Stochastic interest rate and volatility implications for the exposure of FX options

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Abstract

In the aftermath of the global financial crisis, regulators and financial institutions are stressing the importance of the credit value adjustment (CVA) and general future credit exposure evaluation. This study focuses on the exposure analysis of an American put option and up-and-out call option. Future risk is efficiently evaluated using the FDMC method suggested by Graaf et al (2014). For some limiting cases analytical approximations are discussed. The full pipeline from market data to CVA calculation is documented. A comparison is made between models adjusting for calibrated stochastic interest rates and volatility. Significant skew impact for the shorter maturity options is observed and the importance of interest rate risk for longer maturity options is confirmed. Additionally, stochastic volatility impact on the dynamics of the future exposure is shown.

Keywords: Exposure, FX markets, Heston, Hull-White, FDMC, Stochastic dividends
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Chapter 1

Introduction

The 2008 global crisis was initiated by the burst of the US housing market which led to a total financial meltdown. Business cycle theory and history suggest that growth periods and recessions alternate, hence a professional of the field should be aware of and, therefore, be prepared to deal with both. However, many financial institutions failed (most infamously Lehman Brothers) and many had to be bailed out. Basel Committee on Banking Supervision issued the Third Basel Accord [5] suggesting measures how to improve the banking industry. It expanded the risk coverage of the capital framework promoting various measures to manage risks arising from possible decline in counterparty’s credit quality. Furthermore, the importance of credit value adjustment (CVA) for pricing that risk is stressed. This work focuses on the exposure modelling of foreign exchange (FX) and interest rate derivative portfolios which is instrumental for calculating CVA.

As in the majority of financial studies the starting point is the celebrated Black-Scholes model as given in [3]. However, it is commonly accepted that the model is too simplistic. One of the known issues is the inability to capture the so-called volatility smile which is present in most of the markets. In [15] the Heston model was proposed which is able to deal with this to some extent. An in-depth study on exchange rates in [19] reveals the advantages of including the stochastic volatility term in the Black-Scholes model from both the implied and historical perspective.

The no-arbitrage framework implies that exchange rates satisfy the so-called interest rate parity meaning that movements in exchange rates, on average, are driven by the changes in the interest rates (IR). Hence, IR risk assessment is crucial in computing the total risk in the exchange rates. Luckily, in [21] a decent starting point for modelling bond prices is given. However, it lacks the ability to fit the yield curve, but in [16] an extended version, the so-called Hull-White model is derived. Although other models (see eg. [10]) exist, the model is extensively used in practice and, therefore, is used as a building block in this
analysis.

Measuring exposure consist of two stages - scenario generation and estimation of the future value of the portfolio. In [7] the authors propose several ways to calculate the future exposure. One of those is a combination of Monte-Carlo (MC) and Finite Difference (FD) methods. On one hand MC is superior in scenario generation whereas FD is especially useful in pricing derivatives under different initial conditions. Further increase in MC efficiency is achieved by using the scheme as presented in [2].

FD methods are attractive for their robustness and efficiency. However, they suffer from the curse of dimensionality, so solving higher dimensional PDEs becomes unattractively slow if using simple Euler forward/backward or Crank-Nickolson schemes. Speed can be significantly improved by using splitting schemes as shown in [13] and [12].

In this thesis the FDMC method for a 4-dimensional foreign exchange model is implemented. The FD and MC parts incorporate all the techniques suggested in the mentioned papers and the resulting implementation is available upon request. Exposure profiles for an European call option are computed for an arbitrary parameter set and their sensitivity to the parameters are evaluated. To bring it closer to the market the model is calibrated to market data. Afterwards, the impact of stochastic interest rate and stochastic volatility on the exposure profiles of several derivatives are systematically analyzed.

During this study it is observed that stochastic interest rates have a significant impact on the exposure profiles of the American put options, but a negligible in the case of the up-and-out call option. In the case of the former the impact is most profound in options of long maturities.

