Quantum Dynamical $R$-matrices and Quantum Integrable Systems

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Universal $R$-matrices and Drinfeld twisted universal $R$-matrices are studied as solutions to the quantum Yang-Baxter equation. Following a similar procedure, fusion operators are used to define exchange operators or quantum dynamical $R$-matrices that are solutions to the quantum dynamical Yang-Baxter equation. The quantum dynamical $R$-matrices are used to construct a set of transfer operators that describe a quantum integrable system. An elaborate proof of the simultaneous diagonalizability of the transfer operators is provided. This work largely follows a structure outlined by Pavel Etingof.
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1 Introduction

The study of exactly solvable quantum systems, or quantum integrable systems, is one of the most influential and exciting fields in modern mathematical physics. A theory of exactly solvable quantum systems, referred to as the quantum inverse scattering method, was introduced by Faddeev in [Faddeev and Takhtadzhan, 1979]. A short introduction to the quantum inverse scattering method can be found in chapter 12 of [Essler et al., 2005]. The quantum Yang-Baxter equation lies at the basis of this theory of exactly solvable quantum systems.

The aim of this thesis is to define quantum integrable systems through the method of quantum inverse scattering, using solutions to the quantum dynamical Yang-Baxter equation referred to as quantum dynamical $R$-matrices. This process largely follows the structure outlined by Etingof in [Etingof and Latour, 2005].

The following three sections describe the contents of this thesis as presented in chapters 2, 3, and 4 respectively.

1.1 The Quantum Yang-Baxter Equation

As is well known, quantum groups $\mathcal{U}_q$ have a braided Hopf algebra structure (see appendix sections A.3 and A.3.2). Universal $R$-matrices $R \in \mathcal{U}_q \otimes \mathcal{U}_q$ satisfy the quantum Yang-Baxter equation

\begin{equation}
R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}
\end{equation}

on $\mathcal{U}_q \otimes \mathcal{U}_q \otimes \mathcal{U}_q$. For $U, V, W$ finite-dimensional representations of $\mathcal{U}_q$, the operator

\begin{equation}
R_{VW} : V \otimes W \to V \otimes W : (v \otimes w) \mapsto R(v \otimes w)
\end{equation}

satisfies the quantum Yang-Baxter equation on $U \otimes V \otimes W$;

\begin{equation}
R_{VW}^{12} R_{UV}^{13} R_{VW}^{23} = R_{VW}^{23} R_{UV}^{13} R_{VW}^{12}.
\end{equation}

Further solutions to the quantum Yang-Baxter equation may be found by ‘twisting’ universal $R$-matrices. An invertible twist $J \in \mathcal{U}_q \otimes \mathcal{U}_q$ satisfies the twist equation

\begin{equation}
(\Delta \otimes \text{id})(J)(J \otimes 1) = (\text{id} \otimes \Delta)(J)(1 \otimes J).
\end{equation}

Any Hopf algebra can be twisted by $J$ to yield a new Hopf algebra with a new universal $R$-matrix given by

\begin{equation}
R_J := (J^{21})^{-1} R J.
\end{equation}
The twisted universal $R$-matrix $R_J$ is again a solution to the quantum Yang-Baxter equation. Note that if two twists $J$ and $J'$ are gauge equivalent, i.e. there exists an invertible element $x \in U_q$ such that
\begin{equation}
J' = \Delta(x) J (x^{-1} \otimes x^{-1}),
\end{equation}
the twisted universal $R$-matrices $R_J$ and $R_{J'}$ are equal up to a change of basis, and are thus essentially the same solution to the quantum Yang-Baxter equation.

1.2 THE QUANTUM DYNAMICAL YANG-BAXTER EQUATION

Similar to finding solutions to the quantum Yang-Baxter equation on $U_q \otimes U_q \otimes U_q$ by twisting universal $R$-matrices, operators $R_{VW}$ may be twisted by a dynamical twist called the fusion operator in an attempt to find solutions to the quantum dynamical Yang-Baxter equation on $U \otimes V \otimes W$.

A finite-dimensional representation $V$ of $U_q$ can be appended to an irreducible Verma module $M_\lambda$ by a unique intertwiner
\begin{equation}
\Phi^v_\lambda : M_\lambda \rightarrow M_{\lambda - wt} \otimes V
\end{equation}
defined to have the expectation value $\langle \Phi^v_\lambda \rangle = v \in V$. These intertwiners can be concatenated to ‘fuse’ two representations $V$ and $W$ of $U_q$ together, resulting in
\begin{equation}
\Phi^{v,w}_\lambda := (\Phi^v_\lambda - wt \otimes \text{id}) \Phi^w_\lambda : M_\lambda \rightarrow M_{\lambda - wt, wt} \otimes V \otimes W
\end{equation}
and the definition of an invertible dynamical twist called the fusion operator
\begin{equation}
J_{VW}(\lambda) : V \otimes W \rightarrow V \otimes W : v \otimes w \mapsto \langle \Phi^{v,w}_\lambda \rangle
\end{equation}
that satisfies the dynamical twist equation
\begin{equation}
J_{U \otimes V, W}(\lambda) J_{UV}(\lambda - h^3) = J_{U, V \otimes W}(\lambda) J_{VW}(\lambda).
\end{equation}
The parameter $\lambda$ is called the dynamical parameter.

Much like twisting a universal $R$-matrix $R$ into a universal $R$-matrix $R_J$, an operator $R_{VW}$ can be twisted into an operator
\begin{equation}
R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} R_{VW} J_{VW}(\lambda) : V \otimes W \rightarrow V \otimes W
\end{equation}
called the exchange operator. Since the exchange operator depends on the dynamical parameter $\lambda$, it is a solution to the dynamical analog of the quantum
1.3 Quantum Integrable Systems

Yang-Baxter equation on $U \otimes V \otimes W$:
\[
R_{12}^{12}(\lambda - \hbar^3) R_{13}^{13}(\lambda) R_{23}^{23}(\lambda - \hbar^1) = R_{23}^{23}(\lambda) R_{13}^{13}(\lambda - \hbar^2) R_{12}^{12}(\lambda),
\]
called the quantum dynamical Yang-Baxter equation. Because the exchange operator $R_{VW}(\lambda)$ is a solution to the dynamical analog of the quantum Yang-Baxter equation, it is also referred to as a quantum dynamical $R$-matrix.

1.3 Quantum Integrable Systems

Quantum dynamical $R$-matrices may be used to construct a commuting set of operators that describe a quantum integrable system. One way to construct a suitable set of commuting operators is by considering operators $D_C$ associated to central elements $C \in Z(U_q)$. Taking $V$ a finite-dimensional representation of $U_q$, for every $x \in U_q$ the unique operator $D_x$ on functions $f : \mathfrak{h}^* \to V[0]$ is defined by
\[
D_x \text{ tr}|_{M_x}(\Phi_x^\mu q^{2\lambda}) = \text{ tr}|_{M_x}(\Phi_x^\mu x q^{2\lambda}),
\]
independent of the choice of $M_x, V$ and $v \in V[0]$. For central elements $C_W$, associated to finite-dimensional representations $W$ of $U_q$, the set of operators $D_{C_W}$ commutes by definition. Moreover, the trace functions
\[
\Psi^\nu(\lambda, \mu) := \text{ tr}|_{M_\nu}(\Phi_\nu^\mu q^{2\lambda})
\]
satisfy the difference equations
\[
D_{C_W} \Psi^\nu(\lambda, \mu) = \chi_W(q^{2(\bar{\mu} + \bar{\rho})}) \Psi^\nu(\lambda, \mu).
\]
Using the quantum dynamical $R$-matrix $R_{WV}(\lambda)$, another set of operators $D_W$ on functions $f : \mathfrak{h}^* \to V[0]$, called transfer operators, is defined as
\[
(D_W f)(\lambda) := \sum_{\nu \in \mathfrak{h}^*} \text{ tr}|_{W[\nu]} R_{WV}(-\lambda - \rho) T_\nu f(\lambda).
\]
Because of a remarkable relation between the transfer operators $D_W$ and the operators $D_{C_W}$,
\[
D_W = \delta_q(\lambda) D_{C_W} \delta_q(\lambda)^{-1},
\]
where $\delta_q(\lambda)$ is a scaling factor, the transfer operators $D_W$ also commute. Moreover, the trace function $F_V(\lambda, \mu)$, which is a sum of trace functions $\Psi^\nu(\lambda, \mu)$ scaled by the factor $\delta_q(\lambda)$, satisfies the difference equations
\[
D_W F_V(\lambda, \mu) = \chi_W(q^{-2\bar{\mu}}) F_V(\lambda, \mu).
\]
In other words, the trace function \( F_V(\lambda, \mu) \) diagonalizes the set of commuting transfer operators \( D_W \). This means that the set of algebraically independent transfer operators \( \{D_{\Lambda_1}, ..., D_{\Lambda_{n-1}}\} \), with \( \Lambda_i \) the fundamental representations of \( U_q \), define a quantum integrable system.

From a physical perspective, the representation \( V \) is the quantum state space of the system and the operators \( D_{\Lambda_i} \) are conserved quantities of the system.

Finally, two applications of the quantum integrable systems described are discussed shortly. Firstly, quantum spin Calogero-Moser systems described by the Hamiltonian

\[
H = \frac{1}{2} \Delta h^* - \sum_{\alpha \in \Phi^+} \frac{e_\alpha e_{-\alpha}}{(e_{\frac{1}{2}\alpha(\lambda)} - e_{\frac{1}{2}\alpha(\lambda)})^2}
\]

emerge as the term of order \( \hbar^2 \) (with \( q = e^{\frac{\hbar}{2}} \)) in the Taylor expansion of transfer operators \( D_W \). Secondly, Macdonald operators are found to equal

\[
M_r = \phi_{0}^{k}(\lambda)^{-1} \delta_{q}(\lambda)^{-1} D_{\Lambda_r} \delta_{q}(\lambda) \phi_{0}^{k}(\lambda),
\]

where \( \phi_{0}^{k}(\lambda) \) is the vector-valued character, and thus allowing Macdonald polynomials to be defined as a specific case of the trace functions \( \Psi^{(2)}(\lambda, \mu) \), scaled by the vector-valued character \( \phi_{0}^{k}(\lambda) \).
2 Solutions to the QYBE

Quantum groups were introduced in the mid 1980’s by Drinfeld [Drinfeld, 1985, Drinfeld, 1986] and Jimbo [Jimbo, 1985, Jimbo, 1986] independently. Quantum groups appeared as the algebraic formulation of the work of physicists on the Yang-Baxter equation. The braided structure of quantum groups is closely related to solutions of the quantum Yang-Baxter equation, as is emphasised in this chapter.

2.1 Preliminaries

This section introduces the basic notation used throughout this thesis. Familiarity with semi-simple Lie algebras, Hopf algebras, and quantum groups will be assumed. A selection of known results that are used in this thesis is provided in the appendix for reference. A reader less familiar with the material may turn to [Humphreys, 1972] and [Kassel et al., 1997] for a detailed treatment of Lie algebras and quantum groups respectively.

Let \( \mathfrak{g} \) a semi-simple Lie algebra and \( \mathfrak{h} \) its Cartan subalgebra. The quantized universal enveloping algebra \( \mathcal{U}_q(\mathfrak{g}) \) is the corresponding quantum group, and is braided with a universal \( \mathcal{R} \)-matrix \( \mathcal{R} \) (see appendix remark A.35). Throughout this thesis, \( q \in \mathbb{C} \) will be assumed not to be a root of unity. Verma modules of \( \mathcal{U}_q(\mathfrak{g}) \) of highest weight \( \lambda \in \mathfrak{h}^* \) will be denoted by \( M_\lambda \), and are irreducible for generic \( \lambda \) (see appendix definition A.27).

Of particular interest is \( \mathfrak{g} = \mathfrak{sl}_n \), which will be the standard choice throughout this thesis. This choice was made because quantum \( \mathfrak{sl}_n \), or \( \mathcal{U}_q(\mathfrak{sl}_n) \), is more practical and insightful to work with. Moreover, the results for \( \mathfrak{sl}_n \) can still be generalized to any semi-simple Lie algebra \( \mathfrak{g} \) without significant modification of the theory; see [Etingof and Latour, 2005].

Since many elements and operators will exist in tensor products of spaces and representations, a common notation is adopted to indicate on which components of tensor products these elements and operators act. An ordered series of superscript integers indicates in which component of the tensor product the respective components of the element or operator act. See appendix section B for further details and notation.

A result that will be used in many expressions is the identification of the Cartan subalgebra \( \mathfrak{h} \) with its dual \( \mathfrak{h}^* \). The Killing form \( \kappa \) on \( \mathfrak{g} \times \mathfrak{g} \) can be restricted to
\[ (\mathfrak{h} \times \mathfrak{h}) \text{ to yield a non-degenerate bilinear form} \]
\[ \kappa_{\mathfrak{h}} := \kappa|_{\mathfrak{h} \times \mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}. \]

This map gives rise to an orthonormal basis \( \{x_i\} \) of the Cartan subalgebra \( \mathfrak{h} \), i.e.
\[ \kappa_{\mathfrak{h}}(x_i, x_j) = \delta_{ij}, \]
and the identifying isomorphism
\[ (\mathfrak{h} \to \mathfrak{h}^* : x \mapsto \kappa_{\mathfrak{h}}(x, -)). \]

### 2.1 Notation:
For each \( \lambda \in \mathfrak{h}^* \), let \( \overline{\lambda} \in \mathfrak{h} \) denote the element for which
\[ \kappa_{\mathfrak{h}}(\overline{\lambda}, \cdot) = \lambda. \]

The element \( \overline{\lambda} \in \mathfrak{h} \) may be conveniently written as
\[ \overline{\lambda} = \sum_i \lambda(x_i) x_i \]
since
\[ \kappa_{\mathfrak{h}}(\overline{\lambda}, x_j) = \lambda(x_j) = \sum_i \lambda(x_i) \kappa_{\mathfrak{h}}(x_i, x_j) = \kappa_{\mathfrak{h}}\left( \sum_i \lambda(x_i) x_i, x_j \right). \]

Note that this means that \( \mu(\overline{\lambda}) = \sum_i \lambda(x_i) \mu(x_i) = \lambda(\overline{\mu}) \) for all \( \lambda, \mu \in \mathfrak{h}^* \).

### 2.2 The Quantum Yang-Baxter Equation

A universal \( R \)-matrix of \( \mathfrak{u}_q(\mathfrak{sl}_n) \) is a solution to the quantum Yang-Baxter equation.

#### 2.2 Theorem (QYBE):

A universal \( R \)-matrix \( R \) of a Hopf algebra \( H \) satisfies the quantum Yang-Baxter equation on \( H \otimes H \otimes H \):
\[ R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}. \]

**Proof.** Using relation 2 from the definition of the universal \( R \)-matrix, appendix definition A.3, then relation 1, and then relation 2 again provides
\[ R^{12} R^{13} R^{23} = (R \otimes 1) (\Delta \otimes \text{id})(R) \]
\[ = (\Delta^{\text{op}} \otimes \text{id})(R) (R \otimes 1) \]
\[ = (\tau \otimes \text{id})(\Delta \otimes \text{id})(R) R^{12} \]
\[ = (\tau \otimes \text{id})(R^{13} R^{23}) R^{12} \]
\[ = R^{23} R^{13} R^{12}, \]
where \( \tau \) is the swap. \( \square \)
Now recall the definition of the tensor product of two representations (appendix definition A.13). For representations $U, V, W$ of $U_q(\mathfrak{sl}_n)$ write
\begin{equation}
R_{UV}^{12} = R_{UV} \otimes \text{id} : U \otimes V \otimes W \to U \otimes V \otimes W,
\end{equation}
with similar expressions for $R_{VW}^{23}$ and $R_{UW}^{13}$.

2.3 Proposition:
Let $U, V, W$ representations of $U_q(\mathfrak{sl}_n)$, then
\begin{equation}
R_{UV}^{12}R_{UW}^{13}R_{VW}^{23} = R_{VW}^{23}R_{UW}^{13}R_{UV}^{12}
\end{equation}
on $U \otimes V \otimes W$.

Proof.
\begin{equation}
R_{UV}^{12}R_{UW}^{13}R_{VW}^{23} = (\pi_U \otimes \pi_V \otimes \pi_W)R_{12}^{13}R_{23}^{23}
\end{equation}
since $\pi_U, \pi_V, \pi_W$ are linear. Now because $R$ is a solution to the QYBE,
\begin{equation}
(\pi_U \otimes \pi_V \otimes \pi_W)R_{12}^{13}R_{23}^{23} = (\pi_U \otimes \pi_V \otimes \pi_W)R_{23}^{23}R_{13}^{13}R_{12}^{12}
\end{equation}
and the result follows.

2.4 Corollary:
The operator $R_{VV}$ is a solution to the QYBE on $V \otimes V \otimes V$.

The following section will take a closer look at finding solutions to the QYBE, based on the universal $R$-matrix $R$. Chapter 3 will instead consider the operator $R_{WV}$, resulting in solutions to a dynamical analog of the QYBE.

2.3 Twisting
Theorem 2.2 shows that the universal $R$-matrix of $U_q(\mathfrak{sl}_n)$ is a universal solution to the QYBE. This implies that solutions to the QYBE can be found by finding universal $R$-matrices. One way to do this is through twisting.

Consider a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$.

2.5 Definition:
An invertible element $J \in H \otimes H$ is called a twist if
\begin{equation}
(\Delta \otimes \text{id})(J) (J \otimes 1) = (\text{id} \otimes \Delta)(J) (1 \otimes J)
\end{equation}
in $H \otimes H \otimes H$. This condition is also called the non-dynamical twist equation and will be abbreviated as
\begin{equation}
J^{12,3} J^{12} = J^{1,23} J^{23}.
\end{equation}
2.6 Remark:
Note that applying the counit \((\text{id} \otimes \epsilon \otimes \text{id})\) to equation 2.12 provides

\[
((\text{id} \otimes \epsilon)\Delta \otimes \text{id})(J)((\text{id} \otimes \epsilon)J \otimes 1) = (\text{id} \otimes (\epsilon \otimes \text{id})\Delta)(J)(1 \otimes (\epsilon \otimes \text{id})J)
\]

because of the counitality axiom and the fact that \(J\) is invertible. Now applying another counit \((\epsilon \otimes \text{id})\) shows

\[
(\epsilon \otimes \epsilon)(J) = (\epsilon \otimes \text{id})(J)
\]

where simply \((\epsilon \otimes \epsilon)(J) \in \mathbb{C}\). Hence it may be assumed, without loss of generality, that \((\epsilon \otimes \epsilon)(J) = 1\) and thus that \((\epsilon \otimes \text{id})(J) = 1\). By the same assumption, \((\text{id} \otimes \epsilon)(J) = 1\).

2.7 Proposition:
Writing \(J = \sum_i a_i \otimes b_i\), define \(\Delta J : H \rightarrow H \otimes H\) and \(S J : H \rightarrow H\) as

\[
\Delta J(x) := J^{-1} \Delta(x) J \quad \text{and} \quad S J(x) := p^{-1} S(x) p,
\]

with \(p = \sum_i S(a_i) b_i\). Then \((H, \mu, \eta, \Delta J, \epsilon, S J)\) is a Hopf algebra and is called the twist of \(H\) by \(J\), denoted as \(H_J\).

Proof. To be a comultiplication, \(\Delta J\) must satisfy the coassociativity axiom 3.

\[
(\Delta J \otimes \text{id}) \Delta J(x) = (\Delta J \otimes \text{id})(J^{-1} \Delta(x) J)
\]

which becomes, using \((\Delta J \otimes \text{id})(y) = (J^{-1} \otimes 1)(\Delta \otimes \text{id})(J \otimes 1)\),

\[
(\text{id} \otimes J^{-1})(\Delta \otimes \text{id})(J^{-1})(\Delta \otimes \text{id})(\Delta(x))(\Delta \otimes \text{id})(J)(J \otimes 1).
\]

Now applying the condition for a twist, equation 2.12, and axiom 3 turns this into

\[
(1 \otimes J^{-1})(\text{id} \otimes \Delta)(J^{-1})(\text{id} \otimes \Delta)(\Delta(x))(\text{id} \otimes \Delta)(J)(1 \otimes J)
\]

and then reduces to

\[
(\text{id} \otimes \Delta J)(J^{-1})(\text{id} \otimes \Delta J)(\Delta(x))(\text{id} \otimes \Delta J)(J) = (\text{id} \otimes \Delta J)(J^{-1} \Delta(x) J)
\]

as required.

Considering remark 2.6, the counitality axiom is trivially satisfied for \(\epsilon\) since

\[
(\epsilon \otimes \text{id}) \Delta J(x) = (\epsilon \otimes \text{id})(J^{-1} \Delta(x) J)
\]

\[
= (\epsilon \otimes \text{id})(J^{-1})(\epsilon \otimes \text{id}) \Delta(x)(\epsilon \otimes \text{id})(J)
\]

\[
= x.
\]
Finally, $S_J$ must satisfy the convolution inverse axiom \[5\]. However, before being able to show this, it must be proved that the element $p$ is indeed invertible. Writing $J^{-1} = \sum_i a'_i \otimes b'_i$ and using the twist equation \[2.12\] remark \[2.6\] the convolution inverse axiom \[5\] for $S$, and the Sweedler notation (see appendix section \[A.2.2\]), it is shown that

\[
\sum_i a'_i S(b'_i) p = \sum_{i,j} a'_i S(b'_i) S(a_j) b_j
\]

\[
= \sum_{i,j,k} a_k \epsilon(b_k) a'_i S(a_j b'_i) b_j
\]

\[
= \sum_{i,j,k} a_k a'_i S(a_j b'_i) \epsilon(b_k) b_j
\]

\[
= \sum_{i,j,k,(b_k)} a_k a'_i S(a_j b'_i) S(b_k(1)) b_k(2) b_j
\]

\[
= \sum_{i,j,k,(b_k)} a_k a'_i S(b_k(1)) a_j b'_i b_k(2) b_j
\]

\[
= \mu((\mu \otimes \text{id})(\text{id} \otimes S) \otimes \text{id})(J(\Delta \otimes \text{id})(J \otimes (J^{-1} \otimes 1))
\]

\[
= \mu((\mu \otimes \text{id})(\text{id} \otimes S) \otimes \text{id})(\Delta \otimes \text{id})(J \otimes (J \otimes (J^{-1} \otimes 1))
\]

\[
= \mu((\epsilon \otimes \text{id})(J)
\]

\[
= 1
\]

with a similar calculation showing that $p \sum_i a'_i S(b'_i) = 1$, which means that the element $p$ is invertible with inverse

\[
p^{-1} := \sum_i a'_i S(b'_i).
\]

Now $S_J$ may be shown to satisfy the convolution inverse axiom \[5\]. Starting with

\[
\mu(S_J \otimes \text{id})\Delta_J(x) = \mu(S_J \otimes \text{id})(J^{-1} \Delta(x) J)
\]

\[
= \mu((p^{-1} S(\cdot) S \otimes \text{id})(J^{-1} \Delta(x) J)
\]

\[
= \mu((p^{-1} \otimes 1)(S \otimes \text{id})(J^{-1} \Delta(x) J)(p \otimes 1))
\]

\[
= p^{-1} \mu((S \otimes \text{id})(J^{-1} \Delta(x) J)(p \otimes 1))
\]

and writing out $p$, $J$, and $\Delta(x)$ explicitly yields

\[
(S \otimes \text{id})(J^{-1} \Delta(x) J)(p \otimes 1)
\]

\[
= (S \otimes \text{id}) \left( \sum_{j,i,(x)} (a'_j \otimes b'_i)(x_{(1)} \otimes x_{(2)})(a_i \otimes b_i) \right) \left( \sum_k S(a_k) b_k \otimes 1 \right)
\]

\[
= \left( \sum_{j,i,(x)} S(a'_j x_{(1)} a_i) \otimes b'_i x_{(2)} b_i \right) \left( \sum_k S(a_k) b_k \otimes 1 \right)
\]

\[
= \sum_{k,j,i,(x)} S(a_i) S(x_{(1)}) S(a'_j) S(a_k) b_k \otimes b'_i x_{(2)} b_i
\]
which means that
\[ \mu((S \otimes \text{id})(J^{-1} \Delta(x)J)(p \otimes 1)) = \sum_{k,j,i,(x)} S(a_i) S(x_{(1)}) S(a'_j) S(a_k) b_k b'_j x_{(2)} b_i \]
\[ = \sum_{k,j,i,(x)} S(a_i) S(x_{(1)}) S(a_k a'_j) b_k b'_j x_{(2)} b_i \]
\[ = \sum_{i,(x)} S(a_i) S(x_{(1)}) \mu(S \otimes \text{id})(JJ^{-1}) x_{(2)} b_i \]
\[ = \sum_{i,(x)} S(a_i) S(x_{(1)}) \mu(S \otimes \text{id})(JJ^{-1}) x_{(2)} b_i \]
\[ = \sum_{i,(x)} S(a_i) \Delta(x) b_i \]
\[ = \sum_{i,(x)} S(a_i) \mu(S \otimes \text{id}) \Delta(x) b_i \]
\[ = p \epsilon(x) \]
\[
(2.26)
\]
and so finally
\[ \mu(S_J \otimes \text{id}) \Delta_J(x) = p^{-1} p \epsilon(x) = \epsilon(x) \]
\[
(2.27)
\]
as required.

2.8 Proposition:
Let \( H \) a braided Hopf algebra with universal \( R \)-matrix \( R \) and \( J \) a twist, then \( H_J \) with universal \( R \)-matrix
\[ R_J := \tau(J^{-1}) R J \]
is again a braided Hopf algebra.

