Clustered states
in the fractional quantum Hall effect

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In this thesis we construct a new class of non-Abelian quantum Hall states through a generalization of well-known clustered states, the Read-Rezayi series. By assigning a different parafermion field to the electron operator the corresponding electric charge part changes accordingly. A second generalization is obtained by assigning multiple parafermions to the electron operator, which effectively means one takes multiple copies of parafermionic states. The new states are clustered and their clustering properties are explored using a recently developed classification scheme, called a pattern of zeros. We provide explicit formulas for these patterns of zeros, which we derive using CFT. As an extra insight, we also derive the ground-state degeneracy on the torus by counting the number of quasi-hole sectors.

Apart from these new results, this thesis also contains a survey of the fractional quantum Hall effect and its relation to conformal field theory. Starting from the very basics, we repeat the argument to obtain the Laughlin wavefunction and the corresponding description of the quasiholes which are interpreted as magnetic flux insertions. We then move on to a field-theoretic description of the FQHE, based on topological quantum field theory, and explicitly show how a conformal field theory develops on the edge upon quantization. In turn, this CFT is used to obtain expressions for the quantum Hall wavefunctions in terms of chiral correlators. All these steps are done explicitly for the Laughlin series. The relevance of CFT to the FQHE is further explored in the context of non-Abelian statistics.

The survey ends with a description of the Read-Rezayi series, clustered states which are based on parafermionic CFT’s. We explore some of the non-Abelian properties of the Moore-Read state, which is the simplest, non-trivial example of the Read-Rezayi series.
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CHAPTER 1

Introduction

With the discovery of the integer quantum Hall effect in 1980 Klaus von Klitzing [Klitzing et al., 1980] unveiled a new domain in physics which, today, still remains an active area of modern-day research. Their observations included measurements of the electrical Hall conductance which classically correspond to a linear relation with the strength of the applied magnetic field. Instead, what they found is that for certain ranges of the magnetic field strength the Hall conductance does not change. By plotting the conductance versus the magnetic field strength this effect manifests itself as a plateau. The value of the quantized conductance at a plateau is given by an integer number times the fundamental unit of conductance, $\frac{N}{e^2}$. This quickly led to the designation of the effect as the integer quantum Hall effect. As the years progressed improved experiments were performed and the measured quantization reached astonishing precisions. The accuracy in which the system stabilizes its conductance are within error margins of less than $10^{-8}$. This remarkable effect could not be explained in terms of the existing condensed matter theory at the time.

Theorists quickly went to work on modelling this radical new behavior of electrons. Old results were cast in new perspectives and great successes were booked in understanding the integer effect, when out of the blue in 1982 a group of Bell laboratories announced the discovery of quantized Hall conductances where this time the value took on fractional values, $\frac{p}{q} \frac{e^2}{h}$. The fractional quantum Hall effect was realized, [Tsui et al., 1982].

Again, the discovery lead to great amazement among the physics community. The irony of it all was that the success of the models in explaining and understanding the integer effect cannot possibly account for the observations made in the fractional case. The integer effect is largely realized through an interplay of the strong magnetic field and the presence of impurities and defects in the sample. In particular, they lead to the simultaneous existence of extended and localized electron states, and the presence of a mobility gap which arranges the extended states in discrete sets. Impurities therefore tend to enhance the stability of the plateaux. But
the samples in which the fractional effect is realized are extremely clean, and experiments show that impurities rather destroy the fractional effect in the sense that the plateaux are limited to small ranges of the magnetic field strength. As it turns out, the crucial mechanism behind the fractional effect is the Coulomb interaction among the electrons.

The first major step in an understanding of the physics at work in the fractional case was made by Laughlin in 1983. Through an incredible intuitive and compelling train of thought he argued a simple form of an approximate wavefunction for certain cases of the quantized conductivity. The immense precision of his prediction assured him the Nobel prize of 1998. In particular, he presented a clear picture of the $\nu = \frac{1}{\text{odd}}$ cases. This included the prediction of excitations carrying fractional charge and obeying fractional statistics. Such particles are dubbed anyons. The wavefunction serves as a cornerstone for all further developed models that appeared over the course of years.

Quite remarkably, the majority of the fractional cases correspond to $\nu = \frac{p}{q}$ where $q$ is an odd integer. One of the most prominent is the $\nu = \frac{1}{3}$, the first member of the Laughlin sequence. For many years there were no observations that correspond to even denominators. However, in 1987 the first hints appeared of an even denominator plateau, at a filling fraction $\nu = \frac{5}{2}$. In 1999 this plateau was unmistakably confirmed. The puzzle which this state brought along is that it does not seem to fit within the theoretical framework that worked so well to explain the odd-denominator cases.

However, a seminal paper published in 1991 by Moore and Read, took on quite a different approach on describing quantum Hall states. In this paper the author proposed to apply conformal field theory as a theoretical tool to probe the long-range physics present in the samples. It was in that same paper that the authors demonstrated an unorthodox procedure to obtain a new series of quantum Hall states. Although that series was proposed as an academic example, it in fact included a description of the $\frac{1}{2}$ state.

What makes the Moore-Read state so special is its prediction of the corresponding excitations. Similar to the case of the Laughlin states, these excitation are predicted to carry
a fractional charge. But the statistics they obey is a radical new one, called non-Abelian statistics. In the past decade it became clear that systems which display such exotic behavior may very well be prime candidates for the building blocks of a topological quantum computer. These are conjectured to remain relatively immune to decoherence, turning their discovery into a 21st-century quest for the holy grail. Today’s experiments have not been succesful in verifying this new form of statistics. But it is certainly not ruled out either.

This thesis is essentially comprised of two parts. The first part consists out of chapters 2 through 6 and is a fairly large introduction into the fractional quantum Hall effect. In the second part, chapter 7 through 9, we present new material which has only recently been introduced. This part contains a number of new results, not present in the literature.

In chapter 2 we start off with a basic introduction in the quantum Hall effect. The goal is an intuitive understanding of the effect and the formation of plateaux. Chapter 3 is a self-contained introduction, but from a different point of view. In this chapter we aim at a firm understanding of the electron wavefunctions, describing both single- and multiparticle wavefunctions. We also present the argument by Laughlin and the associated wavefunctions and excitations. All these results originate from work done in the early 1980’s.

In chapter 4 we take a jump of roughly 10 years, and find ourself in work done in the early 1990’s. It is a field-theoretic description of the quantum Hall effect, aimed at the understanding of the long-range physics of such samples. We explain how Chern-Simons theory, which is a particular topological quantum field theory, serves as an effective theory of quantum Hall systems, and in term also predicts quantization of the Hall conductance, excitations obeying fractional statistics, and more. This description illustrates the topological nature of the quantum Hall systems. In particular, we describe its relation to the mathematical tool of conformal field theory. The application of CFT to the quantum Hall effect is a miraculous one, which will be clarified in chapter 5. It is essentially this relation which was conjectured by Moore and Read.

Then finally in chapter 6 we arrive at what this thesis aimed for. An understanding of quantum Hall states on the basis of techniques provided by conformal field theory. At
first, this chapter will serve as a demonstration of a certain class of quantum Hall states, the Read-Rezayi states. These states are frequently referred to as clustered quantum Hall states, and they contain the Moore-Read series as subclass. Similar to the Moore-Read states, the Read-Rezayi states also predicts the existance of non-Abelian excitations. Together with the clustering property, which is interpreted as a Coulomb-like interaction, are these the reasons which makes these states so appealing.

The second part of this thesis starts in chapter 7 with a proposal to generalize the Read-Rezayi states in quite a natural way. This generalizations was hinted earlier this year through a classification scheme proposed by Wen and Wang. Based on numerics they showed these states exists, and obey a generalized form of clustering. This complete chapter is new. We present how the states are constructed through CFT, derive the smallest-charge quasiholes and present a possible connection to an existing state called the Haffnian.

This brings us to chapter 8 in which the paper by Wen and Wang is explained. We introduce the notion of a pattern of zeros, which can be used to classify clustered quantum Hall states. The chapter starts off with a summary of the referred paper. But we quickly move on, and approach the situation from a different side. We will give a full description of how CFT can be used to obtain the pattern of zeros of the corresponding state. We will explicitly do this for the new states that we have found. This result is new, with formula (8.97) serving as the prime result of that chapter. This formula is an expression which reproduces a large part of the results obtained by Wen and Wang. However, their results have been obtained numerically as their approach is essentially different.

Finally, chapter 9 is devoted to a lengthy calculation in which we determine the quasihole spectrum of the generalized states. The calculation does not appear in the literature, although the result does match numerical obtained formulas.
A good approach to a new domain of physics always starts with the basics. This chapter is therefore devoted to an elementary introduction of the realm of the quantum Hall effect. We start off with a classical description of the motion of an electron in a two-dimensional plane in the presence of a magnetic field, quickly followed by a quantummechanical approach of the same physical system. By directly solving the Hamiltonian we recover the single-particle eigenstates in the absence of an external potential. Next, we turn on an electric field and via both a classical and quantummechanical sketch end up with a naive understanding of the quantized value of the electrical conductance. This picture is incomplete, however, as the idealized Hamiltonian does not take into account effects of impurities and Coulomb interaction. As it turns out both effects are essential for the formation of plateaux. The remainder of the chapter is dedicated to an intuitive understanding of the quantization and the forming of quantum Hall plateaux for both integer and fractional quantum Hall states.

2.1 Experimental considerations

Before we dive in to a theoretical description of the quantum Hall effect, we first explain the very basic observations and principles that come along the effect.

We deal with a two-dimensional system of electrons. This is achieved, for example, through an interface of layered systems, a metal and a semiconductor separated by a thin oxide layer. Such a material, called Si MOSFET, was used by [Klitzing et al., 1980]. Nowadays, the system is realized by heterojunction comprised of GaAs-AlGaAs [Yoshioka, 2002]. The GaAs has a crystalline structure which is grown layer by layer. During this process one can change the composition such that the compound changes into Al$_x$Ga$_{1-x}$As. The plane separating the two compounds admits a realization of electrons confined to a two-dimensional plane. The
effects of impurities are weaker, allowing for samples with high-mobility which in turn are more suitable for the observation of fractional quantum Hall plateaux.

Figure 2.1: Example of the quantum Hall effect. The diagonal and Hall resistivity are plotted against the magnetic field. At certain ranges of the magnetic field strength the Hall resistivity develops a plateau, together with the vanishing of the diagonal resistivity. The corresponding Hall conductivity is quantized in terms of $\nu = e^2/h$

Let us put these experimental technicalities and layered systems aside. We deal with a two-dimensional system of electrons, subject to a magnetic field directed perpendicular to the plane, see figure 2.1. Along the plane an electric field, or voltage drop, is applied, through which the system develops a response current. The general form of the relation between the current density $J$ and the electric field $E$ is given by the tensor equation

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

(2.1)

The tensor $\sigma$ is called the conductivity tensor. We assume, on average, we have an isotropic sample which causes $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$. In the absence of a magnetic field one usually deals with Ohm’s law, in which the conductivity tensor is a constant times the identity matrix. The magnetic field causes a Lorentz force through which the response current obtains a component perpendicular to the electric field. The quantity $\sigma_{xx}$ is called the diagonal conductivity, while the component $\sigma_{xy}$ is called the Hall conductance.

We also define the resistivity tensor, which is by definition the inverse of the conductivity tensor, $E = \rho J$

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} = \frac{1}{\sigma_{xx}^2 + \sigma_{xy}^2} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

(2.2)

The effect is as follows. The magnetic field strength sweeps through a large interval, while the sample is subject to a (constant) voltage drop. The diagonal and Hall components of the resistivity (or, equivalent, the conductivity) are measured as a function of the magnetic field strength. At certain ranges of the magnetic field strength the system locks its conductance to a certain value which remains constant within this interval. This is called a plateau. Within such a plateau the Hall and diagonal conductance are remarkably precise given by a the following formula

At a plateau:

$$\begin{align*}
\sigma_{xy} &= \sigma_{yx} = \nu \frac{e^2}{h} \\
\sigma_{xx} &= \sigma_{yy} = 0
\end{align*}$$

(2.3) (2.4)
2.2 Classical approach

Figure 2.2: Example of the quantum Hall effect. The diagonal and Hall resistivity are plotted against the magnetic field. At certain ranges of the magnetic field strength the Hall resistivity develops a plateau, together with the vanishing of the diagonal resistivity. The corresponding Hall conductivity is quantized in terms of $\nu = e^2/h$

The quantity $e^2/h$ is the fundamental unit of conductance. The quantity $\nu$ is called the filling factor, for reasons which will become clear later on. The basic observations of the quantum Hall effect is depicted in figure 2.2.

For the integer effect the filling factor $\nu$ is integer, for the fractional effect it is a fraction. Of course, not all fractions occur; the one with a lower numerator and denominator tend to be more stable. By stable we mean the range over which the plateau is present is relatively large.

This is given by

$$H = \frac{1}{2m_e} \sum_i (-i\hbar \nabla + eA)^2 + \frac{e^2}{4\pi e} \int d^2x d^2y \rho(x)\rho(y) \frac{\rho(x)}{|x - x'|} + V(x)$$  \hspace{1cm} (2.5)$$

The coupling of the spin and the magnetic field of all electrons adds an overall, constant contributions which we neglect. Here, and throughout the rest of this thesis we will assume we only deal with spin-polarized electrons, which are all aligned along the direction of the strong magnetic field. The first term is the kinetic energy of the electrons, where $A$ is the electromagnetic potential. The second term describes the Coulomb interaction, where $\rho$ stands for the electron density. The last term stands for the background potential, formed by impurities, the finiteness of the system and other potential defects. The subtle complications which arise in the system become apparent due to this simple Hamiltonian. Namely, by simply increasing or decreasing the magnetic field the system manages through an incredible interplay between all the different contributions to develop the remarkably rich structure associated with the quantum Hall effect.

2.2 Classical approach

Let us start with the classical description of the system at hand. We are dealing with an electron moving in the $(x, y)$-plane in the presence of a magnetic field. The direction of the magnetic field is perpendicular to the plane, thus $B = B\hat{z}$, and in effect the moving electron is subject to a Lorentz force. As a convention, we set the electric charge equal to $-e$, the
2. The quantum Hall effect

electron’s position and velocity are described by the vectors \( \mathbf{r} \) and \( \dot{\mathbf{r}} \) respectively, and therefore the Lorentz force equals \(-\frac{e}{c} \dot{\mathbf{r}} \times \mathbf{B}\). Written out in components, the equation of motion reads

\[
\frac{m}{\text{d}t} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{eB}{c} \\ \frac{eB}{c} & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}
\]

(2.6)

The motion is described by two, coupled differential equations. Its solution is straightforward and the resulting motion of the particle is called a *cyclotron motion*. The trajectory of the electron is a circle around an arbitrary center. The coordinates of the center are defined as \((R_X, R_Y)\), and the circular trajectory has a radius of \(r_0\). The trajectory and velocity of the electron are described by

\[
\mathbf{r} = \begin{pmatrix} R_X \\ R_Y \end{pmatrix} + r_0 \begin{pmatrix} \cos (\omega_c t + \delta) \\ \sin (\omega_c t + \delta) \end{pmatrix}
\]

(2.7)

\[
\dot{\mathbf{r}} = r_0 \omega_c \begin{pmatrix} -\sin (\omega_c t + \delta) \\ \cos (\omega_c t + \delta) \end{pmatrix}
\]

(2.8)

Here, \(\delta\) is an arbitrary phase determined through the initial conditions. Also appearing is \(\omega_c\), which is the angular velocity

\[
\omega_c = \frac{eB}{mc}
\]

(2.9)

This frequency is called the *cyclotron frequency*. The frequency is parametrized by the strength of the magnetic field and as a result electrons moving around the same center but with different radii have an equal angular frequency. This motion is called *isochronous*, because the period is independent of the radius of the trajectory. Instead, the radius is determined through the initial velocity and the cyclotron frequency as

\[
r_0 = \frac{\dot{r}}{\omega_c}
\]

(2.10)

The kinetic energy is thus \(\frac{1}{2}m\omega_c^2 r_0^2\) and we have solved the system classically. Figure 2.3 shows the classical behavior of the particles.

However, when turning to the quantummechanical description we follow the more formal Hamiltonian approach. Already in the classical case this approach reveals some subtleties, which we will make explicit now. The Lagrangian for this system is

\[
L = \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}
\]

(2.11)

where \(\mathbf{A}\) is the vector potential corresponding to the magnetic field, \(\mathbf{B} = \nabla \times \mathbf{A}\). The equations of motion follow directly from the Euler-Lagrange equations. Let us first determine the *canonical momentum* which follows from the Lagrangian via \(p = \frac{\delta L}{\delta \dot{\mathbf{r}}}\)

\[
p = m\dot{\mathbf{r}} - \frac{e}{c} \mathbf{A}
\]

(2.12)

The first term contains the velocity times the electron mass. The second terms contains the gauge potential \(\mathbf{A}\) and in effect the canonical momentum depends on the gauge chosen for
2.2 Classical approach

![Figure 2.3: Examples of the circular, isochronous orbits around an arbitrary center \((R_X, R_Y)\). Electrons encircling the same center move with equal angular frequency, given by \(\omega_c\).](image)

**A.** It is therefore not a physical observable. To derive the Euler-Lagrange equations we also need

\[
\frac{\delta L}{\delta \mathbf{r}} = -\frac{e}{c}(\dot{\mathbf{r}} \cdot \nabla) \mathbf{A}
\] (2.13)

Plugging (2.12) and this expression into the Euler-Lagrange equations returns the equations of motion, that is, expression (2.6)

\[
m\ddot{\mathbf{r}} = \frac{e}{c}(\dot{\mathbf{r}} \wedge \nabla) \wedge \mathbf{A}
\] (2.14)

Let us press on to the Hamiltonian approach. The Hamiltonian is as follows

\[
H = \dot{\mathbf{r}} \cdot \mathbf{p} - L = \frac{1}{2m} \left( \mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2
\] (2.15)

It is of course a gauge invariant quantity. In the Hamiltonian framework the physics is controlled by Hamilton’s equation of motion. One of these equations, namely \(\dot{\mathbf{r}} = \frac{\delta H}{\delta \mathbf{p}}\), introduces a new quantity. It is called the dynamical, or also mechanical, momentum \(\mathbf{\pi}\) and it is related to the velocity \(\dot{\mathbf{r}}\) as

\[
\mathbf{\pi} = m\dot{\mathbf{r}} = \mathbf{p} + \frac{e}{c} \mathbf{A}
\] (2.16)

In the absence of the magnetic potential the dynamical and canonical momentum reduce to the same physical observable, better known as the ordinary momentum \(m\dot{\mathbf{r}}\). Here we have a relatively simple physical system in which the two are different. The most important difference is that as opposed to the canonical momentum, the dynamical momentum is a gauge invariant quantity and therefore an observable. It is also a conserved quantity which follows directly from the equation of motion

\[
\partial_t(\mathbf{\pi}^2) = m^2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{me}{c}\dot{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \mathbf{B}) = 0
\] (2.17)
One might be tempted to therefore just focus on the dynamical momentum and forget all about the canonical one. However, we must be careful when we quantize the theory. In a canonical quantization procedure one introduces canonical commutation relations between the coordinates and their canonical momenta. Quantization is performed easiest in terms of the canonical momentum, which the next subsection will demonstrate. It is especially suitable for a treatment of the response current when we apply an electric field to the system. Quantizing through use of the gauge-invariant operators is also possible, but this will be reserved for the next chapter.

2.3 Quantum mechanical approach

Let us now turn to the quantum mechanical treatment of the problem. Our goal is an initial understanding of the energy spectrum and the single-particle eigenstates of the system. We approach the system by choosing a particular gauge, the so-called Landau gauge

\[ \mathbf{A} = x \mathbf{B} \mathbf{\hat{y}} \]  

(2.18)

treated for instance in [Girvin, 2000]. In this gauge the Hamiltonian reduces to

\[ H = \frac{1}{2m} \left( p_x^2 + (p_y + \frac{eB}{c} x)^2 \right) \]  

(2.19)

In the quantum case we are of course dealing with operators instead of classical quantities. Upon quantization the canonical pair \((\mathbf{r}, \mathbf{p})\) satisfy the canonical commutation relations, namely \([r_i, p_j] = i\hbar\delta_{ij}\). In the coordinate representation the operator \(\mathbf{p}_i\) acts as the differential operator \(-i\hbar \partial_i\). Written in differential form this Hamiltonian calls for a separation of variables. We make the ansatz of writing the time-independent part of the wave function as the following product

\[ \psi_k(x, y) = e^{iky} f_k(x) \]  

(2.20)

\[ \epsilon_k \psi_k(x, y) = H e^{iky} f_k(x) \]  

(2.21)

This \(e^{iky}\)-factor is an eigenfunction of the momentum operator \(p_y\), therefore we can pull it through by replacing \(p_y \rightarrow \hbar k\). The \(y\)-dependent part is parametrized by \(k\). This parameter is a quantum number describing the momentum in the \(y\)-direction. What remains is a one-dimensional, time-independent Schrödinger equation describing the \(x\)-dependent part \(f_k(x)\). A little algebra gives the following energy eigenvalue equation

\[ \epsilon_k f_k(x) = \left[ -\frac{\hbar^2}{2m} \partial_x^2 + \frac{1}{2} m \omega_c^2 (x + k l_B^2)^2 \right] f_k(x) \]  

(2.22)

We have plugged in the cyclotron frequency \(\omega_c = \frac{eB}{mc}\) and introduced a new constant called the magnetic length

\[ l_B = \sqrt{\frac{\hbar c}{eB}} \]  

(2.23)

In effect, the Schrödinger equation is reduced to that of a one-dimensional harmonic oscillator. Its frequency is the classical quantity \(\omega_c\) and it has a shifted origin at \(x_k = -kl_B^2\),
2.3 Quantum mechanical approach

Figure 2.4: A plot of the discrete energy spectrum. The Landau levels correspond to energy levels of the harmonic oscillator, $\epsilon_{k,n} = \left(n + \frac{1}{2}\right)\hbar\omega_c$. Each Landau level is enormously degenerate. In the Landau gauge we can interpret the horizontal axis as the $x$-coordinate of the center of the Hermite polynomials.

which is controlled by the quantum number $k$. This textbook problem has the following energy spectrum.

$$\epsilon_{k,n} = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

We see that the eigenstates of the one-particle wave function fall into a discrete set of energy levels, which are called Landau levels. A Landau level is parametrized by the quantum number $n$, with $n = 0$ corresponding to the lowest Landau level or simply LLL. The energy of the eigenstates are independent of $k$, therefore an entire family of eigenstates corresponds to each energy level $n$ depending on the range of $k$. The Landau levels are enormously degenerate, see figure 2.4.

The (unnormalized) eigenfunctions are [Griffiths, 1995]

$$\psi_{k,n}(x, y) = e^{iky}H_n\left(\frac{x+kl_B^2}{l_B}\right)e^{-\frac{1}{2}l_B^2(x+kl_B^2)^2}$$

where $H_n(x)$ are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

The eigenfunctions are a set of plane waves in the $y$-direction, localized in the $x$-direction centered around $X_k = kl_B^2$. The spread of the wavefunction in the $x$-direction is controlled by the magnetic length. Figure 2.5 shows plots of the first four Hermite polynomials.

Using this expression allows us to determine the degeneracy within the lowest Landau level for finite-size systems. Consider a rectangle of dimension $L_x$ and $L_y$. Imposing periodic boundary conditions along the $y$-direction restricts the allowed values of $k$ to $k = \frac{2\pi}{L_y}n$, $n$ being an integer. Since $k$ also controls the center of the $x$-dependent part of the wavefunction we also demand an upper bound of $k$ to ensure the wavefunction lies well within the rectangle, therefore $k \leq \frac{L_x}{l_B^2}$. In effect the range of the integer $n$ is $0 \leq n \leq \frac{L_xL_y}{2\pi l_B^2}$. Defining $A$ as the total area of the system gives us the following expression for the degeneracy within a Landau level

$$\# \text{ states within a Landau level} = \frac{A}{2\pi l_B^2}$$
2. The quantum Hall effect

Figure 2.5: The first four Hermite polynomials $H_n(x)$. The integer $n$ labels the Landau level, with $n = 0$ corresponding to a simple Gaussian factor: the lowest Landau level. Other eigenstates within a Landau level correspond to shifted centers.

The number of states is determined by the magnetic length, i.e. the magnetic field strength. In fact, from (2.23) we have $l_B^2 = \frac{B}{\Phi_0}$ with the unit flux quantum $\Phi_0 = \frac{e}{c}$. Therefore the number of states within a Landau level is

$$\text{# states within a Landau level} = N_\Phi = \frac{BA}{\Phi_0}$$

(2.28)

The expression on the right hand side is the number of flux quanta penetrating the system.

Let us touch briefly upon the relation between the quantum and classical descriptions. In the classical case the motion of an electron is a circular orbit around an arbitrary center. The period of such an orbit is given by the cyclotron frequency which is parametrized by the magnetic field strength. Because the angular frequency is a constant, independent of the radius of the orbit, electron-orbits with a large radius correspond to large kinetic energies.

When the system is quantized the energy is no longer a continuous spectrum. Rather, we have a discrete spectrum where the spacing between the levels is controlled by the cyclotron frequency. One can interpret this quantization of the energy spectrum as also quantizing the allowed values of the radii of the orbits. The “smallest radius” follows from setting the classical kinetic energy $\frac{1}{2}m\omega_c^2r_0^2$ equal to the smallest energy-quanta $\frac{1}{2}\hbar\omega_c$. We have

$$r_0 = \sqrt{\frac{\hbar}{m\omega_c}}$$

which is in fact just the magnetic length $l_B$. Classically, such an orbit takes up $2\pi r_0^2$ of space ($r_0 = l_B$). A rough estimate on the number of orbits that “fit” in a sample of area $A$ is thus

$$\text{# orbits} = \frac{A}{2\pi l_B^2}$$

This matches the degeneracy per Landau level as found in (2.28).

---

1The unit flux quantum is the elementary unit of flux comprised out of the fundamental constants $e$, $\hbar$ and $c$.  

20
2.4 Electric field

We have a basic understanding of the physics of the electron confined to a two-dimensional plane subject to a perpendicular magnetic field. We now turn on an electric field and describe the response of the system.

2.4.1 Classical approach

We again start off with a classical approach, which means we are dealing with the Lorentz force this time with a non-zero electric field.

\[
m\ddot{\mathbf{r}}(t) = -eE - \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B}
\]

These coupled equations are exactly solvable [Yoshioka, 2002], simply plug them into Mathematica. However, there is another way to obtain the answer using Lorentz covariance. We already know the solution of a particle in the absence of the electric field. By clever use of Lorentz covariance we are able to use those results to our advantage. To be precise, the frame in which the electric field is zero while the magnetic field is not (the “lab frame”) is Lorentz boosted to a moving frame. This results in a non-zero component of the electric field due to the Lorentz covariance of the electromagnetic field. The relative velocity of the moving frame is taken to be \( \mathbf{V} = -V\hat{y} \). To lowest order in \( \frac{V}{c} \) the field components change according to

\[
\begin{align*}
\mathbf{E}' &= 0 \\
\mathbf{B}' &= B\hat{z} \\
\mathbf{E} &= -\frac{V}{c}B\hat{x} \\
\mathbf{B} &= B\hat{z}
\end{align*}
\]

The trajectory and velocity of the particle are recovered by applying the same Lorentz transformation to the trajectory in the lab frame (2.7)

\[
\begin{align*}
\mathbf{r} &= \left( \begin{array}{c} RX \\ RY + c\frac{E}{B}t \end{array} \right) + r_0 \begin{pmatrix} \cos(\omega_c t + \delta) \\ \sin(\omega_c t + \delta) \end{pmatrix} \\
\dot{\mathbf{r}} &= r_0\omega_c \begin{pmatrix} -\sin(\omega_c t + \delta) \\ \cos(\omega_c t + \delta) - c\frac{E}{B} \end{pmatrix}
\end{align*}
\]

By applying an electric field along the \( x \)-direction the electron starts drifting along the \( y \)-direction. The cyclotron motion is still present and along with the drifting of the center of mass the motion is called trochoid. The drifting, which is a movement of the center of the cyclotron motion, takes place with a velocity of

\[
v_{\text{drift}} = \frac{cE}{B}
\]

See figure 2.6 for a graphical interpretation.

Let us assume we are dealing with a classical system of two-dimensional electrons with a density \( n_e \). Ignoring electron-electron interaction allows us to calculate the net current transfer. The cyclotron motion averages to zero, so the only contribution to the average...
velocity originates from, and therefore equals, the drift velocity. The electrical current is defined as $J = -en_e \mathbf{v}$, so we have

$$J = en_e \frac{cE}{B} \hat{y}$$

while the net current in the $x$-direction averages to zero. This is the response current of the electric field. By definition, the relation of the response current and the applied magnetic field is given through use of the resistivity tensor $\rho$. The conductivity tensor $\sigma$ in turn is given by its inverse.

$$\sigma = \sigma_{x} E$$

$$E = \rho J$$

A quick inspection shows that we have

$$\sigma = \frac{en_e c}{B} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho = \frac{B}{en_e} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This answer shows we will have to dig deeper in order to unravel the physics behind the quantum Hall effect. We are still far from the physical observations and conditions present in the lab. For once, there are impurities and defects we have ignored. These random potentials break the Lorentz covariance by scattering the electrons. Classically, the results of the motion given above are still valid in the absence of Lorentz covariance - they just cannot be obtained through the use of Lorentz symmetry. However, introducing a scattering term will affect the physics. Such a scattering can be classically modeled by adding a term $m \dot{\mathbf{r}} \tau$ to the equations of motion; $\tau$ being the scattering relaxation time. Such a term alters the conductivity into

$$\sigma_{xx} = \frac{en_e c}{B} \frac{1}{1 + (\omega c \tau)^2}$$

$$\sigma_{xy} = \frac{en_e c}{B} \frac{(\omega c \tau)^2}{1 + (\omega c \tau)^2}$$

In the impurity-free limit one sends $\omega c \tau \to \infty$ and obtains the scattering-free results (2.37). This shows how random potentials introduce diagonal terms into the conductivity tensor. Terms which are not present in a quantum Hall plateau.
2.4 Electric field

Although this approach is quite limited, it does hint at some of the physics we have ahead of us. First, it indeed explains the fact that the response current is perpendicular to the applied electric field - an effect caused by the Lorentz force. This behavior causes the diagonal components of the conductivity tensor to vanish, an observation which is also made in the quantum Hall regime. On the other hand, this “off-diagonal behavior” of the conductivity tensor is destroyed through the introduction of scattering terms. Such terms arise due to impurities and defects which are always present in samples. To restore the quantum Hall-like behavior, we need some sort of mechanism which blocks the scattering of electrons. Third, the value of the conductivity (tensor) is determined by the ratio between the electron density $n_e$ and the strength of the magnetic field $B$. There is no sign yet of Planck’s constant however. This is why we now turn to quantum mechanics.

2.4.2 Quantum mechanical approach

We start off in a similar way, namely with an alteration of the Hamiltonian. We add an electric field $E = E \hat{x}$ to the expression of the Hamiltonian. In the Landau gauge the Hamiltonian becomes

$$H = \frac{1}{2m} \left( p_x^2 + (p_y + \frac{eB}{c} x)^2 \right) + eEx$$

(2.39)

The last term equals the potential term $qV(x)$ with $-\nabla V = E$. The treatment is very similar to what we have seen before, although the algebra is a bit trickier this time. First, we make the ansatz $\psi_k(x,y) = e^{iky}f_k(x)$ which allows us to replace $p_y$ by $\hbar k$. This gives the following equation for the energy

$$\epsilon_k f_k(x) = \left[ -\frac{\hbar^2}{2m} \frac{\partial}{\partial x}^2 + \frac{1}{2m} \left( \hbar k + \frac{eB}{c} x \right)^2 + eEx \right] f_k(x)$$

(2.40)

This already represents the one-dimensional harmonic oscillator with a shifted origin. The more familiar shape is obtained by completing the square. It is a tricky yet straightforward exercise and eventually one obtains the following expression for the Schrödinger equation

$$\left[ \epsilon_{n,k} - eEX_k - \frac{1}{2}m \left( \frac{E}{B} \right)^2 \right] f_k(y) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_c^2 (x - X_k)^2 \right] f_k(x)$$

(2.41)

Here we once more introduced the magnetic length $l_B^2 = \frac{\hbar^2}{eB}$ and cyclotron frequency $\omega_c = \frac{eB}{mc}$. This is again a harmonic oscillator, this time with a shifted origin $X_k$ which is given by

$$X_k = -kl_B^2 - \frac{eE}{m\omega_c^2}$$

(2.42)

The eigenfunctions are again Hermite polynomials times an exponent.

$$\psi_{k,n}(x,y) = A_k e^{iky}H_n(x - X_k)e^{-\frac{1}{2}l_B^2(x-X_k)^2}$$

(2.43)

where $A_k = \frac{1}{\sqrt{\pi^2}} \left( \frac{1}{2} \right)^{\frac{1}{4}} \left( \frac{1}{2n!} \right)^{\frac{1}{2}}$ is a normalization factor. This time the energy of the state $\psi_{k,n}$ is linearly dependent on $k$ and the large degeneracy is no longer present. Specifically,

$$\epsilon_{n,k} = (n + \frac{1}{2})\hbar\omega_c + eEX_k + \frac{1}{2}m \left( \frac{E}{B} \right)^2$$

(2.44)
The electric field causes an addition of a potential energy term $eEX_k$ and of a kinetic-energy term with velocity $v = \frac{cE}{B}$. The first term is also kinetic and originates from the cyclotron motion.

The classical interpretation is a drifting of the center of the electron-orbit in the direction perpendicular to the electric field. This is reflected in the non-zero expectation value of the current operator. Each $k$-state contributes to the current through the expectation value of the charge times the velocity operator $\frac{e}{m}$. We have

$$\langle J \rangle = -e \frac{1}{m} \langle \psi_{k,n} \mid \left( \mathbf{p} + \frac{e}{c} \mathbf{A} \right) \mid \psi_{k,n} \rangle \quad (2.45)$$

We now assume that the sample is rectangular with dimensions $L_x$ and $L_y$. This way the center is bounded to $0 \leq X_k \leq L_x$. Furthermore, we impose periodic boundary conditions in the $y$-direction, which restricts $k$ to $k = 2\pi n$ for some integer $n$. Using the identity (2.26) for the Hermite polynomials we calculate the expectation value of the velocity in the $x$-direction

$$\langle v_x \rangle = A_k^2 \frac{1}{m} \int dy \left( e^{iky} e^{-iky} \right) \int dx \left( H_n \left( \frac{x-X_k}{l_B} \right) e^{-\frac{1}{2l_B} (x-X_k)^2} \right) \left( -i\hbar \partial_x \right) \left( H_n \left( \frac{x-X_k}{l_B} \right) e^{-\frac{1}{2l_B} (x-X_k)^2} \right) \quad (2.46)$$

Via partial integration we can set $\int (f \partial_x f) = \frac{1}{2} f^2$. Since the exponent drops to zero for $x \to \pm \infty$ the net current and velocity in the $x$-direction is simply zero $\langle J_x \rangle = 0$. As for $J_y$ we have

$$\langle v_y \rangle = A_k^2 L_y \frac{eB}{mc} \int dx \left( H_n \left( \frac{x-X_k}{l_B} \right) e^{-\frac{1}{2l_B} (x-X_k)^2} \right) \left( x + \frac{\hbar c}{eB} \right) \left( H_n \left( \frac{x-X_k}{l_B} \right) e^{-\frac{1}{2l_B} (x-X_k)^2} \right) \quad (2.47)$$

For zero electric field we would have $X_k = -\frac{\hbar c e}{eB}$ and the integral is then zero due to antisymmetry. Not surprisingly there is no net current if there is no electric field. For non-zero electric field, however, we have $X_k = -\frac{\hbar c e}{eB} - \frac{mE^2}{2}$ and the integral is equal to

$$\langle v_y \rangle = -L_y \frac{eB}{c} A_k^2 \int dx \left( H_n \left( \frac{x-X_k}{l_B} \right) e^{-\frac{1}{2l_B} (x-X_k)^2} \right)^2 \quad (2.48)$$

$$= -\frac{eB}{c} \quad (2.49)$$

As was already read off from the energy spectrum, the electron has a drift velocity $-\frac{eB}{c}$ perpendicular to the electric field. The drift velocity for a state $\psi_{n,k}$ is independent of the quantum number $n$ and $k$. However, due to the energy dependence on $k$ there is still a possibility of mixing between Landau levels. The electric field tilts the Landau levels, so in principle states of different Landau levels can have equal energy. Due to tunneling the overlap of such states give an extra contribution to the electrical current (2.45).

For the sake of argument we assume that no such mixing take place. Even better, the tilting due to the electric field is relatively small, such that there is no degeneracy between different levels. For a system of non-interacting electrons with density $n_e$ the current is $\langle J_y \rangle = -en_e v_y = en_e \frac{eB}{c}$. These results translate to the conductivity tensor via $\mathbf{J} = \sigma \cdot \mathbf{E}$.

$$\sigma_{xx} = \sigma_{yy} = 0 \quad (2.50)$$

$$\sigma_{xy} = -\sigma_{yx} = \frac{en_e c}{B} \quad (2.51)$$
2.5 Putting it all together

This corresponds to the classical case of the drifting motion. Let us also incorporate the degeneracy per Landau level. Earlier we found that for a single Landau level there are \( N_\Phi = \frac{BA}{e^2} \) states available. Furthermore the electron density \( n_e \) is given by \( \frac{N_e}{A} \). We combine these two expressions and arrive at a familiar expression for the off-diagonal component of the conductivity tensor

\[
\sigma_{xy} = \nu \frac{e^2}{h}, \quad \nu = \frac{N_e}{N_\Phi}
\]

(2.52)

The dimensionless quantity \( \nu \) is called the filling fraction referring to the ratio between the number of electrons \( N_e \) versus the number of states available within a Landau level \( N_\Phi \).

2.5 Putting it all together

The results obtained so far are aimed in the proper direction. Most importantly, we re-produced the typifying formula of the quantum Hall effect, which is the expression for the conductivity tensor

\[
\sigma_{xx} = \sigma_{yy} = 0 \\
\sigma_{xy} = -\sigma_{yx} = \nu \frac{e^2}{h}
\]

(2.53)\hspace{1cm} (2.54)

Still, there are plenty of topics to be covered, but they are essentially questions on why the quantum Hall effect occurs and what makes the effect measurable. In the introduction we already stated that this thesis is not aimed at an understanding of such complicated mechanisms. Still, to get a picture of what drives the physics at hand, the remainder of this chapter is devoted to a sketching of the essential ingredients responsible for the existence of the quantum Hall effect.

Let us first point out that the result (2.54) is flawed. This formula implies that the off-diagonal structure is valid for all values of the electron density, which is not what is observed in the experiment. The linear relation between the electron density and the conductivity components is caused by the assumption of Lorentz symmetry. Based on general arguments this symmetry will always force this formula to hold, irrespective of the underlying physical details. Luckily, in physical samples this symmetry is not present. Impurities and defects, which are always present, break the Lorentz symmetry and so the formula does not have to hold for all values of \( \nu \). However, we have also seen that impurities are in general capable of destroying the off-diagonal behavior. Ironically, the impurities are a necessary ingredient since they get rid of the Lorentz symmetry, but they are also potentially capable of rendering formula (2.54) completely invalid.

Incompressibility There is an exception to this. Suppose we finetune the chemical potential such that we completely fill an integer \( N \) number of Landau levels. In that case the electrons in the highest filled Landau level carry an energy of \( (N + \frac{1}{2}) \hbar \omega_c \). If we add one extra electron this electron will carry an extra amount of \( \hbar \omega_c \) on top of that energy. This implies a discontinuity in the chemical potential, which suddenly jumps when the transition to the next Landau level occurs. The system is said to be incompressible. An infinitesimal decrease of the area requires a finite amount of energy. The area and the magnetic field strength
determine the number of states available within a Landau level. If all Landau levels are filled then squeezing the system forces some of the electrons to jump to a higher Landau level. This costs a finite amount of energy, no matter how small the compressing is. All excitations with respect to the filled Landau level cost a finite amount of energy. The system is gapped and incompressible.

This incompressibility is the main reason why the off-diagonal form of the conductivity tensor may occur even in the presence of impurities and defects. When an integer number of Landau levels are completely filled, conducting electrons are unaffected by relatively weak scattering potentials. The only way to influence these electrons is by exciting them to a higher Landau level, which costs a relatively large amount of energy in the strong-field limit. This is an intuitive picture of how incompressibility is an essential ingredient to the quantum Hall effect. Without it, impurities and defects destroy the off-diagonal behavior of the conductivity tensor.

Again, we find ourselves at a position in which part of the theory is understood at some level, yet there remain large holes in the presented argument. This time we made use of a finetuning of the chemical potential: we completely filled an integer number of Landau levels, and saw how incompressibility occurs at the precise transition from one Landau level to the next. Such finetuning of only a few electrons is not physical - it cannot occur in an experimental lab where one works with well over $10^{15}$ electrons. Apart from this finetuning, we have also not explained why a plateau is formed around the specific value of the electron density. As it turns out, localization is the key to understanding this effect.

Localization Localization is the effect of turning extended electron states, which are spread out over the entire sample, into localized states centered around scattering peaks and wells. In general, localization is caused by impurities and defects and the essential property of localized electron states is that once occupied such states do not contribute to the current carried by the sample as the electrons can be considered as bound to their position. Furthermore, the energy of such states are somewhat increased or decreased.

As it turns out [Yoshioka, 2002] in the strong magnetic field limit not all states are affected by this mechanism. Although a large amount of the electron states become localized, a significant portion remains delocalized and stretches throughout the sample. Furthermore, the delocalized states are hardly effected by the impurities and their energy remains pretty much the same. The change in energy of the localized states, on the other hand, is substantial as is shown in figure 2.7. It is this broadening of the Landau levels which allows for the forming of plateaux.

The picture is as follows. By increasing the chemical potential electrons are added to the system. If the chemical potential is located between Landau levels the states available for the electrons are all localized. These states do not contribute to the total current, and therefore the current remains the same. Furthermore, the delocalized states carry an energy much smaller than the chemical potential. They are therefore not influenced by the scattering potential, since such scattering processes would require to lift the electron to a much higher energy state above the Fermi level. As long as the chemical potential is somewhere between Landau levels the delocalized states carry the same current and we are dealing with a plateau.

Once the chemical potential “hits” a new Landau level a transition occurs, and the systems obtains extra delocalized electrons which contribute to the current. This is the process where the systems “steps up” to the next plateau. What are in fact describing here is the presence of
2.5 Putting it all together

Figure 2.7: The effect of localization on the Landau levels and the density of states. The localization broadens the levels, and turns a significant portion of the states into localized electron states. Such states do not contribute to the current and are depicted by the gray areas. The white areas represent the delocalized states and are centered around the Landau levels of a clean system. When the chemical potential lies within two Landau levels, i.e., in a gray area, then the system is on a plateau. By increasing the magnetic field strength the chemical potential rises, until at some point overlaps with the energy levels of unoccupied delocalized state. During this phase the system transits to the next plateau.

There is a potential paradox lurking here. If a significant portion of the electrons are bound to the localized states, does this not affect the total current? To be precise, the conductivity tensor equals $\nu \frac{e^2}{h}$, with $\nu$ the electron density. The localized electrons clearly do not contribute to the quantity $\nu$. In a beautiful argument, Laughlin [Prange and Girvin, 1987] showed that the delocalized states precisely compensate this “loss of electron density” and quantize the conductivity tensor in the proper manner. The argument is based on gauge invariance of the system and the presence of a mobility gap. This is the first solid argument which explains the precision of the quantized value of the Hall conductance. See [Laughlin, 1981; Prange and Girvin, 1987; Yoshioka, 2002] for details.

**Edge states** The picture of localization explains the role of impurities in the formation of quantum Hall plateaus. Within a plateau the delocalized states are located at an energy scale well beneath the Fermi level. The Fermi level must be located away from the energy scale of the delocalized states for the quantum Hall effect to occur. If not, the delocalized states are compressible and the conductivity tensor obtains non-zero diagonal components.

What this picture does not take into account is the concept of electrical transport. An electrical current develops once a drop in chemical potential is present in the system. This difference in potential between opposite side of the sample is essentially what is measured in the lab. However, the delocalized states remain well beneath the Fermi level while the localized states do not contribute to the current. Halperin and later Buttiker [Prange and Girvin, 1987] recognized that states located at the edges of the sample are responsible for the
A physical sample is always of finite size. Such a finite size holds the states together by a so-called confining potential. Ideally, this potential, illustrated in figure 2.8, is negligible in the bulk of the sample while it has a steep increment at the edge of the sample. As an effect, the energy of states located near the edge of the sample is lifted and at some point cross the Fermi level. The delocalized states at the edge are able to carry a current, which we will now demonstrate.

**Figure 2.8:** An impression of the effect of a confining potential. The plot gives the energy dependence on the center of the state, $X_k$. The horizontal line represents the chemical potential. The occupied states lie beneath this line. In this picture we have neglected the localized states. Near the edges the confining potential increases rapidly and lifts the energy levels above the chemical potential. The states near the edges are extended along the edge and always have a non-zero velocity. The direction of this velocity is determined by the sign of the slope $dE_k/dX_k$.

We let the edge point along the $y$-direction and we assume the confining potential is only $x$-dependent and smooth. In section 2.4.2 we obtained an expression for the single electron states in the presence of an electric field. Specifically, since the confining potential is only $x$-dependent the momentum along the edge is still conserved and the eigenstate is still peaked near $X_k = -k l_B^2$ (approximately). We write the momentum eigenstate as

$$
\psi_k(x,y) = e^{iX_k y/l_B^2} f_k(x)
$$

(2.55)

The group velocity of these states near the edge is given by

$$
v_k = \frac{1}{\hbar} \frac{dE_k}{dk} = \frac{1}{\hbar} \frac{dE_k}{dX_k} \frac{dX_k}{dk}
$$

(2.56)

For a given value of $k$ (which is positive) the derivative $dX_k/dk$ is negative. The derivative $dE_k/dX_k$ on the other hand is determined by the direction in which is $U(x)$ increases, see figure 2.8. At the “left” edge $U(x)$ increases with decreasing $X_k$. Therefore $dE_k/dX_k$ is negative. On the “right” edge $U(x)$ increases with increasing $X_k$. Thus, $dE_k/dX_k$ is positive. In conclusion, the velocity of the electrons along the edges is positive along the left edge and negative along the right edge. Quite remarkably, the edges contains one-way traffic only.

The electrical current flows along the delocalized states located at the edge. These states are oriented, and flow in particular direction along the sample. These currents always flow and are unaffected by impurities or defects. That is, if they were to affect the current they would need to scatter the electron to the opposite edge. Such tunneling transitions are negligible for well-separated edges.
An electrical current arises due to an imbalance between the states. By applying an electric field the delocalized states allow electrons to be transferred from one edge to the other, creating an imbalance in the chemical potential. This is exactly the voltage drop one measures in the lab. The bulk states do not carry the current themself, but they are responsible for a net current to appear along the edges.

Consider the case where the chemical potential lies well between two Landau levels. Beneath the Fermi level an integer number \( N \) of Landau levels is completely occupied. The delocalized states located within the bulk lie well beneath the Fermi level. Near the edges the confining potential comes into play, lifting the delocalized states near the edge such that they now sit near the Fermi level. This occurs for each filled Landau level. As a corollary, each Landau level contributes one channel to the current transport. The conductivity for each channel is the same, namely \( \frac{e^2}{h} \). The total conductivity is the sum of all these channels, e.g.

\[
\sigma_{xy} = N \frac{e^2}{h} \tag{2.57}
\]

This “explains” the integer effect.

**Fractional effect** The picture presented so far cannot account for what is observed at fractional values of the filling fraction. The reason for this is that, in the picture presented so far, a partially filled Landau level is enormously degenerate. At fractional values of the dimensionless electron density we need to divide \( N_e \) electrons over \( N_\Phi \) available states. In the ideal case these states carry the same amount of energy, so any partitioning will do. Compressing the system reduces the number of states \( N_\Phi \) available, so in general the electrons will need to reorganize (i.e. some angular orbits are no longer available). Since all states carry the same amount of energy an infinitesimal change in area costs an infinitesimal amount of energy. The system is compressible and the quantum Hall effect cannot occur due to impurities.

What we have not taken into account though is the *Coulomb interaction*. It is well known from plenty of other physical systems, such as superfluids and superconductors, that interactions have enormous impact on the physics of the system. In this case, a Coulomb interaction lifts the degeneracy and picks out a specific ground state which becomes the energetically most favorable state available to the system. Furthermore, this ground state is unique and all excitations with respect to it cost a finite amount of energy. It is with respect to this ground state that the system develops a mobility gap and becomes incompressible. Note that in the case of the integer effect the gap is fulfilled by the Landau levels themself.

The complexity associated to the development of a plateau at fractional values of the filling factor is staggering. We will not concern ourselves with the question if or why a certain sample develops a quantum Hall plateaux at fractional filling factors. But keep the following in mind. A plateau at a fractional value of the filling factor can only develop if

- Lorentz symmetry is broken
- The system has a mobility gap and is incompressible
- This gapped state is stable against small deviations of the electron density.

In case of the fractional effect these states are developed because of the Coulomb interaction.
Throughout this thesis we will develop a large number of multiparticle wavefunctions which develop quantum Hall states at fractional values of the filling factor. Such wavefunctions are not realistic physical wavefunctions, but rather eigenstates of idealized Hamiltonians. The physical wavefunctions are far too difficult to obtain, so the idea is that these idealized wavefunctions encapture the essential features of a physical state, namely the topological quantum numbers associated to it.
Quantum Hall wavefunctions

In this chapter we address the idealized quantum Hall system, electrons confined to a two-dimensional plane in the presence of a strong magnetic field, from a different perspective. We will quantize theory in a different matter. First, we show that the energy spectrum can be obtained through a gauge-invariant quantization procedure. This procedure shows that in order to distinguish between the degenerate eigenstates we are forced to choose a gauge, which in this chapter is the symmetric gauge. With this gauge-fixing we are able to introduce a second Hermitian operator which commutes with the Hamiltonian, but who’s eigenstates are not degenerate, at least not in the case for single particles. Simultaneous diagonalization of these operators produces the basis states of the single particle eigenstates.

The operator which is used is the angular momentum operator. Its action on the coordinate representatives of the wavefunction is a simple differentiation followed by multiplication with respect to complex coordinates \( z = x + iy \). This forces the single particle wavefunctions to be simple polynomials in \( z \) times an overall geometric factor. The fact that (part of) the wavefunction only depends on \( z \) and not on \( \bar{z} \) is the main result of this chapter.

From the discussion on the single-particle states we move on to the motivation on multi-electron states. In the absence of Coulomb interaction the eigenbasis of the Hilbert space is obtained and we give an expression for the wavefunction of one filled Landau level. To apply such a procedure to the fractional effect automatically forces us to introduce a Coulomb interaction. Without it, the multiparticle wavefunction would be highly degenerate at fractional filling factors. The Coulomb interaction essentially lifts this degeneracy, and allows for the formation of an energy gap thus restoring incompressibility.

To obtain expressions for exact wavefunctions in the presence of the Coulomb interaction is impossible. This is why we resort to trial wavefunctions, based on severe symmetry constraints which mimic the Coulomb interaction. The simplest example of such a wavefunction is the Laughlin wavefunction, which in fact stands for a whole sequence of wavefunctions with filling...
factor $\nu = 1/M$. We discuss some of its features and interpretation in terms of so-called zero points. The zero points are a reference to the fact that this wavefunction is essentially constructed on the basis of symmetries and its tendency to drop to zero when two or more electrons approach each other.

The chapter is ended with the introduction of quasiholes and quasiparticles, the excitations over quantum Hall states. We show that they are created through the insertion of flux to the system along infinitely thin solenoid, followed by a singular gauge transformation. This allows for the simple interpretation that a quantum Hall system is essentially a relative distribution of zero points/flux tubes and electrons.

### 3.1 Summary of obtained results

In section 2.2 we treated the system of an electron confined to the two-dimensional plane in the presence of a magnetic field, perpendicular to the plane. Let us summarize some of the classical quantities found there

- **Trajectory particle**
  \[
  \mathbf{r} = \left( R_x + \frac{\pi y}{m \omega_c}, R_y - \frac{\pi x}{m \omega_c} \right)
  \]  

- **Velocity particle**
  \[
  \mathbf{v} = \dot{\mathbf{r}}
  \]  

- **Canonical momentum**
  \[
  \mathbf{p} = m \dot{\mathbf{r}} - \frac{e}{c} \mathbf{A}
  \]  

- **Dynamical momentum**
  \[
  \pi = \mathbf{p} + \frac{e}{c} \mathbf{A}
  \]  

- **Hamiltonian**
  \[
  H = \frac{1}{2m} \pi^2
  \]

Also introduced are two parameters which control the dynamics of the system

- **Cyclotron frequency**
  \[
  \omega_c = \frac{eB}{mc}
  \]  

- **Magnetic Length**
  \[
  l_B = \sqrt{\frac{\hbar c}{eB}}
  \]

Upon quantization we find that these quantities control the typical length- and energiescales of the system. The energy levels become discretized in the form of Landau levels and, although electrons are point particles their states take up a finite amount of space, roughly the size of $l_B^2$.

Numerical values that arise in experiment are [Chakraborty and Pietiläinen, 1995]

- **Cyclotron energy**
  \[
  \hbar \omega_c \text{ [meV]} = 1.728B \text{ [T]}
  \]

- **Magnetic length**
  \[
  l_B \text{ [nm]} = 25.64 (B\text{[T]})^{\frac{1}{2}}
  \]

- **Unit flux quantum**
  \[
  \Phi_0 = 4.136 \cdot 10^{-11} \text{ Tcm}^2
  \]

These correspond to samples comprised of GaAs. The magnetic field strength usually ranges from 0 to 30 [T], with the integer-valued $\nu = 1$ plateau arising at around $\sim 10$ [T]. The values can differ, depending on the sample. The reason for this is that the cyclotron frequency depends on the effective mass of the electron.
3.2 Gauge invariant quantization

In this section we will approach the exact same system in a different matter. First, we will use a gauge-invariant approach. This approach will lay down some of the foundations of later chapters. Specifically, we will encounter a general form of the single-particle states in terms of a single, complex variable.

Recall the idealized Hamiltonian

\[ H = \frac{1}{2m} (p + e \mathbf{A})^2 = \frac{1}{2m} \pi^2 \]  

(3.7)

In the quantum case we are of course dealing with operators instead of classical quantities. The canonical momentum \( p \) and the trajectory of the particle \( r \) satisfy the canonical commutation relations \( [r_i, p_j] = i\hbar \delta_{ij} \). But as we have already seen the momentum \( p \) is not gauge invariant. And as can be seen above the gauge field \( A \) enters the Hamiltonian, so we need to pick a gauge in order to solve the Hamiltonian - if we were to stick to the combination \( (r, p) \). We therefore introduce another set of conjugate operators. The first choice is the dynamical momentum \( \pi \), since it is a gauge invariant quantity and the Hamiltonian is completely determined by it. Let us look at the dynamical momentum a bit closer. Using the canonical commutation relations we set

\[ p = -i\hbar \nabla. \]  

Also using \( B = \nabla \wedge A \) we derive

\[ [\pi_x, \pi_y] = \frac{e}{c} ([p_x, A_y] + [A_x, p_y]) = -i\frac{\hbar^2}{l_B^2} \]  

(3.8)

This commutation relation implies that \( \pi_x \) and \( \pi_y \) (or more precise: \( l_B^2 \pi_y \)) are conjugate to each other, and can be used as the first pair of conjugate operators. The second pair of conjugate operators can be read off from the expression of the trajectory of the particle \( (3.1.a) \). In the classical case the trajectory is a circular motion \( (\pi_y m\omega_c, \pi_x m\omega_c) \) relative to some origin described by \( (R_X, R_Y) \). When carried over to the quantum case the coordinates \( (R_X, R_Y) \) become operators, and are in fact conjugate to each other. Their commutation relation is

\[ [R_X, R_Y] = [x - \frac{\pi_y}{m\omega_c}, y + \frac{\pi_x}{m\omega_c}] \]  

\[ = \frac{1}{m\omega_c} ([x, \pi_x] + [y, \pi_y]) + \frac{1}{(m\omega_c)^2} [\pi_x, \pi_y] \]  

(3.10)

\[ = i\frac{l_B^2}{\hbar} \]  

(3.11)

Here we used the relation \( m\omega_c = \frac{\hbar}{l_B^2} \). One can furthermore show that \( R_X \) and \( R_Y \) commute with \( \pi_x \) and \( \pi_y \), completing the discussion that the coordinates \( (R_X, R_Y) \) and momentum \( (\pi_x, \pi_y) \) are two conjugate pairs of operators, just like the operators \( (r, p) \).