On the other hand, the adjustment for stochastic volatility significantly changes the dynamics of both the American put and up-and-out call option. In relation to the put option the volatility skew effect fades with increasing maturity, but for the barrier option the impact is substantial for all maturities, strikes and barriers.

The thesis is organized as follows:

• Chapter 2 gives a brief review of credit risk management and introduces the exposure metrics;

• Chapter 3 defines the models used in this research and reviews their main results;

• Chapter 4 addresses credit value adjustment of European options;

• Chapter 5 gives a detailed explanation of the numeric methods involved in exposure estimation;
• Chapter 6 contains a real life case study where market data is used and the exposure profiles are discussed;

• Chapter 7 overviews the results of this work and summarizes key observations

• Appendices contain all the space consuming tables, figures and derivations which were left out of the main work.
Chapter 2

Credit risk measurement

There were multiple reasons which triggered the 2008 crisis, but the extent to which it did spread was greatly undermined by the misestimation of the risk involved. To prevent a similar situation happening in the future the Basel Committee on Banking Supervision issued The Third Basel Accord (see [5]), which suggested a regulatory framework to help to maintain a healthy banking system. It addressed the issue that losses are incurred whenever the counterparty defaults or when its credit rating is downgraded. To deal with the risk of such losses the counterparty exposure has to be measured and a price tag has to be attached.

2.1 Exposure

Suppose, party A made a trade with some other party B which is obliged to deliver some payoff (which might be negative meaning that party A need to pay) and the exact value of that payoff is dependent on the future market conditions. This contract exposes both parties to market risk i.e. it is not possible to say with certainty what the future market conditions will be. However, financial literature provides several methods to estimate the distribution of the future value of the contract which in turn can be used to evaluate the market risk. Another type of risk that the parties face is that of their counterparty’s default. As noted by [11], the possibility of a default leads to some "asymmetry" in the possible future payoff. Suppose party B defaults, then one of the following happens:

- If the market conditions were such that B had to pay, it will not have the capital to fulfil the obligation completely (otherwise there would be no default). Hence, party A does not receive full payment;

- In case party A had to pay and it did not default, the payment still has to be made as default does not free B’s debtors from paying;
• In case party A also defaults, no cash flows take place.

The risk of having a contract of positive value with a counterparty that cannot deliver on agreed terms in case of default is called the counterparty credit risk. To quantify this risk, \( V(t) \) is defined as the price of the contract of interest at time \( t \) which depends on some asset \( S_t \). Then the counterparty credit risk can be evaluated using the following metrics:

- Expected positive exposure\(^1\) ( \( EPE \) ) at time \( t \) is defined as\(^2\)

\[
EPE(t) = \mathbb{E}[(V(t))^+ | \mathcal{F}_0],
\]

where \( \mathcal{F}_0 \) contains all the observable information at the moment of measurement ( \( t = 0 \) ). This metric measures the expected capital at risk at time \( t \) given that the long party is in-the-money. Discounted expected positive exposure ( \( \tilde{EPE} \) ) gives the estimate in current terms:

\[
\tilde{EPE}(t) = \mathbb{E}[D(t)(V(t))^+ | \mathcal{F}_0],
\]

where \( D(t) \) is the discount factor at time \( t \). This metric plays a major role in the pricing of counterparty credit risk.

- Potential future exposure of \( q\% \) ( \( PFE_q \) ) at time \( t \) is the \( q \)-th quantile of the contract value distribution at time \( t \). This metric assesses capital loss when the counterparty defaults in the worst/best-case market scenario. Therefore, it is used for risk-management, stress-testing and Value-At-Risk estimation. In this research 97.5\% and 2.5\% \( PFEs \) are taken to represent as worst and best case values, respectively. Discounted \( PFEs \) are used in case the capital loss has to be provided in current terms.