Proof. For \( R_J \) to be a universal \( R \)-matrix of \( H_J \), it must satisfy the conditions from definition \( \text{A.3} \). The straightforward computation
\[ R_J \Delta_J(x)(R_J)^{-1} = \tau(J^{-1}) R J \Delta_J(x)J^{-1} R^{-1} \tau(J) \]
\[ = \tau(J^{-1}) R \Delta(x) R^{-1} \tau(J) \]
\[ = \tau(J^{-1}) \Delta^{op}(x) \tau(J) \]
\[ = \tau(J^{-1}) \Delta(x) J \]
\[ = \Delta^{op}(x) \]
shows \( R_J \) to satisfy condition \( \square \). For condition \( \Box \), another computation shows
\[ (\Delta_J \otimes \text{id})(R_J) = (\Delta_J \otimes \text{id})(\tau(J^{-1}) R J) \]
\[ = (\Delta_J \otimes \text{id})(\tau(J^{-1})) (\Delta_J \otimes \text{id})(R) (\Delta_J \otimes \text{id})(J) \]
\[ = (J^{-1} \otimes 1) (\Delta \otimes \text{id})(\tau(J^{-1})) R^{13} R^{23} (\Delta \otimes \text{id})(J) (J \otimes 1). \]
Rewriting
\[(J^{-1} \otimes 1)(\Delta \otimes \text{id})(\tau(J^{-1})) = (J^{-1} \otimes 1)(\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \Delta)(J^{-1})
= (\text{id} \otimes \tau)(\tau \otimes \text{id})(1 \otimes J^{-1})(\text{id} \otimes \Delta)(J^{-1})
= (\tau(J^{-1}))^{13}(\text{id} \otimes \tau)(\tau \otimes 1)(\Delta \otimes \text{id})(J^{-1})
= (\tau(J^{-1}))^{13}(\text{id} \otimes \tau)(\Delta \otimes \text{id})(J^{-1})\]
so that
\[(J^{-1} \otimes 1)(\Delta \otimes \text{id})(\tau(J^{-1})) R_{13} = (\tau(J^{-1}))^{13}(\text{id} \otimes \tau)(\Delta \otimes \text{id})(J^{-1}) R_{13}\]
and
\[R_{23}(\Delta \otimes \text{id})(J)(J \otimes 1) = R_{23}(\text{id} \otimes \Delta)(J)(1 \otimes J)
= (\text{id} \otimes \Delta^{\text{op}})(J) R_{23} J^{23},\]
yield
\[(\Delta_J \otimes \text{id})(R_J) = (\tau(J^{-1}))^{13} R_{23}(\text{id} \otimes \Delta)(J)(J^{-1})(\text{id} \otimes \Delta^{\text{op}})(J) R_{23} J^{23}
= (\tau(J^{-1}))^{13} R_{23}(\text{id} \otimes \tau)[(\Delta \otimes \text{id})(J^{-1})(\text{id} \otimes \Delta)(J)] R_{23} J^{23}\]
where
\[(\Delta \otimes \text{id})(J^{-1})(\text{id} \otimes \Delta)(J) = (\Delta \otimes \text{id})(J^{-1})(\text{id} \otimes \Delta)(J)(1 \otimes J)(1 \otimes J^{-1})
= (\Delta \otimes \text{id})(J^{-1})(\Delta \otimes \text{id})(J)(J \otimes 1)(1 \otimes J^{-1})
= (\Delta \otimes \text{id})(1)(J \otimes 1)(1 \otimes J^{-1})
= J^{12}(J^{-1})^{23}\]
and hence
\[(\Delta_J \otimes \text{id})(R_J) = (\tau(J^{-1}))^{13} R_{23}(\text{id} \otimes \tau)[J^{12}(J^{-1})^{23}] R_{23} J^{23}
= (\tau(J^{-1}))^{13} R_{23} J^{13}(J^{-1})^{32} R_{23} J^{23}
= (\tau(J^{-1}))^{13} R_{23} J^{13}(\tau(J^{-1}))^{23} R_{23} J^{23}
= (\tau(J^{-1}) R J)^{13}(\tau(J^{-1}) R J)^{23}
= R_{J}^{13} R_{J}^{23}\]
as required. \(R_J\) is shown to satisfy condition 3 through a similar computation. \(\square\)

Proposition 2.8 shows that a twist \(J\) can be used to find a new universal \(R\)-matrix \(R_J\), and thus a new solution to the QYBE, from a quantum group with universal \(R\)-matrix \(R\).

2.9 Corollary:
For any braided Hopf algebra \(H\) with universal \(R\)-matrix \(R\), \(R^{-1}\) is a twist.
twist of $H$ by $R^{-1}$ is $H^{\text{op}} = (H, \mu, \eta, \Delta^{\text{op}}, \epsilon, S^{-1})$ with universal $R$-matrix $R^{21}$.

Proof. By definition, $R^{-1}$ is invertible. Moreover, by definition [A.3] and the QYBE, equation 2.6
\[
\left(\Delta \otimes \text{id}\right)(R^{-1})(R^{-1} \otimes 1) = (R^{13}R^{23})^{-1}(R^{12})^{-1}
\]
\[
= (R^{12}R^{13}R^{23})^{-1}
\]
\[
= (R^{13}R^{12})^{-1}(R^{23})^{-1}
\]
\[
= (\text{id} \otimes \Delta)(R^{-1})(1 \otimes R^{-1}),
\]
which confirms that $R^{-1}$ is a twist by definition 2.5.

To find the twist of $H$ by $R^{-1}$, a calculation of the comultiplication and the antipode suffices. The comultiplication as defined in proposition 2.7 becomes
\[
\Delta_{R^{-1}}(x) = R\Delta(x)R^{-1} = \Delta^{\text{op}}(x).
\]
Writing $R^{-1} = \sum_i s'_i \otimes t'_i$, the element $p$ becomes
\[
p = \sum_i S(s'_i) t'_i
\]
\[
= \sum_i S(S^{-1}(t'_i) s'_i)
\]
\[
= S(u^{-1}),
\]
where $u$ is the Drinfeld element (appendix definition [A.6]). The antipode, then, is
\[
S_{R^{-1}} = p^{-1} S(x) p
\]
\[
= p^{-1} S(x) S(u^{-1})
\]
\[
= p^{-1} S(u^{-1} x)
\]
\[
= p^{-1} S(S^{-2}(x)u^{-1})
\]
\[
= p^{-1} S(u^{-1})S^{-1}(x)
\]
\[
= S^{-1}(x).
\]
Finally, proposition 2.8 states that
\[
R_{R^{-1}} = \tau(R) R R^{-1} = R^{21}
\]
is the associated universal $R$-matrix. This completes the proof. \qed

The following proposition shows how any twist $J$ gives rise to other twists in a practical and straightforward way.

2.10 Proposition:
Given a twist $J$ and an invertible element $x \in H$,
\[
J_x := \Delta(x) J(x^{-1} \otimes x^{-1})
\]
is also a twist.

Proof. Clearly, $J_x$ is invertible with inverse $J_x^{-1} = (x \otimes x)J^{-1} \Delta(x^{-1})$. To verify the non-dynamical twist equation, equation 2.12, consider

\[(\Delta \otimes \text{id})(J_x)(J_x \otimes 1)\]

\[= (\Delta \otimes \text{id})(\Delta(x)J(x^{-1} \otimes x^{-1})) (\Delta(x)J(x^{-1} \otimes x^{-1}) \otimes 1)\]

\[= (\Delta \otimes \text{id})(\Delta(x)J)(\Delta(x^{-1}) \otimes x^{-1})(\Delta(x)J(x^{-1} \otimes x^{-1}) \otimes 1)\]

\[= (\Delta \otimes \text{id})(\Delta(x)J)(J(x^{-1} \otimes x^{-1}) \otimes x^{-1})\]

\[(2.43)\]

\[= (\text{id} \otimes \Delta)(\Delta(x)) (\Delta \otimes \text{id})(J)(J \otimes 1)(x^{-1} \otimes x^{-1} \otimes x^{-1})\]

\[= (\text{id} \otimes \Delta)(\Delta(x)) (\Delta \otimes \text{id})(J) (1 \otimes J)(x^{-1} \otimes x^{-1} \otimes x^{-1})\]

\[= (\text{id} \otimes \Delta)(\Delta(x)) (\Delta \otimes \text{id})(J)(x^{-1} \otimes \Delta(x^{-1} x))(1 \otimes J(x^{-1} \otimes x^{-1}))\]

\[= (\text{id} \otimes \Delta)(\Delta(x)) (J(x^{-1} \otimes x^{-1}))(1 \otimes \Delta(x)J(x^{-1} \otimes x^{-1}))\]

\[= (\text{id} \otimes \Delta)(J_x)(J_x \otimes 1)\]

which proves the proposition. \qed

2.11 Corollary:
Setting $J = J'_x := 1 \otimes 1$ shows that for any invertible $x \in H$,

\[(2.44)\]

\[J'_x := \Delta(x)(x^{-1} \otimes x^{-1})\]

is a twist.

Proposition 2.10 and corollary 2.11 provide a way to easily generate a plethora of twists. All that is required to find a new twist, and thus a new solution to the QYBE by proposition 2.8, is an invertible element of the Hopf algebra. However, as the next proposition will demonstrate, the ‘new’ solution to the QYBE found in this way is isomorphic to the solution found through the original twist. Hence proposition 2.10 and corollary 2.11 merely generate trivial variations on the solutions of the QYBE.

2.12 Definition:
Two twists $J$ and $J'$ are said to be gauge equivalent if there exists an invertible element $x \in H$ such that

\[(2.45)\]

\[J' = \Delta(x)J(x^{-1} \otimes x^{-1}).\]

2.13 Proposition:
If $J$ and $J'$ are gauge equivalent, then $H_J \cong H_{J'}$. 
Proof. Using the fact that \( J \) and \( J' \) are gauge equivalent, definition \[2.12\] the relation between \( H_J \) and \( H_{J'} \) may be determined.

Recalling proposition \[2.7\] the comultiplications of \( H_J \) and \( H_{J'} \) relate as
\[
\Delta_{J'}(y) = (J')^{-1} \Delta(y) J' \\
= (\Delta(x) J (x^{-1} \otimes x^{-1}))^{-1} \Delta(y) \Delta(x) J (x^{-1} \otimes x^{-1}) \\
= (x \otimes x) J^{-1} \Delta(x)^{-1} \Delta(y) \Delta(x) J (x^{-1} \otimes x^{-1}) \\
= (x \otimes x) J^{-1} \Delta(x^{-1} y x) J (x^{-1} \otimes x^{-1}) \\
= (x \otimes x) \Delta_J(x^{-1} y x) (x^{-1} \otimes x^{-1}).
\]

(2.46)

To compare the antipodes of \( H_J \) and \( H_{J'} \), also defined in proposition \[2.7\], first consider the relation between the elements \( p \) and \( p' \). Writing \( J = \sum_i a_i \otimes b_i \) and \( J' = \sum_i \alpha_i \otimes \beta_i \), the element \( p' \) may be written as
\[
\begin{align*}
p' &= \sum_i S(\alpha_i) \beta_i \\
&= \sum_i S(x(1) a_i x^{-1}) x(2) b_i x^{-1} \\
&= \sum_i S(x^{-1}) S(a_i) S(x(1)) x(2) b_i x^{-1} \\
&= \sum_i S(x^{-1}) S(a_i) \epsilon(x) b_i x^{-1} \\
&= \epsilon(x) S(x^{-1}) p x^{-1}.
\end{align*}
\]

(2.47)

Now the antipodes of \( H_J \) and \( H_{J'} \) are seen to relate as
\[
\begin{align*}
S_{J'}(y) &= (p')^{-1} S(y) p' \\
&= (\epsilon(x) S(x^{-1}) p x^{-1})^{-1} S(y) \epsilon(x) S(x^{-1}) p x^{-1} \\
&= x p^{-1} S(x^{-1})^{-1} \epsilon(x)^{-1} S(y) \epsilon(x) S(x^{-1}) p x^{-1} \\
&= x p^{-1} S(x^{-1} y x) S(y) S(x^{-1}) p x^{-1} \\
&= x S_J(x^{-1} y x) x^{-1}.
\end{align*}
\]

(2.48)

Moreover, recalling proposition \[2.8\] the \( R \)-matrices of \( H_J \) and \( H_{J'} \) relate as
\[
\begin{align*}
R_{J'} &= \tau((J')^{-1}) R J' \\
&= \tau((\Delta(x) J (x^{-1} \otimes x^{-1}))^{-1}) R \Delta(x) J (x^{-1} \otimes x^{-1}) \\
&= (x \otimes x) \tau(J^{-1}) \Delta^\circ(x)^{-1} \Delta^\circ(x) R J (x^{-1} \otimes x^{-1}) \\
&= (x \otimes x) \tau(J^{-1}) R J (x^{-1} \otimes x^{-1}) \\
&= (x \otimes x) R_J(x^{-1} \otimes x^{-1}).
\end{align*}
\]

(2.49)

The relations between the comultiplications, antipodes, and universal \( R \)-matrices of \( H_J \) and \( H_{J'} \) show that there exists an isomorphism
\[
H_J \rightarrow H_{J'} : y \mapsto x y x^{-1}.
\]

(2.50)

Hence \( H_J \cong H_{J'} \). 

\[ \square \]
2.14 Corollary: \( H_{j'} \cong H \)

Proof. Follows from proposition 2.13 and corollary 2.11.

For \( J \) and \( J' \) gauge equivalent the relation between \( R_J \) and \( R_{J'} \), as described in the proof of proposition 2.13, shows that

\[
\begin{align*}
R_{j'}^{12} R_{j'}^{13} R_{j'}^{23} &= (x \otimes x \otimes x) R_J^{12} R_J^{13} R_J^{23} (x^{-1} \otimes x^{-1} \otimes x^{-1}) \\
R_{j'}^{23} R_{j'}^{13} R_{j'}^{12} &= (x \otimes x \otimes x) R_J^{23} R_J^{13} R_J^{12} (x^{-1} \otimes x^{-1} \otimes x^{-1})
\end{align*}
\]

for some invertible element \( x \). This confirms that these solutions to the QYBE are one and the same solution, up to a change of basis, as claimed before.
3 SOLUTIONS TO THE QDYBE

Chapter 2 has shown that universal $R$-matrices of quantum groups are solutions to the QYBE. Moreover, further solutions to the QYBE were found by twisting the universal $R$-matrices. This chapter introduces a dynamical version of the twist, called the fusion operator, which depends on a dynamical parameter $\lambda$. Twisting a universal $R$-matrix using a fusion operator gives rise to an exchange operator or so-called quantum dynamical $R$-matrix that satisfies the dynamical analog of the QYBE; the quantum dynamical Yang-Baxter equation, or QDYBE for short.

3.1 INTERTWINING OPERATORS

Corollary 2.4 shows that operators $R_{VV}$ on representations $V$ of $\mathfrak{u}_q(\mathfrak{sl}_n)$ are solutions to the QYBE. Analogous to twisting universal $R$-matrices, operators $R_{VW}$ may be twisted by a dynamical twist. A dynamical twist called the fusion operator is constructed from intertwining operators that ‘append’ representations of $\mathfrak{u}_q(\mathfrak{sl}_n)$ to Verma modules.

3.1 DEFINITION:

Let $V, W$ finite-dimensional representations of $\mathfrak{u}_q(\mathfrak{sl}_n)$. An operator $\Phi : V \to W$ that commutes with the action of $\mathfrak{u}_q(\mathfrak{sl}_n)$, i.e.

\[
\Phi(xv) = x\Phi(v) \quad \forall \ x \in \mathfrak{u}_q(\mathfrak{sl}_n), \forall \ v \in V,
\]

is called an **intertwining operator**.

Now recall the definition of the Verma module (appendix definition A.26), and the definition of the tensor product of two representations (appendix definition A.8). For a finite-dimensional representation $V$ of $\mathfrak{u}_q(\mathfrak{sl}_n)$, consider the intertwining operator

\[
\Phi : M_\lambda \to M_\mu \otimes V.
\]

Intertwining operators of this form will be the main subject of interest here.

3.2 DEFINITION:

Let $V$ a finite-dimensional representation of $\mathfrak{u}_q(\mathfrak{sl}_n)$ and $\lambda, \mu \in \mathfrak{h}^*$ weights of $V$, then the map

\[
\langle \cdot \rangle : \text{Hom}_{\mathfrak{u}_q(\mathfrak{sl}_n)}(M_\lambda, M_\mu \otimes V) \to V : \Phi \mapsto (v_\mu^* \otimes \text{id})(\Phi v_\lambda)
\]

is called the **expectation value map**.
3.3 Proposition:
For an intertwiner $\Phi : M_\lambda \to M_\mu \otimes V$,

\[(3.4)\quad \langle \Phi \rangle \in V[\lambda - \mu].\]

Proof. Using the fact that $\Phi$ is an intertwiner, definition 3.1 provides

\[(3.5)\quad K_i \Phi v_\lambda = \Phi q^\lambda(h_i) v_\lambda = q^\lambda(h_i) \Phi v_\lambda,\]

which implies that $\text{wt } \Phi v_\lambda = \lambda$. Now write $\Phi v_\lambda = \sum_i a_i \otimes b_i$ with $a_i \in M_\mu$ and $b_i \in V$, so that the expectation value of $\Phi$ becomes

\[(3.6)\quad \langle \Phi \rangle = (v^*_\mu \otimes \text{id})(\Phi v_\lambda) = (v^*_\mu \otimes \text{id})(\sum_i a_i \otimes b_i) = \sum_i v^*_\mu(a_i) b_i.\]

Since $v^*_\mu \in M^*_\mu$ satisfies $v^*_\mu(v_\mu) = 1$ and $v^*_\mu(w) = 0$ for $\text{wt } w < \mu$, it follows that

\[(3.7)\quad v^*_\mu(a_i) \neq 0 \iff \text{wt } a_i = \mu.\]

Because $\text{wt } a_i + \text{wt } b_i = \text{wt } a_i \otimes b_i = \text{wt } \Phi v_\lambda = \lambda$, this means that $v^*_\mu(a_i) b_i \neq 0$ if and only if $\text{wt } b_i = \lambda - \mu$, proving that $\sum_i v^*_\mu(a_i) b_i \in V[\lambda - \mu].$ \hfill $\square$

3.4 Proposition:
If $M_\mu$ is irreducible, i.e. $\mu$ is generic, the expectation value map

\[(3.8)\quad \langle \cdot \rangle : \text{Hom}_U(q(sl_n))(M_\lambda, M_\mu \otimes V) \to V[\lambda - \mu]\]

is an isomorphism.

Proof. Recall the definition of the Verma module (appendix definition A.26), the one-dimensional representation $\lambda$ of $U_q(b_+)$ (appendix proposition A.23), and the restricted dual (appendix definition A.34). Applying appendix proposition A.25 and the fact that

\[(3.9)\quad \text{Hom}(U, V \otimes W) \cong \text{Hom}(V^\circ \otimes U, W),\]

yields

\[(3.10)\quad \text{Hom}_U(q(sl_n))(M_\lambda, M_\mu \otimes V) \cong \text{Hom}_U(q(sl_n))(U_q(sl_n) \otimes_U b_+, \lambda, M_\mu \otimes V) \cong \text{Hom}_U(q(b_+))(\lambda, M_\mu \otimes V) \cong \text{Hom}_U(q(b_+))(M_\mu^c \otimes \lambda, V) \cong \text{Hom}_U(q(b_+))(M_\mu^c, V \otimes \lambda^c).\]
Now consider the automorphism $\omega : \mathcal{U}_q(\mathfrak{sl}_n) \to \mathcal{U}_q(\mathfrak{sl}_n)$ given by
\[\omega(E_i) = F_i \quad \omega(K^+) = K^+ \quad \omega(F_i) = E_i\]
which, for a representation $(W, \pi_W)$ of $\mathcal{U}_q(\mathfrak{sl}_n)$, defines another representation $W^\omega = (W, \pi_W \circ \omega)$ of $\mathcal{U}_q(\mathfrak{sl}_n)$. Applying the automorphism $\omega$ to the right hand side of equation (3.10) results in
\[\text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(M_\mu, V \otimes \lambda^\circ) \cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(M_\mu^\omega, V^\omega \otimes \lambda),\]
since $\lambda^\omega = (-\lambda)^\omega = \lambda$. Now consider the unique non-degenerated bilinear form, called the Shapovalov form (see theorem 2.43 of [Etingof and Latour, 2005])
\[\phi : v \mapsto S_\mu(v, \cdot)\] on $M_\mu$ is seen to map into $M_\mu^\omega$ since
\[\phi(x v) = S_\mu(x v, \cdot) = S_\mu(v, S(\omega(x)) \cdot) = \omega(x) \phi(v).\]
Because $M_\mu$ is irreducible, this map $\phi$ is an injection. Moreover, because the weight spaces of $M_\mu$ and $M_\mu^\omega$ have the same dimensions, $\phi$ is an isomorphism. Thus,
\[\text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(M_\mu^\omega, V^\omega \otimes \lambda) \cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(M_\mu, V^\omega \otimes \lambda)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathcal{U}_q(\mathfrak{b}_-), \mu, V^\omega \otimes \lambda)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\mathcal{U}_q(\mathfrak{b}_-) \otimes \mathcal{U}_q(\mathfrak{b}_-), \mu, V^\omega \otimes \lambda)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\mu, V^\omega \otimes \lambda)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\mu \otimes \lambda^\circ, V^\omega)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\mu - \lambda, V^\omega)\]
\[\cong \text{Hom}_{\mathcal{U}_q(\mathfrak{b}_-)}(\lambda - \mu, V)\]
\[\cong V[\lambda - \mu].\]
Now putting together equations (3.10), (3.11) and (3.15) yields
\[\text{Hom}_{\mathcal{U}_q(\mathfrak{sl}_n)}(M_\lambda, M_\mu \otimes V) \cong V[\lambda - \mu],\]
where the isomorphism is given by the map $\langle \cdot \rangle$, completing the proof.

**3.5 Corollary:**
If $M_\lambda$ is irreducible, then for all $v \in V$, there exists a unique intertwining operator $\Phi : M_{\lambda + \text{wt} v} \to M_\lambda \otimes V$ such that $\langle \Phi \rangle = v$. This unique $\Phi$ is denoted by $\Phi^{\lambda + \text{wt} v}$. 

\[\]
3.2 FUSION OPERATORS

The intertwining operator $\Phi^\lambda_v$ introduced in corollary 3.5 provides a means to ‘append’ specific Verma modules to a representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$, yielding again a representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$. Iterating this process allows multiple representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$ to be ‘fused’ together.

3.6 Definition:

Let $V, W$ finite-dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$ and $v \in V$, $w \in W$, then

\[ \Phi^\lambda_v \Phi^\lambda_w = (\Phi^\lambda_{-\text{wt}w} \otimes \text{id}) \Phi^\lambda_w : M_\lambda \rightarrow M_{\lambda-\text{wt}w-\text{wt}v} \otimes V \otimes W, \]

for any generic $\lambda$, defines the composition of two intertwining operators.

Note that the expectation value $\langle \Phi^\lambda_v \Phi^\lambda_w \rangle$ of this composition is a bilinear function in $v$ and $w$. Therefore, there exists a linear operator $V \otimes W \rightarrow V \otimes W$, of weight zero, that sends $v \otimes w$ to $\langle \Phi^\lambda_v \Phi^\lambda_w \rangle$.

3.7 Definition:

Let $V, W$ finite-dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$, then the linear operator

\[ J_{VW}(\lambda) : V \otimes W \rightarrow V \otimes W : v \otimes w \mapsto \langle \Phi^\lambda_v \Phi^\lambda_w \rangle \]

is called the fusion operator.

Note that because $J_{VW}(\lambda)(v \otimes w) = \langle \Phi^\lambda_v \Phi^\lambda_w \rangle$, by corollary 3.5 $\Phi^\lambda_v \Phi^\lambda_w = \Phi^\lambda_{J_{VW}(\lambda)(v \otimes w)}$.

3.8 Proposition:

The fusion operator $J_{VW}(\lambda)$ satisfies the following properties:

i. $J_{VW}(\lambda)$ has zero weight.

ii. $J_{VW}(\lambda)$ is lower triangular with respect to the weight decomposition, and has ones on its diagonal. This means that $J_{VW}(\lambda)(v \otimes w) = v \otimes w + \sum_i c_i \otimes b_i$ where $\text{wt} c_i < \text{wt} v$ and $\text{wt} b_i > \text{wt} w$ for all $v, w$.

iii. $J_{VW}(\lambda)$ is invertible when defined.

Proof. The first property follows directly from proposition 3.4 and definition 3.6, which imply that for any weight $\mu$, $J_{VW}(\lambda)$ maps $(V \otimes W)[\mu]$ into itself.

For the second property, consider

\[ \Phi^\lambda_v v_\lambda = v_{\lambda-\text{wt}w} \otimes w + \sum_i a_i \otimes b_i \]
where necessarily \( \text{wt} a_i < \lambda - \text{wt} w \) since \( v_{\lambda - \text{wt} w} \) is a highest weight vector, and, consequentially, \( \text{wt} b_i > \text{wt} w \) since \( \text{wt} a_i + \text{wt} b_i = \lambda \). Applying \( \Phi^{v}_{\lambda - \text{wt} w} \otimes \text{id} \) yields
\[
(3.20) \quad (\Phi^{v}_{\lambda - \text{wt} w} \otimes \text{id}) \Phi^{w}_{\lambda} v_{\lambda} = v_{\lambda - \text{wt} w - \text{wt} v} \otimes v \otimes w + \sum_{i} \Phi^{v}_{\lambda - \text{wt} w} a_i \otimes b_i + T
\]
where the weight of the first component of \( T \) is lower than the highest weight \( \lambda - \text{wt} w - \text{wt} v \). Now applying \( v^{*}_{\lambda - \text{wt} w - \text{wt} v} \otimes \text{id} \otimes \text{id} \) to both sides provides, on the left hand side,
\[
(3.21) \quad (v^{*}_{\lambda - \text{wt} w - \text{wt} v} \otimes \text{id} \otimes \text{id})(\Phi^{v}_{\lambda - \text{wt} w} \otimes \text{id}) \Phi^{w}_{\lambda} v_{\lambda} = v_{\lambda - \text{wt} w - \text{wt} v} \otimes v \otimes w + \sum_{i} \Phi^{v}_{\lambda - \text{wt} w} a_i \otimes b_i + T
\]
while on the right hand side this provides
\[
(3.22) \quad v^{*}_{\lambda - \text{wt} w - \text{wt} v}(v_{\lambda - \text{wt} w - \text{wt} v} \otimes \text{id})(\Phi^{v}_{\lambda - \text{wt} w} a_i) \otimes b_i = v \otimes w + \sum_{i} c_i \otimes b_i
\]
where \( \text{wt} c_i = \text{wt} a_i - (\lambda - \text{wt} w - \text{wt} v) < \lambda - \text{wt} w - \lambda + \text{wt} w + \text{wt} v = \text{wt} v \). Equating the left and right hand sides results in
\[
(3.23) \quad \langle \Phi^{v}_{\lambda} w \rangle = v \otimes w + \sum_{i} c_i \otimes b_i
\]
where \( \text{wt} c_i < \text{wt} v \) and \( \text{wt} b_i > \text{wt} w \), as required.

The third property follows from the second property.

In section 2.3 a twist \( J \), satisfying the non-dynamical twist equation \( 2.13 \), was used to find solutions to the QYBE. The fusion operator \( J_{\nu \lambda}(\lambda) \) satisfies a dynamical analog of equation \( 2.13 \).

3.9 Theorem:
Let \( U, V, W \) finite-dimensional representations of \( \mathfrak{U}_q(\mathfrak{sl}_n) \), then the fusion operator satisfies the dynamical twist equation on \( U \otimes V \otimes W \):
\[
(3.24) \quad J^{123}_{U \otimes V \otimes W}(\lambda) J^{2}_{U \otimes V}(\lambda - h^3) = J^{123}_{U \otimes V \otimes W}(\lambda) J^{23}_{V \otimes W}(\lambda),
\]
using the notation \( J^{1}_{U \otimes V}(\lambda - h^3)(u \otimes v \otimes w) := (J_{UV}(\lambda - \text{wt} w)(u \otimes v)) \otimes w \) and \( J^{123}_{U \otimes V, W}(\lambda) := J_{U \otimes V \otimes W}(\lambda) \).

