frac{14\pi^2}{27} + \frac{22\pi^4}{45}\right)C_F C_A + \left(\frac{-20}{27} - \frac{37\pi^2}{18} + \frac{58\zeta_3}{3}\right)C_F \beta_0
\]

The fixed-order coefficients for the gluon soft function are readily obtained from the quark ones by the replacement \( C_F \to C_A \).
Appendix B.

Plus distribution identities

The used plus distributions appearing in the jet and soft functions in momentum space are defined by (for \(n \geq 0\)) \[38\]:

\[
\mathcal{L}_n(x) = \left[ \frac{\theta(x) \log^n x}{x} \right]_+ = \lim_{\epsilon \to 0} \frac{d}{dx} \left[ \theta(x - \epsilon) \frac{\log^{n+1} x}{n + 1} \right],
\]

where \(x\) is a dimensionless variable. These plus distributions appear for example in the calculation of the jet function where we take the imaginary part of a forward scattering graph. A few identities are

\[
\frac{1}{\pi} \text{Im} \left[ \frac{1}{x + i0} \right] = -\delta(x) \quad (B.2)
\]
\[
\frac{1}{\pi} \text{Im} \left[ \frac{1}{(x + i0)^2} \right] = \delta'(x)
\]
\[
\frac{1}{\pi} \text{Im} \left[ \log(-x - i0) \right] = -\mathcal{L}_0(x)
\]
\[
\frac{1}{\pi} \text{Im} \left[ \log^2(-x - i0) \right] = \frac{\pi^2}{3} \delta(x) - 2\mathcal{L}_1(x).
\]

The plus distributions integrate to:

\[
\int_0^{x_{\text{cut}}} dx \mathcal{L}_n(x) = \frac{\log^{n+1} x_{\text{cut}}}{n + 1} \quad (B.3)
\]

The plus distribution appearing in the evolution factor is given by:

\[
\mathcal{L}^a(x) = \left[ \frac{\theta(x)}{x^{1-a}} \right]_+ = \lim_{\epsilon \to 0} \frac{d}{dx} \left[ \theta(x - \epsilon) \frac{x^a - 1}{a} \right] \quad (B.4)
\]

These two different distributions are related to each other by:

\[
\mathcal{L}_n(x) = \frac{d^n}{da^n} \mathcal{L}^a(x) \bigg|_{a=0} \quad (B.5)
\]

The mixed distributions \(\mathcal{L}_n^a(x)\) are defined as:

\[
\mathcal{L}_n^a(x) = \left[ \frac{\theta(x) \log^n x}{x^{1-a}} \right]_+ = \frac{d^n}{db^n} \mathcal{L}^{a+b}(x) \bigg|_{b=0}, \quad (B.6)
\]
which satisfy $\mathcal{L}_n^0(x) = \mathcal{L}_n(x)$ and $\mathcal{L}_0^a(x) = \mathcal{L}_a(x)$. It is also convenient for the following formulas to define $\mathcal{L}_{-1}(x) = \mathcal{L}_{a-1}(x) = \delta(x)$. In the calculation of the thrust distribution the following rescaling identities were used:

$$\lambda \mathcal{L}_a^a(\lambda x) = \lambda^a \mathcal{L}_a^a(x) + \frac{\lambda^a - 1}{a} \delta(x)$$ (B.7)

$$\lambda \mathcal{L}_n^a(\lambda x) = \sum_{k=0}^{n} \binom{n}{k} \log^k \lambda \mathcal{L}_{n-k}(x) + \frac{\log^{n+1} \lambda}{n + 1} \delta(x)$$ (B.8)

We also needed to calculate convolutions between the plus distributions:

$$\int dy \mathcal{L}_a^a(x - y) \mathcal{L}_n(y) = \frac{1}{a} \sum_{k=-1}^{n+1} V_k^n(a) \mathcal{L}_k^a(x) - \frac{1}{a} \mathcal{L}_n(x)$$ (B.9)

$$\int dy \mathcal{L}_m^a(x - y) \mathcal{L}_n(y) = \sum_{k=-1}^{m+n+1} V_k^{mn}(a) \mathcal{L}_k(x),$$ (B.10)

where the coefficients $V_k^n(a)$ and $V_k^{mn}$ are given by:

$$V_k^n(a) = \begin{cases} a \frac{d}{da} V(a,b) \bigg|_{b=0} & k = -1 \\ \bar{a}(n) \frac{d^{n-k}}{da^{n-k}} V(a,b) \bigg|_{b=0} + \delta_{kn} & 0 \leq k \leq n \\ \frac{a}{n+1} & k = n + 1 \end{cases}$$ (B.11)

$$V_k^{mn} = \begin{cases} a \frac{d}{da} \frac{d}{db} V(a,b) \bigg|_{a=b=0} & k = -1 \\ \sum_{p=0}^{m} \sum_{q=0}^{n} \delta_{p+q,k} \binom{m}{p} \frac{d^{m-p}}{da^{m-p}} \frac{d^{n-q}}{db^{n-q}} V(a,b) \bigg|_{a=b=0} & 0 \leq k \leq m + n \\ \frac{1}{m+1} + \frac{1}{n+1} & k = m + n + 1 \end{cases}$$ (B.12)

with $V(a, b)$ defined as:

$$V(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} - \frac{1}{a} - \frac{1}{b}$$ (B.13)
Appendix C.

General RGE solutions

In this appendix we give the general solution to the multiplicative renormalization group equation and the RGE with a convolution, which appear in the RGE for the hard, jet and soft functions \cite{12}.

C.1. Multiplicative RGE

For the hard function we have to solve a RGE which is a multiplicative equation of the form

$$\mu \frac{d}{d\mu} F(t, \mu) = \gamma_F(t, \mu) F(t, \mu), \quad \gamma_F(t, \mu) = \frac{\rho_F}{\Gamma_{\text{cusp}}[\alpha_s]} \log \left( \frac{\mu^j}{t} \right) + \gamma_F[\alpha_s], \quad (C.1)$$

where $t$ is a variable of mass dimension $j$ and $\rho_F$ is a constant depending on the function $F$. Rearranging Eq. (C.1) we have:

$$\frac{d}{d\log \mu} \log F(t, \mu) = \frac{\rho_F}{j} \Gamma_{\text{cusp}}[\alpha_s] \log \frac{\mu^j}{t} + \gamma_F[\alpha_s] \quad (C.2)$$

This equation can be solved by integrating it from $\mu_0$ to $\mu$ by doing a change of variables to $\alpha_s$, using Eq. (A.1): $d\log \mu = d\alpha_s/\beta(\alpha_s)$:

$$\log F(t, \mu) = \frac{\rho_F}{j} \Gamma_{\text{cusp}}[\alpha_s] \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \log \frac{\mu^j}{t} + \gamma_F[\alpha_s] \quad (C.3)$$

Writing $\log(\mu^j/t) = \log (\mu^j/\mu_0^j) + \log (\mu_0^j/t)$ and using the fact that

$$\log \left( \frac{\mu}{\mu_0} \right) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha'}{\beta(\alpha'_s)}, \quad (C.4)$$

we arrive at:

$$\log \frac{F(t, \mu)}{F(t, \mu_0)} = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \left[ \Gamma[\alpha_s] \left( \frac{\rho_F}{j} \log \frac{\mu_0^j}{t} + \rho_F \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha'_s}{\beta(\alpha'_s)} \right) + \gamma_F[\alpha_s] \right] \quad (C.5)$$

$$= \omega_F(\mu, \mu_0) \log \frac{\mu_0^j}{t} + K_F(\mu, \mu_0).$$
where we have defined

$$\omega_F(\mu, \mu_0) = \frac{\rho_F}{J} \eta(\mu, \mu_0), \quad K_F(\mu, \mu_0) = \rho_F K(\mu, \mu_0) + K_{\gamma_F}(\mu, \mu_0),$$  \hspace{2cm} (C.6)

with

$$\eta(\mu, \mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma_{\text{cusp}}[\alpha_s]$$  \hspace{2cm} (C.7)

$$K(\mu, \mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \gamma_F[\alpha_s] \int_{\alpha_s(\mu_0)}^{\alpha_s} \frac{d\alpha_s'}{\beta(\alpha_s')}$$

$$K_{\gamma_F}(\mu, \mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \gamma_F[\alpha_s]$$