- Expected exposure ( \( EE \) ) is the probability weighted average future value of the portfolio at time \( t \):

\[
EE(t) = \mathbb{E}[V(t)|\mathcal{F}_0].
\]

This metric shows the dynamics of the mean contract value. Note that in case the derivative has just a positive payoff ( e.g. call option ) \( EE \) is equal to \( EPE \). \( EE \) is used in future collateral requirement calculations. The metric in current terms is given by:

\[
\tilde{EE}(t) = \mathbb{E}[D(t)V(t)|\mathcal{F}_0].
\]

\(^1\)For short positions in the contract or to estimate the counterparty’s risk to our credit quality one would similarly look at expected negative exposure

\(^2\)(\( x \)^\+) = \( \max\{x, 0\} \)

\(^3\)Unless stated otherwise, the expectation is taken under the risk-neutral measure
As will be shown, for European options the discounted version of the EE can be retrieved directly from the market without a selection of specific SDE model for the underlying asset $S_t$.

These metrics form the core of this work as it focuses on estimating the impact of stochastic interest rates and volatility on these quantities.

### 2.2 Credit value adjustment

Knowledge of your exposure profile can help to improve the diversification of the portfolio. However, as noted in [5] credit risk has to be included in the price, otherwise a decrease in creditworthiness of one company can negatively affect your portfolio or worse start a chain reaction through the whole market as happened during the 2008 crisis. This adjustment for counterparty credit risk is commonly referred to as credit value adjustment (CVA).

In full form, the price adjustment for possible loss of capital in case of counterparty’s default is given as:

$$\text{CVA} = \mathbb{E}(1 - \delta(\tau))D(\tau)(V(\tau))^+1_{\tau < T},$$

where $T$ is the contract maturity, $\tau$ is the default moment, $\delta(\tau)$ is the recovery rate and $(V(\tau))^+$ is the positive part of the value at the default moment. Equation (2.2) can be rewritten in the following form

$$\text{CVA} = \mathbb{E}\left(\int_0^T (1 - \delta(t))D(t)(V(t))^+dPD(t)\right),$$

with

- $\delta(t)$ as the recovery rate given as a percentage of retrieved capital in case of a default at time $t$;
- $V(t)$ as the underlying contract/ portfolio value at time $t$;
- $D(t)$ is the discount factor;
- $PD(t)$ is the survival probability density of the counterparty.

In the definition there are no restrictions on the variable relations. However, analytic evaluation of it so far has only been done for trivial cases. Following [11] a more "practical" CVA formula is

$$\text{CVA} \approx (1 - \delta)\sum_{i=0}^{m} D(T_i)EPE(T_i)(PD(T_{i+1}) - PD(T_i)),$$
where 0 = T_0 < ... < T_m = T are the monitoring time points. This work considers the inclusion of stochastic interest rates and volatility, hence the metric of interest is an interpolation between (2.2) and (2.3):

\[
CV_A \approx (1 - \delta) \sum_{i=0}^{m} (PD(T_{i+1}) - PD(T_i))ED(T_i)(V(T_i))^+
\]

\[
= (1 - \delta) \sum_{i=0}^{m} (PD(T_{i+1}) - PD(T_i))ED(T_i)PE(T_i),
\]

(2.4)

where \(PE(\cdot)\) stands for positive exposure. This variation relaxes the assumption of deterministic interest rates and allows for correlation between domestic IR and the underlying asset. Comparing (2.2) with (2.4) reveals that in the approximation the following assumptions are implicit:

- The recovery rate is constant;
- The discount factor is independent of default probabilities;
- The underlying asset is independent of default probabilities.

In practice the first one is widely accepted and the latter two are considered important and being worked on. For example, imagine a publicly traded company with a large amount of debt. Then a spike in the interest rates might trigger the company’s default. Also, the trade might suffer from the so-called wrong-way (symmetrically, right-way) risk, meaning that the more in-the-money the trade goes the higher the probability that the counterparty would default.