Having defined our new set of conjugate operators we turn back to discussion of the single-particle states. The first advantage of these new operators is the fact that the Hamiltonian is written entirely in terms of the gauge invariant operators \( \pi_i \).

\[ H = \frac{1}{2m} (\pi_x^2 + \pi_y^2) = \frac{1}{2m} \pi_x^2 + \frac{1}{2m\omega_c} \left( \frac{l_B^2}{\hbar} \pi_y \right)^2 \]  

(3.12)
This expression is that of a harmonic oscillator, with mass $m$ and frequency $\omega_c$. The standard treatment is by introducing ladder operators

$$ a = \frac{l_B}{\sqrt{2\hbar}}(\pi_x - i\pi_y) \quad (3.13.a) $$
$$ a^\dagger = \frac{l_B}{\sqrt{2\hbar}}(\pi_x + i\pi_y) \quad (3.13.b) $$

which satisfy the relation $[a, a^\dagger] = 1$. The Hamiltonian is diagonalized accordingly

$$ H = \hbar \omega_c(a^\dagger a + \frac{1}{2}) \quad (3.14) $$

We recover the results of the Landau levels once more. The eigenstates of the single-particle wavefunctions fall into a discrete set of energy levels

$$ \epsilon_n = (n + \frac{1}{2})\hbar \omega_c \quad (3.15) $$

And so the energy eigenstates carry a quantum number $n$. This completes the discussion on the gauge-invariant approach.

### 3.3 Symmetric gauge and angular momentum

To distinguish the eigenstates within a Landau level we need to introduce other quantum numbers. We have already seen that by choosing a gauge we recover a new set of quantum numbers at the cost of destroying some of the explicit symmetries of the Hamiltonian. By this we mean that the Hamiltonian is no longer invariant under such a symmetry. Rather, these symmetries now correspond to gauge transformations of the Hamiltonian and hence the physics is unchanged. In the case of the Landau gauge the new quantum numbers describe the momentum in some particular direction. Here we will adopt the symmetric gauge. This will eventually lead to quantum numbers associated with angular momentum.

The symmetric gauge is defined as

$$ A = \frac{B}{2}(-y\hat{x} + x\hat{y}) \quad (3.16) $$
$$ = \frac{B}{2} \left[ (-R_Y + \frac{\pi_x}{m\omega_c})\hat{x} + (R_X + \frac{\pi_y}{m\omega_c})\hat{y} \right] \quad (3.17) $$

This gauge is rotationally invariant. We will now demonstrate that with this gauge choice we are able to simultaneously diagonalize the angular momentum operator $L_z$ and the Hamiltonian $H$. The quantum number which we thus introduce is that of the angular momentum in the $z$-direction: $L_z = xp_y - yp_x$. Rewriting this operator in terms of our previously adopted operators $R_X, R_Y, \pi_x$ and $\pi_y$

$$ L_z = \frac{1}{2m\omega_c}(\pi_x^2 + \pi_y^2) - \frac{m\omega_c}{2}(R_X^2 + R_Y^2) \quad (3.18) $$

The last expression is meant to emphasize the fact that $L_z$ consists out of two harmonic oscillators ($L_z = H_1 - H_2$), with mass $M = m\omega_c$ and frequency $\omega = 1$. In any system
where the $L_z$ operator is a 'good' quantum number it is always possible to adopt such a form [Griffiths, 1995].

The first harmonic oscillator $H_1$ is nothing but $\frac{1}{\omega}$ times the Hamiltonian (3.12). We therefore use the ladder operators (3.13.a) and (3.13.b) to diagonalize this term. To diagonalize the second harmonic oscillator $H_2$ we introduce the ladder operators $b$ and $b^\dagger$, such that

$$b = \frac{1}{\sqrt{2l}} (R_X + i R_Y)$$  \hspace{1cm} (3.19.a)

$$b^\dagger = \frac{1}{\sqrt{2l}} (R_X - i R_Y)$$  \hspace{1cm} (3.19.b)

All in all the angular momentum takes on the elegant form

$$L_z = \hbar (a^\dagger a - b^\dagger b)$$  \hspace{1cm} (3.20)

The fact that this operator commutes with the Hamiltonian (3.12) is now manifest. Since $L_z$ is Hermitian its eigenstates form a complete basis, which also diagonalize the Hamiltonian. The states are labeled by two quantum numbers: $n$ and $m$, which are the eigenvalues of the number operators: $a^\dagger a \ket{n,m} = n \ket{n,m}$ and $b^\dagger b \ket{n,m} = m \ket{n,m}$, from which it follows that $n, m \geq 0$. The quantum number $n$ labels the energy; the quantum number $m$ labels the (relative to $n$) angular momentum. The states $\ket{n,m}$ satisfy

$$H \ket{n,m} = \hbar \omega (n + \frac{1}{2}) \ket{n,m}$$  \hspace{1cm} (3.21)

$$L_z \ket{n,m} = \hbar (n - m) \ket{n,m}$$  \hspace{1cm} (3.22)

and the ladder operators act as

$$a \ket{n,m} = \sqrt{n} \ket{n-1,m} \hspace{1cm} b \ket{n,m} = \sqrt{m} \ket{n,m-1}$$  \hspace{1cm} (3.23)

$$a^\dagger \ket{n,m} = \sqrt{n+1} \ket{n+1,m} \hspace{1cm} b^\dagger \ket{n,m} = \sqrt{m+1} \ket{n,m+1}$$  \hspace{1cm} (3.24)

These quantum numbers will frequently return throughout the rest of this thesis. It is therefore informative to study these states in more detail. In particular, we want their coordinate representatives. In standard coordinates the ladder operators are given by

$$a = -\frac{i}{\sqrt{2l_B}} \left[ \frac{1}{2} (x - iy) + l_B^x (\partial_x - i \partial_y) \right] \hspace{1cm} b = \frac{1}{\sqrt{2l_B}} \left[ \frac{1}{2} (x + iy) + l_B^x (\partial_x + i \partial_y) \right]$$  \hspace{1cm} (3.25)

$$a^\dagger = \frac{i}{\sqrt{2l_B}} \left[ \frac{1}{2} (x + iy) - l_B^x (\partial_x + i \partial_y) \right] \hspace{1cm} b^\dagger = \frac{1}{\sqrt{2l_B}} \left[ \frac{1}{2} (x - iy) - l_B^x (\partial_x - i \partial_y) \right]$$  \hspace{1cm} (3.26)

Switching over to complex coordinates

$$z = x - iy \hspace{1cm} \bar{z} = x + iy$$

$$\partial \equiv \partial_z = \frac{1}{2} (\partial_x + i \partial_y) \hspace{1cm} \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2} (\partial_x - i \partial_y)$$  \hspace{1cm} (3.27)

This unusual definition of $z$ (usually we have $z = x + iy$) can be resolved by flipping the magnetic field. For now we will leave it as it is and simply plug this into (3.26). We have

$$a = -\frac{i}{2\sqrt{2l_B}} \left[ z + 4l_B^x \bar{\partial} \right] \hspace{1cm} b = \frac{1}{2\sqrt{2l_B}} \left[ \bar{z} + 4l_B^x \partial \right]$$  \hspace{1cm} (3.29)

$$a^\dagger = \frac{i}{2\sqrt{2l_B}} \left[ \bar{z} - 4l_B^x \partial \right] \hspace{1cm} b^\dagger = \frac{1}{2\sqrt{2l_B}} \left[ z - 4l_B^x \bar{\partial} \right]$$  \hspace{1cm} (3.30)
Other representatives are also possible [Yoshioka, 2002]. In terms of the representatives presented here the angular momentum and the Hamiltonian are given by

\[
H = \frac{\hbar \omega_c}{2} \left( \frac{\bar{z} z}{4l_B^2} - 4l_B^2 \bar{\partial} \partial + \bar{z} \partial - z \bar{\partial} \right) \quad (3.31)
\]

\[
L_z = \hbar \left( \bar{z} \partial - z \bar{\partial} \right) \quad (3.32)
\]

We now take the state \( \psi_{n=0, m=0}(z, \bar{z}) \), which is annihilated by both \( a \) and \( b \). This gives two differential equations, namely

\[
\bar{\partial}\psi_{0,0}(z, \bar{z}) = -\frac{z}{4l_B^2} \psi_{0,0}(z, \bar{z}) \quad (3.33)
\]

\[
\partial\psi_{0,0}(z, \bar{z}) = -\frac{\bar{z}}{4l_B^2} \psi_{0,0}(z, \bar{z}) \quad (3.34)
\]

The normalized solution is then

\[
\psi_{0,0}(z, \bar{z}) = \frac{1}{\sqrt{2\pi l_B}} e^{-\frac{|z|^2}{4l_B^2}} \quad (3.35)
\]

From this state the whole spectra of eigenstates is formed by successive application of the raising operators. In particular, states of the lowest Landau level have \( n = 0 \), and other eigenstates within this level are obtained through the action of \( b^\dagger \). This gives

\[
\psi_{0,m}(z, \bar{z}) = \frac{1}{\sqrt{2\pi 2^m m! l_B}} \left( \frac{z - 4l_B^2 \bar{\partial}}{2l_B} \right)^m e^{-\frac{|z|^2}{4l_B^2}} \quad (3.36)
\]

\[
= \frac{1}{\sqrt{2\pi 2^m m! l_B}} z^m e^{-\frac{|z|^2}{4l_B^2}} \quad (3.37)
\]

We now turn to the construction of single-particle states.

### 3.4 Single particle states

The previous section ended with the spectrum of angular momentum eigenstates, restricted to the lowest Landau level. The unnormalized eigenfunctions are

\[
\psi_m(z, \bar{z}) = z^m e^{-\frac{|z|^2}{4l_B^2}} \quad (3.38)
\]

Such states carry \( \frac{1}{2} \hbar \omega_c \) units of energy and \( m \hbar \) units of angular momentum. Figure 3.1 shows a plot of the probability distribution. General wavefunctions are constructed through linear combination of the eigenstates. With the basis at hand the most general form is quite elegant

\[
\Psi(z) = f(z) e^{-\frac{|z|^2}{4l_B^2}} \quad (3.39)
\]

The function \( f(z) \) is an analytical function. It does not contain any branch cuts or poles which is a desirable behavior since we are describing electrons. An example of such a function
3.4 Single particle states

Figure 3.1: Three-dimensional plot of the eigenfunction $|\psi_m|^2$. The factor $|z|^{2m}$ diverges when $|z|$ is taken to infinity. The exponent $e^{-\frac{|z|^2}{4l_B^2}}$ cancels this behavior, localizing the state along a circle with a width of order $l_B$.

is the localized wavepacket, constructed as a so-called coherent state

$$f_\lambda(z) = \frac{1}{\sqrt{2\pi l_B^2}} e^{\frac{1}{2}z^* \lambda} e^{-\frac{1}{2l_B^2} \lambda^* z}$$ (3.40)

$$|\Psi_\lambda(z)|^2 = \frac{1}{2\pi l_B^2} e^{-\frac{1}{2l_B^2} |z-\lambda|^2}$$ (3.41)

This wavefunction has the smallest spread possible and shows a Gaussian behavior localized around the (complex-valued) coordinate $\lambda$.

Let us study some of the details of the eigenstates (3.38) and the single-particle wavefunctions produced by them. Some of these results have already passed by in the previous section, but we will repeat them here since the setting is different.

From figure 3.1 but also from the eigenstates themselves it is clear that the eigenfunctions constitute a Gaussian-like behavior in the radial ($|z|$) direction. The maximum occurs at the radial distance $|z| = \sqrt{2ml_B}$ with a spread of order $l_B$. In the radial direction away from this maximum the eigenstate quickly tends to zero. This is a balancing between the factors $z^m$ and $e^{-\frac{|z|^2}{4l_B^2}}$. The first blows up far away from the origin, while the latter cancels this behavior regardless of the angular momentum $m$. Thus the latter factor, which we emphasize is the same for all eigenstates, is an overall factor and serves as a sort of 'confining mechanism' by canceling the divergence behavior of the polynomial part.

Suppose we are dealing with a circular system of area $A = \pi R^2$. Such a system is realized by adding a confining potential to the Hamiltonian. Alternatively, we may also impose an upper bound to the angular momentum, meaning we only allow eigenstates up to a certain order $m_{\text{max}}$. The maximum of an eigenstates occurs at around $|z|_{\text{max}} = \sqrt{2ml_B}$, so if we only consider eigenstates of order $m \leq m_{\text{max}} = \frac{R^2}{2l_B^2}$ we effectively confine the electron to a disk of radius $R$. Note that it is the geometrical factor which is effectively responsible for simulating this finite-size effect.

We may assume that in the presence of a physical confining potential the eigenstates with angular momentum $m \leq m_{\text{max}}$ are still a good approximation of the true eigenstates of the
system. This means that there are in total $m_{\text{max}}$ eigenstates available. That is, the number of single-electron states within the lowest Landau level of a circular system equals

$$N_\Phi = \frac{R^2}{2l_B^2} = \frac{A}{2\pi l_B^2}$$ \hspace{1cm} (3.42)

This is in agreement with formula (2.28).

The end of this subsection is ended with some words on the function $f(z)$ and the wavefunctions that follow from it. It is very important to keep in mind that this function is **holomorphic** and has a highest degree for finite-size samples. By holomorphic we mean that the function $f$ depends on $z$ alone, and not on $\bar{z}$. In contrast, the factor $e^{-\frac{|z|^2}{4l_B^2}}$ does depend on $z$ and $\bar{z}$ (although not on their arguments). This factor arises because of the planar geometry we have been assuming. This allowed for the simultaneous diagonalization of the angular momentum operator $L_z$ and the Hamiltonian. However, this went at the cost of translational invariance.

One of the great novelties of the quantum Hall regime appears when such states are considered on different types of topologies and geometries. So far we have been working on a quite trivial surface, namely a compact surface with no holes and (flat) geometry. In appendix A a different type is considered, namely that of a spherical geometry. This surface has the same topology as the planar case, but comes equipped with a different geometry. Specifically, it is a compact surface with no boundary and of finite size. The main result, which is sketched in the appendix, is that such conditions automatically lead to a finite number of eigenstates - much like the finite-size 'disk' we have considered above. The difference, however, lies in their geometrical part. Specifically, wavefunctions are of the type

$$\Psi(\theta, \varphi) = \frac{e^{-\frac{i}{2}\varphi}}{(1 + |z|^2)^{N_\Phi/2}} \sum_{m=0}^{N_\Phi} a_m z^m$$ \hspace{1cm} (3.43)

$$z = \frac{1}{\tan\frac{\pi}{2}} e^{i\varphi}$$ \hspace{1cm} (3.44)

This arises due to a stereographical projection onto the compactified plane. Note the similarity between (3.43) and (3.39), if we identify

$$e^{-\frac{i}{2}\varphi} \left(1 + |z|^2\right)^{N_\Phi/2} \iff e^{-\frac{|z|^2}{4l_B^2}}$$ \hspace{1cm} (3.45)

$$\sum_{m=0}^{N_\Phi} z^m \iff f(z)$$ \hspace{1cm} (3.46)

As we mentioned before, we can also consider surfaces with non-trivial topologies. The simplest example is that of a torus, which has one handle. The torus can be viewed as a parallelogram with opposite ends identified. It is has a flat geometry, meaning the single-particle wavefunctions are similar to that of the planar case (3.39). Indeed, the geometrical factor is carried over to the wavefunctions on the torus. However, the periodic boundary conditions of the torus are an essential symmetry of the system and as a result rotational symmetry is no longer present. This means the function $f(z)$ is no longer restricted to ordinary polynomials, but rather have a richer structure in order to restore the periodic behavior. See for instance [Haldane and Rezayi, 1985].

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3. Quantum Hall wavefunctions
3.5 From single to multiparticle states

In this thesis we will not be considering explicit expressions of wavefunctions on the torus. However, we do emphasize the richness of the physics that arises when considering such non-trivial topologies. We therefore will be (re)viewing such results when we use other techniques, such as conformal field theory and topological field theory. Luckily, these mathematical structures allow us to bypass the explicit form of the wavefunction on the torus and still extract relevant information from the system. This will become more clear later on.

For now we end this with the general form of quantum Hall wavefunctions

$$\Psi(x, y) = f(z)F_g(z, \bar{z})$$ (3.47)

In this product the function $f(z)$ is a 'topological factor' while the function $F_g$ is a 'geometrical factor'. Although both are essential for a full understanding of the quantum Hall system, we will frequently omit the geometrical part when we move over to the conformal field theory description in chapter 5.

3.5 From single to multiparticle states

So far we have only been dealing with single-particle eigenstates. The corresponding wavefunctions consist out of polynomials times an overall factor. When constructing multiparticle wavefunctions (at least) two effects have to be taken into account, both of which have a significant impact on our physical description so far.

First of all we have to take the statistics of the electrons into account. Electrons are indistinguishable and fermionic and as a result the wavefunction of any multi-electron state is subject to such constraints. The construction of such a multiparticle Hilbert space is a tedious manner [Yoshioka, 2002]. Second, multiparticle electrons come equipped with a Coulomb interactions among the electrons. These interactions are the essential driving force behind the fractional effect.

Let us first consider a basis for the multiparticle states. For that, we make an important assumption. We restrict the Hilbert space of each individual electron to that of the lowest Landau level. Such a restriction leads to the following general form of the multiparticle wavefunction [Girvin, 2000]

$$\Psi(z_1, \ldots, z_{N_e}) = f(z_1, \ldots, z_{N_e})e^{\frac{j}{e\hbar} \sum_{i=1}^{N_e} |z_i|^2}$$ (3.48)

Such a form always describes multiparticle wavefunctions as long as there is no Landau level mixing. We will assume this to hold. It is reasonable to expect, as long as the interaction potential $V(|z_1 - z_2|)$ is relatively weak. That is, the difference in energy between Landau levels is of order $\hbar \omega_c$. Therefore any mixing between Landau levels needs to cross this energy gap.

With this restriction in mind we write down a basis for the multiparticle case. This follows from the discussion of the single-particle case as treated in section 3.4. We have

$$\prod_{i=1}^{N_e} z_i^{m_i}$$ (3.49)

These states are eigenstates of the total angular momentum operator. We interpret them as the $i$'th electron is in the state with angular momentum $m_i$. The total angular momentum
is the total degree of the term, namely $\sum_i m_i$. This basis comes into play by demanding the multiparticle wavefunction to be an eigenstate of the total angular momentum operator, which is for instance accomplished by putting the system on a sphere. For example, the wavefunction for a completely filled Landau level is constructed as follows. We have $N_e$ electrons that are divided over an equal number of states. Due to statistics there is only one, unique way of accomplishing this, which is given by

$$f(z_1, \ldots, z_{N_e}) = \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \cdots \\ 1 & z_3 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$ (3.50)

This determinant is called a Slater determinant [Yoshioka, 2002] and, including the geometrical factor, it is usually written as follows

$$\Psi(z_1, \ldots, z_{N_e}) = \prod_{i<j} (z_i - z_j) e^{\frac{i}{4\hbar} \sum_{i=1}^{N_e} |z_i|^2}$$ (3.51)

Since all states are occupied we have a filling factor of $\nu = \frac{N_e}{N_\Phi} = 1$. This is the (idealized) wavefunction of a completely filled Landau level.

Turning now to the discussion on Coulomb interactions. As stated in section 2.5 the Coulomb interaction is the essential mechanism behind the forming of an energy-gap and the existence of an incompressible state at fractional filling factors. Adding such a potential term to the Hamiltonian

$$H_{\text{kin}} \to H_{\text{kin}} + H_{\text{int}}$$ (3.52)

will affect the multivariable states significantly. The usual treatment of such an additional term in the Hamiltonian would proceed via a perturbative expansion. However, the quantum Hall case is special. This is because the kinetic energy term is completely fixed and the same for all particles present if we assume a strong-magnetic-field limit and neglect inter-Landau level mixing. So although the interaction term is relatively weak, it cannot be treated perturbatively.

However, this does not mean we are completely in the dark here. We have already seen some of the general features concerning the quantum Hall regime. Specifically, imposing additional symmetries such as rotational symmetries and restriction to the lowest Landau levels have an enormous impact on the general structure of the wavefunctions. Through educated guesses one still can come up with expressions for wavefunctions.

For example, suppose we want to incorporate the Coulomb interaction into our description. We know that such an interaction is long-ranged and is strongly repulsive for short distances. Furthermore, such a potential term is a function of the separation between electrons alone, i.e. $V(|z_i - z_j|)$. We may write

$$f(z_1, \ldots, z_{N_e}) = \prod_{i<j} g(z_i - z_j)$$ (3.53)

In this expression $g(z)$ is a function with the property that it contains a zero at $z = 0$. We have effectively factorized the wavefunction into products of two-body wavefunctions. When any
two electrons approach each other the complete wavefunction vanishes, due the factorization. The wavefunction naturally tends to configurations in which the electrons are kept away from each other.

Written this way \( f \) is called a Jastrow function, and it has been around since the 1960’s. It serves as a strong starting point for quantum Hall wavefunctions, which we will demonstrate soon. However, it is important to realize that this function only takes into account two-body interactions. Higher orders correlations are neglected which at first is a reasonable assumption. Later on our starting point will be different, and we will also take higher order correlations into account. In conclusion, the Jastrow-function is a very useful form, but not applicable to the entire quantum Hall regime.

3.6 Laughlin’s wavefunction

With all approximations in hand we are now in a position to derive the first great success of a theoretical approach to quantum Hall wavefunctions within the fractional quantum Hall effect, which is Laughlin’s wavefunction. We approach the situation via a variational approach. For that, we first list the physical considerations [Prange and Girvin, 1987]

1. The wavefunctions is constructed out of products of single-electron wavefunctions in the lowest Landau level.
2. The wavefunction describes electrons, which are fermions. The function \( f \) is therefore totally antisymmetric with respect to all its coordinates.
3. The wavefunction is described by a Jastrow-type function, keeping the electrons apart.
4. The wavefunction is an eigenstate of the total angular momentum operator. The holomorphic function \( f \) is therefore a homogeneous polynomial (homogeneous meaning that all terms have the same degree).

Based on these ingredients and inspired by the expression for the polynomial which describes a filled Landau level, Laughlin came up with a variational wavefunction satisfying these requirements

\[
\Psi(z_1, \ldots, z_{N_e}) = \prod_{i<j} (z_i - z_j)^M e^{\sum_i |z_i|^2 / \mu_B}, \quad M \text{ odd and positive} \quad (3.54)
\]

This wavefunction is called Laughlin’s wavefunction. It will frequently appear throughout this thesis, with the function \( f = \prod_{i<j}(z_i - z_j)^M \) denoted as a Laughlin-Jastrow factor.

Quite remarkably, these functions are the only functions which satisfy these requirements. Even more striking is that they are characterized by a single parameter, \( M \), which turns out not even to be a variational parameter since it is completely fixed by the angular momentum condition and the statistics of the electrons. So although this approach is variational, all possible functions are potential (trial) quantum Hall wavefunctions. The argumentation presented here leads to a sequence of quantum Hall systems, one for each parameter \( M \).

What does the parameter \( M \) represent? As we will demonstrate now, it fixes (or is fixed by) the filling fraction. First, notice that the maximum degree of each coordinate \( z_i \) appearing equals

\[
\text{Maximum degree} = M(N_e - 1) \quad (3.55)
\]
From the perspective of one electron and results from section 3.3 we view this maximum degree as the maximum angular-momentum of a single electron. Keeping the discussion of section 3.4 in mind it is clear these results correspond to a circular sample of finite size with radius $\sqrt{2M(N_e - 1)}l_B$. The maximum degree is therefore identified with the number of flux quanta, $N_\Phi$. This gives

$$N_\Phi + 1 = MN_e - M \quad (3.56)$$

Assuming the density of the electrons is uniform allows us to compute the filling factor (i.e. the dimensionless electron density). In the thermodynamic limit we take $N_e$, and as an effect $N_\Phi$ as well, to infinity. This gives the filling factor

$$\nu = \frac{1}{M}, \quad M \text{ odd and positive} \quad (3.57)$$

In effect, Laughlins wavefunction constitutes a whole family of wavefunctions. Although the construction based on symmetries serves as an elegant way of deriving Laughlins wavefunctions, it is in fact an analogy due to Laughlin himself that led to a better understanding of these wavefunctions. This analogy is called the plasma analogy, and it is usually presented as follows [Prange and Girvin, 1987], [Girvin, 2000].

The problem is mapped to that of a classical statistical physics problem by interpreting the probability distribution as the Boltzmann weight of a particular 2 dimensional classical plasma. One writes the partition function as

$$Z = \int \cdots \int dz_1 \cdots dz_{N_e} |\Psi(z_1, \ldots z_{N_e})|^2 \quad (3.58)$$

and the Boltzmann weight is therefore

$$|\Psi(z_1, \ldots z_{N_e})|^2 = e^{-\beta U_{\text{plasma}}}, \quad \beta = \frac{1}{M} \quad (3.59)$$

$$U_{\text{plasma}} = \frac{m_e^2}{l_B^2} \sum_{i<j} (- \log |z_i - z_j|) + \frac{m_e}{4l_B^2} \sum_k |z_k|^2 \quad (3.60)$$

The strength of the analogy lies in the fact that the fake classical potential $U_{\text{plasma}}$ mimics that of a single-component plasma with charge $m_e$, confined to the two-dimensional plane with in the presence of a neutralizing background. Using known techniques from classical statistical physics and plasma physics one can say a great deal about these wavefunctions. In particular, one can derive the filling fraction, i.e. the electron density and show that any deviation from a homogeneous distribution results in a large energy cost. The wavefunction is said to be locally neutral. We refer to the literature for such discussions.

### 3.7 Exact Hamiltonian and zero points

Laughlin’s wavefunction serves as a very good approximation of the true ground state, especially at a filling fraction of $\nu = \frac{1}{3}$. However, it is important to keep in mind that it is not the true ground state, which is impossible to write down explicitly. Still, numerical studies [Prange and Girvin, 1987] have shown that the overlap of Laughlin’s wavefunction with exact ground states has an error margin less than 1 percent for low values of $M$. It is a
3.8 Excitations

remarkably successful wavefunction for all sorts of two-body interaction terms one can come up with (e.g. $\frac{1}{r}$, $-\log r$ or $e^{-r^2}$), even though there are no variational parameters present in the wavefunction.

It is possible to construct Hamiltonians (more specific, interaction potentials) for which Laughlin’s wavefunctions are the exact ground state. These correspond to leading potentials of the more general class of Haldane’s pseudopotentials. In some extent, these leading potentials mimic the Coulomb interaction in the lowest Landau level. For our discussion, and in view of what is coming ahead, it is more instructive to approach the wavefunction on a more analytical level. By that we mean that we look at the zero points associated to this wavefunction.

The wavefunction has the clear property that when two electrons are brought together, the wavefunction rapidly vanishes with a factor $(z_i - z_j)^M$. This behavior is essential for the Pauli principle to hold, but it is also a feature which favors a reduction of the Coulomb potential.

Recall that the angular momentum of one particle wavefunction is given by its maximum degree, i.e. the number of flux quanta $N_\Phi$. Since the Laughlin-Jastrow factor is a finite polynomial of complex coordinates the fundamental theorem of algebra states that the wavefunction has $N_\Phi$ roots with respect to all its coordinates. Laughlin’s wavefunction is special in the sense that all roots are attached to the electrons. Only when two electrons are brought together does the wavefunction tend to zero.

Consider a state in which this distribution of zeroes is different. For such a state to describe fermions it must contain at least one factor of $\prod_{i<j}(z_i - z_j)$ to satisfy the Pauli principle. That means that, with respect to a coordinate $z_i$, $(N_e - 1)$ of its roots are attached to the remaining fermions. The remaining $(M - 1)(N_e - 1)$ roots may be placed anywhere else. Compared to Laughlin’s wavefunction such a distribution is less favorable for a reduction of the Coulomb interaction. First off all, there is now a larger probability that the electrons are close to each other, and this causes an increase in the potential energy. Furthermore, electrons tend to be repelled from the position of the roots. This causes a decrease in density around those positions, which again causes an increase in the potential energy - this time due to the charge deficit and the positively charged, neutralizing background.

Laughlin’s wavefunction has an energetically optimal distribution of the zeroes available and the electrons which form the state.

3.8 Excitations

So far we have seen in what manner Laughlin’s wavefunction describes an (idealized) system of electrons with zeroes attached to them. The number of zeroes is given by the number of flux units, and the ratio between the number of electrons and the number of zeroes equals the filling factor. There is a sequence of such filling factors, given by one over an odd number, and each element of this sequence corresponds to a Laughlin state. Although not proven in this thesis, but what is clear from experiment, is that these states form plateaux meaning they are incompressible and thus fall into the category of quantum Hall states. Such a plateau is characterized by the fact that over a certain range of the magnetic field strength the resistance of the sample hardly deviates from the value of $1/\nu$ times the elementary quantum of resistance.

\[ \text{Laughlin’s wavefunctions are a family of wavefunctions, but they will frequently be addressed as simply “the” wavefunction.} \]
However, the filling factor $\nu$ coincides with the dimensionless electron density only at a very specific value of the magnetic field strength. By tweaking this field strength one adds or removes a certain number of flux quanta. The ratio between the number of electrons and the number of flux quanta is no longer given by the proportionality constant $\nu$, but rather by some number close to it, which we call $\nu^*$. The true dimensionless electron density of the system is given by $\nu^*$ and along the plateaux this quantity changes according to change in the magnetic field strength. Still, there is a mechanism which causes the conductivity to pinpoint onto the ideal value $\nu$. This implies we are dealing with a system which somehow takes care of the extra flux quanta by reorganizes itself such that its conductivity is still determined by $\nu$. To state it once more, we do not focus on the question what kind of mechanism is behind this phenomenon, but rather take it as a given and ask ourselves what the consequences are for such a state to exist.

What we will address in this subsection is the following. When we increase the magnetic field strength the electron density $\nu^*$ deviates from the idealized value $\nu$. The increase introduces extra flux quanta to the system, which, in the spirit of the discussion in the previous section, causes an imbalance between the number of electrons versus the number of zeroes. We will now show that this change in electron density corresponds to the introduction of quasiparticles in the system. These quasiparticles are called anyons.

We will follow an argument originally given by Laughlin [Laughlin, 1983]. See also the paper by [Stern, 2007] for a more pedagogical discussion on the subject. We start with a fractional quantum Hall state with filling factor $\nu$ on an annulus-shaped sample. The annulus has a hole in the middle, and in this hole we place a solenoid. Through the solenoid runs a magnetic field, perpendicular to the sample. This magnetic field carries the extra flux quanta we introduce to the system. This system strongly resembles the one used to describe the Aharonov-Bohm effect [Wilczek, 1990]. Not surprisingly, it is precisely this effect which plays a central role in this discussion.

It is well known that the magnetic field, which we is confined to the solenoid, is described by a vector potential outside of the solenoid

$$\mathbf{A} = \frac{\Phi_s(t)}{2\pi r} \hat{\theta}$$

$\Phi_s(t)$ stands for the flux inserted into the solenoid at time $t$. At $t = t_0$ we set $\Phi_s(t)$ to zero. As time progresses we start to increase the flux through the solenoid. This process is done adiabatically, meaning that if the system was in an eigenstate at $t < t_0$ it remains in the eigenstate of the changing Hamiltonian for $t > t_0$. If the system is in some particular linear combination of eigenstates, then incompressibility and adiabaticity ensure the state remains in the same state with respect to these changing eigenstates.

A change in the vector potential causes an electric field in the same direction. In turn, the electric field induces a current in the sample. This is known as Faraday’s law of induction. The electric field is given by

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{2\pi c} \frac{\partial \Phi_s(t)}{\partial t} \frac{\hat{\theta}}{r}$$

But we are still dealing with the quantum Hall effect. Therefore the response current is perpendicular to the applied electric field (i.e. radially outwards or inwards, depending on
3.8 Excitations

the direction of the magnetic field). The conductivity tensor is also set, therefore the current density is

\[ j_r = \nu \frac{e^2}{\hbar} E_\theta \]  

(3.64)

to obtain the net current, which flows along the radial direction, we integrate the radial component along a circle whose origin is the solenoid, e.g.

\[ I = \int_{0}^{2\pi} d\theta \left( r j_r \right) \]

\[ I = \nu \frac{e^2}{\hbar c} \frac{\partial \Phi_s(t)}{\partial t} \]  

(3.65)

Finally, the total charge at time \( t \) which is transferred this way is obtained through integrating over time, starting when the flux was zero \( (t_0) \) up to a time \( t \), e.g.

\[ Q(t) = \int_{t_0}^{t} dt I \]

\[ Q(t) = \nu \frac{e^2}{\hbar c} \Phi_s(t) \]  

(3.66)

This is the amount of charge lost obtained in the center of the annulus through the action of flux insertion, depending on the sign of the magnetic field relative to the fixed background field. In effect, flux insertion causes a deficit of charge located at the insertion point, and flux extraction causes a lump of charge. The first is referred to as a quasihole, the latter as a quasiparticle, referring to the fact that there is a surplus of electrons located at the origin.

What is the effect of the presence of the flux tube on the system, after this flux insertion has taken place? In other words, what is the effect of the gauge field on the wavefunction? There is the possibility that the eigenstates of the new system has no correspondence to the quantum Hall system whatsoever, making this an important question to address.

First notice that outside of the flux tube this gauge field is pure gauge. This means that locally it is equal to a total derivative of a smooth function. For our wavefunction at hand we may write

\[ A = \nabla \chi \]  

(3.67)

\[ \chi(z) = \frac{\Phi_s}{2\pi} \theta \]  

(3.68)

The coordinate \( \theta \) is discontinuous, since \( 2\pi \) and 0 constitute the same angle, therefore the function \( \chi \) is multivalued and singular at the origin. This reflects why this function can only be valid at local, open patch since the branch cut of \( \theta \) can always be put outside this patch.

Because the gauge field is a pure gauge we may apply gauge transformations to gauge it to zero. However, as an effect of the topology we can only perform this locally, but not globally. Let us press on to see what conflicts arise.

The gauge field \( A \) can be gauged away via \( A \rightarrow A - \nabla \chi \). Globally such an action is forbidden, at least in principle. Recall that when performing a gauge transformation the wavefunction changes according to \( \Psi(z) \rightarrow \Psi(z) e^{ie\chi(z)/\hbar} \). Stubborn as we are, we perform the gauge transformation globally. The result is a multivalued wavefunction

\[ \Psi(z) \rightarrow \Psi(z) e^{ie\Phi_0/\hbar}, \quad \Phi_0 = \frac{\hbar c}{e} \]  

(3.69)

If we restrict this wavefunction to an open patch it is single-valued again. But even the global version has an intuitive interpretation. Outside the solonoid, and inside the annulus the gauge
field is gauged away at the cost of the introduction of a multivalued wavefunction. The cause of this phase factor stems from the Aharonov-Bohm effect. By encircling the solonoid the electron experiences a Berry curvature and produces a phase factor. The Berry curvature is nothing but the vector potential $A_\mu$. In general, however, other effects may also serve as a Berry curvature, for example a geometric curvature. By gauging the field away the Berry curvature is removed and its effect is incorporated into the multivalued phase factor of (3.69).

There is an exception to this story about multivaluedness though. It arises when the inserted flux is a multiple of the unit flux quantum, e.g. $\Phi_s = n\Phi_0$. In this case $\theta$ and $\theta + 2\pi$ produce the same phase factor and the wavefunction is again single-valued. Gauging the gauge field to zero produces a phase factor $e^{in\theta}$.

The point to this story is as follows. The insertion of the flux has two effects. First off all, it creates a lump of charge located at the origin. Second, it introduces a vector potential to the system. This means the new Hamiltonian describing this particular system is not the same as the Hamiltonian present before flux insertions. The exception takes place when the inserted flux equals an integer number of flux units. In that case we are allowed to gauge the solenoid’s vector potential away. This gauge transformation restores the Hamiltonian back to its original form, that is, as it was before the flux insertions. As a corollary the spectrum of eigenstates is the same as before. Near and in the solenoid the gauge potential cannot be gauged away due to the presence of the magnetic field. Within that region the Hamiltonian is not valid. But such a region is smaller than the typical length scale $l_B$, and in view of what is be treated in the next chapter, we typically only care about length scales larger than $l_B$.

From now on the statement of flux insertion is always referred to the case where an integer number of flux quanta is inserted in the system. Its net effect is the creation of a quasiparticle through an adiabatic reorganization of the eigenstates. The charge of the quasihole is

$$Q_{q.h} = e\nu$$

while for quasiparticles it is $-e\nu$. This is partially expected. The quasihole refers to a deficit of charge, and is therefore positively charged. What is perhaps suprising is that, since $\nu$ is fractional the quasiparticle carry a fractional charge which is smaller than the electron charge. This is one of the remarkable properties of the fractional quantum Hall systems. Through some underlying mechanism the electrons are able to form states containing fractionally charged excitations. As a corollary, fractionally charged particles obey a statistics different from ordinary fermions and boson. We will return to this matter in the next section.

Based on these argument Laughlin proposed the following operator which produces a quasihole at the origin

$$\Psi(z_1, \ldots, z_{N_e}) \longrightarrow \prod_i z_i \Psi(z_1, \ldots, z_{N_e})$$

$$= \left( \prod_i |z_i| \right) e^{i\sum_i \theta_i} \Psi(z_1, \ldots, z_{N_e})$$

Such an operation indeed matches the description so far, as we will prove now. The operator contains a factor of $z_i$ for each electron coordinate. Since $z_i = |z_i| \exp(i \arg z_i) = r_i e^{i\theta_i}$, its action is as follows. Multiplication through the factor of $|z_i|$ pushes each electron radially outwards. This accounts for the effect of inserting an infinitesimal flux tube with one unit of flux. Multiplication of the phase factor $e^{i\theta_i}$ is equivalent of gauging this flux tube away.
3.8 Excitations

thus introducing additional angular momenta for each electron. One could view this as an adiabatic relabeling of the single-particle eigenstates

\[ z_i^m \rightarrow z_i^{m+1} \]  

(3.73)

Direct expansion of (3.72) indeed shows that these are related. The resulting electron state is indeed an eigenstate of the Hamiltonian, different from the ground state.

There is also an analog operator for the quasiparticle. Its given by the conjugate operator of the quasihole operator, i.e.

Quasihole operator: \[ \prod i z_i \]  

(3.74)

Quasiparticle operator: \[ \prod i -\partial z_i \]  

(3.75)

These operators create a quasihole/-particle at the origin. Through a translation we can create the quasihole/-particle anywhere we like. Throughout this thesis we will only deal with the quasiholes, as the quasiparticle’s properties are obtained through conjugation.\(^2\)

Having identified the operator which creates a quasihole, we can obtain an expression for the wavefunction containing \(N_q\) quasiholes. It follows from the insertion of the quasihole operator into the expression for the wavefunction [Laughlin, 1983]

\[
\Psi(z_1, \ldots, z_{N_e}) = \prod_i \prod_j (z_i - w_j) \prod_{i<j} (z_i - z_j)^M e^{\sum_i |z_i|^2/M} \]  

(3.76)

To summarize, the adiabatic insertion of a single unit of flux through an infinitely thin solonoid causes the creation of a deficit of charge located at the solenoid. Accompanied by a singular gauge transformation the flux is removed and its effect on the electron state is mediated through the addition of angular momenta to all electrons present. The deficit of charge is now a quasihole. The amount of charge removed equals \(\nu e\), meaning the quasihole is positively charged. Apart from charge the quasiparticles also carry vorticity. When an electron encircles the quasihole once it picks up a phase of \(2\pi\), which is the characteristic of a topological defect. The fractionally charged quasiparticles are one of the most important characterizations of the fractional quantum Hall effect.

Remarks

This final section of this chapter is devoted to a brief survey of some important aspects.

Let us mention that the Laughlin wavefunction is the first, prime example of what is called an Abelian quantum Hall state. During the 1980’s schemes were developed which, on the basis of the Laughlin wavefunction, which proposed new ways of obtaining quantum Hall states. The first of which is called the hierarchy scheme.

It is postulated as follows. One starts out with a Laughlin wavefunction at filling fraction \(1/M\). The next step is to insert a very large number of quasiholes. As we have seen, these insertions correspond to local deficits in the electron density. If a significant amount of quasiholes are present in the system, the electron density will start to increase on a local

\(^2\)Apologies for all the quasiholes that are mistakenly refered to as quasiparticles throughout this thesis...
scale. The next step is to condense the quasiholes into a Laughlin state, such that they form an incompressible state. The electrons started out in an incompressible state, and remain in such a state throughout the process. However, the presence of the quasiholes cause a shift in the electron density. This mixed state is interpreted as a “daughter state”. Since the quasiholes are also in a Laughlin state, we can repeat the procedure for that state as well. This way a whole family of states is obtained. This is basically the hierarchy approach, \cite{Haldane1983a,Halperin1984}.

Later on Jain developed another approach, called composite fermion theory. It turned out to include the states described by the hierarchy scheme, and also accounted for other filling fractions as well. The procedure is as follows.

One starts out with a number of electrons in a magnetic field. Through the application of a singular gauge transformations we can “attach” a number of flux quanta to the electrons. This turns the electrons into flux-carrying particles, called composite fermions. If the number of flux quanta per electron equals $2p\Phi_0$, then the resulting magnetic field is reduced by

$$B^* = B - 2p\Phi_0 n_e$$

(3.77)

where $n_e$ is the electron density. We view this system much like the old one, in the sense that we deal with (composite) fermions in the presence of a (reduced) magnetic field. With respect to this reduced magnetic field the system develops Landau levels. By attaching the right amount of flux to the electrons, we can obtain composite fermions who’s effective filling fraction is such that they occupy an integer number of effective Landau levels. This filling fraction is given by

$$\nu^* = \frac{n_e\Phi_0}{|B^*|}, \quad \nu^* \in \mathbb{Z}_{>0}$$

(3.78)

In turn, this corresponds to a different filling fraction for the electrons, namely

$$\nu = \frac{n_e\Phi_0}{B} = \frac{\nu^*}{2p\nu^* \pm 1}$$

(3.79)

These are called Jain’s states \cite{Jain1998}. The name “composite particle” will also appear later on in this thesis, but these do not correspond to these particles described here.

The hierarchy and Jain states are another example of Abelian quantum Hall states. This is the only part of the thesis in which they appear, but it is good to understand that there are a lot of examples of quantum Hall states available. In almost all these versions the Laughlin series appears as a (sometimes trivial) realization.

Another remark which we need to make is that of the bosonic version of the quantum Hall effect. Indeed, we have dealt with two-dimensional systems of electrons subject to magnetic field. In recent years it has become clear that other systems also exhibit some of the typifying behavior of the quantum Hall regime. For example, trapped Bose-Einstein condensates can be trapped in rotating potential. The bosons are weakly interacting, confined to a trap which rotates with some angular frequency. Quite magically, the system develops a gap and the corresponding Hamiltonian can be mapped to that of the quantum Hall effect.

The reasons why we mention this is that we are not restricted to fermions alone. In later chapters we will work primarily with bosons in the understanding that they are easily turned
into fermions. By that we mean that we can account for the statistics through multiplication of an overall factor. This will all be explained in the relevant chapters.

Still, the rotating BEC’s show that there we should not restrict ourself to the quantum Hall regime alone, as there might be plenty enough alternative physical realizations.
3. Quantum Hall wavefunctions
Having set the stage of the quantum Hall system from a first quantized perspective, we now move on to a field theoretic description. Our first aim is to obtain an effective theory describing the physics that typify the quantum Hall effect. As it turns out, this theory is Chern-Simons theory, which is an example of a topological quantum field theory. Topological field theories exhibit all kind of exotic behavior. They do not possess any local degrees of freedom, yet the physics associated to them is highly non-trivial. We will show how this theory reproduces such effects as conductivity quantization, the energy gap and the fractionally charged quasiholes, and also describe some of the peculiarities that come along with the theory, among of which is the concept of fractional statistics.

The second part of this chapter is an understanding of the quantization of Chern-Simons theory, and its results are quite staggering, to say the least. What probably makes Chern-Simons theory so mysterious is its relation to dimensional reduction. In the late 1980’s the famous Witten showed [Witten, 1989] that Chern-Simons theory has a strong relation with conformal field theory in two dimensions. This essentially arises due to the gauge invariance of the Chern-Simons theory. With this link in place we can bypass the (2+1)-dimensional description of quantum Hall systems (which is much harder to fully grasp) and make use of the highly structured and well-developed theory of CFT.

4.1 Chern-Simons theory as an effective field theory

In setting up a connection between the quantum Hall effect and conformal field theory, we automatically arrive in the realm of topological quantum field theories. Ironically, these theories were originally comprised as simple toy models in an attempt to get a better understanding of quantum field theory in general, long before their connection to the quantum Hall
regime was made. The area of TQFT has been developed to such a large extent, that even mathematicians appreciate their structure.

Their relevance to quantum Hall systems is largely based on the principle of renormalization. But first, let us go back to some of the basic principles of the Quantum Hall regime. Following the discussion by [Fröhlich and Zee, 1991] (see also [Zee, 1995] and [Ardonne, 2002])

(i) We are describing a system of electrons restricted to (2+1)-dimensional spacetime.

(ii) The electromagnetic current $J^\mu$ is conserved: $\partial_\mu J^\mu = 0$.

(iii) Parity and time reversal are broken by the external magnetic field.

(iv) Our goal is a description of the long distance (low energy) physics in terms of an effective, local Lagrangian.

The first two statements allow us to write the electromagnetic current as the curl of another field

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$$

(4.1)

The $\epsilon^{\mu\nu\lambda}$ stands for the antisymmetric tensor density, which, as it stands here, is only valid in three dimensions. This is one of the special features of the dimensionality we are working in.

Written out in this form there is a redundancy present in the field $a^\mu$. Replacing $a_\mu$ by $a_\mu + \partial_\mu \Lambda$ for any smooth, differentiable function $\Lambda$, leaves the conserved current $J^\mu$ intact. This redundancy incorporates $a$ with a gauge structure. Through the remaining of this chapter we will describe the physics through the statistical gauge field $a$, while keeping in mind that this is ultimately a description of the electromagnetic current $J$. The reason for this, which becomes clear later on, is that theory is much better understood and manageable in terms of $a$.

Having said that, we now turn to the question: what effective action do we need to write down in order to describe the essential features of the quantum Hall effect in terms of $a$? We first notice that since $a$ is a gauge potential the action can only consist out of gauge-invariant terms, such as the Maxwell term. As it turns out, the action relevant to the quantum Hall effect is the so-called Chern-Simons term. Chern-Simons theory is an example of a topological quantum field theory. Chern-Simons theory comes in many varieties, depending on the number of fields, the number of dimensions and the underlying gauge structure. For the quantum Hall effect we only consider (2+1)-dimensional case, and for now we specialize its gauge group to be Abelian. The corresponding Chern-Simons action is

$$S_{\text{eff}} = \int \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \ldots$$

(4.2)

In this action $k$ is the coupling constant and its value is always integer. This is a result of gauge invariance, and the proof is given later on. For now we will simply take this as a given. Before we dive in to a description of this action, we first motivate why this term is allowed to be written down in the first place (see [Zee, 1995] and [Fröhlich and Zee, 1991] for a more comprehensive review on these matters).

The precise microscopic details of the system are obscure, since it is too difficult to keep track of all degrees of freedom the systems exhibits. To bypass this descriptive barrier we focus
4.1 Chern-Simons theory as an effective field theory

on the physics of long distances instead. Since the quantum Hall effect is a macroscopic, bulk effect we expect that the relevant aspects are essentially described by the long-range modes alone. From a field theoretical point of view the relevant, long-range physics is approached using the scheme of renormalization. We can, at least in principle, integrate out the short distance modes of the system causing the theory to develop an effective action. This effective action describes the remaining long-distance modes. In general, under renormalization coupling constants of the interaction terms start “running” and terms which were previously absent are introduced to the action.

The integration of short-distance modes causes some terms in the action to scale at a relatively higher rate than other terms. In the end, only those terms which scale the “fastest” (the relevant terms) are kept; the long-range physics of the system is essentially encaptured by them. The influence of the other (irrelevant) terms become negligible, and upon renormalization they are simply thrown away. To be complete, there might also be terms present which are not affected by the renormalization procedure at all, which are called the marginal terms.

In any case, what remains is an effective action, valid for a long-range description of the system at hand, an approach often used in the systems exhibiting criticality. We stress, however, that the Quantum Hall states we are after are by no means critical. Rather, we expect that some features of the effective theory are carried over to the true physical state. An example of these features are so-called topological quantum numbers, such as the filling fraction, the shift and properties of the quasiparticle excitations.

The get a more intuitive feeling for this discussion, recall that the typical length scale of the system is given by the magnetic length. For typical samples this length scale is roughly valued around $l_B \sim 2.5 \cdot 10^2$ [nm]. The effective description given here is essentially obtained by integrating out the degrees of freedom with a length scale smaller than $l_B$. The theory we obtain is valid at distances larger than this typical length scale. In practice, when working with this effective theory the constraint of a minimal length scale is relaxed and only introduced at the very end of the calculations.

A few more words on why exactly the Chern-Simons theory is the effective theory that emerges from the renormalization procedure is in order. One way of determining whether a term appearing in the action is relevant relative to another is through an analysis of their scaling dimension. In our system the conserved current $J$ has mass dimension 2, the derivative $\partial$ has mass dimension 1 leaving the gauge potential $a$ to also have mass dimension 1. The simplest, lowest-dimensional interaction term is $a_\mu a^\mu$, but since it is not gauge invariant, it is of no use to us. The simplest term which matches all criteria posted before is the Chern-Simons term $\epsilon^{\mu\nu\lambda} a_\nu \partial_\nu a_\lambda$, which has a mass dimension of 3. As a comparison, the Maxwell term $f_{\mu\nu} f^{\mu\nu}$ has mass dimension 4.

Do note that it is certainly not the only possible topological interaction. There are a lot of variations available, which all go by the name of Chern-Simons theory. For example, we can introduce multiple gauge fields $a_i$, each corresponding to a separate conserved current. This is used if we are dealing with multiple filled Landau levels, which are taken to be separately conserved. See also [Wen, 1995]. The generalization we are eventually after is the case where the gauge structure of $a$ is taken to be non-Abelian. We will come to this in chapter 4.6.

These generalizations are quite relevant, as they contain a far richer structure than the ordinary Abelian Chern-Simons theory. This structure comes also at a cost of their interpretation. In the case of the Abelian Chern-Simons theory we are dealing with an Abelian gauge field, which, through the picture of renormalization, is viewed as an emergent degree of
freedom. This means that at the microscopic level the effective theory breaks down and the
gauge field becomes meaningless. Still, at larger scales its relation to the underlying degrees
of freedom is quite straightforward, as it describes the charge density and its flow through the
system. However, we must emphasize that this relation is not always so simple. Even worse,
in this thesis we will eventually move on to quantum Hall fluids who’s effective theory cannot
even be obtained through a renormalization procedure. Therefore the connection between
the gauge field and the electron degrees of freedom is not always as simple as depicted here.

4.2 Chern-Simons theory and the quantum Hall effect

As stated, from now on we will approach the quantum Hall regime from an effective field
theory point of view. This means we assume Chern-Simons theory effectively describes the
long-range physics. Such systems might not always correspond to true physical systems,
which is why we address them as so-called topological Hall fluids [Zee, 1995].

To clarify the relation to the quantum Hall system we will now reproduce the conductivity
tensor from the Abelian Chern-Simons Lagrangian. For that we introduce an external vector
potential and let it couple to the gauge field $a$. The resulting Lagrangian density is

$$L = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A^\text{ext}_\mu \partial_\nu a_\lambda \quad (4.3)$$

There is possible point of confusion arising here, so we make the following statement: the
external field $A^\text{ext}_\mu$ does not describe the fixed magnetic background field. What this vector
potential does describe is the electric field applied to the sample and small perturbations
around the fixed magnetic field. The effect of the static magnetic field, which serves as a
fixed background field, has already been processed through the renormalization procedure.

To realize why this is so, we have to understand that the renormalization approach is
carried out in the presence of a background field, the magnetic field. This is comparable to
the case of integrating out the degrees of freedom associated to an ionic lattic (phonons),
and incorporating them into the effective mass of an electron. The same happens here. For
instance, the strong-field limit causes the kinetic energy of the electrons to be fixed, due to
the presence of the Landau levels. Such properties are automatically incorporated into the
effective theory. The effect of a static and classic magnetic field is taken into account by the
Chern-Simons term itself and no longer through the interaction with some external potential.

Having said that, we now apply some basic machinery to this Lagrangian. The response
current due to the external field is determined by varying the action with respect to $a_\mu$. These
classical Euler-Lagrange equation of motions are

$$\partial_\sigma \frac{\delta L}{\delta (\partial_\sigma a_\mu)} - \frac{\delta L}{\delta a_\mu} = -\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A^\text{ext}_\lambda = 0 \quad (4.4)$$

$$\Rightarrow \quad \frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A^\text{ext}_\lambda \quad (4.5)$$

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4.3 Quasiparticles and fractional statistics

Next, plug in the conserved current $J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$. Written out in components

\begin{align*}
J^0 &= \frac{1}{2\pi k} (\partial_x A_y - \partial_y A_x) = \frac{1}{2\pi k} \delta B_z \quad (4.6.a) \\
J^x &= \frac{1}{2\pi k} (\partial_t A_y - \partial_y A_0) = \frac{1}{2\pi k} E_y \quad (4.6.b) \\
J^y &= \frac{1}{2\pi k} (\partial_t A_x - \partial_x A_0) = \frac{1}{2\pi k} E_x \quad (4.6.c)
\end{align*}

The first term describes a relation between the charge density and (local) perturbation of the magnetic field. By locally increasing (decreasing) the amount of flux we can create a lump (deficit) of charge. This is the classical analog of the quasiparticle excitation described in chapter 3.8. The (local) change in electron density $n$ is

$$\delta n = \frac{1}{2\pi k} \delta B_z \quad (4.7)$$

The remaining two terms (4.6.b) and (4.6.c) reproduce the Hall conductivity tensor. We set $E_y = 0$, and plug in the proper units ($e = 1$ and $h = 2\pi$)

\begin{align*}
\sigma_{xy} &= \frac{1}{2\pi k} = \frac{1}{k} \frac{e^2}{\hbar} \quad (4.8) \\
\sigma_{xx} &= 0 \quad (4.9)
\end{align*}

The coupling strength $k$ fixes the value of the quantization of the Hall conductance. Since $k$ is integer we recognise the Laughlin sequence as the odd-valued cases. As a bonus we are also handed a sequence for even-valued $k$. These also correspond to Laughlin wavefunctions, but this time the fundamental particles are bosonic.

This is of course not a derivation of the existence of a plateau and the proper quantization of the conductance. What we have shown is a consistency check of using a Chern-Simons term to describe the physics of the quantized conductance. The questions remain whether it is possible to (a) explain the stability of the coupling constant $k$ as to why and when a plateau may occur and (b) given this term, what kind of physics can be extracted from this seemingly trivial action. Both issues deserve a proper justification, and it will be the second question on which the focus of this thesis will be reserved for.

Let us end this chapter by mentioning the paper [Fröhlich and Kerler, 1991] in which the authors proved, on general grounds, that the features of any quantum Hall system, such as incompressibility, the existence of a gapped phase and quantization of the conductivity tensor can effectively always be described by a pure Chern-Simons term. There are always higher order terms present in the action, but they are not required to account for the essential features of the quantum Hall effect. This justifies, at least enough for us, why we focus our attention solely on the Chern-Simons term and forget about the remaining perturbations all together. However, it does not explain in any way if the Chern-Simons theory is stable or if other mechanisms cause the energy-gap to close thus destroying the effect.

4.3 Quasiparticles and fractional statistics

Apart from the conductivity tensor, the Chern-Simons theory also incorporates a description of the quasiparticles described in chapter 3.8. They are introduced as sources $j^\mu$ coupled
minimally with the gauge potential [Nayak et al., 2008]. We will ignore any kinetic terms of the quasiparticles, and simply let them move along fixed trajectories. The Lagrangian is

\[ \mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda - j^\mu a_\mu \]  

(4.10)

and the corresponding equations of motion

\[ \tilde{j}^\mu = \frac{k}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu a_\lambda \]  

(4.11)

For a point particle at rest the spatial components \( \tilde{j}^i \) vanish, and the equation of motion reduces to \( j^0 = \frac{k}{2\pi} \epsilon^{ij} \partial_i a_j \equiv \frac{k}{2\pi} b \). We introduced \( b \) as the Chern-Simons analog of a two-dimensional magnetic potential. Similar to the electromagnetic case we calculate the Chern-Simons flux carried by the particle by integrating over a small area containing the particle.

\[ \int_A d^2x \tilde{j}^0 = \frac{k}{2\pi} \int d^2x b = \frac{k}{2\pi} \oint_C \vec{a} \cdot d\vec{s} \]  

(4.12)

The second equality is the two-dimensional divergence theorem.