Exponentiating Eq. (C.5) we have now expressed the solution in terms of an evolution kernel and a boundary condition:

$$F(t, \mu) = U_F(t, \mu, \mu_0) F(t, \mu_0), \quad U_F(t, \mu, \mu_0) = e^{K(\mu, \mu_0)} \left( \frac{\mu_0^2}{t} \right)^{\omega_F(\mu, \mu_0)}$$  \hspace{2cm} (C.8)

Using the expansions in $\alpha_s$ for $\beta[\alpha_s], \Gamma_{\text{cusp}}[\alpha_s]$ and $\gamma_F[\alpha_s]$ in Eqs. (A.2) and (A.5) and Eq. (A.8), the $\eta(\mu, \mu_0), K(\mu, \mu_0)$ and $K_{\gamma_F}(\mu, \mu_0)$ can be solved order by order, giving a solution that is correct at each order to all orders in perturbation theory. Up to NNLL order they are [12]:

$$\eta(\mu, \mu_0) = -\frac{\Gamma_0}{\beta_0}\left[ \log(r) + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\Gamma_1}{\beta_0} - \frac{\beta_1^2}{\beta_0} \right) (r - 1) \right]$$

$$+ \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left( \frac{\beta_1^2}{\beta_0} - \frac{\beta_2}{\beta_0} + \frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1}{\Gamma_0} \right) \frac{r^2 - 1}{2}$$  \hspace{2cm} (C.9a)

$$K(\mu, \mu_0) = -\frac{\Gamma_0}{4\beta_0^2} \left[ \frac{4\pi}{\alpha_s(\mu_0)} \left( 1 - \frac{1}{r} - \log r \right) + \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (1 - r + \log r) \right]$$

$$+ \frac{\beta_1}{2\beta_0} \log^2 r + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\beta_1^2}{\beta_0} - \frac{\beta_2}{\beta_0} \right) \left( \frac{1 - r^2}{2} + \log r \right)$$

$$+ \left( \frac{\beta_1 \Gamma_1}{\beta_0 \Gamma_0} - \frac{\beta_1^2}{\beta_0^2} \right) (1 - r + r \log r) - \left( \frac{\Gamma_2}{\Gamma_0} - \frac{\beta_1 \Gamma_1}{\beta_0 \Gamma_0} \right) \frac{(1 - r)^2}{2} \right]$$  \hspace{2cm} (C.9b)

$$K_{\gamma_F}(\mu, \mu_0) = -\frac{\gamma_0}{2\beta_0} \left[ \log r + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1) \right]$$  \hspace{2cm} (C.9c)
Appendix C. General RGE solutions

where \( r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \). For the Wilson coefficients of the two quark and gluon operators in SCET the RGE-equation is of this form, so for the hard function we have \( \rho_H = -2 \) and \( t = Q^2 \) and find for the hard function evolution kernel

\[
U_H(Q, \mu, \mu_0) = \left| e^{-2K_F(\mu, \mu_0) + K_\gamma(\mu, \mu_0)} \left( \frac{-Q^2 - i0}{\mu_0^2} \right)^{n_H(\mu, \mu_0)} \right|^2 \tag{C.10}
\]

C.2. RGE with a convolution

For the RGE of the jet and soft function we have to solve the following general anomalous dimension equation which involves a convolution:

\[
\mu \frac{d}{d\mu} F(t, \mu) = \int dt' \gamma_F(t - t', \mu) F(t', \mu), \tag{C.11}
\]

where the anomalous dimension has the form:

\[
\gamma_F(t, \mu) = -\rho_F \Gamma_{cusp}^{\alpha_s} \frac{1}{j} \frac{1}{\mu^j} \mathcal{L}_0 \left( \frac{t}{\mu^j} \right) + \gamma_F^{\alpha_s}[\delta(t)]. \tag{C.12}
\]