Proof. Consider the iterated composition
\[
(3.25) \quad (\Phi^{v}_{\lambda - \text{wt} w - \text{wt} v} \otimes \text{id} \otimes \text{id})(\Phi^{v}_{\lambda - \text{wt} w} \otimes \text{id}) \Phi^{w}_{\lambda}
\]
that maps
\[
(3.26) \quad M_{\lambda} \rightarrow M_{\lambda - \text{wt} w - \text{wt} v - \text{wt} u} \otimes U \otimes V \otimes W
\]
from two different perspectives. Firstly, write equation (3.25) as
\[
(\Phi^u_{\lambda-\text{wt } w} \otimes \text{id}) \Phi^w_{\lambda} = (\Phi^u_{\lambda-\text{wt } w} \otimes v) \otimes \text{id}) \Phi^w_{\lambda} \\
= \Phi^u_{\lambda-\text{wt } w}(u \otimes v, w) \\
= \Phi^u_{\lambda}(u \otimes v) \otimes w).
\]
(3.27)

Secondly, write equation (3.25) as
\[
(\Phi^u_{\lambda-\text{wt } w-\text{wt } v} \otimes \text{id} \otimes \text{id}) \Phi^{v,w}_{\lambda} = (\Phi^u_{\lambda-\text{wt } w-\text{wt } v} \otimes \text{id} \otimes \text{id}) \Phi^v_{\lambda}(v \otimes w) \\
= \Phi^u_{\lambda}(v \otimes w) \\
= \Phi^u_{\lambda}(v \otimes w).
\]
(3.28)

Taking the expectation value of equations (3.27) and (3.28) and equating them, yields
\[
(J_{U \otimes V, W}(\lambda)(J_{UV}(\lambda-\text{wt } w)(u \otimes v) \otimes w)) = (J_{UV}(\lambda)(u \otimes J_{WW}(\lambda)(v \otimes w))
\]
for all \( u \in U, v \in V \) and \( w \in W \). Hence
\[
J^{1,2,3}_{U \otimes V, W}(\lambda) \ J^{2,3}_{UV}(\lambda - \hbar^2) = J^{2,3}_{U, V \otimes W}(\lambda) \ J^{2,3}_{WW}(\lambda)
\]
on \( U \otimes V \otimes W \) as required. \( \square \)

3.2.1 The Universal Fusion Operator

The fusion operator \( J_{WW}(\lambda) \) can be generalized to a universal fusion operator \( J(\lambda) \) which specializes to the normal fusion operator \( J_{WW}(\lambda) \) on every \( V \otimes W \). This universal fusion operator \( J(\lambda) \) will be especially useful in chapter 4 when it is necessary to consider manipulations of operators without restricting to specific representations.

3.10 Theorem:

There is a unique \( J(\lambda) \in \mathfrak{U}_q(\mathfrak{sl}_n) \otimes \mathfrak{U}_q(\mathfrak{sl}_n) \) (see appendix remark A.35) with the properties from proposition 3.8 that satisfies the equation
\[
J(\lambda) \left( \text{id} \otimes q^{\theta(\lambda)} \right) = R^{21} q^{-\sum_i x_i} \left( \text{id} \otimes q^{\theta(\lambda)} \right) J(\lambda),
\]
where \( \theta(\lambda) = 2(\lambda + \bar{\lambda}) - \sum_i x_i^2 \). This unique solution \( J(\lambda) \) of zero weight is called the universal fusion operator, which has the property that it specializes to the fusion operator \( J_{WW}(\lambda) \) on \( V \otimes W \) for \( V, W \) finite-dimensional representations of \( \mathfrak{U}_q(\mathfrak{sl}_n) \).

The idea of the proof is to define an expression
\[
F(\lambda) := (\nu^*_{\lambda-\text{wt } w-\text{wt } v} \otimes \text{id} \otimes \text{id})((\Phi^u_{\lambda-\text{wt } w} \otimes \text{id}) (u \otimes q^{-2\bar{\lambda}} \otimes \text{id}) \Phi^w_{\lambda} v_{\lambda}),
\]
(3.32)
where \( u \) is the Drinfeld element (appendix definition A.6) and \( u q^{-2\rho} \) is central in \( \mathfrak{U}_q(\mathfrak{sl}_n) \). \( F(\lambda) \) will then be computed in two different ways. First, it is shown that \( u q^{-2\rho}|_M = q^{-\lambda(\lambda-2\rho)}id \), i.e. it acts as straightforward multiplication, so that

\[
\begin{align*}
(3.33) \\
F(\lambda) = q^{-\lambda(wtw-\lambda+2\rho)}J_{VW}(\lambda)(v \otimes w).
\end{align*}
\]

The second way to compute \( F(\lambda) \) is to pull the element \( u q^{-2\rho} \) through the intertwiner \( \Phi_{\lambda-wtw}^\chi \) and work out the expression from there. The two resulting expressions for \( F(\lambda) \) are then equated to show that \( J_{VW}(\lambda) \) is a solution to the equation for every \( V, W \). What is left is to show that there exists a solution \( J(\lambda) \) as in the theorem that specializes to \( J_{VW}(\lambda) \) on every \( V, W \).

A full proof of theorem 3.10 can be found in [Etingof and Schiffmann, 2002].

3.2.2 Example: \( \mathfrak{U}_q(\mathfrak{sl}_2) \)

The fusion operator \( J_{VW}(\lambda) \) can be explicitly computed for finite-dimensional representations \( V \) of \( \mathfrak{U}_q(\mathfrak{sl}_2) \); for example the irreducible two-dimensional representation \( V = V_4 \). Appendix section A.3.3 states that this representation of highest weight \( \lambda = 1 \) equals \( V[+1] \oplus V[-1] = \mathbb{C} \, v_+ \oplus \mathbb{C} \, v_- \) under the actions

\[
\begin{align*}
K^\pm v_+ &= q^\pm v_+ & K^\pm v_- &= q^\pm v_- \\
F v_+ &= v_- & F v_- &= 0 \\
E v_+ &= 0 & E v_- &= v_+.
\end{align*}
\]

Now consider \( V \otimes V \) with ordered basis consisting of:

\[
\begin{align*}
v_+ \otimes v_+ & \text{ of weight 2,} \\
v_+ \otimes v_- & \text{ of weight 0,} \\
v_- \otimes v_+ & \text{ of weight 0,} \\
v_- \otimes v_- & \text{ of weight -2.}
\end{align*}
\]

Proposition 3.8 states that the fusion operator is lower triangular with ones on the diagonal, and thus that \( J_{VW}(\lambda) \) fixes each basis element except \( v_+ \otimes v_- \). This leaves \( J_{VW}(\lambda)(v_+ \otimes v_-) = \langle(\Phi_{\lambda+1}^\chi \otimes id_v) \Phi_{\lambda}^{v_-}\rangle \) to be determined. Corollary 3.5 and the implication of definition 3.2 determine

\[
\begin{align*}
(3.36) \quad \Phi_{\lambda}^{v_-} v_{\lambda} &= v_{\lambda+1} \otimes v_- + \psi(q, \lambda) F v_{\lambda+1} \otimes v_+ \\
\end{align*}
\]

for \( \Phi_{\lambda}^{v_-} : M_{\lambda+1+wtv_-} \rightarrow M_{\lambda+1} \otimes V \), where \( \psi \) is some unknown function of \( q \) and \( \lambda \). The next step then is to determine

\[
\begin{align*}
(3.37) \quad (v_+ \otimes id_v)(\Phi_{\lambda+1}^{v_-} v_{\lambda+1}) &= \langle \Phi_{\lambda+1}^{v_-} \rangle = v_+ \\
\end{align*}
\]
and
\[
(v^*_\lambda \otimes \text{id}_\nu)(\Phi_{v_{\lambda+1}}^{v_\mu} F v_{\lambda+1}) = (v^*_\lambda \otimes \text{id}_\nu)(\Delta(F) \Phi_{v_{\lambda+1}}^{v_\mu} v_{\lambda+1}) \\
= (v^*_\lambda \otimes \text{id}_\nu)((F \otimes 1 + K^{-1} \otimes F) \Phi_{v_{\lambda+1}}^{v_\mu} v_{\lambda+1}) \\
= (v^*_\lambda \otimes \text{id}_\nu)((K^{-1} \otimes F) \Phi_{v_{\lambda+1}}^{v_\mu} v_{\lambda+1}) \\
= q^{-\lambda} F (v^*_\lambda \otimes \text{id}_\nu)(\Phi_{v_{\lambda+1}}^{v_\mu} v_{\lambda+1}) \\
= q^{-\lambda} F v_+ \\
= q^{-\lambda} v_-
\]

These three results yield
\[
J_{\nu\nu}(\lambda)(v_+ \otimes v_-) = \langle (\Phi_{v_{\lambda+1}}^{v_\mu} \otimes \text{id}_\nu) \Phi_{v_{\lambda-1}}^{v_-} \rangle \\
= (v^*_\lambda \otimes \text{id}_\nu)(\Phi_{v_{\lambda+1}}^{v_\mu} \otimes \text{id}_\nu)(\Phi_{v_{\lambda-1}}^{v_-} v_\lambda) \\
= (v^*_\lambda \otimes \text{id}_\nu)(\Phi_{v_{\lambda+1}}^{v_\mu} \otimes \text{id}_\nu)(\Phi_{v_{\lambda-1}}^{v_-} v_\lambda) \\
= (v^*_\lambda \otimes \text{id}_\nu)(\Phi_{v_{\lambda+1}}^{v_\mu} v_{\lambda+1} \otimes v_- + \psi(q, \lambda) F v_{\lambda+1} \otimes v_+) \\
= v_+ \otimes v_- + \psi(q, \lambda) q^{-\lambda} v_- \otimes v_+
\]

Finally, to determine \(\psi(q, \lambda)\) consider
\[
0 = \Phi_{v_{\lambda-1}}^{v_-} E v_\lambda \\
= \Delta(E) \Phi_{v_{\lambda-1}}^{v_-} v_\lambda \\
= (E \otimes K + 1 \otimes E)(v_{\lambda+1} \otimes v_- + \psi(q, \lambda) F v_{\lambda+1} \otimes v_+) \\
= E v_{\lambda+1} \otimes K v_- + v_{\lambda+1} \otimes E v_- \\
\quad + \psi(q, \lambda) (EF v_{\lambda+1} \otimes K v_+ + F v_{\lambda+1} \otimes E v_+)
\]

which, using relations specified in appendix equations A.43 and A.44 and equation 3.34 reduces to
\[
0 = 0 + v_{\lambda+1} \otimes v_+ + \psi(q, \lambda) \left( \frac{K - K^{-1}}{q - q^{-1}} + FE \right) v_{\lambda+1} \otimes q v_+ + 0 \\
= v_{\lambda+1} \otimes v_+ + \psi(q, \lambda) \left( \frac{q^{\lambda-1} - q^{-(\lambda+1)}}{q - q^{-1}} v_{\lambda+1} \otimes q v_+ \right) \\
= \left( 1 + \psi(q, \lambda) \frac{q^{\lambda-1} - q^{-(\lambda+1)}}{q - q^{-1}} \right) v_{\lambda+1} \otimes v_+ \\
\Rightarrow 0 = 1 + \psi(q, \lambda) \frac{q^{\lambda-1} - q^{-(\lambda+1)}}{q - q^{-1}} q
\]

and thus provides
\[
\psi(q, \lambda) = \frac{q - q^{-1}}{q^{-\lambda-1} - q^{\lambda+1}} q^{-1}.
\]

Substituting this result into the expression for \(J_{\nu\nu}(\lambda)(v_+ \otimes v_-)\) yields
\[
J_{\nu\nu}(\lambda)(v_+ \otimes v_-) = v_+ \otimes v_- + \frac{q - q^{-1}}{1 - q^{2(\lambda+1)}} v_- \otimes v_+.
\]
The fusion operator $J_{VV}(\lambda)$ can now be written as the matrix

\[(3.44)\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q^{-1} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with $(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)$ the ordered basis of $V \otimes V$.

### 3.3 Exchange Operators

Proposition 2.8 shows how a universal $R$-matrix, or solution to the QYBE, can be constructed by twisting another universal $R$-matrix. This suggests, in light of theorem 3.9, that twisting an operator $R_{VW}$ with the fusion operator, a dynamical twist, might yield a solution to the quantum dynamical Yang-Baxter equation.

#### 3.11 Definition:

Let $V, W$ finite-dimensional representations of $U_q(\mathfrak{sl}_n)$ and $R$ its universal $R$-matrix, then the operator

\[(3.45)\]

\[R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} R_{WV}^{21}(\lambda) : V \otimes W \rightarrow V \otimes W\]

is called the exchange operator.

#### 3.12 Theorem (QDYBE):

Let $U, V, W$ finite-dimensional representations of $U_q(\mathfrak{sl}_n)$, then the exchange operator satisfies the quantum dynamical Yang-Baxter equation on $U \otimes V \otimes W$.

\[(3.46)\]

\[R_{UU}^{12}(\lambda - h^3) R_{UV}^{13}(\lambda) R_{VW}^{23}(\lambda - h^1) = R_{VV}^{23}(\lambda) R_{WV}^{12}(\lambda) R_{WW}^{13}(\lambda - h^2) R_{UU}^{12}(\lambda)\]

Proof. Recall the dynamical twist equation, theorem 3.9 which states:

\[(3.47)\]

\[J_{U \otimes V, W}^{12,3}(\lambda) J_{UW}^{12}(\lambda - h^3) = J_{U,V \otimes W}^{13,2}(\lambda) J_{VV}^{12}(\lambda),\]

and by permutation (see Notation B) implies

\[(3.48)\]

\[J_{U \otimes W, V}^{13,2}(\lambda) J_{UV}^{13}(\lambda - h^2) = J_{U,W \otimes V}^{13,2}(\lambda) J_{WW}^{13}(\lambda),\]

\[(3.49)\]

\[J_{V \otimes W, U}^{23,1}(\lambda) J_{VW}^{23}(\lambda - h^1) = J_{V,W \otimes U}^{23,1}(\lambda) J_{UU}^{23}(\lambda).\]

By definition of the exchange operator, definition 3.11, the left hand side of...
3.3 Exchange Operators

Equation [3.46] becomes

\[
R^{12}_{UU}(\lambda - h^3) R^{13}_{UV}(\lambda) R^{23}_{VV}(\lambda - h^1)
\]

(3.50)

\[
= J^{12}_{UU}(\lambda - h^3)^{-1} R^{21}_{VV}(\lambda - h^3) J^{13}_{UV}(\lambda)^{-1} R^{31}_{UU}(\lambda) J^{23}_{VV}(\lambda - h^1)^{-1} R^{32}_{UU}(\lambda - h^1).
\]

Rewriting equations [3.47] and [3.49] as

\[
J^{12}_{UU}(\lambda - h^3)^{-1} = J^{23}_{VV}(\lambda)^{-1} J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda),
\]

(3.51)

\[
J^{21}_{VV}(\lambda - h^3) = J^{21}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) J^{23}_{VV}(\lambda),
\]

(3.52)

\[
J^{23}_{VV}(\lambda - h^1)^{-1} = J^{31}_{UU}(\lambda)^{-1} J^{23}_{UU,V,W}(\lambda)^{-1} J^{31}_{VV,W,U}(\lambda),
\]

(3.53)

\[
J^{32}_{VV}(\lambda - h^1) = J^{21}_{UU,V,W}(\lambda)^{-1} J^{31}_{UU,V,W}(\lambda) J^{21}_{VV}(\lambda)
\]

(3.54)

and substituting those into equation [3.50] yields

\[
R^{12}_{UU}(\lambda - h^3) R^{13}_{UV}(\lambda) R^{23}_{VV}(\lambda - h^1)
\]

(3.55)

\[
= J^{23}_{VV}(\lambda)^{-1} J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) R^{21}_{VV}(\lambda - h^3) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{13}_{UU}(\lambda) J^{12}_{UU,V,W}(\lambda)^{-1} R^{31}_{UU}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{31}_{UU}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)^{-1} J^{32}_{UU,V,W}(\lambda) J^{13}_{VV}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{23}_{UU}(\lambda) J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{23}_{UU}(\lambda) J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
= J^{23}_{VV}(\lambda)^{-1} J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) R^{21}_{VV}(\lambda - h^3) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{31}_{UU}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)^{-1} J^{32}_{UU,V,W}(\lambda) J^{13}_{VV}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{23}_{UU}(\lambda) J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

\[
\times J^{23}_{UU}(\lambda) J^{12}_{UU,V,W}(\lambda)^{-1} J^{13}_{UU,V,W}(\lambda) J^{23}_{VV}(\lambda)^{-1} J^{21}_{UU,V,W}(\lambda)
\]

Here part 1 of definition [A.3] was used to find

\[
R^{12}_{UU,V,W}(\lambda) = J^{21}_{UU,V,W}(\lambda) R^{12}.\]

Analogously, the right hand side of equation [3.46] becomes

\[
R^{23}_{VV}(\lambda) R^{13}_{UV}(\lambda - h^2) R^{12}_{UU}(\lambda)
\]

(3.56)

\[
= J^{23}_{VV}(\lambda)^{-1} R^{32}_{UV}(\lambda) J^{13}_{UV}(\lambda - h^2)^{-1} R^{31}_{UU}(\lambda - h^2) J^{23}_{VV}(\lambda)^{-1} R^{32}_{UU}(\lambda - h^2)
\]

Rewriting equation [3.48] as

\[
J^{13}_{UU}(\lambda - h^2)^{-1} = J^{32}_{VV}(\lambda)^{-1} J^{13}_{UU,W,V}(\lambda)^{-1} J^{13}_{UU,W,V}(\lambda),
\]

(3.57)

\[
J^{21}_{WW}(\lambda - h^2) = J^{31}_{UU,W,V}(\lambda)^{-1} J^{31}_{UU,W,V}(\lambda) J^{31}_{WW}(\lambda)
\]

(3.58)
and substituting those into equation 3.56 yields

\[ R_{23}^{12} V W (\lambda - h^2) R_{13}^{12} U W (\lambda) \]

\[ = J_{V W}^{23} (\lambda - h^2) J_{W V}^{32} (\lambda - h^2) J_{U V}^{12} (\lambda - h^2) \]

\[ \times J_{U W, V}^{31} (\lambda - h^2) J_{W U, V}^{31} (\lambda - h^2) \]

\[ \times J_{W, U, V}^{31} (\lambda - h^2) J_{U, W, V}^{31} (\lambda - h^2) \]

\[ = R_{23}^{12} V W (\lambda) R_{13}^{12} U W (\lambda) R_{23}^{12} V W (\lambda) R_{13}^{12} U W (\lambda) \]

Now because of the QYBE, equation 2.6, the expressions in 3.55 and 3.59 are equal, leading to:

\[ R_{23}^{12} U V (\lambda - h^3) R_{13}^{12} U W (\lambda - h^2) R_{23}^{12} V W (\lambda - h^1) = R_{23}^{12} V W (\lambda) R_{13}^{12} U W (\lambda - h^2) R_{23}^{12} V W (\lambda) \]

as required.

3.13 Corollary:

The exchange operator \( R_{V V} (\lambda) \) is a solution to the QDYBE on \( V \otimes V \otimes V \), and is called a quantum dynamical \( R \)-matrix.

Note that the approach taken to find solutions to the QDYBE in this chapter is slightly different from the approach taken in chapter 2. In chapter 2, a twist is defined as an invertible element satisfying the twist equation. These twists are then used to twist universal \( R \)-matrices and find new solutions to the QYBE. In this chapter, however, one specific dynamical twist called the fusion operator is shown to satisfy the dynamical twist equation. Twisting a universal \( R \)-matrix by this dynamical twist results in a solution to the QDYBE. Alternatively, mimicking chapter 2, a general dynamical twist could be defined as satisfying the dynamical twist equation and then used to find general solutions to the QDYBE.

3.3.1 Example: \( \mathfrak{U}_q (\mathfrak{sl}_2) \)

Like the fusion operator, the exchange operator \( R_{V V} (\lambda) \) can be explicitly computed for finite-dimensional representations of \( \mathfrak{U}_q (\mathfrak{sl}_2) \). Consider again the irreducible two-dimensional representation \( V = V_1 \) of \( \mathfrak{U}_q (\mathfrak{sl}_2) \), described in appendix section A.3.3 and section 3.2.2.

The universal \( R \)-matrix \( R \mid_{V \otimes V} \) can be computed using the general expression for \( R \) given in appendix equation A.46. In the case of the irreducible two-dimensional
representation \( V \) the expression reduces to:

\[
R_{|V \otimes V} = q^{\frac{1}{2}(h \otimes h)} \sum_{n=0}^{1} (q - q^{-1})^n (E^n \otimes F^n).
\]

Simple calculations yield

\[
\begin{align*}
R_{|V \otimes V}(v_+ \otimes v_+) &= q^{\frac{1}{2}} v_+ \otimes v_+ \\
R_{|V \otimes V}(v_+ \otimes v_-) &= q^{-\frac{1}{2}} v_+ \otimes v_- \\
R_{|V \otimes V}(v_- \otimes v_-) &= q^{-\frac{1}{2}} v_+ \otimes v_- + q^{-\frac{1}{2}} (q - q^{-1}) v_+ \otimes v_- \\
R_{|V \otimes V}(v_- \otimes v_+) &= q^{\frac{1}{2}} v_- \otimes v_-
\end{align*}
\]

so that \( R_{|V \otimes V} \) can be expressed in matrix form as

\[
[R_{|V \otimes V}] = \begin{pmatrix}
q^{\frac{1}{2}} & 0 & 0 & 0 \\
0 & q^{-\frac{1}{2}} & q^{-\frac{1}{2}}(q - q^{-1}) & 0 \\
0 & 0 & q^{-\frac{1}{2}} & 0 \\
0 & 0 & 0 & q^{\frac{1}{2}}
\end{pmatrix}
\]

with \((v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)\) the ordered basis of \( V \otimes V \).

Since \( J_{VV}(\lambda) \) has already been determined, in equation 3.44, \( R_{VV}(\lambda) \) can now be determined by straightforward matrix multiplication:

\[
[R_{VV}(\lambda)] = [J_{VV}(\lambda)]^{-1} [R_{|V \otimes V}] [J_{2V}^{21}(\lambda)]
\]

to find

\[
[R_{VV}(\lambda)] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \frac{q-q^{-1}}{1-q^{2(\lambda+1)}} & 0 \\
0 & \frac{q-q^{-1}}{1-q^{2(\lambda+1)}} & \frac{(q^{2(\lambda+1)}-q^{-2})(q^{2(\lambda+1)}-q^{-2})}{(1-q^{2(\lambda+1)})^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with \((v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)\) the ordered basis of \( V \otimes V \).
4 QUANTUM INTEGRABLE SYSTEMS

In this chapter, the exchange operators or quantum dynamical $R$-matrices introduced in section 3.3 are used to construct a set of commuting operators that describes a quantum integrable system. This quantum integrable system will give rise to the quantum spin Calogero-Moser system, treated in section 4.4, and Macdonald operators, treated in section 4.5.

A quantum integrable system is a quantum system whose time-evolution can be exactly solved. Since time-evolution is governed by the Schrödinger equation, one way of describing a quantum integrable system involves diagonalizing the energy function or Hamiltonian of the system. This requires a maximal set of commuting linear operators that is simultaneously diagonalized. See [Caux and Mossel, 2011] for a discussion on the definition of quantum integrable systems.

4.1 TRANSFER MATRIX CONSTRUCTION

This section presents a variation on the construction of a quantum integrable system as outlined in chapter 6 of [Stokman, 2016].

Recall, from proposition 2.3, that the operator $R_{WV} : W \otimes V \rightarrow W \otimes V$ for $W, V$ representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$ is a solution to the QYBE, and consider the tensor representation $V^{\otimes n-1} = V^{(1)} \otimes \cdots \otimes V^{(n-1)}$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$.

4.1 Definition:

For $W$ a representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$, the operator

\begin{equation}
T_W := R_{WV(V)}^{12} \cdots R_{WV(V)}^{1n} : W \otimes V^{\otimes n-1} \rightarrow W \otimes V^{\otimes n-1}
\end{equation}

is called the monodromy matrix associated to $R_{WV}$.

4.2 Definition:

The operator

\begin{equation}
\mathcal{T}_W := \text{tr}_W T_W = \text{tr}_W \left(R_{WV(V)}^{12} \cdots R_{WV(V)}^{1n}\right) : V^{\otimes n-1} \rightarrow V^{\otimes n-1}
\end{equation}

is called a transfer matrix of the system.

The transfer matrices $\mathcal{T}_W$ can be thought of as acting on the quantum state space of the system.

4.3 Theorem:

For $U, W$ representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$, the transfer matrices $\mathcal{T}_U$ and $\mathcal{T}_W$ commute
and satisfy
\( T_{U \otimes W} = T_U T_W \).

From a physical perspective, the transfer matrices \( T_W \) represent conserved quantities on the quantum state space \( V^{\otimes n-1} \) of the system. Taking \( \Lambda_i \) the fundamental representations of \( \mathfrak{U}_q(\mathfrak{sl}_n) \) (see appendix definition A.32), the set of transfer matrices \( \{ T_{\Lambda_1}, \ldots, T_{\Lambda_{n-1}} \} \) describes a quantum integrable system.

4.4 Remark: In [Stokman, 2016], the operators \( R_{WV} \) depend on a so-called spectral parameter. Instead of choosing different representations \( W \) for the transfer matrices to find a set of commuting operators, the spectral parameter may be varied to find a commuting set of transfer matrices with fixed \( W \). This leads to a quantum integrable system known as the Heisenberg spin chain of \( n-1 \) lattice sites. The spin at the \( i \)-th site is then described by the representation \( V_{(i)} \).

4.2 Quantum Inverse Scattering

This section introduces a set of algebraically independent commuting operators \( D_{\Lambda_i} \) with common eigenvectors given by the trace function \( F_V(\lambda, \mu) \), that describes a quantum integrable system on a quantum state space \( V \). The first part of this section demonstrates a method for finding a set of operators \( D_C \) based on central elements \( C \in Z(\mathfrak{U}_q(\mathfrak{sl}_n)) \). It is easily shown that these operators \( D_C \) form a set of commuting difference operators. In the second part of this section, operators \( D_W \) associated to finite-dimensional representations \( W \) of \( \mathfrak{U}_q(\mathfrak{sl}_n) \), called transfer operators, are constructed from the exchange operators or quantum dynamical \( R \)-matrices \( R_{WV}(\lambda) \). The operators \( D_C \) and \( D_W \) are found to relate to each other, implying that the operators \( D_W \) also form a set of commuting difference operators. The subset of operators \( D_{\Lambda_i} \) is then found to describe a quantum integrable system.

Section 4.3 provides an elaborate proof of how the operators \( D_C \) and \( D_W \) relate to each other, proving the main theorem for the operators \( D_W \).

4.2.1 Commuting Operators and Central Elements

A quantum integrable system may be described by a maximal set of commuting operators that share common eigenfunctions given by a trace function. One way to construct such a set of commuting operators, using central elements of \( \mathfrak{U}_q(\mathfrak{sl}_n) \), is described in [Etingof and Kirillov, 1994].
4.5 Definition:
Define the ring of difference operators

\[ D := \left\{ D = \sum_{\nu \in h^\ast} A(\nu) T_\nu \mid \text{almost all } A(\nu) = 0 \right\} \]

acting on functions \( f \) on \( h^\ast \), where \((T_\nu f)(\lambda) := f(\lambda + \nu)\) and the \( A(\nu) \) are meromorphic functions on \( h^\ast \), with multiplication given by

\[ (A(\nu) T_\nu)(B(\mu) T_\mu) = A(\nu) B(\mu + \nu) T_{\mu + \nu}. \]

Now fix a representation \( V \) of \( \mathfrak{U}_q(\mathfrak{sl}_n) \) with \( V[0] \neq 0 \).

4.6 Theorem:
Let \( \nu \in V[0] \), and \( \mu \) generic. Then for every \( x \in \mathfrak{U}_q(\mathfrak{sl}_n) \) there exists a unique operator \( D_x \in \mathcal{D} \otimes \mathfrak{U}_q(\mathfrak{sl}_n) \) such that

\[ D_x \text{ tr} |_{M_\mu}(\Phi^\nu_q q^{2\lambda}) = \text{ tr} |_{M_\mu}(\Phi^\nu_q x q^{2\lambda}), \]

independent of the choice of representations \( M_\mu, V \) and \( \nu \in V[0] \).

Proof. See chapter 6 of [Etingof and Kirillov, 1994].

Considering the operators defined in theorem 4.6 for central elements of \( \mathfrak{U}_q(\mathfrak{sl}_n) \) leads to an important corollary.