In effect the Chern-Simons term has attached a certain amount of CS-flux to all quasiparticles which couples minimally with the gauge field. A particle carrying minus one unit of Chern-Simons charge (= \( j^0 \)) also drags along \( \frac{2\pi}{k} \) units of Chern-Simons flux.\(^1\) The equation of motion (4.11) serves as a constraint: each quasiparticle drags along the field \( b \), and, vice versa, this field is only present at the particle’s position.

As a result of this flux attachment the quasiparticles exhibit quite interesting features. Consider the case of two quasiparticles. Each particle carries a certain amount of flux, tracing out a fluxtube through space-time. The magnetic potential \( b \) is zero everywhere, except at the particles’ location. However, the gauge field \( a \) is not. As a result of the coupling between the quasiparticles and the gauge field, two particles encircling each other will pick up a phase due to the (Chern-Simons version of the) Aharanov-Bohm effect. The amount of phase picked up by a particle, carrying one unit of CS-charge, encircling another identical particle at rest is \( e^{i\Phi} \), where \( \Phi \) equals the flux of the particle: \( \frac{2\pi}{k} \). A clockwise interchange of the two particles is equivalent to half this action. Therefore the phase acquired by the particle is also halved upon such an action. We write this interchange as

\[ \psi(r_1, r_2) = e^{i\theta} \psi(r_2, r_1) \]  

(4.13)

where \( \theta = \frac{\pi}{k} \). As a result the particles obey a new type of statistics called fractional statistics [Wilczek, 1990; Lerda, 1992]. It is a generalization of Fermi and Bose statistics, hence the corresponding particles are referred to as anyons (as in “any phase”). The angle \( \theta \) is called the statistical angle of the particle, and we see that the case \( \theta = 0 \) and \( \theta = \pi \) describe bosons and fermions respectively. In general, \( \theta \) is given by \( \nu\pi \) if the corresponding Chern-Simons theory is also Abelian.

\(^1\)We explicitly state that we are dealing with CS-charge and CS-flux, similar to the electromagnetic case. These charges and fluxes are not associated with the ordinary electromagnetic field. We are merely borrowing terminology.
Topological quantum field theory - a closer look

We mentioned at the beginning of this chapter that Chern-Simons theory is an example of topological field theory. But what exactly do we mean by that? The first aspect which is quite unconventional about the Chern Simons terms is the fact that all contractions are done through use of the fully antisymmetric tensor density $\epsilon^{\mu\nu\lambda}$. It is defined as a fully antisymmetric tensor density and $\epsilon^{012} = 1$. A tensor density is different from a normal tensor, in the sense under a coordinate transformation the $\epsilon$-object picks up an additional determinant. Together with the measure $d^3x$ the Chern-Simons term is a scalar with respect to local coordinate transformations

$$\left(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right) \, d^3x = \left( \det \left[ \frac{\partial x'}{\partial x} \right] \right) \epsilon^{\mu'\nu'\lambda'} a_{\mu'} \partial_{\nu'} a_{\lambda'} \, d^3x'$$

(4.14)

Since the metric never enters the action the Chern-Simons theory can be defined on any compact manifold $M$, while remaining invariant under small coordinate transformations. This is an example of a topological quantum field theory which does not care about the geometry of spacetime. It has the peculiar property of metric-independence.

The immediate consequence of this is a vanishing of the energy-momentum tensor. The energy-momentum tensor is defined as the variation of the action with respect to the metric. Consequently, this theory has zero Hamiltonian and all eigenstates have zero energy. To make this explicit, notice that the momentum conjugate to $a_1$ is $\delta \mathcal{L} / \delta (\partial_\nu a_1) = k / 4\pi a_2$ and to $a_2$ it is $-k / 4\pi a_1$. The Hamiltonian indeed identically vanishes

$$\mathcal{H} = \frac{k}{4\pi} (a_2 \partial_t a_1 - a_1 \partial_t a_2) - \mathcal{L} = 0$$

(4.15)

At first sight this might seem rather trivial. A theory with no Hamiltonian does not contain any local degrees of freedom. This is also reflected in the equations of motion (4.5). In the absence of an external field they read

$$\partial_\mu a_\nu - \partial_\nu a_\mu = 0$$

(4.16)

This states that in the absence of any sources or external fields the field strength of the gauge potential is always zero. Solutions of the gauge field which correspond to a vanishing field strength are known as flat connections. For a compact manifold with no boundary this means that locally the gauge field can always be written as a gauge transformation. The gauge field is said to be pure gauge, its local degrees of freedom correspond to gauge transformation and hence they are not physical.

But even a gauge field which is pure gauge can turn non-trivial upon quantization. The key ingredient here is the topology of the manifold on which the theory is defined. For non-trivial compact manifolds, such as the torus, there are no open subsets which cover the entire manifold. On such spaces solutions appear which are pure gauge but still different from the vacuum solution $a_\mu = 0$. This means that locally such a gauge field can be gauged away, but not globally. The corresponding Hilbert space is degenerate and the theory contains so-called topological degrees of freedom.
Let us make this more explicit, and also clarify the situation when the manifold does have a boundary. Following [Wen, 1995; Nayak et al., 2008; Elitzur et al., 1989] we consider the Abelian Chern-Simons action with the corresponding gauge group $U(1) \cong \mathbb{R}/\mathbb{Z}$

$$S = \frac{k}{4\pi} \int_{\mathcal{M}} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$  \hspace{1cm} (4.17)

In this case $a$ is a connection, i.e. a Lie-algebra valued one-form: $a = a_\mu dx^\mu$. Elements of the gauge group correspond to $e^{i\Lambda}$. Furthermore, we specialize to the compact manifold $\mathcal{M} = \Sigma \times \mathbb{R}$, where $\Sigma$ denotes the (compact) spatial surface and $\mathbb{R}$ the time direction. Let us perform a $U(1)$ gauge transformation on the gauge field

$$a_\mu \rightarrow e^{i\Lambda(x)} a_\mu e^{-i\Lambda(x)} - ie^{-i\Lambda(x)} \partial_\mu e^{i\Lambda(x)} = a_\mu + \partial_\mu \Lambda$$ \hspace{1cm} (4.18)

For “small” gauge transformations\(^2\) this gauge transformation can be plugged into the action. The corresponding change is

$$S \rightarrow \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} (a_\mu + \partial_\mu \Lambda) \partial_\nu (a_\lambda + \partial_\lambda \Lambda)$$ \hspace{1cm} (4.19)

$$= \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} [a_\mu \partial_\nu a_\lambda + (\partial_\mu \Lambda)(\partial_\nu a_\lambda) + a_\mu \partial_\nu \partial_\lambda \Lambda + (\partial_\mu \Lambda)(\partial_\nu \partial_\lambda \Lambda)]$$ \hspace{1cm} (4.20)

The first term is the original action and the last two terms vanish due to the antisymmetry of $\epsilon$. The second term is partially integrated and gives a full derivative minus a term which, again, vanishes due to antisymmetry. In the end, the gauge transformation induces a boundary term in the action

$$\Delta S = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} \partial_\mu [\Lambda \partial_\nu a_\lambda]$$ \hspace{1cm} (4.21)

The action is therefore gauge invariant only if we consider the class of gauge transformations which have the property $\Lambda|_{\partial \mathcal{M}} = 0$.

This is a strong statement. By definition, a gauge theory comes equipped with a gauge structure. The redundancy in the gauge potential $a$ is reflected by the (unphysical) gauge degrees of freedom. In Chern-Simons theory these gauge transformation allow us to locally gauge the field to zero. But in the presence of a boundary this is no longer true. The only gauge transformations allowed are those that leave the boundary intact. At the boundary the gauge degrees of freedom turn into dynamical degrees of freedom, and hence they are physical. The gauge degrees of freedom at the edge have been traded in. Different choices of boundary conditions for the gauge field are essentially different gauge fixings on the boundaries. But since gauge transformations are not allowed on the boundary, different gauge fixings are physically inequivalent. We will momentarily see what exactly distinguishes them.

The relation with the quantum Hall effect is quite clear. At the boundary the confining potential lifts the energy of the electron states residing near the edge of the system. Near the edge the energy gap vanishes and the theory contains gapless degrees of freedom. In chapter 2.5 we explained that its these edge modes which are responsible for carrying the current. Here we see that the edge modes automatically arise when the effective theory is studied at a boundary.

\(^2\)There exist gauge transformation which are not deformable to the identity. These are called “large” gauge transformations and are treated at the end of this section.
We are now in a position to motivate quantization of the coupling constant $k$ and hence the rigid character of Chern-Simons theory. The gauge transformation (4.18) are taken to be small. However, in general a gauge transformation of the Chern-Simons term induces an extra term in the action. This a special property of Chern-Simons theory. The correct change of the action is \cite{Dunne et al., 1989}

$$S \rightarrow S + \Delta S + 2\pi km$$

(4.22)

where $\Delta S$ is defined as above and $k$ is the coupling constant. The variable $m$ is an integer and it stands for the “winding number” of the corresponding gauge transformation. A gauge transformation is a smooth function of $\mathcal{M} \rightarrow U(1)$. Since the homotopy group of $U(1)$ is non-trivial such smooth functions will in general not be deformable to the identity, since the function can wind around the circle a couple of times. These functions can be classified according to the number of times such a function winds around the circle, the winding number. Functions with the same winding number are continuously deformable into each other, hence this classification makes sense.

The extra term caused by the “large” gauge transformation destroys the gauge invariance of the action for non-zero $m$. However, for quantum mechanics it is not the action but rather $\exp(iS)$ which has to remain gauge invariant. This automatically fixes $k$ to be an integer which is a statement we have been frequently using. The fact that $k$ is integer is typifying statement on the nature of topological field theories in general, and Chern-Simons theory in particular. The introduction of perturbative terms to the action will have no effect on the coupling $k$ for relatively weak perturbations. The gauge invariance of the system is a reflection of the conservation law for the electric current. Therefore it cannot be broken by perturbative terms. Terms can only influence the coupling constant $k$ if they shift it by at least an integer.

### 4.5 Quantization of Abelian Chern-Simons theory

In the previous section we demonstrated the difficulty that comes equipped with Chern-Simons theory. In a first attempt, we have separated the physical (topological) degrees of freedom from the unphysical gauge degrees of freedom. For instance, the gauge degrees of freedom residing on the edge have been traded in and now correspond to physical degrees of freedom. It turns the physical system on the edge into a dynamical one, which in some sense compensates for the failure of full gauge invariance of the action. Through a procedure called dimensional reduction we can obtain an effective theory which describes these edge modes. This (1+1)-dimensional theory turns out to be a chiral conformal field theory, which we will prove in this and the following section.

The procedure to obtain an effective effective edge theory is quite elegant, and was first shown in the renowned paper by \cite{Witten, 1989}. The main result of that paper, however, is that this procedure is also applicable to obtain a theory describing the bulk of the system. With the proper modification we obtain a (2+0)-dimensional chiral conformal theory, which we refer to as the bulk theory. The bulk theory is, in some sense, the spatial version of the chiral conformal field theory describing the edge of the system. For this thesis it is the bulk theory which will be of far greater importance, because this conformal field theory of the bulk allows us to compute electron wavefunctions of the quantum Hall system.
Apart from the edge degrees of freedom there are also topological degrees of freedom which manifest themself through non-trivial solutions of the gauge field inside the bulk. An example of such a manifestation is the degeneracy of the corresponding Hilbert space. We will return to a special case of this topic in chapter 9.

As stated, the correspondence between topological field theory and the conformal theory was explicitly shown by [Witten, 1989]. His approach is a bit too mathematical though, so we follow some different sources [Nayak et al., 2008; Elitzur et al., 1989; Wen, 1995; Pilo, 1997].

The basic idea to obtain the effective theory describing the edge is as follows. We choose a particular boundary condition by fixing the gauge field. Classically, the system is fixed. But in quantum mechanics deviations from the classical solution also contribute to the path integral. Therefore, the gauge fixing comes equipped with a set of constraints, which literally constrain the remaining degrees of freedom such that the gauge choice is a proper one.

Let us set the scene first. We put the theory on a manifold \( M = \Sigma \times \mathbb{R} \), with \( \mathbb{R} \) describing the direction of time. \( \Sigma \) describes the spatial components of the manifold, and for now we take it to be a compact surface with a boundary. Locally, we use a coordinate system: \( (t, x, y) \), and the components of \( a \) are written as \( a_t, a_x \) and \( a_y \). Furthermore, we let \( x \) denote the coordinate parallel along the boundary while \( y \) denotes the direction perpendicular to the boundary. To quantize the theory we choose a gauge. The gauge which can always be chosen is the so-called temporal gauge

\[
a_t = 0 \tag{4.23}
\]

This is both a boundary conditions and a gauge fixing of the bulk. The corresponding classical equation of motion which follows from varying \( a_t \) is \( f_{ij} = 0 \), i.e.

\[
\partial_x a_y - \partial_y a_x = 0 \tag{4.24}
\]

In the path integral approach this equation of motion serves as a constraint on the remaining degrees of freedom. By substituting (4.23) and (4.24) into the action we obtain the path integral

\[
\int D a_x D a_y \delta (\partial_x a_y - \partial_y a_x) \exp \left( \frac{k}{2\pi} \int_{\Sigma \times \mathbb{R}} a_y \partial_t a_x - a_x \partial_t a_y \right) \tag{4.25}
\]

The constraint (4.24) is automatically solved by writing the spatial derivatives as derivatives of a smooth function

\[
a_x = \partial_x \varphi, \quad a_y = \partial_y \varphi \tag{4.26}
\]

\[
\varphi = \varphi + 2\pi \tag{4.27}
\]

The second line is a condition on \( \varphi \), originating from the fact that the gauge group is \( U(1) \) which is compact.

This transformation comes equipped with a Jacobian of the integration measure \( D a_i \). However, the theory is Abelian therefore this Jacobian is an overall constant and in the end canceled by normalization. Substituting (4.26) into the path integral gives

\[
\int D \varphi \exp \left( \frac{k}{2\pi} \int_{\Sigma \times \mathbb{R}} (\partial_y \varphi)(\partial_t \partial_x \varphi) - (\partial_x \varphi)(\partial_t \partial_y \varphi) \right) \tag{4.28}
\]
The constraint is dropped, since it is automatically satisfied by $\varphi$. A little partial integration and clever reshuffling of derivatives turns this into

$$\int D\varphi \exp \left( \frac{k}{2\pi} \int_{\Sigma \times \mathbb{R}} \partial_x (\partial_y \varphi)(\partial_t \varphi) - \partial_y (\partial_x \varphi)(\partial_t \varphi) \right)$$

(4.29)

As stated $\Sigma$ is a compact space with a boundary, for instance $[0, 1] \times S^1$, turning the manifold into a cylinder. The second term vanishes, because it is an overall derivative and $y$ does not have a boundary. The first term is precisely the action at the boundary. We integrate over the overall derivative and end up with the action describing the boundary. The explicit action is

$$S_{\text{edge}} = \frac{k}{2\pi} \int_{S^1 \times \mathbb{R}} dt \, dx \, \partial_x \varphi \partial_t \varphi$$

(4.30)

This action describes a two-dimensional conformal field theory. It is the so-called compactified chiral boson and we will frequently use it throughout the rest of this thesis. We have already seen that the corresponding Chern-Simons theory reproduces the proper conductivity tensor and quasiparticle excitations of the Laughlin states. In the remainder of this chapter we will see how this theory is related to the bulk and edge modes of the system. In the next chapter we will drop the (2+1)-dimensional theory all together and solely focus on the conformal field theory description.

The quantization of the chiral boson is done in appendix B. There, it is also shown why this theory is conformal.

### 4.5.1 Edge theory

The full theory of the edge modes is quite an interesting model, as it is the quickest way to explicitly apply the techniques of conformal field theory. However, throughout the remainder of this thesis we are primarily concerned with wavefunction of the bulk theory. Nevertheless, since we are only a few steps away from writing the edge theory down, the temptation of being somewhat complete is hard to refuse.

The problem with the action (4.30) is that it does not directly apply to the edge modes. As it turns out, one can identify the field $\varphi$ with the electron density at the edge through $\rho = \frac{1}{2\pi} \partial_x \varphi$ [Wen, 1992]. This shows in some sense how excitations of $\varphi$ correspond to edge modes. In any case, the classical equation of motion for $\varphi$ is

$$\partial_t \partial_x \varphi = 0$$

(4.31)

In this theory the edge modes do not move. That can hardly represent an electrical current, let alone the fact that in the quantum Hall system there is always a chiral current at the edge. Luckily, there is a way out. We are free to choose a different gauge fixing to obtain the effective theory [Wen, 1992]. Namely, let us set

$$a_t + va_x = 0$$

(4.32)

The parameter $v$ has dimension ‘velocity’, which already gives away its interpretation. To continue, we switch to different coordinates

$$\tilde{x} = x - vt$$

(4.33)

$$\tilde{t} = t, \quad \tilde{y} = y$$

(4.34)
The components of the gauge field are given by

\[ \tilde{a}_t = a_t + v a_x \]  
\[ \tilde{a}_x = a_x \]  
\[ \tilde{a}_y = a_y \]  

(4.35) (4.36) (4.37)

The change of the action is trivial

\[ S_{CS} = \frac{k}{4\pi} \int \epsilon^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\lambda}} \]  

(4.38)

With this new coordinate system the gauge choice (4.32) corresponds to the temporal gauge (4.23) only now with respect to the new coordinates. We can follow the same procedure described before to obtain an effective action in terms of the new coordinates. The result is equation (4.30), only written in terms of the coordinates \((\tilde{t}, \tilde{x}, \tilde{y})\). Switching back to the ordinary coordinates gives the following action

\[ S_{\text{edge}} = \frac{k}{2\pi} \int_{S^1 \times \mathbb{R}} dt \, dx \, (\partial_t + v \partial_x) \varphi \partial_x \varphi \]  

(4.39)

This theory is again a massless scalar field, the chiral boson. This time its equations of motions are

\[ (\partial_t - v \partial_x)(\partial_x \varphi) = 0 \]  

(4.40)

The modes are dynamical, and move around with a velocity \(v\). Furthermore, they are chiral, meaning the modes only move in one particular direction along the edge, depending on the sign of the product \(kv\). This is a reflection of the parity and time symmetry breaking caused by the magnetic field. The story so far seems to imply that we are free to choose our gauge, as in equation (4.32). This is not true, as the edge system is determined by the physical system itself. In fact, this is what we also mentioned in chapter 2: the slope of the confining potential determines the velocity of the edge state, provided it is steep enough. We now turn to the discussion on the bulk theory, [Wen, 1992].

From now on we will work only on manifolds which have no boundary. Our main goal is an understanding of quantum Hall wavefunctions. For that we construct an \((2+0)\)-dimensional effective theory which describes the bulk of the system.

### 4.5.2 Bulk theory

At first it is not clear why the procedure presented above would also be of use for a description of the bulk of a quantum Hall system. When the system has an edge the effective theory already describes a \((1+1)\)-dimensional edge theory. The solution to this is actually quite elegant. We can imagine slicing the manifold in two, with a spatial slice separating the two pieces. The spatial slice serves as a boundary of the two pieces, and, keeping causality in mind, we apply the same procedure to obtain an “edge theory” of this spatial slice.

Before we come to it we first set the proper scene. First off all, we again define the theory on a manifold \(\mathcal{M} = \Sigma \times \mathbb{R}\), but this time let the spatial part represent a torus. This means the manifold does not have a boundary.

Let us sketch the idea. The key-insight provided by [Witten, 1989] (see also [Nayak et al., 2008]) is that we view the manifold as essentially two separate manifolds glued together with
4.5 Quantization of Abelian Chern-Simons theory

Figure 4.1: The slicing of a manifold $\mathcal{M}$. Through this procedure we are able to obtain an effective “edge theory” on the spatial slice.

proper boundary conditions. See figure 4.1. The cutting takes place along a spatial slide. Let us be specific and choose the slide at $t = 0$, i.e.

$$
\mathcal{M} \sim \mathcal{M}_1 \cup_f \mathcal{M}_2 \quad (4.41)
$$

$$
\mathcal{M}_1 = T \times (-\infty, 0] \quad (4.42)
$$

$$
\mathcal{M}_2 = T \times [0, \infty) \quad (4.43)
$$

Strictly speaking, the manifold $\mathcal{M}$ is not simply the union of the two sub-manifolds. The submanifolds contain a boundary, and as we have seen the configuration of the field $a$ on this boundary is “frozen in”. That means that if the two manifolds are glued together the resulting manifold contains a spatial slice with a particular configuration.

Still, let us press on and see how far we get. We may choose a particular configuration of $a$ residing on the spatial surface. This corresponds to a boundary condition on both submanifolds as described in the case of the edge modes. Such a boundary condition is a fixed one for the submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, and the corresponding surgery. However, it is not fixed for the original manifold $\mathcal{M}$ therefore the corresponding “edge theory” has a different interpretation.

Witten’s insight in this matter is the fact that we may assign a 3-manifold invariant to $\mathcal{M}$, which is nothing more but the partition function.

$$
Z(\mathcal{M}) = \int Da \exp(iS_{CS}) \quad (4.44)
$$

Such an invariant is interpreted as the vacuum-to-vacuum amplitude, $\langle 0|0 \rangle$. When we slice the manifold into two the two submanifold inherit this partition function. By fixing the gauge field $[A]$ at the boundary we obtain a partition function. This is a functional of the boundary configuration [Nayak et al., 2008; Lankvelt, 2004].

$$
\Psi([A]) = \int_{a(x,y,t=0)=A(x,y)} Da \exp \left( i \int_{-\infty}^{0} dt \int T^2 L_{CS} \right) \quad (4.45)
$$

This partition function represents a path integral which sums over all histories leading to the particular configuration $[A]$.

To obtain an effective theory on this spatial slice we repeat the steps done at the beginning of this section. First we switch to complex coordinates, $z = x + iy$ and $\bar{z} = x - iy$. Keep in mind that the boundary is now described by $t = 0$. Our gauge fixing is the so-called holomorphic gauge which sets $a_{\bar{z}} = 0$. Its corresponding constraint is again $f_{ij} = 0$, only now $i, j$ run over $z, t$. Repeating all steps automatically leads to the following theory describing
4. Chern-Simons theory

the bulk of the system

\[ S_{\text{CS}} = \frac{k}{4\pi} \int_{T^2} dz \, d\bar{z} \, \partial_z \varphi \partial_{\bar{z}} \varphi \]  (4.46)

This is the complex coordinate version of the chiral boson. It describes the bulk theory of a (2+1)-dimensional Chern-Simons theory. The Chern-Simons theory is an effective, low-energy field theoretic description of a sequence of quantum Hall series. We have seen in what way the conductivity tensor and quasiparticle excitations of the Laughlin wavefunctions are taken into account by this theory. What remains to be done is a proper procedure which will produce electron wavefunctions for the bulk system.

4.6 Generalization

Up till now we have only worked with a specific Abelian Chern-Simons theory. Its great accomplishment is a field theoretic description of a certain sequence of quantum Hall systems, the Laughlin sequence. We reproduced numerous properties, and also showed how we can obtain a lower-dimensional effective theory, the application of which will be done in the next chapter.

However, there are for more quantum Hall systems around, therefore we require more room on the TQFT-side. There are a few ways we can generalize the preceding picture. The first one is the introduction of multiple gauge fields. A simple example of such a theory is

\[ L = \frac{1}{4\pi} K_{IJ} a_I \partial a_J \]  (4.47)

The indices \( I \) and \( J \) run over the number of gauge fields present. Each gauge field represents a separate conserved current and the matrix \( K_{IJ} \) is a generalized version of the coupling constant \( k \), containing the coupling between the gauge fields. Such a description is a first approach to an effective theory for, for example, multilayered systems, hierarchy states or multiple filled Landau levels. See the literature for more details on this matter [Wen, 1992, 1995; Zee, 1995].

For this thesis the more relevant generalization is one where the gauge structure of \( a \) is taken to be non-Abelian (but still compact). Up till now we have only considered the gauge group \( U(1) \), but in principle there are other choices available. The non-Abelian case has a far richer structure and also leads to far more interesting topological Hall fluids. The results obtained so far generalize to the non-Abelian case, but are usually more complex and more difficult to obtain. Some of the derivations are useful to repeat, in particular the dimensional reduction of the effective theory.

A non-Abelian gauge field is a Lie-algebraic valued one form, this time written as \( a = a_\mu T^a \, dx^\mu \). \( T^a \) are the generators for the Lie-algebra and \( a \) is the corresponding index. The non-Abelian Chern-Simons therm is more complex, as it contains third order terms needed for gauge invariance.

\[ S = \frac{k}{4\pi} \int_M \epsilon^{\mu\nu\lambda} \mathrm{Tr} \left( a_\mu \partial_\nu a_\lambda + \frac{2}{3} f_{ijk} a_\mu^i a_\nu^j a_\lambda^k \right) \]  (4.48)

where \( f_{ijk} \) are the structure constants. To obtain an edge theory we fix a gauge and solve the theory according to its constraints. Without proof we merely mention the results [Nayak et al., 2008; Lankvelt, 2004]. The amplitude for a particular configuration \(|A|\)
at $t = 0$ is given by the following
\[ \Psi([A]) = \int_{a(x,y,t=0)=A(x,y)} \mathcal{D}a \exp \left( i \int_{-\infty}^{0} dt \int_{T^2} \mathcal{L}_{\text{WZW}} \right) \] (4.49)
\[ \mathcal{L}_{\text{WZW}} = \frac{k}{4\pi} \int_{\text{edge}} d^2 x \, \text{Tr}(g^{-1} \partial_\mu gg^{-1} \partial_\mu g) \] (4.50)
\[ + \int d^3 x \, \epsilon^{\mu\nu\lambda} \text{Tr}(\tilde{g}^{-1} \partial_\mu \tilde{g} \tilde{g}^{-1} \partial_\nu \tilde{g} \tilde{g}^{-1} \partial_\lambda \tilde{g}) \] (4.51)

This theory is called a Wess-Zumino-Witten model, and it is well known that it constitutes a conformal field theory. In the conformal field theory approach we will primarily work with such models, although their reference to this action is implicit. Still, it is good to know where the description comes from.

The WZW-models correspond to TQFT’s with non-Abelian compact gauge groups. As such groups possess a lot more structure than their Abelian counterparts, we expect the corresponding quantum Hall fluids to obtain more exotic features as well. This is indeed the case. The quasiparticles of non-Abelian quantum Hall fluids exhibit a generalized form of statistics. That is, it has an even greater structure than the fractional statistics we described earlier. The exact meaning will be treated in the next chapter. For now we merely emphasize the relevance of non-Abelian WZW-models.

### 4.7 Quantum Hall wavefunctions as conformal blocks

To properly obtain wavefunctions in terms of electron coordinates we will need a better reference to what constitutes an electron in our theory. The gauge potential $a$ and its effective description in terms of Chern-Simons theory does at first not reference to single electrons. Quite surprisingly, it is easier to understand this relation if we look at the quasiparticles first.

In the Chern-Simons description the quasiparticles of the quantum Hall systems are described in terms of source-terms coupled with the gauge potential $a$ through a minimal coupling term,
\[ \mathcal{L}_{\text{coupling}} = \text{Tr}(\tilde{g}^\mu a a^\mu) \] (4.52)

The quasiparticles thus carry some representation $(j)$ of the gauge algebra. The insertion of a number of quasiparticles into the system is interpreted as the insertion of Wilson lines in the path integral. Wilson lines constitute the gauge-invariant observables of a gauge theory and we view them as the worldlines of the quasiparticles when they move through the manifold. Witten’s invariant of the 3-manifold in the presence of quasiparticles is interpreted as the expectation value of the Wilson operators
\[ Z(M; \{ \gamma, j \}) = \int \mathcal{D}a \ W_{\gamma_1,j_1} \cdots W_{\gamma_n,j_n} \exp(iS_{\text{CS}}) \] (4.53)
\[ W_{\gamma,j} = \text{Tr}_f \left[ \mathcal{P} \exp(i \oint_{\gamma} a^\mu T a^\mu dx^\mu) \right] \] (4.54)

The interpretation is the vacuum-to-vacuum amplitude, where at some point quasiparticle are created, move around and in the end annihilate. Readers familiar with knot theory might
be amused to see that this is the path integral version of the Reshetikhin-Turaev invariants of which the Jones polynomial is a particular example.

We now perform the same dimensional reduction to obtain an effective theory in the bulk. That is, we take a spatial slide and a particular gauge field configuration and calculate the corresponding amplitude. However, the Wilson lines run through this spatial surface so the effective theory we obtain will contain quasiparticles which puncture the plane, see figure 4.2. What we obtain is an amplitude for a specific configuration of the spatial slide, in the presence of quasiparticles.

\[ \Psi([A]) = \int_{a(x,y,t=0)=A(x,y)} D\alpha W_{\gamma,j} \exp \left( i \int_{-\infty}^{0} dt \int_{T^2} L_{WZW} \right) \]  

We emphasize that in this amplitude the trajectories of the quasiparticles (the Wilson lines) are fixed.

\[ \langle V_{qp}(z_1) \cdots V_{qp}(z_n) \rangle = \sum_{\{\gamma\}} Z(\{\gamma, j\}) \]  

**Figure 4.2:** Insertion of quasiparticles. The end point of the Wilson lines correspond to charged particles living on the spatial slice. The cause the insertion of operators into the correlators of the corresponding CFT.

This last restriction can be relaxed. We can take a particular configuration on the spatial slides, which includes the puncturing of the quasiparticles. To these insertions \( V_i(z_i) \) we assign a correlation function. Its amplitude is given, from a TQFT perspective, by the sum over all possible histories that lead to this particular configuration. These histories correspond to all possible Wilson lines which contain loops ending at the surface. Since the (2+1)-dimensional theory is topological the precise alignment of the lines is not important (as long as they do not intersect each other). We are allowed to simply sum over the equivalence classes instead.

\[ \langle V_{qp}(z_1) \cdots V_{qp}(z_n) \rangle = \sum_p A_p F_p(z_1, \ldots, z_n) \]  

However, we already know what theory describes the correlator on the left-hand side. It is precisely the WZW-model given in (4.51). Therefore the correlators also correspond to the conformal blocks of the corresponding edge theory.

\[ \langle V_{qp}(z_1) \cdots V_{qp}(z_n) \rangle = \sum_{\{\gamma\}} Z(\{\gamma, j\}) \]

The function \( F_p \) are called conformal blocks and the \( A_p \) are some coefficients. These conformal blocks have a richer structure than ordinary correlators, as the corresponding correlators can

---

3For readers familiar with knot theory, the equivalence classes are, in some sense, defined as (ribbon) Reidemeister equivalence classes.
be multivalued with respect to these blocks. This means they are sensitive to the relative positions of the inserted operators allowing the possibility of so-called braiding operations.

We have obtained an expressions which shows how the insertions of quasiparticles are given by the conformal blocks of the CFT. But where does the electron come into play? This insight was provided by the paper [Moore and Read, 1991], see also [Read and Moore, 1992]. The presence of electrons in the system also corresponds to the insertion of Wilson lines. However, the wavefunction is always singlevalued with respect to its electron coordinates. This means the electrons have trivial braiding properties with each other and with respect to all quasiparticles present in the theory. The corresponding Wilson loops are completely topologically trivial.

The master formula to obtain electron wavefunctions is given in the next chapter, along with its relation to the conformal field theory. For now we end this chapter with some final remarks. First off all, we just stated for an effective theory to describe a quantum Hall fluid there must be some reference to the electron. That means there is some operator present in the theory which has all the characteristics of the electron, such as spin and charge, and, most importantly, singlevaluedness with respect to all operators present. This greatly constrains the corresponding CFT since most models do not contain such an operator. See [Fröhlich et al., 2001] (or also [Ardonne, 2002]) for an overview of these restrictions.

This chapter was devoted to a brief introduction of Chern-Simons theory and its relation to the quantum Hall system. Our main result is a dimensional reduction of this effective theory, a form of holography. The (2+0)-dimensional theory describes the bulk of the system and we can use it to obtain electron wavefunctions. We will now drop the (2+1)-dimensional description all together, in favor of this (2+0)-dimensional approach. This is because this theory is conformal, and comes equipped with a formidable, rigid structure.
4. Chern-Simons theory
In this chapter we explain the connection between conformal field theory and the quantum Hall effect. This thesis so far has been a large introduction into this matter, so it pays off the spend a whole chapter on it.

The basis of the connection should be clear by now. The Chern-Simons theory is the low-energy effective field theory of the quantum Hall system. Upon quantization it develops a Hilbert space which is intimately connected with a certain CFT. This bulk theory can be used to obtain electron wavefunctions, which we will demonstrate in this chapter. The first example will be the reproduction of the Laughlin state from the CFT side. We will make use of the fact that the Abelian Chern-Simons term reproduces the quantization of the conductivity tensor, while at the same time its bulk theory is described by a compactified chiral boson. This suggests that the correlators of the chiral boson are candidates for the Laughlin wavefunctions.

Apart from this construction, we will also motivate why CFT is such an important tool for the quantum Hall fluids. This is mainly due to the existence of non-Abelian quasiholes, which greatly enhances the structure of the quantum Hall fluid. Non-Abelian quasiholes satisfy a new kind of statistics, different from (Abelian) fractional statistics. It is mainly due to these excitations that cause the fluid to obtain such a complex structure, that we are forced to depart from the realm of first quantization. Indeed, CFT provides many tools through which we are able to come with sensible statements about the nature of non-Abelian quantum Hall fluids.

But before we come to the connection itself it is wise to spend some words on conformal field theory. As this subject has been under development for quite some time, it is impossible to treat it thoroughly here. Instead, we will treat some selected topics of which the author claims thinks they are most relevant to this thesis.
5. Conformal field theory and the quantum Hall effect

5.1 Conformal field theory - Selected topics

A conformal field theory is the collective noun for (quantum) field theories that behave covariantly under the conformal group. The conformal symmetry is comprised out of the general coordinate transformations which preserve angles of any two vectors. Such coordinate transformations act as local scale transformations

\[ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \omega(x)g_{\mu\nu}(x) \]  

(5.1)

In general, such quantum theories can form sorts of structures. For \( n \)-dimensional Euclidean manifolds the corresponding group of conformal transformations is isomorphic to \( SO(n+1,1) \), with the \((n+2)(n+1)/2\) generators corresponding to

- **Translations**
  \[ x'_{\mu} = x_{\mu} + a_{\mu} \]  
  (5.2.a)

- **Rotations**
  \[ x'_{\mu} = M_{\mu\nu}x_{\nu} \]  
  (5.2.b)

- **Dilations**
  \[ x'_{\mu} = e^\lambda x_{\mu} \]  
  (5.2.c)

- **Special Conformal Transformation**
  \[ x'_{\mu} = \frac{x_{\mu} - b_{\mu}|x|^2}{1 - 2b_{\mu}x_{\mu} + |b|^2|x|^2} \]  
  (5.2.d)

For this thesis the relevant theories emerge when we specialize to CFT’s defined on a two dimensional space-time. In this case the conformal mappings (5.1) have a richer structure than their higher-dimensional counterparts. Specifically, the coordinate transformations are generated by an infinite dimensional algebra. Most of these generators are only locally well-defined, therefore the corresponding structure is known as the **conformal algebra**. The coordinate transformations defined above are also present and correspond to the only generators which are globally well-defined. Hence, they form a group, the conformal group, which is a proper subgroup of the conformal algebra. The theory itself is local and therefore must be sensitive to local coordinate transformations. The concept of local conformal covariance and the associated infinite dimensional structure is the source of the highly rigid properties associated with two dimensional CFT’s.

**Holomorphic mappings** To make this more quantitative, its convenient to work with complex coordinates. For that we introduce holomorphic and antiholomorphic coordinates \( z = x + iy \) and \( \bar{z} = x - iy \). In this case, the conformal transformations correspond to (anti-)holomorphic mappings from the complex plane onto itself.

\[ z \rightarrow w(z) \]  

(5.3)

\[ \bar{z} \rightarrow \bar{w}(\bar{z}) \]  

(5.4)

Holomorphic mappings are complex analytic mappings which only depend on \( z \). This means the function is differentiable and the only singularities it possible has are a finite number of poles. As we by know, we are only concerned in CFT’s which do not depend on \( \bar{z} \). Such CFT’s are called **chiral** and with them we are allowed to drop all references to the anti-chiral part.

**Virasoro algebra** The generators for the conformal transformations form an infinite dimensional Lie algebra, known as the Witt algebra. For quantum field theories the conformal symmetry turns out to be anomalous and the corresponding symmetry of the theory is not the
Witt algebra but rather its central extension. In addition to the infinite number of generators that induce conformal transformations the central extension also contains a central element $C$. Since this element is central it usually replaced by its eigenvalue $c$ of the corresponding CFT. This quantity is called the **central charge**. The complete symmetry is generated by this element plus the quantum generators $L_n$ ($n \in \mathbb{Z}$). They obey what is known as the **Virasoro algebra**

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \quad (5.5)$$

The central charge is a reflection of the quantization procedure. Conformal invariance implies a scale invariant theory, meaning there is no reference to a distance scale whatsoever. Yet any sensible quantum field theory requires the introduction of a minimum length scale, in order to make the theory renormalizable. This seeming contradiction is resolved through the introduction of a conformal anomaly. This anomaly comes into play, for example, when the system is put on a finite or curved system, as it is a measure for the response on the scales introduced to the system. For this thesis it mostly serves as a label. Any conformal field theory contains a specific central charge, thus allowing for a simple early check to see whether two CFT’s constitute the same physical system.

**Primary and secondary fields**  
The information off a conformal quantum field theory contains a collection of a number of fields and the correlations functions between them. The theory is said to be covariant with respect to the conformal algebra. This means that its fields transform in a particular way under the action of this algebra. Alternatively we can state that the fields carry a representation (form a module) of the Virasoro algebra. For example, the action of the central element $C$ on a field always yields the same eigenvalue, given a particular CFT. What distinguishes the fields is the module which they carry. Its this module which determines in what matter the field transforms under a conformal coordinate transformation.

The most important classes of fields are the ones which carry a highest-weight module of the Virasoro algebra. The highest weight property is reflected by the action of the quantum generators on this field

$$L_n \phi = 0 \quad \text{for } n > 0 \quad (5.6)$$

$$L_0 \phi = \Delta \phi \quad (5.7)$$

Such a field is called a **primary field**. The real-valued constant $\Delta$ is called the **scaling dimension** of the field $\phi$. For chiral fields it can also be referred to as the **conformal dimension** of the field.

An example of a primary field is the identity operator $1$. The identity operator is also known as the vacuum operator, as it is the only operator which always has a non-zero expectation value, $\langle 1 \rangle$. The identity operator has scaling dimension $\Delta_1$ zero, therefore it carries the trivial representation of the Virasoro algebra.

In general, under a conformal transformation (5.3) primary fields transform as

$$\phi'(w) = \left( \frac{dw}{dz} \right)^{-\Delta} \phi(z) \rightarrow \phi(w) \quad (5.8)$$

In any non-trivial case there are also fields present which correspond to non-highest weight vectors, called **descendants**. Non-highest weight vectors are obtained through proper ladder
operators, in this case \(L_{-n}\) \((n > 0)\), hence these descendants are associated with a particular primary field. An example is the derivative of any primary, \(L_{-1}\phi = \partial \phi\). The primary field is a highest-weight state in the literal sense: all its descendants are obtained through the action of the ladder operators \(L_{-n}\) on the primary field. A primary field together with all its descendants is called a conformal family. In this thesis we will limit the applications only to the class of Rational Conformal Field Theory (RCFT). Such models have the special property of containing only a finite number of conformal families with respect to an extended chiral algebra. The word “Rational” stems from the fact that the conformal dimensions and the central charge in such models are rational-valued. In the final chapter we will clarify these remarks.

For descendants formula (5.8) only applies if the transformation is globally well-defined. These transformations correspond to the two-dimension versions of (5.2.a), (5.2.b), (5.2.c) and (5.2.d). For the remaining local transformation there is no general transformation rule for the descendants.

**Correlations functions** The transformation rule (5.8) has a variety of applications. Its simple structure causes a series of constraints on the correlation functions of the theory. For that we first introduce the notion of a chiral correlator. For correlators of a quantum field theory to make sense one has to introduce the concept of a time direction. In CFT this direction is somewhat arbitrary, and the most convenient choice happens to be the radial direction, \(|z|\). This lead to the concept of radial ordering, which is in some sense the complex analog of time ordering

\[
\mathcal{R}\phi_1(z)\phi_2(w) = \begin{cases} 
\phi_1(z)\phi_2(w) & |z| > |w| \\
\phi_2(w)\phi_1(z) & |z| < |w|
\end{cases}
\]  

(5.9)

The chiral correlator is then defined as the radial ordered vacuum expectation value of a product of fields

\[
\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = \langle 0 | \mathcal{R} \phi_1(z_1) \cdots \phi_n(z_n) | 0 \rangle
\]

(5.10)

The great success of CFT is the large number of constraints the symmetries force upon these expressions.

Consider for instance the correlation function of \(n\) primary fields. Under a conformal transformation (5.8) such a transformation leads to

\[
\langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = \prod_i \left( \frac{dw_i}{dz} \right)^{-\Delta_i} \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle
\]

(5.11)

This simple form completely fixes the general expression for two- and three point correlators.

\[
\langle \phi_1(z)\phi_2(w) \rangle = C_{12} \frac{1}{(z - w)^{2\Delta}}
\]

(5.12)

where \(\Delta = \Delta_{\phi_1} = \Delta_{\phi_2}\)

(5.13)

If relation (5.13) is not satisfied the correlation function is zero. This is a direct consequence from the demand that the expressions left and right in (5.12) both satisfy the transformation rule (5.11). The three point correlator is also highly rigid

\[
\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = C_{123} \frac{1}{(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta_3}(z_2 - z_3)^{\Delta_2 + \Delta_3 - \Delta_1}(z_1 - z_3)^{\Delta_1 + \Delta_3 - \Delta_2}}
\]

(5.14)
Again, this is the only expression for three point correlators which satisfies the transformation rule (5.11). We mention here that the constants $C_{12}$ and $C_{123}$ are real-valued constants, known as the structure constants. They can be zero, in which case the correlator is not allowed. The structure constants are part of the “definition of a certain CFT, as they determine what three-point functions are allowed. They are partially defined by the normalization of the fields.

**Ward identity** The impact of the symmetry on all correlators is neatly summarized through use of the so-called Ward identity. Through Noether’s theorem we know that the presence of a continuous symmetry leads to a conserved current. In the case of the conformal symmetry this symmetry is associated with the energy-momentum tensor, defined as

$$T^{\mu \nu} = -\eta^{\mu \nu} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \partial_\nu \phi$$

(5.15)

$$\partial_\mu T^{\mu \nu} = 0$$

(5.16)

where the indices run over the “true” space-time coordinates. In complex coordinates the chiral energy-momentum essentially has one component, $T_{zz}$. The remaining components vanish due to the traceless and holomorphic property of this tensor.

In any case, the presence of such a conserved current leads to a series of constraint imposed on the correlation functions the theory. These constraints are known as Ward identities, and for conformal field theory they are highly rigid. Such identities can even be used to solve the correlation functions for a higher number fields, as they are essentially a set of differential equations. The Ward identity comes into play when we apply an infinitesimal transformation to a correlator. We thus let $z \to z + \epsilon(z)$ the resulting variation is

$$\delta_\epsilon \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = -\frac{1}{2\pi i} \oint C \epsilon(z) \langle T(z) \phi_1(z_1) \cdots \phi_n(z_n) \rangle$$

(5.17)

The demand of conformal symmetry manifests itself through the Ward identities. Alternatively, we have also seen how the conformal transformations are generated by the operators $\{L_n\}$. It should not come as a surprise that the two are related. Indeed, the mode expansion, and its inverse, of the energy-momentum tensor reads

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$$

(5.18)

$$L_n = \frac{1}{2\pi i} \oint dz \, z^{n+1} T(z)$$

(5.19)

**Operator Product Expansion and Operator Algebra** A typical behavior of quantum field theory is the emergence of singularities and infinities. For instance, the expression of the two- and three-point correlator is singular when we bring two or more of the operators together. In this limit quantum fluctuations emerge which express themselves through the singularities of the correlator.

This singular behavior, which arises when two operators are brought together within a correlator, is summarized through their Operator Product Expansion. For two fields $\phi$ and $\chi$
it has the following general form [Noyvert, 2007]

\[
\phi(z)\chi(w) \sim \frac{1}{(z-w)^\alpha} \left( [\phi\chi]_\alpha (w) + (z-w)[\phi\chi]_{\alpha-1} (w) + (z-w)^2 [\phi\chi]_{\alpha-2} (w) + \ldots \right)
\]  

(5.20)

This expression requires some clarification. The Operator Product Expansion, or OPE in short, only makes sense when we consider it within a correlation function. The “∼”-sign is there to remind us of this fact. Furthermore, on the right hand side expression there appears a sum over fields. Each term \([\phi\chi]_i\) is nothing more but a normal ordered product of modes appearing in the theory. In general the most singular terms correspond to primary fields, while higher order terms are the corresponding descendants. We assume that the overall singularity in front \(1/(z-w)^\alpha\) might have a fractional power \(\alpha\), while the powers of \((z-w)\) appearing in the brackets are strictly positive. Usually, only the singular terms of an OPE are given, but we should keep in mind that there are in principle an infinite number of singular terms present.

The power \(\alpha\) itself is determined through the conformal dimensions of the operators appearing left and right.

\[
\alpha = \Delta_\phi + \Delta_\chi - \Delta_{[\phi\chi]}
\]  

(5.21)

This form assures the expressions left and right behave in the same way under conformal transformations.

An example of an OPE is that of the energy-momentum tensor. Its OPE with that of a primary field has a fixed singular form, which is a reflection of the conformal Ward identity.

\[
T(z)\phi(w) \sim \frac{\Delta_\phi}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w \phi(w)
\]  

(5.22)

Another example is that of the OPE of \(T(z)\) with itself, which, again, has a fixed singular form

\[
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w)
\]  

(5.23)

Non-singular terms do appear on the right hand side, but are omitted.

We can turn this train of thought around. The conformal field theory itself is primarily meant for one thing: a tool to compute correlators. The correlators themself consist out of a product of fields, so in essence their behavior is completely determined through the OPE. Therefore, the knowledge of all possible OPE’s in the theory, including their regular terms, implies that we have all information needed to solve every possible correlators. This is known as the operator algebra, and its alternative way to “define” a theory, i.e. this algebra essentially fixes all correlation functions.

**Fusion** In general, the OPE of two fields creates a summation of terms. Each terms contains one field times some coefficient and a factor of \(1/(z-w)^\gamma\). Since the operator algebra is closed, these new fields are either primary or secondary fields. A general rule is that when a primary field enters the summation, it automatically implies the presence of all its descendants in the OPE as well. This is why we adopt the following notation

\[
\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k
\]  

(5.24)
This is called the fusion product of \( \phi_i \) and \( \phi_j \) with the non-negative integers \( N^k_{ij} \) named fusion numbers. It has the following interpretation. The fields \( \phi_i \) are all primary fields. If a primary field \( \phi_k \) appears in the OPE of \( \phi_i \) and \( \phi_j \) then consequently its fusion number \( N^k_{ij} \) is non-zero. In that case the primary field and all its descendants appear in the OPE.

Fusion is an associative and commutative operation, meaning

\[
(\phi_1 \times \phi_2) \phi_3 = \phi_1 (\times \phi_2 \phi_3) \quad (5.25) \\
\phi_1 \times \phi_2 = \phi_2 \times \phi_1 \quad (5.26)
\]

An example is the fusion product of the identity operator with the remaining fields, which always takes the form

\[
1 \times \phi = \phi \quad (5.27)
\]

In this case its fusion product with all fields is unique. We will come across fields, different from the vacuum operator, which have this same behavior. That is, their fusion is unique with all fields in the theory. Such a field is called a simple current

\[
\psi = \text{simple current} \implies \psi \times \phi = \phi' \quad (5.28)
\]

which basically means that the fusion of \( \psi \) with any other field is always unique. If the fusion product of two or more fields contains more than one conformal family we will instead speak of multiple fusion channels.

Conformal blocks The direction application of the fusion rules are, of course, the correlation functions. In the relevant CFT at hand there is only one field which has a non-zero vacuum expectation value, which is the identity operator \( 1 \). This means that if we have a correlator of any number of fields, then this correlator is non-zero only if the total fusion product (of all fields) contains the vacuum operator,

\[
\langle \phi_1 \cdots \phi_n \rangle \neq 0 \quad \text{only if} \quad \phi_1 \times \cdots \times \phi_n = 1 + \cdots \quad (5.29)
\]

Things become quite interesting when there are multiple ways for the fields to fuse to the identity. In that case, each channel for which the fields fuse to the identity contributes to the correlator. In general, the correlator is some linear combination over all these contributions.

\[
\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = \sum_p A_p F_p(z_1, \ldots, z_n) \quad (5.30)
\]

In the context of the quantum Hall effect the coefficients cannot be chosen completely arbitrary. Note that any correlator comprised solely out of simple currents has only one conformal block.

Actually, this expression is a bit too general as the effect of conformal symmetry has not been fully incorporated yet. Nevertheless, the functions \( F_p \) are called conformal blocks. They form a basis of an \( n \)-dimensional vector space, \( n \) being the number of conformal blocks appearing in the summation. The correlator is a linear combination of the conformal blocks. This combination can be changed through the action of braiding. Through analytic continuation we can let a field \( \phi_i \) in the correlator wind around another field back to its original position. This is called braiding, and it causes a unitary transformation on the conformal blocks. This is also called a monodromy operation\(^1\). An example of this will be given in chapter 6.

\(^1\)Monodromy is literally the study of how mathematical objects behave as they run around a singularity.
5. Conformal field theory and the quantum Hall effect

5.2 Laughlin’s wavefunction and the chiral boson

Before we describe what the rules of the game are when we apply CFT to the quantum Hall regime it is better to start with the simplest example, the Laughlin wavefunction. In chapter 4 we showed how we can describe the physics on long-range distances, and high energy through use of a topological field theory, Chern-Simons theory. Starting with an Abelian Chern-Simons theory, we explicitly derived the quantization of the conductivity tensor for the Laughlin sequence. This Chern-Simons theory is characterized by its compact gauge group, $U(1)$. Upon quantization we are able to produce an effective theory describing the bulk of the system. This theory is the compactified chiral boson, and its action is given by (4.46). In appendix B this theory is quantized, and its important features are explained and proven in some cases. This chapter is mainly a summary of that appendix and its application to the quantum Hall effect.

This section is treated in any standard reference on this matter. It is the prime example of the merging of CFT and quantum Hall systems. Still, it all began with the paper [Moore and Read, 1991] and everything is essentially treated there. Other treatments include [Read and Moore, 1992; Nayak et al., 2008; Ardonne, 2002; Slingerland, 2002; Lankvelt, 2004].

5.2.1 Chiral boson

First off, there is the action

$$S_{\text{CS}} = \frac{1}{4\pi} \int dz d\bar{z} \partial_z \varphi \partial_{\bar{z}} \varphi$$

(5.31)

(5.32)

This definition differs from (4.46) through a rescaling of the field by a factor of $\frac{1}{\sqrt{k}}$. This rescaling changes the periodic identification of $\varphi$. To match with the literature we switch to a different notation by replacing $k$ by $M$. The periodicity constraint on $\varphi$ is

$$\varphi(z) = \varphi(z) + 2\pi \sqrt{M}$$

(5.33)

Which shows that we are dealing with a compactification radius $M$ (for more details see appendix B).

The action (5.31) describes a boson. The propagator of its chiral part is

$$\langle \phi(z)\phi(w) \rangle = -\log(z - w)$$

(5.34)

The field is not a primary or secondary field itself, since it does not transform under the standard transformation rule. However, the field can be used to construct other fields which are conformal. The first operator we need is the current operator,

$$\partial_z \varphi(z)$$

(5.35)

Its important properties are

$$\partial_{\bar{z}} \varphi(z) = 0$$

(5.36)

$$\Delta_{\partial \varphi} = 1$$

(5.37)
The first line implies that the current $\partial \varphi$ is conserved. Any conserved current has an associated conserved charge, which allows us to associate a charge operator

$$Q = \frac{1}{2\pi} \oint \! \! \! \! \! \! d\omega J(\omega)$$  \hspace{1cm} (5.38)

Any operator which does not commute with $Q$ carries an amount of charge. So one way to measure this charge is by commutating it with $Q$. For details see the appendix. The measured charge is called the $u(1)$ charge. We will come back to its interpretation.

The important class of primary fields we are after are the vertex operators. These operators, their conformal dimensions and their charges are

$$V_\alpha(z) = :e^{i \alpha \varphi(z)}:$$  \hspace{1cm} (5.39)

$$\Delta_\alpha = \frac{\alpha^2}{2}$$  \hspace{1cm} (5.40)

$$Q_\alpha = \alpha$$  \hspace{1cm} (5.41)

Normal ordering is implicitly assumed from now on. The conformal dimension can be determined, for instance, through a scale transformation $z \rightarrow \lambda z$. This expression is too general, as not all operators are compatible with the compactification. The class we are after are

$$e^{i \frac{m}{\sqrt{M}} \varphi(z)}, \quad m \in \mathbb{Z}_{\geq 0}$$  \hspace{1cm} (5.42)

$$\Delta = \frac{m^2}{2M}$$  \hspace{1cm} (5.43)

$$Q = \frac{m}{\sqrt{M}}$$  \hspace{1cm} (5.44)

The reason why we take $m \geq 0$ will be explained in the next subsection.

Also relevant are the OPE between two vertex operators

$$e^{i \alpha \varphi(z)} e^{i \beta \varphi(w)} \sim (z - w)^{\alpha \beta} e^{i (\alpha + \beta) \varphi(w)}$$  \hspace{1cm} (5.45)

The general form of correlation functions of vertex operators are

$$\lim_{w \to \infty} w^{\beta^2} \langle e^{i \beta \varphi(w)} \prod_{i=1}^{n} e^{i \alpha_i \varphi_i(z_i)} \rangle = \prod_{i<j}^{n} (z_i - z_j)^{\alpha_i \alpha_j} \beta = - \sum_{i=1}^{n} \alpha_i$$  \hspace{1cm} (5.46) \hspace{1cm} (5.47)

This expression requires some clarification. A more extensive one is given in the appendix. Strictly speaking, any correlator of vertex operators is subject to a neutrality condition. This neutrality condition states that the total $u(1)$ charge of the operators must equal zero. If not, the correlator would not be invariant under the identification (5.33). To circumvent this problem we have inserted a background charge operator

$$O_{bg} = \lim_{w \to \infty} w^{\beta^2} e^{i \beta \varphi(w)}$$  \hspace{1cm} (5.48)

$$\beta = - \sum_{i=1}^{n} \alpha_i$$  \hspace{1cm} (5.49)
This charge is defined such that it literally neutralizes the correlator. Furthermore, by sending the charge to infinity we render its remaining contribution to zero. See the appendix for the proof of this statement.

We now have all the ingredients necessary to reproduce the Laughlin state.

5.2.2 Laughlin state

The identification with the Laughlin state is as follows. First, we look at the spectrum of vertex operators available, equation (5.42). From this spectrum we pick a suitable operator, namely, one which is single-valued with respect to the remaining operators. The operator under consideration is

\[ m = M \]  

\[
V_{el} = e^{i\sqrt{M} \varphi(z)}
\]  

(5.50)

Indeed, the OPE of this operator with one of the remaining vertex operators is

\[
e^{i\sqrt{M} \varphi(z)} e^{i\frac{m}{\sqrt{M}} \varphi(w)} = (z - w)^m e^{i\frac{m+M}{\sqrt{M}} \varphi(w)}, \quad m \in \mathbb{Z}_{\geq 0}
\]  

(5.51)

If we perform a monodromy operation, in which we wind the electron around the other operator, this OPE obtains a phase factor of \( e^{i2\pi m} \). This is best seen by setting \( w = 0 \) and let \( z \rightarrow e^{i2\pi} z \). Since \( e^{i2\pi m} \) is a single-valued phase factor, this statement means that the electron operator is relatively local with respect to all other operators. Any correlator containing the electron operator is necessary singlevalued with respect to that operator.

The conformal dimension and \( u(1) \) charge of the electron operator are

\[
\Delta = \frac{M}{2}, \quad M \in \mathbb{Z}_{\geq 0}
\]  

(5.52)

\[
Q_{u(1)} = \sqrt{M}
\]  

(5.53)

The fact that \( M \) is integer and larger than zero originates from our TQFT description.

And finally, we can write down the wavefunction as a correlator

\[
\Psi_M^{L}(z_1, \ldots, z_N_e) = \left< O_{bg} \prod_{i=1}^{N_e} V_{el}(z_i) \right>
\]  

(5.54)

\[
= \lim_{w \rightarrow \infty} w^{MN_e^2} \left< e^{-i\sqrt{M} N_e \varphi(w)} \prod_{i=1}^{N_e} e^{i\sqrt{M} \varphi(z_i)} \right>
\]  

(5.55)

Direct application of expression (5.46) reproduces the results of chapter 3

\[
\Psi_M^{L}(z_1, \ldots, z_N_e) = \prod_{i<j} (z_i - z_j)^M
\]  

(5.56)

We have reproduced the topological part of the Laughlin wavefunction, including the bosonic versions. By inserting the geometric factor by hand we obtain the full expression for the Laughlin wavefunction. This is the first great accomplishment of our work so far.