However, these relaxations comes at a cost. In [11] it is noted that the advantage of is its modularity. There is no need for a separate CVA department as everything is already available at a modern financial institution. For example, the management team can provide exposure profiles of the portfolio, credit department could provide the counterparty’s default probabilities and the interest rate desk would be responsible for the discount factors. For academic purposes the focus is on (2.4), but possible future study could explore its benefits over (2.3).
Chapter 3

FX and IR models

The widely celebrated Black-Scholes\(^1\) (BS) theory as given in [3] assumes the following stochastic differential equation (SDE) for the underlying \(S_t\):

\[
\frac{dS_t}{S_t} = (r - d)dt + \sigma dW_t,
\]

where \(r\) is the constant continuously compounded domestic interest rate (IR), \(d\) is the dividend/foreign IR yield, \(\sigma\) is the volatility and \(W_t\) is a standard Wiener process\(^2\). This model forms the core of modern financial theory, but in this chapter defines and presents the main results for several extensions to FX option pricing. Also, the focus of this work being the FD methods, for each model an associated option pricing PDE is described.

3.1 Hull-White

According to interest rate parity together with no-arbitrage arguments, interest rates are suspected to be the core drivers in the foreign exchange markets. Therefore, before moving on to the full FX model, we start by having a more detailed look into the Hull-White model which is assumed throughout this work to be the model for the interest rates. The following parametrisation of the Hull-White SDE is used:

\[
dr_t = \lambda(\theta(t) - r_t)dt + \eta dW_t,
\]

where

- \(\theta(t)\) is the deterministic function setting the drift of the short-rate process. In case

\(^1\)The application of Black-Scholes model with dividends to price FX derivatives is also know as Garman–Kohlhagen model.

\(^2\)Unless stated otherwise, the terms are given under risk neutral measure
\( \theta_t \equiv \theta \) the model is known as the Vasicek model. This time dependent term allows to fit the implied zero coupon bond prices to the observed yield curves in the market;

- \( \lambda > 0 \) is the reversion to \( \theta_t \) speed. Higher values imply that the process will retract to \( \theta_t \) faster;
- \( \eta \) governs the volatility level of the short rate.

Suppose one is interested in pricing some contingent claim \( g(r_T) \) for some maturity \( T \). \( V(\cdot) \) such that

\[
V(r, t) = \mathbb{E} \left[ e^{-\int_t^T r_t dt} g(r_T) \big| r_t = r \right].
\]

Then the usual financial argumentation implies that the maturity-reversed version \( u(r, \tau) = V(r, T - \tau) \) has to satisfy the following PDE

\[
\frac{\partial u}{\partial \tau} = \frac{1}{2} \eta^2 \frac{\partial^2 u}{\partial r^2} + \lambda (\theta(T - \tau) - r_\tau) \frac{\partial u}{\partial r} - r_\tau u
\]

usually in fixed income markets, interest rate derivatives are defined in terms of bond prices. However, the Hull-White model is defined in terms of the short rate \( r_t \). In the next section formulas to describe zero coupon bonds are given.

**Zero coupon bond**

For the Hull-White short-rate model it is possible to obtain zero coupon bond prices (ZCB) analytically. In [10] it is shown that the price \( P(t, T) \) of a \( T \)-maturity ZCB at time \( t \) is equal to

\[
P(t, T) = \frac{P(0, T)}{P(0, t)} \exp[-A(t, T) - B(t, T)r_t],
\]

with

\[
B(t, T) = -\frac{1}{\lambda} \left( e^{-\lambda(T-t)} - 1 \right)
\]

\[
A(t, T) = -B(t, T) f(0, t) + \frac{\eta^2}{4\lambda} B^2(t, T)(1 - e^{-2\lambda t}).
\]

The availability of an analytical expression of ZCB price allows to efficiently price fixed and floating coupon bonds, swaps, etc.
3.2 Black-Scholes-Hull-White-Hull-White

Incorporating stochastic interest rates into the Black-Scholes world yields the Black-Scholes-Hull-White-Hull-White (BS2HW) model:

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r^d - r^f)dt + \sigma dW^S_t \\
\frac{dr^d_t}{r^d_t} &= \lambda_d(\theta_d(t) - r^d_t)dt + \eta_d dW^d_t \\
\frac{dr^f_t}{r^f_t} &= \left[\lambda_f(\theta_f(t) - r^f_t) - \eta_f \rho_{S,r}^d \sigma\right] dt + \eta_f dW^f_t \\
d[W^S_t, W^d_t] &= \rho_{S,r}^d dt, d[W^S_t, W^f_t] = \rho_{S,r}^f dt, d[W^f_t, W^d_t] = \rho_{r^d,r^f} dt.
\end{align*}
\]

where the sub/superscripts \( f \) and \( d \) are shorthand for foreign and domestic respectively. Note that foreign short rate is defined under foreign risk neutral, hence the additional term \( \eta_f \rho_{S,r}^d \sigma \) correcting for the measure change from foreign into domestic risk neutral as given in [22]. This model allows to price hybrid products whose payoff \( g(S_T, r^d_T, r^f_T) \) can depend on all three stochastic components of the model. The pricing of such a contingent claim means finding \( V(\cdot) \) such that

\[
V(s, r^d, r^f, t) = \mathbb{E} \left[ e^{-\int_t^T r^f_u \, du} g(r_T) | S_t = s, r^d_t = r^d, r^f_t = r^f \right].
\]

The time-reversed function \( u(s, r^d, r^f, \tau) = V(s, r^d, r^f, T - \tau) \) satisfies the following PDE

\[
\begin{align*}
\frac{\partial u}{\partial \tau} &= \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \eta^d_d \frac{\partial^2 u}{\partial (r^d)^2} + \frac{1}{2} \eta^f_f \frac{\partial^2 u}{\partial (r^f)^2} \\
&\quad + (r^d - r^f) s \frac{\partial u}{\partial s} + \lambda_d(\theta^d(T - \tau) - r^d) \frac{\partial u}{\partial r^d} + \lambda_f(\theta^f(T - \tau) - r^f - \rho_{S,r}^d \eta_f \sigma) \frac{\partial u}{\partial r^f} \\
&\quad + \rho_{S,r}^d \eta_d s \sigma \frac{\partial^2 u}{\partial s \partial r^d} + \rho_{S,r}^f \eta_f s \sigma \frac{\partial^2 u}{\partial s \partial r^f} + \rho_{r^d,r^f} \eta_f \eta_d \frac{\partial u}{\partial r^d \partial r^f} \\
&\quad - r^d u
\end{align*}
\]

\[
u(s, r^f, r^d, 0) = g(s, r^f, r^d).
\]

In the original Black-Scholes world there is only one domestic risk-free interest rate, but in the markets there are multiple candidates (e.g. government bond or overnight interbank rates) and most of them are maturity dependent. BS2HW partially reduces the problem by incorporating the yield curves within its interest rate parts. Also, the model becomes more risk-aware, since contrary to the regular BS it contains domestic and foreign interest rate risks. The incorporation of additional variables leads to the PDE being 3 dimensional in space, implying heavier computational load. Luckily, for European options the model
can be reduced to a 1 dimensional one using measure change theory and, most importantly, analytical prices from BS can be reused to price European options under BS2HW model analytically.

**European options**

An attractive advantage of the BS2HW model is that with appropriate measure changes one can use the formulas derived under BS to price options under BS2HW. Suppose one is interesting in pricing a claim $g(S_T)$ where $S_T$ is the exchange rate at maturity $T$. First of all define the forward exchange rate by

$$F_t = S_t \frac{P^f(t, T)}{P^d(t, T)}$$

, where $P^f(\cdot, \cdot)$ and $P^d(\cdot, \cdot)$ are respectively the foreign and domestic ZCB processes. Note that this definition yields that $S_T = F_T$. Moreover, one can quickly check that the bonds in the Hull-White based world satisfy

$$\frac{dP^d(t, T)}{P^d(t, T)} = r_d^d dt - \eta_d b_d(t, T) dW^d_t$$

$$\frac{dP^f(t, T)}{P^f(t, T)} = (r_f^f + \eta_f \rho_{S,r} \sigma b_f(t, T)) dt - \eta_d b_f(t, T) dW^f_t,$$

where

$$b_\ast(t, T) = \frac{1 - e^{-\lambda_\ast(T-t)}}{\lambda_\ast}.$$
Hence the forward exchange rate satisfies the following SDE