4.7 Corollary:
If \( C \in Z(\mathfrak{U}_q(\mathfrak{sl}_n)) \), then the trace function \( \Psi^\nu(\lambda, \mu) := \text{ tr} |_{M_\mu}(\Phi^\nu_q q^{2\lambda}), \nu \in V[0] \), satisfies the difference equation

\[ D_C \Psi^\nu(\lambda, \mu) = C_\mu \Psi^\nu(\lambda, \mu), \]

where \( C_\mu \in \mathbb{C}(q) \) such that \( C|_{M_\mu} = C_\mu \cdot \text{id} \).

Proof. The corollary follows easily from theorem 4.6;

\[ D_C \Psi^\nu(\lambda, \mu) = D_C \text{ tr} |_{M_\mu}(\Phi^\nu_q q^{2\lambda}) \]
\[ = \text{ tr} |_{M_\mu}(\Phi^\nu_q C \cdot q^{2\lambda}) \]
\[ = \text{ tr} |_{M_\mu}(\Phi^\nu_q C_\mu \cdot q^{2\lambda}) \]
\[ = C_\mu \text{ tr} |_{M_\mu}(\Phi^\nu_q q^{2\lambda}) \]
\[ = C_\mu \Psi^\nu(\lambda, \mu). \]

\[ \square \]
For any finite-dimensional representation $W$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$, a central element $C_W$ can be defined. The associated operators $D_{C_W}$ will play an important role in describing the quantum integrable system.

**4.8 Theorem:**
For $U, W$ finite-dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$, the element

$$C_W := \left(\text{id} \otimes \text{tr} |_W\right)(R_{21} R(1 \otimes q^{2\tau}))$$

is central in $\mathfrak{U}_q(\mathfrak{sl}_n)$ and satisfies

$$C_{U \otimes W} = C_U C_W$$

and

$$C_W|_{M_\mu} = \chi_W(q^{2(\mu + \rho)}) \text{id}.$$  

**Proof.** See section 4.3.1.

**4.9 Corollary:**
The operator $D_{C_W}$ satisfies

$$D_{C_{U \otimes W}} = D_{C_U} D_{C_W} = D_{C_W} D_{C_U},$$

and the trace functions $\Psi^\nu(\lambda, \mu)$, $v \in V[0]$, satisfy the difference equation

$$D_{C_W} \Psi^\nu(\lambda, \mu) = \chi_W(q^{2(\mu + \rho)}) \Psi^\nu(\lambda, \mu).$$

**Proof.** Follows from corollary 4.7 and theorem 4.8.

**4.2.2 Transfer Operators and the Trace Function**

A transfer operator is constructed from an exchange operator or quantum dynamical $R$-matrix $R_W(\lambda)$, similar to the construction of a transfer matrix from an operator $R_{WV}$ as shown in definition 4.2.

Again, fix a finite-dimensional representation $V$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$ with $V[0] \neq 0$. From a physical perspective, this representation $V$ is the quantum state space of the system, resembling the quantum integrable system described in section 4.1.

**4.10 Definition:**
Let $W$ a finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$, then the **transfer operator** $D_W \in \mathfrak{D} \otimes \mathfrak{U}_q(\mathfrak{sl}_n)$ is defined on functions $f : \mathfrak{h}^* \to V[0]$ as

$$\left( D_W f \right)(\lambda) := \sum_{\nu \in \mathfrak{h}^*} \text{tr} |_{W[\nu]} R_W(-\lambda - \rho) T_\nu f(\lambda).$$
The transfer operators $D_W$ have a remarkable relation to the operators $D_{C_W}$ that were constructed from central elements $C_W$ in section 4.2.1.

4.11 Proposition:

\[(4.15)\]  
\[D_W = \delta_q(\lambda) D_{C_W} \delta_q(\lambda)^{-1},\]

where

\[(4.16)\]
\[\delta_q(\lambda) := \prod_{\alpha \in \Phi^+} (q^{\alpha(\lambda)} - q^{-\alpha(\lambda)}) = q^{2\rho(\lambda)} \prod_{\alpha \in \Phi^+} (1 - q^{-2\alpha(\lambda)}).\]

Proof. See section 4.3 and section 4.3.6 in particular.

Proposition 4.11 is an amazing result and implies that the transfer operators $D_W$ satisfy the same properties, stated in corollary 4.9, as the operators $D_{C_W}$. In particular, the set of transfer operators $D_W$ has common eigenfunctions given by a normalized trace function. This trace function is constructed as a sum of the trace functions $\Psi^v(\lambda, \mu)$ defined in corollary 4.7, defining

\[(4.17)\]  
\[\Psi^v(\lambda, \mu) := \sum_i \Psi^{v_i} \otimes v^*_i \in V[0] \otimes V^*[0],\]

where the $v_i$ form a basis of $V[0]$ and the $v^*_i$ are dual to the $v_i$. The summed trace function $\Psi^v(\lambda, \mu)$ will play an important role in the proof of the main theorem. However, the trace function that provides common eigenfunctions of the transfer operators $D_W$ requires scaling by the factor $\delta_q(\lambda)$.

4.12 Definition:

For $\lambda, \mu \in \mathfrak{h}^*$, $\mu$ generic, the trace function $F^v(\lambda, \mu)$ is defined as

\[(4.18)\]  
\[F^v(\lambda, \mu) := \Psi^v(\lambda, -\mu - \rho) \delta_q(\lambda).\]

4.13 Theorem (Main Theorem):

The trace function $F^v(\lambda, \mu)$ satisfies the difference equations

\[(4.19)\]  
\[D_W F^v(\lambda, \mu) = \chi_W(q^{-2\mu}) F^v(\lambda, \mu),\]

where the transfer operators $D_W$ act in the $F^v(\cdot, \mu) : \mathfrak{h}^* \to V[0]$ component.

Proof. Follows from proposition 4.11; see section 4.3.6.

Theorem 4.13 shows that the transfer operators $D_W$ form a maximal set of commuting operators with common eigenfunctions given by the trace function $F^v(\lambda, \mu)$. 
Now consider the transfer operators $D_{\Lambda_1}, \ldots, D_{\Lambda_{n-1}}$, where the $\Lambda_i$ are the fundamental representations of $\mathcal{U}_q(\mathfrak{sl}_n)$. By proposition 4.11 and corollary 4.9, the transfer operators $D_{\Lambda_1}, \ldots, D_{\Lambda_{n-1}}$ form a set of algebraically independent commuting operators. Moreover, because of corollary 4.9 and appendix proposition A.33, every transfer operator $D_W$ may be written as an algebraic combination of transfer operators $D_{\Lambda_i}$. From a physical perspective, the transfer operators $D_{\Lambda_1}, \ldots, D_{\Lambda_{n-1}}$ represent conserved quantities of the system. This means that the set of transfer operators $\{D_{\Lambda_1}, \ldots, D_{\Lambda_{n-1}}\}$ describes a quantum integrable system on the quantum state space $\mathcal{V}$.

4.3 Proof of the Main Theorem

This section largely follows the structure of sections 7.5, 7.6, 7.7, and 7.8 of [Etingof and Latour, 2005] and sections 1 and 2 of [Etingof and Varchenko, 2000].

The proof of theorem 4.13 (and proposition 4.11) follows the procedure outlined in section 4.2.1. It is extensive, but the computations are worked out in detail and organized into lemmas so that it should be relatively easy to follow.

4.3.1 Central Elements

The goal of this section is to determine the central elements $C_W \in Z(\mathcal{U}_q(\mathfrak{sl}_n))$ related to finite-dimensional representations $W$ of $\mathcal{U}_q(\mathfrak{sl}_n)$, which were introduced in [Drinfeld, 1990]. Interestingly, these central elements are constructed using universal $R$-matrices.

4.14 Proposition:

Let $x \in H$, then

i. $\sum_{(a)} a^{(1)} x S(a^{(2)}) = \epsilon(a) x$ for all $a \in H$,

ii. $x \in Z(H)$,

are equivalent.

Proof. Suppose ii., then

(4.20) $\sum_{(a)} a^{(1)} x S(a^{(2)}) = x \sum_{(a)} a^{(1)} S(a^{(2)}) = x \epsilon(a) 1 = \epsilon(a) x$. 

Now suppose and let then
\[
xa = x(\epsilon \otimes \text{id})\Delta(a) \\
= x \sum (a) \epsilon(a_1)a_2(2) \\
= \sum (a) \epsilon(a_1)x a_2(2) \\
= \sum (a) a_1 x S(a_2) a_3(2) \\
= \sum (a) a_1 \epsilon(a_2)x \\
= (\text{id} \otimes \epsilon)\Delta(a)x \\
= ax
\]
(4.21)

4.15 Definition:
An $H$-bimodule of an algebra $H$ is a vector space $V$ with left and right actions $H \otimes V \to V : h \otimes v \mapsto hv$ and $V \otimes H \to V : v \otimes h \mapsto vh$ that are compatible, i.e. $(hv)h' = h(vh')$.

Note that an $H$-bimodule is the same as an $H \otimes H^{\text{op}}$-module.

4.16 Proposition:
Let $V$ an $H$-bimodule and $v \in V$, then
\begin{enumerate} [i.]
\item $\sum (a) a_1 v S(a_2) = \epsilon(a)v$ for all $a \in H$,
\item $va = av$ for all $a \in H$,
\end{enumerate}
are equivalent.

Proof. Use the proof of proposition 4.14 and replace $x$ by $v$. □

4.17 Definition:
A linear functional $\theta : H \to \mathbb{C}$ such that $\theta(xy) = \theta(yS^2(x))$ for all $x, y \in H$ is called a quantum trace.

4.18 Proposition:
Let $\theta : H \to \mathbb{C}$ a quantum trace and $z \in H \otimes H$ such that $\Delta(a)z = z\Delta(a)$ for all $a \in H$, then $C = (\text{id} \otimes \theta)z \in Z(H)$.

Proof. Consider the $H$-bimodule $V = H \otimes H$ with $H \otimes V \to V : a \otimes v \mapsto \Delta(a)v$ and $V \otimes H \to V : v \otimes a \mapsto v\Delta(a)$. Then for all $a \in H,$
\[
za = z\Delta(a) = \Delta(a)z = az.
\]
(4.22)
This means proposition 4.16 can be applied to find

\[
\epsilon(a) C = (\text{id} \otimes \theta) \epsilon(a) z
\]

\[
= (\text{id} \otimes \theta) \sum (a) a(1) z S(a(2))
\]

\[
= (\text{id} \otimes \theta) \sum (a) \Delta(a(1)) z \Delta(S(a(2)))
\]

\[
= (\text{id} \otimes \theta) \sum (a_1) (a(1) \otimes a(2)) z (S(a(4)) \otimes S(a(3))^2(a(2)))
\]

\[
= (\text{id} \otimes \theta) \sum (a_1) (a(1) \otimes 1) z (S(a(4)) \otimes S(S(a(2))a(3)))
\]

\[
= (\text{id} \otimes \theta) \sum (a_1) (a(1) \otimes 1) z (S(a(3)) \otimes S(\epsilon(a(2)) 1))
\]

\[
= \sum (a_1) a(1) \epsilon(a(2)) S(a(2))
\]

which, by proposition 4.14, implies that \(C \in \mathcal{Z}(H)\) as required.

4.19 Corollary:

Let \(W\) a finite-dimensional representation of \(U_q(\mathfrak{sl}_n)\), then

\[
C_W := (\text{id} \otimes \text{tr} |_W)(R^{21}R(1 \otimes q^{2\rho}))
\]

is a central element of \(U_q(\mathfrak{sl}_n)\).

Proof. Recall that \(S^2(x) = q^{2\rho} x q^{-2\rho}\) (appendix proposition A.16). The function

\[
\theta(x) := \text{tr} |_W(x q^{2\rho})
\]

is a quantum trace by definition 4.17 since

\[
\theta(xy) = \text{tr} |_W(xy q^{2\rho})
\]

\[
= \text{tr} |_W(y q^{2\rho} x)
\]

\[
= \text{tr} |_W(yS^2(x) q^{2\rho})
\]

\[
= \theta(yS^2(x)).
\]

Because \(R^{21}R \Delta(x) = R^{21} \Delta^\text{op}(x)R = \Delta(x)R^{21}R\), proposition 4.18 can be applied to provide that

\[
(id \otimes \theta)R^{21}R \in \mathcal{Z}(U_q(\mathfrak{sl}_n))
\]

and hence that

\[
(id \otimes \text{tr} |_W)(R^{21}R(1 \otimes q^{2\rho})) \in \mathcal{Z}(U_q(\mathfrak{sl}_n))
\]

as required.

4.20 Theorem:
For all representations $V, W$ of $\mathcal{U}_q(\mathfrak{sl}_n)$,

$$C_{V \otimes W} = C_V C_W.$$  

\textbf{Proof.} Writing out the definition of $C_{V \otimes W}$ as given in corollary 4.19 becomes

$$C_{V \otimes W} = (\text{id} \otimes \text{tr}|_V \otimes \text{tr}|_W)((\text{id} \otimes \Delta)(R^{21})(\text{id} \otimes \Delta)(R)(1 \otimes q^{2\pi} \otimes q^{2\pi}))$$

$$= (\text{id} \otimes \text{tr}|_V \otimes \text{tr}|_W)(R^{21}R^{31}R^{13}R^{12}(1 \otimes q^{2\pi} \otimes q^{2\pi}))$$

Now, since $C_W$ is central, this further reduces to

$$C_{V \otimes W} = C_W(\text{id} \otimes \text{tr}|_V)(R^{21}R^{12}(1 \otimes q^{2\pi}))$$

as required. \hfill $\Box$

\textbf{4.21 Theorem:}

\textbf{On the Verma module $M_\mu$,}

$$C_W|_{M_\mu} = \chi_W(q^{2(\pi+\pi)})(\text{id}).$$

\textbf{Proof.} First note that central elements $C \in Z(\mathcal{U}_q(\mathfrak{sl}_n))$ always act as a constant on Verma modules as these are cyclic, i.e. they are generated by a highest weight vector $v_\mu$.

Now consider the central element $C_W$. Looking at the explicit expression for a universal $R$-matrix as given in section 8.3 of [Klimyk and Schmüdgen, 1997], an $R$-matrix $R$ of $\mathcal{U}_q(\mathfrak{sl}_n)$ can be written as

$$R = q^{\sum_i x_i \otimes x_i} + \sum_j a_j \otimes b_j$$

where $\text{wt } a_j > 0$ and $\text{wt } b_j < 0$. This means that on a highest weight vector $v_\mu$,

$$C_W v_\mu = (\text{id} \otimes \text{tr}|_W)(R^{21}R(1 \otimes q^{2\pi})) v_\mu$$

$$= (\text{id} \otimes \text{tr}|_W)(q^{\sum_i x_i \otimes x_i} q^{\sum_i x_i \otimes x_i} (1 \otimes q^{2\pi})) v_\mu$$

$$= \text{tr}|_W(q^{2(\pi+\pi)} v_\mu$$

$$= \chi_W(q^{2(\pi+\pi)}) v_\mu.$$ Again, because $C_W$ is central and $M_\mu$ is cyclic, it follows that

$$C_W x = \chi_W(q^{2(\pi+\pi)}) x$$

for all $x \in \mathcal{U}_q(\mathfrak{sl}_n)$. This completes the proof. \hfill $\Box$
4.3.2 Difference Equation

Now that a suitable central element of $\mathfrak{U}_q(\mathfrak{sl}_n)$ has been found in $C_W$, the next step is to apply the corresponding operator $D_{C_W}$ to the trace function $\Psi_V(\lambda, \mu)$. On the one hand, by corollary 4.7, this will equal $C|_{M_\mu} \Psi_V(\lambda, \mu)$. On the other hand, using theorem 4.6 will lead to an explicit expression for the operator $D_{C_W}$.

Translating to the transfer operator $D_W$ and the trace function $F_V(\lambda, \mu)$ satisfying the difference equations as they appear in theorem 4.13 will be a trivial effort later on in section 4.3.6.

4.22 Notation:

The expressions found in this section live in either $M_\mu \otimes V \otimes V^*$, $M_\mu \otimes V \otimes V^* \otimes \mathfrak{U}_q(\mathfrak{sl}_n)$, or $M_\mu \otimes V \otimes V^* \otimes \mathfrak{U}_q(\mathfrak{sl}_n) \otimes \mathfrak{U}_q(\mathfrak{sl}_n)$. Adopting the notation used in [Etingof and Latour, 2005], the components of such expressions will be labelled with sub- or superscript 0, 1, 1*, 2, and 3 respectively.

Applying $D_{C_W}$ to the trace function $\Psi_V(\lambda, \mu)$ and writing out its definition, definition 4.17, and the action of $D_{C_W}$ from theorem 4.6 yields

$$D_{C_W} \Psi_V(\lambda, \mu) = D_{C_W} \Psi_V^{11^*}(\lambda, \mu)$$
$$= D_{C_W} \sum_i \Psi_V^{i1}(\lambda, \mu) \otimes v_i^*$$
$$= D_{C_W} \sum_i \text{tr} \left| \phi_{\mu,01}^{V,01} \right|_0 (q_0^{2\lambda}) \otimes v_i^*$$
$$= \sum_i \text{tr} \left| \phi_{\mu,01}^{V,01} C_W \right|_0 (q_0^{2\lambda}) \otimes v_i^*.$$  \hspace{1cm} (4.36)

Setting

$$\Phi_{\mu,011^*}^{V,01} := \sum_i (\phi_{\mu,01}^{V,01} \otimes v_i^*) : M_\mu \rightarrow M_\mu \otimes V \otimes V^*,$$

using the fact that $C_W$ is central, and writing out its definition from corollary 4.19, the expression can be made explicit as

$$D_{C_W} \Psi_V(\lambda, \mu) = \sum_i \text{tr} \left| C_W \phi_{\mu,01}^{V,01} \right|_0 (q_0^{2\lambda}) \otimes v_i^*$$
$$= \sum_i \text{tr} \left| \phi_{\mu,01}^{V,01} \right|_0 (q_0^{2\lambda}) \otimes v_i^*$$
$$= \text{tr} \left| \phi_{\mu,01}^{V,01} \right|_0 (q_0^{2\lambda}) \otimes v_i^*$$
$$= \text{tr} \left| \phi_{\mu,01}^{V,01} \right|_0 (q_0^{2\lambda}) \otimes v_i^*.$$  \hspace{1cm} (4.38)

The proof of theorem 4.13 proceeds by further rewriting this expression for the action of $D_{C_W}$ on the trace function $\Psi_V(\lambda, \mu)$. This is done in three steps, each formulated as a proposition and treated in a separate section to maintain overview. The proof of each of these three propositions depends on a series of lemmas, and is given at the end of each respective section.
4.3.3 First Proposition

Part of the expression in equation 4.38 still contains $R$-matrix coefficients that will need to be reworked. The lemmas in this section represent steps taken to arrive at an expression that is based only on the trace function $\Psi_V$ and a modified version of the universal fusion operator.

4.23 Definition:

The operator

\[
\mathcal{J}(\lambda) := J(-\lambda - \rho + \frac{1}{2}(h^1 + h^2))
\]

in $\mathfrak{U}_q(\mathfrak{s\ell}_n) \otimes \mathfrak{U}_q(\mathfrak{s\ell}_n)$ is called the **modified fusion operator**.

The final result of this section is formulated in the following proposition.

4.24 Proposition:

\[
D_C \Psi_V(\lambda, \mu) = \text{tr}|W_2| \mathcal{J}^{3,12}(\lambda) \mathcal{J}^{12}(\lambda - \frac{1}{2} h^3) \Psi^{11*}_V(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) \mathcal{J}^{32}(\lambda)^{-1} q^{2\eta}(q^{-2\lambda} J_{3,12}(\lambda) J_{12}(\lambda - \frac{1}{2} h^3)) \Psi^{11*}_V(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) J_{32}(\lambda)^{-1}.
\]

To get to the proof of proposition 4.24 start by considering the expression on the right hand side of equation 4.38;

\[
D_C \Psi_V(\lambda, \mu) = \text{tr}|W_2| \mathcal{J}^{3,12}(\lambda) \mathcal{J}^{12}(\lambda - \frac{1}{2} h^3) \Psi^{11*}_V(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) \mathcal{J}^{32}(\lambda)^{-1} q^{2\eta}(q^{-2\lambda} J_{3,12}(\lambda) J_{12}(\lambda - \frac{1}{2} h^3)) \Psi^{11*}_V(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) J_{32}(\lambda)^{-1}.
\]

Comparing to the right hand side of proposition 4.24, it is clear that an equality may be found by considering only the factor

\[
\text{tr}|0|(R^{20} R^{02} \Phi^{V,011*}_{\mu} q_0^{2\eta}).
\]

After a small reformulation, using appendix proposition A.3, this becomes

\[
(\mu_{23} \circ S_3) \left( \text{tr}|0|(R^{20} (R^{03})^{-1} \Phi^{V,011*}_{\mu} q_0^{2\eta}) \right).
\]

Now since

\[
\text{tr}|0|(R^{20} (R^{03})^{-1} \Phi^{V,011*}_{\mu} q_0^{2\eta}) = \text{tr}|0|(R^{03})^{-1} \Phi^{V,011*}_{\mu} q_0^{2\eta} R^{20}
\]

it suffices to show that

\[
\text{tr}|0|(\Phi^{V,011*}_{\mu} R^{20} q_0^{2\eta} (R^{03})^{-1}) = \mathcal{J}^{3,12}(\lambda) \mathcal{J}^{12}(\lambda - \frac{1}{2} h^3) \Psi^{11*}_V(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) \mathcal{J}^{32}(\lambda)^{-1}.
\]
4.3 Proof of the Main Theorem

Defining $X_V(\lambda, \mu) := \text{tr} |_0 (\Phi_{\mu}^{V,011^*} R_{20}^{23} q_6^{2\lambda} (R_{03})^{-1})$, this requirement becomes

$$X_V(\lambda, \mu) = \mathcal{J}^{3,12}(\lambda) \mathcal{J}^{12}(\lambda - \frac{1}{2} h^3) \psi_V^{11^*}(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu) \mathcal{J}^{32}(\lambda)^{-1}.$$  \hspace{1cm} (4.46)

Moving the two outer modified fusion terms from the right hand side of equation (4.46) to the left, and defining $Y_V(\lambda, \mu) := \mathcal{J}^{3,12}(\lambda)^{-1} X_V(\lambda, \mu) \mathcal{J}^{32}(\lambda)$, rewrites the requirement as

$$Y_V(\lambda, \mu) = \mathcal{J}^{12}(\lambda - \frac{1}{2} h^3) \psi_V^{11^*}(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^3, \mu).$$  \hspace{1cm} (4.47)

This equation can now be shown to hold by studying the functions $X_V$ and $Y_V$.

4.25 Lemma:

$$X_V(\lambda, \mu) = R_{12,3}^{2\lambda} q_3^{2\lambda} X_V(\lambda, \mu) q_3^{-2\lambda} (R_{23})^{-1}$$  \hspace{1cm} (4.48)

Proof. The lemma follows from a straightforward calculation using the definition of $\Phi_{\mu}^{V,011^*}$, the cyclic property of the trace, and properties of the universal $R$-matrix.

$$X_V(\lambda, \mu) = \text{tr} |_0 (\Phi_{\mu}^{V,011^*} R_{20}^{23} q_6^{2\lambda} (R_{03})^{-1})$$

$$= \text{tr} |_0 ((R_{03})^{-1} \Phi_{\mu}^{V,011^*} R_{20}^{23} q_6^{2\lambda})$$

$$= \text{tr} |_0 (R_{13}^{13} (R_{013})^{-1} \Phi_{\mu}^{V,011^*} R_{20}^{23} q_6^{2\lambda})$$

$$= R_{13}^{13} \text{tr} |_0 (\Phi_{\mu}^{V,011^*} (R_{03})^{-1} R_{20}^{23} q_6^{2\lambda})$$

$$= R_{13}^{13} \text{tr} |_0 (\Phi_{\mu}^{V,011^*} R_{20}^{23} (R_{03})^{-1} q_6^{2\lambda})$$

$$= R_{12,3}^{13} \text{tr} |_0 (\Phi_{\mu}^{V,011^*} R_{20}^{23} q_6^{2\lambda} q_3^{2\lambda} (R_{03})^{-1} q_3^{-2\lambda})$$

$$= R_{12,3}^{13} q_3^{2\lambda} X_V(\lambda, \mu) q_3^{-2\lambda} (R_{23})^{-1}.$$  \hspace{1cm} (4.49)

This calculation makes use of

$$(R_{03})^{-1} = R_{13}^{13} (R_{13})^{-1} (R_{03})^{-1}$$

$$= R_{13}^{13} (R_{03} R_{13})^{-1}$$

$$= R_{13}^{13} (((\Delta \otimes \text{id})(R))^{013})^{-1}$$

$$= R_{13}^{13} (R_{013})^{-1},$$

$$(R_{03})^{-1} R_{20}^{23} = R_{23}^{23} (R_{23})^{-1} (R_{03})^{-1} R_{20}^{23}$$

$$= R_{23}^{23} (R_{03} R_{23})^{-1} R_{20}^{23}$$

$$= R_{23}^{23} (((\Delta \otimes \text{id})(R))^{023})^{-1} R_{20}^{23}$$

$$= R_{23}^{23} R_{20}^{23} (((\Delta \otimes \text{id})(R))^{203})^{-1}$$

$$= R_{23}^{23} R_{20}^{23} (R_{23} R_{03})^{-1}$$

$$= R_{23}^{23} R_{20}^{23} (R_{03})^{-1} (R_{23})^{-1}.$$  \hspace{1cm} (4.50)
and

\[(R^{03})^{-1} q_0^{2\lambda} = (R^{03})^{-1} q_0^{2\lambda} q_3^{2\lambda} q_3^{-2\lambda} = q_0^{2\lambda} q_3^{2\lambda} (R^{03})^{-1} q_3^{-2\lambda}.\]

\[\square\]

4.26 Lemma:

\[(J(\lambda) q_i^{2\lambda} q_{\sum_i x_i} = R^{21} q_i^{2\lambda} J(\lambda))\]

**Proof.** Equation 3.31 provides that

\[(J_{\nu\omega}(\lambda)(\text{id} \otimes q^{\theta(\lambda)}) = R^{21} q^{-\sum_i x_i} (\text{id} \otimes q^{\theta(\lambda)}) J_{\nu\omega}(\lambda))\]

where \(\theta(\lambda) = 2(\bar{\lambda} + \bar{\rho}) - \sum_i x_i^2\). Since \(J_{\nu\omega}\) has zero weight, it commutes with \(q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-2(\bar{\lambda} + \bar{\rho})}\). Right multiplying equation 4.54 by this term thus yields

\[(J_{\nu\omega}(\lambda)(q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-\sum_i x_i^2}) J_{\nu\omega}(\lambda)).\]