But there is far more information to extract. First off, take a look at the conformal dimension of the electron operator. This tells us that under a rotation the operator transforms as

\[
z \rightarrow w(z) = e^{i2\pi} z
\]  

(5.57)

\[
V_{el}(z) \rightarrow V_{el}(w) = e^{-i2\pi \Delta_{el}} V_{el}(z)
\]  

(5.58)
5.2 Laughlin’s wavefunction and the chiral boson

This is why we identify the conformal dimension also to the spin of the operator. Since $\Delta_{el} = \frac{M}{2}$ we conclude that the spin is integer for $M$ even, and half-integer for $M$ odd. In other words, $V_{el}(z)$ is bosonic for $M$ even and fermionic for $M$ odd. This is also reflected in the wavefunction (5.56). Its easy to check that under the interchange $z_i \leftrightarrow z_j$ the wavefunction is symmetry for $M$ even and antisymmetric for $M$ odd. This is of course nothing but quantum statistics at play.

The $u(1)$ charge also has a natural meaning. The OPE of two electrons is

$$e^{i\sqrt{M}\varphi(z)} e^{i\sqrt{M}\varphi(w)} \sim (z - w)^M e^{i2\sqrt{M}\varphi(w)}$$ \hspace{1cm} (5.59)

For $M$ positive this reflects a repulsive interaction. The electrons are kept away from each other through the factor $(z - w)^M$. Furthermore, if we wind an electron around the other the wavefunction picks up a phase of $e^{2\pi M}$. The electron thus carries charge and vorticity, concepts we treated in chapter 3.

In standard units the electrons each electron carries a charge of $-e$ and the strength of the repulsion is $e^2$. Since the electrons carry a $u(1)$ charge of $\sqrt{M}$ we can identify $\sqrt{M} = 1 [-e]$. It is convenient to work in natural units where $-e = 1$. This can be accomplished by renormalizing the current operator. In this case, the charge operator becomes

$$Q_e = \frac{1}{\sqrt{M}} Q_{u(1)} = \frac{1}{2\pi \sqrt{M}} \oint dw J(w)$$ \hspace{1cm} (5.60)

In effect, the vertex operator $e^{i\sqrt{M}\varphi}$ carries a charge of $\frac{M}{\sqrt{M}}$ and a vorticity of $m$. For the electron operator we have $m = M$, therefore it carries a unit charge and a vorticity $M$.

The fact that we may relate the $u(1)$ charge with the electric charge also gives a natural interpretation to the background charge operator. In a quantum Hall sample there is an ionic lattic present, which compensates the total electron charge rendering the sample electrically neutral. Hence we identify the background charge with the true background of the quantum Hall sample.

We note that in the original paper [Moore and Read, 1991] constructed a different method which uses a smeared out background charge. This can be seen as adding a curvature term to the action [Di Francesco et al., 1995]. This procedure is a bit more involved, so we will not go into it here. However, its advantage is that it reproduces the geometric factor thus obtaining the full wavefunction.

The great thing about the CFT description is that it also gives a decent description of the quasiholes. Similar to the electrons we identify a quasihole with an operator from the corresponding CFT. For that we will take the operator with the smallest charge. This operator is

$$V_{qh} = e^{i\sqrt{M}\varphi(z)}$$ \hspace{1cm} (5.61)

It has conformal dimension $\Delta = \frac{1}{2M}$. Its spin is fractional which is in agreement with the spin-statistics theorem and the fractional statistics of the quasiholes. Furthermore, it carries a vorticity of 1 and according to the current operator it carries an electric charge of $\frac{1}{M}$. However, we must be careful with this last statement, as the sign of the quasihole’s charge is actually opposite to that of the electron. This can be seen as follows.

From the OPE of a quasihole with an electron (5.51) we see that the wavefunction vanishes by a factor of $(z - w)$ as an electron approaches a quasihole. This means the electrons feels
as if the charge of the quasi-hole is negative, and it is therefore repelled from its location. The outside observer, however, measures a deficit of electrons at the quasi-hole’s location. This deficiency of electrons is positively charged for the outside observer.

To construct wavefunctions containing quasiholes we compute the correlator containing $N_e$ electrons and $N_q$ quasiholes.

$$\Psi^M(z_1, \ldots z_{N_e}, w_1, \ldots, w_{N_q}) = \lim_{u \to \infty} u^{Q_{bg}^2} \left< e^{-iQ_{bg}\varphi(u)} \prod_{i=1}^{N_e} e^{i\sqrt{M}\varphi(z_i)} \prod_{i=1}^{N_q} e^{i\frac{m}{\sqrt{M}}\varphi(w_i)} \right> \quad (5.62)$$

$$= N_q \prod_{i<j} (w_i - w_j) \prod_{i}^{N_e} \prod_{j}^{N_q} (z_i - w_j) \prod_{i<j}^{N_e} (z_i - z_j)^M \quad (5.63)$$

where the background charge $Q_{bg}$ is given by $\sqrt{MN_e + N_q \frac{m}{\sqrt{M}}}$. This expression is the second main result of this subsection.

All in all, throughout this thesis we have come across the Laughlin sequence in a number of ways. We have seen how the state can be obtained from first quantization, largely on the basis of symmetries of the system. Second, we have seen how Abelian Chern-Simons theory serves as an effective low-energy theory for this sequence and can be used for a field-theoretic description of the states. Last but not least, the connection with CFT is made with its correlators describing the quantum Hall wavefunctions.

We will discuss the wavefunction in the presence of quasiholes in the next subsection. Before we come to that, let us reflect on our treatment so far and set up a more rigorous framework.

### 5.3 Conformal field theory and the quantum Hall effect

Let’s give a description on how CFT and the quantum Hall effect. In the lowest Landau level for a circular FQH droplet the general form for the wavefunction

$$\Psi(z_1, \ldots, \bar{z}_{N_e}) = f(z_1, \ldots, z_{N_e}) e^{-\frac{1}{4\pi B} \sum_i |z_i|^2} \quad (5.64)$$

In this expression the function $f(z_1, \ldots, z_{N_e})$ is analytical and holomorphic. The fact that $f$ is holomorphic suggests that the chiral correlators of a CFT may serve as candidates to obtain the wavefunction of a quantum Hall system at a particular filling fraction. This is largely based on the low-energy effective field theory description, which is given by a Chern-Simons theory.

From the part of CFT, we start off by choosing a chiral Rational CFT. Such theories always contain the Virasoro algebra as a symmetry. However, there are plenty of theories in which there are additional structures present. This is already the case with the compactified chiral boson, which has a gauge structure. Naturally, the WZW-models also fall into this class, and in turn they can be used to construct Coset theories [Di Francesco et al., 1995].

In any case, whatever theory is picked it is still bound to some restrictions. The most important one is the identification of the electron. When we wish to identify an operator with the electron it should carry all characteristics of the electron. For a formal introduction to this matter and its implication on the chiral algebra see [Fröhlich et al., 2001]. Here, we list some of the more obvious features.
We set $V_{el}$ to represent our electron operator with scaling dimension $\Delta_{el}$. The first property it should satisfy is that it represents either bosons or fermions. Via the spin-statistics theorem we can relate this condition to its behavior under rotations. Upon a rotation we send $z \to e^{i2\pi z}$, and so the operator picks up a phase $e^{2\pi i \Delta_{el}}$. For bosons we need integer spin, therefore $\Delta_{el} \in \mathbb{N}$. Consequently, for fermions we need $\Delta_{el} \in \mathbb{N} + \frac{1}{2}$.

A second condition we implement is that the electron operator is relatively local with respect to all other operators. This condition is met if the OPE of the electron with any other operator $V_q$ is singlevalued. From expression (5.20) we see that this is satisfied if

$$\Delta_{el} + \Delta_q - \Delta_{q'} = 0 \mod 1 \tag{5.65}$$

where $V_{q'}$ is defined as the fusion product of the two operators.

A third property is more related to the chiral algebra itself. The characteristic of the FQHE is the behavior of the electrical current. Hence, the chiral algebra must contain such a current which is described by a $u(1)$ current algebra. This current algebra is in fact the compactified chiral boson, only we allow rational values of the compactification radius $M$ as well. This construction will become clear in the next chapter, where we introduce a more complicated CFT, the parafermions. In general, the electron operator will take the following form

$$V_{el} = \psi e^{i\alpha \varphi(z)} \tag{5.66}$$

The vertex operator part describes the $u(1)$ current. Although not proven here, one can show that $\alpha = \frac{1}{\sqrt{\nu}}$. This takes into account the electrical current. The operator $\psi$ is an operator yet to be determined. Its structure can be quite rich. However, it is still restricted to certain rules. The most important one is that it must be a simple current. This means that its fusion product with all remaining operators in the theory is unique.

Having identified the electron operator we obtain an expression for the wavefunction through the following correlator

$$\Psi_L^M(z_1, \ldots, z_{N_e}) = \left\langle O_{bg} \prod_{i=1}^{N_e} V_{el}(z_i) \right\rangle \tag{5.67}$$

We have inserted a background charge, since the theory always contains at least one $u(1)$ current algebra. This is the “master formula” of the CFT-QHE connection. Note that the geometric factor has to be attached by hand.

The question remains, why are chiral correlators suitable candidates to construct quantum Hall wavefunctions with? This is reflected in a number of properties. The first argument we already mentioned, which is the chirality of the correlators and the wavefunctions. Second, if the RCFT is chosen properly the operator identified with the electron exhibits all characteristics of the electron. This includes (anti)symmetry and spin.

Consider the OPE of our (fictitious) electron operator. Since it is a simple current its OPE has the general form

$$V_{el}(z)V_{el}(w) \sim (z-w)^{2\Delta_{el}-\Delta_{f}} V_{f}(w) + \ldots \tag{5.68}$$

From a CFT perspective the electrons fuse together to form a composite operator, $V_f(w)$. This is quite an unusual statement, as the electrons repel obey the Pauli principle, which
forbids them to come that close. We interpret this as bringing the electrons relatively close to each other, while the remaining electrons are kept far away. The electrons kept far away will not be able to distinguish the two electrons and simply “view” them as a composite particle. Furthermore, at length scales smaller than the magnetic length we expect the CFT-description to essentially break down. So the true fused state is some local description of the two-electron compositon which is difficult to describe. For long distances, however, its effects are encaptured through the operator $V_{f}(w)$.

Looking at expression (5.68) we conclude that the process of fusing two electrons is suppressed if the sign of $2\Delta_{el} - \Delta_{f}$ is positive. It depends on the details of the CFT if this requirement is met. If that is the case, then this OPE resembles the Coulomb interaction quite well, as it keeps the electrons separated from each other.

Based on chapter 4 we do not expect that our theory neatly describes the Coulomb interaction or other microscopic details of the system. The CFT description is an idealized approach, which, based on renormalization, cannot account for all details of the system. What it does contain are all the features of the system which “survive” the renormalization procedure. In other words, the CFT model is a description of the fixed point of the universality class of the quantum Hall fluid at hand.

In the next chapter we will give the first, and most famous example of a CFT description of fraction quantum Hall states. These states are called the Read-Rezayi states, named after their founders [Read and Rezayi, 1999]. This includes a description of the Moore-Read state [Moore and Read, 1991], which was discovered a few years prior to the Read-Rezayi series. These states are based on a CFT which contains a parafermionic structure, which are a certain class of non-Abelian CFT’s. There are plenty of other CFT description around which we will not address. We refer to the literature for such overviews [Nayak et al., 2008].

The reason why these states are so interesting is mostly due to their non-Abelian nature and the fact that they have quite physical characteristics. Specifically, they have a property called clustering which greatly modifies the way in which the wavefunctions behave as multiple electrons are brought together. The Laughlin state only deals with simple 2-body interaction. The parafermionic states also takes into account higher order correlations.

However, we cannot continue before we address the concept of non-Abelian statistics which is what makes the parafermionic states so interesting. The remainder of this chapter is a brief introduction to this concept.

### 5.4 Berry phase, non-Abelian statistics and quasiholes

Recall that the expression of the Laughlin wavefunction in the presence of quasiholes, equation (5.63), is in agreement with the results obtained for the Laughlin state in chapter 3, equation (3.76). That is, up to an overall factor of $\prod_{i<j}^{N_{q}}(w_{i} - w_{i})^{1/\ell}$. Where did this factor come from and what does it represent?

Let us state that there is good explanation for this discrepancy. First, observe that expression (5.63) is singlevalued with respect to the electron coordinates $z_{i}$, but not with respect to the quasihole’s $w_{i}$. The first is a physical demand. Electrons are real particles and a single-valued wavefunction is therefore a solid, physical demand. But the quasiholes are not physical particles. They are a reflection of the collective behavior of the electrons, so it is reasonable to allow exotic features such as multivalued wavefunctions and fractional quantum numbers to emerge.
5.4 Berry phase, non-Abelian statistics and quasiholes

Having said that, we interpret the difference of expression (5.63) and (3.76) as follows [Moore and Read, 1991; Read, 2008]. The factor \( \prod_{i<j} (w_i - w_j) \) takes into account the statistics of the quasiparticles. The action of winding one quasiparticle around another is described by a monodromy operation. As a sidenote, there is also a “dynamical phase” associated to this action. It is determined by the energy and total time in which the action takes place. This contribution completely decouples from the one considered here, so we do not take it into account.

In any case, a monodromy operation causes the wavefunction (5.63) to pick up a phasefactor of \( e^{i \frac{2\pi}{M}} \). This is nothing more but the fractional statistics of the quasiholes. We conclude the statistics is described by a statistical angle of \( \frac{2\pi}{M} \), in agreement with the results of chapter 4.3.

However, the assumption that the monodromy operation equals the physical action of moving a quasihole around another is not always valid. This is because the monodromy is essentially an analytical continuation of the wavefunction. It does not take into account the possible existence of a other contributions, such as gauge fields. To incorporate this we have to take into account the Berry phase [Wilczek, 1990]. This phase, along with the effect of the monodromy, is determined by parallel transporting the quasihole. For Abelian quasiholes the Berry phase is [Nayak et al., 2008]

\[
\alpha = \int d\mathbf{R} \cdot \langle \Psi(\mathbf{R}) | \nabla_{\mathbf{R}} | \Psi(\mathbf{R}) \rangle
\]  

(5.69)

The term \( \langle \Psi(\mathbf{R}) | \nabla_{\mathbf{R}} | \Psi(\mathbf{R}) \rangle \) is called the Berry connection and it is the cause of the Berry phase in essentially the same manner as the Aharonov-Bohm effect. In chapter 3 we showed how we can absorb the effect of the Berry connection through a singular gauge transformation at the cost of a multivalued wavefunction. Formula (5.63) exhibits this behavior, as the Berry connection is zero in this system. This is not the case with Laughlin’s expression (3.76) though. If we parallel transport a quasihole around another in this expression we cannot simply apply the monodromy operation. In this expression the Berry connection is non-zero, and therefore we also have to take the Berry phase into account. See [Read, 2008] for more information on this matter and also a calculation showing that applying the technique of parallel transport to Laughlin’s expression (3.76) one can obtain the same phase factor as presented in this chapter.

The fact that CFT automatically gives wavefunctions in which the Berry curvature is zero for all particles is quite suprising and noteworthy. However, it a conjecture which is not rigorously proved. For certain classes of quantum Hall states it has explicitly been shown that the statistics of the quasiholes is completely determined through the chiral correlators, [Blok and Wen, 1992].

It is a reflection of the fact that the CFT automatically form representations of the Braid group, with the monodromy serving as the action of a braid operator. Fractional statistics is also a kind of representation theory of the Braid group which makes the correspondence with CFT quite natural [Nayak et al., 2008; Slingerland, 2002].

This is a good time to address the issue of non-Abelian statistics. This concept is a direct generalization of fractional statistics, which also accounts for Bose and Fermi statistics. One naturally interprets statistics as a representation theory of the Braid group. The Braid group contains an infinite number of elements and its representation theory is incredibly structured. The term actually corresponds to a whole class of groups, denoted by \( B_N \). This group is
5. Conformal field theory and the quantum Hall effect

Figure 5.1: Braiding operation of two particles.

algebraically generated through the generators $\sigma_i$, $i \in \{1, \ldots, N-1\}$. Pictorially, $\sigma_i$ is represented by the counter-clockwise exchange of the $i$’th and $i+1$’th particle, see figure 5.1. The defining relations of the braid group are given by

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2 \]  
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq N - 1 \]

Boson and fermions correspond to simple representations of the braid group. Namely, the counter-clockwise interchange of particles for the $\sigma$ operators act on the wavefunction as

\[ \Psi \rightarrow \rho(\sigma_i) \Psi \]
\[ \rho(\sigma_i) = \rho(\sigma_i^{-1}) = \begin{cases} 1 & \text{for bosons} \\ -1 & \text{for fermions} \end{cases} \]

The order in which interchanges are performed has no influence on the statistics. Furthermore, we cannot distinguish between clockwise and counter-clockwise interchanges. This is characteristic for bose and fermi statistics.

For Abelian anyons the situation is slightly more complicated. We know that the interchange of two Abelian anyons leads to the multiplication with an overall phase-factor, characterized by the statistical angle. The order of interchanges is still not important, but there is a difference between a counter-clockwise ($\sigma_i$) and clockwise ($\sigma_i^{-1}$) interchange. This type of statistics also corresponds to one-dimensional representations of the braid group

\[ \rho(\sigma_i) = e^{i\theta} \]
\[ \rho(\sigma_i^{-1}) = e^{-i\theta} \]

where $\theta$ is the statistical angle.

The generalization to non-Abelian statistics should come as no surprise by now. Namely, these cases correspond to higher-dimensional representations of the braid group. In that case, wavefunctions of the particles are degenerate.

For the sake of argument, consider $N$ particles all fixed at a certain location, $R_1, \ldots, R_N$. The corresponding wavefunction $\Psi$ is $g$-fold degenerate and can be viewed as a superposition of some orthonormal basis $\psi_\alpha$

\[ \Psi = \sum_{\alpha=1}^{g} a_\alpha \psi_\alpha \]

In that case, the action of the braid group on this element is represented by a $g \times g$ unitary matrix acting on the basisstates

\[ \psi_\alpha \rightarrow \rho(\sigma_1)_{\alpha\beta} \psi_\alpha \]
In this case we have interchanged particles one and two, in a counterclockwise manner. This can also be done for particles two and three

\[ \psi_\alpha \rightarrow \rho(\sigma_2)_{\alpha\beta} \psi_\alpha \] (5.78)

Here is the essential part of the story: these two interchanges do not commute. So in general, we have

\[ \rho(\sigma_1)\rho(\sigma_2) \neq \rho(\sigma_2)\rho(\sigma_1) \] (5.79)

This is precisely the characteristic of non-Abelian statistics. In general, such braiding operations lead to rotation in the \( g \)-dimensional vector space which is spanned by the \( \psi_\alpha \). On top of that, there is no other way of inducing such transition. Only through the physical, non-local action of braiding of particles are we able to accomplish such a rotation. This is the prime example of the topological nature of non-Abelian quasiparticles. Local interactions are not capable of influencing this internal vector space.

By definition, non-Abelian quantum Hall states contain quasihole excitations which exhibit non-Abelian statistics. Using CFT or TQFT to describe these particles leads to the possibility that the non-Abelian nature is only present in the idealized (mathematical) effective description of the quantum Hall states. However, we expect that the essential properties, such as the braiding behavior and the electrical charge, are carried over to the physical system as well and the non-Abelian statistics is maintained as long as the quasiparticles do not approach each other too close. In physical samples the different ground states will have a small energy difference, which becomes noticeable only if the quasiparticles are close to one another. The braiding behavior on the other hand is an example of a topological property, as it survives small perturbations imposed on the system and remains present over long distances.

For the possibility of non-Abelian statistics to occur there are some requirements which have to be fulfilled by the quasiparticles. Specifically, in the presence of a significantly large number of quasiparticles the corresponding ground state is degenerate. The wavefunction is essentially a vector of a vector space. Second, by adiabatically braiding the quasiparticles we can induce rotations within this vector space. For this to be accomplished we will need a sufficiently large energy gap separating the degenerate ground state from the excited states. Last but not least, braiding of the quasiparticles is the only way through which these rotations can be accomplished. One can think of many examples in which the ground state is degenerate, but most of the time one can transfer from one state to another via local interactions. It is precisely these local perturbation which have no influence on the non-Abelian nature of quasiparticles.
In this chapter we will describe the most famous example of a series of quantum Hall states, which are based on an underlying CFT. Contrary to the Laughlin states, whose relation to CFT was proven afterwards, these states are obtained on the basis of a CFT, called the $\mathbb{Z}_k$ parafermionic conformal field theory [Fateev and Zamalodchikov, 1985]. The states obtained this way are called the Read-Rezayi states [Read and Rezayi, 1999].

What makes these states so interesting is their physical interpretation. The states are exact FQH ground states of idealized interaction Hamiltonians. These Hamiltonians represent higher-order correlations among the electrons, which is why we expect these states to correspond to universality classes of physical quantum Hall states.

The Read-Rezayi states are a generalization of another quantum Hall sequence, which are called the Moore-Read states [Moore and Read, 1991]. The Moore-Read states was the first example of a non-Abelian quantum Hall fluid, and we will study some of its properties in detail. Specifically, we will obtain expressions for the electron wavefunction in the presence of zero and two quasiholes. These expressions are obtained using techniques provided by conformal field theory, which demonstrates the rigid structure of this mathematical tool. We also obtain an expression for a “bare” wavefunction of four quasiholes. That is, we evaluate and study the correlator of four quasiholes in the absence of electrons. This correlator is the simplest example of a correlator which equals a sum of conformal blocks.

### 6.1 The parafermion CFT

Parafermionic quantum Hall states are based on a conformal field theory introduced in the mid-80’s by [Fateev and Zamalodchikov, 1985]. This CFT is called the $\mathbb{Z}_k$ parafermionic conformal field theory and in this section we will give a brief introduction to its structure.
6. Read-Rezayi quantum Hall states

The original paper by [Fateev and Zamalodchikov, 1985] is a formal introduction to a slightly different CFT, as it concludes also the anti-chiral part and contains a larger symmetry group, namely $\mathbb{Z}_k \otimes \mathbb{Z}_k$. This is why the sources for this chapter are scattered around a number of papers. These include [Read and Rezayi, 1999; Ardonne and Schoutens, 2007; Barkeshli and Wen, 2008; Slingerland, 2002].

6.1.1 Parafermions

The basic property of the CFT is the presence of a $\mathbb{Z}_k$ symmetry. The space of fields is build up as a direct sum of subspaces, with each subspace characterized by a so-called $\mathbb{Z}_k$ charge. The $\mathbb{Z}_k$ charge of a field $\Phi_{lm}$ is denoted by its label $l$. The label $m$ is an additional label which we treat later on. Under $\mathbb{Z}_k$ transformations they transform as

$$\Phi^l(z) \xrightarrow{\omega} \omega^r \Phi^l(z), \quad \omega \in \mathbb{Z}_k \quad (6.1)$$

In total, the $\mathbb{Z}_k$-parafermion consists out of $\frac{1}{2}k(k+1)$ Virasoro primary fields. Among those fields are $k-1$ parafermions, denoted by $\psi_a(z)$ (with $a \in \{0, \ldots, k-1\}$). These fields are a natural generalization of Majorana fermions and will, with the proper adjustments, play the role of the fermions appearing in the CFT description of the FQHE states. Let us therefore study them in a bit more detail. They are the operators which have a trivial $\mathbb{Z}_k$ charge. Their operator algebra is as follows

$$\psi_a(z)\psi_b(w) \sim C_{a,b}(z-w)^{-(\Delta_a+\Delta_b-\Delta_{a+b})}[\psi_{a+b}(w)+\ldots] \quad a + b < k \quad (6.2)$$

$$\psi_a(z)\psi_b(w) \sim C_{a,b}(z-w)^{-(\Delta_a+\Delta_b-\Delta_{a+b}-k)}[\psi_{a+b-k}(w)+\ldots] \quad a + b > k \quad (6.3)$$

$$\psi_a(z)\psi_{k-a}(w) \sim (z-w)^{-2\Delta_a}[1+(z-w)\frac{2\Delta_a}{c}T(w)+\ldots] \quad (6.4)$$

$$T(z)\psi_a(w) \sim \frac{\Delta_b}{(z-w)^2}\psi_a(w) + \frac{1}{z-w}\partial \psi_a(w) \quad (6.5)$$

The central charge is $c = 2(k-1)/(k+2)$ and the $C_{a,b}$ are called the structure constants of the algebra - they can be determined through use of consistency relations. Note that we have normalized the fields such that $C_{a,a} = 1$. Furthermore, the conformal weights $\Delta_a$ are defined as

$$\Delta_a = \frac{a(k-a)}{k} \quad (6.6)$$

which satisfy the following identities, useful for later purposes

$$\Delta_a = \Delta_{k-a} \quad (6.7)$$

$$\Delta_a + \Delta_k - \Delta_{a+b} = \frac{2ab}{k} \quad (6.8)$$

$$\Delta_a + \Delta_b - \Delta_{a+b-k} = \frac{2(k-a)(k-b)}{k} \quad (6.9)$$

The parafermions $\psi_a$ carry a label $a$, which we will call the $\mathbb{Z}_k$ quantum number of the parafermion. There is no standard convention on how this label should be addressed though. Do keep in mind that this label is not the same as the $\mathbb{Z}_k$ charge mentioned above.
6.1 The parafermion CFT

The $\mathbb{Z}_k$ quantum number has a modulo additive property, meaning that when we fuse two parafermions their $\mathbb{Z}_k$-charges add up modulo $k$: $\psi_a \times \psi_b = \psi_{a+b \mod k}$. This automatically leads to the following identity

$$ (\psi_a)^{\times k} = 1 \quad (6.10) $$

From that point of view it is natural to see that $\psi_0 = \psi_k = 1$. Note that this greatly influences the correlators, as these are non-zero only if the fusion product of its operators contains the identity operator.

The operation $a \mod k$ frequently appears throughout this thesis. This is why we introduce the following notation

$$ [a]_k \equiv a \mod k \quad (6.11) $$

With this notation the fusion rules of the parafermions are quite simple

$$ \psi_{[a]_k} \times \psi_{[b]_k} = \psi_{[a+b]_k} \quad (6.12) $$

### 6.1.2 Operator content and field identification

Apart from the parafermions there are also a number of other fields present in the system. We give a brief overview of the basics. However, these fields and their properties will only seriously come into play in chapter 9. In this chapter we will give a brief overview of the remaining fields.

The fields in the $\mathbb{Z}_k$ theory are written as

$$ \Phi_{lm} = \begin{cases} l & \in \{1, \ldots, k-1\} \\ m & \in \{-l+2, -l+4, \ldots, l\} \end{cases} \quad (6.13) $$

In this case $l$ constitutes the $\mathbb{Z}_k$ charge of the field. The label $m$ is their $\mathbb{Z}_k$ quantum number. The conformal dimension is

$$ \Delta_{lm} = \frac{l(k-l)}{2k(k+2)} + \frac{(l-m)(l+m)}{4k}, \quad l \in \{1, \ldots, k\} \quad m \in \{-l+2, \ldots, l\} \quad (6.14) $$

Some of these fields have their own notation, namely

- Parafermions $\psi_a = \Phi_{2a-k}^k$  $\Delta_{2a} = \frac{a(k-a)}{k}$  
- Spin fields $\sigma_a = \Phi_a^a$  $\Delta_a = \frac{a(k-a)}{2k(k+2)}$  
- Neutral fields $\epsilon_a = \Phi_{2a}^0$  $\Delta_{2a} = \frac{a(a+1)}{k+2}$

Where the range of $a$ is such that the labels of the fields fall inside the range of (6.13).

Due to the strict range of the labels $l$ and $m$ it is quite cumbersome to explicitly write down the fusion rules of the fields using this convention. It is much more convenient to apply a different approach, namely, we relax the ranges of $m$ and $l$ to the extent that $\Phi_{lm}^a$ represents a primary field if $l$ and $m$ satisfy

$$ l \in \mathbb{Z} \quad (6.18) $$

$$ m \in \mathbb{Z} \quad (6.19) $$

$$ l + m \mod 2 = 0 \quad (6.20) $$
The ranges of the labels are now infinite. Next we invoke a field identification among these infinite number of fields.

\[ \Phi^l_m = \Phi^{l+2k}_m = \Phi^{k-l}_{k+m} \]  

(6.21)

In other words \( \Phi^l_m, \Phi^l_{n+2k} \) and \( \Phi^{k-l}_{k+n} \) all represent the same primary field. For instance, parafermions correspond to \( \psi_a = \Phi^1_{2a} = \Phi^{0}_{2a} = \ldots \). A better terminology refers to this as a primary sector as all these fields have the same conformal properties. We mention that throughout this thesis the basic range (6.13) is referred to as the “unit cell”.

With the use of these field identifications the fusion rules take on the following form

\[ \Phi^l_m \times \Phi^{l'}_{m'} = \sum_{l'' \in E \times E} \Phi^{l''}_{m+m'} \]  

(6.22)

where the range of \( l'' \) runs over \( \{ |l-l'|, |l-l'|+2, \ldots, \min(l+l', 2k-l-l'') \} \). It takes some time to get acquainted with the rules though.

This field identification originates from a particular coset construction of the parafermionic CFT. Namely, as is shown in many references (see the beginning of this chapter) the \( \mathbb{Z}_k \) CFT admits a coset construction in terms of \( su(2)_k/u(1)_2 \). From this point of view the two labels correspond to an \( su(2) \) label \( (l) \) and a \( u(1) \) label \((l)\). See [Di Francesco et al., 1995, final chapter] for more info on this matter. We will not refer to this coset construction, but it helps to know where the field identification originates from.

Naturally, fields that are identified have the same conformal dimension. The conformal dimension of any field can be computed by taking the labels in the principle series, that is \( l \in \{1, 2, \ldots, k\} \) and \( m \in \{-l+2, -l+4, \ldots, l\} \). This is accomplished by applying the field identifications. The conformal dimension is then given by formula (6.14).

We now turn to the construction of FQHE states.

### 6.2 \( \mathbb{Z}_k \) quantum Hall states - Read-Rezayi series

The Read-Rezayi states are a prime example of how conformal field theory is used to derive all major features of the state. Unlike the Laughlin series, the state does not follow from elementary considerations. Rather, the states are defined on the basis of CFT and afterwards, on the basis of the symmetries the resulting state exhibits, their physical relevance is justified. Indeed, some of the Read-Rezayi states are prime candidates for certain quantum Hall states.

#### 6.2.1 Electron operator and filling fraction

In the case of Read-Rezayi we identify the operator \( \psi_1(z) \) of the \( \mathbb{Z}_k \) parafermionic CFT as (part of) the electron operator. This operator is a simple current and is therefore a suitable candidate for the electron operator. However, this operator by itself cannot be the full electron operator. First, this operator does not carry any electric \((u(1))\) charge (the parafermionic operator algebra does not contain a \(\hat{u}(1)\) subalgebra). Second, the fusing of two parafermions is singular \((\sim (z-w)^{-\frac{2}{k}})\), which is not a desirable property. Third and last, the operator \( \psi_1(z) \) is not a local operator. If we perform a monodromy action where \( z \) goes around \( w \), then the fusion product \( \sim (z-w)^{-\frac{2}{k}} \) picks up a phase \(-\frac{2}{k}\), which is fractional for \( k > 2 \).
As it turns out, we can fix all these problems by the following modification. We attach a \( u(1) \) vertex operator \( \exp(i\alpha\varphi)(z) \) to \( \psi_1(z) \) and tune \( \alpha \) in such a way that the combined operator is a local one and the fusion product does not behave singular.

To make this explicit we make the ansatz where \( \psi_1 \exp(i\alpha\varphi)(z) \) plays the role of the electron operator. To determine \( \alpha \) we consider the fusion of two electron operators

\[
(\psi_1 e^{i\alpha\varphi}(z)) (\psi_1 e^{i\alpha\varphi}(w)) = (\psi_1(z) \psi_1(w)) (e^{i\alpha\varphi(z)} e^{i\alpha\varphi(w)})
\]

In the first step we made use of the fact that the \( u(1) \) vertex part and the parafermionic part of the CFT decouple. In the second line we applied the OPE’s of the parafermionic CFT as described in section 6.1 and of the vertex operator as described in the appendix B.

We want to impose the condition that the electron operator is a local operator and does not behave singular anywhere. This sets \( \alpha^2 - \frac{2}{k} = M \), where \( M \) is an integer \( \geq 0 \). The electron operator of the Read-Rezayi states is thus

\[
V_{el} = \psi_1 e^{i\sqrt{\frac{2M+2}{k}} \varphi}
\]

The operator \( \psi_1 \) has conformal dimension \( \frac{k-1}{k} \), while for the vertex operator it is \( \frac{kM+2}{2k} \). The total conformal dimension, which is also the spin of the particle, is simply the sum of the parafermionic part and the \( u(1) \) part: \( \Delta_{el} = \Delta_{\psi_1} + \Delta_{u(1)} = \frac{M+2}{2} \). This operator is therefore \textit{bosonic} for \( M \) \textit{even}, and \textit{fermionic} for \( M \) \textit{odd}. We can also read off the \( u(1) \) charge: \( \sqrt{\frac{kM+2}{k}} \). We normalize the \( u(1) \)-current such that this charge is set to 1. The electric charge is then determined via the operator \( \frac{k}{kM+2} \partial \phi \).

Now that we have identified the proper electron operator, we can begin constructing electron wavefunctions. We plug it in the general expression for the wavefunction of \( N_e \) electrons (5.67). We take \( N_e \) to be a multiple of \( k \) (in the absence of quasiholes), or else the parafermions cannot fuse to the identity and the correlator comes out identically zero. The wavefunction for the Read-Rezayi state is

\[
\Psi_{RR}(z_1, \ldots, z_{N_e}) = \lim_{w \to \infty} w^{N_e^2 \left( \frac{2M+2}{k} \right)} \langle \psi_1(z_1) \cdots \psi_1(z_{N_e}) \rangle \langle e^{i\sqrt{\frac{2M+2}{k}} \varphi(z_1)} \cdots e^{i\sqrt{\frac{2M+2}{k}} \varphi(z_{N_e})} e^{-iN_e \sqrt{\frac{2M+2}{k}} \varphi(w)} \rangle
\]

Recall that the operator \( \lim_{w \to \infty} w^{N_e^2 \left( \frac{kM+2}{k} \right)} e^{-iN_e \sqrt{\frac{kM+2}{k}} \varphi(w)} \) serves as the background charge which is needed to satisfy charge neutrality. The \( u(1) \) part can be computed using results from chapter 5. We have

\[
\Psi_{RR}(z_1, \ldots, z_{N_e}) = \langle \psi_1(z_1) \cdots \psi_1(z_{N_e}) \rangle \prod_{i<j} (z_i - z_j)^{M+\frac{2}{k}}
\]

These states are called the Read-Rezayi states. It consists out of a parafermionic correlator times a peculiar Laughlin-Jastrow factor. We will study it in more detail in the next section.
6. Read-Rezayi quantum Hall states

For now we mention that for $k > 2$ the computation of the parafermionic correlator is quite hard. Exact expressions do exist however, and we will deal with them soon. The case of $k = 2$ is special, as the parafermionic CFT is much better understood in this case. In this case the parafermions reduce to the Ising model. This quantum Hall state is called the Moore-Read state [Moore and Read, 1991], and they were found prior to the Read-Rezayi states. This example will be treated at the end of this chapter.

The strength of the CFT approach is that we are still able to come up with sensible statements about these states even if the exact expressions of the wavefunctions are not known.

A first example is the computation of the filling fraction. Consider the limit where we take one electron and send it to infinity, $z_i \to \infty$. The separation of the remaining $N_e - 1$ parafermionic operators become relatively small, so they fuse together to form a single operator. Since all $N_e$ parafermions fuse to the identity, we know that $N_e - 1$ parafermions fuse to the operator $\psi_{k-1}(w)$. In this limit the parafermionic part behaves as $z^{-2\Delta_1}$:

$$\langle \psi_{1}(z)\psi_{k-1}(w) \rangle \sim z_i^{-\frac{2k-1}{k}}, \quad z_i \to \infty$$

(6.29)

The leading part of the Laughlin-Jastrow factor, on the other hand, is

$$\prod_{j=1, j \neq i}^{N_e} (z_i - z_j)^{M + \frac{2}{k}} \sim z_i^{(M + \frac{2}{k})(N_e - 1)}, \quad z_i \to \infty$$

(6.30)

Combining these two results gives us the behavior of the wavefunction itself in this limit

$$\Psi(z_1, \ldots, z_{N_e}) \sim z_i^{(M + \frac{2}{k})N_e - (M + 2)}, \quad z_i \to \infty$$

(6.31)

We know that the highest degree of any electron coordinate is equal to the number of fluxes, $N_\Phi$. By letting $z_i \to \infty$ it is exactly this highest power of $z_i$ which dominates. We therefore set $N_\Phi = (M + \frac{2}{k})N_e - (M + 2)$. This gives the following relation

$$N_\Phi = \frac{1}{\nu_{k,M}}N_e - S_{k,M}$$

(6.32)

where

$$\nu_{k,M} = \frac{k}{kM + 2}$$

(6.33)

$$S_{k,M} = M + 2$$

(6.34)

There is an imbalance between the ratio $\frac{N_\Phi}{N_e}$ and the filling fraction $\nu$. This ratio is restored in the thermodynamic limit where $N_\Phi, N_e \to \infty$.

The quantity $S$ is called the shift factor. This shift factor arises because in the above limit we have in fact made use of a spherical geometry by sending an electron to infinity. The shift factor is a topological quantum number, and plays an important role in the classification of quantum Hall states. See [Wen, 1992; Zee, 1995] and appendix A for more information on this matter.
6.2.2 Quasihole operator

Finding the quasiholes in the spectrum is a procedure very similar to that of finding the electron. In addition to the parafermions the CFT contains a set of parafermionic primary fields, which are denoted by $\sigma_i$. The quasihole we want to consider are built up out of these operators multiplied by a $u(1)$ vertex part. In the Read-Rezayi case we make the following ansatz for the smallest charge operator

$$V_{qh} = \sigma_1 e^{i \beta \varphi};$$

(6.35)

The parameter $\beta$ is determined through physical conditions. First, the electron operator $V_{el}$ was constructed such that it was a local one. In addition, we demand that it stays local with respect to all other operators present in the theory. To satisfy this condition, we will not require the quasihole operator itself to be local, and as it turns out it indeed never is. A second condition we implement is that the fusion of an electron with a quasihole is non-singular. This conditions assures us we are dealing with a quasihole rather than a quasiparticle.

Using the fusion rules (6.22), we can determine the fusion of an electron with a quasihole. To write down the fusion product for general $k$ we will to switch to the notation of chapter 6.1. In this notation $\psi_1 = \Phi_0^2$ and $\sigma_1 = \Phi_1^3$. The operator product expansion reads

$$V_{el}(z)V_{qh}(w) = \left( \Phi_0^2(z)\Phi_1^3(w) \right) e^{i \frac{M+2}{k} \varphi(z)} e^{i \beta \varphi(w)}$$

(6.36)

$$\sim (z-w)^{-\delta h + \frac{k-1}{2}} e^{i \frac{M+2}{k} \varphi_3(w)} e^{i (\beta + \frac{M+2}{k}) \varphi(w)}$$

(6.37)

The parameter $\delta h$ equals $\Delta_{\psi_1} + \Delta_{\sigma_1} = \Delta_{\Phi_1^3}$. We have

$$\Delta_{\psi_1} = \frac{k-1}{k},$$

(6.38)

$$\Delta_{\sigma_1} = \frac{k-1}{2k(k+2)},$$

(6.39)

$$\Delta_{\Phi_1^3} = \frac{k-1}{2k(k+2)} + \frac{(k-2)}{k},$$

(6.40)

Therefore $\delta h = \frac{1}{k}$ and on the basis of the physical conditions we conclude that $\beta = \frac{1}{\sqrt{k}} \frac{1}{\sqrt{kM+2}}$. The quasihole operator is

$$V_{qh} = \sigma_1 e^{i \frac{1}{\sqrt{k}} \frac{1}{\sqrt{kM+2}} \varphi}$$

(6.41)

The conformal dimension (and spin) of the quasihole equals the sum $\frac{k-1}{2k(k+2)} + \frac{1}{2k(kM+2)} = \frac{(k-1)M+3}{2k(kM+2)}$. Furthermore the quasihole carries an electric charge of $Q_{qh} = \frac{1}{kM+2}$. Although the operator is mutually local with respect to the electron operator, it is in fact not local with respect to itself.

The construction of wavefunctions with both electrons and quasiholes can now be considered. The wavefunction proposed by Read and Rezayi for $N_q$ quasiholes and $N_e$ electrons is

$$\Psi_{R.R}(z_1, \ldots, z_{N_q}, w_1, \ldots, w_{N_e}) = \langle \sigma_1(w_1) \cdots \sigma_1(w_{N_e}) \psi_1(z_1) \cdots \psi_1(z_{N_q}) \rangle \times$$

$$\left( e^{-iN_\varphi(w)} \prod_{i=1}^{N_e} e^{i \frac{M+2}{k} \varphi(z_i)} \prod_{j=1}^{N_q} e^{i \frac{1}{\sqrt{k}} \frac{1}{\sqrt{kM+2}} \varphi(w_j)} \right)$$

(6.42)
where $N$ is the corresponding background charge, equal to $N = N_e \frac{kM+2}{k} + n \frac{1}{kM+2}$. The correlator of the boson vertex operators can be calculated using results from appendix B. We have

$$\Psi_{R,R}(z_1, \ldots, z_{N_e}, w_1, \ldots, w_n) = \langle \sigma_1(w_1) \cdots \sigma_1(w_n) \psi_1(z_1) \cdots \psi_1(z_{N_e}) \rangle \times \prod_{i<j} (z_i - z_j)^{M+\frac{2}{k}} \prod_{i<j} (w_i - w_j)^{\frac{1}{kM+2}} \prod_{i}^{N_e} \prod_{j}^{n} (z_i - w_j)^{\frac{1}{k}} \quad (6.43)$$

Earlier we stated that to construct wavefunctions the sum of the $Z_k$ quantum numbers has to add up to 0 mod $k$. The electron’s $Z_k$ quantum number is 2, while the quasihole’s is 1. Therefore we need

$$N_q + 2N_e = 0 \text{ mod } k \quad (6.44)$$

Since the fusion products of the spin fields are a sum over multiple primaries, there will in general be multiple ways to fuse to the identity. In fact, the number of ways the operators can fuse to the identity grows quite fast. For trivial topologies this number equals the degeneracy of the state in the presence of the quasiholes.

### 6.3 Clustering property

In describing the electron wave functions of the Read-Rezayi quantum Hall states we have come across a family of quantum Hall wavefunctions, characterized by the number $k$ and $M$. For $M$ even these states are bosonic, while for $M$ odd they are fermionic. In this section we will take a closer look at the case where $M = 0$. This bosonic state is given by

$$\Psi^{M=0}_{RR}(z_1, \ldots, z_{N_e}) = \langle \psi_1(z_1) \cdots \psi_1(z_{N_e}) \rangle \prod_{i<j} (z_i - z_j)^{\frac{2}{k}} \quad (6.45)$$

Written this way we can emphasize some important aspects concerning also the wavefunctions for non-zero $M$. Equation (6.45) contains a parafermionic conformal block times a Laughlin-Jastrow factor. It is clear that the L.J. factor present in this expression contains branch cuts for $k > 2$. As a result it is not single-valued with respect to the electron coordinates. The parafermionic conformal block is also multivalued and contains both branch cuts and singularities, which can be seen from the parafermionic operator algebra.

However, when combined to form the bosonic wavefunction (6.45) all branch cuts and singularities are canceled out. What remains is a nice single-valued expression for a bosonic wavefunction. This is not too surprising, since the electron operator was constructed precisely to fulfill these conditions. The bosonic wavefunction serves as the cornerstone of a whole family of both bosonic and fermionic wavefunctions. By multiplication with the factor $\prod_{i<j} (z_i - z_j)^{M}$ with $M \in \mathbb{Z}_{\geq 0}$ we can construct a whole family of wavefunctions - the simplest fermionic wavefunction corresponding to the case $M = 1$.

The overall factor $\prod_{i<j} (z_i - z_j)^{M}$ is a simple Laughlin-Jastrow factor. Its structure is not too complex. It is thus the wavefunction $\Psi^{M=0}_{RR}$ which contains all defining properties of

---

1The case $k = 2$ is special in the sense that neither factors contain any branch cuts. However, the correlator does contain poles which in turn are canceled by the Laughlin-Jastrow factor. We will treat this example at the end of this section.
the Read-Rezayi states, so let us study it in more detail. In particular, it satisfies what is called a clustering condition. The way this condition works out for the Read-Rezayi series is as follows. Up to \( k \) bosons can be brought to the same point, and the wavefunction remains non-zero. A \( k + 1 \)’th boson extra, on the other hand, will always give zero. This property is characterized as [Ardonne, 2002].

\[
\Psi_{\text{Bose}} = \begin{cases} \Psi_{\text{Bose}}(z_1 = \ldots = z_k \neq z_{k+1}, \ldots) & \neq 0 \\ \Psi_{\text{Bose}}(z_1 = \ldots = z_k = z_{k+1}, \ldots) & = 0 \end{cases} \quad (6.46)
\]

This quantum Hall state is said to be clustered at order \( k \). In chapter 7 we will rephrase the clustering condition to a more general form such that it also applies to other states. When applied to the Read-Rezayi states the general definition reduces to the one considered here.

We know that, due to conformal invariance, the bosonic wavefunction is a polynomial with finite degree in all its variables. One aspect of classifying its properties is by means of the order of its zeroes. We therefore ask the following: what is the power of the zero when an electron, say \( z_j \), approaches a cluster of \( k \) electrons located at \( z_1 = \ldots z_k \neq z_j \)? By the clustering condition we know the wavefunction will tend to zero. But in what way does it approach this value?

To answer this question we will first ‘construct’ a cluster of \( k \) electrons. Let us fuse the electrons, with coordinates \( z_1, \ldots z_k \), together one by one. Starting with the fusion of the first two, which is actually a calculation we performed earlier in determining the electron operator characterized as \( k \) non-zero. A cluster of \( k \) electrons, with coordinates \( z \), approaches this value?

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We have

\[
\psi_{n-1} e^{i(n-1)\sqrt{\frac{\varphi}{\pi}}(z_{n-1})} \psi_1 e^{i\sqrt{\frac{\varphi}{\pi}}(z_1)} \sim (z_1 - z_2)^{-\frac{2(n-1)}{k}} + (z_2 - z_3)^{-\frac{2(n-1)}{k}} + \cdots + \left. \frac{2(n-1)}{k} \right| \psi_{n} e^{i n \sqrt{\frac{\varphi}{\pi}}(z_n)} \sim \psi_{n} e^{i n \sqrt{\frac{\varphi}{\pi}}(z_n)} \quad (6.49)
\]

Again, the zero of the parafermionic part is canceled by the bosonic part. Via induction we see that the leading term remains constant. This can be repeated \( k - 1 \) times. However, a \( k \)’th electron will fuse differently, owing to the fact that the OPE is different (6.4).

We have

\[
\psi_{k-1} e^{i(k-1)\sqrt{\frac{\varphi}{\pi}}(z_{k-1})} \psi_1 e^{i\sqrt{\frac{\varphi}{\pi}}(z_k)} \sim 1 e^{i k \sqrt{\frac{\varphi}{\pi}}(z_k)} \quad (6.51)
\]

The leading term is still constant, but the parafermions have fused to the identity operator. This is a crucial part of the derivation. A cluster of \( k \) electrons is a lump of charge described by a \( u(1) \) operator with the same amount of charge. The cluster has a trivial parafermionic part.
If we now take any other electron and bring it to the cluster of \( k \) electrons the wavefunction will always vanish. Up till now the fusion of the parafermionic part canceled the contribution of the vertex operators. But of the parafermionic part is trivial, so it does not create a singular part. If we bring \( z_j \) to the cluster (with \( k < j \leq N_e \)) we have

\[
e^{i k \sqrt{2} \varphi(z_k)} \psi_1 e^{i \sqrt{2} \varphi(z_j)} \sim (z_k - z_j)^2 \psi_1 e^{i (k+1) \sqrt{2} \varphi(z_k)} \quad (6.52)
\]

So to come back to our original question, the wavefunction vanishes according to \((z_k - z_j)^2\). There is a very neat way we can write this factorization of the ground-state wavefunction. Setting \( z_1 = z_2 = \ldots = z_k \) we have for our bosonic wavefunction [Cappelli et al., 2001]

\[
\Psi_{\text{bose}}^{N_e}(z_k, z_k, \ldots, z_k, z_{k+1}, \ldots z_{N_e}) \propto \Psi_{\text{bose}}^{N_e-k}(z_{k+1}, z_{k+2}, \ldots z_{N_e}) \prod_{j=k+1}^{N_e} (z_k - z_j)^2 \quad (6.53)
\]

This implies that from the wavefunction containing \( N_e \) electrons we obtain the wavefunction for \((N_e - k)\) electrons by considering the limit used above. Equation (6.53) can be seen as a characterization of the Read-Rezayi states. This equation tells us that the first zero of the polynomial pops up when any \( k + 1 \) electrons are brought together.

On the basis of this discussion we can write down a Hamiltonian, for which the bosonic wavefunction is the zero energy ground state. The Hamiltonian describes an idealized interaction between the electrons and is given by

\[
H = V \sum_{i_1 < i_2 \ldots < i_{k+1}} \delta^2(z_{i_1} - z_{i_2}) \delta^2(z_{i_2} - z_{i_3}) \cdots \delta^2(z_{i_k} - z_{i_{k+1}}) \quad (6.54)
\]

The bosonic wave-function is an exact ground state wavefunction of this Hamiltonian [Read and Rezayi, 1999]. This Hamiltonian embodies a \( k + 1 \)-point interaction among the electrons.

This is what makes the Read-Rezayi series so interesting. There are many CFT’s around, and the possibility to construct quantum Hall fluids is quite large. But that does not imply that these states correspond to ground states of simple Hamiltonians. The Read-Rezayi series does have this property, and the associated Hamiltonians even have a natural interpretation. They are essentially higher order correlations among the electrons. The Hamiltonian itself is fictitious; it cannot describe a physical sample. Rather, what we assume is that is encaptures essential properties of the universality class of the corresponding physical quantum Hall state. Quantum numbers of the quasiholes, such as charge and braiding properties, are expected to remain valid for the physical states as well. As long as we do not use the Hamiltonian to make statements about the short-range physics, the approximation assumed by applying CFT remains valid.

### 6.4 Exact electron wavefunctions

So far we have identified the electron and quasihole operators for a class of quantum Hall states. On the basis of this operator we have constructed an expression which, in principle, describes a family of wavefunctions. The downside is that this expression contains a parafermionic correlator which in general is very hard to solve to nearly impossible. Still, using techniques from conformal field theory we have been able to extract various aspects such as the corresponding Hamiltonian and its clustering property.
Luckily, explicit expressions of the quantum Hall wavefunctions in the absence of quasi-holes do exist. They are not constructed through direct computation of the correlator but rather via inspired guesswork [Read and Rezayi, 1999] and proven to be exact afterwards [Gurarie and Rezayi, 2000; Read and Rezayi, 1999].

Following the reasoning of [Gurarie and Rezayi, 2000; Read and Rezayi, 1999] we partition the electrons up into \( \frac{N_e}{k} \) clusters of \( k \) electrons. For this to work we need \( N_e \) to be a multiple of \( k \), say \( N_e = ck \). A way of dividing up the electrons is then for instance,

\[
(z_1, \ldots, z_k), (z_{k+1}, \ldots, z_{2k}), \ldots, (z_{(c-1)k+1}, \ldots, z_{ck})
\] (6.55)

This way the \( a \)’th cluster consists out of \((z_{(a-1)k+1}, \ldots, z_{ak})\). Next, we take two distinct clusters and define the following associated factor

\[
\chi_{a,b} = \left[ \frac{(z_{(a-1)k+1} - z_{(b-1)k+1}) (z_{(a-1)k+1} - z_{(b-1)k+2}) \cdots (z_{ak} - z_{bk}) (z_{ak} - z_{(b-1)k+1})}{(z_{ak} - z_{ak+2}) (z_{ak+2} - z_{(b-1)k+3}) \cdots (z_{ak} - z_{bk}) (z_{ak} - z_{bk+1})} \right]
\] (6.56)

The wavefunction for the Read-Rezayi states can be obtained by taking the product of all possible factors of \( \chi_{a,b} \) and sum over all possible ways of dividing the \( N_e \) electrons over \( c \) clusters of \( k \) electrons. The summation over all electrons is a step taken to ensure the wavefunction is symmetric in all its coordinates. We can replace this step by applying the symmetrizer \( S_{N_e} \) to the product of factors of \( \chi \). The wavefunction is

\[
\Psi_{R.R.}(z_1, \ldots, z_{N_e}) = S_{N_e} \left[ \prod_{a<b} \chi_{a,b} \right]
\] (6.57)

To proof the equivalence of this expression with equation (6.28) for \( M = 0 \) was done in [Read and Rezayi, 1999] by showing that (6.57) satisfies the same factorization property as in (6.53). This includes the fact that there no sub-leading terms of the type \((z_{k-1})\), for \( z_1 = z_2 = \ldots = z_{k-1} \).

A second, equivalent expression also exist, as was shown and proven in [Cappelli et al., 2001]. It uses the concept of coloring. We divide the \( N_e \) electrons into \( k \) groups (colors) of \( c \) electrons each. The electrons carrying the \( u \)’th color are denoted as

\[
(z^{(u)}_1, z^{(u)}_2, \ldots, z^{(u)}_c)
\] (6.58)

To each color \((u)\) we assign a Laughlin wavefunction

\[
\prod_{i<j} (z^{(u)}_i - z^{(u)}_j)^2
\] (6.59)

The wavefunction is constructed by multiplying all these Laughlin factors and symmetrize over all possible ways of dividing the electrons into groups of \( c \) electrons each.

\[
\Psi_{R.R.}(z_1, \ldots, z_{N_e}) = S \left[ \prod_{a=1}^k \prod_{i<j} (z^{(u)}_i - z^{(u)}_j)^2 \right], \quad N_e = ck
\] (6.60)

It is important to keep in mind that we do not mix any of the colors. That, is there are no factors of \((z^{(u)}_i - z^{(u)}_j)\) appearing within the symmetrizer. Proving that this expression equals
the more abstract form (6.28) is largely based on showing that both expressions satisfy the clustering condition (6.53).

Unfortunately, no general expression for the wavefunction in the presence of quasiholes exist. There are special cases where they are obtainable, and the difficulty in obtaining them demonstrates their complex structure.

6.5 Example: $Z_2$ - Moore-Read states

The simplest example of the $Z_k$ Read Rezayi states is the case where $k = 2$. The parafermionic sector is described by the $Z_2$ conformal field theory, more frequently known as the Ising model. In this theory we are dealing with one type of parafermion, better-known as a Majorana (massless) fermion [Di Francesco et al., 1995]. This sequence of states have a filling fraction of

$$\nu = \frac{1}{M + 1}, \quad M \in \mathbb{Z}_{\geq 0}$$

and it is in fact one of the first non-Abelian fractional quantum Hall state which is constructed on the basis of a CFT. It appeared in the seminal paper by Moore and Read [Moore and Read, 1991], see also [Read and Moore, 1992]. It is in this paper where the authors first emphasized the relationship between the underlying topological effective field theory and a description of electron wavefunctions in terms of conformal blocks.

Although the authors discussed the $Z_2$ state as a mere demonstration of the techniques at hand, it later became clear that this state might very well be a good candidate for the $\nu = \frac{5}{2}$ state found in particular samples. This was later on supported, largely on the basis of numerical studies. Furthermore, recent measurements of the elementary quasihole’s charge, through so-called shot-noise experiments, certainly do not rule out the Moore-Read state as a prime candidate. However, in recent years more states were developed and proposed, all for the specific $\nu = \frac{5}{2}$ state. They are all in agreement with numerical and experimental results. This makes the nature of the $\frac{5}{2}$-state still indecisive.

Even if the Moore-Read state turns out not to be the proper $\nu = \frac{5}{2}$ description, it still serves as the best pedagogical example of a non-Abelian quantum Hall state in which the conformal field theory is an adequate theoretical probing tool. This is because the underlying conformal field theory is the Ising model, which has been a well-known CFT for quite some time. It is is an example of a theory which has been completely solved. In the next few paragraphs we will give a demonstration of the CFT techniques available to us and in what way we can make proper use of them to extract all sorts of information. In particular, we will, using CFT, derive an exact expression for a wavefunction containing two quasiholes. This in turn allows us to determine the braiding matrices and demonstrate the principle of non-Abelian braiding from a CFT point of view.