\[ \frac{dF_t}{F_t} = \left( \sigma \eta_d b_d(t, T) \rho_{S,r} - \eta_d \eta_f b_d(t, T) b_f(t, T) \rho_{r^d,r^f} + \eta_d^2 (b_d(t, T))^2 \right) dt \\
+ \sigma dW^S_t + \eta_d b_d(t, T) dW^d_t - \eta_f b_f(t, T) dW^f_t \\
= \sigma dW^{S,\text{forward}}_t + \eta_d b_d(t, T) dW^{d,\text{forward}}_t - \eta_f b_f(t, T) dW^{f,\text{forward}}_t \\
F_0 = S_0 \frac{P^f(0, T)}{P^d(0, T)} . \]

where the superscript forward means that the process is a standard Wiener process under the forward measure (i.e. \( \frac{P^d(t, T)}{P^f(t, T)} \) is the numeraire). Rewriting the latter SDE in integral form one can deduce that\(^3\) \( F_T = \int F_t^* \) with \( F_t^* \) defined as

\[ \frac{dF_t^*}{F_t^*} = \sigma^*_t dW^{F,\text{forward}}_t \\
F_t^* = F_t = S_t \frac{P^f(t, T)}{P^d(t, T)} , \]

where \( W^{F,\text{forward}} \) is another independent Brownian motion under the forward measure and

\[ (\sigma^*_t)^2 = \frac{1}{t} \int_0^t \left[ \sigma^2 + \eta_d^2 (b_d(t, T))^2 + \eta_f^2 (b_f(t, T))^2 + 2\sigma \eta_d b_d(t, T) \right. \\
- 2\sigma \eta_f b_f(t, T) - 2\eta_d \eta_f b_d(t, T) b_d(t, T) \left. \right] dt \\
= \frac{1}{t} \int_0^t \left[ (\sigma^2 t + \eta_d^2 B_{dd}(t, T) + \eta_f^2 B_{ff}(t, T) + 2\sigma \eta_d \rho_{S,r} B_d(t, T) \\
- 2\sigma \eta_f \rho_{r^d,r^f} B_f(t, T) - 2\eta_d \eta_f B_{df} \rho_{r^d,r^f}(t, T) \right] dt , \]

\(^3\)\(^d\) in this context means equality in distributions.
with

\[
B_d(t, T) = \frac{1}{\lambda_d} \left( e^{-\lambda_d t} - 1 \right) + t
\]

\[
B_f(t, T) = \frac{1}{\lambda_f} \left( 1 - e^{-\lambda_f t} \right) + t
\]

\[
B_{dd}(t, T) = \frac{1}{\lambda_d^2} \left( 2\lambda_d - e^{-2\lambda_d t} + 4e^{-\lambda_d t} - 3 \right)
\]

\[
B_{ff}(t, T) = \frac{1}{\lambda_f^2} \left( 2\lambda_f t - e^{-2\lambda_f t} + 4e^{-\lambda_f t} - 3 \right)
\]

\[
B_{df}(t, T) = \frac{1}{\lambda_d \lambda_f} \left( 1 - e^{-(\lambda_d+\lambda_f)t} \right) + \frac{e^{-\lambda_d t} - 1}{\lambda_d} + \frac{e^{-\lambda_f t} - 1}{\lambda_f} + t.
\]

Observe that although \( S_T \) is only equal to \( F^*_T \) in distribution, the prices of \( g(S_T) \) and \( g(F^*_T) \) at the initial time are equal since the payoffs are equal in distribution. Combining all the former steps yields that

\[
V_0 = \mathbb{E} e^{-\int_0^T r_s \, ds} g(S_T)
\]

\[
= \frac{P^d(0, T)}{P^f(0, T)} \mathbb{E}_{\text{forward}} g(S_T)
\]

\[
= \frac{P^d(0, T)}{P^f(0, T)} \mathbb{E}_{\text{forward}} g(F_T)
\]

\[
= \frac{P^d(0, T)}{P^f(0, T)} \mathbb{E}_{\text{forward}} g(F^*_T).
\]

This result implies that the pricing formulas and practices for European derivatives of BS world can be reused to price those options in BS2HW world. The only difference with BS being that the price depends on three state variables - FX rate, domestic and foreign short rates.