Here \( F = J, S \) and \( t, t' \) are again variables of mass-dimension \( j \) and \( \rho_F \) a constant depending on \( F \). For \( F = J \) we have \( j = 2, \rho_F = 4 \) and for \( F = S \) we have \( j = 1, \rho_F = -4 \). This equation can be solved by going to Laplace space, where the RGE with a convolution then turns into a multiplicative RGE. Defining the Laplace transformed functions by

\[
\tilde{F}(\nu, \mu) = \int_0^\infty dt e^{-\nu t} F(t, \mu),
\]

we get:

\[
\mu \frac{d}{d\mu} \tilde{F}(\nu, \mu) = \tilde{\gamma}_F(\nu, \mu) \tilde{F}(\nu, \mu), \tag{C.13}
\]

where

\[
\tilde{\gamma}_F(\nu, \mu) = \frac{\rho_F \Gamma_{cusp}^{\alpha_s}}{j} \log(\nu \mu e^{\gamma_E}) + \gamma_F^{\alpha_s}[\delta(t)]. \tag{C.14}
\]

This equation now has the same form as the multiplicative RGE of the Wilson coefficients, so we have found the solution in Laplace space:

\[
\tilde{F}(\nu, \mu) = U_{\tilde{F}}(\nu, \mu, \mu_0) \tilde{F}(\nu, \mu_0), \quad U_{\tilde{F}}(\nu, \mu, \mu_0) = e^{K_F(\mu, \mu_0)}(\nu \mu_0^j e^{\gamma_E})^{\omega_F(\mu, \mu_0)} \tag{C.15}
\]

To get the solution in \( t \)-space, we take the inverse Laplace transform:

\[
F(t, \mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\nu e^{\nu t} \left[ e^{K_F(\mu, \mu_0)}(\nu \mu_0^j e^{\gamma_E})^{\omega_F(\mu, \mu_0)} \tilde{F}(\nu, \mu_0) \right] \tag{C.16}
\]

Writing \( \tilde{F}(\nu, \mu_0) \) as the Laplace transform of \( F(t', \mu_0) \) and rearranging gives:

\[
e^{K_F(\mu, \mu_0)} \int_0^\infty dt' F(t', \mu_0) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\nu e^{\nu(t-t')} (\nu \mu_0^j e^{\gamma_E})^{\omega_F(\mu, \mu_0)} \tag{C.17}
\]
Using the inverse Laplace transform identity Eq. (D.2) we arrive at the solution in $t$ space:

$$F(t, \mu) = \int_0^\infty dt' U_F(t - t', \mu, \mu_0) F(t', \mu_0),$$

(C.19)

where the evolution kernel is given by

$$U_F(t, \mu, \mu_0) = \frac{e^{\gamma \omega_F(\mu, \mu_0)} e^{K_F(\mu, \mu_0)}}{\Gamma(-\omega_F(\mu, \mu_0))} \left[ \frac{1}{\mu^3} L^{-\omega_F(\mu, \mu_0)} \left( \frac{t}{\mu^3} \right) - \frac{1}{\omega_F(\mu, \mu_0)} \delta(t) \right].$$

(C.20)
Appendix D.

Laplace transforms

Below are given the Laplace transforms of the plus distributions $L_n$ which occur in the jet and soft function up to two-loop order:

\[
\int_0^\infty dt e^{-\nu t} \frac{1}{\mu^j} L_{-1} \left( \frac{t}{\mu^j} \right) = 1 \\
\int_0^\infty dt e^{-\nu t} \frac{1}{\mu^j} L_0 \left( \frac{t}{\mu^j} \right) = - \log \left( \nu e^{\gamma_E} \mu^j \right) \\
\int_0^\infty dt e^{-\nu t} \frac{1}{\mu^j} L_1 \left( \frac{t}{\mu^j} \right) = \frac{\log^2(\nu e^{\gamma_E} \mu^j)}{2} + \frac{\pi^2}{12} \\
\int_0^\infty dt e^{-\nu t} \frac{1}{\mu^j} L_2 \left( \frac{t}{\mu^j} \right) = - \frac{\log^3(\nu e^{\gamma_E} \mu^j)}{6} - \frac{\pi^2}{6} \log(\nu e^{\gamma_E} \mu^j) - \frac{2 \zeta_3}{3} \\
\int_0^\infty dt e^{-\nu t} \frac{1}{\mu^j} L_3 \left( \frac{t}{\mu^j} \right) = \frac{\log^4(\nu e^{\gamma_E} \mu^j)}{4} + \frac{\pi^2}{4} \log^2(\nu e^{\gamma_E} \mu^j) + 2 \zeta_3 \log(\nu e^{\gamma_E} \mu^j) + \frac{3 \pi^4}{80}
\]

(D.1)

The general inverse Laplace transform identity is

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dv e^{vt}(v \mu^j_0 e^{\gamma_E})^{-a} = \frac{1}{\Gamma(a)} \left[ \frac{1}{\mu^j} L^a \left( \frac{t}{\mu^j} \right) + \frac{1}{a} \delta(t) \right].
\]

(D.2)
Appendix E.