6.5.1 The Ising model and electron wavefunctions

As stated before, the $Z_2$ parafermions reduce to that of ordinary Majorana fermions. This model contains 3 primary fields: a vacuum operator $1$, a Majorana fermion $\psi$ and a spin field $\sigma$. The fusion rules and conformal dimensions are listed in table (6.5.1). To make the
operator algebra more explicit we also mention all relevant operator product expansions
\[
\psi(z)\psi(w) \sim \frac{1}{z-w} + \ldots
\]  
\[\psi(z)\sigma(w) \sim C^\sigma_{\psi\sigma} \frac{1}{(z-w)^{\frac{3}{8}}} \sigma(z) + \ldots \]  
\[
\sigma(z)\sigma(w) \sim C^1_{\sigma\sigma} \frac{1}{(z-w)^{\frac{1}{8}}} + C^\psi_{\sigma\sigma} (z-w)^{\frac{3}{8}} \psi(z) + \ldots
\]

The constants \(C\) are the structure constants of this algebra and the Majorana fields have been normalized such that \(C^\psi_{\psi\psi} = 1\). Let us stress that the fusion product of two sigma fields has two fusion channels. These multiple fusion channels form the very core of non-Abelian statistics from a conformal field theory point of view. We will treat this in the next subsection.

A second point of attention is the OPE of two Majorana fields. The leading order term has a simple pole with all higher order terms being regular. Furthermore, the resulting field is the identity operator. These properties imply that the Majorana fields are free fields and thus allows us to apply Wick’s theorem to any conformal block consisting out of an even number of Majorana fields (any odd number would always give zero due to the \(\mathbb{Z}_2\) structure). Let us demonstrate this by direct computation of the electron wavefunction.

The electron operator is
\[
V_{\text{el}} = \psi e^{i\sqrt{M+1}\varphi}, \quad \Delta_{\text{el}} = \frac{M+1}{2}
\]  

Two topological quantum numbers are the filling fraction and the shift
\[
\nu = \frac{1}{M+1} \quad (6.66)
\]
\[
S = M + 2 \quad (6.67)
\]

The expression for our electron wavefunction containing an even number of electrons \((N_e)\) is
\[
\Psi_{\text{M.R.}}(z_1, \ldots, z_{N_e}) = \langle \psi(z_1) \cdots \psi(z_{N_e}) \rangle_{\mathbb{Z}_2} \lim_{x \to \infty} x^M e^{(M+1)N_e^2} (e^{i\sqrt{M+1}\varphi(z_1)} \cdots e^{i\sqrt{M+1}\varphi(z_{N_e})}) e^{-i\sqrt{M+1}N_e\varphi(x)} u(1) \]

The computation of the bosonic correlator is straightforward. The \(\mathbb{Z}_2\) correlator can be evaluated using Wick’s theorem [Di Francesco et al., 1995]. Keep in mind that the operators are fermions, so we have to keep track of the order of contractions.

\[
\langle \psi(z_1) \cdots \psi(z_{N_e}) \rangle_{\mathbb{Z}_2} = \text{Pfaff} \left( \frac{1}{z_i - z_j} \right)
\]  

where
\[
\text{Pfaff}(M_{i,j}) = \frac{1}{2^N (N_e)!} \sum_{\pi \in S_{N_e}} \text{sgn}(\pi) \prod_{i=1}^{N_e} M_{\pi(1), \pi(2)}
\]  

\[\text{Table 6.1:}\] The \(\mathbb{Z}_2\) Ising model - conformal dimensions and fusion rules

| \(\psi\) | \(\Delta\) | \(\times\) | \(\psi\) | \(\sigma\)
| \(\sigma\) | \(\frac{1}{2}\) | | \(1\) |  
| \(\frac{1}{16}\) | | | |  

---

99
and $M$ is an $N_e \times N_e$ matrix. The sum is taken over all possible permutations. This leads to an overcounting of all possible, unique ways of contracting the fermions. The normalization factor in front is due to this overcounting. The second expression on the right-hand side is known as the Pfaffian, which is equal to the anti-symmetrized product of an anti-symmetric matrix $M$, i.e. the square of its determinant

$$\text{Pfaff}(M_{i,j}) = A(M_{1,2}M_{3,4} \ldots) = \sqrt{\det(M)} \quad (6.71)$$

This leads to the following expression for the Moore-Read wavefunction

$$\Psi_{\text{MR}}(z_1, \ldots, z_{N_e}) = \text{Pfaff}(\frac{1}{z_i - z_j}) \prod_{i<j} (z_i - z_j)^{M+1} \quad (6.72)$$

which is also known as the Pfaffian or BCS wavefunction [Moore and Read, 1991]. This is equivalent to the expressions found in the previous section.

For $M = 0$ this wavefunction is an exact ground state of the following three body Hamiltonian [Greiter et al., 1991]

$$H = V \sum_{i<j<k} \delta^2(z_i - z_j)\delta^2(z_i - z_k) \quad (6.73)$$

6.5.2 Non-Abelian quasiholes

In the Moore-Read wavefunction the smallest charge quasihole operator is found to be

$$V_{\text{qh}} = \sigma e^{i \frac{1}{2} \sum_{i<j} \sqrt{M+1} \phi}, \quad \Delta_{\text{qh}} = \frac{M+3}{16(M+1)}, \quad q_{\text{qh}} = \frac{1}{2(M+1)} \quad (6.74)$$

Using this expression the wavefunction in the presence of quasiholes derived in equation (6.43) becomes

$$\Psi_{\text{MR}}(z_1, \ldots, z_{N_e}, w_1, \ldots, w_n) = \langle \sigma(w_1) \cdots \sigma(w_n)\psi(z_1) \cdots \psi(z_{N_e}) \rangle \times \prod_{i<j} (z_i - z_j)^{M+1} \prod_{i<j} (w_i - w_j)^{\frac{1}{2}} \prod_{i}^{N_e} n_{i} \prod_{i}^{n} (z_i - w_j)^{\frac{1}{2}} \quad (6.75)$$

Unfortunately, the spin fields are interacting fields as can be seen from their OPE. This means we cannot apply Wick’s theorem directly to obtain an expression for the quasiholes. Still, there are other methods available which rely on the highly rigid structure of CFT.

Let us begin with the correlator containing an even number of electrons and two quasiholes. The correlator which we have to solve is

$$\langle \psi(z_1) \cdots \psi(z_{N_e})\sigma(w_1)\sigma(w_2) \rangle \quad (6.76)$$

with an even number of Majorana fermions. Based on the fusion rules we see that there is only one way in which this correlator fuses to the identity. That is $\sigma \times \sigma \Rightarrow 1$.

The calculation of this correlator is done in two steps. First, we reduce the number of $\psi$ fields such that we are left with a 4-point correlator. The motivation for this step is the analytical structure of the correlator. For this correlator we are able to identify all branch
6.5 Example: $\mathbb{Z}_2$ - Moore-Read states

cuts, singularities and limits. By matching these with the reduced correlator we essentially prove their equivalence. The second step is to solve the remaining 4-point correlator.

On the basis of the operator algebra we can determine the limits of expression (6.76). Those limits are

$$
\lim_{z_1 \to z_2} (z_1 - z_2) \langle \psi(z_1) \psi(z_2) \psi(z_3) \cdots \psi(z_N) \sigma(w_1) \sigma(w_2) \rangle = \langle \psi(z_3) \cdots \psi(z_N) \sigma(w_1) \sigma(w_2) \rangle
$$

(6.77)

$$
\lim_{z_1 \to z_2} (z_1 - z_2) \frac{\langle \psi(z_1) \psi(z_2) \sigma(w_1) \sigma(w_2) \rangle}{\langle \sigma(w_1) \sigma(w_2) \rangle} = 1
$$

(6.78)

Furthermore, the Majorana fields are antisymmetric with respect to each other, so interchanging once or twice gives a factor of $-1$ and $1$ respectively. They also pick up a phase of $-1$ if we braid them around a spin field. The last property we need is that if we take a $\psi(z)$ $(\sigma(w))$ field and send it to infinity the correlator (6.76) behaves as $z^{-2\Delta}$, $w^{-2\Delta}$.

The conjecture is that the following identity holds, on the basis that the left- and right-hand side obey the same limits and braiding rules.

$$
\langle \psi(z_1) \cdots \psi(z_N) \sigma(w_1) \sigma(w_2) \rangle = \langle \sigma(w_1) \sigma(w_2) \rangle \text{Pfaff} \left( \frac{\langle \psi(z_1) \psi(z_2) \sigma(w_1) \sigma(w_2) \rangle}{\langle \sigma(w_1) \sigma(w_2) \rangle} \right)
$$

(6.79)

This expression appears for instance in [Chung and Stone, 2007].

**Proof** First of all, the Pfaffian is an antisymmetric function therefore the antisymmetry of the fermions is automatically satisfied. Second, the braiding behavior of the spin fields is completely determined by the two-point correlator in front, since all phases picked up by the denominator and numerator inside the Pfaffian cancel each other out. Furthermore, braiding of a fermionic field with a spin field is also the same, on the basis of the second identity in equation (6.71). That same identity also acknowledges the corresponding behavior for $z_1 \to \infty$, given the fact that the expression inside the Pfaffian is a CFT correlator.

What remains to be proven are the identities (6.77) and (6.78). The proof is via induction. For an antisymmetric $(2 \times 2)$ matrix $A$ the expression Pfaff$(A)$ is equal to the matrixcomponent $A_{1,2}$. For two electrons expression (6.79) therefore reduces to (6.76) and the limit in equation (6.78) is satisfied. This proofs the “base case”.

For the inductive step we assume that expression (6.79) satisfies limit (6.77) for $N_e - 2$ electrons, and from there proof that it holds also for $N_e$ electrons.

Given a $(2n \times 2n)$ antisymmetric matrix $A$ it can be shown the Pfaffian satisfies the following recursion relation

$$
\text{Pfaff}(A) = \sum_{m=2}^{2n} (-1)^m A_{1m} \text{Pfaff}(A_{[1\bar{m}]})
$$

(6.80)

where $A_{1m}$ is the $(1, m)$'th entry in $A$ while $A_{[1\bar{m}]}$ is an improvised way of denoting the $((2n - 2) \times (2n - 2))$ matrix obtained by removing the 1st and $m$'th columns and rows. Using this identity for the case of $N_e$ electrons and taking the limit of equation (6.77) we get

$$
\lim_{z_1 \to z_2} (z_1 - z_2) \langle \sigma(w_1) \sigma(w_2) \rangle \text{Pfaff} \left( \frac{\langle \psi(z_1) \psi(z_j) \sigma(w_1) \sigma(w_2) \rangle}{\langle \sigma(w_1) \sigma(w_2) \rangle} \right) = \langle \sigma(w_1) \sigma(w_2) \rangle \times
$$

(6.81)
Notice that $z_1$ is not present in any of the Pfaffians. The only non-zero contribution of this limit comes from the case where $m = 2$; all other terms vanish due to the factor $(z_1 - z_2)$. What remains is exactly the Pfaffian of $N_e - 2$ electrons, thus completing the inductive step.

In effect, we have shown that the conjecture holds on the basis that both expressions satisfy the same limits.

In effect we have reduced the correlator to a 4-point function. To solve this equation we again use the operator algebra to determine the limits (poles) of this expression. In particular, we have

\[
\lim_{z_1 \to z_2} \frac{\langle \psi(z_1)\psi(z_2)\sigma(w_1)\sigma(w_2) \rangle}{\langle \sigma(w_1)\sigma(w_2) \rangle} \propto \frac{1}{z_1 - z_2} \quad (6.82)
\]

\[
\lim_{w_1 \to w_2} \frac{\langle \psi(z_1)\psi(z_2)\sigma(w_1)\sigma(w_2) \rangle}{\langle \sigma(w_1)\sigma(w_2) \rangle} \propto \frac{1}{(z_1 - w_1)^\frac{1}{2} \langle \sigma(w_1)\sigma(w_2) \rangle} \quad (6.83)
\]

\[
\lim_{z_1 \to w_1} \frac{\langle \psi(z_1)\psi(z_2)\sigma(w_1)\sigma(w_2) \rangle}{\langle \sigma(w_1)\sigma(w_2) \rangle} \propto \langle \psi(z_1)\psi(z_2) \rangle = \frac{1}{z_1 - z_2} \quad (6.84)
\]

In [Di Francesco et al., 1995] this correlator is solved and gives

\[
\frac{\langle \psi(z_1)\psi(z_2)\sigma(w_1)\sigma(w_2) \rangle}{\langle \sigma(w_1)\sigma(w_2) \rangle} = \frac{1}{2(z_1 - z_2)} \left[ \frac{(z_1 - w_1)(z_2 - w_2)}{(z_1 - w_2)(z_2 - w_1)} + \frac{(z_1 - w_2)(z_2 - w_1)}{(z_1 - w_1)(z_2 - w_2)} \right] \quad (6.85)
\]

It is not hard to prove that, again, both expression agree on all limits. Therefore on the basis of analyticity, this equality holds. We leave this check as an exercise for the enthusiastic reader.

What remains is combining all results, that is plugging (6.85) into (6.79), and use the resulting expression for equation (6.75). This gives

\[
\Psi_{\text{MR}}(z_1, \ldots, z_{N_e}, w_1, w_2) = (w_1 - w_2)^{N_e-1}(w_1 - w_2)^{-\frac{1}{2}} \prod_{i<j} (z_i - z_j)^{M+1} \times \text{Pfaff} \left( \frac{1}{2(z_i - z_j)} \right) \left[ \frac{(z_i - w_1)(z_j - w_2)}{(z_i - w_2)(z_j - w_1)} + \frac{(z_i - w_2)(z_j - w_1)}{(z_i - w_1)(z_j - w_2)} \right] \prod_{i}^{N_e} (z_i - w_1)^{\frac{1}{2}} (z_i - w_2)^{\frac{1}{2}} \quad (6.86)
\]

The second line can be rewritten quite nicely. For that we use the general expression for the Pfaffian (6.70). Suppressing the normalization and summation over permutations gives

\[
\prod_{i=1}^{N_e} \left[ \frac{(z_{\pi(2i-1)} - w_1)(z_{\pi(2i-1)} - w_2)}{2(z_{\pi(2i-1)} - z_{\pi(i)})} \times \left[ \frac{(z_{\pi(2i-1)} - w_1)(z_{\pi(i)} - w_2)}{(z_{\pi(2i-1)} - w_2)(z_{\pi(i)} - w_1)} + \frac{(z_{\pi(2i-1)} - w_2)(z_{\pi(i)} - w_1)}{(z_{\pi(2i-1)} - w_1)(z_{\pi(i)} - w_2)} \right] \right] \quad (6.87)
\]
Note that the final product in (6.86) has been rewritten to make the cancellation of branch cuts and poles explicit. We see that all factors of \((z_i - w_j)^{1/2}\), i.e. branch cuts with respect to the electron coordinates, cancel each other out. When the smoke clears we are left with a regular expression in terms of the electrons coordinates, that is no poles and branch cuts and of finite degree.

\[
\Psi_{\text{M.R.}}(z_1, \ldots, z_{N_e}, w_1, w_2) = (w_1 - w_2)^{\frac{1}{4M+4} - \frac{1}{8}} \text{Pfaff}\left( \frac{(z_i - w_1)(z_j - w_2) + (z_i - w_2)(z_j - w_1)}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)^{M+1} \tag{6.88}
\]

This expression first appeared in the paper of Moore and Read [Moore and Read, 1991]. A quite remarkable result is that for the fermionic state \(M = 1\) the \(z_i\)-independent prefactor, \((w_1 - w_2)^{\frac{1}{4M+4} - \frac{1}{8}}\), vanishes. We can apply the conjecture that braiding of two quasiholes in the CFT language is equivalent to analytic continuation of the coordinates \(w_i\). Using this conjecture we conclude that the double interchange of the two spin fields does not induce a phase factor. This of course does not mean that they are bosonic or fermionic. On the contrary, we will demonstrate that in the presence of other quasiholes the chiral correlator turns into an element of a higher-dimensional vector space.

In [Greiter et al., 1991] an argumentation is presented which obtains expression (6.88) on the basis of “flux insertion”, similar to that presented in chapter 3.8. However, the reasoning is more subtle and in fact is not based on flux insertion at all. Rather, an adiabatic action is performed which relabels the momentum eigenstates - something which also occurs through flux insertion in the Abelian case. We refer to the paper for details.

### 6.5.3 Four point correlator

We can go further than the two-point correlator and try to write down a general expression for a correlator containing four spin fields. The four-point correlator which we need to solve is then

\[
\Psi_{\text{MR}}(z_1, \ldots, z_{N_e}, w_1, w_2, w_3, w_4) = \langle \sigma(w_1)\sigma(w_2)\sigma(w_3)\sigma(w_n)\psi(z_1)\cdots\psi(z_{N_e}) \rangle \tag{6.89}
\]

What makes this correlator interesting is the presence of multiple fusion channels that lead to the identity operator. These multiple fusion channels result in two conformal blocks, which form the basis of a two-dimensional vector space. A nice pictorial way of visualizing this argumentation is the through a “Bratelli diagram”.

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This figure can also be used to count the dimension of a correlator containing \( N_q = 2m \) quasiholes. Each step to the right corresponds to an extra quasihole present in the correlator. The first quasihole fuses with the identity, therefore its fusion is unique. The next quasihole has two choices, it can fuse into \( \psi \) or into \( 1 \). In both cases, the next quasihole will fuse to the field \( \sigma \). And so on. This implies that all “even-numbered” quasiholes have two fusion channels, while all “odd-numbered” have only one. This is true, except for the final quasihole. This “even-numbered” quasihole must fuse to the identity, which is a unique fusion channel. All in all, the dimensionality of the space spanned by the conformal blocks is

\[
\dim V = 2^{n-1} \quad (6.90)
\]

Determining the exact expression for the wavefunction in the presence of quasihole turns into a difficult task very quickly. A very sensible approach is to use a technique called bosonization. In this approach the CFT is “mapped” to one comprised out of free bosons. In the case of the Majorana fermion we can take two copies of the CFT to form a chiral boson. This turns all operators into bosonic expressions including the spin fields. The expressions can be difficult, but if manageable one can apply Wick’s theorem to solve the correlators. This program is carried for the wavefunctions presented here in the paper [Nayak and Wilczek, 1996]. There, the authors compute, among other things, a number of correlators, including the two- and four-point function.

We will not carry out this program. Instead, we will only calculate the four-point correlation function of the quasiholes in the absence of electrons

\[
\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle \quad (6.91)
\]

This expression by itself is not a legitimate quantum Hall state. The spin fields are essentially topologically defects in the quantum Hall fluid, therefore electrons are needed to turn them into physical objects.

Nevertheless, is is a useful demonstration the determine the conformal blocks. This is the only chapter in which the explicit form of the conformal blocks is discussed, so for the sake of argument we will use this ‘bare’ expression. There is some relevance to the equivalent correlator in the presence of electrons, which we will come back to at the end of this chapter.

The calculation of the conformal block is quite lengthy, and moved to the appendix. To perform it we first need to introduce some new techniques of CFT. See appendix C for an extensive review on this matter, and also [Nayak et al., 2008] for a related discussion. The final result is

\[
\lim_{w \to \infty} \langle \sigma(0)\sigma(z)\sigma(1)\sigma(w) \rangle = A_+ F_+(z) + A_- F_-(z) \quad (6.92)
\]

\[
F_+ = \lim_{w \to \infty} (wz(1-z))^{-\frac{1}{2}} \sqrt{\frac{1 + \sqrt{1-z}}{2}} \quad (6.93)
\]

\[
F_- = \lim_{w \to \infty} (wz(1-z))^{-\frac{1}{2}} \sqrt{\frac{1 - \sqrt{1-z}}{2}} \quad (6.94)
\]

The \( F \) functions are the conformal blocks we have so frequently been mentioning. We can identify them with the different fusion channels in the following matter. By letting \( z \to 0 \) we
fuse two of the spin fields $\sigma(0) \times \sigma(z)$. Its effect on the blocks is

$$\lim_{z \to 0} (z(1-z))^{-\frac{1}{8}} \sqrt{\frac{1 \pm \sqrt{1 - z}}{2}} = \begin{cases} \lim_{z \to 0} z^{-\frac{1}{8}} & \text{for } F_+ \\ \lim_{z \to 0} z^{\frac{3}{8}} & \text{for } F_- \end{cases} \quad (6.95)$$

If we now take a look at the OPE of two spin fields again,

$$\sigma(z)\sigma(0) \sim 1 + z^\frac{3}{8} \psi(z) + \ldots \quad (6.96)$$

We conclude that $F_+$ corresponds to the fusion channel where the fields fuse to the identity, while $F_-$ corresponds to fusion in to the fermion $\psi$.

Continuing on the discussion of section 5.4 we demonstrate the concept of braiding. A specific braiding operation is accomplished by the monodromy operation, in which the field at $z$ winds around the spin field located at $z = 1$. This is accomplished through analytic continuation.

$$z \circlearrowleft 1 : (z(1-z))^{-\frac{1}{8}} \sqrt{\frac{1 \pm \sqrt{1 - z}}{2}} \rightarrow e^{\frac{2\pi i}{8}} (z(1-z))^{-\frac{1}{8}} \sqrt{\frac{1 \pm e^{\pi i} \sqrt{1 - z}}{2}} \quad (6.97)$$

The monodromy operation interchanges the two conformal blocks, and multiplies the wavefunction with a phase factor.

We view expression (6.92) as a two-dimensional vector in the space of the conformal blocks. The $F$ functions form a basis of this space, which is why we view the correlator as the vector $\begin{pmatrix} A_+ \\ A_- \end{pmatrix}$. In this picture the monodromy operation is seen as a unitary transformation of this vector space

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \xrightarrow{z \circlearrowleft 1} e^{\frac{2\pi i}{8}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} \quad (6.98)$$

The reason why this property of non-Abelian quantum Hall fluids is so interesting is the topological nature of this operation. The only way to perform a basis transformation is through the physical braiding of the quasiholes. In the effective field theory limit, small perturbations or thermal excitations will not induce these transitions which makes the vector space of conformal blocks ‘protected’ from local interactions. This is the very basic idea of topological quantum computation. See [Nayak et al., 2008].

Still, the correlator we have determined does not correspond to the expression obtained if one takes the electrons into account. Without proof, we mention that the true correlator, which includes the bosonic $c = 1$ part and where $M$ is set to 1, is given by [Nayak and Wilczek, 1996]

$$\langle \sigma(w_1)\sigma(w_2)\sigma(w_3)\sigma(w_4)\psi(z_1)\cdots\psi(z_N) \rangle \prod_{i<j} (z_i - z_j)^2 \prod_{i<j} (w_i - w_j)^\frac{1}{M} \prod_i \prod_j (z_i - w_j)^\frac{1}{2} = A_1 \Psi^{(1)} + A_\psi \Psi^{(\psi)} \quad (6.99)$$

The constants $A_1$ are coefficients, while the functions $\Psi^{(1,\psi)}$ constitute the conformal blocks with the labels denoting the corresponding fusion channels. The conformal blocks are given
by
\[
\Psi^{(1)} = \left( \frac{(w_1 - w_3)(w_2 - w_4)}{(1 + \sqrt{x})^{1/2}} \right)^{1/4} \left( \Psi^{(13)(24)} + \sqrt{x} \Psi^{(13)(23)} \right) 
\]

(6.100)
\[
\Psi^{(\psi)} = \left( \frac{(w_1 - w_3)(w_2 - w_4)}{(1 - \sqrt{x})^{1/2}} \right)^{1/4} \left( \Psi^{(13)(24)} - \sqrt{x} \Psi^{(14)(23)} \right) 
\]

(6.101)

where in turn
\[
\Psi^{(13)(24)} = \prod_{i<j} (z_i - z_j)^2 \text{Pfaff} \left( \frac{(z_i - w_1)(z_i - w_3)(z_j - w_2)(z_j - w_4) - (i \leftrightarrow j)}{z_i - z_j} \right) 
\]

(6.102)
\[
\Psi^{(14)(23)} = \prod_{i<j} (z_i - z_j)^2 \text{Pfaff} \left( \frac{(z_i - w_1)(z_i - w_4)(z_j - w_2)(z_j - w_3) - (i \leftrightarrow j)}{z_i - z_j} \right) 
\]

(6.103)

This time, the equivalent braiding operation of winding \( \sigma_2 \) around \( \sigma_3 \) results is accomplished by replacing \( \sqrt{x} \) by \( -\sqrt{x} \) in all of the expressions above. It corresponds to the following unitary transformation
\[
\begin{pmatrix} A_1 \\ A_{\psi} \end{pmatrix} \xrightarrow{\sigma_2 \circ \sigma_3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_{\psi} \end{pmatrix} 
\]

(6.104)

The presence of the electrons and the \( u(1) \) part of the theory have, in fact, canceled the overall, Abelian phase factor which appears in (6.98). The non-Abelian part, however, is unaffected and the monodromy operation still induces a 90 degrees rotation by interchanging the two conformal blocks.

This result is best interpreted from a topological point of view. Namely, the electrons are physical particles meaning that their topological nature should be trivial. Braiding of electrons, either with each other or with a quasihole can only induce Abelian phase factors.

This means that the non-Abelian nature of the quasiholes is an intrinsic property of the excitations alone. Winding two quasiholes around each other will, in general, imply that the quasiholes also encircle a large number of electrons. But the braiding with respect to the quasiholes is trivial, and cannot induce any rotations within the vector space.

Put differently, the braiding of two quasihole causes a rotation in the space of conformal blocks. This space is completely accounted for by the spin fields in the \( Z_2 \) part of the CFT. The presence of the electrons, and the attachment of a chiral boson to all operators can only change the Abelian part of the theory. This is why the non-Abelian part of the Ising model and the \( Z_2 \) quantum Hall state coincide.

We end this chapter by stating that the true correlator is obtained through bosonization [Nayak and Wilczek, 1996]. For the remaining \( Z_k \) the bosonization procedure is a lot more complicated. In these cases it is more convienent to apply the CFT techniques, such as the ones used in the appendix. The full correlator of four quasiholes in the presence of electrons and the braiding matrices for all \( Z_k \) states have been obtained in [Ardonne and Schoutens, 2007]. In this paper the technique used is essentially the one depicted in the appendix. This is why in this thesis we have chosen for this approach in favor of bosonization: it also applies, in a more general form, to the higher clustered states.
In two papers by Wen and Wang [Wen and Wang, 2008a,b] the authors introduced a classification procedure of possible clustered quantum Hall states on the basis of zero points for symmetric polynomials. For that they generalized the notion of clustering as it appears in the Read-Rezayi case and introduced a set of restrictions. Symmetric polynomials which obey these definitions define a clustered quantum Hall state. Then, through use of consistency relations the author numerically obtained a whole new range of clustered quantum Hall state. The numerical results reached up to a “clustering of order 10”. We will come back to this treatment in the next chapter.

Among the numerical results the authors identified the Laughlin states and the Read-Rezayi states. They also found new states which they related to generalizations of the Read-Rezayi states. The first generalization is called the $\mathbb{Z}_k^{(p)}$ quantum Hall States or generalized parafermionic states. The second are the $\bigotimes_i \mathbb{Z}_k^{(p_i)}$ quantum Hall states or composite parafermionic states. In this chapter we will present a CFT description of both these states. We identify the electron and quasihole operators and give some some examples.

In the summer of 2008 a paper appeared by [Barkeshli and Wen, 2008] in which the CFT description of these generalized quantum Hall states appeared. This paper matches all results presented in this chapter. Do note that this chapter was completed before this paper came out.

### 7.1 $\mathbb{Z}_k^{(p)}$ quantum Hall States

The first generalization we treat is that of the $\mathbb{Z}_k^{(p)}$ quantum Hall states. We will use the parafermionic CFT presented in chapter 6.1.
7. Generalized clustered quantum Hall states

7.1.1 Electron Operator and filling fraction

We now propose a way to generalize the results obtained from the Read Rezayi states. The basic idea is quite simple. In the case of Read-Rezayi the electron operator was identified as the vertex operator \( \psi_1 e^{i \frac{1}{\nu} \phi} \). Our proposal is to replace \( \psi_1 \) by another parafermion present in the spectrum, and modify the \( u(1) \) part accordingly. To accomplish this we start off with the ansatz \( \psi_p e^{i \alpha \phi} \) and simply repeat all steps taken in the previous chapter, yielding similar results. The operator product expansion of two \( \psi_p e^{i \alpha \phi} (p < k/2) \) operators is

\[
\psi_p e^{i \alpha \phi}(z) \times \psi_p e^{i \alpha \phi}(w) \sim (z - w)^\alpha \frac{2p^2}{k} \psi_{2p} e^{2i \alpha \phi}(w) \tag{7.1}
\]

The electron operator is a local, non-singular one. Therefore we set \( \alpha \frac{2p^2}{k} = M \in \mathbb{Z}_{\geq 0} \) in order to get rid of any unwanted poles and branch cuts. The electron operator is

\[
V_{el} = \psi_p e^{i \frac{kM + 2p^2}{k} \phi}, \quad M \in \mathbb{Z}_{\geq 0} \tag{7.2}
\]

This operator has conformal dimension (and spin) \( \frac{M + 2p^2}{2} \), so it is fermionic for \( M = \text{odd} \) and bosonic for \( M = \text{even} \). The \( u(1) \) current operator is normalized such that the electron operator has charge 1, i.e.

\[
Q_e = \frac{1}{\sqrt{kM + 2p^2}} \int \frac{dw}{2\pi} J_\phi(w) \tag{7.3}
\]

The FQHE wavefunction follows from the general expression (5.67)

\[
\Psi(z_1, \ldots, z_{N_e}) = \langle \psi_p(z_1) \cdots \psi_p(z_{N_e}) \rangle \prod_{i<j} (z_i - z_j)^{M + 2p^2/k} \tag{7.4}
\]

We will refer to the states as the \( \mathbb{Z}_{k}^{(p)} \) parafermion states.

The filling fraction can be obtained by considering the limit of taking one electron and sending it to infinity. On the one hand we know that the power of the leading term equals the number of states available, i.e. this limit behaves as \( z^{N_e} \). On the other hand, within the parafermionic correlator the remaining parafermions fuse together to form a single operator, \( \psi_{k-p} \). Therefore the parafermionic correlator will behave as \( z^{-2\Delta_p} \). The LJ-factor behaves as \( (N_e - 1)(M + \frac{2p^2}{k}) \). By matching the powers we obtain the following expressions for the filling fraction and the shift

\[
N_\phi = \nu^{-1} N_e - S \tag{7.5}
\]

\[
\nu_{z_k}^{(p)} M = \frac{k}{kM + 2p^2} \tag{7.6}
\]

\[
S = M + 2p \tag{7.7}
\]

This calculation was done more explicit for the Read-Rezayi states in chapter 6.2.1. The filling fraction and electron operator appear in [Barkeshli and Wen, 2008].

In the derivation given above we chose \( p \) to be smaller than \( k/2 \). Had we chosen \( p \geq k/2 \) then the fusion product in 7.1 would have been different. A quick calculation shows that for
the case of \( p = \frac{k}{2} \), for \( k = \) even, our results still apply. When the appropriate fusion rule is employed we have
\[
\psi_{\frac{k}{2}} e^{i\alpha\phi}(z) \psi_{\frac{k}{2}} e^{i\alpha\phi}(w) \sim (z - w)^{\Delta_{\psi}} e^{i2\alpha\phi}(w)
\]
(7.8)

From this it follows that we must set \( \alpha = \sqrt{M + \frac{k}{2}} \) in order to maintain locality. This indeed corresponds to the electron operator (7.2) by setting \( p = \frac{k}{2} \).

Should we then consider also the possibility of \( p > \frac{k}{2} \)? The answer is no. If we instead of \( \psi_p \) with \( p < \frac{k}{2} \) would choose \( \psi_{k-p} \) as our electron operator we would gain nothing new. The conformal dimensions of both operators are the same, since \( \Delta_{\psi_p} = \Delta_{\psi_{k-p}} \), and also the fusion rules map into each other when we interchange \( \psi_p \leftrightarrow \psi_{k-p} \). This implies that if we would use \( \psi_{k-p} \) as the operator representing an electron, its physical properties, such as conformal dimension and filling fraction, are already described by another operator present in the spectrum, namely \( \psi_p \). We can safely restrict ourself to \( p \leq k/2 \).

### 7.1.2 Quasiholes

The next step in our description is the identification of the quasihole operator. We start off as general as possible, by taking the spin field \( \sigma_l \) and an associated \( u(1) \) vertex part to represent our quasihole.
\[
V_{qh} = \sigma_l e^{i\beta\phi}
\]
(7.9)

Following the steps of chapter 6.2.2 we demand the physical condition of a local, non-singular electron operator. To determine the fusion of a quasihole with an electron we switch to the notation of chapter 6.1. This means we write \( \sigma_l = \Phi_{l}^{\phi} \) and \( \psi_p = \Phi_{2p}^{\phi} \). The fusion of the two gives
\[
V_{el}(z) \times V_{qh}(w) = \left( \Phi_{2p}^{0} \times \Phi_{l}^{k} \right) e^{i\sqrt{\frac{4kM+2p^2}{k}}} \times e^{i\beta\phi}
\]
(7.10)
\[
\sim (z - w)^{\delta h + \beta} e^{i\sqrt{\frac{4kM+2p^2}{k}}} \left( \Phi_{l+2p}^{0} e^{i(\beta + \sqrt{\frac{4kM+2p^2}{k}})\phi} \right)(w)
\]
(7.11)

where \( \delta h = \Delta_{\psi_p} + \Delta_{\sigma_l} - \Delta_{\Phi_{l+2p}^{k}} \). The corresponding conformal dimensions are treated in chapter 6.1, so we simply copy/paste them here.
\[
\Delta_{\Phi_{l+2p}^{k}} = \begin{cases} \frac{p(k-p)}{k} & p \leq k-l \\ \frac{l(k-l)}{2(k+2)} & p > k-l \end{cases}
\]
(7.12)
\[
\Delta_{\psi_p} = \frac{p(k-p)}{k}
\]
(7.13)
\[
\Delta_{\sigma_l} = \frac{l(k-l)}{2k(k+2)}
\]
(7.14)

The conformal dimensions of the spin fields and parafermions are given by a single formula. The conformal dimension of the resulting field is divided into two cases. This is a result of the field identification, treated in chapter 6.1. See also chapter 9 in which we will determine all the charges of all quasiholes.
As a corollary we have

\[ \delta h = -nl_p - \frac{lp}{k} \quad (7.15) \]

\[ n_p = \begin{cases} 0 & p \leq k - l \\ k - l - p & p > k - l \end{cases} \quad (7.16) \]

The integer \( n_p \) is non-negative. Plugging this expression back into (7.11) allows us to derive an expression for \( \beta \). Locality and finiteness of the wavefunction forces

\[ \beta = \frac{lp + \sqrt{k}n_p}{\sqrt{k} \sqrt{kM + 2p^2}} \]

Clearly, the smallest charge is obtained by setting \( l = 1 \). We have arrived at the expression for our quasihole operator

\[ V_{\text{qh}} = \sigma_1 e^{i \frac{\sqrt{k}p}{\sqrt{kM + 2p^2}}} \quad (7.17) \]

In order for the \( \mathbb{Z}_k \) correlator to be non-zero the total \( \mathbb{Z}_k \) charge must be 0 mod \( k \). This restricts the number of electrons \( N_e \) and number of quasiholes \( n \) to satisfy \( 2pN_e + n = 0 \) mod \( k \). Note also that by setting \( p = 1 \) we recover the results of the Read-Rezayi states, equation (6.43).

### 7.2 \( \bigotimes_i \mathbb{Z}_{k_i}^{(p_i)} \) quantum Hall States

We now move on to the second generalization. It is in fact a generalization of the \( \mathbb{Z}_k^{(p)} \) quantum Hall states, which, as is clear by now, includes the Read-Rezayi states. We will call these states \( \bigotimes_i \mathbb{Z}_{k_i}^{(p_i)} \) quantum Hall states or “product parafermion quantum Hall states”.

#### 7.2.1 Electron Operator and filling fraction

In the composite parafermion quantum Hall states the electron operator is represented by a product of multiple parafermionic fields along with a single \( u(1) \) vertex operator which takes care of the electrical current. The parafermionic fields may be taken from different \( \mathbb{Z}_k \) theories, i.e. correspond to different values of \( k \). Furthermore, \( p_i \) is always taken smaller than the corresponding \( k_i/2 \), for each \( i \). The electron operator becomes

\[ V_{\text{el}} = \left[ \prod_i \psi_{p_i} \right] e^{\frac{i}{\sqrt{p} \sqrt{k}}} \quad (7.19) \]
The electron operator is formed out of the direct product of the $\mathbb{Z}_{k_i}$ theories. The different $\mathbb{Z}_{k_i}$ decouple, meaning that the fusion of two electrons is as follows

$$\Psi_e(z_1) \times \Psi_e(w) = \prod_i \psi_{p_i}(z_1) \times \psi_{p_i}(w) \left( e^{i \frac{1}{2} \varphi(z_1)} \times e^{i \frac{1}{2} \varphi(w)} \right)$$

(7.20)

$$\sim (z - w)^{1/2 - \sum_i \frac{p_i^2}{k_i}} \prod_i \psi_{2p_i}(w) \left( e^{i \frac{2}{2} \varphi(w)} \right)$$

(7.21)

The filling fraction $\nu$ is derived on the basis that the power of the factor of $(z - w)$ is a non-negative integer $M$. This leads to

$$\nu = \frac{1}{M + 2 \sum_i \frac{p_i^2}{k_i}}$$

(7.22)

This expression along with definition (7.19) reveals the exact form of the electron operator. Its conformal dimension is the sum of the conformal dimensions of all its constituents.

$$\Delta_{el} = \frac{1}{2\nu} + \sum_i \Delta_{\{p_i,k_i\}}$$

$$= \frac{M + 2 \sum_1 p_i}{2}$$

(7.23)

This shows that the electron operator is bosonic for $M = \text{even}$ and fermionic for $M = \text{odd}$.

Wavefunctions are obtained through CFT correlators, one for each $\mathbb{Z}_{k_i}$ theory and one for the electrical current part. To maintain charge neutrality a background charge is inserted in the correlator of the electrical current. This leads to the following abstract expression

$$\Psi_{\bigotimes_{i} \mathbb{Z}_{k_i}^{(p_i)}}(z_1, \ldots, z_{N_e}) = \prod_i (\psi_{p_i}(z_1) \cdots \psi_{p_i}(z_{N_e})) \prod_{j < i} (z_i - z_j)^{M + \sum_i \frac{p_i^2}{k_i}}$$

(7.24)

These states are called product parafewon quantum Hall states and throughout this thesis they are denoted by $\bigotimes_{i} \mathbb{Z}_{k_i}^{(p_i)}$ for multiple values of $p$ and $k$, or simply $(\mathbb{Z}_{k}^{(p)})^\otimes n$ if we have $n$ copies of the same $\mathbb{Z}_{k}$ theory.

As an example we consider the “product Pfaffian state” which consists out of two copies of the $\mathbb{Z}_2$ Ising model, e.g. $\mathbb{Z}_2^{(1)} \otimes \mathbb{Z}_2^{(1)}$. In this case the electron operator and corresponding filling fraction are

$$V_{el} = \psi(z) \psi(z) e^{i \frac{1}{2} \varphi}$$

$$\nu = \frac{1}{M + 2}$$

(7.25)

and the resulting wavefunction consists out of two copies of the Pfaffian wavefunction

$$\Psi_{\mathbb{Z}_2^{(1,1)}} = \text{Pfaff} \left( \frac{1}{z_i - z_j} \right)^2 \prod_{j < i} (z_i - z_j)^{M + 2}$$

(7.26)

We have already studied the properties of the Pfaffian in some detail in chapter 6.5. One identity that did not come up was the square of the Pfaffian equals what is called the Haffnian, $\text{Pfaff}(A)^2 = \text{Haff}(A^2)$. Equation (7.26) can also be written as

$$\Psi_{\mathbb{Z}_2^{(1,1)}} = \text{Haff} \left( \frac{1}{z_i - z_j} \right)^2 \prod_{j < i} (z_i - z_j)^{M + 2}$$

(7.27)
where the Haffnian is defined in term of a \((2n \times 2n)\) symmetric matrix \(B\) as [Di Francesco et al., 1995]

\[
\text{Haff}(B_{ij}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \prod B_{\sigma(2i-1)\sigma(2i)}
\]

This wavefunction has been studied in [Green, 2002] and appears there as a \(d\)-wave pairing wavefunction for bosons at filling fraction \(\frac{1}{2}\). They also predict a quasihole charge of \(\frac{1}{4}\), which matches is also predicted by this CFT (see following section). However, in this reference this wavefunction was not related to any underlying CFT, as is shown here. In [Read, 2008] the author does provide a (different) CFT description of this state. The electron operators match in OPE and conformal dimension. But quite remarkably, a twist field is introduced through which the quasihole is constructed. This twist field does not appear in the CFT description of the \(\mathbb{Z}_{(1,1)}^2\). It is unclear what the nature is of this mismatch.

We can go one step further and derive a whole series of paired quantum Hall states. This is realized by taking \(n\) copies of the Majorana fermion \(\psi\) out of \(\mathbb{Z}_2\) and form the electron operator accordingly. The resulting quantum Hall state is the state \((\mathbb{Z}_2^{(1)})^\times_n\)

\[
V_{el} = (\psi)^n e^{i\frac{1}{\sqrt{\nu}}\phi}, \quad \nu = \frac{1}{M + n}, \quad S = M + 2n
\]

\[
\Psi_{(\mathbb{Z}_2^{(1)})^\times_n} = \text{Pfaff}\left(\frac{1}{z_i - z_j}\right)^n \prod_{j<i} (z_i - z_j)^{M+n}
\]

These are just an example of the many states that we can form using out of the generalized form. Wether such states correspond to physical systems remains to be seen. So far not even the Read-Rezayi states have been confirmed experimentally.

Still, the states serve as a decent “testing ground” for applying CFT techniques and also stretch the boundaries of what kind of quantum Hall states are allowed on a more abstract level.

### 7.2.2 Quasiholes

To construct the quasiholes of the product parafermionic case, we first argue that, in general, all possible fields that can possibly be formed are of the following type

\[
\prod_i \left[ \Phi_{m_i}^i e^{i\frac{Q_i}{\sqrt{\nu}} \phi} \right]
\]

This is not too suprising. One simply allows all possible combinations of fields appearing in the \(\mathbb{Z}_k\) theories. The \(u(1)\) part is, in turn, defined such that the electron operator is a local one. This amounts to determining the allowed values of the quasihole’s charge \(Q_i\).

We have seen that in the \(\mathbb{Z}_k^{(p)}\) case the smallest-charge operator is always formed through use of the spin field \(\sigma_1\). This justifies the following ansatz for the smallest-charge operator

\[
V_{qh} = \prod_i \sigma_{i,1} e^{i\frac{Q_i}{\sqrt{\nu}} \phi}
\]
7.3 Correspondence

The charge $Q_\gamma$ is defined such that the electron operator is local. To determine its value we switch to the notation where $\sigma_{i,1}^1 = \Phi_{i,1}^1$ and $\psi_p = \Phi_{i,2p}^0$. The fusion with the electron operator results in

$$V_{\text{qh}}(z)V_{\text{el}}(w) \sim (z - w)^{\sum_i \delta \Delta_i + \frac{Q_\gamma}{\nu}} \prod_i \left( \Phi^1_{i,1+2p_i} \right) e^{\frac{Q_\gamma + 1}{\sqrt{\nu}} \phi}(w)$$

(7.33)

$$\delta \Delta_i = \Delta_{\Phi_{i,1+2p_i}^1} - \Delta_{\psi_p} - \Delta_{\Phi_{i,1}^1} = -\frac{p_i}{k_i}$$

(7.34)

We are being a little creative with the label $i$, but from the context it should be clear that it simply refers to the corresponding $Z_k$ theory. The answer $\delta \Delta_i = -\frac{p_i}{k_i}$ was already obtained in the $Z_{k(p)}^p$ case, which is why we merely state it here.

In effect, the quasiholes charge must be tuned such that the fusion process is regular. This means

$$Q_\gamma = \nu \sum_i \frac{p_i^2}{k_i}$$

(7.35)

which is the electric charge of the quasihole in units of the electric charge.

7.3 Correspondence

With the two types of generalizations the number of filling fractions we can obtain has exploded. A lot of these filling fractions are not unique. First off all, there are the $Z_k^{(p)}$ states

$$V_{\text{el}} = \psi_p e^{\frac{i}{\sqrt{\nu}} \phi}$$

(7.36)

$$\nu = \frac{k}{kM + 2p^2}$$

(7.37)

where $p$ and $k$ are taken to be coprime. We can construct a product parafermion quantum Hall state with the same filling fraction. For that we take $n$ copies of the parafermion $\psi_1$, all taken from $Z_k$. This is the $(Z_k^{(1)})^n$ state and is described by

$$V_{\text{el}} = (\psi_1)^n e^{i\frac{1}{\sqrt{\nu}} \phi}$$

(7.38)

$$\nu = \frac{k}{kM + 2n}$$

(7.39)

We might be tempted to think these states are identical if we set $n = p^2$. But this is not the case, and it is a good example which shows that the filling fraction alone is not enough to fully describe a quantum Hall state. The easiest way to see that these states do not resemble the same quantum Hall state is on the basis of their different conformal dimension.

$$\Delta_{Z_k^{(p)}} = \frac{p(k - p)}{k} + \frac{1}{2\nu}$$

$$\Delta_{(Z_k^{(1)})^n} = \frac{n(k - 1)}{k} + \frac{1}{2\nu}$$

(7.40)

Furthermore, the central charge of the corresponding CFT’s also differ

$$c_{Z_k^{(p)}} = \frac{2k - 2}{k + 2} + 1$$

$$c_{(Z_k^{(1)})^n} = \frac{2nk - 2n}{k + 2} + 1$$

(7.41)
The fact that the underlying CFT’s and corresponding conformal operators differ eventually impacts the quantum Hall state on a less abstract level. For instance, if we put these quantum Hall states on a spherical topology, then the ratio $N_\phi$ and $N_e$ no longer equals the filling fraction but also picks up an extra shift. The shift is also a topological quantum number, and a quick calculation shows that they are not equal

$$S_{Z_k(p)} = M + 2p \frac{2p + k}{k}$$

$$S_{Z_k^{(1)}(n)} = M + 2n \frac{2 + k}{k}$$

(7.42)

The states have the same filling fraction, but are in different topological phases.

How do such subtle differences present themselves in the resulting quantum Hall state? Topological phases are characterized by the fact that one cannot construct a local operator which measures the type of phase we deal with. Instead, one needs to classify the according to their topological properties, such as ground state degeneracy and braiding matrices. In some sense, such topological quantum numbers correspond to measurements of non-local operators.

The ideal case would be to come up with a simple list of topological quantum numbers, such as the shift which was just presented, to classify all these states. But at the moment it is not completely resolved what truly defines a topological phase and what the minimal input is needed to fully describe it. Is there any other way to classify the corresponding wavefunction which allows for more physical arguments? Since these states describe different phases of topological order but share the same filling fraction, then there must be some way to make the distinction more apparent. We now turn to this matter and present a treatment of classifying wavefunctions. For that we use the properties of the underlying CFT’s to calculate less abstract physical quantities, which make the subtle differences in the topological phases more transparent. It is not a full classification scheme, but it does serve as step towards such a program.
CHAPTER 8

Classification and Pattern of zeros

In chapter 6.3 we treated the concept of clustering associated with the Read-Rezayi states or, in our newly adopted lexicon, $Z_k^{(1)}$ quantum Hall states. The idea is that for $M = 0$ the Read-Rezayi states reduce to a bosonic wavefunction which is characterized by the so-called Read-Rezayi clustering condition of order $k$. This condition states that we may bring together (fuse) $k$ electrons while the wavefunction remains non-zero. Bringing a $k + 1$'th electron to this cluster will cause the wavefunction to vanish, and it does so with order 2.

We have also seen that this condition is naturally implemented through a factorization of the $M = 0$ wavefunction, which appears in chapter 6. This formula reads

$$\Psi_{N_e}^{M=0}(z_k, z_{k+1}, \ldots z_{N_e}) \propto \Psi_{M=0}^{N_e-k}(z_{k+1}, z_{k+2} \ldots z_{N_e}) \prod_{j=k+1}^{N_e} (z_k - z_j)^2 \quad (8.1)$$

The newly found quantum Hall states are also based on the $Z_k$ parafermionic CFT. Hence, we expect them to obey a similar, though perhaps more complex, clustering behavior as the Read-Rezayi case. The natural question that follows is in what way we can characterize the clustering behavior of the new states, and if they allow for a similar factorization as equation 8.1.

The classification itself is based on what is called a pattern of zeros and we will start by introducing some new concepts and terminology, needed to clarify the classification. The classification will follow after that. The next section is a summary of the paper [Wen and Wang, 2008a]. We do not follow the paper in the literal sense though. The papers [Wen and Wang, 2008b; Barkeshli and Wen, 2008] also deal with this subject.
8. Derived Polynomials

Consider a bosonic FQHE wavefunction defined on a sphere, which is a compact surface with no boundary. In chapter 3 (see also appendix A) we investigated this wavefunction in quite some detail. We list some of its important properties here.

We are dealing with a multivariable wavefunction which is analytic and holomorphic. It does not contain any branch cuts or poles. Furthermore, it is a translational invariant function and it is an eigenstate of the SU(2) spherical symmetry. This last property arises due to type of surface we are dealing with - it is not an intrinsic property of a FQHE wavefunction. But we can certainly use it to our advantage. The function is well-defined as the number of electrons is taken to infinity \( N_e \to \infty \). All in all, we are dealing with a fully symmetric, translational invariant, of finite degree, analytic, multivariable wavefunction. A serious mouthful indeed.

Let us stress that we are not dealing with the general case of a symmetric polynomial with an infinite number of variables. These polynomial expressions we are dealing with are states such as the Read-Rezayi states and the Laughlin wavefunction. We have seen that these states are characterized by the way they behave when a number of electrons are brought together. See chapter 3.7 for a discussion on the Laughlin wavefunctions on this matter and chapter 6.3 for the Read-Rezayi states. This behavior, in which the wavefunction vanishes when a number of electrons are brought together, mimics the Coulomb interaction among the electrons and turns the related universality classes into reasonable candidates for quantum Hall fluids.

We will now give a more formal treatment of this characterization.

Suppose we start off with a wavefunction of \( N_e \) electrons \( \Psi(z_1, \ldots, z_{N_e}) \). The function \( \Psi \) is a multivariable polynomial. We can obtain another polynomial by letting one electron adiabatically approach another arbitrarily close. That is, we fuse the electrons together to form a new, composite particle. This fusion process results in another polynomial \( P' \) which is called a derived polynomial of the original wavefunction. This polynomial is a function of the remaining \((N_e - 2)\) electron coordinates and the obtained two-electron composite. For example, let \( z_1 \to z_2 \equiv z_2^{(2)} \)

\[
\lim_{z_1 \to z_2} \Psi(z_1, z_2, \ldots, z_{N_e}) \quad \longrightarrow \quad (z_1 - z_2)^{D_{11}} P'(z_2^{(2)}, z_3, \ldots, z_{N_e}) + \ldots
\]  

(8.2)

The notation \( z_2^{(2)} \) signifies that this coordinate describes a composite particle obtained by fusing two electrons together. In this limit the right hand side generally contains a higher order terms, which we drop. The derived polynomial is thus the leading term of this expression.

We can perform a similar operation with the derived polynomial, for example by letting \( z_2^{(2)} \to z_3 \) or \( z_3 \to z_4 \). In both limits we will obtain new, derived polynomials

\[
\lim_{z_2^{(2)} \to z_3} P'(z_2^{(2)}, z_3, \ldots, z_{N_e}) \quad \longrightarrow \quad (z_2^{(2)} - z_3)^{D_{21}} P''(z_3^{(3)}, z_4, \ldots, z_{N_e})
\]  

(8.3)

\[
\lim_{z_3 \to z_4} P'(z_2^{(2)}, z_3, \ldots, z_{N_e}) \quad \longrightarrow \quad (z_3 - z_4)^{D_{11}} P'''(z_2^{(2)}, z_4^{(2)}, \ldots, z_{N_e})
\]  

(8.4)

By iterating this process we obtain an (infinite) set of derived polynomial \( P'(\{z_i^{(a)}\}) \) and a set of corresponding powers \( \{D_{ab}\} \). The derived polynomials are all functions of the variables \( z_i^{(a)} \). With this notation we mean that this coordinate represents a ‘type-a composite’, obtained by fusing \( a \) electrons together. A type-1 particle naturally corresponds to the electron.
8.1 Derived Polynomials

The derived polynomials are a set of analytical functions with respect to their coordinates. As a natural consequence of the symmetries of the quantum Hall wavefunctions, the derived polynomials also possess (part of) these symmetries. They can therefore in some sense be interpreted as wavefunctions for the (different) type-a particles appearing in their arguments. The corresponding set \{D_{ab}\} is called a 'pattern of zeros'. It is completely determined through the derived polynomial, although the converse is not always true. The expressions for the derived polynomials are complex and if they can be found remains to be seen. However, the pattern of zeros are just a set of numbers. Furthermore, we can apply a number of restrictions which greatly reduces the number of allowed patterns. This is why the pattern of zeros are a useful tool for classifying the FQHE wavefunctions.

We end this part with a final clarification. We will only deal with bosonic wavefunctions, meaning that the wavefunction of the fundamental particles (the type-1 particles) satisfy Bose statistics. The reason for this is that the fermionic versions of the wavefunctions are automatically classified as well. Namely, for every antisymmetric wavefunction we can construct a unique symmetric polynomial, via

\[
\Psi_{\text{Sym}} = \Psi_{\text{Anti-symm}} \prod_{i<j} (z_i - z_j) \quad (8.5)
\]

The antisymmetric wavefunction always contains this factor at least once.

8.1.1 Pattern of zeros: \(D_{ab}\) characterization

The pattern of zeros are a mathematical feature, suitable for all sorts of functions. We do not want to approach the situation too general, which is why we implement some of the features that come along with our kind of polynomials. For instance, the symmetries described earlier correspond to a set of criteria, namely

\[
\begin{align*}
D_{ab} \in \mathbb{Z}_{\geq 0} & \quad \text{Wavefunction is singlevalued and regular} \quad (8.6) \\
D_{ab} = D_{ba} & \quad \text{Fusion product is commutative} \quad (8.7) \\
D_{aa} = \text{even} & \quad \text{Coordinates represent bosons} \quad (8.8) \\
D_{a1} = 0 & \quad \text{Translational invariance} \quad (8.9)
\end{align*}
\]

Some of these features are quite natural to understand. A singlevalued and regular wavefunction forbids any appearance of fractional or negative powers when fusing electrons together. The commutativity of fusion manifests itself in the way that taking the limit \(z_i \to z_j\) generates the same derived polynomial as the 'reverse' limit \(z_j \to z_i\). Last not but least, we will assume the composite particles formed through the fusion of the fundamental bosonic electrons, are also bosonic themself. The last criterium states that the roots can only arise when at least two electrons are brought together.

Also of great importance is the so-called unique-fusion condition. Fusing \(a\) electrons together can be done in many different ways, and each way may result in a different derived polynomial. This is why we implement the unique-fusion condition, which states that all these derived polynomials obtained by fusing \(a\) electrons together are always linearly dependent. As a corollary, the power \(D_{ab}\) does not depend on the preceded fusion steps taken.

A consequence of this unique fusion condition is that the set \(\{D_{ab}\}\) is overdetermined. We can express any element \(D_{ab}\) in terms of elements from the subset \(\{D_{a1}\}\). If we fuse a type-\(a\) particle with a type-\(b\) particle the resulting derived polynomial is linearly dependent to the
case in which we fuse one type-
particle with type-1 particles. This leads to the following expression

\[ D_{ab} = \sum_{i=1}^{a+b-1} D_{i1} - \sum_{j=1}^{b-1} D_{j1} - \sum_{j=1}^{a-1} D_{j1} \] (8.10)

A corollary is that the subset \( \{D_{a1}\} \) contains all the information about the pattern of zeros \( \{D_{ab}\} \).