Applying both sides to \(v \otimes w \in V[\nu] \otimes W[\mu]\) means that the left hand side becomes

\[(J_{\nu\omega}(\lambda)(q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-\sum_i x_i^2})(v \otimes w)\]

\[(= J_{\nu\omega}(\lambda)(q^{-2\nu(\bar{\lambda} + \bar{\rho})} \otimes q^{-\sum_i \mu(x_i)^2})(v \otimes w)\]

\[(= J_{\nu\omega}(\lambda)(q^{-2\nu(\bar{\lambda} + \bar{\rho})} \otimes q^{-\mu(\bar{\mu})})(v \otimes w)\]

while the right hand side becomes

\[R^{21} q^{-\sum_i x_i} (q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-\sum_i x_i^2}) J_{\nu\omega}(\lambda)(v \otimes w)\]

\[(= R^{21} q^{-\frac{1}{2} \sum_i \nu(h_i) \mu(h_i)} (q^{-2\nu(\bar{\lambda} + \bar{\rho})} \otimes q^{-\sum_i \mu(x_i)^2}) J_{\nu\omega}(\lambda)(v \otimes w)\]

\[(= R^{21} q^{-\mu(\bar{\mu})} (q^{-2\nu(\bar{\lambda} + \bar{\rho})} \otimes q^{-\mu(\bar{\mu})}) J_{\nu\omega}(\lambda)(v \otimes w)\]

\[(= R^{21} (q^{-2\nu(\bar{\lambda} + \bar{\rho})} \otimes q^{-\mu(\bar{\mu})}) J_{\nu\omega}(\lambda)(v \otimes w),\)

or in other words, equation 4.55 becomes

\[(J_{\nu\omega}(\lambda)(q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-\bar{\mu}}) = R^{21} (q^{-2(\bar{\lambda} + \bar{\rho})} \otimes q^{-\bar{\mu}}) J_{\nu\omega}(\lambda)\]

on \(V[\nu] \otimes W[\mu]\). Acting on the right of equation 4.58 with \(q^{\bar{\tau} + \bar{\mu}} \otimes q^{\bar{\tau} + \bar{\mu}}\), which again commutes with \(J_{\nu\omega}\) because of its zero weight, yields

\[(J_{\nu\omega}(\lambda)(q^{-2(\bar{\lambda} + \bar{\rho})} + (\bar{\tau} + \bar{\mu})) \otimes q^{\bar{\tau}}) = R^{21} (q^{-2(\bar{\lambda} + \bar{\rho})} + (\bar{\tau} + \bar{\mu}) \otimes \text{id}) J_{\nu\omega}(\lambda)\]
so that substituting $\lambda \rightarrow \left( -(\lambda + \rho) + \frac{1}{2}(\nu + \mu) \right)$ turns it into

\begin{equation}
J_{\mathcal{VW}}(-\lambda - \rho + \frac{1}{2}(\nu + \mu)) (q^{2\lambda} \otimes q^\nu)
= R^{21} \left( q^{2\lambda} \otimes \text{id} \right) J_{\mathcal{VW}}(-\lambda - \rho + \frac{1}{2}(\nu + \mu))
\end{equation}

on $\mathcal{V}[\nu] \otimes \mathcal{W}[\mu]$. In general, so not specifically on $\mathcal{V}[\nu] \otimes \mathcal{W}[\mu]$, equation \ref{Jvwm} may be written as

\begin{equation}
J_{\mathcal{VW}}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2)) q^{2\lambda} q^\nu 
= R^{21} q^{2\lambda} J_{\mathcal{VW}}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2))
\end{equation}

since $q^{\sum_{i,j}\nu_i\nu_j}(\nu \otimes \omega) = q^{\nu(\sigma)}(\nu \otimes \omega)$. Equation \ref{Jvwm} implies the result

\begin{equation}
\mathcal{J}(\lambda) q^{2\lambda} q^{\sum_{i,j}\nu_i\nu_j} = R^{21} q^{2\lambda} \mathcal{J}(\lambda).
\end{equation}

\[\square\]

4.27 Lemma:

\begin{equation}
q^{2\lambda} q_{123}^{\sum_{i,j}(\nu_i\nu_j+1)} Y_{\mathcal{V}}(\lambda, \mu) = Y_{\mathcal{V}}(\lambda, \mu) q^{2\lambda} q_{123}^{\sum_{i,j}\nu_i\nu_j}
\end{equation}

Proof. Applying the converse of lemma \ref{Xvlemb} to $X_{\mathcal{V}}(\lambda, \mu)$ in the definition of $Y_{\mathcal{V}}(\lambda, \mu)$ yields

\begin{equation}
Y_{\mathcal{V}}(\lambda, \mu) = J^{3,12}(\lambda)^{-1} \left( q_3^{2\lambda} (R^{12,3})^{-1} X_{\mathcal{V}}(\lambda, \mu) R^{23} q^{3\lambda} \right) J^{32}(\lambda)
= (R^{12,3} q_3^{2\lambda} J^{3,12}(\lambda))^{-1} X_{\mathcal{V}}(\lambda, \mu) R^{23} q_3^{2\lambda} J^{32}(\lambda).
\end{equation}

Now applying lemma \ref{Xvlem3} twice, once on each term on either side of $X_{\mathcal{V}}(\lambda, \mu)$, provides

\begin{equation}
Y_{\mathcal{V}}(\lambda, \mu) = \left( J^{3,12}(\lambda) q_3^{2\lambda} q_{123}^{\sum_{i,j}(\nu_i\nu_j+1)} \right)^{-1} \times
\end{equation}

\[\begin{aligned}
& X_{\mathcal{V}}(\lambda, \mu) J^{32}(\lambda) q_3^{2\lambda} q_{123}^{\sum_{i,j}\nu_i\nu_j} \\
& = q_{123}^{\sum_{i,j}(\nu_i\nu_j+1)} q_3^{2\lambda} J^{3,12}(\lambda)^{-1} \times
\end{aligned}\]

\[\begin{aligned}
& X_{\mathcal{V}}(\lambda, \mu) J^{32}(\lambda) q_3^{2\lambda} q_{123}^{\sum_{i,j}\nu_i\nu_j} \\
& = q_{123}^{\sum_{i,j}(\nu_i\nu_j+1)} q_3^{2\lambda} Y_{\mathcal{V}}(\lambda, \mu) q_3^{2\lambda} q_{123}^{\sum_{i,j}\nu_i\nu_j}
\]

which proves the lemma by shifting the terms left of $Y_{\mathcal{V}}(\lambda, \mu)$ to the left hand side of equation \ref{Xvlemb}.

\[\square\]

4.28 Lemma:

\begin{equation}
Y_{\mathcal{V}}(\lambda, \mu) = \text{tr}_0 \left( \Phi_{\mu}^{V,011*} R^{20} q_0^{2\lambda} q_0^\nu \right)
\end{equation}
Proof. Similar to equation (4.33), the inverse of the $R$-matrix $R^{03}$ may be written
\begin{equation}
(R^{03})^{-1} = q_{03}^{-\sum_i x_i \otimes x_i} + M_{03}
\end{equation}
where $M_{03}$ is a term with negative weight in the component labelled as 3. This means that
\begin{equation}
X_V(\lambda, \mu) = \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} (R^{03})^{-1})
= \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} q_{03}^{-\sum_i x_i \otimes x_i} + M_{03}))
= \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} q_{03}^{-\sum_i x_i \otimes x_i} + M'_{11\ast 3})
= \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} q_{03}^{-\sum_i x_i \otimes x_i} + M'_{11\ast 3})
\end{equation}
where $M'_{11\ast 3}$ is a term with negative weight in the component labelled as 3. Now since proposition 3.8 states that $J$, and therefore $J$, is lower triangular with ones on its diagonal, it follows from the definition of $Y_V(\lambda, \mu)$ that also
\begin{equation}
Y_V(\lambda, \mu) = \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} q_{03}^{-\sum_i x_i \otimes x_i} + M'_{11\ast 3})
\end{equation}
where $M'_{11\ast 3}$ is a term with negative weight in the component labelled as 3. However, it turns out that $M''_{11\ast 3} = 0$. To see this, consider a term $Y'_V(\lambda, \mu)$ of weight $\beta = (\beta_1, \beta_2, \beta_3)$ in $Y_V(\lambda, \mu)$, and any element $\nu \in V \otimes V^* \otimes \mathfrak{sl}_n \otimes \mathfrak{sl}_n$ of weight $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. See appendix section A.4 for the $Q$-grading of $\mathfrak{sl}_n$. Applying both sides of lemma 4.27, considering only the term $Y'_V(\lambda, \mu)$ of $Y_V(\lambda, u)$, to $\nu$ yields
\begin{equation}
q_3^{\lambda} q_{123}^{\sum_i (x_i \otimes x_i + 1 \otimes x_i \otimes x_i)} Y'_V(\lambda, \mu) \nu = q_{\sum_i 2M(x)(\beta_3 + \gamma_3)(\beta_1 + \gamma_1)(\beta_2 + \gamma_2)(\beta_3 + \gamma_3)(\beta_1 + \gamma_1)(\beta_2 + \gamma_2) Y'_V(\lambda, \mu) \nu
= \sum_i q_{\beta_3 + \gamma_3}^{2M(x)(\beta_3 + \gamma_3)(\beta_1 + \gamma_1)(\beta_2 + \gamma_2) Y'_V(\lambda, \mu) \nu
= q_{\beta_3 + \gamma_3}^{2M(x)(\beta_3 + \gamma_3)(\beta_1 + \gamma_1)(\beta_2 + \gamma_2) Y'_V(\lambda, \mu) \nu
= q_{\gamma_3(2\lambda + 2\gamma)} Y'_V(\lambda, \mu) \nu
\end{equation}
for generic $\lambda$, which means that either $\beta_3 = 0$ or $Y'_V(\lambda, \mu) = 0$. So in fact,
\begin{equation}
Y_V(\lambda, \mu) = \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda} q_{03}^{-\sum_i x_i \otimes x_i})
\end{equation}
proving the lemma. □

Defining $Z_V(\lambda, \mu) := \text{tr} |_0 (\Phi^V_{\mu} \rho^{011*} R^{03} q_0^{2\lambda})$, lemma 4.28 states that
\begin{equation}
Y_V(\lambda, \mu) = Z_V(\lambda - \frac{1}{2} h^3, \mu).
\end{equation}

4.29 Lemma:
\begin{equation}
Z_V(\lambda, \mu) = R^{21} q_1^{2\lambda} Z_V(\lambda, \mu)
\end{equation}

Proof. The lemma follows from a straightforward calculation using the definition
of $\Phi^\mu_{\nu,11}$ and the cyclic property of the trace.

\[
Z_V(\lambda, \mu) = \text{tr}_0[\Phi_{\mu}^{V,011^*} R^{20} q_0^{2\lambda}] \\
= \text{tr}_0[R^{2,01} \Phi_{\mu}^{V,011^*} q_0^{2\lambda}] \\
= \text{tr}_0[R^{21} R^{20} \Phi_{\mu}^{V,011^*} q_0^{2\lambda}] \\
= R^{21} \text{tr}_0[R^{20} \Phi_{\mu}^{V,011^*} q_0^{2\lambda}] \\
= R^{21} \text{tr}_0[\Phi_{\mu}^{V,011^*} q_0^{2\lambda} R^{20}] \\
= R^{21} \text{tr}_0[Q_0^{2\lambda} \Phi_{\mu}^{V,011^*} R^{20}] \\
= R^{21} Q_0^{2\lambda} \text{tr}_0[\Phi_{\mu}^{V,011^*} R^{20} q_0^{2\lambda}] \\
= R^{21} Q_0^{2\lambda} Z_V(\lambda, \mu).
\]

(4.74)

4.30 Lemma:

(4.75) $Z_V(\lambda, \mu) = J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$

Proof. Right multiplying both sides of lemma 4.26 by $\Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$ results in

(4.76) $J^{12}(\lambda) q_1^{2\lambda} q_{12}^{2\lambda} \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu) = R^{21} q_1^{2\lambda} J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$.

Since $\Psi_V^{11^*} \in V[0] \otimes V^*[0]$, which means that $q_1^2 \Psi_V^{11^*} = q_0^1 \Psi_V^{11^*}$ for all $a \in h$, equation 4.76 reduces to

(4.77) $J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu) = R^{21} q_1^{2\lambda} J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$.

By equation 4.77 and lemma 4.29, the functions $J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$ and $Z_V(\lambda, \mu)$ are seen to satisfy the same equation for $X$:

(4.78) $X = R^{21} q_1^{2\lambda} X$.

Now because $J^{12}(\lambda) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2, \mu)$ and $Z_V(\lambda, \mu)$ share the same zero weight term, satisfying equation 4.78 implies that they are equal, proving the lemma. A more detailed description of solutions to equation 4.78 and the zero weight terms can be found in lemma 7.39 and proposition 7.40 of [Etingof and Latour, 2005].

Proof of Proposition 4.24. As remarked at the beginning of this section, leading up to equations 4.46 and 4.47, it suffices to show that

(4.79) $X_V(\lambda, \mu) = J^{3,12}(\lambda) J^{12}(\lambda - \frac{1}{2} h^2) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^2, \mu) J^{32}(\lambda)^{-1}$

or

(4.80) $Y_V(\lambda, \mu) = J^{12}(\lambda - \frac{1}{2} h^2) \Psi_V^{11^*}(\lambda + \frac{1}{2} h^2 - \frac{1}{2} h^2, \mu)$. 
By lemma 4.28
\begin{equation}
Y_V(\lambda, \mu) = Z_V(\lambda - \frac{1}{2} h^3, \mu)
\end{equation}
and by lemma 4.30
\begin{equation}
Z_V(\lambda, \mu) = J^{12}(\lambda) \psi_Y^{11}(\lambda + \frac{1}{2} h^2, \mu)
\end{equation}
so that indeed equation 4.80 holds. This completes the proof. □

4.3.4 Second Proposition
After applying proposition 4.24, the \( R \)-matrix coefficients have vanished. Now the next step is rewriting the right hand side of the expression so that a single operator, acting on the trace function, emerges.

4.31 Proposition:
\begin{equation}
D_{C_W} \psi_V(\lambda, \mu) = \sum_{\nu} \text{tr} |W_{[\nu]}(G(\lambda, \nu) \otimes \text{id}) R_{WV}(-\lambda - \rho)| T_{\nu} \psi_V(\lambda, \mu)
\end{equation}
for a certain \( G(\lambda, \nu) \in U_q(\mathfrak{sl}_n) \) that is to be determined.

Of crucial importance to the proof of this proposition is a good understanding of shifts in the dynamical parameter \( \lambda \), and especially how operators affect these shifts. See appendix section A.4 for details.

4.32 Notation:
Throughout the rest of this section, let \( R = \sum_i s_i \otimes t_i \), \( J(\lambda) = \sum_i a_i \otimes b_i(\lambda) \), \( J(\lambda) = \sum_i c_i \otimes d_i(\lambda) \), and their inverses denoted by accents, e.g. \( R^{-1} = \sum_i s'_i \otimes t'_i \).

4.33 Lemma:
The element
\begin{equation}
Q(\lambda) := \mu(S^{-1} \otimes \text{id})(J^{21}(\lambda)),
\end{equation}
is invertible with inverse
\begin{equation}
Q^{-1}(-\lambda - \rho - h) = \sum_i d'_i(\lambda) S^{-1}(c'_i)
\end{equation}

Proof. First, rewrite the dynamical twist equation, equation 3.24 as
\begin{equation}
J^{12,3}(\lambda) = J^{1,23}(\lambda) J^{23}(\lambda) (J^{12})^{-1}(\lambda - h^3)
\end{equation}
4.3 Proof of the Main Theorem

Now applying $\mu_{321} \circ (S_1^{-1} S_3^{-1})$ to both sides, on the left hand side,

$$
\mu_{321} \circ S_1^{-1} \circ S_3^{-1} \left( J^{12,3}(\lambda) \right) \\
= \mu_{321} \circ S_1^{-1} \circ S_3^{-1} \left( \sum_{i(a)} a_i(1) \otimes a_i(2) \otimes b_i(\lambda) \right) \\
= \mu_{321} \left( \sum_{i(a)} S^{-1}(a_i(1)) \otimes a_i(2) \otimes S^{-1}(b_i(\lambda)) \right) \\
= \sum_{i(a)} S^{-1}(b_i(\lambda)) a_i(2) S^{-1}(a_i(1)) \\
= \sum_i S^{-1}(b_i(\lambda)) \epsilon(a_i) \\
= 1
$$

and, on the right hand side,

$$
\mu_{321} \circ S_1^{-1} \circ S_3^{-1} \left( J^{1,23}(\lambda) J^{23}(\lambda) (J^{12})^{-1}(\lambda - h^3) \right) \\
= \mu_{321} \circ S_1^{-1} \circ S_3^{-1} \left( \sum_{j,k,l,(d)} a_j \otimes b_j(1)(\lambda) a_k b_l(\lambda - h^3) \otimes b_{j(2)}(\lambda) b_k(\lambda) \right) \\
= \mu_{321} \left( \sum_{j,k,l,(d)} S^{-1}(b_j(1)) S^{-1}(a_j) \otimes b_{j(4)}(\lambda) a_k b_l(\lambda + h^3) \otimes S^{-1}(b_{j(2)}(\lambda)) \right) \\
= \sum_{j,k,l,(d)} S^{-1}(b_{j(2)}(\lambda)) S^{-1}(b_{j(4)}(\lambda)) b_{j(1)}(\lambda) a_k b_l(\lambda + h) \times S^{-1}(a_j) \\
= \sum_{j,k} S^{-1}(b_{j(2)}(\lambda)) \epsilon(d_j(\lambda)) a_k b_l(\lambda + h) S^{-1}(a_j) \\
= \sum_{j,k} S^{-1}(b_{j(2)}(\lambda)) a_k b_l(\lambda + h) S^{-1}(a_j) \\
= Q(\lambda) \sum_i b_i(\lambda + h) S^{-1}(a_i').
$$

Equating both sides then provides

$$
1 = Q(\lambda) \sum_i b_i(\lambda + h) S^{-1}(a_i').
$$

Since $Q(\lambda)$ and $\sum_i b_i(\lambda + h) S^{-1}(a_i')$ both have zero weight, and

$$
\sum_j d_j'(\lambda) S^{-1}(c_j') = \mu_{21}(\text{id} \otimes S^{-1})(J^{-1}(\lambda)) \\
= \mu_{21}(S^{-1} \otimes \text{id})(J^{-1}(\lambda - \rho + \frac{1}{2} h^2)) \\
= \mu_{21} \left( \sum_i S^{-1}(a_i') \otimes b_i(-\lambda - \rho + \frac{1}{2} h^2) \right) \\
= \sum_i b_i'(-\lambda - \rho) S^{-1}(a_i'),
$$

a change of variables $\lambda \to (-\lambda - \rho - h)$ is possible and results in

$$
1 = Q(-\lambda - \rho - h) \sum_i d_j'(\lambda) S^{-1}(c_j').
$$
Similarly,
\[ 1 = \sum_i d'_i(\lambda) S^{-1}(c'_i) Q(-\lambda - \rho - h), \]
proving the lemma. \(\square\)

**4.34 Lemma:**
\[ \sum_i d'_i(\lambda) q^{2\lambda} S(c'_i) = q^{\sum_i x_i^2} Q^{-1}(-\lambda - \rho - h) S(u) q^{2\lambda} \]
where \(u = \sum_i S(t_i)s_i\) is the Drinfeld element.

*Proof.* Rewriting lemma 4.26 as
\[ J^{-1}(\lambda) q^{-2\lambda} = q^{-\sum_i x_i \otimes x_i} q_1^{-2\lambda} J^{-1}(\lambda) R^{21} \]
and then applying \(\mu_{21} \circ S_1\) yields, on the left hand side,
\[ \mu_{21} \circ S_1 \left( J^{-1}(\lambda) q^{-2\lambda} \right) = \mu^{\text{op}} \left( \sum_i S(c'_i q^{-2\lambda}) \otimes d'_i(\lambda) \right) = \sum_i d'_i(\lambda) q^{2\lambda} S(c'_i), \]
and, on the right hand side,
\[ \mu_{21} \circ S_1 \left( q^{-\sum_i x_i \otimes x_i} q_1^{-2\lambda} J^{-1}(\lambda) R^{21} \right) \]
\[ = \mu^{\text{op}} \left( \sum_{j,k} q^{\sum_i x_i \otimes x_i} \left( S(q^{-2\lambda} c'_j t_k) \otimes d'_j(\lambda) s_k \right) \right) \]
\[ = \sum_{j,k} q^{\sum_i x_i^2} d'_j(\lambda) s_k S(t_k) S(c'_j) q^{2\lambda} \]
\[ = \sum_j q^{\sum_i x_i^2} d'_j(\lambda) S(u) S(c'_j) q^{2\lambda} \]
\[ = \sum_j q^{\sum_i x_i^2} d'_j(\lambda) S^{-1}(c'_j) S(u) q^{2\lambda}, \]
since
\[ S(u) S(x) = S(x u) = S(u S^{-2}(x)) = S(S^{-2}(x)) S(u) = S^{-1}(x) S(u). \]
(4.97)

Now applying lemma 4.33 to the right hand side provides
\[ q^{\sum_i x_i^2} \sum_j d'_j(\lambda) S^{-1}(c'_j) S(u) q^{2\lambda} = q^{\sum_i x_i^2} Q^{-1}(-\lambda - \rho - h) S(u) q^{2\lambda} \]
and completes the proof. \(\square\)

**4.35 Lemma:**
\[ S(u) q^{2\lambda} S(c) q^{2\rho} z = q^{2\lambda} S^{-1}(c) z S(u) q^{2\rho} \]
for all \( z, c \in \mathfrak{U}_q(\mathfrak{sl}_n) \).

**Proof.** From \( S^2(c) = q^{2\tau} c q^{-2\tau} \) (appendix proposition \ref{app:prop:A.16}), it follows that

\[
q^{-2\tau} S^2(c) = c q^{-2\tau}
\]

(4.100)

\[
S^{-1}(q^{-2\tau} S^2(c)) = S^{-1}(c q^{-2\tau})
\]

Further, \( u q^{-2\tau} \) is central in \( \mathfrak{U}_q(\mathfrak{sl}_n) \) since

\[
z u q^{-2\tau} = u S^{-2}(z) q^{-2\tau}
\]

(4.101)

\[
= u q^{-2\tau} q^{2\tau} S^{-2}(z) q^{-2\tau}
\]

\[
= u q^{-2\tau} z,
\]

which means that

\[
S^{-1}(u q^{-2\tau}) = S(S^{-2}(u q^{-2\tau}))
\]

(4.102)

\[
= S(q^{-2\tau} u q^{-2\tau} q^{2\tau})
\]

\[
= S(u) q^{2\tau}
\]

is also central in \( \mathfrak{U}_q(\mathfrak{sl}_n) \).

Using these identities, it is easy to find that

\[
S(u) q^{2\tau} S(c) q^{2\tau} z = S(u) q^{2\tau} q^{2\tau} S^{-1}(c) z
\]

(4.103)

\[
= S(u) q^{2\tau} q^{2\tau} S^{-1}(c) z
\]

\[
= q^{2\tau} S^{-1}(c) z S(u) q^{2\tau}
\]

as required. \( \square \)

**4.36 Lemma:**

\[
\sum_{i,(d_i)} d_{i(1)}(\lambda) \otimes q^{2\tau} S^{-1}(c_i) q^{-2\tau} d_{i(2)}(\lambda)
\]

(4.104)

\[
\otimes q^{-\sum_{x_i \otimes x_i \otimes x_i}^2} \sum_{j,k,(s'_j),(d_j)} s'_{j(1)}(\lambda) \otimes S^{-1}(c_k) S^{-1}(t'_j) s'_{j(2)}(\lambda) d_k(\lambda)
\]

**Proof.** Rewriting lemma \[4.26\] as

\[
q_{1}^{2\tau} \mathcal{J}(\lambda) q_{1}^{2\tau} = (R^{21})^{-1} \mathcal{J}(\lambda) q_{1}^{2\tau} \mathcal{J}(\lambda)
\]

(4.105)

and then applying \( \mu_{13} \circ S_{1}^{-1} \circ \Delta_{2} \) (for \( \mu_{13} = \mu_{23} \circ \tau_{12} \)) yields, on the left hand side,

\[
\mu_{13} \circ S_{1}^{-1} \circ \Delta_{2} \left( \sum_{i,(d_i)} q^{2\tau} c_i q^{-2\tau} \otimes d_i(\lambda) \right)
\]

(4.106)

\[
= \mu_{13} \circ S_{1}^{-1} \left( \sum_{i,(d_i)} q^{2\tau} c_i q^{-2\tau} \otimes d_{i(1)}(\lambda) \otimes d_{i(2)}(\lambda) \right)
\]

\[
= \mu_{13} \left( \sum_{i,(d_i)} q^{2\tau} S^{-1}(c_i) q^{-2\tau} \otimes d_{i(1)}(\lambda) \otimes d_{i(2)}(\lambda) \right)
\]

\[
= \sum_{i,(d_i)} d_{i(1)}(\lambda) \otimes q^{2\tau} S^{-1}(c_i) q^{-2\tau} d_{i(2)}(\lambda),
\]
and, on the right hand side,

\[
\mu_{13} \circ S^{-1} \circ \Delta_2 \left( \sum_{j,k} \left( t_j' \ c_k \otimes s_j' \ d_k(\lambda) \right) q^{\sum_i (x_i \otimes x_j)} \right)
\]

\[
= \mu_{13} \circ S^{-1} \left( \sum_{j,k,(s_j'),(d_j)} \left( t_j' \ c_k \otimes s_j' \ d_{k(1)}(\lambda) \otimes s_j' \ d_{k(2)}(\lambda) \right) \times q^{\sum_i (x_i \otimes x_j \otimes 1 + x_i \otimes 1 \otimes x_i)} \right)
\]

(4.107)

\[
= \mu_{13} \left( q^{-\sum_i (x_i \otimes 1 + 1 \otimes x_i)} \sum_{j,k,(s_j'),(d_j)} \left( S^{-1}(c_k) \ S^{-1}(t_j') \otimes s_j' \ d_{k(1)}(\lambda) \otimes S^{-1}(c_k) \ S^{-1}(t_j') \otimes s_j' \ d_{k(2)}(\lambda) \right) \right)
\]

proving the lemma. \qed

4.37 Lemma:

(4.108)

\[
\sum_{i,(s_i')} s_i' \otimes S^{-1}(t_i') \ S^{-1}(t_i') = u_2^{-1} R
\]

Proof. From the definition of the universal $R$-matrix, definition \[A.3\], it follows that

(4.109)

\[
(\Delta \otimes \text{id})(R^{-1}) = (R^{23})^{-1}(R^{13})^{-1}.
\]

Applying $\mu_{32} \circ S^{-1}$ to the left hand side of equation (4.109) provides

(4.110)

\[
\mu_{32} \circ S^{-1} \left( \sum_{i,(s_i')} s_i' \otimes S^{-1}(t_i') \right) = \sum_{i,(s_i')} s_i' \otimes S^{-1}(t_i') \ s_i'(2),
\]

while on the right hand side this provides,

(4.111)

\[
\mu_{32} \circ S^{-1} \left( \sum_{j,k} s_k' \otimes s_j' \otimes t_j' \right) = \sum_{j,k} s_k' \otimes S^{-1}(t_j') \ s_j'
\]

\[
= \sum_{j,k} s_k' \otimes S^{-1}(t_j') \ S^{-1}(t_j') \ s_j'
\]

\[
= \sum_{j,k} s_k' \otimes S^{-1}(t_j') \ u^{-1}
\]

\[
= \sum_{j,k} s_k' \otimes u^{-1} \ S(t_j')
\]

\[
= u_2^{-1} (\text{id} \otimes S)(R^{-1})
\]

\[
= u_2^{-1} R
\]

which proves the lemma. \qed

4.38 Lemma:

(4.112)

\[
\sum_i \left( 1 \otimes S(c_i) \right) \Delta^{op}(d_i) = \left( 1 \otimes S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \right) (J^{21})^{-1}(\lambda + \frac{1}{2} h^2)
\]