QCD gauge group

SU(3) is the gauge group of QCD. The special unitary group SU(N) consists of all unitary matrices which have determinant 1. It has $N^2$ complex entries, thus $2N^2$ real parameters. But the constraints of a special unitary matrix set the number of independent real parameters to $N^2 - 1$. SU(N) is also a $(N^2 - 1)$-dimensional manifold and thus a Lie group. The generators of SU(N) are Hermitian and traceless matrices. These generators form a vector space $su(N)$. We call a basis of $N^2 - 1$ linear independent Hermitian traceless matrices $\{T^a\}$, so $a$ goes from 1 to $N^2 - 1$. They form a Lie algebra:

$$[T^a, T^b] = if^{abc}T^c,$$

(E.1)

where $f^{abc}$ are the structure constants. The structure constants are real and antisymmetric, which can be easily shown using that the generators are Hermitian and the anti-commutativity of the bracket.

In the fundamental representation of the Lie algebra of $SU(N)$, the generators are given by $T^a = \frac{1}{2}\lambda_a$, where $\lambda_a$ are the Gell-Mann matrices. The Casimir operator in the fundamental representation, $C_F$, is given by

$$C_F\delta_{ij} = (T^a T^a)_{ij} = T^a_{ik} T^a_{kj} = \frac{1}{2} \left( \delta_{ij} \delta_{kk} - \frac{1}{N} \delta_{ik} \delta_{kj} \right) = \frac{1}{2} \left( \delta_{ij} N - \frac{1}{N} \delta_{ij} \right) = \delta_{ij} \frac{N^2 - 1}{N},$$

(E.2)

In the adjoint representation the generators are defined by the structure constants: $(F^a)_{bc} = -if^{abc}$. These $F^a$ matrices are $(N^2 - 1)$ by $(N^2 - 1)$ matrices. The Casimir operator in the adjoint representation is given by:

$$\delta_{ab} C_A = (F^c F^c)_{ab} = F^c_{ad} F^c_{db} = -f^{cad} f^{cdd} = f^{acd} f^{bcd} = \delta_{ab} N.$$

(E.3)

So for $SU(3)$, the gauge group of QCD, we find $C_F = \frac{4}{3}$ and $C_A = 3$. 

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Appendix F.

Helicity spinor amplitudes

In this thesis we use the same notation and conventions for the helicity amplitudes as in [30]. The spinors are denoted as

\[ \langle p^\pm \rangle = \frac{1 \pm \gamma^5}{2} u(p), \quad |p^\pm \rangle = \text{sgn}(p^0) \bar{u}(p) \frac{1 \mp \gamma^5}{2}. \quad (F.1) \]

The spinor products are:

\[ \langle pq \rangle = \langle p^-|q^+ \rangle, \quad [pq] = \langle p^+|q^- \rangle. \quad (F.2) \]

For absolute squaring the helicity amplitudes, we use the identity:

\[ \langle pq \rangle [qp] = 2p \cdot q, \quad (F.3) \]

and the shorthand notation:

\[ \langle ij \rangle \equiv \langle pi|pj \rangle, \quad [ij] \equiv [pi|pj]. \quad (F.4) \]

We also use:

\[ s_{ij} \equiv (p_i + p_j)^2. \quad (F.5) \]

For massless particles, we have \( s_{ij} = 2p_i \cdot p_j \). Parity transformations have the effect of \( \langle .. \rangle \leftrightarrow [..] \) on the amplitudes. But since we are always squaring amplitudes, we can effectively multiply by a factor of 2 and forget about the parity related amplitudes, since a parity transformed amplitude gives the same result when you square it. The helicity amplitudes written down in this thesis actually also have a phase (see [30] for the full expression), but since we are only interested in squared amplitudes and the helicity amplitudes correspond to distinct final states, we have no interference and the phase does not matter.
Bibliography