8.1.2 Pattern of zeros: \( S_a \) characterization

To fully make contact with the paper by Wen and Wang we also introduce a second, equivalent characterization scheme based on similar principle [Wen and Wang, 2008a]. The characterization is also based on a sequence of integers denoted by \( \{S_a\} \). The integer \( S_a \) is defined as the order of zeros in \( \Psi \) as we simultaneously bring \( a \) electrons together to the point \( z^{(a)} \).

\[
\lim_{\lambda \to 0} \lambda^{-S_a} \Psi(z_1(\lambda), \ldots, z_a(\lambda), z_{a+1}, \ldots) = P(z^{(a)}, z_{a+1}, \ldots) \] (8.11)

where \( z_i(\lambda) = \lambda \xi_i + z^{(a)} \). In analogy with the sequence \( \{D_{ab}\} \) the integers \( \{S_a\} \) are also subject to a number of constraints which follow from the symmetries of the quantum Hall wavefunctions. First off, \( S_a \) is independent of the choice of which \( a \) electrons are brought together. This is the equivalent statement of the unique-fusion condition. We have in fact already assumed this condition to hold in expression (8.11). Strictly speaking, the polynomial \( P \) in (8.11) also depends on the set \( \{\xi_i\} \) since different choices of \( \xi_i \) lead to different ways in which the electrons are fused together. We will however assume the resulting polynomials are always linearly dependent.

Apart from the unique-fusion condition, we also have

\[
S_1 = 0 \quad \text{Translational invariance} \] (8.12)
\[
S_a \in \mathbb{Z}_{\geq 0} \quad \text{Wavefunction is singlevalued and non-singular} \] (8.13)
\[
S_{2a} = \text{even} \quad \text{Coordinates represent bosons} \] (8.14)

Furthermore, \( S_0 \) is defined to be zero. The first relation follows from the fact that in the absence of quasiholes all zeros are “attached” to the remaining electrons. Singlevaluedness obviously implies that all \( S_a \) are non-negative and integer-valued and the last restriction is also straightforward.

The sequences \( \{D_{ab}\} \) and \( \{S_a\} \) are closely related. The polynomial \( P \) in equation (8.11) is linearly dependent on a derived polynomial obtained by fusing the electrons one-by-one. We write this in the suggestive manner

\[
\lim_{z_a \to z_{a-1}} (z_a - z_{a-1})^{-D_{a-1,1}} \left[ \cdots \lim_{z_3 \to z_2} (z_2 - z_3)^{-D_{21}} \left[ \lim_{z_2 \to z_1} (z_1 - z_2)^{-D_{11}} \Psi(z_1) \right] \cdots \right] \propto P(z^{(a)}, z_{a+1}, \ldots) \] (8.15)

where the order of the limits is taken “from the inside out”. We now make a handwaving argument. By substituting \( z_i = \lambda \xi_i + z^{(a)} \) we have the conversion of \((z_i - z_{i+1})^{-D_{i1}}\) into
\( \lambda^{-D_{11}}(\xi_i - \xi_{i+1})^{-D_{11}} \). The limits “combine” into one limit, namely the limit where \( \lambda \to 0 \). All in all, we rewrite (8.15) into

\[
(\xi_1 - \xi_2) \cdots (\xi_{a-1} - \xi_a) \lim_{\lambda \to 0} \left[ \lambda^{-\sum_{i=1}^{a-1} D_{i1}} \Psi(\{z_i\}) \right] = P(z^{(a)}, z_{a+1}, \ldots)
\]  

(8.16)

This substitution is not quite mathematically rigorous. However, we merely want to show that the total degree in which the wavefunction vanishes equals the sum \( \sum_{i=1}^{a-1} D_{i1} \). Comparing with (8.11) we have the following correspondence between the sequences \( \{S_a\} \) and \( \{D_{ab}\} \).

\[
S_a = \sum_{i=1}^{a-1} D_{i1}
\]

(8.17)

And together with (8.10) we can also invert

\[
D_{ab} = S_{a+b} - S_a - S_b
\]

(8.18)

(8.19)

A special case of this expression is \( D_{a1} = S_{a+1} - S_a \). Using expression (8.18) the restrictions (8.9) translate into (8.14) and vice versa. Note that these conversion equations automatically imply (8.10).

Since we can map from one sequence to the other, both describe equally well the (local) conditions of a quantum Hall wavefunctions. Both sequences will therefore be referred to as a pattern of zeros characterizing symmetric polynomials.

### 8.1.3 Example: Laughlin wavefunction

Let us start with a simple example to clarify some of the definitions and results we have obtained so far. We look at the bosonic quantum Hall wavefunctions of the Laughlin sequence, with filling fraction \( \nu = \frac{1}{M} \), namely

\[
\Psi_L(z_1, \ldots) = \prod_{i<j}(z_i - z_j)^{\nu^{-1}}, \quad \nu^{-1} = M, \text{ even integer}
\]

(8.20)

This wavefunction indeed satisfies all the symmetries mentioned at the beginning of this chapter. What happens when we start fusing electrons together? Consider, for instance \( z_1 \to z_2 \). It is evident the wavefunction vanishes with an order \( M \), from which we conclude \( D_{11} = M \). The resulting derived polynomial is

\[
\lim_{z_1 \to z_2} [(z_1 - z_2)^{-M} \Psi_L] = P'(z^{(2)}_2, z_3, \ldots) = \prod_{2<i<j}(z_2^{(2)} - z_i)^{2M} \prod_{2<i<j}(z_i - z_j)^{M}
\]

(8.21)

When we fuse \( z^{(2)}_2 \) and \( z_3 \) in this derived polynomial, only the first product in the above expression contributes since the second product does not contain any factors of \( (z_2 - z_3) \). This sets \( D_{21} = 2M \) and the corresponding derived polynomial is

\[
\lim_{z_2 \to z_3} [(z_2 - z_3)^{-2M} P'] = P''(z^{(3)}_3, z_4, \ldots) = \prod_{3<i<j}(z_3^{(3)} - z_i)^{2M} \prod_{3<i<j}(z_i - z_j)^{M}
\]

(8.22)
It is not hard to see that this generalizes to \( D_{q1} = qM \). By plugging this into expression (8.10) we have

\[
D_{ab} = abM
\]  

(8.23)

This is the pattern of zeros that describes the Laughlin wavefunction. One can check that it satisfies all the criteria as appearing in expression (8.9). The integers \( \{S_a\} \) follow from expression (8.17)

\[
S_a = \frac{a(a - 1)}{2} M
\]  

(8.24)

As a corollary of the simple form of (8.23), the derived polynomials can also be written down explicitly [Wen and Wang, 2008a]

\[
P(\{z_i^{(a)}\}) = \left[ \prod_{a < b} \prod_{i,j} \left( z_i^{(a)} - z_j^{(b)} \right)^{abM} \right] \left[ \prod_{a} \prod_{i<j} \left( z_i^{(a)} - z_j^{(a)} \right)^{a^2M} \right]
\]  

(8.25)

For the more complicated states such expressions are far more difficult to accomplish.

The Laughlin wavefunction has a very peculiar property. Its expression is sort of invariant under the fusion of electrons. By this we mean the following. Suppose we fuse all electrons into type-\( a \) electrons. Provided that \( N_e \) is a multiple of \( a \), we end up with a derived polynomial which only depends on the coordinates \( \{z_i^{(a)}\} \). What makes the Laughlin wavefunction so special is that this derived polynomial is also a Laughlin wavefunction

\[
\Psi_{\text{Laughlin}}(z_1, \ldots, z_{N_e}) \rightarrow P'(z_{a1}^{(a)}, z_{a2}^{(a)}, \ldots, z_{aN_e}^{(a)}) = \prod_{i<j} \left( z_i^{(a)} - z_j^{(a)} \right) \frac{a^2}{\tau}
\]  

(8.26)

The Laughlin wavefunction is an expression for an incompressible quantum Hall state. We observe that not only the electrons form an incompressible state, but also all of the composite particles that follow from fusing the electrons together. This is a highly non-trivial matter as for general symmetric polynomials we would not expect such behavior. It also brings up an interesting question: are there any other polynomials of which the derived polynomials of the composite particles have the same expression as their parent state? Or is the Laughlin wavefunction the only type of polynomial which has this property?

As it turns out, other quantum Hall states are intimately related to the Laughlin wavefunction. The Laughlin wavefunction turns an infinite sequence of derived polynomials and pattern of zeros into a very restricted and simple form.

8.1.4 Clustering condition

The pattern of zeros so far are already restricted in quite a large manner. Still, the approach is far too general to make sensible statements about the structure of the polynomials. Specifically, the infinite number of variables are too difficult to keep track off. This is why we introduce the notion of \( k \)-clustering condition. It relates the wavefunction with an infinite number of coordinates to a Laughlin wavefunction. The Laughlin wavefunction also has an infinite number of coordinates, but its classification is completely fixed. As the authors put it, the clustering condition makes “a polynomial with an infinite number of variables behave more like a polynomial with a finite number of variables”.

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In this section we will give definitions of the clustering conditions in three different ways: (1) in terms of the wavefunctions and their derived polynomial (2) through a set of restrictions on the pattern of zeros \( \{D_{ab}\} \) and (3) in the same manner for \( \{S_a\} \). The three definitions are equivalent.

We consider the symmetric polynomial \( \Phi(z_1, \ldots, z_{Ne}) \). We take \( Ne \) to be a multiple of \( k \).

Next, we divide the electrons up into \( Ne/k \) clusters of \( k \) electrons each. In each cluster we fuse all electrons together. We thus obtain a derived polynomial \( P_c \) which is a function of type-\( k \) particles alone. A symmetric polynomial \( \Phi(z_1, \ldots, z_{Ne}) \) satisfies the \( k \)-cluster condition if the resulting derived polynomial \( P_c \) is a singlevalued Laughlin wavefunction [Wen and Wang, 2008a]

\[
\Psi(z_1, z_2, \ldots, z_{Ne}) \longrightarrow P_c(z_k^{(1)}, z_2^{(k)}, \ldots, z_{Ne}^{(k)}) \sim \prod_{i<j} (z_{ik}^{(k)} - z_{jk}^{(k)})^L \tag{8.27}
\]

for some positive integer \( L \). That is, the wavefunction describing the type-\( k \) composite particles is interpreted as an ordinary Laughlin wavefunction. We know that the behavior of the Laughlin wavefunction is completely fixed. Fusing the composite particles \( z_i^{(k)} \) into larger composites, e.g. \( z_i^{(ak)} \), we again obtain Laughlin-type wavefunctions. This is not yet “the” definition of clustering, but it is the most important property.

What is the effect of this condition? We have seen that the Laughlin wavefunction has a very simple pattern of zeros, namely that of equation (8.23). This pattern is also valid for equation (8.27), only this time it is the type-\( k \) composites which satisfy this pattern of zeros. Therefore

\[
D_{ka,kb} = abL \tag{8.28}
\]

\[
S_{ka} = \frac{ka(ka-1)}{2}L \tag{8.29}
\]

For the type-\( k \) composites pattern of zeros is completely determined through the parameter \( L \), which is given by \( D_{kk} \). We will now show that the infinite pattern of zeros \( \{D_{ab}\} \) is completely determined through its subset \( \{D_{11}, \ldots, D_{k-1,1}\} \).

The authors proposed the following general form for a clustered FQHE wavefunction

\[
\Psi(z_1, \ldots, z_{Ne}) = G(z_1, \ldots, z_{Ne}) \prod_{i<j} (z_i - z_j)^{\nu-1} \tag{8.30}
\]

This form is on the basis that the total order of all coordinates of the wavefunction \( \Psi \) satisfies

\[
S_{Ne} = \frac{1}{2\nu} N_e^2 + O(N_e) \tag{8.31}
\]

The Laughlin-Jastrow factor in (8.30) has a total order which, up to \( O(N_e) \) equals the leading term of the total order of the complete wavefunction \( \Psi \)

\[
\text{Total order of } \prod_{i<j} (z_i - z_j)^{\nu-1} \text{ equals } \frac{1}{2\nu} N_e^2 - \frac{1}{2\nu} N_e = S_{Ne} - O(N_e) \tag{8.32}
\]

Combine this result with (8.31) and we conclude that the total order of the function \( G \) must be \( O(N_e) \). Furthermore, since \( \Psi \) is translational invariant, then so is \( G \). Therefore \( G \) also satisfies

\[
G(\lambda z_1, \ldots, \lambda z_{Ne}) = \lambda^{s_{Ne}} G(z_1, \ldots, z_{Ne}), \quad s_{Ne} = O(N_e) \tag{8.33}
\]
8. Classification and Pattern of zeros

where \( s_{Ne} \) is the total order of \( \mathcal{G} \).

On the basis of this general solution of clustered quantum Hall state we can translate the cluster condition into restrictions on the pattern of zeros. Namely, suppose we fuse the electrons into type-\( k \) particles. This is the same limit considered in \((8.27)\), only applied to the LJ-factor instead of the entire wavefunction \( \Psi \). Using \((8.25)\), the resulting derived polynomial of the Laughlin-Jastrow factor is

\[
P_{LJ}^{c} (z_{k}^{(k)}, z_{2k}^{(k)}, \ldots, z_{Ne}^{(k)}) = \prod_{i<j} (z_{ik}^{(k)} - z_{jk}^{(k)})^{k^2} \tag{8.34}
\]

We will now make a statement. We identify this expression with that of equation \((8.25)\) for the entire wavefunction \( \Psi \). That is, the limit where all electrons fuse into type-\( k \) particles, equation \((8.27)\), leads to the same derived polynomial for the wavefunction \( \Psi \) and the Laughlin-Jastrow factor.

\[
\Psi(z_1, z_2, \ldots, z_{Ne}) \rightarrow P_{LJ}^{c} (z_{k}^{(k)}, z_{2k}^{(k)}, \ldots, z_{Ne}^{(k)}) = \prod_{i<j} (z_{ik}^{(k)} - z_{jk}^{(k)})^{k^2} \tag{8.35}
\]

Put differently, the derived polynomial obtained by taking this limit for the function \( \mathcal{G} \) is constant

\[
\mathcal{G}_{c} (z_{k}^{(k)}, z_{2k}^{(k)}, \ldots, z_{Ne}^{(k)}) = 1 \tag{8.36}
\]

We also implement the following into the function \( \mathcal{G} \)

\[
\mathcal{G}(\ldots, z_{i}^{(a \mod k)} \ldots) = \mathcal{G}(\ldots, z_{i}^{(a \mod k)} \ldots) \tag{8.37}
\]

\[
\mathcal{G}(\ldots, z_{i}^{(nk)} \ldots) = \mathcal{G}(\ldots, \ldots) \tag{8.38}
\]

By this we mean that the derived polynomials of the function \( \mathcal{G} \) do not depend on type-\( nk \) particles. All the derived polynomials have in some sense a “cyclic property”. For this function a composite particle consisting out of \( a \) particles is the same as one consisting out of \( a + k \) particles.

The general form \((8.30)\) for the wavefunctions together with these properties of \( \mathcal{G} \) is the defining property of clustering of order \( k \) for a symmetric polynomial \( \Psi \). If these definitions and implementations seem rather vague, fear not. We will make all expressions presented here more explicit when we turn to the CFT description of our newly acquired states.

Given the above implementations, we now turn to the question what this implies for the pattern of zeros. We first note that given the results presented above the pattern of zeros \( \{D_{ab}\} \) splits into

\[
D_{ab} = \frac{ab}{\nu} + d_{ab} \tag{8.39}
\]

The first term is the contribution of the Laughlin Jastrow factor (compare with expression \((8.23)\)). The second term \( d_{ab} \) is defined as the pattern of zeros satisfied by the function \( \mathcal{G} \). Note that since this function is not a quantum Hall wavefunction this pattern does not satisfy the same properties as \( D_{ab} \). What it does satisfy is

\[
d_{ab} = d_{ba} \tag{8.40.a}
\]

\[
d_{ab} = 0 \quad \text{if } b \pmod k = 0 \tag{8.40.b}
\]

\[
d_{a,b+k} = d_{ab} \tag{8.40.c}
\]
8.1 Derived Polynomials

These relations follow from the definition of \( G \), (8.36), (8.37) and (8.38). We define \( D_{ab} \) to satisfy the clustering condition of order \( k \) if it satisfies these restrictions on \( d_{ab} \) plus the general form (8.39).

With these definitions the infinite set \( D_{ab} \) is completely determined by the finite subset \( \{D_{a1}\}_{a=1}^{k-1} \), through the following relation

\[
D_{a,b+k} = \frac{ab}{\nu} + \frac{ak}{\nu} + d_{ab} \\
= D_{ab} + \frac{k}{\nu}
\]  
(8.41)

By iterating this formula we obtain an expression which relates arbitrary \( D_{ab} \), for arbitrary \( a, b \), to the subset \( \{D_{a1}\}_{a=1}^{k-1} \)

\[
D_{ab} = D_{a,[b]_k} + \frac{b-\lfloor b \rfloor_k k}{\nu} + b \left( \frac{a-\lfloor a \rfloor_k k}{k} - \frac{1}{\nu} \right) \\
= D_{a,[b]_k} + \frac{ab - [a]_k [b]_k}{\nu} + \frac{a-\lfloor a \rfloor_k k}{k} \frac{b-\lfloor b \rfloor_k k}{\nu}
\]  
(8.42)

The expression \( \frac{b-\lfloor b \rfloor_k k}{\nu} \) is actually the floor function \( \lfloor \frac{a}{k} \rfloor \), i.e. the largest integer which is equal or smaller than \( \frac{a}{k} \). A special case is the following

\[
D_{k1} = \frac{k}{\nu}
\]  
(8.43)

In [Wen and Wang, 2008a] this quantity is denoted as \( m \).

For the pattern of zeros \( \{S_a\} \) we follow a similar approach. First off all, a \( k \)-clustering property means that \( \{S_a\} \) satisfies

\[
S_a = \frac{a(a-1)}{2\nu} + s_a
\]  
(8.44)

where

\[
s_{a+k} - s_a - s_k = 0
\]  
(8.45)

\[
s_{a+k} - s_a = s_{k} - s_b
\]  
(8.46)

which follows from the relation \( d_{ab} = s_{a+b} - s_a - s_b \). This can be derived by using relation (8.18) and definitions (8.41) and (8.44). Due to translational invariance we know that \( S_1 = 0 \).

This implies \( s_1 = 0 \) Therefore we find

\[
s_{1+k} - s_1 - s_k = 0
\]  
(8.47)

\[\implies s_{1+k} = s_k
\]  
(8.48)

\[\implies s_{a+k} - s_a = s_k
\]  
(8.49)

This last line is then used to obtain an expression for \( S_a \)

\[
S_{a+bk} = \frac{(a+bk)(a+bk-1)}{2\nu} + s_{a+bk}
\]  
(8.50)

\[
= \frac{(a+bk)(a+bk-1)}{2\nu} + s_a + bsk
\]  
(8.51)

\[
= \left\{ \frac{a(a-1)}{2\nu} + s_a \right\} + b \left\{ \frac{k(k-1)}{2\nu} + s_k \right\} + \frac{k}{2\nu} (2ab + bk(b-1))
\]  
(8.52)

\[
= S_a + bS_k + \frac{k}{2\nu} (2ab + bk(b-1))
\]  
(8.53)
This relation allows us to express an arbitrary element $S_b \in \{S_a\}$ in terms of elements from the finite subset $\{S_{\alpha'}\}_{\alpha'=2}^k$.

We have already come accross an example of a clustered quantum Hall state. Namely, the Laughlin wavefunction itself always satisfies the clustering condition for any $k$. This is what we saw when we treated this example. The derived polynomials of the composite particles are Laughlin wavefunctions of these composites with some lower filling fraction $\nu' - 1 = \frac{2}{\nu'}$. However, the Laughlin wavefunction does not contain any extra structure, in the sense that the patterns $d_{ab}$ and $s_a$ are all zero.

This has been quite a lengthy formulation of the clustering condition, but it is important to get it right the first time. The paper [Wen and Wang, 2008a] is completely built upon this concept.

This is also the moment where we depart from the treatment of that paper and head in a new direction. In [Wen and Wang, 2008a] the authors follow a numerical treatment. The pattern of zeros we have obtained are subject to a series of consitency relations which greatly reduce the number of possible patterns. Then, through a numerical approach, the authors list all possible solutions of patterns of zeros which satisfy the clustering condition, up to $k = 10$, $k$ being the size of a cluster. Among these results the authors identified the Laughlin states, the Read-Rezayi states, the $Z_{k}^{(p)}$ states, the composite $\bigotimes_{i} Z_{k_{i}}^{(p_{i})}$ states and a few new states that have not been classified yet.

We will follow a different treatment. Starting from CFT we will show that the pattern of zeros of $Z_{k}^{(p)}$ and $\bigotimes_{i} Z_{k_{i}}^{(p_{i})}$ indeed satisfy the clustering condition as defined above. We will accomplish this by deriving an explicit formula for the pattern of zeros.

8.2 CFT formulation of the pattern of zeros

In this section we will approach the pattern of zeros from a different perspective. Namely, we will start with a given CFT and obtain from it the information for the pattern of zeros. This reversed procedure is clearly not suitable for a classification of possible quantum Hall states, but it does provide a decent tool for exploring the new obtained quantum Hall states.

8.2.1 Pattern of zeros from CFT

The situation is as follows. We obtain the pattern of zeros from the quantum Hall wavefunctions through the limits of bringing the coordinates $z_{i}^{(a)}$ together. From the CFT connection we know that these wavefunctions are expressed in terms of correlators of the operators which represent the electrons. Therefore the CFT equivalent action of bringing two electrons together is replacing the product of two operators by their operator product expansion. The extra step which we have to take is that we also must identify the operator which represent the composite particles and determine all possible OPE’s. The CFT we will study all provide us with this information.

Suppose we have this information. This means we have an infinite set of operators $\mathcal{V}_{a}$, where $\mathcal{V}_{a}$ represents the type-$a$ particle, and all the possible OPE’s among these operators. Then in particular we must have

$$\mathcal{V}_{a} \times \mathcal{V}_{b} = \mathcal{V}_{a+b} \quad (8.54)$$
This unique, Abelian fusion rule follows from the fact that we only deal with simple currents and the unique fusion condition. The explicit OPE of this fusion rule is

\[
V_a(z)V_b(w) \sim \frac{1}{(z-w)^{\Delta_a+\Delta_b-\Delta_{a+b}}} V_{a+b} + \ldots
\]  

(8.55)

By inserting this expression in the correlator we obtain an expression for the derived polynomials

\[
\langle \cdots V_a(z)V_b(w) \cdots \rangle = \frac{1}{(z-w)^{\Delta_a+\Delta_b-\Delta_{a+b}}} \langle \cdots (V_{a+b} + \ldots) \cdots \rangle
\]  

(8.56)

Such a replacement is only valid within a correlator. But the wavefunctions are identified with correlators, so this substitution is legitimate. From this expression we identify

\[
P(\{z^{(a)}\}) = \langle \cdots V_{a+b} \cdots \rangle
\]  

(8.57)

\[
D_{ab} = \Delta_{a+b} - \Delta_a - \Delta_b
\]  

(8.58)

Since we also have \(D_{ab} = S_{a+b} - S_a - S_b\) we see that the scaling dimensions and the set \(\{S_a\}\) satisfy similar restrictions. However, they are not equals since \(S_1 = 0\) while \(\Delta_1 = \Delta_{el}\). To obtain an expression for \(S_a\) we use the relation between the two patterns

\[
D_{a1} = S_{a+1} - S_a
\]

(8.59)

Since \(S_1 = 0\) we have \(D_{11} = S_2\, and therefore we identify

\[
S_2 = \Delta_2 - 2\Delta_1
\]

(8.60)

Since \(D_{21} = S_3 - S_2\) we have

\[
S_3 = \Delta_3 - \Delta_2 - \Delta_1 + S_2 = \Delta_3 - 3\Delta_1
\]

(8.61)

Iteration of this process gives us the following expression

\[
S_a = \Delta_a - a\Delta_1
\]

(8.62)

In general the expressions for the derived polynomials (8.57) are hard to determine. They are given by the correlator

\[
P(\{z^{(a)}\}_{a \in A}) = \lim_{w \to \infty} w^Q \left\langle \mathcal{O}_{bg}(w) \prod_{i,a} V_a(z_i^{(a)}) \right\rangle
\]

(8.63)

with the background charge \(Q\) properly defined. The pattern of zeros are easily identified, provided we have identified the CFT and the corresponding operators which represent the composites.

**8.2.2 Example: chiral boson and Laughlin**

To give an example we will again start with the Laughlin wavefunction. In that case the electron operator is given by the vertex operator \(e^{\frac{i}{\sqrt{\nu}} \varphi}\), where for the Laughlin states we set
\( \nu = \frac{1}{M} \). The identification of the operator which represent a composite particle is straightforward. We have for a type-\( a \) particle

\[
\mathcal{V}_a = e^{i \frac{a}{\sqrt{\nu}} \varphi} \tag{8.64}
\]

\[
\Delta_a = \frac{a^2}{2} \tag{8.65}
\]

The pattern of zeros are

\[
D_{ab} = \left( a + b \right)^2 - \frac{a^2}{2} \nu^{-1} - \frac{b^2}{2} \nu^{-1} = ab \nu^{-1} \tag{8.66}
\]

and

\[
S_a = \frac{a^2}{2} \nu^{-1} - \frac{1}{2} \nu^{-1} = \frac{a(a-1)}{2} \nu^{-1} \tag{8.67}
\]

This matches the results of section 8.1.3. We can also obtain the expression for the derived polynomials, using result (B.53) from the appendix

\[
P(\{z(a)\}_{a \in A}) = \lim_{w \to \infty} w^Q \left( V(w) \prod_{i,a} e^{i \frac{a}{\sqrt{\nu}} \varphi(z(a))} \right)
\]

\[
= \prod_{a,b,i,j} \left( z_i^{(a)} - z_j^{(b)} \right) \prod_{a} \prod_{i<j} \left( z_i^{(a)} - z_j^{(a)} \right)^2 \tag{8.68}
\]

Which again matches the previously obtained results. What is new is that these identities extend to cases where the chiral boson is a subalgebra of the complete CFT. This allows for the possibility of \( \nu^{-1} \) being fractional. The contributions of the vertex operator to the pattern of zeros in the case where \( \nu^{-1} \) is fractional is still given by formula (8.65) and (8.67).

### 8.2.3 Extracting the conformal dimensions

We can also try to invert the procedure. That is, we would also like to extract the CFT from the pattern of zeros. However, one quickly finds that there is an ambiguity present if we try to determine \( \Delta_a \) from the sets \( D_{ab} \) or \( S_a \). From (8.62) we have

\[
\Delta_a = S_a + a \Delta_1 \tag{8.69}
\]

Also

\[
D_{ab} = \Delta_{a+b} - \Delta_a - \Delta_b = S_{a+b} - S_a - S_b \tag{8.70}
\]

Since \( S_1 \) is zero we are able to compute \( S_a \) from the pattern \( D_{ab} \). A priori \( \Delta_1 \) is not restricted. We see that multiple \( \Delta_a \) lead to the same pattern of zeros. In particular, the simultaneous shift of all conformal dimensions

\[
\Delta_a \rightarrow \Delta_a + a \gamma \tag{8.71}
\]

lead to the same expressions for \( D_{ab} \) and \( S_a \). We are not completely free in choosing \( \Delta_1 \). The first is the spin-statistics theorem, treated in chapter 5.3 and reference [Fröhlich et al.,...}

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It states that the conformal dimension of the bosons and the composites are all integer valued. We also expect the conformal dimension to be positive and non-zero. If not, the CFT is automatically non-unitary. Non-unitary CFT’s which describe quantum Hall states do exist though, although these states are currently under debate [Read, 2008]. It is therefore not completely clear if this is a correct implementation. Still, we will require
\[ \Delta_1 \in \mathbb{Z}_{>0} \]  
(8.72)

This also limits \( \gamma \) in (8.71).

There is still an ambiguity left though, which implies that we are not able to extract the conformal dimensions from the pattern of zeros alone. For that we have to implement more conditions which we know through the general CFT description of the quantum Hall fluids. This was treated in chapter 5.3. Specifically, the electron operator consists out of a simple current times a \( u(1) \) vertex operator, equation (5.66). The vertex operator is present due to the electrical current. This definition can be extended to the composite particles giving
\[ V_a = \psi\lbrack a\rbrack_k e^{\frac{ia^2}{\nu} \phi} \]  
(8.73)

\[ \Delta_a = \Delta_{\text{sc}\lbrack a\rbrack}_k + \frac{a^2}{2\nu} \]  
(8.74)

Where \([a]_k \equiv a \mod k\). This is the starting point of the appendix in [Wen and Wang, 2008a]. Note that \(\psi\lbrack a\rbrack_k\) satisfies an Abelian fusion rule, i.e. \(\psi\lbrack a\rbrack_k \times \psi\lbrack b\rbrack_k = \psi\lbrack a+b\rbrack_k\). In addition, there are a finite number of \(\psi_a\). Defining \(k\) as the order, e.g. all fields satisfy \((\psi_a)^k = 1\), we have \(\psi_a \times \psi_{k-a} = 1\) and \(\Delta_{\text{sc}\lbrack a\rbrack}_k = \Delta_{\text{sc}\lbrack k-a\rbrack}_k\), which, again, follows from the definition of a simple current [Fuchs, 1992, page 328].

To determine an expression which expresses \(\Delta_a\) in terms of \(S_a\) we make use of the clustering definition, obtained in chapter 8.1.4. There we obtained the following condition on \(S_a\)
\[ S_{ak} = aS_k + \frac{(ak)^2}{2\nu} - \frac{ak^2}{2\nu} \]  
(8.75)

Now, using the relation (8.62), the definition (8.74) and the fact that \(\Delta_{\text{sc}\lbrack ak\rbrack}_k = \Delta_{\text{sc}\lbrack 1\rbrack}_k = 0\) we also have
\[ S_{ak} = -ak\Delta_{\text{sc}\lbrack a\rbrack}_k + \frac{(ak)^2}{2\nu} - \frac{ak}{2\nu} \]  
(8.76)

Equating these two expression gives
\[ \Delta_{\text{sc}\lbrack a\rbrack}_k = \frac{S_k}{k} + \frac{k-1}{2\nu} \]  
(8.77)

This in turn allows us to determine all the conformal dimensions of the simple currents through equation (8.69) and (8.74)
\[ \Delta_a = S_a - \frac{aS_k}{k} + \frac{ak}{2\nu} \]  
(8.78)

In the appendix of [Wen and Wang, 2008a] the symbols are slightly different, namely \(\Delta_a = h_a\), \(\Delta_{\text{sc}\lbrack a\rbrack}_k = h_{\text{sc}\lbrack a\rbrack}_k\), \(\Delta_1 = \frac{C_1}{\nu}\) and \(\frac{k}{\nu} = m\).
The authors present more implementations which we will not go into. We mention that
\[ S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0 \quad \text{and even} \quad (8.79) \]
\[ S_{a+b} - S_a - S_b \geq 0 \quad \text{and even} \quad (8.80) \]
These in turn can be related to generalized parafermion CFT’s, treated for instance in [Noyvert, 2007]. See the appendix [Wen and Wang, 2008a] for details.

In effect, by assuming the physical conditions such as clustering and the presence of a vertex operator we get rid of the unwanted ambiguity (8.71) and are able to uniquely identify the conformal dimensions. The justification of these assumptions is not completely rigorous though. For now, we reverse the procedure by assuming a certain CFT and from there describe the pattern of zeros.

\section*{8.3 \( \mathbb{Z}_k^{(p)} \) - Pattern of zeros}

We will now apply our tools to the case of the \( \mathbb{Z}_k^{(p)} \) quantum Hall states. These states are clustered according to the definition of chapter 8.1.4, which we show now. A recent paper appeared which treats these states [Barkeshli and Wen, 2008]. However, they do not recover the pattern of zeros.

\subsection*{8.3.1 Introduction}

In chapter 7 we treated the construction of the wavefunctions of the \( \mathbb{Z}_k^{(p)} \) states. They are written as
\[ \Psi(z_1, \ldots, z_{N_e}) = \langle \psi_p(z_1) \cdots \psi_p(z_{N_e}) \rangle \prod_{j<i} (z_i - z_j) \frac{z_i^{2p}}{N_e} \quad (8.81) \]
In this expression and for the remainder of this chapter we set \( M = 0 \); we will come back to the non-zero cases in the next section. We identify the parafermionic correlator with the function \( G \) in equation (8.30). Since it is a conformal correlator it indeed satisfies (8.33).

On the level of the parafermion CFT the clustering of order \( k \) is reproduced due to the modular property of the \( \mathbb{Z}_k \) charge combined with the contribution of the electric charge sector. The parafermions have the important property that fusing \( k \) of the same parafermions together always results in the identity operator.
\[ (\psi_p)^{\times k} \sim 1 \quad \text{where } 0 \leq p < k \quad (8.82) \]
\[ e^{i\alpha \varphi} \times e^{i\beta \varphi} \sim e^{i(\alpha + \beta) \varphi} \quad (8.83) \]
These properties of the OPE has the following impact on the electron operator. By fusing \( k \) electrons together we obtain the operator
\[ \mathcal{V}_{kp} \equiv (V_{el})^{\times k} = 1 e^{ikp\sqrt{2k}} \quad (8.84) \]
This operator describes the type-\(k\) particles of the \(\mathbb{Z}_k^{(p)}\) states. Their parafermionic part is trivial, therefore the wavefunction of these composite is given by

\[
\Phi(z_k, z_{2k}, \ldots, z_{N_e}) = \lim_{w \to \infty} w^{2p^2 N_e^2} e^{-ip N_e \sqrt{\frac{2}{k}} \varphi(z)} \prod_{i=1}^{N_e} e^{ik p \sqrt{\frac{2}{k}} \varphi(z_{ik})}
\]

Indeed, the derived polynomials, which we view as wavefunctions for the composite particles, are Laughlin-like. This expression matches definition (8.27). If we continue on fusing the type-\(k\) particles together the parafermionic part of the operator remains the identity. This means that all derived polynomials which are obtained this way are Laughlin-like. We also identify \(L = 2p^2 k\) in equation (8.27). Furthermore, the properties of the function \(G\) (equation (8.36), (8.37) and (8.38)) are also recognised.

In conclusion, the definition of the clustering condition is indeed satisfied by these states. A small word of caution is in order though. The cyclic property (8.82) is characteristic for all parafermionic operators. However, notice that for a general combination of \(p\) and \(k\) we have the possibility that we need less than \(k\) parafermions to fuse to the identity. In that case the bosonic wavefunction also satisfies a clustering condition, but with clusters containing less than \(k\) electrons. Note that the \(k\) clustering condition is still satisfied though.

If we take for instance \(p = \frac{1}{2}\) and \(k = \text{even}\), we always have a pairing property \((\psi_\varphi)^2 = 1\). These cases can be circumvented by taking \(p\) and \(k\) to be coprime i.e. their lowest common divisor is one. The results obtained in the next section still apply to the non-coprime cases. The reason why we mention this is that in the numerical results of [Wen and Wang, 2008a] the non-coprime results are all absent! We will come back to this.

### 8.3.2 Operator algebra

To derive an exact expression for the pattern of zeros we need to

First, some notation. The \(\mathbb{Z}_k\) charge of a parafermion is always taken to be its principal value, i.e. modulo \(k\). We write

\[
[a]_k \equiv a \mod k, \quad \text{with } 0 \leq [a] < k
\]

We define the vertex operator \(\mathcal{V}_m(z)\) as

\[
\mathcal{V}_m(z) = \psi_{[m]_k}(z) e^{im \sqrt{\frac{2}{k}} \varphi(z)}, \quad m \in \mathbb{Z}_{\geq 0}
\]

\[
\Delta_m = \frac{m^2}{k} + \frac{[m]_k (k - [m]_k)}{k}
\]

This operator carries an (unnormalized) \(u(1)\) charge of \(m \sqrt{\frac{2}{k}}\) and a \(\mathbb{Z}_k\) charge of \([m]_k\). For \(m = p\) this operator corresponds to the electron operator (7.2) for the \(\mathbb{Z}_k^{(p)}\) state and for \([m]_k = 0\) it reduces to \(1 e^{im \sqrt{\frac{2}{k}}}\) which is the operator describing type-\(k\) particles. The fusion rules of these operators are

\[
\mathcal{V}_m \times \mathcal{V}_l = \mathcal{V}_{m+l}
\]

\[
= \psi_{[m+l]_k} e^{i(m+l) \sqrt{\frac{2}{k}} \varphi(z)}
\]
On the basis of the parafermionic operator algebra (chapter 6.1) and the $u(1)$ operator algebra (appendix B) one can derive the following operator algebra satisfied by these operators

\[
\mathcal{V}_l \times \mathcal{V}_m \sim (z-w)^{\delta \Delta} \mathcal{V}_{m+l}
\]

\[
\delta \Delta = \frac{2}{k} (lm + [l+m]_k (k - [l+m]_k) - [l]_k (k - [l]_k) - [m]_k (k - [m]_k))
\]

This compact notation simplifies a little if we split up the OPE’s into two cases, namely the cases $[l+m]_k = [l]_k + [m]_k$ and $[l+m]_k = [l]_k + [m]_k - k$. This gives

\[
\delta \Delta = \begin{cases} 
\frac{2}{k} (lm - [l]_k [m]_k) & [l]_k + [m]_k \leq k \\
\frac{2}{k} (lm - (k - [l]_k)(k - [m]_k)) & [l]_k + [m]_k > k 
\end{cases}
\]

One of the special features of this algebra is that by fusing vertex operators their $u(1)$ charge always adds up, while their $\mathbb{Z}_k$ charge only adds up modulo $k$. This means, for example, that the vacuum operator $1$ and the type-$k$ composite $\mathcal{V}_p$ have different $u(1)$ charge yet equal $\mathbb{Z}_k$ charge. The reason to emphasize this property is because from a CFT point of view it is the reason why these non-Abelian quantum Hall states develop zeros in the first place. To see this, consider the fusion of two operators $\mathcal{V}_l$ and $\mathcal{V}_m$ into a third, $\mathcal{V}_{l+m}$. If, for all these operators, the $u(1)$ charge equals the $\mathbb{Z}_k$ charge, then on the basis of the first case in (8.93) no zero is developed in this fusion process.

### 8.3.3 Pattern of zeros: exact expressions

On the basis of the operator algebra (8.93) the pattern of zeros $\{D_{ab}\}$ is easily derived. For that we need to identify the electron and composite operators. For the $\mathbb{Z}_k^{(p)}$ state the electron operator is identified with $\mathcal{V}_p$. Therefore the operator which represents the type-$a$ particle, i.e. an $a$-electron composite particle is identified as

\[
\begin{align*}
V_{\text{type-}a} &= \mathcal{V}_p \\
V_{\text{el}} &= \mathcal{V}_p
\end{align*}
\]

The pattern of zeros is now directly determined through the operator algebra (8.93) and (8.94) and the identity (8.58)

\[
D_{ab} = \begin{cases} 
\frac{2}{k} (abp^2 - [ap]_k [bp]_k) & \text{if } |ap|_k + |bp|_k \leq k \\
\frac{2}{k} (abp^2 - (k - [ap]_k)(k - [bp]_k)) & \text{if } |ap|_k + |bp|_k > k 
\end{cases}
\]

In particular, the “minimal input” is

\[
D_{a1} = \begin{cases} 
\frac{2}{k} (ap^2 - [ap]_k p) & \text{if } |ap|_k + p \leq k \\
\frac{2}{k} (ap^2 - (k - p)(k - [ap]_k)) & \text{if } |ap|_k + p > k 
\end{cases}
\]

Using the floor function, i.e. $|\frac{ap}{k}|$ is the largest integer $n$ such that $n \leq \frac{ap}{k}$, this is rewritten as

\[
D_{a1} = \begin{cases} 
2p|\frac{ap}{k}| & \text{if } |ap|_k + p \leq k \\
2p|\frac{ap}{k}| + 2((a + 1)p) & \text{if } |ap|_k + p > k 
\end{cases}
\]
8.4 $\bigotimes_i \mathbb{Z}_{k_i}^{(p_i)}$ - Pattern of zeros

To arrive at this expression we used $\frac{ap-[ap]_k}{k} = [\frac{ap}{k}]$. We have for instance

$$D_{ak,bk} = 2abkp^2$$

$$D_{k,1} = 2p^2$$

(8.100) (8.101)

The quantity $D_{k,1}$ is defined as $m$ in [Wen and Wang, 2008a]. The quantity $D_{k,k} = 2kp^2 = \frac{2k^2}{\nu}$ is defined as $L$ in this reference and in chapter 8.1.4.

Note that this expression for $D_{ab}$ satisfies

$$D_{ab} = D_{[a]_k|b]_k} + [a]_k[b]_k + \frac{ab}{\nu}$$

(8.102)

Which is the clustering condition on $D_{ab}$ as defined in chapter 8.1.4. We identify the first two terms with the pattern $d_{ab}$. One can easily verify that this expression satisfies all the identities of $d_{ab}$.

The other pattern of zeros is given by (8.62)

$$S_a = \frac{a(a-1)p^2}{k} - ap + [ap]_k + \frac{ap^2 - [ap]_k [ap]_k}{k}$$

(8.103)

We identify $s_a$ with the final three terms, as defined in chapter 8.1.4. Specifically, $s_k = -p(k-p) = -k\Delta_d$, which is an identity appearing in [Wen and Wang, 2008a].

The formulas for the pattern of zeros presented here are new and do not appear in the paper [Wen and Wang, 2008a]. As already mentioned their approach is more general as they numerically obtain the set $\{S_2, \ldots, S_k\}$ for $1 \leq k \leq 10$. Using Mathematica we have checked that indeed all their numerical results match these exact formulas.

8.4 $\bigotimes_i \mathbb{Z}_{k_i}^{(p_i)}$ - Pattern of zeros

We proceed with a similar treatment of the product quantum Hall states.

The states we are describing are the product quantum Hall states, denoted by $\mathbb{Z}_{k_1}^{p_1} \times \mathbb{Z}_{k_2}^{p_2} \times \ldots \times \mathbb{Z}_{k_n}^{p_n}$. The filling fraction is

$$\nu = \frac{1}{M + 2 \sum_i \frac{p_i^2}{k_i}}$$

(8.104)

To study its pattern of zeros we set $M + 0$. The state is defined through the set $\{p_i\} \equiv \{k_i, p_i\}$. We also define the $k_i$-modulo function

$$[l]_i \equiv l \mod k_i$$

(8.105)

With this notation we introduce the following operator

$$\psi_{(l)} = \left[ \prod_i \psi_{[l]_i} \right] e^{iV \sum_i \frac{\sigma^2}{k_i} \phi}$$

(8.106)

$$\Delta_{(l)} = \sum_i \frac{p_i^2}{k_i} + \sum_i [l]_i (k_i - [l]_i)$$

(8.107)

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The label of $\mathcal{V}_{\{l_i\}}$ corresponds to a set of integers $l_i$ - one for each $\mathbb{Z}_{k_i}$ theory appearing in the product quantum Hall state. This is a straightforward generalization of the operator introduced in the previous section. Notice that each parafermion has its own modulo function which resembles the modulo arithmetic behavior of each independent $\mathbb{Z}_{k_i}$ theory. We identify the electron and composite electron operators with

$$V_{\text{type } a} = \mathcal{V}_{\{a p_i\}}, \quad a \in \mathbb{Z}_{>0} \quad (8.108)$$

$$V_{\text{el}} = \mathcal{V}_{\{p_i\}} \quad (8.109)$$

Each parafermionic CFT has its own modulo arithmetic behavior. This reflects itself in the pattern of zeros and basically gives one big mess. To circumvent the mainly notational complications we introduce the following function

$$G(l,m,k_i) = \begin{cases} 2 [l_i][m_i] & [l_i] + [m_i] \leq k_i \\ \frac{(k_i-[l_i])(k_i-[m_i])}{k_i} & [l_i] + [m_i] > k_i \end{cases} \quad (8.110)$$

This function represents the contribution of each parafermionic theory to a fusion process. For the pattern of zeros we have

$$D_{ab} = 2ab \sum_i \frac{p_i^2}{k_i} - \sum_i G(a p_i, b p_i, k_i) \quad (8.111)$$

$$G(a p_i, b p_i, k_i) = \begin{cases} 2 \frac{[a p_i][b p_i]}{k_i} & [a p_i] + [b p_i] \leq k_i \\ \frac{(k_i-[a p_i])(k_i-[b p_i])}{k_i} & [a p_i] + [b p_i] > k_i \end{cases} \quad (8.112)$$

which is the pattern of zeros for the $\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n} = \mathbb{Z}_{\{k_i\}}$ clustered quantum Hall states. This pattern of zeros matches all criteria defined in chapter 8.1.4. Specifically, we have

$$D_{g,1} = 2g \sum_i \frac{p_i^2}{k_i} \quad (8.113)$$

where $g$ is the least common multiple of the set $\{k_i\}$. In the case where all $k_i$ are taken coprime with respect to each other it is simply the product of all $k_i$’s, $g = \prod_i k_i$. The number $g$ is the lowest number of electrons needed to fuse the parafermions of all $\mathbb{Z}_{k_i}$ contributions to the identity. The product quantum Hall states are therefore clustered states of order $g$.

From a CFT perspective the product quantum Hall states are formed through the use of multiple $\mathbb{Z}_{k_i}$ theories and the resulting CFT is a product of these theories. This relation carries over to the pattern of zeros in the following way. The pattern of zeros of the product quantum Hall states equals the sum of the pattern of zeros of the underlying $\mathbb{Z}_{k_i}$ theories. To make this explicit we write the pattern of zeros (8.112) as

$$D^i_{ab} = \sum_i D^i_{ab} \quad (8.114.a)$$

$$D^i_{ab} = 2ab \frac{p_i^2}{k_i} - G(a p_i, b p_i, k_i) \quad (8.114.b)$$
The definition of (8.114.b) matches with that of the pattern of zeros of a single $\mathbb{Z}_k$ theory (8.97). This is not too surprising. If two states are both clustered, then the product of the two states will also be a clustered state.

The pattern of zeros $S_a$ are easily recovered using this notion. It is the sum of the pattern of zeros of the separate $\mathbb{Z}_{k_i}$ theories.

$$S_a = \sum_i S^i_a \quad (8.115)$$

$$S^i_a = \frac{a(a - 1)p_i^2}{k_i} - ap_i + [ap_i]_i + \frac{ap_i^2 - [ap_i]_i[ap_i]_i}{k_i} \quad (8.116)$$

On the basis of the general definition we know that the Laughlin wavefunctions are also clustered states, although their clustering is trivial (“of order 1”). Still, it does mean that any clustered state of order $k$ can be multiplied with any Laughlin wavefunction and it still remains a clustered state of order $k$. The pattern of zeros add up. We have already used this property, since a quantum Hal wavefunction multiplied with a Laughlin-Jastrow factor is also a quantum Hall wavefunction.

$$\Psi \rightarrow \Psi \prod_{i<j} (z_i - z_j)^M \quad (8.117)$$

$$\nu' = \nu^{-1} + M^{-1} \quad (8.118)$$

This is the reason why we could safely restrict to $M = 0$. The higher $M$ cases can be viewed as products of clustered states.

Again, the formulas of the pattern of zeros do not appear in the literature.

### 8.5 The case of $k$ and $p$ not coprime

Up till now we only considered states of the type $\mathbb{Z}_k^{(p)}$, where $p$ and $k$ are taken coprime. What happens when they are not? We already mentioned that these cases are peculiar in the sense that they are still clustered state, but at a lower clustering order than $k$.

The conjecture is that such a state can always be described by a product state of the type $(\mathbb{Z}_s^{(q)})^n$, where $q$ and $s$ are taken coprime. To be completely precise, the content

$$\mathbb{Z}_k^{(p)}: \quad V_{el} = \psi_p e^{i \frac{1}{\sqrt{\nu'}} \varphi}, \quad \nu = \frac{k}{kM + 2p^2} \quad (8.119)$$

$$S = M + 2p\left(\frac{2p + k}{k}\right) \quad (8.120)$$

describes the $\mathbb{Z}_k^{(p)}$ state. Define $n$ as the smallest common divisor of $p$ and $k$. Since $p$ and $k$ are not coprime there exist positive integers $q > 0$ and $s > 0$ such that $nq = p$ and $ns = k$. The following content

$$(\mathbb{Z}_s^{(q)})^n: \quad V_{el} = (\psi_q)^n e^{i \frac{1}{\sqrt{s\nu}} \varphi}, \quad \frac{q}{s} = \frac{p}{k} \quad (8.122)$$

$$\nu = \frac{s}{sM + 2nq^2} \quad (8.123)$$

$$S = M + 2qn\left(\frac{2q + s}{s}\right) \quad (8.124)$$
specifies the \( (\mathbb{Z}_s^{(q)})^n \) state and describes the same quantum Hall state as the \( (\mathbb{Z}_s^{(q)})^n \) state. The first, crucial observation is the equality of the conformal dimensions

\[
\Delta_{Z_{k}^{(p)}} = \frac{p(k - p)}{k} + \frac{1}{2\nu} \\
\Delta_{(\mathbb{Z}_s^{(q)})^n} = \frac{nq(s - s)}{s} + \frac{1}{2\nu} = \frac{nq(ns - nq)}{ns} + \frac{1}{2\nu} = \Delta_{Z_{k}^{(p)}}
\]

It is on the basis of this equality that also the filling fraction and shift of both theories match. The fact that the conformal dimensions match already proves the equivalence between the pattern of zeros. For completeness, we will prove this here. The pattern of zeros which arises in the \( Z_{k}^{(p)} \) case is

\[
D_{ab} = \frac{2abp^2}{k} - G(ap, bp, k) \\
G(ap, bp, k) = \begin{cases} 
2 \frac{|ap|_k |bp|_k}{k} & [ap]_k + [bp]_k \leq k \\
(k - [ap]_k)(k - [bp]_k) & [ap]_k + [bp]_k > k
\end{cases}
\]

The pattern of zeros of the \( (\mathbb{Z}_s^{(q)})^n \) states, on the other hand, are given by.

\[
D_{ab} = 2ab \frac{nq^2}{s} - nG(aq, bq, s)
\]

Now note that \( \frac{nq^2}{s} = \frac{p^2}{k} \). Furthermore, as is well-known from modular arithmetic

\[
n[aq]_s = [anq]_{ns} = [ap]_k
\]

Therefore \( nG(aq, bq, s) = G(ap, bp, k) \). With this we have proven that the pattern of zeros of both states coincide.

The two states share the same properties such as filling fraction, pattern of zeros and shift. It is tempting to conclude that they are indeed the same. However, this conclusion would be too hasty. From the CFT point of view an equality in the pattern of zeros, shift and filling fraction, does not automatically imply an equality in terms of conformal dimensions. In this case the best way to see it is through the central charges as these are not equal

\[
c_{Z_{k}^{(p)}} = \frac{2k - 2}{k + 2} \\
c_{(\mathbb{Z}_s^{(q)})^n} = n \frac{2s - 2}{s + 2}
\]

Note that \( c_{Z_{k}^{(p)}} < c_{(\mathbb{Z}_s^{(q)})^n} \).

What in some sense happened is that the parafermionic operator which represents the electron in the \( \mathbb{Z}_s^{(p)} \) does not generate all the parafermions. That is, upon fusion of \( \psi_p \) with itself only \( q \) of the \( k \) parafermions are generated. Only a subalgebra is generated. It is not completely clear what the precise interpretation is of this discrepancy. It probably means that the \( Z_{k}^{(p)} \) CFT is able to generate the same pattern of zeros as the \( (\mathbb{Z}_s^{(q)})^n \) CFT, but is itself not a proper CFT to describe quantum Hall states with. A conjecture which we make is that the identification of the electron operator is illegal in the case of \( Z_{k}^{(p)} \). Unfortunately, we have no concrete indication of this.

In [Wen and Wang, 2008b] the authors pursue an approach based on modular tensor categories to extract more information from the pattern of zeros, such as the quasihole spectra. This method can also be applied here, although we have not pursued this method.
8.6 Example: $\mathbb{Z}_5^{(2)}$

This section is quite extensive in the number of definitions and properties that have come along. This is why we present an easy example in which we explicitly derive the pattern of zeros through simple fusion steps. The example is not completely pedagogical, as we will obtain some new results.

We consider the $\mathbb{Z}_5^{(2)}$ quantum Hall state. Its filling fraction is $\nu = \frac{5}{5M+8}$, with $\nu = \frac{5}{8}$ and $\nu = \frac{5}{13}$ being the first candidate for respectively a bosonic and a fermionic quantum Hall state. Its electron operator is given by

$$V_{el} = \psi_2 e^{i\sqrt{5M+8}q}$$

The pattern of zeros for the underlying bosonic wavefunction ($M = 0$) is given by

$$\{D_{11}, D_{21}, D_{31}, D_{41}\} = \{0, 2, 4, 4\}$$

$$\{S_2, S_3, S_4, S_5\} = \{0, 2, 6, 10\}$$

In addition we have $D_{22} = 4$ and $D_{32} = 6$. These cases saturate all possible ways of fusing (up to) 5 electrons together. Last but not least, we also have $D_{k1} \equiv D_{51} = 8$.

8.6.1 Pattern of zeros - an example

The bosonic wavefunction ($M = 0$) for 5 electrons is given by the following CFT correlator

$$\tilde{\Psi}^{(N_e = 5)}_{\mathbb{Z}_5^{(2)}}(z_1, \ldots, z_5) = \langle V_2(z_1) V_2(z_2) V_2(z_3) V_2(z_4) V_2(z_5) \rangle$$

For aesthetic reasons we have suppressed the background charge. It is crucial to keep in mind that it is still there though.

By applying the operator algebra we can determine the zeros of this 5-point correlator. The first step is fusing two electrons, for instance $z_1 \to z_2$. Since $D_{11}$ is zero, this gives

$$\lim_{z_1 \to z_2} \langle V_2(z_1) V_2(z_2) V_2(z_3) V_2(z_4) V_2(z_5) \rangle = \langle V_4(z_2) V_2(z_3) V_2(z_4) V_2(z_5) \rangle$$

This is our first derived polynomial, written in terms of a four-point correlator. The operator $V_4$ resembles the two-electron composite particle. An important aspect to keep in mind, is that the right hand side contains terms of order $O(z_1 - z_2)^2$ and higher. In the limit where $z_1 \to z_2$ these terms vanish, and we are left with the leading term.

We now have two choices

(i) fuse two other electrons together, say $z_3 \to z_4$. This corresponds to $D_{11} = 0$.

(ii) fuse the composite particle at $z_2$ with an electron, say $z_2 \to z_3$, which corresponds to $D_{21} = 0$.

In both cases we end up with a three point parafermion correlator. The leading term of limit (i) gives

$$\lim_{z_3 \to z_4} \langle V_4(z_2) V_2(z_3) V_2(z_4) V_2(z_5) \rangle = \langle V_4(z_2) V_4(z_3) V_2(z_5) \rangle$$

(8.138)
The final product is a three-point correlator. This means we are in business, as a three-point correlator is automatically fixed due to conformal invariance. For the parafermionic part we apply the general rule for three-point correlators, see equation (5.14). For the \( u(1) \) vertex part we apply (B.53). Inserting back the background term gives

\[
\lim_{w \to \infty} w^{5/3} e^{-i5\sqrt{5/3} \varphi(w)} e^{i3\sqrt{5/3} \varphi(z_2)} e^{i\sqrt{5/3} \varphi(z_4)} e^{i\sqrt{5/3} \varphi(z_5)} \langle \psi_4(z_2) \psi_4(z_4) \psi_2(z_5) \rangle = (z_2 - z_4)^6 (z_2 - z_5)^2 (z_4 - z_5)^2 \quad (8.139)
\]

In this final product the limits \( z_2 \to z_4, z_2 \to z_5 \) and \( z_4 \to z_5 \) are zeros of order 6, 2 and 2. These cases correspond to \( D_{22}, D_{21} \) and \( D_{21} \). This takes care of limit (i).

The leading term of limit (ii) \( (z_2 \to z_3) \) has a zero of order \( D_{21} = 2 \). The derived polynomial is

\[
\lim_{z_2 \to z_3} (z_2 - z_3)^{-2} \langle V_4(z_2) V_2(z_3) V_2(z_4) V_2(z_5) \rangle = \langle V_0(z_3) V_2(z_4) V_2(z_5) \rangle \quad (8.140)
\]

The right hand side contains two correlators, which we compute directly using CFT.

\[
\lim_{w \to \infty} w^{5/3} e^{-i5\sqrt{5/3} \varphi(w)} e^{i3\sqrt{5/3} \varphi(z_2)} e^{i\sqrt{5/3} \varphi(z_4)} e^{i\sqrt{5/3} \varphi(z_5)} \langle \psi_1(z_3) \psi_2(z_4) \psi_2(z_5) \rangle = (z_3 - z_4)^4 (z_3 - z_5)^4 \quad (8.141)
\]

In this product the limits \( z_3 \to z_4 \) (\( D_{31} \)), \( z_3 \to z_5 \) (\( D_{31} \)) and \( z_4 \to z_5 \) (\( D_{11} \)) are of order 4, 4 and 0 respectively. We have thus completely saturated all possible way of fusing the electrons together.