Proof. Substituting $\lambda \rightarrow (-\lambda - \rho + \frac{1}{2}(h^1 + h^2 + h^2))$ in the dynamical twist equation for the fusion operator $J(\lambda)$, equation \[3.24\], which is possible since $J(\lambda)$ has zero
First, rewrite the right hand side of lemma 4.36 as
\[
J^{12,3}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2 + h^3)) J^{12}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2 - h^3)) = J^{1.23}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2 + h^3)) J^{23}(-\lambda - \rho + \frac{1}{2}(h^1 + h^2 - h^3)),
\]
which, using the definition of the modified fusion operator \( \mathcal{J}(\lambda) \), as given in proposition 4.24, becomes
\[
\mathcal{J}^{12,3}(\lambda) J^{12}(\lambda + \frac{1}{2} h^3) = \mathcal{J}^{1.23}(\lambda) J^{23}(\lambda - \frac{1}{2} h^1).
\]
Now applying \( \mu_{12} \circ S_1 \) to the left hand side of equation 4.114 provides
\[
\mu_{12} \circ S_1 \left( \sum_{i,j,(c_i)} c_{i(1)} c_{i(2)} d_j(\lambda + \frac{1}{2} h^3) \otimes d_i(\lambda) \right)
\]
\[
= \mu_{12} \left( \sum_{i,j,(c_i)} S(c_{i(1)}) c_{i(2)} d_j(\lambda + \frac{1}{2} h^3) \otimes d_i(\lambda) \right)
\]
\[
= \sum_{i,j} S(c_i) c_{i(2)} d_j(\lambda + \frac{1}{2} h^3) \otimes d_i(\lambda)
\]
\[
= \mu_{12} \left( \sum_{i,j,(c_i)} S(c_{i(1)}) c_{i(2)} d_j(\lambda + \frac{1}{2} h^3) \otimes d_i(\lambda) \right)
\]
\[
= \sum_{i,j} S(c_j) \epsilon(c_i) d_j(\lambda + \frac{1}{2} h^3) \otimes d_i(\lambda)
\]
\[
= \sum_{i,j} S(c_j) d_j(\lambda + \frac{1}{2} h^3) \otimes 1
\]
\[
= \sum_{i,j} S(S^{-1}(d_j(\lambda + \frac{1}{2} h^3))) c_j(1) \otimes 1
\]
\[
= \sum_{i,j} S(S^{-1}(b_i(-\lambda + \frac{1}{2} h^3) - \rho)) a_j(1) \otimes 1
\]
\[
= S(Q)(-\lambda - \rho - \frac{1}{2} h^3) \otimes 1,
\]
and, on the right hand side,
\[
\mu_{12} \circ S_1 \left( \sum_{k,l,(d_k)} c_k \otimes d_{k(1)}(\lambda) c_l \otimes d_{k(2)}(\lambda) d_l(\lambda - \frac{1}{2} h^1) \right)
\]
\[
= \mu_{12} \left( \sum_{k,l,(d_k)} S(c_k) d_{k(1)}(\lambda) c_l \otimes d_{k(2)}(\lambda) d_l(\lambda + \frac{1}{2} h^1) \right)
\]
\[
= \sum_{k,l} S(c_k) d_{k(1)}(\lambda) c_l \otimes d_{k(2)}(\lambda) d_l(\lambda + \frac{1}{2} h^1)
\]
\[
= \sum_k \left( S(c_k) \otimes 1 \right) \Delta(d_k(\lambda)) \mathcal{J}(\lambda + \frac{1}{2} h^1).
\]
Equating the new right hand side and left hand side results in
\[
\sum_k \left( S(c_k) \otimes 1 \right) \Delta(d_k(\lambda)) \mathcal{J}(\lambda + \frac{1}{2} h^1) = S(Q)(-\lambda - \rho - \frac{1}{2} h^3) \otimes 1
\]
and the result follows by moving \( \mathcal{J}(\lambda + \frac{1}{2} h^1) \) to the other side and then applying the swap \( \tau \) to the entire equation. \( \square \)

**4.39 Lemma:**
\[
\sum_{i,(d_i)} d_{i(1)}(\lambda) \otimes q^{2X} S^{-1}(c_i) q^{-2X} d_{i(2)}(\lambda)
\]
\[
= q^{-\sum_{k,(x)} x \otimes x} u_{x(1)} S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \mathcal{J}^{21} \mu_{12} \left( \sum_{i,j,(c_i)} c_{i(1)} c_{i(2)} \mathcal{J}(\lambda + \frac{1}{2} h^1) \right)
\]

**Proof.** First, rewrite the right hand side of lemma 4.36 as
\[
q^{-\sum_{k,(x)} x \otimes x} \sum_{j,k,(s_j)} \left( 1 \otimes S^{-1}(c_j) \right) \left( s_{j(1)}^{(1)} \otimes S^{-1}(c_k) s_{j(2)}^{(2)} \right) \Delta(d_k(\lambda))
\]
so that applying lemma 4.37 yields

\[ q^{-(\sum_{x_i \otimes x_i + 1 \otimes x_i^2} \sum_k (1 \otimes S^{-1}(c_k)) u_2^{-1} R \Delta(d_k(\lambda)))} = q^{-\sum_{x_i \otimes x_i + 1 \otimes x_i^2} \sum_k u_2^{-1} (1 \otimes S(c_k)) \Delta^{\text{op}}(d_k(\lambda))) R}. \]

(4.120)

Now lemma 4.38 can be applied to obtain

\[ q^{-\sum_{x_i \otimes x_i + 1 \otimes x_i^2} u_2^{-1} (1 \otimes S(Q)(-\lambda - \rho - \frac{1}{2} h^2)) (J^{21})^{-1}(\lambda + \frac{1}{2} h^2) R} \]

and the result follows since this equals the left hand side of lemma 4.36.

Proof of Proposition 4.31: Starting from proposition 4.24, the first step is to write out the right hand side explicitly, obtaining

\[
\begin{align*}
\text{tr} |_{W_2} \left( (\mu_{23} \circ S_3) \left( a_{2-2X}^{-1} \left( \sum_{i,j,(d)} d_{i1}(\lambda) c_j \otimes d_{i2}(\lambda) d_j(\lambda - \frac{1}{2} h^2) \right) \right) \times 
\Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \left( \sum_k 1 \otimes d_k^*(\lambda) \otimes c_k^* q_2^{2\pi} q_2^{2\pi} \right) \right) \\
= \text{tr} |_{W_2} \left( (\mu_{23} \circ S_3) \left( \sum_{i,j,k,(d)} d_{i1}(\lambda) c_j \Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \otimes 
\Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \right) \left( q^{-2X} d_{i2}(\lambda) d_j(\lambda - \frac{1}{2} h^2) \right) \left( d_k^*(\lambda) q_2^{2\pi} \otimes c_k^* q_2^{2\pi} \right) \right) \\
= \text{tr} |_{W_2} \left( \sum_{i,j,k,(d)} d_{i1}(\lambda) c_j \Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \otimes 
\Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \right) \left( q^{-2X} d_{i2}(\lambda) d_j(\lambda + \frac{1}{2} h^2) \right) \\
= \sum_{i,j,k,(d)} d_{i1}(\lambda) c_j \Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \times 
\text{tr} |_{W_2} \left( q^{-2X} d_{i2}(\lambda) d_j(\lambda + \frac{1}{2} h^2) \right) .
\end{align*}
\]

Using the cyclic property of the trace turns the expression into

\[ \sum_{i,j,k,(d)} d_{i1}(\lambda) c_j \Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \times 
\text{tr} |_{W_2} \left( d_k^*(\lambda) q_2^{2\pi} S(c_k^*) S(c_k) q_2^{2\pi} q^{-2X} d_{i2}(\lambda) d_j(\lambda + \frac{1}{2} h^2) \right) . \]

(4.123)

Applying lemma 4.34 allows rewriting as

\[ \sum_{i,j,(d)} d_{i1}(\lambda) c_j \Psi^{11}_{\text{V}}(\lambda + \frac{1}{2} h^2, \mu) \times 
\text{tr} |_{W_2} \left( q^{\sum_{x_i}^2} Q^{-1}(-\lambda - \rho - h^2) S(u) q_2^{2\pi} \times 
S(c_k^*) q_2^{2\pi} q^{-2X} d_{i2}(\lambda) d_j(\lambda + \frac{1}{2} h^2) \right) . \]

(4.124)
Lemma 4.39 then allows further re-writing as
\[
\sum_{i,j,(d)} d_i(1) c_j \Psi_V^{11^*}(\lambda + h^2, \mu) \times
\]
\[
\text{tr} |_{W_2} \left( q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) q^{2X} S^{-1}(c_i) \times q^{-2X} d_i(2) (\lambda) d_j(\lambda + \frac{1}{2}h^2) S(u) q^{2\eta} \right).
\] (4.125)

Using the cyclic property of the trace again and rearranging the expression yields
\[
\text{tr} |_{W_2} \left( \sum_{i,j,(d)} d_i(1) c_j \Psi_V^{11^*}(\lambda + h^2, \mu) \times q^{2X} S^{-1}(c_i) q^{-2X} d_i(2) (\lambda) \times d_j(\lambda + \frac{1}{2}h^2) S(u) q^{2\eta} q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) \right)
\]
\[
= \text{tr} |_{W_2} \left( \sum_{i,j,(d)} \left( d_i(1) (\lambda) \otimes q^{2X} S^{-1}(c_i) q^{-2X} d_i(2) (\lambda) \right) \times \left( c_j \otimes d_j(\lambda + \frac{1}{2}h^2) S(u) q^{2\eta} q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) \right) \times \left( \Psi_V^{11^*}(\lambda + h^2, \mu) \otimes 1 \right) \right),
\] (4.126)

so that lemma 4.39 turns the expression into
\[
\text{tr} |_{W_2} \left( q^{-\sum(x \otimes x_i + 1 \otimes x_i^2)} \left( 1 \otimes u^{-1} S(Q)(\lambda - \rho - \frac{1}{2}h^2) \right) \times \right.
\]
\[
\left( (J^{21})^{-1}(\lambda + \frac{1}{2}h^2) R J(\lambda + \frac{1}{2}h^2) \times \left( 1 \otimes S(u) q^{2\eta} q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) \right) \right) \Psi_V^{11^*}(\lambda + h^2, \mu).
\] (4.127)

Every term in this expression is of zero weight, either in the $W_2$ component or in both components, which means that the term $q^{-\sum x \otimes x_i}$ may be moved to yield
\[
\text{tr} |_{W_2} \left( q^{-\sum x \otimes x_i} \left( 1 \otimes u^{-1} S(Q)(\lambda - \rho - \frac{1}{2}h^2) \right) \times \right.
\]
\[
\left( (J^{21})^{-1}(\lambda + \frac{1}{2}h^2) R J(\lambda + \frac{1}{2}h^2) \times \left( 1 \otimes S(u) q^{2\eta} q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) \right) q^{-\sum x \otimes x_i} \right) \Psi_V^{11^*}(\lambda + h^2, \mu)
\] (4.128)

\[
= \text{tr} |_{W_2} \left( q^{-\sum x \otimes x_i} \left( 1 \otimes u^{-1} S(Q)(\lambda - \rho - \frac{1}{2}h^2) \right) \times \right.
\]
\[
\left( (J^{21})^{-1}(\lambda + \frac{1}{2}h^2) R J(\lambda + \frac{1}{2}h^2) \times \left( 1 \otimes S(u) q^{2\eta} q^{\sum r^2} Q^{-1}(\lambda - \rho - h^2) \right) \right) \Psi_V^{11^*}(\lambda + h^2, \mu).
\]

Once again using the cyclic property of the trace, the fact that $Q$ is of zero weight,
the fact that $S(u) q^{2p}$ is central, and appendix proposition A.17 provides
\[
\text{tr} |_{W_2} \left( q^{\sum_i 1 \otimes x_i^2} \left( 1 \otimes q^{\sum_i x_i^2} Q^{-1}(-\lambda - \rho - h^2) \right) \times 
\left( 1 \otimes u^{-1} S(u) q^{2p} S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \right) \times 
(J^{21})^{-1} (\lambda + \frac{1}{2} h^2) R J (\lambda + \frac{1}{2} h^2) \right) \Psi_{\nu}^{11^*} (\lambda + h^2, \mu) 
\right) = \text{tr} |_{W_2} \left( \left( 1 \otimes Q^{-1}(-\lambda - \rho - h^2) \right) \times 
\left( 1 \otimes q^{-2p} q^{2p} S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \right) \times 
(J^{21})^{-1} (\lambda + \frac{1}{2} h^2) R J (\lambda + \frac{1}{2} h^2) \right) \Psi_{\nu}^{11^*} (\lambda + h^2, \mu) \right)
\] (4.129)

Because a trace is taken over $W_2$, the component labelled as 2 may be moved to
the first position of the tensor product, effectively performing a swap. Moreover,
taking the trace over $W$ is the same as taking the trace over the sum of its weight
spaces. Making these changes, the expression becomes
\[
\sum_\nu \text{tr} |_{W}[\nu] \left( \left( q^{2p} Q^{-1}(-\lambda - \rho - \nu) S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \otimes 1 \right) \times 
J^{-1} (\lambda + \frac{1}{2} h^2) R^{21} J^{21} (\lambda + \frac{1}{2} h^2) \right) \Psi_{\nu}^{11^*} (\lambda + \nu, \mu) \right) = \sum_\nu \text{tr} |_{W}[\nu] \left( \left( q^{2p} Q^{-1}(-\lambda - \rho - \nu) S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \otimes 1 \right) \times 
J^{-1} (\lambda - \rho + \frac{1}{2} h^1) R^{21} J^{21} (\lambda - \rho + \frac{1}{2} h^1) \right) \Psi_{\nu}^{11^*} (\lambda + \nu, \mu) \right) = \sum_\nu \text{tr} |_{W}[\nu] \left( \left( q^{2p} Q^{-1}(-\lambda - \rho - \nu) S(Q)(-\lambda - \rho - \frac{1}{2} h^1) \otimes 1 \right) \times 
R^{21}_{WW} (\lambda - \rho + \frac{1}{2} h^1) \right) \Psi_{\nu}^{11^*} (\lambda + \nu, \mu) \right)
\] (4.130)

Now since $\Psi_{\nu}^{11^*} \in V[0] \otimes V^* [0]$, a shift by $h^1$ equals the zero shift. Using this fact,
the expression reduces to
\[
\sum_\nu \text{tr} |_{W}[\nu] \left( \left( q^{2p} Q^{-1}(-\lambda - \rho - \nu) S(Q)(-\lambda - \rho) \otimes 1 \right) \times 
R^{21}_{WW} (\lambda - \rho) \right) \Psi_{\nu}^{11^*} (\lambda + \nu, \mu) \right) = \sum_\nu \text{tr} |_{W}[\nu] \left( (G(\lambda, \nu) \otimes 1) R^{21}_{WW} (\lambda - \rho) \right) T_\nu \Psi_{\nu}^{11^*} (\lambda, \mu),
\] (4.131)
as stated by the proposition, with $G(\lambda, \nu) = q^{-2p} Q^{-1}(-\lambda - \rho - \nu) S(Q)(-\lambda - \rho)$. \qed
4.3 Proof of the Main Theorem

4.3.5 Third Proposition

Proposition 4.31 has reduced the expression for $D_{CW}$ $\Psi_V(\lambda, \mu)$ as given by proposition 4.24 to again a single operator

\begin{equation}
\sum_{\nu} \text{tr}_{W[\nu]} \left( (G(\lambda, \nu) \otimes 1) R_{WW}^{21}(-\lambda - \rho) \right) T_{\nu}
\end{equation}

acting on the trace function $\Psi_V(\lambda, \mu)$. The conclusion is that

\begin{equation}
D_{CW} = \sum_{\nu} \text{tr}_{W[\nu]} \left( (G(\lambda, \nu) \otimes \text{id}) R_{WW}(-\lambda - \rho) \right) T_{\nu}.
\end{equation}

What remains is to rewrite the function $G(\lambda, \nu)$ in this expression for $D_{CW}$.

4.40 Proposition:

\begin{equation}
G(\lambda, \nu) = \frac{\delta_q(\lambda + \nu)}{\delta_q(\lambda)}
\end{equation}

Proof. By corollary 2.16 from [Etingof and Varchenko, 2000], $G(\lambda, \nu) = \frac{f(\lambda + \nu)}{f(\lambda)}$ for a suitable function $f$. This leaves the function $f$ to be determined.

Consider the equation

\begin{equation}
\chi_W(q^{2(\mu + \rho)}) \Psi_V(\lambda, \mu) = D_{CW} \Psi_V(\lambda, \mu),
\end{equation}

which holds for all representations $V$ of $U_q(sl_n)$, and so must also hold for $V = \mathbb{C}$. In this simple case $M_\mu \otimes \mathbb{C} \cong M_\mu$, and hence the intertwiners $\Phi_\mu$ are scalar. This means that

\begin{equation}
\begin{split}
\Psi_V(\lambda, \mu) &= \text{tr}_{M_\mu} q^{2\lambda} \\
&= \chi_{M_\mu}(q^{2\lambda}) \\
&= \frac{1}{\prod_{\alpha \in \Phi^+} (1 - q^{-2\alpha(\lambda)})} \\
&= \frac{q^{\mu(x)} \prod_{\alpha \in \Phi^+} (1 - q^{-2\alpha(\lambda)})}{\delta_q(\lambda)} \\
&= \frac{q^{\mu(x)} \prod_{\alpha \in \Phi^+} (1 - q^{-2\alpha(\lambda)})}{\delta_q(\lambda)}.
\end{split}
\end{equation}

since the character $\chi_{M_\mu}$ of a Verma module $M_\mu$ is given by

\begin{equation}
\begin{split}
\chi_{M_\mu}(q^x) &= \sum_{\beta \in \Phi^+} \dim M_\mu[\beta] q^{\beta(x)} \\
&= q^{\mu(x)} \prod_{\alpha \in \Phi^+} (1 + q^{-\alpha(x)} + q^{-2\alpha(x)} + \ldots) \\
&= q^{\mu(x)} \prod_{\alpha \in \Phi^+} \sum_{n=0}^{\infty} (q^{-\alpha(x)})^n \\
&= \prod_{\alpha \in \Phi^+} (1 - q^{-\alpha(\lambda)}).
\end{split}
\end{equation}
Moreover, because $V = \mathbb{C}$ the operator $D_{C_W}$ simplifies to
\begin{equation}
D_{C_W} = \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] G(\lambda, \nu) T_\nu.
\end{equation}

Now using this and equation\footnote{4.136} equation\footnote{4.135} can be written as
\begin{equation}
\chi_W(q^{2(\pi+\rho)}) \frac{\delta_{q^{\nu_2(\pi+\rho)}}}{\delta_q(\lambda)} = \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] G(\lambda, \nu) T_\nu \frac{q^{2(\nu_2(\pi+\rho))}}{\delta_q(\lambda)},
\end{equation}

since $\mu(\nu) = \nu(\mu)$ for all $\mu, \nu \in \mathfrak{h}^*$. But the character on the left hand side equals
\begin{equation}
\chi_W(q^{2(\pi+\rho)}) = \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] q^{\nu_2(2(\pi+\rho))} G(\lambda, \nu) \frac{q^{2(\nu_2(\pi+\rho))}}{\delta_q(\lambda + \nu)},
\end{equation}

so equation\footnote{4.139} must imply that $f = \delta_q$ and that
\begin{equation}
G(\lambda, \nu) = \frac{\delta_q(\lambda + \nu)}{\delta_q(\lambda)}.
\end{equation}

This explicitly determines $f$ for the case $V = \mathbb{C}$, and thus also for any other representation $V$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$, completing the proof. \hfill \Box

4.3.6 FINAL PROOF

Thanks to proposition\footnote{4.40} the operator $D_{C_W}$ can now be written as
\begin{equation}
\begin{aligned}
D_{C_W} &= \sum_\nu \text{tr}_{W[\nu]} \left( \left( \frac{\delta_q(\lambda + \nu)}{\delta_q(\lambda)} \otimes \text{id} \right) R_{WV}(\lambda + \nu) \right) T_\nu \\
&= \delta_q(\lambda)^{-1} \sum_\nu \text{tr}_{W[\nu]} \left( R_{WV}(\lambda + \nu) \right) \delta_q(\lambda + \nu) T_\nu \\
&= \delta_q(\lambda)^{-1} D_W \delta_q(\lambda).
\end{aligned}
\end{equation}

\textbf{Proof of the Main Theorem}~\footnote{4.13} By corollary\footnote{4.9} the trace function $\Psi_V(\lambda, \mu)$ satisfies the difference equations
\begin{equation}
D_{C_W} \Psi_V(\lambda, \mu) = \chi_W(q^{2(\pi+\rho)}) \Psi_V(\lambda, \mu).
\end{equation}

Acting on $\Psi_V(\lambda, \mu)$ with the difference operator $D_{C_W}$, as described in theorem\footnote{4.6} and demonstrated in equation\footnote{4.36} and then consecutively applying propositions\footnote{4.24} and\footnote{4.31} to rewrite the expression now allows to state the difference equations\footnote{4.143} for the transfer operators $D_W$:
\begin{equation}
\delta_q(\lambda)^{-1} D_W \delta_q(\lambda) \Psi_V(\lambda, \mu) = \chi_W(q^{2(\pi+\rho)}) \Psi_V(\lambda, \mu).
\end{equation}
Moving $\delta_q(\lambda)^{-1}$ to the right hand side and substituting $\mu \to (-\mu - \rho)$ results in

$$D_W \Psi_V(\lambda, -\mu - \rho) \delta_q(\lambda) = \chi_W(q^{-2\pi}) \Psi_V(\lambda, -\mu - \rho) \delta_q(\lambda)$$

or

$$D_W F_V(\lambda, \mu) = \chi_W(q^{-2\pi}) F_V(\lambda, \mu)$$

as the theorem states.

### 4.4 Quantum Spin Calogero-Moser System

Quantum spin Calogero-Moser systems are examples of Calogero-Moser systems (see [Calogero, 2008]). Calogero-Mosersystemsemergedintheearly1970’swhenit
was discovered that such a system is exactly solvable; they are integrable systems. Since then Calogero-Moser systems have found a role in research areas ranging
from physics to pure mathematics. Remarkably, Calogero-Moser systems were first
solved in the quantal context in [Calogero, 1971], with solvability in the classical
context following in 1975.

From a physical point of view, the quantum spin Calogero-Moser system des-
cribes a system of $n$ equal quantum particles on a line interacting pairwise through
a repulsive force and with a common external potential. The Hamiltonian, or en-
ergy function, of the quantum spin Calogero-Moser system is defined as follows
(see section 7.3 of [Etingof and Latour, 2005]).

**4.41 Definition:**
The quantum spin Calogero-Moser Hamiltonian is given by the operator

$$H = \frac{1}{2} \Delta_{h^*} - \sum_{\alpha \in \Phi^+} \frac{e_{\alpha} e_{-\alpha}}{(e^{\frac{1}{2}\alpha(\lambda)}) - (e^{-\frac{1}{2}\alpha(\lambda)})^2},$$

where $\Delta_{h^*} := \sum_i \partial_{x_i}^2$ and $e_{\alpha} \in \mathfrak{sl}_n$ as in appendix section A.1. For $V$ a finite-
dimensional representation of $\mathfrak{sl}_n$, $H \in C^\infty(h^*) \otimes V[0]$, since $e_{\alpha} e_{-\alpha} : V[0] \to V[0]$.

The first term in the Hamiltonian corresponds to the kinetic energy of the sys-
pond, while the second term corresponds to the potential energy of the system. The Hamiltonian as presented in definition 4.41 follows from the operator $D_W^{(\lambda)}$ in
a natural way. The rest of this section demonstrates how a single Hamiltonian,
without dependence on the representation $W$, is constructed.

**4.42 Definition:**
Let \( q = e^{\frac{h}{\hbar}} \) and \( \iota \) the substitution \( \lambda \to \frac{\lambda}{\hbar} \), then

\[
\tilde{D}_{W}^{(\lambda)} := \iota \circ D_{W}^{(\lambda)} \circ \iota^{-1}.
\]

4.43 Remark:

\[
\tilde{D}_{W}^{(\lambda)} = \sum_{\nu \in \mathfrak{h}^{\ast}} \text{tr} [w_{\nu}] R_{WW} \left( -\frac{\lambda}{\hbar} - \rho \right) T_{\nu
u}
\]

since for \( f : \mathfrak{h}^{\ast} \to \mathbb{V}[0] \)

\[
\tilde{D}_{W}^{(\lambda)} f(\lambda) = \sum_{\nu \in \mathfrak{h}^{\ast}} \iota \circ \text{tr} [w_{\nu}] R_{WW} \left( -\lambda - \rho \right) T_{\nu} f(\lambda)
\]

The Hamiltonian of the quantum spin Calogero-Moser system, definition 4.41, is found by writing out the Taylor expansion of the operator \( \tilde{D}_{W}^{(\lambda)} \) and picking out the term of order \( \hbar^2 \). Surprisingly, up to a constant, this term will be completely independent of the representation \( W \).

Since \( \tilde{D}_{W}^{(\lambda)} \) is constructed from the exchange operator \( R_{WW} \), its Taylor expansion is computed using the Taylor expansions of the universal \( R \)-matrix \( R \) and the fusion operator \( J(\frac{h}{\hbar}) \).

4.44 Lemma:

The universal \( R \)-matrix \( R \) and the fusion operator \( J(\frac{h}{\hbar}) \) have Taylor expansions

\[
R = 1 + \hbar r + O(\hbar^2)
\]

\[
J(\frac{h}{\hbar}) = 1 + \hbar j(\lambda) + O(\hbar^2)
\]

where

\[
r = \frac{1}{2} \sum_{i} x_{i} \otimes x_{i} + \sum_{\alpha \in \Phi^{+}} e_{\alpha} \otimes e_{-\alpha} \in \mathfrak{U}(\mathfrak{s}\mathfrak{l}_{n}) \otimes \mathfrak{U}(\mathfrak{s}\mathfrak{l}_{n})
\]

and

\[
j(\lambda) = \sum_{\alpha \in \Phi^{+}} \frac{e_{-\alpha} \otimes e_{\alpha}}{1 - e^{\alpha(\lambda)}} \in \mathfrak{U}(\mathfrak{s}\mathfrak{l}_{n}) \otimes \mathfrak{U}(\mathfrak{s}\mathfrak{l}_{n}).
\]

Proof. It follows from the explicit expression for a universal \( R \)-matrix, as given in section 8.3 of [Klimyk and Schmüdgen, 1997], that with \( q = e^{\frac{h}{\hbar}} \),

\[
R = 1 + \frac{h}{2} \left( \sum_{i} x_{i} \otimes x_{i} + 2 \sum_{\alpha \in \Phi^{+}} e_{\alpha} \otimes e_{-\alpha} \right) + O(\hbar^2).
\]
To determine the term $j(\lambda)$ in the Taylor expansion of $J(\frac{\lambda}{\hbar})$, recall equation 3.31. It follows from
\begin{equation}
q^{-\sum_i x_i \otimes x_i} = 1 - \frac{\hbar}{2} \sum_i x_i \otimes x_i + O(\hbar^2)
\end{equation}
and equation 4.154 that
\begin{equation}
R^{21} q^{-\sum_i x_i \otimes x_i} = 1 + \hbar \sum_{\alpha \in \Phi^+} e_{-\alpha} \otimes e_\alpha + O(\hbar^2).
\end{equation}
Using this, the Taylor expansion of equation 3.31 for $J(\frac{\lambda}{\hbar})$ becomes
\begin{equation}
1 + \hbar j(\lambda) (1 \otimes e^\lambda) + O(\hbar^2) = 1 + \hbar \sum_{\alpha \in \Phi^+} e_{-\alpha} \otimes e_\alpha (1 \otimes e^\lambda) j(\lambda) + O(\hbar^2),
\end{equation}
and thus
\begin{equation}
j(\lambda) (1 \otimes e^\lambda) = \sum_{\alpha \in \Phi^+} (e_{-\alpha} \otimes e_\alpha) (1 \otimes e^\lambda) j(\lambda)
\end{equation}
or
\begin{equation}
j(\lambda) - (1 \otimes e^\lambda) j(\lambda) (1 \otimes e^{-\lambda}) = \sum_{\alpha \in \Phi^+} e_{-\alpha} \otimes e_\alpha.
\end{equation}
Now since $j(\lambda) = \sum_{\mu > 0} j^\mu(\lambda)$ with $j^\mu(\lambda) \in \mathfrak{U}_q(\mathfrak{sl}_n)[\mu] \otimes \mathfrak{U}_q(\mathfrak{sl}_n)[\mu]$, the second term becomes
\begin{equation}
(1 \otimes e^\lambda) j(\lambda) (1 \otimes e^{-\lambda}) = \sum_{\mu > 0} (1 \otimes e^\lambda) j^\mu(\lambda) (1 \otimes e^{-\lambda}) = \sum_{\mu > 0} e^{\mu(\lambda)} j^\mu(\lambda)
\end{equation}
and the equation reduces to
\begin{equation}
\sum_{\mu > 0} (1 - e^{\mu(\lambda)}) j^\mu(\lambda) = \sum_{\alpha \in \Phi^+} e_{-\alpha} \otimes e_\alpha.
\end{equation}
Equation 4.161 implies that for $\alpha \in \Phi^+$,
\begin{equation}
j^\alpha(\lambda) = \frac{e_{-\alpha} \otimes e_\alpha}{1 - e^{\alpha(\lambda)}}
\end{equation}
and $j^\alpha(\lambda) = 0$ otherwise, leading to the expression
\begin{equation}
j(\lambda) = \sum_{\alpha \in \Phi^+} \frac{e_{-\alpha} \otimes e_\alpha}{1 - e^{\alpha(\lambda)}}
\end{equation}
as stated by the lemma. \qed
4.45 Proposition:

\[
\tilde{D}_W^{(\lambda)} = (\dim W) \text{id} + \hbar^2 \left( \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \left( \frac{e_{-\nu} e_{\nu}}{e_{\nu} e_{-\nu}} \right) \right) + O(\hbar^3),
\]

where \( \partial_\nu \) is the directional derivative with respect to \( \nu \).