### 3.3 Heston

Studies like [19] provide evidence of strong volatility skew presence in the FX markets. Several models have been proposed to take it into account. Most popular is SABR introduced in [14], the local volatility model defined in [4] and the Heston model suggested in [15]. We chose Heston because of its intuitive definition. The Heston model extends Black-Scholes by
replacing constant volatility with a CIR process:

\[ dS_t = (r - d)dt + \sqrt{v_t}dW_t^S \]
\[ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \]

where

- \( W_t^S, W_t^v \) are standard Wiener processes with correlation \( \rho \). This correlation controls the skew of the smile;
- \( \kappa \) is the mean reversion speed and \( \gamma \) is the volatility of volatility. These parameters have an impact on the curvature of the smile;
- \( \bar{v} \) is the long term mean and together with \( v_0 \) shifts the smile down or up.

Suppose one is interested in pricing a European option with maturity \( T > 0 \) and payoff \( g(S_T, v_T) \) at time \( t \in [0, T) \) i.e. finding \( V(\cdot) \) such that

\[ V(s, v, t) = e^{-r(T-t)}\mathbb{E}[g(S_T, v_T)|S_t = s, v_t = v]. \] (3.2)

As shown in [20] that \( u(s, v, t) = V(s, v, T - t) \) has to satisfy the following PDE

\[ \frac{\partial u}{\partial t} = \frac{1}{2}s^2v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\gamma^2 v \frac{\partial^2 u}{\partial v^2} + \rho \gamma sv \frac{\partial^2 u}{\partial s \partial v} + rs \frac{\partial u}{\partial s} + \kappa(\bar{v} - v) \frac{\partial u}{\partial v} - ru \]

\[ u(s, v, 0) = g(s, v). \]

Inclusion of stochastic volatility relieves the normality assumption by allowing for heavier tails and allows for clusters of high or low volatility to occur. The next section deals with the inclusion of stochastic volatility in the BS2HW model defined previously.
3.4 Heston-Hull-White-Hull-White

The most complete model in this work combines the BS2HW with the Heston yielding the Heston-Hull-White-Hull-White (H2HW):

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r^d - r^f)dt + \sqrt{v_t}dW^S_t \\
\frac{dv_t}{v_t} &= \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW^v_t \\
\frac{dr^d_t}{r^d_t} &= \lambda_d(\theta_d(t) - r^d_t)dt + \eta_dW^d_t \\
\frac{dr^f_t}{r^f_t} &= \left[\lambda_f(\theta_f(t) - r^f_t) - \eta_f\rho_{S,r}\sqrt{v_t}\right]dt + \eta_fW^f_t \\
d[W^S_t, W^d_t] &= \rho_{S,S}dt, d[W^S_t, W^v_t] = \rho_{S,v}dt, d[W^f_t, W^d_t] = \rho_{r,r}dt, \\
d[W^v_t, W^d_t] &= \rho_{v,v}dt, d[W^S_t, W^v_t] = \rho_{S,v}dt.
\end{align*}
\]

The time-reversed pricing mechanism \(u\) of a contingent claim \(g(S_T, v_T, r^d_T, r^f_T)\) has to satisfy the following PDE

\[
\frac{\partial u}{\partial \tau} = \frac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \gamma^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2} \eta^2 d \frac{\partial^2 u}{\partial (r^d)^2} + \frac{1}{2} \eta^2 f \frac{\partial^2 u}{\partial (r^f)^2} \\
+ (r^d_T - r^d_t)s \frac{\partial u}{\partial s} + \kappa(\bar{v} - v) \frac{\partial u}{\partial v} + \lambda_d(\theta^d(T - \tau) - r^d_t) \frac{\partial u}{\partial r^d_t} \\
+ \lambda_f(\theta^f(T - \tau) - r^f_t - \rho_{S,r} \eta_f \sqrt{v}) \frac{\partial u}{\partial r^f_t} \\
+ \rho_{S,v} \gamma s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{S,v,a} \eta_d s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r^d} + \rho_{S,v} \eta_f s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r^f} \\
+ \rho_{v,v,a} \eta_d v \frac{\partial^2 u}{\partial v \partial r^d} + \rho_{v,v} \gamma \gamma \eta_d \sqrt{v} \frac{\partial^2 u}{\partial v \partial r^f} + \rho_{r,v} \eta_f \eta_d \frac{\partial u}{\partial r^d \partial r^f} \\
- r^d_T u(s, v, r^d, r^f, 0) = g(s, v, r^d, r^f).
\]