We also give a factorization formula, which was also used to characterize the behavior of Read-Rezayi states. Such a formula links the wavefunctions for \( N_e \) electrons to that of \( N_e - 5 \) electrons

\[
\lim_{z_4 \to z_5} (z_4 - z_5)^{-D_{41}} \left[ \lim_{z_3 \to z_4} (z_3 - z_4)^{-D_{31}} \left[ \lim_{z_2 \to z_3} (z_2 - z_3)^{-D_{21}} \left[ \lim_{z_1 \to z_2} \psi_{N_e}^{Z_5^{(2)}}(z_1, \ldots, z_{N_e}) \right] \right] \right] = \prod_{i=6}^{N_e} (z_5 - z_i)^{D_{51}} \psi_{Z_5^{(2)}}^{N_e-k}(z_6, \ldots, z_{N_e}) \quad (8.142)
\]

where \( D_{21} = 2, D_{31} = D_{41} = 4 \) and \( D_{51} = 8 \). We see that such an expression is quite obscure and not as simple as the Read-Rezayi case, where all \( D_{a1} \) are zero for \( a < k \).

This is the simplest example of different derived polynomials and the obtained pattern of zeros \( \mathbb{Z}_k^{(p)} \).

The results obtained do not provide us any new information on the structure of the \( \mathbb{Z}_5^{(2)} \) state. We already fully exploited the relevant information contained in the conformal dimensions of the electron and composite electron operators. Nevertheless, it is a useful exercise to get acquainted with the material and get some hands-on feeling of what is going on on the level of the wavefunctions themselves. Still, we obtained some explicit results for the three-point correlators which we will now turn to our advantage.
8.6.2 Exact Wavefunction

A particular useful aspect we have unveiled in the previous paragraph is an exact expression for the derived polynomial of three composite particles. Using these expressions and the order of zeros used to obtain them we are able to “reverse engineer” the process and come up with an exact expression for the 5-point wavefunction. The conjecture is that the following wavefunction obeys all the limits considered above, plus the fact that it is a symmetric polynomial.

\[ \Psi_{Z_5^{(2)}}^{(N_e=5)}(z_1, \ldots, z_5) = S_5 \left[ (z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_4)^2 (z_4 - z_5)^2 (z_5 - z_1)^2 \right] \]  

(8.143)

The operator \( S_5 \) is a properly normalized symmetrizer of the 5 coordinates.

**Proof** It is sufficient to show that all roots of this wavefunction coincide with the pattern of zeros. Let us check the limits one by one. The first limit we consider is by taking \( z_1 \to z_2 \). The leading term of this limit should be a constant. From the expression between the brackets in equation (8.143) we observe that the leading term is obtained by permuting \( z_2 \) with either \( z_3, z_4 \) or \( z_5 \). The remaining permutations are dropped, since these contain factors of \( (z_1 - z_2) \).

The resulting derived polynomial is

\[ \lim_{z_1 \to z_2} \Psi_{Z_5^{(2)}}^{(N_e=5)}(z_1, \ldots, z_3) = (z_2 - z_3)^4 (z_2 - z_4)^2 (z_4 - z_5)^2 (z_2 - z_5)^2 \]

+ \[ (z_2 - z_4)^4 (z_2 - z_3)^2 (z_3 - z_5)^2 (z_2 - z_5)^2 \]

+ \[ (z_2 - z_5)^4 (z_2 - z_3)^2 (z_2 - z_4)^2 (z_3 - z_4)^2 \]  

(8.144)

If our conjecture is correct (which it is) this expression is just that of equation (8.137). Let us press on, and consider the fusing of two of the remaining three electrons \( (z_3, z_4 \text{ and } z_5) \). Such a process would generate a constant leading term. Indeed, if, for instance, we let \( z_3 \to z_4 \) the third line vanishes, while the second and third line in equation (8.144) combine into

\[ (z_2 - z_3)^2 (z_4 - z_5)^2 (z_2 - z_4)^6 \]  

(8.145)

This polynomial is precisely that of equation (8.139). Since (8.144) is symmetric with respect to \( z_3, z_4 \) and \( z_5 \) we have successfully proven the first limits of the wavefunction.

We now move on to the other two limits. The first one is \( z_1 \to z_2 \) followed by \( z_2 \to z_3 \). The first limit leads to expression (8.144). For the second limit we observe that the second and third line in this expression contains a factor of \( (z_2 - z_3)^2 \), while the first has a factor of \( (z_2 - z_3)^4 \). This implies that the leading term has a pole of order 2 in the limit \( z_2 \to z_3 \), exactly as predicted. The remainder is our last derived polynomial

\[ (z_3 - z_4)^4 (z_3 - z_5)^4 \]  

(8.146)

This expression matches that of equation (8.140). And again, due to symmetry of the electron coordinates this limit holds for all other cases.

The equivalence can also be checked using the pattern of zeros \( \{S_a\} \). We have also checked all limits with Mathematica. The \( Z_5^{(2)} \) state is treated in [Barkeshli and Wen, 2008]. However, the wavefunction presented here does not appear in the literature.
8.6.3 General $k$ case

The expression for $k$ electrons in the $Z^{(2)}_k$ state is quite elegant. We have been able to generalize and proven it for arbitrary $k$ (with $k$ even). First, let us give the pattern of zero for the $Z^{(2)}_k$ case with $k = \text{even}$.

$$\{D_{11}, D_{21}, \ldots, D_{k-1,1}, D_{k1}\} = \{0, 0, \ldots, 0, 2, 4, 4, \ldots, 4, 8\}$$  \hfill (8.147)

Note that for $a = \frac{k-1}{2}$ we have $D_{a1} = 0$.

The exact wavefunction for $k$ electrons is given by the following expression

$$\Psi^{(2)}_{k} = S_k \left[ (z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_4)^2 \cdot \ldots \cdot (z_{k-1} - z_k)^2(z_k - z_1)^2 \right]$$  \hfill (8.148)

Here, the operator $S_k$ is a properly normalized symmetrizer, which symmetrizes over all $k$ coordinates.

**Proof** Observe that within the symmetrizer there are no terms of the type $(z_i - z_i + 2)$. We now fuse all odd-labeled coordinates together (at $z_1$), except for $z_k$. Since the odd-labeled coordinates form no zeros with respect to each other, the wavefunction remains non-zero with respect to this fusion process. That is, the majority of the terms appearing due to the symmetrizer will vanish. The leading term is still constant though.

We have formed a composite electron cluster containing $\frac{k-1}{2}$ electrons and the wavefunction remains non-zero. Due to the symmetrizer this can be accomplished for any $\frac{k-1}{2}$ electrons. All in all there are no zeros when $\frac{k-1}{2}$ electrons are fused together. It is also independent of the order of fusion. Therefore $D_{ab} = 0$ for $a + b < \frac{k+1}{2}$, and specifically $D_{a1} = 0$ for $a < \frac{k-1}{2}$.

The expression we are left with is a derived polynomial

$$P'(z_1, z_2, z_4, \ldots, z_{k-1}, z_k) = S_{\frac{k+1}{2}} \left[ (z_1 - z_{k-1})^2(z_{k-1} - z_k)^2(z_k - z_1)^2 \prod_{i=1}^{\frac{k-1}{2}} (z_1 - z_{2i})^4 \right]$$  \hfill (8.149)

To understand where this expression comes from, observe that in the fusion process products combine as

$$(z_i - z_{i+1})^2(z_{i+1} - z_i + 2)^2 \rightarrow (z_i - z_{i+1})^2(z_{i+1} - z_1)^2 = (z_1 - z_{i+1})^4$$  \hfill (8.150)

From the derived polynomial we press on, and fuse $z_k$ with $z_1$. Due to the symmetrizer we have terms that contain factors $(z_1 - z_k)$ of order 2 and of order 4. The leading term is identified with the case $D_{a1} = 2$ for $a = \frac{k-1}{2}$. The derived polynomial we are left with is

$$P''(z_1, z_2, z_4, \ldots, z_{k-1}) = S_{\frac{k+1}{2}} \left[ \prod_{i=1}^{\frac{k-1}{2}} (z_1 - z_{2i})^4 \right]$$  \hfill (8.151)

There are no factors of $(z_i - z_j)$ with $i$ and $j$ even, not even after symmetrization. This ensures that $D_{a1} = 4$ for $\frac{k-1}{2} < a < k - 1$ each time $z_1$ is fused with any other coordinate.
8.7 Exact wavefunction: $Z_k^{(p)}$, $rp + 1 = k$

Inspired by the example of the $Z_3^{(2)}$ state we now propose a generalized form of the exact wavefunction. We generalize in two ways. First, we consider the $Z_k^{(p)}$ state, where $p$ and $k$ satisfy the relation $rp + 1 = k$ for some integer $r$. Second, we will write down wavefunctions for $N_e$ electrons with $N_e$ a multiple of $k$.

The relevant data for the $M = 0$ state is

$$V_{el} = V_2$$
$$\nu = \frac{k}{2p^2}$$

(8.152)
(8.153)

The expression for the $M = 0$ bosonic wavefunction for $k$ electrons is given by the following correlator

$$\tilde{\Psi}^{(N_e = k)}_{Z_k^{(p)}}(z_1, \ldots, z_k) = \langle V_2(z_1) \cdots V_2(z_k) \rangle$$

(8.154)

One can generalize to positive $M$ by multiplying with an appropriate Laughlin-Jastrow factor. We have $rp + 1 = k$. This means we can fuse $r$ electrons together without creating any zeros. Upon fusing the $r + 1$'th electron the first zero is encountered. It equals

$$D_{r+1,1} = 2p\lfloor \frac{rp}{k} \rfloor + 2((r + 1)p - (rp + 1)) = 2p - 2$$

(8.155)

The composite particle that emerges from this process is obtained by fusing $r + 1$ electrons. Fusing an extra electron, and assuming $r + 2 < k$, gives

$$D_{r+1,1} = 2p\lfloor \frac{(r + 1)p}{k} \rfloor$$
$$= 2p$$

(8.156)
(8.157)

We iterate this process and obtain the pattern of zeros for the $rp + 1 = k$ case. It is the set $\{D_{11}, \ldots, D_{k1}\} = \{0, \ldots, 0, 2p - 2, 2p, \ldots, 2p, 4p - 4, 4p, \ldots, 2p^2 - 2, 2p^2\}$. For example

$Z_k^{(2)}: \{0, \ldots, 0, 2, 4, \ldots, 4, 8\}$, such that $k = 2r + 1$

(8.158)

$Z_k^{(3)}: \{0, \ldots, 0, 4, 6, \ldots, 6, 8, 12, \ldots, 12, 18\}$, such that $k = 3r + 1$

(8.159)

$Z_k^{(4)}: \{0, \ldots, 0, 6, 8, \ldots, 8, 12, 16, \ldots, 16, 18, 24, \ldots, 24, 32\}$, such that $k = 4r + 1$

(8.160)

In these pattern of zeros we notice a staircase-like behavior. Sequences of the same integer alternate with a single number. We can understand this behavior as follows. Recall the exact expression for the pattern of zeros

$$D_{a1} = \begin{cases} 2p\lfloor \frac{ap}{k} \rfloor & \text{if } [ap]_k + p \leq k \\ 2p\lfloor \frac{ap}{k} \rfloor + 2((a + 1)p)_k & \text{if } [ap]_k + p > k \end{cases}$$

(8.161)

The floor function is the cause of the sequences of the same integer. The modulo function is the cause of the small increasement.
To write down the exact wavefunction we follow a similar treatment as presented in [Cappelli et al., 2001] (see also chapter 6). That is, we introduce the notion of coloring. In the Read-Rezayi case we deal with a clustering of order $k$ and an equal number of colors, $k$. In this case we also have a clustering of order $k$, but we will only need $r$ colors (with $r$ defined as $rp + 1 = k$).

We have $N_e = nk$ electrons available. We divide these electrons into $r$ groups. However, we cannot evenly distribute the electrons among these groups. Instead, we will have $r - 1$ groups containing $np$ electrons each. We will denote an electron coordinate from such a group as $z_i^{(a)}$ where $1 \leq i \leq np$ and $1 \leq a \leq r - 1$. These groups of electrons are

$$\left( z_1^{(1)}, \ldots, z_{np}^{(1)} \right), \left( z_1^{(2)}, \ldots, z_{np}^{(2)} \right), \ldots, \left( z_1^{(r-1)}, \ldots, z_{np}^{(r-1)} \right)$$

(8.162)

The label (1) is called the color of the group and we have simply chosen a particular way of coloring the electrons.

There are still $n(p+1)$ electrons left. We group these together with the color of the group denoted by “0”. However, in addition we add a second coloring to the coordinates. That is, the electron coordinates carry three labels: $z_i^{(0,u)}$. The label 0 represents the color of the group, the label $u$ represents the subcolor within this group and the label $i$ in turn distinguishes the electrons with the same subcoloring. The number of subcolors is taken to be $p$, i.e. $1 \leq u \leq p + 1$, which means there are $c$ electrons per subcolor. The zero’th color contains the electrons

$$\left( z_1^{(0,1)}, \ldots, z_c^{(0,1)} \right), \left( z_1^{(0,2)}, \ldots, z_c^{(0,2)} \right), \ldots, \left( z_1^{(0,p+1)}, \ldots, z_c^{(0,p+1)} \right)$$

(8.163)

This brings the total amount of electrons to $(r - 1)np + np + n = nk = N_e$, as it should be.

Apart from this different coloring scheme we do take the same steps as in the original paper [Cappelli et al., 2001]. Namely, we now assign to all colors and subcolors a Laughlin factor. That is, to the $a$’th color we associate a factor

$$\prod_{1 < j}^{np} \left( z_i^{(a)} - z_j^{(a)} \right) D_{r+1,1} = 2p$$

(8.164)

While for the zero’th color we assign a Laughlin product which relates electrons from different subcolors

$$\prod_{u < v}^{p+1} \prod_{i,j=1}^{c} \left( z_i^{(0,u)} - z_j^{(0,v)} \right) D_{r,1} = 2p - 2$$

(8.165)

And a Laughlin product among the electrons from the same subcolor

$$\prod_{i,j=1}^{c} \left( z_i^{(0,u)} - z_j^{(0,u)} \right) D_{r+1,1} = 2p$$

(8.166)

The complete wavefunction is obtained by taking the product of all these Laughlin wavefunction

$$\Psi_{\mathcal{Z}^p}^{N_e}(\{z_i\}) = S_{N_e} \left[ \prod_{a=1}^{r-1} \prod_{i=1}^{np} \left( z_i^{(a)} - z_j^{(a)} \right)^2 p \prod_{u < v}^{p+1} \prod_{i,j=1}^{c} \left( z_i^{(0,u)} - z_j^{(0,v)} \right)^{2p-2} \prod_{u=1}^{p+1} \prod_{i=1}^{c} \left( z_i^{(0,u)} - z_j^{(0,u)} \right)^{2p} \right]$$

(8.167)
Unfortunately, we do not have fully rigorous proof that this wavefunction does indeed have the specified pattern of zeros. Explicit calculations show that this function is correct, at least for small values of \( k, p \) and \( N_e \). We have explicitly checked this for several cases. Therefore we conjecture it to be the correct wavefunction.

We must confess though that the notion of subcoloring is perhaps misleading. One can also view it from a different perspective. Namely, we can think of the electrons carrying the \( \langle 0, u \rangle \) as only carrying one color, \( \langle u \rangle \). Again, we assign a Laughlin product to each color. In addition, we also assign a Laughlin product among different colors \( \langle u \rangle, \langle v \rangle \). The most general form one can write down is then

\[
\prod_{u,v} \prod_{i<j} (z_{ij}^{(u)} - z_{ij}^{(v)})^{m_{u,v}} \prod_{i<j} (z_{ij}^{(u)} - z_{ij}^{(u)})^{m_{u,u}}
\] (8.168)

The wavefunction is then obtained by applying the symmetrizer. Notice that the conjectured wavefunction above is indeed such an expression.

A generalization to the general case \( rp + s = k \) has not been successful so far. We believe that we are on the right track of coloring the electrons, and assigning a Laughlin wavefunction to each color.

8.8 Discussion

There is one issue which we have not addressed so far. Up till now we have been determining the pattern of zeros on the basis of the operator algebra satisfied by the electron operator and its composites. The orders \( \{ D_{ab} \} \) can be written in terms of the scaling dimensions. The question is then, are the pattern of zeros sufficient to completely determine the corresponding conformal field theory? If this would not be the case, then the wavefunction we have found might not be unique and there could exist many linearly independent polynomials with the same pattern of zeros.

We already encountered this problem when we extracted the conformal dimensions from the pattern of zeros. It has become clear that the presence of a \( u(1) \) charge is a necessary requirement to obtain a unique set of conformal dimensions corresponding to the different types of fused electrons. But even in that case, we only end up with the conformal dimensions of the CFT.

To address the issue of uniqueness of the CFT it is imperative to understand that a CFT is completely determined by the operators present in the theory and the operator algebra among these operators. These in turn are fully controlled by the structure constants and scaling dimensions. The question is then, do the scaling dimensions uniquely define the structure constants of the CFT? There is no general answer to this question. Luckily for us it was already shown in the original paper [Fateev and Zamalodchikov, 1985] that by choosing the scaling dimensions of the simple currents to be \( \Delta_{\psi_l} = l(k - l)/k \), the structure constants of the \( \mathbb{Z}_k \) CFT are completely fixed.

This is not always the case though. In the same paper [Fateev and Zamalodchikov, 1985] the authors explored other possible CFT’s that also possess a \( \mathbb{Z}_k \) symmetry. In particular, they found that by setting \( \Delta_{\psi_l} = 2l(k - l)/k \) the corresponding structure constants are no longer uniquely determined, but rather depend on a continuous parameter \( \lambda \). This parameter induces a large variation of the operator algebra, all with the same scaling dimensions. Since correlation functions depend on both the structure constants and scaling dimensions,
the pattern of zeros are not sufficient to fully classify the corresponding polynomial. This means the wavefunction we have found is unique and completely determined by the scaling dimensions of the $\mathbb{Z}_k^{(p)}$ theory.

### 8.9 Numerical Results

In this chapter we present some numerical results for the simplest cases available. We have used the following formula

$$D_{ab} = \begin{cases} \frac{2}{k} (abp^2 - \lceil ap \rceil \lceil bp \rceil) & \text{if } \lceil ap \rceil + \lceil bp \rceil \leq k \\ \frac{2}{k} (abp^2 - (k - \lceil ap \rceil)(k - \lceil bp \rceil)) & \text{if } \lceil ap \rceil + \lceil bp \rceil > k \end{cases}$$

$$D_{a1} = \begin{cases} \frac{2}{k} (ap^2 - \lceil ap \rceil p) & \text{if } \lceil ap \rceil + p \leq k \\ \frac{2}{k} (ap^2 - (k - p)(k - \lceil ap \rceil)) & \text{if } \lceil ap \rceil + p > k \end{cases}$$

First off all, the Read-Rezayi series which, as we have seen, have a rather trivial pattern of zeros

$$\Phi_{\mathbb{Z}_k^{(p)}} : (\nu|S_2, \ldots, S_n) = \begin{cases} k \frac{1}{2} & (0, 0, \ldots, 0) \\ (D_{11}, \ldots, D_{k1}) = (0, \ldots, 0, 2) \end{cases}$$

The first $\mathbb{Z}_k^{(p)}$ states, for $p > 1$ and $p, k$ coprime.

| $\Phi_{\mathbb{Z}_k^{(p)}}$ | $\nu|S_2, \ldots, S_n$ | $(D_{11}, \ldots, D_{k1})$ |
|---------------------------|------------------------|--------------------------|
| $\Phi^2_{\mathbb{Z}_5}$   | $(\nu|S_2, \ldots, S_n) = \frac{5}{8}(0, 2, 6, 10)$ | $(D_{11}, \ldots, D_{k1}) = (0, 2, 4, 4, 8)$ |
| $\Phi^3_{\mathbb{Z}_7}$   | $(\nu|S_2, \ldots, S_n) = \frac{7}{18}(0, 4, 10, 18, 30, 42)$ | $(D_{11}, \ldots, D_{k1}) = (0, 4, 6, 8, 12, 12, 18)$ |
| $\Phi^3_{\mathbb{Z}_9}$   | $(\nu|S_2, \ldots, S_n) = \frac{9}{8}(0, 0, 0, 2, 6, 10, 14, 18)$ | $(D_{11}, \ldots, D_{k1}) = (0, 0, 0, 2, 4, 4, 4, 8)$ |
| $\Phi^4_{\mathbb{Z}_{10}}$| $(\nu|S_2, \ldots, S_n) = \frac{5}{9}(0, 0, 4, 10, 16, 24, 36, 48, 60)$ | $(D_{11}, \ldots, D_{k1}) = (0, 0, 4, 6, 8, 12, 12, 12, 18)$ |
| $\Phi^2_{\mathbb{Z}_{11}}$ | $(\nu|S_2, \ldots, S_n) = \frac{11}{8}(0, 0, 0, 0, 2, 6, 10, 14, 18, 22)$ | $(D_{11}, \ldots, D_{k1}) = (0, 0, 0, 0, 2, 4, 4, 4, 4, 8)$ |
| $\Phi^3_{\mathbb{Z}_{11}}$ | $(\nu|S_2, \ldots, S_n) = \frac{11}{18}(0, 0, 2, 8, 14, 20, 30, 42, 54, 66)$ | $(D_{11}, \ldots, D_{k1}) = (0, 0, 2, 8, 6, 6, 10, 12, 12, 12, 18)$ |
| $\Phi^4_{\mathbb{Z}_{11}}$ | $(\nu|S_2, \ldots, S_n) = \frac{11}{32}(0, 2, 10, 18, 30, 46, 62, 84, 108, 132)$ | $(D_{11}, \ldots, D_{k1}) = (0, 2, 8, 8, 12, 16, 16, 22, 24, 24, 32)$ |
| $\Phi^5_{\mathbb{Z}_{11}}$ | $(\nu|S_2, \ldots, S_n) = \frac{11}{50}(0, 8, 10, 16, 20, 24, 30, 32, 40, 40, 50)$ | $(D_{11}, \ldots, D_{k1}) = (0, 8, 10, 16, 20, 24, 30, 32, 40, 40, 50)$ |
Also available are the first non-coprime cases

\[
\Phi_{Z_4^2} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (\frac{1}{2}, \frac{1}{2}|0, 4, 8) \\
(D_{11}, \ldots, D_{k1}) = (0, 4, 4, 8)
\]
\[
\Phi_{Z_4^3} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (\frac{3}{4}, \frac{1}{3}|0, 0, 4, 8, 12) \\
(D_{11}, \ldots, D_{k1}) = (0, 0, 4, 4, 8)
\]
\[
\Phi_{Z_8} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (\frac{1}{4}, \frac{1}{2}|0, 8, 16, 32, 48, 72, 96) \\
(D_{11}, \ldots, D_{k1}) = (0, 8, 16, 16, 16, 24, 24, 32)
\]

\[
\Phi_{Z_6^2} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (\frac{1}{3}, \frac{1}{2}|0, 6, 12, 24, 36) \\
(D_{11}, \ldots, D_{k1}) = (0, 6, 6, 12, 12, 18)
\]
\[
\Phi_{Z_6^3} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (1, \frac{1}{3}|0, 0, 0, 4, 8, 12, 16) \\
(D_{11}, \ldots, D_{k1}) = (0, 0, 0, 4, 4, 4, 8)
\]
\[
\Phi_{Z_8^2} : (\nu, \frac{p}{k}|S_2, \ldots, S_n) = (\frac{1}{2}, \frac{1}{3}|0, 0, 6, 12, 18, 30, 42, 54) \\
(D_{11}, \ldots, D_{k1}) = (0, 0, 6, 6, 6, 12, 12, 18)
\]

Product states $\bigotimes_i Z_{k_i}^{(p_i)}$ can be obtained by simply adding the corresponding pattern of zeros of the $Z_{k_i}^{(p_i)}$ theories together. These numerical results match those of the paper [Wen and Wang, 2008a].
CHAPTER 9

Quasiholes

This chapter is intended only for one purpose. We determine the number of allowed quasihole sectors of the newly found quantum Hall states. This number is in fact equal to the ground-state degeneracy on the torus. The most important results are formulas (9.42) and (9.69).

9.1 Counting of sectors

The electron operators we have come across so far have always been of the following type.

\[
V_{el}(z) = \psi(z)e^{i\frac{1}{\nu}\phi(z)}
\]  

(9.1)

Specifically, it is composed out of a simple current \(\psi(z)\) and a \(u(1)\) vertex operator. The simple current operator carries some representation of an electrically neutral chiral algebra \(\mathcal{P}\). In addition, for the operator to represent a valid electron operator it has to carry a unit charge and integer or half-integer valued spin. The algebra \(\mathcal{P}\) forms, together with the \(u(1)\) current algebra, what is called a chiral algebra

\[
\mathcal{A} = \mathcal{P} \otimes u(1)
\]

(9.2)

When using CFT to describe a quantum Hall system it is this chiral algebra which is referred to. The Virasoro algebra, which is the algebra of conformal transformations, is contained in the enveloping algebra. The electron operator in turn is labeled by a unitary representation of this chiral algebra, which automatically ensures the operator carries a representation of the Virasoro algebra as well. The class of representations that can be carried by the electron operator is quite limited. For example, the electric neutral part must correspond to a
simple current which causes the fusion of the electron with any other operator to be unique. Furthermore, the electron operator is local with respect to itself and all other operators.

The Quasiholes are formed in a very similar way to the that of the electron. Each one is labeled by a unitary representation of the chiral algebra \( A \), which is why they also carry a representation of the Virasoro algebra. However, unlike the electron, their electrical neutral part is not limited to the class of simple currents.

\[
V_\gamma(w) = \Phi_\gamma(w)e^{iQ_\gamma \sqrt{\nu} \varphi(w)} \tag{9.3}
\]

A priori the quantity \( Q_\gamma \) can take on any value. Written this way it conveniently labels the electric charge carried by the quasihole. The operator \( \Phi_\gamma \) carries a unitary representation of the chiral algebra, but its fusion products with the remaining fields can have multiple channels. Furthermore, the fields need not be local with respect to each other - only the electron operator has this restriction.

How does locality translate to conformal field theory? Recall that the electron operator consists out of a simple current. One of its properties is its unique fusion property. The fusion of a simple current with any other field always results in a single unique fusion product. There is only one fusion channel available, therefore

\[
V_{el} \times V_\gamma(w) \sim (z - w)^{\Delta_\gamma - \Delta_{el} - \Delta_\gamma} V_\gamma(w) \tag{9.4}
\]

Relative locality translates into the requirement

\[
\Delta_\gamma' - \Delta_{el} - \Delta_\gamma = 0 \pmod{1} \tag{9.5}
\]

This has been one of the pillars which we have frequently used to construct the various quasihole and electron operators.

After this rather formal outlook, we are now in a better position of handling the original problem, counting the various types of quasiparticles present in a quantum Hall system. The idea is to find all possible operators of the type (9.3) satisfying the locality condition (9.5). Let us therefore fuse an electron with a quasihole.

\[
\left( \psi(z)e^{\frac{i}{\sqrt{\nu}} \varphi(z)} \right) \times \left( \Phi_\gamma(w)e^{iQ_\gamma \sqrt{\nu} \varphi(w)} \right) \sim (z - w)^{\delta h + \nu^{-1}Q_\gamma} [\psi \Phi_\gamma](w)e^{i(Q_\gamma + 1) \frac{1}{\sqrt{\nu}} \varphi(w)} \tag{9.6}
\]

where

\[
\delta h = \Delta_{[\psi \Phi_\gamma]} - \Delta_\psi - \Delta_\Phi_\gamma \tag{9.7}
\]

The requirement of locality (9.5) translates into the following formula for the quasihole’s charge

\[
Q_\gamma = \frac{C + \delta h}{\nu^{-1}}, \quad C \in \mathbb{Z}_{\geq 0} \tag{9.8}
\]

A first look at this formula suggests that the number of quasiholes is in fact infinitely large. This is indeed true, since \( C \) can take on arbitrary large values. Negative values are excluded. Such quasiholes lead to poles in the wavefunction which are unphysical and excluded from the Hilbert space.

Most of these quasiholes are related with each other. A lot of the quasiholes can be obtained from another quasihole by fusing with a certain number of electrons. For instance, the composite particles that we encountered in the previous section are all constructed through the consecutive fusion of multiple electrons. All the composite particles which arise this way
are Virasoro primary fields. We conclude that there are an infinite number of Virasoro primary fields.

But using the Virasoro algebra to distinguish the particles from each other is not completely the right way to approach the physics. In chapter 4 we described in what way a topological quantum field theory describes the effective field theory of quantum Hall fluids. In particular, the observables of these TQFT’s are given by Wilson loop operators. Upon quantization we are able to extract a conformal field theory which describes the (1+1)- or (0+2)-dimensional “edges” of the system. From a topological perspective the Wilson loops are distinct only through their braiding properties. Electron operators are topologically trivial, since they are constructed as local operators with respect to all operators present in the theory.

This is why we introduce equivalence classes of quasiholes. Two fields are defined to be equivalent if one can be obtained from the other by fusing with a number of electron operators

\[ V_\gamma \sim V_{\gamma'} \quad \leftrightarrow \quad V_\gamma \sim V_{\gamma'} (V_{\text{el}})^n \]  

(9.9)

These equivalence classes are called superselection sectors. What we are after is the number of equivalence classes of quasiholes or, equivalently, the number of superselection sectors. This equivalence relation allows us to restrict the domain of \( Q_\gamma \) to its principle domain, that is

\[ 0 \leq Q_\gamma < 1 \]  

(9.10)

This restriction can be made because the electron operator has charge one, so any quasihole is always equivalent to one with a charge that falls inside this domain. For instance, the 'bare' vacuum operator \( 1 \) has zero charge therefore making it equivalent to the electron operator. From now on we will simply call these equivalence classes the quasiholes.

From a CFT point of view this identification is quite natural to consider. The equivalence classes of quasiholes are known as sectors, and they arise as chiral algebra-analogous of conformal families. A conformal family is the set of a Virasoro primary field along with all its Virasoro descendants. These descendants are obtained by fusing the primary field with the energy-momentum tensor. It is the energy-momentum tensor which generates the CFT symmetry. In the chiral algebra we also deal with the extended symmetry generator, which together with the energy-momentum tensor generates the symmetry of the chiral algebra.

In the case of the chiral algebra the role of the symmetry generator is fulfilled by the electron operator. Therefore by applying the electron operator to a field which is primary with respect to the chiral algebra we generate all its descendants. The primary fields can be considered as the 'fundamental quasihole types'. The set of such a primary together with its (chiral algebra) descendants are the equivalence classes of quasiholes. In addition, the electron operator can be obtained by fusing the electron operator with the vacuum. It is therefore not a primary field with respect to the chiral algebra.

The number of superselection sectors is a topological quantum number. It is interpreted as the groundstate degeneracy which arises when the system is put on the torus in the absence of quasiholes. As treated in chapter 4 non-trivial topologies lead to the existence of non-trivial topological quantum numbers.
9. Quasiholes

solutions of the ground state, even for Abelian TQFT’s. This is another example of the peculiar properties of topological field theories. This is a well-known identity from conformal field theory, which goes back to the paper [Verlinde, 1988]. For a related discussion on the groundstate degeneracy on the torus see [Oshikawa et al., 2007]

9.1.1 Laughlin states

The easiest example is that of our main example, the Laughlin states. For filling fraction $\nu^{-1} = M \in \mathbb{Z}_{>0}$ the electron operator is

$$V_{el} \equiv V_M = e^{i\sqrt{M}\varphi} \quad (9.11)$$

Other operators present in the theory are anyonic operators; they represent the quasiholes.

$$V_m = e^{im\sqrt{M}\varphi}, \quad m \in \mathbb{Z}_{\geq 0} \quad (9.12)$$

These operators are the only type of operators which are single-valued and non-singular with respect to the electron operator. The fusing of an electron with a quasihole $V_m$ results in the operator $V_{m+M}$. We therefore have the following quasihole sectors

$$V_m, \quad m \in \{0, \ldots, M - 1\} \quad (9.13)$$

The number of quasiholes therefore equals

$$\#[M]_{\text{Laughlin}} = M \quad (9.14)$$

This is the degeneracy of the ground state on the torus, see for instance [Oshikawa et al., 2007].

9.2 $\mathbb{Z}_k^{(p)}$ - Quasiholes and sector counting

In this chapter we turn our attention to the new $\mathbb{Z}_k^{(p)}$ states. We algebraically derive the number of quasihole equivalence classes, referred to as simple the number of quasiholes, which we know is equal to the ground state degeneracy on the torus.

The algebra is a bit obscure, so we try to present it as clear as possible. This section is basically a long derivation of a simple expression for the ground state degeneracy, which can be found at the end of this chapter. The approach is the same as treated for in [Barkeshli and Wen, 2008]. However, they ultimately solve the problem on a computer, while the steps take here are all done analytical.

9.2.1 Operator algebra and field identification

Before we come to counting of the fields we first recall some known properties of the $\mathbb{Z}_k$ theory. This is already treated extensively in chapter 6.1, so we merely present the most important results here.

The primary sectors in the $\mathbb{Z}_k$ theory are defined through the following field identifications, see for instance [Di Francesco et al., 1995, chapter 18.5]

$$\Phi^l_m = \Phi^l_{m+2k} \quad (9.15)$$

$$\Phi^l_m = \Phi^l_{k+l} \quad (9.16)$$

$$l - m = 0 \mod 2 \quad (9.17)$$
There in total \( \frac{1}{2}k(k+1) \) primary fields. The conformal dimension of the field \( \Phi_{lm}^l \) is given by

\[
\Delta_{lm}^l = \frac{(k-l)}{2k(k+2)} + \frac{(l-m)(l+m)}{4k}, \quad l \in \{1, \ldots, k\} \quad m \in \{-l+2, \ldots, l\}
\]  

Note that we restrict the range of the labels. If we take a field with labels which fall outside of these ranges, we invoke the field identification (9.17) to obtain an equivalent field which does carry the proper labels. The two fields are equivalent, and therefore have the same conformal dimensions. The conformal dimension of any field \( \Phi_{lm}^l \) is given by

\[
\Delta_{\Phi_{lm}} = \Delta_{lm} + n_{lm}
\]

where \( n_{lm} \) is an integer.

Of particular importance are the spin fields \( \sigma_l \equiv \Phi_{ll}^l \) with conformal dimension

\[
\Delta_{\sigma_l} = \frac{l(k-l)}{2k(k+2)}
\]

and the parafermions \( \psi_i \equiv \Phi_{02i}^0 \) with conformal dimension

\[
\Delta_{\psi_i} = \frac{i(k-i)}{k}.
\]

9.2.2 Fusing electrons and quasiholes

For the \( \mathbb{Z}_k(p) \) quantum Hall states the electron operator and filling fraction are

\[
\begin{align*}
V_{el} &= \Phi_{2pe}^0 e^{i\sqrt{\nu} \phi} \\
\nu &= \frac{1}{M + \frac{2pe}{k}}
\end{align*}
\]  

While the quasiholes are represented by

\[
V_\gamma = \Phi_{me}^l e^{iQ_\gamma \sqrt{\nu} \phi}, \quad l \in \{1, \ldots, k\} \quad m \in \{-l+2, \ldots, l\}
\]

The operator product expansion of an electron and a quasihole is given by

\[
\left( \Phi_{2pe}^0 e^{i\sqrt{\nu} \phi} \right) (z) \left( \Phi_{me}^l e^{iQ_\gamma \sqrt{\nu} \phi} \right) (w) = (z-w)^{C_\gamma} \Phi_{m+2pe}^l e^{i(Q_\gamma+1)\frac{1}{\nu}}
\]

where

\[
C_\gamma = \delta \Delta + Q_\gamma \frac{1}{\nu},
\]

\[
\delta \Delta = \Delta_{\phi_{m+2p}} - \Delta_{\phi_m} - \Delta_{\phi_p}
\]

On the basis of these OPE’s we want to determine all allowed values of the quasihole’s charge \( Q_\gamma \), such that the resulting quasihole is relatively local with respect to the electron operator. The quasihole’s charge is given by

\[
Q_\gamma = \frac{C_\gamma + \delta \Delta}{\nu^{-1}}
\]

The constant \( C_\gamma \) is the exponent which determines wether the two operators are relatively local. We require it to be integer and non-negative. Furthermore, based on the discussion of the beginning of this chapter we also restrict the domain of the allowed charges.

\[
0 \leq Q_\gamma < 1 \quad C \in \mathbb{Z}_{\geq 0}
\]
The program is now clear. The exponent $C_\gamma$ takes on non-negative integer values. Given $C_\gamma$, we can compute the quasihole’s charge through (9.26) and (9.25). The question is, what values of $C_\gamma$ are allowed such that the range (9.27) is satisfied? For that we need to compute $\delta \Delta$.

The quantity $\delta \Delta$ is determined through three conformal dimensions. One of those, namely $\Delta_{\phi^0_{2p}}$ is easily determined. However, to identify the conformal dimensions of the fields $\Phi^l_{m+2p}$ and $\Phi^l_m$ we have to be careful. This is due to the field identification among the $\mathbb{Z}_k$-fields. Given a field $\Phi^l_m$ we know that its conformal dimension is given by (9.18) if and only if the labels $l$ and $m$ fall inside the proper range. If they do not, we use the field identifications (9.17) to find the equivalent field which does have the proper labels.

Those same field identifications allow us to modify the “unit cell” of $l$ and $m$. As will become clear later on, for this derivation it is convenient not to work with the standard range as appearing in (9.18), but rather with a block-form which we define now. We replace the fields with $m \leq 0$ by the field $\Phi^k_{-|m|}$. The ranges of $l$ and $m$ change accordingly

\begin{align}
&l \in \{1, \ldots, k\} & l \in \{0, \ldots, k\} \\
&m \in \{-l+2, \ldots, l\} & m \in \{1, \ldots, k\} \\
&(l-m) = 0 \mod 2 & (l-m) = 0 \mod 2
\end{align}

(9.29) (9.30) (9.31)

Figure (9.1) shows this identification graphically.

For each field $\Phi^l_m$, represented by a black dot in figure (9.1), we apply the fusion rule (9.23). We end up with the field $\Phi^l_{m+2p}$. To determine the allowed charges we need

\[
\delta \Delta = \Delta_{\phi^l_{m+2p}} - \Delta_{\phi^l_m} - \Delta_{\phi^0_{2p}}
\]

(9.32)

The conformal dimension of the electron operator is given by $\Delta_{\phi^0_{2p}} = \frac{p(k-p)}{k}$. The conformal dimension $\Delta_{\phi^l_m}$, e.g. the second term, fall into two cases

\[
\Delta_{\phi^l_m} = \begin{cases} 
\Delta^l_m = \Delta^l_l + \frac{(l-m)(l+m)}{4k} & 1 \leq m \leq l \\
\Delta^{k-l}_{k-m} = \Delta^l_l + \frac{(2k-l-m)(m-l)}{4k} & l < m \leq k
\end{cases}
\]

(9.33)

where $\Delta^l_l$ is the conformal dimension of a spin field $\sigma_l$, and we have made use of the fact that $\Delta^l_l = \Delta_{\sigma_l}$. This follows directly from the unit cell we have been using.

**Figure 9.1:** Left and right show two different choices of unit cells among the $\mathbb{Z}_5$ fields. Both cells contain one primary field for each $\mathbb{Z}_5$ sector. The two cells are related via field identifications.
The conformal dimension of the resulting field \( \Delta_{\Phi l m+2p} \) takes on three different expressions:

\[
\Delta_{\Phi l m+2p} = \begin{cases} 
\Delta_l + \frac{(l-m-2p)(l+m+2p)}{4k} & -l < m + 2p \leq l \quad (a) \\
\Delta_{l-m+2p-k} + \frac{2k-l-m-2p}{4k} & l < m + 2p \leq -l + 2k \quad (b) \\
\Delta_{l-m+2p-2k} + \frac{(l-m-2p+2k)(l+m+2p-2k)}{4k} & 2k - l < m + 2p \leq l + 2k \quad (c)
\end{cases}
\]

We now fill these three cases, the two cases in expression (9.33) and the conformal dimension of the electron, into the general expression for \( \delta \Delta \), equation (9.32). This means we must consider six cases in total. However, the combination of (ii) and (a) is excluded as they are not compatible (just compare the ranges of \( m \)). A bit of algebra reduce \( \delta \Delta \) into the following form:

\[
\delta \Delta = -n'_l - \left[ \frac{mp}{k} \right], \quad \left[ \frac{mp}{k} \right] \equiv mp \mod k
\]

(9.35)

Where \( n'_l \) is an integer, given by

\[
n'_l = \begin{cases} 
\left\lfloor \frac{mp}{k} \right\rfloor + p & (ia) \\
\left\lfloor \frac{mp}{k} \right\rfloor + \frac{1}{2}(l - m) & (ib) \\
\left\lfloor \frac{mp}{k} \right\rfloor + k - m - p & (ic) \\
\left\lfloor \frac{mp}{k} \right\rfloor & (iib) \\
\left\lfloor \frac{mp}{k} \right\rfloor + k + p + \frac{1}{2}(m - l) & (iic)
\end{cases}
\]

(9.36)

Recall that \( (l - m) \) is always even, so that these are indeed all integers.

Figure (9.2) shows an example of how the parafermionic part of the electron operator acts on the fields present in the \( \mathbb{Z}_2^{(2)} \) theory and how the different cases come into play. For instance fusion of \( \psi_5 \), (case (i)) with the electron, \( \psi_2 \), results in \( 1 \), (case (a)). The integer \( n'_l \) is thus determined by case (ia). Likewise, fusing the electron with \( \sigma_5 \) is an example of (ib); \( \sigma_6 \) of (ic); \( \psi_1 \) of (iib) and \( \Psi_7^2 \) of (iic).

### 9.2.3 Quasihole charges

We are now in a position to determine the allowed quasihole charges. For that we use fill the expression for \( \delta \Delta \) (9.35) into the expression for the charges (9.26). Together with the restrictions on \( C_\gamma \) and \( Q_\gamma \), we have

\[
Q_\gamma = \frac{C_\gamma + n'_m + \left\lfloor \frac{mp}{k} \right\rfloor + \frac{2p^2}{k}}{M + \frac{2p^2}{k}}, \quad C_\gamma \in \mathbb{Z}
\]

(9.37)

\[
0 \leq Q_\gamma < 1
\]

(9.38)

Note that we restrict \( C_\gamma \) to integer values, but do allow it to take on negative-values as well. We will come back to this matter.

To comply with the range of \( Q_\gamma \) we need the numerator in (9.37) to be positive and smaller than the denominator. The integer \( n'_m \), (9.36), does not effect the number of allowed charges.
which satisfy the above relations. What does effect the number of allowed charges is the fractional value of \( \frac{mp}{k} \). There are two cases that vary in the number of allowed charges

\[
C_{\gamma} + n_m^l \in \begin{cases} 
\{0, \ldots, M + \lfloor \frac{2p^2}{k} \rfloor \} & [mp] < \lfloor 2p^2 \rfloor \\
\{0, \ldots, M + \lfloor \frac{2p^2}{k} \rfloor - 1 \} & [mp] \geq \lfloor 2p^2 \rfloor 
\end{cases}
\]  \hspace{1cm} (9.39)

With this information in hand we are able to compute the allowed charges of all quasiholes present in the system, and as a corollary the number of quasihole sectors.

But there is still an issue to resolve here. Namely, the exponent \( C_{\gamma} \) is not allowed to take on negative values. This leads to singular OPE’s between the electron and the quasihole operators. Still, such a quasihole represents an equivalence class of quasiholes. By fusing with an electron we can always obtain a quasihole which has a charge \( Q_{\gamma} \) larger than 1. We conclude that not all “primitive” quasiholes have a charge that lies within the range \( 0 \leq Q_{\gamma} < 1 \).

### 9.2.4 Number of quasihole sectors

What remains is determining the number of quasihole sectors in the combined \( \mathbb{Z}_k \times u(1) \) theory. From (9.39) it follows that the number of quasihole sectors that are formed by use of the field \( \Psi_m^l \) depends only on \( m \):

\[
\text{# quasiholes formed with } \Psi_m^l = \begin{cases} 
M + \lfloor \frac{2p^2}{k} \rfloor + 1 & [mp] < \lfloor 2p^2 \rfloor \\
M + \lfloor \frac{2p^2}{k} \rfloor & [mp] \geq \lfloor 2p^2 \rfloor 
\end{cases}
\]  \hspace{1cm} (9.40)

where \( m \in \{1, \ldots, k\} \). The total number of \( \Psi_m^l \) fields is \( \frac{1}{2}k(k+1) \). We need to determine how many of those fields form one extra quasihole sector, i.e. the number of fields that satisfy \([mp] < \lfloor 2p^2 \rfloor \). The derivation is a bit cloudy, so we first give the answer

\[
\text{# fields which satisfy } [mp] < \lfloor 2p^2 \rfloor = \frac{1}{2}(k+1)[2p^2] 
\]  \hspace{1cm} (9.41)
Proof

Recall that \( m \in \{1, \ldots, k\} \). Based on this we first note that the number of values of \( m \) which satisfy \( |mp| < [2p^2] \) equals \( [2p^2] \). Next we need the number of \( \Phi_m \) fields for a given value of \( m \). This is determined through what values the label \( l \) takes on. For \( m = \text{odd} \) we have \( l \in \{1, 3, \ldots, k\} \) and for \( m = \text{even} \) we have \( l \in \{0, 2, \ldots, k-1\} \). In the \( \mathbb{Z}_k \) case where \( k \) is odd there are thus \( \frac{1}{2}(k+1) \) fields for each value of \( m \). This gives \( \frac{1}{2}(k+1)[2p^2] \) fields which satisfy \( |mp| < [2p^2] \).

For \( \mathbb{Z}_k \) with \( k \) even the situation is a bit more subtle. There are now \( \frac{k}{2} + 1 \) fields for \( m = \text{even} \) and \( \frac{k}{2} \) for \( m = \text{odd} \). Since \( k \) is even the value of \([2p^2]\) is also even (in the case of \( k = \text{odd} \) \([2p^2]\) can be either odd or even). Furthermore, if \( m \) is odd/even, then so is \(|mp|\). In conclusion, there are \([2p^2]\) values of \( m \in \{1, \ldots, k\} \) which satisfy the relation \(|mp| < [2p^2]\). Half of those values of \( m \) are even. This brings the total number of fields which satisfy \(|mp| < [2p^2]\) to:
\[
\frac{1}{2}\left(\frac{k}{2} + 1\right)[2p^2] + \frac{k}{2}[2p^2] = \frac{1}{2}(k+1)[2p^2].
\]
This completes the proof.

We now combine the results (9.40) and (9.41). Using the relation \( k[2\frac{p^2}{k}] + [2p^2] = 2p^2 \) the two expression combine into the compact form
\[
\# \{k, p, M\}_{\mathbb{Z}_k^{(p)}} = \frac{1}{2}(k+1)(kM + 2p^2)
\]
(9.42)

This the number of quasihole equivalence classes for the \( \mathbb{Z}_k^{(p)} \) cases. For \( p = 1 \) we obtain the results for the Read-Rezayi states as appearing for instance in [Ardonne, 2002]. In the figures below three cases are presented of the distribution of quasihole sectors with respect to the \( \mathbb{Z}_k \) theory, \( \mathbb{Z}_5^{(2)} \), \( \mathbb{Z}_7^{(2)} \) and \( \mathbb{Z}_7^{(3)} \). Not all of these fields satisfy \( C_\gamma > 0 \), though.

This result also appears in the recent paper [Barkeshli and Wen, 2008]. Their approach is the same, in the sense that they have the same expression for the quasihole charge in terms of the filling fraction and conformal dimensions. However, ultimately their derivation is done on a computer while the derivation here is algebraic. Also, expression (9.40) for the degeneracy per \( \mathbb{Z}_k \) field does not appear in this article.

Furthermore, also in [Wen and Wang, 2008b] this result is recovered. Quite remarkably, the result is obtained in quite a different manner, based on modular tensor categories.

Let us address the non-coprime cases once more, treated in chapter 8.5. Recall that in these cases the parafermion associated with the electron does not generate all the parafermions present in the corresponding \( \mathbb{Z}_k \) theory. Upon fusing of the electron, the parafermions we obtain are \( \psi_p, \psi_{2p}, \ldots, \psi_{k-p} \). Since \( p \) and \( k \) are coprime we obtain a subset of the complete set of algebras.

The fact that we deal with a subalgebra of the parafermions extends to the quasihole sectors as well. In this chapter we have determined the maximal number of quasihole sectors we can form. But it is very well possible that a subset of these sectors form a subalgebra in the \( \mathbb{Z}_k \) part of theory. Indeed, the general fusion rule of two \( \mathbb{Z}_k \) primaries is
\[
\Phi_m^l \times \Phi_m^{l'} = \sum_{l''} \Phi_m^{l''} \]  
(9.43)

That is, the labels \( m \) and \( m' \) add up - just like the parafermions. Therefore the number of quasiholes appearing in the non-coprime cases is smaller than formula (9.42) gives.
9.3 $\bigotimes_{i=1}^{n} \mathbb{Z}_{k_i}^{(p_i)}$ - Quasiholes and sector counting

The lengthy derivation of the previous section paid off in the resulting expression of the number of quasiholes. The second type of generalization that we have come across are the product states. We will generalize the results obtained in the previous section and come up with expressions for the quasiholes, smallest charge operator and the number of quasihole sectors. We will be a bit blunt in the algebra, since the steps are essentially the same.

The states are denoted as $\bigotimes_{i=1}^{n} \mathbb{Z}_{k_i}^{(p_i)}$, and are characterized by the set $\{k_i, p_i\}$. Let us
repeat the expressions for the electron operator and filling fraction

\[ V_{\text{el}} = \prod_i \psi_{p_i} e^{i \frac{1}{\sqrt{\nu}} \varphi} \]  
\[ \nu^{-1} = M + 2 \sum_i \frac{p_i^2}{k_i} \]  

Quasihole operators consist out of the product of \( Z_k \) primaries together with a \( u(1) \) vertex operator

\[ V_{\gamma} = \prod_i \Phi_{l_i m_i}^{l_i} \]  
\[ \nu \frac{1}{\nu} = M + 2 \sum_i \frac{p_i^2}{k_i} \]  

The field \( \Phi_{l_i m_i}^{l_i} \) denotes the primary field \( \Phi_{l_i m_i} \) from the \( Z_k \) theory. Quasiparticles are thus labeled by their charge and the set \( \{l_i, m_i\} \). From now on we will write a quasihole in terms of their labels, that is

\[ V_{\gamma} \equiv \{Q_{\gamma}; \{l_i, m_i\}\} \]  

Alternatively, \( \gamma \) will sometimes be used as an abstract label to denote such a quasiparticle.

### 9.3.1 Fusion of quasiparticles

We repeat all steps taken in the previous section, and generalize them accordingly. The fusion of two quasiparticles is dictated by the fusion of each underlying \( Z_k \) theory. Specifically, in terms of the just-defined labels the fusion rules satisfy

\[ \{Q_{\gamma}; \{l_i, m_i\}\} \times \{Q'_{\gamma}; \{l'_i, m'_i\}\} = \prod_{i \in A} \left[ \sum_{l''_i = \max(0, |l_i - l'_i|)}^{\min(l_i + l'_i, 2k_i - l_i - l'_i)} \right] \{Q_{\gamma} + Q'_{\gamma}; \{l''_i, m''_i\}\} \]  

In particular, the operator product expansion with the electron operator is

\[ \{Q_{\gamma}; \{l_i, m_i\}\} \times \{1; \{0, 2p_i\}\} \sim z^C \{Q_{\gamma} + 1; \{l_i, m_i + 2p_i\}\} \]  
\[ C = \frac{Q_{\gamma}}{\nu} + \delta \Delta \]  
\[ \delta \Delta = \Delta_{\{Q_{\gamma} + 1; \{l_i, m_i + 2p_i\}\}} - \Delta_{\{Q_{\gamma}; \{l_i, m_i\}\}} - \Delta_{\{1; \{0, 2p_i\}\}} \]

For the electron to be single-valued we require \( C \) to be integer-valued.

Also applicable are the field identifications for each \( Z_k \) theory.

\[ \{Q_{\gamma}; \{\cdots, l_i, m_i, \cdots\}\} \sim \{Q_{\gamma}; \{\cdots, k_i - l_i, l_i - m_i, \cdots\}\} \]  
\[ \sim \{Q_{\gamma}; \{\cdots, l_i, m_i - 2k_i, \cdots\}\} \]  

By use of these field identifications a quasiparticle carrying labels \( \{Q_{\gamma}; \{l'_i, m'_i\}\} \) is equivalently described by \( \{Q_{\gamma}; \{l_i, m_i\}\} \) where the range of all \( m_i \) and \( l_i \) fall inside \( m_i \in \{1, \ldots k_i\} \) and

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\( l_i \in \{0, \ldots, k_i \} \). In terms of these labels the conformal dimension of the combined \( \mathbb{Z}_{k_i} \) part of the quasihole (excluding the \( u(1) \) part) is expressed as

\[
\Delta\{Q_\gamma;\{l_i,m_i\}\} = \sum_i \Delta\{l_i,m_i\}
\]

(9.54)

\[
\Delta\{l_i,m_i\} = \begin{cases} 
\Delta_{l_i}^i + \frac{(l_i-m_i)(l_i+m_i)}{4k_i} & 1 \leq m_i \leq l_i \\
\Delta_{l_i}^i + \frac{(2k_i-l_i-m_i)(m_i-l_i)}{4k_i} & l_i < m_i \leq 2k_i - l_i 
\end{cases}
\]

(9.55)

where \( \Delta_{l_i}^i \) is the conformal dimension of the spin field \( \sigma_{l_i} \). Fusing an electron with a quasihole which labels fall inside this range, results in the quasihole described by \( \{Q_\gamma+1;\{l_i,m_i+2pi\}\} \).

Its labels may fall outside the ranges considered above, therefore a proper field identifications has to be employed. The conformal dimension of this quasihole is expressed as

\[
\Delta\{Q_\gamma+1;\{l_i,m_i+2pi\}\} = \sum_i \Delta\{l_i,m_i+2pi\}
\]

(9.56)

\[
\Delta\{l_i,m_i+2pi\} = \begin{cases} 
\Delta_{l_i}^i + \frac{(l_i-m_i-2pi)(l_i+m_i+2pi)}{4k_i} & -l_i < m_i + 2pi \leq l_i \\
\Delta_{l_i}^i + \frac{(2k_i-l_i-m_i-2pi)(m_i+2pi-l_i)}{4k_i} & l_i < m_i + 2pi \leq l_i + 2k_i \\
\Delta_{l_i}^i + \frac{(l_i-m_i-2pi+2k_i)(l_i+m_i+2pi-2k_i)}{4k_i} & 2k_i - l_i < m_i + 2pi \leq l_i + 2k_i
\end{cases}
\]

(9.57)

Combining the results gives an expression for \( \delta \Delta \)

\[
\delta \Delta = \sum_i -n_i^{m_i} - \frac{[m_ip_i]_{k_i}}{k_i}
\]

(9.58)

where \( n_i^{m_i} \) is an integer, namely

\[
n_i^l = \begin{cases} 
\left\lfloor \frac{mp}{k} \right\rfloor + p & (ia) \\
\left\lfloor \frac{mp}{k} \right\rfloor + \frac{1}{2}(l - m) & (ib) \\
\left\lfloor \frac{mp}{k} \right\rfloor + k - m - p & (ic) \\
\left\lfloor \frac{mp}{k} \right\rfloor & (iib) \\
\left\lfloor \frac{mp}{k} \right\rfloor + k + p + \frac{1}{2}(m - l) & (iic)
\end{cases}
\]

(9.59)

These results, singlevaluedness and expression (9.26) lead to the following expression for the quasihole’s charge

\[
Q_\gamma = \frac{C + \sum_i \left( n_i^{m_i} + \frac{[m_ip_i]_{k_i}}{k_i} \right)}{M + \sum_j \frac{2p_j}{\kappa_j}} \quad 0 \leq Q_\gamma < 1
\]

(9.60)

The fractions appearing in the summation in both the numerator and denominator are fractional valued. However, the sum of these fractions is generically larger than 1. We need an expression appearing in the numerator and denominator of the type 'integer' + 'fraction', where the fraction is positive and < 1. For that we introduce a constant \( K \)

\[
K = \prod_i k_i
\]

(9.61)
This constant allows us to rewrite the denominator and numerator as follows

\[
\sum_i \frac{[m_ip_i]}{k_i} = \left[ K \sum_i \frac{[m_ip_i]}{k_i} \right] + \left[ \frac{K \sum_i \frac{[m_ip_i]}{k_i}}{K} \right] K \tag{9.62}
\]

\[
\sum_j \frac{2p_j^2}{k_j} = \left[ K \sum_j \frac{2p_j^2}{k_j} \right] + \left[ \frac{K \sum_j \frac{2p_j^2}{k_j}}{K} \right] K \tag{9.63}
\]

This is an awkward looking expression. But it is the most general way of writing both denominator and numerator as a sum of 'integer' + 'fraction', where the fraction is positive and < 1. Note specifically that \( K \sum_j \frac{2p_j^2}{k_j} \) is integer valued which allows us to apply the modulo function.