Proof. Using lemma 4.44 to compute the Taylor expansion of \( R_{WV} \left( -\frac{1}{\hbar} - \rho \right) \) yields

\[
\text{tr} |W\rangle R_{WV} \left( -\frac{1}{\hbar} - \rho \right) = \dim W + \hbar^2 \sum_{\alpha \in \Phi^+} (\text{tr} |W\rangle e_{-\alpha} e_{\alpha}) \frac{e_{\alpha} e_{-\alpha}}{e_{\nu} e_{-\nu}} + O(\hbar^3).
\]

Multiplying by the Taylor expansion

\[
T_{\nu} = e^{h \partial_\nu} = 1 + \hbar \partial_\nu + \hbar^2 \frac{1}{2} \partial^2_\nu + O(\hbar^3)
\]

and taking the sum over all \( \nu \in \mathfrak{h}^* \) provides the result. See section 7.3 of [Etingof and Latour, 2005] for a detailed computation.

Since \( \mathfrak{sl}_n \) is a semi-simple Lie algebra, every bilinear symmetric invariant form on \( \mathfrak{sl}_n \) equals the Killing form up to a constant. In particular, since the Killing form allows identification of \( \mathfrak{sl}_n \) with \( \mathfrak{sl}_n^* \) as illustrated in section 2.1, a bilinear symmetric invariant form \( \kappa_W : \mathfrak{sl}_n \times \mathfrak{sl}_n \to \mathbb{C} : a, b \mapsto \text{tr} |W\rangle (a b) \) may be written as \( \kappa_W(\cdot, \cdot) = c_W \sum_i x_i^* (\cdot) x_i^*(\cdot) \) for a certain constant \( c_W \) and \( \{ x_i^* \} \) the basis of \( \mathfrak{h}^* \) dual to the basis \( \{ x_i \} \) of \( \mathfrak{h} \).

4.46 Lemma:

\[
\sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \partial_\nu^2 = c_W \sum_i \partial^2_{x_i^*},
\]

where \( c_W \) is the constant such that \( \text{tr} |W\rangle (a b) = c_W \sum_i x_i^*(a) x_i^*(b) \) for all \( a, b \in \mathfrak{h} \).

Proof. For all \( a, b \in \mathfrak{h} \),

\[
\left( \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \nu \otimes \nu \right) (a \otimes b) = \sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \nu(a) \nu(b) = \text{tr} |W\rangle (a b) = c_W \sum_i x_i^*(a) x_i^*(b) = \left( c_W \sum_i x_i^* \otimes x_i^* \right) (a \otimes b),
\]

implying that

\[
\sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \nu \otimes \nu = c_W \sum_i x_i^* \otimes x_i^*.
\]
The lemma is a direct result of this equality. \( \square \)

Applying lemma \ref{lemma:4.46} to the Taylor expansion of \( \bar{D}^{(\lambda)}_W \), proposition \ref{proposition:4.45} leaves the constant \( c_W \) as the only factor in the term of order \( \hbar^2 \) that depends on the representation \( W \). Picking out this term of order \( \hbar^2 \) and dividing by \( c_W \) leaves an operator independent of the representation \( W \); the Hamiltonian of the quantum spin Calogero-Moser system.

**4.47 Proposition:**

\[
\lim_{\hbar \to 0} \frac{D^{(\lambda)}_W - \dim W \id}{\hbar^2 c_W} = \frac{1}{2} \sum_i \partial^2_{x_i^*} - \sum_{\alpha \in \Phi^+} \frac{e_{\alpha} e_{-\alpha}}{(e^{\frac{1}{2} \alpha(\lambda)} - e^{-\frac{1}{2} \alpha(\lambda)})^2}
\]

**Proof.** Starting with the Taylor expansion of \( \bar{D}^{(\lambda)}_W \), proposition \ref{proposition:4.45}, and subtracting the term of order zero in \( \hbar \) leaves

\[
D^{(\lambda)}_W - \dim W \id = \hbar^2 \left( \sum_{\nu \in h^*} \dim W[\nu] \frac{1}{2} \partial^2_{x^\nu} - \sum_{\alpha \in \Phi^+} (\text{tr} | W e_{-\alpha} e_{\alpha}) \frac{e_{\alpha} e_{-\alpha}}{(e^{\frac{1}{2} \alpha(\lambda)} - e^{-\frac{1}{2} \alpha(\lambda)})^2} \right) + O(\hbar^3).
\]

By lemma \ref{lemma:4.46} and because \( \text{tr} | W (e_{-\alpha} e_{\alpha}) = c_W \kappa(e_{-\alpha}, e_{\alpha}) = c_W \), this becomes

\[
D^{(\lambda)}_W - \dim W \id = \hbar^2 \left( c_W \sum_i \frac{1}{2} \partial^2_{x_i^*} - \sum_{\alpha \in \Phi^+} c_W \frac{e_{\alpha} e_{-\alpha}}{(e^{\frac{1}{2} \alpha(\lambda)} - e^{-\frac{1}{2} \alpha(\lambda)})^2} \right) + O(\hbar^3),
\]

so that

\[
\frac{D^{(\lambda)}_W - \dim W \id}{\hbar^2 c_W} = \frac{1}{2} \sum_i \partial^2_{x_i^*} - \sum_{\alpha \in \Phi^+} \frac{e_{\alpha} e_{-\alpha}}{(e^{\frac{1}{2} \alpha(\lambda)} - e^{-\frac{1}{2} \alpha(\lambda)})^2} + O(\hbar).
\]

Now taking the limit \( \lim_{\hbar \to 0} \) proves the proposition. \( \square \)

**4.5 Macdonald Operators**

Macdonald polynomials \( P_\lambda(q, k) \), first presented in \cite{Macdonald, 1988}, are a family of orthogonal symmetric polynomials in \( n - 1 \) variables, and generalize a number of other families of orthogonal polynomials. One way to define Macdonald polynomials is as common eigenfunctions of a commutative set of \( n - 1 \) commuting difference operators \( M_r \) called Macdonald operators. This is an interesting definition, since theorem \ref{theorem:1.13} provides an integrable system of partial transfer operators \( D^{(\lambda)}_W \). Indeed, the Macdonald operators whose common eigenfunctions define the
Macdonald polynomials are found to be a specific case of the partial transfer operators $D_W^{(\lambda)}$. This is described in [Etingof and Kirillov, 1994].

4.48 Remark:
Note that in the notation of [Macdonald, 1988], the Macdonald polynomial $P_\lambda(q, k)$ would be written as $P_\lambda(q^2, (q^k)^2)$.

4.49 Definition:
The Macdonald operators on polynomials in $\mathbb{C}(q, q^k)[x_1, \ldots, x_{n-1}]$ are defined as

\begin{equation}
M_r = \sum_{i_1 < \cdots < i_r} \left( \prod_{j \notin \{i_1, \ldots, i_r\}} \frac{q^k x_i - q^{-k} x_j}{x_i - x_j} \right) T_{q^2, x_{i_1}} \cdots T_{q^2, x_{i_r}},
\end{equation}

where $(T_{q^2, x_i} f)(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, q^2 x_i, \ldots, x_{n-1})$.

Consider the finite-dimensional representation $U$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$ that is defined as the $q$-analog of the representation $S^{(k-1)n}(\mathbb{C}^n)$ of $\mathfrak{sl}_n$ (the symmetric power).

4.50 Proposition:
For $\mu$ dominant integral (appendix definition A.30) and $V_{\mu+(k-1)\rho}$, $k \in \mathbb{N}$, the irreducible finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$ (appendix proposition A.29), there exists an intertwiner

\begin{equation}
\Phi_{\mu, k}^u : V_{\mu+(k-1)\rho} \to V_{\mu+(k-1)\rho} \otimes U,
\end{equation}

with $u \in U[0]$, which is unique op to a factor.

Proof. See lemma 1 of [Etingof and Kirillov, 1994].

4.51 Definition:
The vector-valued character $\phi_k^{\mu}(\lambda) : U[0] \to U[0]$ is defined as

\begin{equation}
\phi_k^{\mu}(\lambda)(u) := \text{tr} |_{V_{\mu+(k-1)\rho}} (\Phi_{\mu, k}^u q^{2\lambda}),
\end{equation}

with $u \in U[0]$ and $\Phi_{\mu, k}^u$ as in proposition 4.50.

4.52 Theorem:
\begin{equation}
\phi_k^{0}(\lambda) = \prod_{m=1}^{k-1} \prod_{\alpha \in \Phi^+} \left( e^{i\alpha(\lambda)} - q^{-2m} e^{-i\alpha(\lambda)} \right)
\end{equation}

Proof. See the main lemma of [Etingof and Kirillov, 1994].
As described in [Etingof and Kirillov, 1994], the Macdonald operators $M_r$ that the Macdonald polynomials $P_\lambda(q, k)$ are common eigenfunctions of may be defined in terms of the transfer operators $D_W$ for specific representations $W$. To show this, first consider the transfer operators $D_{C_W}$ defined in theorem 4.6 with $C_W$ central elements of $\mathfrak{U}_q(\mathfrak{sl}_n)$ defined in corollary 4.19.

4.53 Theorem:
Let $\Lambda_r$ a fundamental representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$, then for
\[ C_{\Lambda_r} = (\text{id} \otimes \text{tr}|_{\Lambda_r})(R^{21} R(1 \otimes q^{2\rho})) \]
the Macdonald operator $M_r$ equals
\[ M_r = \phi^k_0(\lambda)^{-1} D_{C_{\Lambda_r}} \phi^k_0(\lambda), \]
where $\phi^k_0(\lambda)$ is seen as the operator of multiplication by $\phi^k_0(\lambda)$.

Proof. See theorem 4 of [Etingof and Kirillov, 1994].

4.54 Corollary:
\[ M_r = \phi^k_0(\lambda)^{-1} \delta_q(\lambda)^{-1} D_{\Lambda_r} \delta_q(\lambda) \phi^k_0(\lambda) \]
Proof. This is a direct consequence of theorem 4.53 and proposition 4.11.

Theorem [4.53] implies that the Macdonald polynomials, that satisfy difference equations for the Macdonald operators, can now be found as the eigenfunctions satisfying the difference equations for $D_{C_{\Lambda_r}}$, corollary 4.9:
\[ M_r \phi^k_0(\lambda)^{-1} \phi^k_\mu(\lambda) = \chi_{\Lambda_r}(q^{2(\mu^+ + \rho)}) \phi^k_0(\lambda)^{-1} \phi^k_\mu(\lambda), \]
and the definition follows.

4.55 Definition:
The Macdonald polynomials $P_\mu(q, k)$ are defined as
\[ P_\mu(q, k) = \frac{\phi^k_\mu(\lambda)}{\phi^k_0(\lambda)}. \]
Popular Summary

Quantum integrable systems are quantum systems whose time-evolution can be exactly solved. Since time-evolution is governed by the Schrödinger equation, one way of describing quantum integrable systems involves diagonalizing the energy function or Hamiltonian of the system. This requires a maximal set of commuting linear operators that is simultaneously diagonalized. These commuting operators can be thought of as conserved quantities of the system. One method of defining quantum integrable systems is referred to as the quantum inverse scattering method.

At the basis of this theory lies the quantum Yang-Baxter equation, which operates in the framework of quantum groups. An example of a quantum group is the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a semi-simple Lie algebra $\mathfrak{g}$. Important is that quantum groups carry a Hopf algebra structure, which means that they are algebras with the addition of a comultiplication, counit, and antipode map. In particular, quantum groups are braided Hopf algebras, which means that there is a so-called universal $R$-matrix $R \in \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ that satisfies the quantum Yang-Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$  

New universal $R$-matrices, and thus solutions to the quantum Yang-Baxter equation, can be constructed by ‘twisting’ a universal $R$-matrix $R$.

A universal $R$-matrix $R$ can also be seen as an operator $R_{VW}$ on two finite-dimensional representations $V$ and $W$ of the quantum group. Similar to twisting a universal $R$-matrix $R$, an operator $R_{VW}$ can be twisted by a dynamical twist to obtain an operator $R_{VW}(\lambda)$. The parameter $\lambda$ is called a dynamical parameter. This results in a solution to the quantum dynamical Yang-Baxter equation

$$R_{UV}(\lambda - h^3)R_{UW}^{13}(\lambda)R_{VW}^{23}(\lambda - h^1) = R_{VW}^{23}(\lambda)R_{UV}^{13}(\lambda - h^2)R_{UV}^{12}(\lambda).$$

A solution $R_{VW}(\lambda)$ is also referred to as a quantum dynamical $R$-matrix.

Fixing a representation $V$ of the quantum group, quantum dynamical $R$-matrices $R_{WV}(\lambda)$ can be used to construct a set of commuting transfer operators that is simultaneously diagonalized by a trace function. These transfer operators then describe a quantum integrable system. From a physical perspective, the transfer operators are the conserved quantities of the system, acting on a quantum state space given by $V$. 
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A Appendix

Most of the following is found in [Humphreys, 1972] and [Kassel et al., 1997].

A.1 The Lie Algebra $\mathfrak{sl}_n$

$\mathfrak{sl}_n$ is the Lie algebra of complex $n \times n$ matrices with zero trace. It is generated by

\begin{equation}
\begin{aligned}
e_i &= M_{i,i+1}, & f_i &= M_{i+1,i}, & h_i &= M_{i,i} - M_{i+1,i+1},
\end{aligned}
\end{equation}

where $M_{i,j}$ is the matrix with a 1 at position $(i,j)$ and zeros elsewhere. The Cartan subalgebra is the subalgebra $\mathfrak{h}$ of diagonal matrices, and the positive and negative nilpotent subalgebras are $\mathfrak{n}_+ = \langle e_i \rangle$ and $\mathfrak{n}_- = \langle f_i \rangle$ respectively. Consequently, the positive and negative Borel subalgebras are $\mathfrak{b}_+ = \langle e_i, h_i \rangle$ and $\mathfrak{b}_- = \langle h_i, f_i \rangle$ respectively.

Consider the linear forms $\epsilon_1, \ldots, \epsilon_n$ on $\operatorname{Mat}(n, \mathbb{C})$ defined by $\epsilon_i(M_{j,j}) = \delta_{ij}$. The set of simple roots $\Delta$ is formed by the linear forms $\alpha_i = \epsilon_i - \epsilon_{i+1}$, which also form a basis of $\mathfrak{h}^*$. The set of positive roots $\Phi^+$ is formed by sums of the simple roots $\sum_{i=j}^k \alpha_i$ where $1 \leq j \leq k \leq n - 1$. This is the type $A_{n-1}$ root system with a total of $n^2 - n$ roots in $\Phi$.

The $h_i$ are the simple co-roots and form a basis of the Cartan subalgebra $\mathfrak{h}$. The elements $\omega_i \in \mathfrak{h}^*$ that form a basis of $\mathfrak{h}^*$ dual to the basis $\{h_i\}$ of $\mathfrak{h}$, which means that $\omega_i(h_j) = \delta_{ij}$, are the fundamental weights.

The root space decomposition of $\mathfrak{sl}_n$ is given by

\begin{equation}
\begin{aligned}
\mathfrak{sl}_n = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{sl}_n[\alpha],
\end{aligned}
\end{equation}

where the root spaces $\mathfrak{sl}_n[\alpha] = \{y \in \mathfrak{sl}_n \mid [x,y] = \alpha(x)y \text{ for all } x \in \mathfrak{h}\}$ are one dimensional, i.e. $\mathfrak{sl}_n[\alpha] = \mathbb{C}e_\alpha$ for some $e_\alpha \in \mathfrak{sl}_n$. Using the Killing form $\kappa$ on $\mathfrak{sl}_n \otimes \mathfrak{sl}_n$, the $e_\alpha$ are normalized as $\kappa(e_\alpha, e_{-\alpha}) = 1$. Note that $e_i = e_\alpha$, and $f_i = e_{-\alpha}$.

The generators $e_i, f_i, h_i$ $(1 \leq i \leq n-1)$ of $\mathfrak{sl}_n$ are subject to the following relations:

\begin{equation}
\begin{aligned}
[h_i, h_j] &= 0 \\
[h_i, e_j] &= \alpha_j(h_i) e_j \\
[h_i, f_j] &= -\alpha_j(h_i) f_j \\
e_i, f_j &= \delta_{ij} h_i \\
[e_i, e_j] &= 0 = [f_i, f_j] \quad \text{if } |i - j| > 1 \\
e_i, [e_i, e_{j \pm 1}] &= 0 = [f_i, [f_i, f_{i \pm 1}]].
\end{aligned}
\end{equation}
A.1 Definition:
An important element in Lie algebra theory is
(A.4) \[ \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \]

Since the positive roots of \(\mathfrak{sl}_n\) are formed by sums of consecutive simple roots, the element \(\rho\) may in this case alternatively be written as
(A.5) \[ \rho = \frac{1}{2} \sum_{1 \leq j \leq k \leq n-1} \sum_{i=j}^{k} \alpha_i, \quad \alpha_i \in \Delta. \]

A.1.1 Example: \(\mathfrak{sl}_2\)
In the case \(n = 2\), the Cartan subalgebra \(\mathfrak{h}\) and both the positive and negative nilpotent subalgebras \(\mathfrak{n}_+\) and \(\mathfrak{n}_-\) are one dimensional:
(A.6) \[ \mathfrak{n}_+ = \langle e \rangle, \quad \mathfrak{h} = \langle h \rangle, \quad \mathfrak{n}_- = \langle f \rangle \]
with
(A.7) \[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Linear maps on the Cartan subalgebra are just (multiplication by) complex numbers, i.e. \(\mathfrak{h}^* \cong \mathbb{C}\). There is only one simple root, \(\{2\} = \Delta\), so the root system is \(A_1\) with \(\Phi = \{2, -2\}\). Hence the defining relations of the generators are:
(A.8) \[ [e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \]
The element \(\rho\) from definition A.1 simply becomes \(\rho = 1\) for \(\mathfrak{sl}_2\).

A.2 Hopf Algebras
Hopf algebras are algebras. A Hopf algebra is an associative algebra with a co-
multiplication and counit, called a bialgebra, and the addition of an antipode.

A.2.1 Definition of a Hopf Algebra
An associative algebra \(A\) is a triple \((A, \mu, \eta)\) where \(A\) is a vector space over some
field \(k\), and \(\mu\) and \(\eta\) are \(k\)-linear maps. These two operators are defined as:

i. Multiplication;
\[ \mu : A \otimes A \to A. \]
ii. **Unit;**
\[ \eta : k \to A. \]

The operators \( \mu \) and \( \eta \) satisfy the following two axioms:

1. **Associativity;**
\[ \mu(\mu \otimes \text{id}) = \mu(\text{id} \otimes \mu). \]

2. **Unitality;**
\[ \mu(\eta \otimes \text{id}) = \text{id} = \mu(\text{id} \otimes \eta). \]

A **bialgebra** \( A \) is a quintuple \( (A, \mu, \eta, \Delta, \epsilon) \), which is an associative algebra \( (A, \mu, \eta) \) equipped with two additional algebra operators:

iii. **Comultiplication;**
\[ \Delta : A \to A \otimes A. \]

iv. **Counit;**
\[ \epsilon : A \to k. \]

The operators \( \Delta \) and \( \epsilon \) satisfy the following two axioms:

3. **Coassociativity;**
\[ (\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta. \]

4. **Counitality;**
\[ (\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta. \]

A **Hopf algebra** \( H \) is a sextuple \( (H, \mu, \eta, \Delta, \epsilon, S) \), which is a bialgebra \( (H, \mu, \eta, \Delta, \epsilon) \) equipped with an additional invertible linear operator:

v. **Antipode;**
\[ S : H \to H. \]

The operator \( S \) satisfies the following axiom:

5. **Convolution inverse;**
\[ \mu(S \otimes \text{id})\Delta = \eta \circ \epsilon = \mu(\text{id} \otimes S)\Delta. \]

A last important addition to these definitions are the concepts of opposite and coopposite bialgebras and Hopf algebras.

**A.2 Notation:**

\( \mu^{\text{op}} = \mu \circ \tau \) is the opposite multiplication, where the linear map \( \tau \) is the swap; \( \tau(x \otimes y) = y \otimes x. \) Analogously, \( \Delta^{\text{op}} = \tau \circ \Delta \) is the opposite comultiplication.
A.2 Hopf Algebras

For a bialgebra \( (A, \mu, \eta, \Delta, \epsilon) \), this notation gives rise to three related bialgebras:

\[
\begin{align*}
A^{\text{op}} &= (A, \mu^{\text{op}}, \eta, \Delta, \epsilon) \\
A^{\text{cop}} &= (A, \mu, \eta, \Delta^{\text{op}}, \epsilon) \\
A^{\text{op,cop}} &= (A, \mu^{\text{op}}, \eta, \Delta^{\text{op}}, \epsilon).
\end{align*}
\]

(A.9)

These \( A^{\text{op}} \), \( A^{\text{cop}} \), and \( A^{\text{op,cop}} \) are again all bialgebras. The same notation is adopted for Hopf algebras, with

\[
\begin{align*}
H^{\text{op}} &= (H, \mu^{\text{op}}, \eta, \Delta, \epsilon, S^{-1}) \\
H^{\text{cop}} &= (H, \mu, \eta, \Delta^{\text{op}}, \epsilon, S^{-1}) \\
H^{\text{op,cop}} &= (H, \mu^{\text{op}}, \eta, \Delta^{\text{op}}, \epsilon, S)
\end{align*}
\]

(A.10)

also again all Hopf algebras.

A.2.2 The Sweedler Notation

Let \( A = (A, \mu, \eta, \Delta, \epsilon) \) be a bialgebra. For any \( x \in A \) the element \( \Delta(x) \) in \( A \otimes A \) is of the form

\[
\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}.
\]

(A.11)

This sum is conveniently written as

\[
\sum_{(xy)} (xy)_{(1)} \otimes (xy)_{(2)} = \sum_{(x)(y)} x^{(1)} y^{(1)} \otimes x^{(2)} y^{(2)}.
\]

(A.12)

The fact that \( \Delta \) is a morphism of algebras implies that

\[
\sum_{(x)} (x^{(1)})_{(1)} \otimes (x^{(1)})_{(2)} \otimes x^{(2)} = \sum_{(x)} x^{(1)} \otimes \left( \sum_{(x^{(2)})} (x^{(2)})_{(1)} \otimes (x^{(2)})_{(2)} \right),
\]

(A.13)

it is simply written as

\[
\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}.
\]

(A.14)

Because of the coassociativity axiom, there is no confusion in this notation. The counitality axiom may be expressed as

\[
\sum_{(x)} \epsilon(x^{(1)}) x^{(2)} = x = \sum_{(x)} x^{(1)} \epsilon(x^{(2)})
\]

(A.15)

for all \( x \in A \).
Furthermore, the opposite comultiplication $\Delta^\text{op}$ may be expressed as

\[(A.16) \quad \Delta^\text{op}(x) = \sum_{(x)} x^{(2)} \otimes x^{(1)}.\]

A.2.3 **BRAIDED HOPF ALGEBRAS**

Braided bialgebras, also known as quasi-triangular bialgebras, are bialgebras with a so-called universal $R$-matrix. Braided bialgebras that also have an invertible antipode are braided Hopf algebras.

A.3 **Definition:**

A Hopf algebra $H$ is **braided** if and only if there exists an invertible element $R \in H \otimes H$ such that

1. $\Delta^\text{op}(x) = R\Delta(x)R^{-1}$ for all $x \in H$,
2. $(\Delta \otimes \text{id})(R) = R^{13}R^{23}$,
3. $(\text{id} \otimes \Delta)(R) = R^{13}R^{12},$

where $R^{12} = R \otimes 1$, $R^{23} = 1 \otimes R$, and $R^{13} = (\text{id} \otimes \tau)(R \otimes 1) = (\tau \otimes \text{id})(1 \otimes R)$.

Such an element $R \in H \otimes H$ is called a **universal** $R$-matrix.

By definition, a universal $R$-matrix is invertible. Its inverse is explicitly given in the following proposition.

A.4 **Proposition:**

Let $R$ be a universal $R$-matrix of a Hopf algebra $H$ with antipode $S$ and counit $\epsilon$.

\[(A.17) \quad (S \otimes \text{id})(R) = R^{-1} = (\text{id} \otimes S)(R)\]

\[(\epsilon \otimes \text{id})(R) = 1 = (\text{id} \otimes \epsilon)(R).\]

A.5 **Corollary:**

A direct consequence of proposition [A.4] is that

\[(A.18) \quad (S \otimes S)(R) = (S \otimes \text{id})(\text{id} \otimes S)(R) = (S \otimes \text{id})(R^{-1}) = R.\]

A.6 **Definition:**

Let $H$ a braided Hopf algebra with universal $R$-matrix $R$ and write $R = \sum_i s_i \otimes t_i$, then the element

\[(A.19) \quad u := \sum_i S(t_i) s_i \in H\]

is called the **Drinfeld element**.
The Drinfeld element has the following useful properties.

**A.7 Proposition:**
Let $u$ be the Drinfeld element from definition A.6 and write $R^{-1} = \sum_i s'_i \otimes t'_i$, then $u$ is invertible and

$$u^{-1} = \sum_i S^{-1}(t'_i) s'_i$$

$$u x = S^2(x) u.$$ 

**A.2.4 Representations of Hopf Algebras**
When considering representations of a Hopf algebra $H$, it is regarded as an algebra, taking only $(H, \mu, \eta)$ into account. The additional structure of the Hopf algebra $H$, consisting of the comultiplication $\Delta$, the counit $\epsilon$, and the antipode $S$, all provide opportunity to define additional representations of $H$.