Let us press on and continue our program. The expression for the quasihole charge is

\[
Q_\gamma = \frac{C'}{D'} + \frac{f}{k} \quad f, g < K \tag{9.64}
\]

where \( C', D', f \) and \( g \) all integer valued. Specifically

\[
C' = C + \left[ K \sum_i \frac{[m_ip_i]}{k_i} \right] + \sum_i m_i \quad D' = M + \left[ K \sum_j \frac{2p_j^2}{k_j} \right] K \tag{9.65}
\]

\[
f = \left[ K \sum_i \frac{[m_ip_i]}{k_i} \right] K \quad g = \left[ K \sum_j \frac{2p_j^2}{k_j} \right] K \tag{9.66}
\]

Solving for all possible values of \( C \), or, equivalently, \( C' \) such that \( 0 \leq Q_\gamma < 1 \) gives us the spectrum of quasiholes and also the number of quasihole sectors that are formed per \( \bigotimes_i \mathbb{Z}_{k_i} \) field. This gives

\[
\text{# quasiholes formed with} \quad \prod_i \Phi_{m_i,k_i} = \begin{cases} 
D' + 1 & f < g \\
D' & f \geq g 
\end{cases} \tag{9.67}
\]

The total number of \( \bigotimes_i \mathbb{Z}_{k_i} \) fields equals \( \prod_i \frac{1}{2} k_i(k_i + 1) \). The total number of fields which satisfy \( f < g \) equals \( \prod_i \frac{1}{2}(k_i + 1)g \). This brings the total number of quasiholes to the sum

\[
\prod_i \frac{1}{2} k_i(k_i + 1) \left( M + \left[ \frac{K \sum_j \frac{2p_j^2}{k_j}}{K} \right] + \left[ \frac{K \sum_j \frac{2p_j^2}{k_j}}{K} \right] K \right) \tag{9.68}
\]

which combines into

\[
\# [k,M] \bigotimes_i \mathbb{Z}_{k_i} = \prod_i \frac{1}{2} k_i(k_i + 1) \left( M + \sum_j \frac{2p_j^2}{k_j} \right) \tag{9.69}
\]

This is the main result of this chapter. This formula also appears in [Barkeshli and Wen, 2008].
We have discussed multiple aspects of the fractional quantum Hall effect. Starting from a classical description, we gave a description of the single-particle behavior. Upon quantization we obtained the single particle eigenstates and described there highly structured form. In addition, we also described certain multiparticle states, the Laughlin sequence, thus obtaining a first hands-on wavefunction of certain fractional quantum Hall states.

We also showed in what manner conformal field theory can be used to extract all sort of information from the states. We described under what circumstances a CFT is a proper candidate to encapture the long-range physics of a quantum Hall state. This includes the identification of the electron and the construction of electron wavefunctions through the conformal blocks of the CFT. In turn, CFT allows us to get a grip on both Abelian and non-Abelian quantum Hall state candidates, the Read-Rezayi and Moore-Read states in particular. We obtained expressions for the electron wavefunctions, properties of the quasiholes, including statistics and quantum numbers, and topological quantum numbers, with the groundstate degeneracy on the torus in particular.

New results are also obtained. We generalized the Read-Rezayi states and obtained new candidates for quantum Hall states. The first are called the generalized parafermion states, and are constructed by identifying a different simple current from the same $Z_k$ parafermionic CFT with the electron operator. The second generalization is done by tensoring multiple parafermionic CFT’s. This is accomplished effectively by taking a product of decoupled parafermionic operators to represent the electron operator, hence the name product parafermion states.

The new states are a direct generalization of the Read-Rezayi series. We obtained exact expressions for the wavefunction in terms of correlators, the smallest charge quasihole and the ground state degeneracy, with each expression appearing as a natural generalization.

Still, the behavior of the new states is more complex. They satisfy a clustering behavior,
much like the Read-Rezayi case, but not as simple. To effectively classify the new states we used a new classification scheme, the pattern of zeros. The operator algebra allows us to formulate exact expressions for these pattern of zeros, and in doing so we verified some numerical results which appear in the literature. Through these pattern of zeros, examples and educated guesses we conjectured exact expressions for certain classes of the generalized parafermion states. They seem to be related to a generalized version of a coloring-scheme, first introduced for the Read-Rezayi series.

The new states are still far from classified though. The pattern of zeros admits an intuitive and useful picture to classify states with and we have shown that the new states are indeed more complex than their Read-Rezayi counterparts. However, the program is not completely rigid as the identification of a CFT that belongs to a specific pattern of zeros is not unique. Introducing assumptions on the CFT side does allow us to identify certain classes of clustered states. At the moment it is not quite clear what the basic restrictions on the pattern of zeros are, but we have showed that a proper definition of the electron operator within the CFT does resolve some of the problems. Still, this issue remains open to debate.

Putting the classification of pattern of zeros aside, the new found states does provide us with interesting open questions. The generalization of the exact expressions for the wavefunctions for all new cases should be possible. Also, some issues should be addressed. For instance, we found that the pattern of zeros for the non-coprime generalized parafermions states coincide with certain product parafermion states. We also showed that in these non-coprime cases we deal with a subalgebra of the CFT. The question is, does this provide a proper CFT description of a quantum Hall state? The results presented here suggests no, but this is subject to debate.

Much more information can ofcourse be extracted from the new states. The tools are already available. Braiding behavior, degeneracies in the presence of quasiholes, exact expressions for the wavefunctions: the pioneering work has already been done, as the literature is full of examples on how these can be obtained using CFT. Whether these states are physically realized remains to be seen. For now they serve as an excellent pedagogical example on which the tools of CFT, TQFT and much more, can be let loose upon.
Appendices
In this appendix we study the effects of putting the quantum Hall system on a spherical geometry. What is special about the sphere is its property that it is of finite size, yet it does not have any boundaries. Furthermore, its geometry is essentially different from that of the plane since it possesses, after all, a curvature. The effect of the curvature manifests itself through the introduction of a topological quantum number, the shift. As an example we define the Laughlin wavefunction on the sphere and calculate its shift.

A.1 Single particle wavefunctions

By putting the wavefunction on a sphere a number of properties pop up. First off all, the wavefunction will pick-up on the spherical symmetry present and as an effect will itself live in a representation of the rotation group $SU(2)$. As this representation is assumed to be finite, this in turn has an effect that there are a finite number of single-particle eigenstates, which is different from the infinite plain-case. The spherical symmetry allows us to simultaneously diagonalize the Hamiltonian and the angular momentum, thus labeling the eigenstates with two quantum numbers. The coordinate representation of these states are best described via a spinor representation. In the end, via stereographical projection the wavefunction is projected onto the compactified plane. This turns the spinor representation into a polynomial one.

Let us start with the basics. We essentially follow [Chakraborty and Pietiläinen, 1995], although the original derivation was done by [Haldane, 1983b]. An electron is confined to a finite two-dimensional surface in the presence of a magnetic field. Specifically, the surface is a sphere with radius $R$ and a magnetic monopole is located at the center. The magnetic field therefore only has a radial component, e.g. $\vec{B} = B\hat{r}$ where $\hat{r} = \frac{\vec{r}}{R}$. Due to quantization of the monopole’s charge, an argument provided by Dirac, the strength of the magnetic field is
limited to

\[ B = N_\Phi \frac{\Phi_0}{4\pi R^2}, \quad N_\Phi \in \mathbb{Z} \] \hspace{1cm} (A.1)

Here, \( N_\Phi \) equals the number of flux quanta piercing through the surface. Due to Gauss’ law it equals the monopole’s charge.

The Hamiltonian for a single electron, neglecting spin-coupling, is given by the kinetic energy. Because of the curved surface and the presence of a magnetic vector potential \( \mathbf{A} \) the kinetic energy is determined through the gauge-invariant angular momentum operator, \( \mathbf{\Lambda} = \mathbf{r} \wedge \mathbf{\pi} \), in terms of

\[ H = \frac{1}{2m} |\mathbf{\Lambda}|^2 \] \hspace{1cm} (A.2)

\[ \mathbf{\Lambda} = \mathbf{r} \wedge (-i\hbar \nabla + e\mathbf{A}) \] \hspace{1cm} (A.3)

The components \( \Lambda^i \) generate a Lie algebra with the commutator given by

\[ [\Lambda^i, \Lambda^j] = i\hbar \epsilon^{ijk} \Lambda^k - \hbar \mathbf{S} \cdot \epsilon \wedge \mathbf{r} \] \hspace{1cm} (A.4)

Let us define the operator \( L^i \) as

\[ L^i = \Lambda^i + \hbar \mathbf{S} \cdot \mathbf{r} \] \hspace{1cm} (A.5)

These operators generate an \( SU(2) \) Lie algebra, as can be seen from their commutation relations. In fact, the commutator with the remaining operators is

\[ [L^i, X^j] = i\epsilon^{ijk} X^k \] \hspace{1cm} (A.6)

where \( \mathbf{X} = \mathbf{L}, \mathbf{A} \) or \( \hat{\mathbf{r}} \). From this it follows that \( \mathbf{L} \) commutes with the Hamiltonian \( (\mathbf{H} \times \Lambda' \Lambda') \) and thus the wavefunction form a representation of these operators. The best approach to obtain the eigenstates is to introduce quantum numbers which label their relative angular momentum. The eigenstates thus form a basis for the representation of \( SU(2) \).

Representations of \( SU(2) \) are labeled by their eigenvalue of the Casimir operator \( |\mathbf{L}|^2 \). In general, this eigenvalue takes on the values \( \hbar^2 J(J+1) \), where \( J \in \mathbb{Z}_{>0} \) and the dimension of the rep equals \( 2J + 1 \). As a corollary we deduce the spectrum of the Hamiltonian

\[ |\Lambda|^2 |n\rangle = (|\mathbf{L}|^2 - \hbar^2 S^2) |n\rangle = \hbar^2 (J(J+1) - S^2) |n\rangle \] \hspace{1cm} (A.7)

\[ \epsilon_n = (n(n + 1) + (2n + 1)S) \frac{\hbar \omega_c}{2}, \quad n = 0, 1, 2 \ldots \] \hspace{1cm} (A.8)

Since the Hamiltonian is positive we have \( J \geq S \). We see a discrete energy spectrum arising, similar to the planar case, by replacing \( J \) by \( n = J - S \)

The \( n \)’th Landau level corresponds to the \( 2J + 1 \) dimensional representation. The degeneracy per Landau level is therefore

\[ \text{Degeneracy}[n] = 2n + N_\Phi + 1 \] \hspace{1cm} (A.10)
A.1 Single particle wavefunctions

From now on we will set \( n = 0 \) which corresponds to an energy \( \epsilon_0 = \frac{1}{2} \hbar \omega_c \). It is the lowest energy level, i.e. the lowest Landau level.

To differentiate between the \( N_\Phi + 1 \) eigenstates within the lowest Landau level Haldane introduced a particular gauge choice. In the usual spherical coordinate system the unit vector \( \hat{r} \) is parametrized as \((\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). The gauge introduced by Haldane in [Haldane, 1983b] is then written as

\[
A = \frac{\hbar S \cos \theta}{e R \sin \theta} \hat{\varphi}
\]  

(A.11)

To identify the eigenstates we switch to a new coordinate system, namely that of a spinor representation. By definition

\[
u = \cos\frac{\theta}{2} e^{i\frac{\varphi}{2}}
\]  

(A.12)

\[
v = \sin\frac{\theta}{2} e^{-i\frac{\varphi}{2}}
\]  

(A.13)

In terms of these coordinates the \( L^z \) component of the angular momentum operator equals

\[
L^z = \frac{1}{2} \left( u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \right)
\]  

(A.14)

From group theory we know that \( SU(2) \) eigenstates of a particular representation are orbitals labeled by their angular momentum component along a particular axes, e.g. their \( L^z \) eigenvalue. With this in mind, the choice of a spinor coordinate system is quite natural. The \( u \) and \( v \) coordinates behave as boson creation operators with the destruction operators given by \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial v} \). The \( L^z \) operator is thus a sum of two number operators, very much like in the planar case. We do not have to bother writing down any complicated differential equations, because eigenstates of number operators are easily constructed. They are

\[
|N_\Phi, m> = v^{N_\Phi - m} u^m, \quad m \in \{0, \ldots N_\Phi\}
\]  

(A.15)

\[
L^z |N_\Phi, m> = (m - \frac{N_\Phi}{2}) |N_\Phi, m>
\]  

(A.16)

In general, the wavefunction of one electron has the following form

\[
\Phi(\theta, \varphi) = \sum_{m=0}^{N_\Phi} c_m v^{N_\Phi - m} u^m
\]  

(A.17)

The wavefunction of one electron in the lowest Landau level on a sphere is given by the elegant form (A.17) in terms of the spinor coordinates (A.13). Since the wavefunction is that of spherical system we may use a stereographical projection to map the system onto the compactified plane. Introducing a complex variable \( z \) as

\[
z = \frac{u}{v}
\]  

(A.18)

\[|z|^2 = |u|^2 + |v|^2
\]  

(A.19)

Using \( v = |v| e^{-i\frac{\varphi}{2}} \) we write the eigenstates (A.15) as

\[
\Phi(\theta, \varphi) = v^{N_\Phi} \sum_{m=0}^{N_\Phi} \left( \frac{u}{v} \right)^m e^{-i\frac{\varphi}{2}}
\]  

\[= \frac{e^{-i\frac{\varphi}{2}}}{(1 + |z|^2)^{N_\Phi/2}} \sum_{m=0}^{N_\Phi} z^m
\]  

(A.21)
Similar to what is observed in the planar case, we see that wavefunction of one electron splits into a geometrical part \( F_g(z, \bar{z}) \) and a polynomial function \( f(z) \) independent of \( \bar{z} \)

\[
\Phi(\theta, \varphi) = f(z)F_g(z, \bar{z})
\]

(A.22)

\[
f(z) = \sum_{m=0}^{N_\Phi} c_m z^m
\]

(A.23)

The strength of the approaches used throughout this thesis, be it the CFT approach or the classification of symmetric polynomials, is based on the restrictive form of \( f(z) \).

### A.2 Shift

The effect of the curvature of the sphere manifests itself through the introduction of a geometric factor (A.21) and the finiteness of the energy eigenstates in a given Landau level.

Much more relevant to this thesis is its effect on the relation between the filling factor \( \nu \) and the ratio \( N_e/N_\Phi \). Let us make this quantitative first.

The energy spectrum of the Landau levels is given by

\[
\epsilon_n = (n(n + 1) + (2n + 1)S) \frac{\hbar \omega_c}{2}, \quad n = 0, 1, 2 \ldots
\]

(A.24)

The quantum number \( n \) correspond to the different Landau levels. Each Landau level is degenerate, with its degeneracy determined by the corresponding representation of \( SU(2) \). Namely, for each level there are \( 2n + N_\Phi + 1 \) states available.

We now fill up the \( L \) lowest Landau levels. That means, we add \( N_e \) electrons to the lowest energy states available, with \( N_e \) given by

\[
N_e = \sum_{n=0}^{L-1} 2n + N_\Phi + 1 = L\Phi_0 + L^2
\]

(A.25)

From this we see that the ratio of the number of electrons and flux quanta does not give the proper filling factor \( L \). Instead we find the modified relation

\[
N_\Phi = \nu^{-1}N_e - S
\]

(A.26)

where \( \nu^{-1} = L \) is the proper filling factor and \( S \) equals \( L \). The quantity \( S \) is an example of a topological quantum number [Zee, 1995]. It is a reflection of the effect of the curvature on the system. One way to view this is that an electrons takes up the area of a circle with radius \( l_B \). On the plain its area is given by \( 2\pi l_B^2 \), but in the presence of the positive curvature this area is larger. The curvature is determined through the strength of the monopole, which essentially fixes the radius of the sphere. Note that in the thermodynamic limit, the ratio \( N_e/N_\Phi \) indeed tends to \( \nu \) quite fast. This is to be expected since we essentially recover the plain geometry in this limit.

Physical samples are of course flat, therefore the quantity \( S \) will not pop up. However, it is a useful quantity as it is essentially a topological quantum number which need not be the same for different quantum Hall systems. And in general it is indeed different. In fact, in the final chapters there are examples of systems which have the same filling fractions yet different shifts. That already shows that the two systems correspond to different topological order. Note that for more complicated geometries with different curvatures \( S \) will also be more complex. In agreement with this line of thought, \( S \) also vanishes for non-curved spaces such as the plain and the torus.
A.3 Laughlins wavefunction

As an example of the shift factor let us obtain the Laughlin wavefunction on the sphere. We make use of its representation on the plain

\[ \Psi_{\text{Laughlin, plane}}(z_1, \ldots, z_{N_e}) = \prod_{i<j} (z_i - z_j)^M e^{i \Phi \sum_i |z_i|^2} \]  

We now make use of the discussion presented in chapter 3.4. That is, the exponent is essentially a geometric factor while the polynomial in front is determined through the topology of the plain. Since the topology of the (compactified) plane and the sphere are the same, all we need to change is the geometric part. That is, we strip off the exponent and replace it by its spherical equivalent \((A.21)\).

\[ \Psi_{\text{Laughlin, sphere}}(z_1, \ldots, z_{N_e}) = \prod_{i<j} (z_i - z_j)^M \prod_k \frac{e^{-i \frac{1}{2} \phi}}{(1 + |z_k|^2)^{N\Phi/2}} \]  

Its spinor representative is

\[ \Psi_{\text{Laughlin, sphere}}(u_1, v_1, \ldots, u_{N_e}, v_{N_e}) = \prod_{i<j} (u_i - u_j)^M \prod_k v_k^{N\Phi} \]

\[ = \prod_{i<j} (u_i v_j - u_j v_i)^M \]  

This is not a derivation of the Laughlin wavefunction on the sphere. In the second line we have made use of the fact that the number of flux quanta \(N\Phi\) must equal the highest degree of each variable of the Laughlin wavefunction,

\[ N\Phi = M(N_e - 1) \]  

This essentially an input we put in by hand. It matches that of [Haldane, 1983b].

When we calculate the filling factor we identify the highest degree with the number of flux quanta (which is circular argument as presented here, but still true in a more rigorous treatment). This is essentially equation \((A.31)\). Comparing with equation \((A.26)\) gives

\[ \nu = \frac{1}{M} \]  

\[ S = M \]

For Abelian quantum Hall states the shift on the sphere is always \(\nu\).
The chiral boson

This appendix is devoted to a self-contained description of a $\hat{u}(1)$ conformal field theory, the so-called chiral boson. The main source for these derivations is the book [Di Francesco et al., 1995], although the conventions such as mode expansions and the normalization of the action differ. This treatment shows up in any other standard reference on conformal field theory. Our goal is a firm understanding in what way the bose field plays is related to conformal field theory.

B.1 Quantization

We define $\varphi(x,t)$ to be a free boson field, living on a cylinder of circumference $R$. That is, we introduce a periodic conditions $\varphi(x,t) = \varphi(x + R, t)$. The action for a Bose field is as follows

$$S[\varphi] = \frac{1}{8\pi} \int d^2x \partial_u \varphi \partial^u \varphi = \frac{1}{8\pi} \int dx dt \left( (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right)$$

We will now quantize this action using a canonical approach. The conjugate momentum field follows from the Lagrangian density.

$$\Pi(x,t) = \frac{\delta L}{\delta (\partial_t \varphi(x,t))} = \frac{1}{4\pi} \dot{\varphi}(x,t)$$

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Keeping the periodic boundary conditions in mind, the mode expansion of the free boson field and its conjugate are

\[ \varphi(x,t) = \sum_n e^{2\pi i n x R} \varphi_n(t) \]
\[ \varphi_n(t) = \frac{1}{R} \int dx \, e^{-2\pi i n x R} \varphi(x,t) \]  
\[ \Pi(x,t) = \sum_m e^{2\pi i m x R} \pi_m(t) \]
\[ \pi_m(t) = \frac{1}{R} \int dx \, e^{-2\pi i m x R} \Pi(x,t) \]  

(B.3.a)  
(B.3.b)

Quantization follows from the equal time commutation relations. The commutator of main importance is that of the canonical pair

\[ [\varphi(x,t), \Pi(y,t)] = i \delta(x-y) \]  

(B.4)

Such commutation relations are naturally carried over to the modes (B.3.a) and (B.3.b). The commutators satisfied by the modes are

\[ [\varphi_n(t), \pi_m(t)] = \frac{1}{R^2} \int dx \, e^{-2\pi i \frac{n+m}{R}} [\varphi(x,t), \Pi(y,t)] \]
\[ = \frac{i}{R} \delta_{n,-m} \]  

(B.5)  
(B.6)

From which we conclude that the momentum mode conjugate to \( \varphi_n \) is \( R \pi_{-n} \). Through equation (B.2), (B.3.a)) and (B.3.b)) the modes of the derived field \( \dot{\varphi} \) are completely fixed, namely \( \dot{\varphi}_n(t) = 4\pi \pi_n(t) \). The fields are Hermitian, therefore the modes satisfy \( \varphi_n^\dagger(t) = \varphi_{-n}(t) \) and \( \pi_n^\dagger(t) = \pi_{-n}(t) \). These result are now used to construct an expression for the Hamiltonian in terms of these modes. For that we first need an expression for the Lagrangian density

\[ L = \frac{1}{8\pi} \int dx \left( (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right) \]
\[ = \frac{1}{8\pi} \int dx \sum_{n,m} e^{2\pi i \frac{n+m}{R}} \left\{ \dot{\varphi}_n \dot{\varphi}_m + nm \left( \frac{2\pi}{R} \right)^2 \varphi_n \varphi_m \right\} \]
\[ = \frac{1}{8\pi} R \sum_n \left\{ \dot{\varphi}_n \dot{\varphi}_{-n} - \left( \frac{2\pi n}{R} \right)^2 \varphi_n \varphi_{-n} \right\} \]  

(B.7)  
(B.8)

Next, we plug everything into the general expression for the Hamiltonian.

\[ H = \sum_n \varphi_n R \pi_{-n} - L \]
\[ = 4\pi R \left( \sum_n \pi_n \pi_{-n} \right) - 2\pi R \sum_m \left\{ \pi_m \pi_{-m} - \left( \frac{m}{2R} \right)^2 \varphi_m \varphi_{-m} \right\} \]
\[ = 2\pi R \sum_{n=-\infty}^{\infty} \left\{ \pi_n \pi_{-n} + \left( \frac{n}{2R} \right)^2 \varphi_n \varphi_{-n} \right\} \]  

(B.9)  
(B.10)  
(B.11)

This Hamiltonian represents an infinite sum of decoupled harmonic oscillators with mass \( R \) and frequency \( \omega_n = \frac{2\pi |n|}{R} \) (one has to keep careful track of which operators are each others

\[ \text{This convention is different from } [\text{Di Francesco et al., 1995}], \text{where the canonical momentum is identified through use of the time-derivative } \dot{\varphi}(x,t) \]
conjugate). The standard treatment is to diagonalize the Hamiltonian via the introduction of ladder operators from which the solutions of the field automatically follow. Let \( \varphi_n := \varphi_n(0) \).

In the Schrödinger picture we have

\[
\begin{align*}
    a_n &= R\pi_n - \frac{in}{2} \varphi_n, \\
    \bar{a}_n &= R\pi_n - \frac{in}{2} \varphi_n, \\
    \varphi_n &= \frac{1}{n} (a_n + \bar{a}_{-n}) \\
    a_0 &= \bar{a}_0 = R\pi_0
\end{align*}
\]

(B.12)

The operators then satisfy the following commutation relations

\[
[a_n, a_m] = [\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0} \\
[a_n, \bar{a}_m] = 0
\]

(B.13)

Note that the zero mode \( \varphi_0 \) cannot be defined in terms of \( a_n \) and \( a_{-n} \). But as is clear from (B.11), this term does not enter the Hamiltonian, so \( H \) can be expressed completely in terms of the new operators.

\[
H = \frac{\pi}{R} (a_0^2 + \bar{a}_0^2) + \frac{2\pi}{R} \sum_{n>0} (a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)
\]

(B.15)

We have successfully diagonalized the Hamiltonian. Excitations of the system are created through application of the creation operators, and in principle we are able to construct the Hilbert space from this. For now, it is more practical to derive the mode expansion of the Bose field in terms of these ladder operators. In the Schrödinger picture the field operator follows by plugging in (B.12)

\[
\varphi(x,0) = \varphi_0 + i \sum_{n>0} \frac{1}{n} (a_n - \bar{a}_{-n})e^{2\pi in^2 x R}
\]

(B.16)

Switching to the Heisenberg picture is straightforward. The time evolution of an operator \( A \) in the Heisenberg picture is generated by the Hamiltonian as \( \frac{dA}{dt} = i[H, A] \). Then we have

\[
\begin{align*}
    \frac{da_n}{dt} &= -\frac{2\pi in}{R} a_n, \\
    \frac{d\bar{a}_n}{dt} &= -\frac{2\pi in}{R} \bar{a}_n \\
    \frac{d\varphi_0}{dt} &= 4\pi \varphi_0 \quad \Rightarrow \quad \varphi_0(t) = \varphi_0(0) + 4\pi \varphi_0 t
\end{align*}
\]

(B.17)

The expression for the field operator in terms of constant mode operators at arbitrary time is then

\[
\varphi(x, t) = \varphi_0 + 4\pi \varphi_0 t + i \sum_{n \neq 0} \frac{1}{n} \left( a_n e^{2\pi in (x-t) R} - \bar{a}_{-n} e^{2\pi in (x+t) R} \right)
\]

(B.18)

As a final step we will transform to conformal coordinates. For that we first perform a Wick rotation, which means replacing \( t \) by \(-it\). After this transformation to Euclidean space-time we introduce conformal coordinates

\[
\begin{align*}
    z &= e^{2\pi \left(\frac{r-ix}{R}\right)} \\
    \bar{z} &= e^{2\pi \left(\frac{r+ix}{R}\right)}
\end{align*}
\]

(B.19)
B. The chiral boson

The mode expansion of the free boson field in terms of these conformal transformation is then

\[ \varphi(z, \bar{z}) = \varphi_0 - i R \pi_0 \log (z \bar{z}) + i \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \]  

(B.20)

Note how the field naturally splits into two decoupled types of excitations, namely holomorphic and antiholomorphic dependent. Such excitations are usually referred to as chiral or right-moving (holomorphic) and antichiral or left-moving (anti-holomorphic). This decoupling is a direct consequence of the periodicity built in by the cylinder. Throughout this thesis we focus completely on the chiral part.

A second conclusion we draw from the mode expansion is that the Bose field is itself not a quasi-primary field. Under conformal transformations the field does not transform in the usual way, meaning correlators which consist out of Bose field will in general not satisfy the Ward identity. However, it is possible to construct various fields consisting out of multiple (derived) Bose fields which are conformal. One of those operators is the current operator.

B.2 Current operator

In complex coordinates the action (B.1) looks as

\[ S = \frac{1}{4\pi} \int dzd\bar{z} \partial \varphi \partial \bar{\varphi} \]  

(B.21)

We have already seen from the mode expansion how the bosonic field splits into a holomorphic and anti-holomorphic part. We focus on the holomorphic part, which follows from the above through the relation \( \varphi(z, \bar{z}) \rightarrow \varphi(z) + \bar{\varphi}(\bar{z}) \). The propagator of this chiral field is [Di Francesco et al., 1995]

\[ \langle \varphi(z) \varphi(w) \rangle = -\log (z - w) \]  

(B.22)

Differentiating with respect to \( z \) gives the propagator of the derived fields

\[ \langle i \partial_z \varphi(z, \bar{z}) i \partial_w \varphi(w, \bar{w}) \rangle = \frac{1}{(z - w)^2} \]  

(B.23)

The factor of \( i \) is introduced to compensate for some awkward minus signs. As stated before, we will focus our discussion on the holomorphic part. From the propagator it follows that the most singular term is

\[ i \partial \varphi(z) i \partial \varphi(w) \sim \frac{1}{(z - w)^2} \]  

(B.24)

This is in fact the only singular term, since the antisymmetric term \( \frac{1}{z - w} \) drops out due to the bosonic properties. Equation (B.24) is therefore the Operator Product Expansion of the holomorphic field \( i \partial \varphi \) with itself. The energy-momentum tensor is defined as the conserved current associated with infinitesimal conformal variations of the action [Di Francesco et al., 1995].

\[ T(z) = \frac{1}{2} : i \partial \varphi i \partial \varphi : \]  

(B.25)
B.2 Current operator

The double-dot notation stands for a normal ordering of the mode operators. That is, all annihilation operators to the right and the creation operators to the left. Using Wick’s theorem we derive the OPE of \( T(z) \) with the operator \( i\partial \varphi(w) \)

\[
T(z)\varphi(w) \sim :i\varphi(z)i\varphi(z): \Rightarrow \frac{i\varphi(z)}{(z-w)^2}
\]

\[
\sim \frac{i\varphi(w)}{(z-w)^2} + \frac{i\varphi(w)}{(z-w)^2}
\]

In the last line \( \varphi(z) \) was replaced by its expansion around \( w \). This expression indeed verifies that \( i\partial \varphi \) is a primary field. It has conformal dimension \( \Delta = 1 \) (\( \bar{\Delta} = 0 \)). Using the same method the OPE of \( T(z) \) with itself is also deduced.

\[
T(z)T(w) \sim \frac{1}{2}:i\varphi(z)i\varphi(z): + :i\varphi(z)i\varphi(w): + :i\varphi(w)i\varphi(w): + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}
\]

From the first term we read off that the central charge \( c \) equals 1. To get a better interpretation of what this representation of the conformal algebra exemplifies, consider the mode expansion of the primary field \( i\partial \varphi \).

\[
i\partial \varphi(z) = \sum_n a_n z^{-n-1}
\]

where the mode operators \( a_n \) satisfy the following commutation relations

\[
[a_n, a_m] = n\delta_{n+m,0}
\]

This algebra is called the Heisenberg algebra. It is the simplest example of an affine extension of a Lie algebra, namely the extension of \( u(1) \), denoted by \( \hat{u}(1) \). This is one way to understand in what way the free boson field is associated to the gauge group \( U(1) \). Let us make this more explicit.

The action (and Lagrangian) (B.21) is invariant under translations. Namely, we may shift the field \( \varphi \) with a constant to \( \varphi + A \) without affecting the Lagrangian. In view of the fractional quantum Hall effect, this symmetry is related to the gauge group of the underlying topological field theory. In that case, it is better to talk of a redundancy present in the field \( \varphi \). We may take either value of \( \varphi \) and still expect the same dynamics and there is no ‘physical way’ of making a distinction between \( \varphi \) and \( \varphi + A \). Such a redundancy is referred to as a gauge structure of the theory with the action of translation representing a certain element of the gauge group.

In principle, we have some freedom in choosing this gauge group (as long as it is Abelian for bosons). In view of the discussion on topological field theory we choose it to be compact and continuous. This automatically leads to the gauge group \( U(1) \). Elements of this group still act as translations, i.e. \( \varphi \rightarrow \varphi + A \). However, the compactness of the group imply that we restrict the domain of translations by identifying \( \varphi \) and \( \varphi + 2\pi M \). With this identification we state that even with respect to the gauge structure there is no distinction between \( \varphi \) and \( \varphi + 2\pi M \). The constant \( M \) is referred to as the compactification radius.
We should keep in mind, however, that this story is not automatically carried over to the edge theory of a topological field theory. On the edge the unphysical gauge degrees of freedom freeze in and become physical. The transformation $\varphi \rightarrow \varphi + A$ of the bose field is in that case a dynamical one, as the zero’th mode of the bose field is shifted. Nevertheless, one still deals with the compactification radius as the dynamical theory on the edge originates from the $U(1)$ gauge group of the TQFT.

With this in mind we continue our discussion on the operator $i\partial \varphi$. This operator forms what is called a conserved (holomorphic) current. Its scaling and conformal dimension $\Delta$ equals 1. This makes the operator holomorphic and conserved

$$J(z) \equiv i\partial \varphi(z)$$

$$\partial_{\bar{z}} J(z) = 0$$

To such a conserved current we associate a charge operator $Q$, called the holomorphic charge.

$$Q = \frac{1}{2\pi} \oint dw J(w)$$

This notion of charge makes sense, since the integral on the left is invariant under conformal transformations. The charge of any operator $O(z)$ is defined (or ‘measured’) via the commutator $[Q, O(z)] = qO(z)$. In conformal field theory the commutator between fields is calculated through a contour integration [Di Francesco et al., 1995]

$$[Q, O(z)] = \oint w d\bar{z} J(z) \frac{1}{2\pi} O(z)$$

The operator $O(z)$ is then said to carry a $u(1)$ charge $q$, referring to the underlying gauge group. A charge can be interpreted as a label stating in what way the representation transforms under gauge transformations.

### B.3 Vertex Operators

There is a particular interesting class of fields formed with the use of the free bose fields, which are first of all primary fields and, second, they carry a $u(1)$ charge. Such fields are known as vertex operators and throughout this thesis they play a fundamental role. The vertex operator is defined as the normal ordered product of the formal exponential of the chiral bose field

$$V_{\alpha}(z) \equiv :e^{i\alpha \varphi(z)} := \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \varphi(z)^n :$$

By normal ordering we mean that all operators which annihilate the vacuum are located to the right

$$V_{\alpha}(z) = \exp \left( i\alpha \varphi_0 + \sum_{n>0} \frac{1}{n} a_{-n} z^n \right) \exp \left( a_0 \alpha \log z - \alpha \sum_{n>0} \frac{1}{n} a_n z^{-n} \right)$$

From now on normal ordering of vertex operators will implicitly be assumed.
Vertex operators are an interesting class of operators. Specifically they are primary fields, which follows from their OPE with the energy momentum tensor

\[ T(z) V_\alpha(w) \sim \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{2n!} \left[ i \partial \bar{\varphi} i \partial \varphi (z) : \varphi^n : (w) \right] \]  

Again, we apply Wick’s theorem. The right hand side splits into two sums

\[ \left[ i \partial \bar{\varphi} i \partial \varphi (z) : \varphi^n : (w) \right] = \sum_{n=2}^{\infty} \frac{(i\alpha)^n}{(n-2)!} \varphi^{n-2} : (w) + \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{(n-1)!} \partial_z \varphi^{n-1} : (w) \]  

A proper relabeling gives the following OPE

\[ T(z) V_\alpha(w) \sim \frac{\alpha^2}{2} \frac{V_\alpha(w)}{(z-w)^2} + \frac{\partial_w V_\alpha(w)}{z-w} \]  

From which we read off the scaling dimension of the vertex operators. These are determined by the label \( \alpha \) through

\[ \Delta_{V_\alpha} = \frac{\alpha^2}{2} \]  

We always take \( \alpha \) real, which means the conformal dimension is non-negative. The vertex operator also carries a \( u(1) \) charge \( \alpha \), namely

\[ [Q, V_\alpha(z)] = \frac{1}{2\pi} \oint_z dw \left( i \partial \bar{\varphi} (w) \sum_m \frac{(i\alpha \varphi(z))^m}{m!} \right) \]  

As stated, the charge labels in what way the operator transforms under a gauge transformation. Such a transformation is of the type \( \varphi \rightarrow \varphi + A \), therefore

\[ V_\alpha(z) = e^{i\alpha \varphi(z)} \rightarrow e^{i\alpha A} e^{i\alpha \varphi(z)} \]  

If we restrict the domain of \( A \) to some interval \([0, 2\pi R]\) then the object \( e^{i\alpha A} \) is indeed an element of the gauge group \( U(1) \). The vertex operator carries a \( U(1) \) representation labeled by their charge \( \alpha \). Note however, that we could have chosen a different gauge group, such as \( \mathbb{R} \), and these results would still be valid.

The choice of \( U(1) \) leads to a severe restriction of the allowed vertex operators. Namely, a priori the label \( \alpha \) may take on any (real) value. Compactification of the boson identifies \( \varphi \) and \( \varphi + 2\pi M \) as the same operator, therefore this identity must be reflected by the vertex operators. The following identity must hold

\[ V_\alpha(z) = e^{i\alpha \varphi(z)} = e^{i\alpha \varphi(z) + 2\pi i \alpha M} \]  

This restricts the allowed vertex operators to

\[ V_\alpha(z) = e^{i\frac{m}{2\pi} \varphi(z)} \quad m \in \mathbb{Z} \]  

The identification of \( U(1) \) is the underlying gauge group of the bose theory leads to a discretization of the number of allowed vertex operators. Note that the results in the next section are also valid if this restriction is not made.
B.4 Correlation functions and background charge

Our goal is an understanding of the correlation functions of an arbitrary number of vertex operators. For that, we first give the OPE of a finite number of vertex operators. That is, we want to write the following product:

\[ e^{i\alpha \varphi(z)} \cdots e^{i\beta \varphi(z)} \]

as a single operator. The vertex operator consists out of an infinite number of decoupled harmonic oscillators. The normal ordered product follows from direct application of the commutation rules. For a proof see [Di Francesco et al., 1995]. In any case,

\[ V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \cdots V_{\alpha_n}(z_n) \sim V_{\alpha_1+\alpha_2+\cdots+\alpha_n}(z_n) \exp \left( -\sum_{i<j} \alpha_i \alpha_j \langle \varphi(z_i) \varphi(z_j) \rangle \right) \]  

(B.46)

The \( \sim \)-symbol is there to remind us that this identity only holds within a correlation function. Consider also the OPE of two vertex operators

\[ V_{\alpha}(z)V_{\beta}(w) \sim (z-w)^{a\beta} V_{\alpha+\beta}(w) \]  

(B.47)

in which we plugged in the two-point correlator of the free bose field (B.22). Equation (B.46) also follows by repetitive application of (B.47). From the OPE we are in a position to determine the correlators. Global conformal invariance tells us that the only operator which has a non-zero vacuum expectation value is the vacuum operator itself \( 1 \) \((\equiv V_{\emptyset}(z))\). This has a direct consequence for a two-point correlator of vertex operators, namely

\[ \langle V_{\alpha}(z)V_{\beta}(w) \rangle = \begin{cases} (z-w)^{-\alpha^2} & \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \]  

(B.48)

The correlator of an arbitrary number of vertex operators obeys a similar restriction. Namely, the correlator is non-zero only if the neutrality condition is satisfied.

\[ \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \cdots V_{\alpha_n}(z_n) \rangle = \exp \left( \sum_{i<j} \alpha_i \alpha_j \log(z_i - z_j) \right) \]  

(B.49)

\[ = \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \]  

(B.50)

Where

\[ \sum_{i=1}^{n} \alpha_i = 0 \]  

(B.51)

The easiest argument for this is the following. Consider the gauge transformation \( \varphi \rightarrow \varphi + a \), where \( a \) is a constant. The correlator accumulates a phase of the form \( \exp (ia \sum_{i=1}^{n} \alpha_i) \). Imposing single-valuedness on the correlator then leads to the neutrality condition.

The neutrality condition (B.51) is quite a restriction on the correlation functions of vertex operators. To be of any use we circumvent this problem by introducing the notion of a background charge. The effect of such a background charge is that it completes the neutrality condition (B.51) while not effecting the general formula (B.50). This is accomplished by the introduction of a vertex operator, located at \( w = \infty \) with a holomorphic charge \( \beta \) such that

\[ \lim_{w \to \infty} w^{\beta^2} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \cdots V_{\alpha_n}(z_n) V_{\beta}(w) \rangle \iff \beta = -\sum_{i=1}^{n} \alpha_i \]  

(B.52)
The charge of the inserted operator is tweaked such that it literally ‘neutralizes’ the correlator, rendering it non-zero. The factor $w^{\beta^2} (= \prod_{k=1}^{n} w^{-\beta \alpha_k})$ serves to cancel the coordinate contribution of the inserted operator. Direct application of (B.50) gives

$$\lim_{w \to \infty} w^{\beta^2} \left\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n)V^{\beta(w)} \right\rangle = \lim_{w \to \infty} \prod_{k=1}^{n} w^{-\beta \alpha_k} \prod_{k \leq n} (z_k - w)^{\alpha_k \beta} \prod_{i < j \leq n} (z_i - z_j)^{\alpha_i \alpha_j}$$

$$= \prod_{i < j \leq n} (z_i - z_j)^{\alpha_i \alpha_j} \quad \text{(B.53)}$$

The general correlator (B.50) is only valid if the neutrality condition is valid. This condition can be circumvented through the insertion of a background charge. By construction, the inserted operator does not effect the “naive” outcome (B.50).
B. The chiral boson
APPENDIX C

Quasiparticles and Conformal blocks

In this section we wish to determine the following correlation function

\[ \langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle \]  \hspace{1cm} (C.1)

The relevance of this correlator and its interpretation is explained in chapter 6.5.3. Here, we will merely explain how we, using CFT, we can compute this correlator.

The important property of this correlator is that it is a sum of conformal blocks. This is pictorially represented in figure C.1.

![Figure C.1](image)

**Figure C.1:** Vacuum to vacuum amplitude of the four-point correlator. The vacuum starts out with the identity operator 1. Then four consecutive fusion with spin fields occur, with the restriction that in the end the correlator fuses to the identity again. The field \( \Phi_p \) admits two channels, via \( \psi \) or via 1.

To determine this correlation function we make use of a general technique appearing in CFT. We basically need two ingredients. The first is the presence of so-called null states. These states decouple from the CFT in the sense that any correlator containing them is automatically zero. The second is the use of the conformal Ward identity. Through use of this identity we are able to map any correlator to one which contains a null state. The operator which causes this action is a differential operator. Hence, we are lead to a differential equation, which in turn is exactly solvable.

We follow a number of sources, which all treat the subject in essentially the same manner. These are [Belavin et al., 1986; Di Francesco et al., 1995; Ginsparg, 1988].
C.1 Ward identity and descendant states

In chapter 5 we introduced the concept of a Ward identity. A Ward identity is a reflection of the symmetries present in the theory. It expresses in what manner the symmetries impose restrictions on all correlation functions. By inserting the energy-momentum tensor one can derive a set of equations which all correlation functions must obey. See [Di Francesco et al., 1995] and [Ginsparg, 1988] for details.

All restrictions can be neatly brought together to form one relation. This is the (unintegrated) expression of the conformal Ward identity

\[ (T(w)\phi(z_1, \bar{z}_2) \cdots \phi_n(z_n, \bar{z}_n)) = \sum_{j=1}^{n} \left( \frac{h_j}{(w-z_j)^2} + \frac{1}{w-z_j} \partial_{z_j} \right) \langle \phi_1(z_1, \bar{z}_2) \cdots \phi_n(z_n, \bar{z}_n) \rangle \]  

(C.2)

This is a heavy restriction on the correlation functions, and we can make clever use of it. For that we need the connection with descendant states. In a representation of the Virasoro algebra one deals with a set of primary fields and their descendants. A field \( \phi \) is primary if and only if its operator product expansion with the energy momentum tensor is of the following form

\[ T(w)\phi(z, \bar{z}) \equiv \sum_{n=0}^{\infty} (w-z)^{n-2}(L_{-n})\phi(z, \bar{z}) \]

\[ \sim \frac{h_{\phi}}{(w-z)^2} \phi(z, \bar{z}) + \frac{1}{w-z} \partial \phi(z, \bar{z}) + \ldots \]  

(C.4)

In this expression we have identified \((L_0\phi) = h_{\phi}\phi\) and \((L_1\phi) = \partial \phi\). All higher order terms are descendant states of a primary field. Using an appropriate contour integration one has the following expression for a descendant field \( \phi^{(-n)} \).

\[ \phi^{(-n)}(z) \equiv (L_n)\phi(z) = \oint_{z_1} \frac{dw}{2\pi i (w-z)^{n-1}} T(w)\phi(z) \quad (n \geq 2) \]

(C.5)

This expression generalises to arbitrary descendants of the form \( \phi^{(-n_1, \ldots, -n_l)}(w, \bar{w}) \). We quickly mention that the sum \( \sum_{i} n_i \) is called the level of the descendant field. By use of the energy-momentum tensor we can therefore express correlation functions containing descendant fields in terms of only primary fields. For example, consider the following correlation function containing the descendant field \( \phi_1^{(-m)} \) and primaries \( \phi_i \).

\[ \langle \phi_1^{(-m)}(z_1)\phi_2(z_2) \cdots \phi_l(z_n) \rangle = \oint_{z_1} \frac{dw}{2\pi i (w-z_1)^{m-1}} \langle T(w)\phi_1(z_1) \phi_2 \cdots \phi_l \rangle \]

(C.6)

In effect, we have mapped a correlation function containing a descendant state to an expression in which an operator acts on a correlator containing only primary fields.

We now apply the Ward identity to this expression. Abbreviating the left hand side to \( \langle X \rangle \) and inserting the Ward identity C.2 into the right hand side gives us the following expression.

\[ \langle X \rangle = \sum_{j=2}^{n} \oint_{z_j} \frac{dw}{2\pi i (w-z_j)^{m-1}} \left( \frac{h_j}{(w-z_j)^2} + \frac{1}{z-w_j} \partial_{w_j} \right) \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle \]

\[ \equiv \mathcal{L}_m \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle \]  

(C.7)
Note that the contour in expression (C.6) is encircled around $z_1$, while in expression (C.7) it is deformed such that it encircles the poles located at $\{z_j\}_{j=2}^n$. The contour integral can be carried out in a straightforward manner. The operator $L_{-m}$ is a differential operator and is expressed as

$$L_{-m} = \sum_j \left\{ \frac{(m-1)h_j}{(w_j-w_1)^m} + \frac{1}{(w_j-w_1)^{m-1}} \frac{\partial}{\partial w_j} \right\}$$  \hspace{1cm} (C.9)$$

For future reference we explicitly state the expression for a correlation function containing one descendant field

$$\langle \phi_1^{(-n)}(z_1) \cdots \phi_m(z_m) \rangle = L_{-n} \langle \phi_1(z_1) \cdots \phi_m(z_m) \rangle = \sum_{j=2}^m \left\{ \frac{(n-1)h_j}{(z_j-z_1)^n} + \frac{1}{(z_j-z_1)^{n-1}} \frac{\partial}{\partial z_j} \right\} \langle \phi_1(z_1) \cdots \phi_m(z_m) \rangle$$  \hspace{1cm} (C.11)$$

### C.2 Null states

The introduction of null states in this theory is best done in the context of the Hilbert space of a CFT. We will not fully discuss this context, but merely mention some of the important results. Specifically, states in the Hilbert space correspond to fields in the following manner \(\text{Di Francesco et al., 1995}\).

First of all, we assume that our Hilbert space contains a unique, conformally invariant vacuum denoted as $|0\rangle$ and satisfying $L_n|0\rangle = 0$ for $n > 0$. The (Virasoro) highest weight states are then defined as

$$|h\rangle = \phi(0)|0\rangle$$  \hspace{1cm} (C.12)$$

$$L_0|h\rangle = h|h\rangle, \hspace{1cm} L_n|h\rangle = 0 \text{ for } n > 0$$ \hspace{1cm} (C.13)$$

where $\phi$ is a primary field. With this definition we see that the identity operator $1$ corresponds to the vacuum $|0\rangle$.

Descendant states are obtained by acting with $L_{-n}$ ($n > 0$) on the highest weight states. The operators $L_{\pm n}$’s act as an infinite set of raising and lowering operators. The spectrum generated this way contains all states of the form

$$L_{-n_1}L_{-n_2} \cdots L_{-n_k}|h\rangle = |h + N\rangle, \hspace{1cm} n_1 \geq n_2 \geq \ldots \geq n_k$$ \hspace{1cm} (C.14)$$

The state on the right hand side is again an eigenstate of $L_0$ with eigenvalue $h + N$, where $N = \sum n_i$ is the level of the descendant state.

We will assume we are dealing with CFT’s in which the Hilbert space is unitarizable. In that case we can define a hermitian conjugation and thus an inner product. Conjugation is defined as $L_k^\dagger = L_{-k}$ and

$$\langle h| = |h\rangle^\dagger = (\phi(0))^\dagger|0\rangle$$  \hspace{1cm} (C.15)$$

$$\langle \phi(z)\rangle^\dagger = z^{-2h}\phi(\frac{1}{z})$$ \hspace{1cm} (C.16)$$

$$\langle h|h\rangle = \lim_{z \to \infty} z^{2h} \langle \phi(z)\phi(0)|0\rangle$$ \hspace{1cm} (C.17)$$

After this rather formal introduction to the structure of the Hilbert space we can now explain the nature of a null state. A null state is a descendant (of level $N$) of some highest weight
state (of dimension $h$), but is itself also a primary field (with dimension $h + N$). It is in general a linear combination of states, all at level $N$, and is annihilated by all raising operators $L_n$. Furthermore, it is an eigenstate of $L_0$.

$$L_n|\chi\rangle = 0 \quad n > 0 \quad (C.18)$$

$$L_0|\chi\rangle = (h + N)|\chi\rangle \quad (C.19)$$

These properties show that the descendant $\chi$ is in fact a primary field [Belavin et al., 1986]. Its conformal family, consisting out of all descendants of $\chi$ form a submodule, which means the module $V(c, h)$ is reducible. By modding out the null states we construct what is called a Verma module, which is then an irreducible module (does not contain any submodules) of the Virasoro algebra. Formally this means we can set a null state equal to zero

$$|\chi\rangle = 0 \quad \langle \phi |\chi\rangle = 0 \quad (C.20)$$

$$\langle \chi |\chi\rangle = 0 \quad (C.21)$$

As a consequence, any correlation function containing a null state will be zero

$$\langle \chi \phi_1 \cdots \phi_n \rangle = 0 \quad (C.22)$$

### C.3 Restrictions on the correlator

Having introduced the concept of null states and the identity (C.11) brings us in a position in which the correlator can be computed.

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle \quad (C.23)$$

A well known property of the $\mathbb{Z}_2$ CFT is that the primary field $\sigma$ is degenerate at level 2 [Di Francesco et al., 1995]. We will denote this null state as $|\chi\rangle$. This state is some linear combination of all possible descendants at level 2.

$$|\chi\rangle = [L_{-2} + aL_{-1}^2] |h_{2,1}\rangle \quad (C.24)$$

This null state is annihilated by both $L_1$ and $L_2$. We can apply these operators to the state, and, using the Virasoro algebra compute the coefficient $a$. We have

$$0 = L_1|\chi\rangle = L_1[L_{-2} + aL_{-1}^2] |h_{2,1}\rangle$$

$$= [L_{-2}L_1 + 3L_{-1} + a(L_{-1}^2L_1 + 4L_{-1}L_0 + 2L_{-1})] |h_{2,1}\rangle$$

$$= [3 + 4ah_{2,1} + 2a] L_{-1} |h_{2,1}\rangle \quad (C.25)$$

$$0 = L_2|\chi\rangle = L_2[L_{-2} + aL_{-1}^2] |h_{2,1}\rangle$$

$$= [\frac{c}{2} + 4L_0 + L_{-2}L_2 + a(6L_0 + 6L_{-1}L_1 + L_{-1}^2L_2)] |h_{2,1}\rangle$$

$$= [\frac{c}{2} + 4h_{2,1} + 6ah_{2,1}] |h_{2,1}\rangle \quad (C.26)$$

Note that this equation has to be valid for any field which is degenerate at level 2. Solving for $a$ gives us an expression for the state $|\chi\rangle$. The answer is $a = -\frac{3}{2(2h+1)}$. 

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This in turn allows us to write the following general expression for a null field at level 2.

\[
0 = \chi_h(z) = \phi_{h}^{(-2)}(z) - \frac{3}{2(2h + 1)} \frac{\partial^2}{\partial w^2} \phi_h(z) \tag{C.27}
\]

We have successfully identified the null state at level 2. We can now make use of the results explained in appendix C.1 and C.2. The idea is as follows. We can apply the right combination of \(L_{-n}\) such that a field inside the correlation function 'hits' a null state. Since the null states decouples from the spectrum the resulting correlation function is zero. Furthermore, since the null state is a descendant we can use the Ward identity to turn the expression into a differential equation, as is explained in Appendix C.1. From (C.22) we have the following

\[
\left\langle \left( (L_{-2}\sigma)(z_1) - \frac{4}{3}(L_{-1}\sigma)(z_1) \right) \sigma(z_2)\sigma(z_3)\sigma(z_4) \right\rangle = 0 \tag{C.28}
\]

In this expression \(L_{-1}\) can be replaced by \(\frac{\partial}{\partial z_1}\). For \(L_{-2}\) we use expression (C.11).

\[
\left\{- \frac{4}{3} \frac{\partial}{\partial z_1} + \sum_{j=2}^{4} \left\{ \frac{1}{16(z_j - z_1)^2} + \frac{1}{z_j - z_1} \frac{\partial}{\partial z_j} \right\} \right\} \left\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \right\rangle = 0 \tag{C.29}
\]

A tremendous result indeed. We have acquired a differential equation from which we can derive an exact expression for the correlation function. Solving the equation will be the subject of the next subsection.

### C.4 Obtaining the conformal blocks

The solution of the differential equation is not completely straightforward. In fact, it is quite a tedious task as the differential equation is essentially one of four coordinates.

Therefore, it is wise to use some other restrictions on the correlation functions. We have seen how global invariance greatly restrains the two- and three-point functions. Global invariance therefore also restricts the four-point functions, although its form is less restrictive. Its result is that the correlation function has the following form

\[
\left\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \right\rangle = (z_{13}z_{24})^{-\frac{1}{8}} Y(x) \tag{C.30}
\]

In this expression the notation \(z_{12}\) stands for \(z_1 - z_2\). Furthermore, \(x = \frac{z_{13}z_{24}}{z_{14}z_{23}}\) is the anharmonic ratio. The function \(Y(x)\) is still arbitrary. Do notice that it only depends one variable, instead of four.

This form can be used to turn the differential equation (C.29) which depends on four variables, into one which only depends on one. We “only” have to fill in (C.30) into (C.29). This is probably the most annoying step of this whole calculation, so we merely state the result. The differential equation solved by \(Y(x)\) is

\[
\left\{ \frac{4}{3} \frac{d^2}{dx^2} - \frac{1}{16} \left[ \frac{1}{x^2} + \frac{1}{(x - 1)^2} \right] + \frac{1}{8} \frac{1}{x(x - 1)} + \left[ \frac{1}{x} + \frac{1}{x - 1} \right] \frac{d}{dx} \right\} Y(x) = 0 \tag{C.31}
\]

Having stepped over this bridge, we will continue on a more easy pace. We substitute \(Y(x) = (x(1 - x))^{-\frac{1}{8}} u(x)\). This cancels the poles present in the differential equation.

\[
\left\{ x(1 - x) \frac{d^2}{dx^2} + \left( \frac{1}{2} - x \right) \frac{d}{dx} + \frac{1}{16} \right\} u(x) = 0 \tag{C.32}
\]
Step by step, we break the differential equation down to a less sophisticated expression. One final substitution will do the trick

\[ x = \sin^2 \theta \implies \frac{d}{dx} = \frac{1}{\sin 2\theta} \frac{d}{d\theta} \]
\[ \frac{d^2}{dx^2} = \frac{1}{\sin^2 2\theta} \frac{d^2}{d\theta^2} - \frac{2\cos 2\theta}{(\sin 2\theta)^3} \frac{d}{d\theta} \]  

(C.33) 

(C.34)

This gives the equation

\[ \left( \frac{d^2}{d\theta^2} + \frac{1}{4} \right) u(\theta) = 0 \]  

(C.35)

This second order differential equation has the following general form

\[ u(\theta) = A_+ \sin \frac{1}{2} \theta + A_- \cos \frac{1}{2} \theta \]  

(C.36)

By reversing all substitutions we end up with the following expression for the four-point correlator (for the record, \( x = \frac{z_{12} z_{34}}{z_{13} z_{24}} \) and \( z_{ij} = z_i - z_j \)).

\[ \langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle = (z_{13} z_{24} x(1-x))^{-\frac{1}{2}} \left( A_+ \sqrt{\frac{1 + \sqrt{1-x}}{2}} + A_- \sqrt{\frac{1 - \sqrt{1-x}}{2}} \right) \]  

(C.37)

This is an example of a hypergeometric function. In general, the conformal blocks correspond to such functions and they are frequently obtained using the method depicted here.

This expression is exact. However, to compute quantities such as braiding relations its more convenient to apply a global conformal transformations to this expression. In that case we send three of the four fields to the points 0, 1 and \( \infty \). The anharmonic ratio \( x \) will equal the remaining coordinate. This coordinate cannot be fixed through global invariance, and is thus the “free variable”. We have

\[ \lim_{w \to \infty} \langle \sigma(0)\sigma(z)\sigma(1)\sigma(w) \rangle = A_+ F_+(z) + A_- F_-(z) \]  

(C.38)

\[ F_+ = \lim_{w \to -\infty} (wz(1-z))^{-\frac{1}{2}} \sqrt{\frac{1 + \sqrt{1-z}}{2}} \]  

(C.39)

\[ F_- = \lim_{w \to -\infty} (wz(1-z))^{-\frac{1}{2}} \sqrt{\frac{1 - \sqrt{1-z}}{2}} \]  

(C.40)
Bibliography


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