**A.8 Definition:**
For two representations $(V, \pi_V)$ and $(W, \pi_W)$ of $H$, the comultiplication $\Delta$ defines the tensor product of these representations on $V \otimes W$ as $\pi_V \otimes \pi_W = (\pi_V \otimes \pi_W) \Delta$.

With this definition, the coassociativity axiom 3 implies

**A.9 Proposition:**
$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ for all representations $U, V, W$ of $H$.

**A.10 Definition:**
Using the counit $\epsilon$, define the trivial representation

$$V_{tr} = \mathbb{C}, \quad \pi_{V_{tr}}(x) = \epsilon(x).$$

With this definition, the counitality axiom 4 implies

**A.11 Proposition:**
$V_{tr} \otimes W \cong W \cong W \otimes V_{tr}$ for all representations $W$ of $H$.

Finally, the antipode $S$ can be used to define the dual representation as follows:

**A.12 Definition:**
Provided a representation $(V, \pi_V)$, define $\pi_V^*: H \to \text{End}(V^*) : x \mapsto \left(\pi_V(S(x))\right)^*.$

In other words, the dual representation $(V^*, \pi_{V^*})$ is defined as

$$\pi_{V^*}(x)(\phi)(v) = \phi(\pi_V(S(x))(v))$$
for all $x \in H$, $v \in V$, and $\phi \in V^*$, which is also simply written as
\[(A.23) \quad (x \phi)(v) = \phi(S(x) v).\]

If the Hopf algebra $H$ is braided, its universal $R$-matrix can be used to define maps between tensor products of representations.

### A.13 Definition:
Provided two representations $(V, \pi_V)$ and $(W, \pi_W)$ of $H$, define
\[(A.24) \quad R_{VW} = (\pi_V \otimes \pi_W)R : V \otimes W \to V \otimes W\]

The map $R_{VW}$ is of special interest when considering the quantum Yang-Baxter equation.

### A.3 The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_n)$
Recall the description of $\mathfrak{sl}_n$ as given in section [A.1]. The Lie algebra is generated by $e_i, f_i, h_i$ ($1 \leq i \leq n - 1$), and has Cartan subalgebra $\mathfrak{h}$ of diagonal matrices generated by the $h_i$. Its root system $\Phi$ contains the simple roots $\alpha_i \in \Delta$, which form a basis for $\mathfrak{h}^*$. Now let $q \in \mathbb{C}$, assumed not to be a root of unity.

### A.14 Definition:
The quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$ is generated by $q^{\pm h_i} = K_i^\pm$, $E_i$, and $F_i$ ($1 \leq i \leq n - 1$) with defining relations
\[(A.25) \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i \]
\[K_i K_j = K_j K_i \]
\[K_i E_j = q^{\alpha_j(h_i)} E_j K_i \]
\[K_i F_j = q^{-\alpha_j(h_i)} F_j K_i \]
\[[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \]
\[[E_i, E_j] = 0 = [F_i, F_j] \quad \text{if } |i - j| > 1 \]
\[E_i^2 E_{i\pm 1} + E_{i\pm 1} E_i^2 = (q + q^{-1}) E_i E_{i\pm 1} E_i \]
\[F_i^2 F_{i\pm 1} + F_{i\pm 1} F_i^2 = (q + q^{-1}) F_i F_{i\pm 1} F_i \]

Important is that quantum groups carry the Hopf algebra structure as described in section [A.2].
A.15 Theorem:
There exists a unique Hopf algebra structure on $U_q(sl_n)$, given by

\[
\Delta K_i^\pm = K_i^\pm \otimes K_i^\pm, 
E_i \otimes K_i + 1 \otimes E_i, 
F_i \otimes 1 + K_i^{-1} \otimes F_i
\]

The square of the antipode, $S^2$, is of particular interest. On the generators of $U_q(sl_n)$ it acts as

\[
S^2(K_i^\pm) = S(K_i^\mp) = K_i^\pm \\
S^2(E_i) = S(-E_i K_i^{-1}) = -S(K_i^{-1})S(E_i) = K_i E_i K_i^{-1} \\
S^2(F_i) = S(-K_i F_i) = -S(F_i)S(K_i) = K_i F_i K_i^{-1}.
\]

A.16 Proposition:
In general,

\[
S^2(x) = q^{2\rho} x q^{-2\rho}, \quad x \in U_q(sl_n).
\]

Alternatively, the square of the antipode may be expressed in terms of the $K_i$.

Writing $\rho$ as in equation (A.5) and separating the exponent of $q$ yields

\[
q^{2\rho} = \prod_{1 \leq j \leq k \leq n-1} \prod_{i=j}^k q^{h_i} = \prod_{i=1}^{n-1} q^{(n-i)h_i}
\]

so that

\[
S^2 = \hat{K} \times \hat{K}^{-1}, \quad x \in U_q(sl_n).
\]

where $\hat{K} : = \prod_{i=1}^{n-1} K_i^{i(n-i)}$.

A.17 Proposition:

\[
u^{-1} S(\nu) = q^{4\rho}
\]

Proposition A.17 follows from that fact that $U_q(sl_n)$ is a ribbon algebra with ribbon element $\theta : = u q^{-2\rho}$ for which $S(\theta) = \theta$, see [Kassel, 1995].

A.3.1 Representations of $U_q(sl_n)$
Let $V$ a representation of $U_q(sl_n)$.

A.18 Definition:
For $\lambda \in \mathfrak{h}^*$, define

\[
V[\lambda] : = \{ v \in V \mid K_i \nu = q^{\lambda(h_i)} \nu, \quad \forall 1 \leq i \leq n-1 \}.
\]
If $V[\lambda] \neq \{0\}$, then $\lambda$ is called a **weight** and $V[\lambda]$ is called a **weight subspace** of the representation $V$.

Weight subspaces provide a way of defining the character of a representation.

**A.19 Definition:**

The **character** $\chi_V$ of a representation $(V, \pi_V)$ of $U_q(\mathfrak{sl}_n)$ is defined as

$$\chi_V(q^x) := \text{tr}(\pi_V(q^x)) = \sum_{\lambda \in \mathfrak{h}^*} \dim V[\lambda] q^{\lambda(x)}, \quad x \in \mathfrak{h}.$$  

**A.20 Definition:**

If there exists a vector $v_\lambda \in V$ such that for all $1 \leq i \leq n - 1$,

$$K_i v_\lambda = q^{\lambda(h_i)} v_\lambda$$

$$E_i v_\lambda = 0,$$

then $v_\lambda$ is called a **highest weight vector** of $V$, and $\lambda$ its **highest weight**.

The following theorem states an important fact regarding highest weight representations that is integral to the representation theory of $U_q(\mathfrak{sl}_n)$.

**A.21 Theorem:**

For each weight $\lambda \in \mathfrak{h}^*$ there exists a unique, up to isomorphism, irreducible representation of highest weight $\lambda$, denoted as $V_\lambda$. Moreover, the highest weight vector $v_\lambda$ of weight $\lambda$ is unique, up to scalar multiples, in $V_\lambda$.

The unique irreducible representations described in theorem A.21 may be described in terms of a single family of representations, called Verma modules. To define these, a quantum analog of the Borel subalgebras is required.

**A.22 Definition:**

The positive and negative quantized Borel subalgebras are defined as

$$U_q(b^+) = U_q(n^+) U_q(\mathfrak{h}) \quad \text{and} \quad U_q(b^-) = U_q(n^-) U_q(\mathfrak{h}).$$

Note that $U_q(b^\pm)$ are Hopf subalgebras of $U_q(\mathfrak{sl}_n)$ while $U_q(n^\pm)$ are not.

**A.23 Proposition:**

Any $\lambda \in \mathfrak{h}^*$ can be regarded as a one-dimensional representation of $U_q(b^+)$, which is defined by

$$\lambda(K_i) = q^{\lambda(h_i)}$$

$$\lambda(E_i) = 0.$$
A.3 The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_n)$

The following definition states how representations on Hopf subalgebras of $\mathcal{U}_q(\mathfrak{sl}_n)$ induce representations of $\mathcal{U}_q(\mathfrak{sl}_n)$ itself.

**A.24 Definition:**

Let $a \subset \mathcal{U}_q(\mathfrak{sl}_n)$ a Hopf subalgebra and $W$ a representation of $a$, then

(A.36) \[ \mathcal{U}_q(\mathfrak{sl}_n) \otimes_a W \]

is the **induced representation** of $\mathcal{U}_q(\mathfrak{sl}_n)$.

**A.25 Proposition:**

Let $V$ a representation of $\mathcal{U}_q(\mathfrak{sl}_n)$ and $W$ a representation of a Hopf subalgebra $a \subset \mathcal{U}_q(\mathfrak{sl}_n)$, then the obvious map

(A.37) \[ \text{Hom}_{\mathcal{U}_q(\mathfrak{sl}_n)}(\mathcal{U}_q(\mathfrak{sl}_n) \otimes_a W, V) \to \text{Hom}_a(W, V) \]

is an isomorphism.

The one-dimensional representation $\lambda$ of $\mathcal{U}_q(\mathfrak{b}_+)$ described in proposition A.23 induces a representation on $\mathcal{U}_q(\mathfrak{sl}_n)$, called a Verma module.

**A.26 Definition:**

The representation

(A.38) \[ M_\lambda := \mathcal{U}_q(\mathfrak{sl}_n) \otimes_{\mathcal{U}_q(\mathfrak{b}_+)} \lambda = \mathcal{U}_q(\mathfrak{n}_-) \nu_\lambda \]

with $\nu_\lambda$ a highest weight vector, is called the quantized **Verma module** of highest weight $\lambda$, with highest weight vector $\nu_\lambda$.

The definition of the Verma module $M_\lambda$ means that it may be thought of as the highest weight vector $\nu_\lambda$ along with all its ‘reductions’ by the $F_\alpha$, $\alpha \in \Phi^+$.

Since theorem A.21 states that there is a unique irreducible representation for each weight $\lambda \in \mathfrak{h}^*$, an important question to ask is if the Verma module of a certain weight is irreducible.

**A.27 Definition:**

A weight $\lambda$ is said to be **generic** if

(A.39) \[ \alpha(\lambda + \rho) \neq \frac{n}{2} \alpha(\rho) \quad \forall \, \alpha \in \Phi^+ \quad \forall \, n \in \mathbb{Z}^+, \]

where $\rho$ is the element from definition A.1.

**A.28 Proposition:**

The Verma module $M_\lambda$ is irreducible for generic $\lambda$. 
When the Verma module $M_\lambda$ is reducible, it contains a maximal proper sub-module $N_\lambda := \sum_{N \subseteq M_\lambda} N$. This leads to an expression of irreducible highest weight representations in terms of Verma modules, as shown by the following proposition.

A.29 Proposition:
For each $\lambda \in \mathfrak{h}^*$, the associated unique irreducible highest weight representation $V_\lambda$ from theorem A.21 is given by

\[
\begin{align*}
V_\lambda &= M_\lambda & \text{for generic } \lambda \\
V_\lambda &= M_\lambda / N_\lambda & \text{otherwise}.
\end{align*}
\]

If the highest weight $\lambda$ is not generic, the irreducible representation $V_\lambda$ may be finite-dimensional. The next proposition specifies when this is the case.

A.30 Definition:
A weight $\lambda \in \mathfrak{h}^*$ is called dominant integral if

\[
\lambda(h_i) \in \mathbb{Z}^+ \quad \forall \ 1 \leq i \leq n - 1.
\]

A.31 Proposition:
$V_\lambda$ is finite-dimensional if and only if $\lambda$ is dominant integral.

A.32 Definition:
The irreducible finite-dimensional representations $\Lambda_i := V_{\omega_i}$, with $\omega_i$ fundamental weights, are called the fundamental representations of $\mathfrak{U}_q(\mathfrak{sl}_n)$.

A.33 Theorem:
If $V$ is a finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$ with $V = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$, then $V$ is a direct sum of the fundamental representations $\Lambda_i$.

A.34 Definition:
The restricted dual of a highest weight representation $V$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$ is defined as

\[
V^* := \bigoplus_{\nu} V[\mu]^*
\]

and is a representation of $\mathfrak{U}_q(\mathfrak{sl}_n)$.

A.3.2 Braided Structure

A.35 Remark:
Even though $\mathfrak{U}_q(\mathfrak{sl}_n)$ is equipped with a Hopf algebra structure, as detailed in theorem A.15, it does not admit a braided structure as described in section A.2.3.
specific completion \( \mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathcal{U}_q(\mathfrak{sl}_n) \) does, however, admit a braided structure. Details of this completion can be found in section 4.11 of [Etingof and Latour, 2005].

The completion contains infinite sums, which could potentially be an issue since these are not in \( \mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathcal{U}_q(\mathfrak{sl}_n) \). For example, the universal \( R \)-matrix contains an infinite sum over terms containing tensor products \( E^i_n \otimes F^i_n \). However, when working on a Verma module, \( E^i_n \) acts as zero on the module for large enough \( n \). Thus, when working on Verma modules the infinite sum becomes a finite sum. Therefore, the completion may be assumed without any consequences to the calculations and results in this thesis.

Quantum groups are braided Hopf algebras, which means they contain an element \( R \), called a universal \( R \)-matrix, that satisfies the conditions of definition A.3. Universal \( R \)-matrices play an important role in the theory of quantum groups, as they have many useful properties. Most notably, they provide solutions to both the quantum Yang-Baxter and quantum dynamical Yang-Baxter equations (see chapters 2 and 3), as do they constitute central elements of \( \mathcal{U}_q(\mathfrak{sl}_n) \) that play an essential role in quantum integrable systems (see chapter 4).

An explicit expression for the universal \( R \)-matrix of a quantum group can be found in section 8.3 of [Klimyk and Schmüdgen, 1997]. However, as this explicit expression for the universal \( R \)-matrix is unwieldy, it rarely makes an appearance in calculations.

### A.3.3 Example: \( \mathcal{U}_q(\mathfrak{sl}_2) \)

The quantum group \( \mathcal{U}_q(\mathfrak{sl}_2) \) is generated by \( K^\pm, E, \) and \( F \) with defining relations

\[
\begin{align*}
KE &= q^2EK, & KF &= q^{-2}FK, \\
KK^{-1} &= 1 = K^{-1}K, & [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}.
\end{align*}
\]

The unique Hopf algebra structure on \( \mathcal{U}_q(\mathfrak{sl}_2) \) is given by

<table>
<thead>
<tr>
<th></th>
<th>( K^\pm )</th>
<th>( E )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>( K^\pm \otimes K^\pm )</td>
<td>( E \otimes K + 1 \otimes E )</td>
<td>( F \otimes 1 + K^{-1} \otimes F )</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( S )</td>
<td>( K^\pm )</td>
<td>( -EK^{-1} )</td>
<td>( -KF )</td>
</tr>
</tbody>
</table>

and \( S^2(x) = K x K^{-1} \) for \( x \in \mathcal{U}_q(\mathfrak{sl}_2) \).

The quantized Verma module is given by \( M_\lambda = \text{span}\{v_\lambda, F v_\lambda, F^2 v_\lambda, \ldots\} \) with \( \text{wt} F^k v_\lambda = \lambda - 2k \), where \( v_\lambda \) is its highest weight vector. The actions of the
generators of $\mathfrak{U}_q(\mathfrak{sl}_2)$ on $M_\lambda$ are
\begin{align}
K^\pm F^k v_\lambda &= q^{\pm (\lambda - 2k)} F^k v_\lambda, \\
FF^k v_\lambda &= F^{k+1} v_\lambda,
\end{align}
(A.44)
with $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$. Note that in particular $E v_\lambda = 0$.

For $\lambda$ dominant integral, which means $\lambda = n \in \mathbb{Z}^+$ for $\mathfrak{sl}_2$, the unique irreducible finite-dimensional representation $V_n$ of dimension $n+1$ with highest weight vector $v_n$ is given by $\text{span}\{v_n, F v_n, \ldots, F^n v_n\}$ with $\text{wt} F^k v_n = n - 2k$. The actions of the generators of $\mathfrak{U}_q(\mathfrak{sl}_2)$ on $V_n$ are the same as those in equation A.44, supplemented by
\begin{equation}
F^{n+1} v_n = 0.
\end{equation}
(A.45)

The universal $R$-matrix $R$ of $\mathfrak{U}_q(\mathfrak{sl}_2)$ takes the explicit form
\begin{equation}
R = q^\frac{1}{2} h \otimes h \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(q - q^{-1})^n}{[n]_q!} (E^n \otimes F^n),
\end{equation}
(A.46)
where $[n]_q! := \prod_{k=1}^n \frac{q^k - q^{-k}}{q - q^{-1}}$.

### A.4 Shifts in Dynamical Parameters
The notation $A(\lambda + h)(\nu)$ is defined as $A(\lambda - \text{wt} \nu)(\nu)$ for $A(\lambda) \in \mathfrak{U}_q(\mathfrak{sl}_n)$ acting on a weight vector $\nu$ of a representation $V$. The term $h$ marks a shift in the dynamical parameter $\lambda$. This shift depends on the element $\nu$ that is acted on. There is, however, also a rigorous definition of $A(\lambda - h)$ independent of its action on specific representations.

The quantum group $\mathfrak{U}_q(\mathfrak{sl}_n)$ is generated by the $K_i, E_i, F_i$ under the defining relations described in A.25. Let $Q = \mathbb{Z} \alpha_i \otimes \cdots \otimes \mathbb{Z} \alpha_{n-1}$ where the $\alpha_i \in \Delta$ are the simple roots. Recall that the $\alpha_i$ form a basis for $\mathfrak{h}^*$.

### A.36 Proposition:
The quantum group $\mathfrak{U}_q(\mathfrak{sl}_n)$ is $Q$-graded, which means that
\begin{equation}
\mathfrak{U}_q(\mathfrak{sl}_n) = \bigoplus_{\alpha \in Q} \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha],
\end{equation}
(A.47)
where
\begin{equation}
\mathfrak{U}_q(\mathfrak{sl}_n)[\alpha] = \{ x \in \mathfrak{U}_q(\mathfrak{sl}_n) \mid K_i x K_i^{-1} = q^{\alpha(h)} x \ \forall i \}.
\end{equation}
(A.48)
with
\[(A.49)\]
\[\mathfrak{U}_q(\mathfrak{sl}_n)[\alpha] \mathfrak{U}_q(\mathfrak{sl}_n)[\beta] \subseteq \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha + \beta]\]
and \(1 \in \mathfrak{U}_q(\mathfrak{sl}_n)[0]\).

The Hopf algebra structure on \(\mathfrak{U}_q(\mathfrak{sl}_n)\) acts on the \(Q\)-grading as
\[(A.50)\]
\[\Delta(\mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]) \subseteq \bigoplus_{\beta} \mathfrak{U}_q(\mathfrak{sl}_n)[\beta] \otimes \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha - \beta]\]
\[
\epsilon|_{\mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]} = 0 \quad \text{unless} \quad \alpha = 0 \\
S(\mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]) = \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha].
\]

Now formally define an algebra
\[(A.51)\]
\[\mathcal{F}(\mathfrak{h}^*) := \{ f : \mathfrak{h}^* \to \mathbb{C} \mid f \text{ has finite support} \},\]
which is a Hopf-like algebra (see remark \[A.35]\).

The Hopf algebra-like structure on \(\mathcal{F}(\mathfrak{h}^*)\) is given by
\[(A.52)\]
\[\Delta : \mathcal{F}(\mathfrak{h}^*) \rightarrow \mathcal{F}(\mathfrak{h}^*) \otimes \mathcal{F}(\mathfrak{h}^*) \quad (\Delta f)(\mu, \nu) := f(\mu + \nu)\]
\[\epsilon : \mathcal{F}(\mathfrak{h}^*) \rightarrow \mathbb{C} \quad (\epsilon f) := f(0)\]
\[S : \mathcal{F}(\mathfrak{h}^*) \rightarrow \mathcal{F}(\mathfrak{h}^*) \quad (S f)(\mu) := f(-\mu),\]
and \(\mathcal{F}(\mathfrak{h}^*)\) can be written as
\[(A.53)\]
\[\mathcal{F}(\mathfrak{h}^*) = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbb{C} P_{\mu},\]
where \(P_{\mu} \in \mathcal{F}(\mathfrak{h}^*)\) is defined as \(P_{\mu}(\nu) := \delta_{\mu\nu}\).

By definition of \(P_{\mu}\),
\[(A.54)\]
\[P_{\mu} P_{\nu} = \delta_{\mu\nu} P_{\mu} \quad \sum_{\mu \in \mathfrak{h}^*} P_{\mu} = 1.\]

Moreover, the Hopf algebra-like structure on \(\mathcal{F}(\mathfrak{h}^*)\) implies
\[(A.55)\]
\[\Delta(P_{\mu}) = \sum_{\nu} P_{\nu} \otimes P_{\mu - \nu} \quad \epsilon(P_{\mu}) = \delta_{\mu,0} \quad S(P_{\mu}) = P_{-\mu} .\]

The Hopf-like algebra \(\mathcal{F}(\mathfrak{h}^*)\) can be appended to the quantum group \(\mathfrak{U}_q(\mathfrak{sl}_n)\) by formally defining
\[(A.56)\]
\[\mathfrak{U}_q^{ext}(\mathfrak{sl}_n) := \mathcal{F}(\mathfrak{h}^*) \otimes \mathfrak{U}_q(\mathfrak{sl}_n)\]
as vector spaces. Then \(\mathfrak{U}_q^{ext}(\mathfrak{sl}_n)\) is also a Hopf-like algebra (see remark \[A.35]\).
defined by the imbeddings
\begin{align}
\mathcal{F}(h^*) & \hookrightarrow \mathfrak{U}^\text{ext}_q(\mathfrak{sl}_n) : f \mapsto f \otimes 1 \\
\mathfrak{U}_q(\mathfrak{sl}_n) & \hookrightarrow \mathfrak{U}^\text{ext}_q(\mathfrak{sl}_n) : x \mapsto 1 \otimes x
\end{align}

and the cross-relation
\begin{align}
P_{\mu+\alpha} \otimes x & := (P_{\mu+\alpha} \otimes 1)(1 \otimes x) = (1 \otimes x)(P_{\mu} \otimes 1)
\end{align}

for $x \in \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]$ and $\mu \in h^*$.

The cross-relations are written
\begin{align}
P_{\mu} x & := (P_{\mu} \otimes 1)(1 \otimes x) \\
x P_{\mu} & := (1 \otimes x)(P_{\mu} \otimes 1)
\end{align}

for short, so that
\begin{align}
P_{\mu+\alpha} x & = x P_{\mu}
\end{align}

for $x \in \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]$.

Representations $V = \bigoplus_{\nu} V[\nu]$ of $\mathfrak{U}_q(\mathfrak{sl}_n)$ define representations of $\mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n)$ with
\begin{align}
P_{\mu}|_{V[\nu]} &= \delta_{\mu\nu} \text{id}_{V[\nu]},
\end{align}

i.e. $P_{\mu}|_{V}$ is projection on $V[\mu]$. The action of $P_{\mu}$ on $\mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n)$ is well-defined since
\begin{align}
P_{\mu+\alpha} (x \nu) & = \delta_{\mu+\alpha,\nu+\alpha} x \nu = \delta_{\mu,\nu} x \nu = x P_{\mu} \nu
\end{align}

for $x \in \mathfrak{U}_q(\mathfrak{sl}_n)[\alpha]$ and $\nu \in V[\nu]$.

Now the shift in dynamical parameter $\lambda$ of an element $A(\lambda) \in \mathfrak{U}_q(\mathfrak{sl}_n)$ can be defined in terms of the Hopf-like algebra $\mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n)$ (again, see remark A.35).

**A.37 Definition:**

Let $A(\lambda) \in \mathfrak{U}_q(\mathfrak{sl}_n)$, then
\begin{align}
A(\lambda + h) & := \sum_{\mu} A(\lambda + \mu) P_{\mu} \in \mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n) \\
A(\lambda + \hat{h}) & := \sum_{\mu} P_{\mu} A(\lambda + \mu) \in \mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n) \\
A(\lambda + h^2) & := \sum_{\mu} A(\lambda + \mu) \otimes P_{\mu} \in \mathfrak{U}_q(\mathfrak{sl}_n) \otimes \mathfrak{U}_q^\text{ext}(\mathfrak{sl}_n)
\end{align}

It follows that
\begin{align}
B A(\lambda + h) &= A(\lambda + \beta + h) B
\end{align}

for $B \in \mathfrak{U}_q(\mathfrak{sl}_n)[\beta]$, and that
\begin{align}
\Delta(A(\lambda + h)) &= \Delta(A)(\lambda + h^1 + h^2) \\
S(A(\lambda + h)) &= S(A)(\lambda - \hat{h}).
\end{align}
B Notation

The following notation is adopted throughout this thesis.

\( \mathfrak{sl}_n \) The complex Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \).

\( \tau \) The swap \( \tau(v \otimes w) = w \otimes v \).

\( \tau_{12} \) The swap \( \tau_{12}(u \otimes v \otimes w) = v \otimes u \otimes w \).

\( R^{21} \) \( \tau(R) \) in \( H \otimes H \) or \( \tau(R) \otimes 1 \) in \( H \otimes H \otimes H \).

\( R^{12} \) \( R \otimes 1 \) in \( H \otimes H \otimes H \).

\( R^{13} \) \( \tau_{23}(R^{12}) = \tau_{12}(R^{23}) \) in \( H \otimes H \otimes H \).

\( R^{12,3} \) \( (\Delta \otimes \text{id})(R) = R^{13}R^{23} \) in \( H \otimes H \otimes H \).

\( J^{21} \) \( \tau(J) \) in \( H \otimes H \) or \( \tau(J) \otimes 1 \) in \( H \otimes H \otimes H \).

\( J^{12} \) \( J \otimes 1 \) in \( H \otimes H \otimes H \).

\( J^{12,3} \) \( (\Delta \otimes \text{id})(J) \) in \( H \otimes H \otimes H \).

\( J_{\mathbb{W}}^{21}(\lambda) \) \( \tau J_{\mathbb{W}}^{21}(\lambda) \tau \) on \( V \otimes \mathbb{W} \).

\( J_{U \otimes V, \mathbb{W}}^{12,3}(\lambda) \) \( J_{U \otimes V, \mathbb{W}}^{12,3}(\lambda) \) on \( (U \otimes V) \otimes \mathbb{W} \).

\( J_{U \otimes W, V}^{13,2}(\lambda) \) \( \tau_{23}J_{U \otimes W, V}^{12,3}(\lambda) \tau_{23} \) on \( U \otimes V \otimes \mathbb{W} \).

\( J_{U \otimes V, \mathbb{W}}^{12}(\lambda - h^3)(u \otimes v \otimes w) \) \( (J_{U \otimes V, \mathbb{W}}^{12}(\lambda - h^3)(u \otimes v)) \otimes w \) on \( U \otimes V \otimes \mathbb{W} \).

\( J_{U \otimes \mathbb{W}}^{13}(\lambda - h^2) \) \( \tau_{23}J_{U \otimes \mathbb{W}}^{12}(\lambda - h^3) \tau_{23} \) on \( U \otimes V \otimes \mathbb{W} \).

\( J_{U \otimes \mathbb{W}}^{23}(\lambda - h^1) \) \( \tau_{12}\tau_{23}J_{U \otimes \mathbb{W}}^{12}(\lambda - h^3) \tau_{23}\tau_{12} \) on \( U \otimes V \otimes \mathbb{W} \).
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