The implementation used in this research is based on the work provided in [12]. Note that pricing with this model requires solving a 4 dimensional PDE which might result in large computational requirements. However, in case one is dealing with European options the PDE can be reduced to a 2 dimensional one which is shown in the next section.

**European options**

In the case of BS2HW, the switch from spot to forward exchange rate allowed to use analytic formulas derived for BS to price options under BS2HW. However, analytic formulas are not available for the Heston model yet and, therefore, have to be approximated. By
nature Heston is a 2 dimensional model of the asset where one dimension is the stock and the other is its volatility. When stochastic interest rates are added, the problem becomes 4 dimensional which in case of solving PDEs significantly reduces the speed and increases the computational load. In [18] the author has derived a faster method for solving Heston-Hull-White based on the tree method. However, at the time of writing the extensions to H2HW were not yet documented. In this section it is shown how (using change of measure arguments) we can reduce the pricing of European options to solving a 2-dimensional PDE. This provides a significant speed up compared to tackling it via 4 dimensional system.

The derivation starts by repeating the steps as in the BS2HW case, such that one finds that the forward exchange rate satisfies the following PDE under the forward measure.

\[
\frac{dF_t}{F_t} = \sqrt{v_t}dW_{t}^{\text{forward},S} + \eta_d b_d(t, T)dW_{t}^{\text{forward},d} - \eta_f b_f(t, T)dW_{t}^{\text{forward},f}
\]

\[
F_0 = S_0 \frac{P^f(0, T)}{P^d(0, T)}.
\]

with the variance process satisfying:

\[
dv_t = \kappa(\bar{v} - v_t)dt + \gamma \sqrt{v_t}dW_{t}^{\text{forward},v}.
\]

In order to derive the PDE, the steps from [20] are repeated in our context. To start, construct a portfolio \( \Pi \) consisting of 1 unit of the option with value\(^4\) \( V_t \), \( \Delta_t \) units of forward rate \( F_t \) and \( \phi_t \) units of another option \( U_t \) used in hedging the volatility. Then the portfolio value dynamics are given by

\[
\Pi_t = V_t + \Delta_t F_t + \phi_t U_t
\]

Assuming self-financing portfolio the following holds:

\[
d\Pi_t = dV_t + \Delta_t dF_t + \phi_t dU_t
\]

\(^4\)Note that \( V_t = V(t, F_t, v_t) \) and \( U_t = U(t, F_t, v_t) \)

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Then by Itô for $V$ we arrive at

$$dV_t = \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial F_t} dF_t + \frac{\partial V_t}{\partial v_t} dv_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial F_t^2} d[F_t, F_t]_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial v_t^2} d[v_t, v_t]_t + \frac{\partial^2 V_t}{\partial F_t \partial v_t} d[F_t, v_t]_t$$

$$= \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial F_t} dF_t + \frac{\partial V_t}{\partial v_t} dv_t$$

$$+ \frac{1}{2} F_t^2 (v_t + \eta_d^2 (b_d(t, T))^2 + \eta_f^2 (b_f(t, T))^2 + 2 \sqrt{v_t} \eta_d b_d(t, T) \rho_{S, r, d})$$

$$- 2 \sqrt{v_t} \eta_f b_f(t, T) \rho_{S, r, f} - 2 \eta_d b_d(t, T) \eta_f b_f(t, T) \rho_{r, d, r, f} \frac{\partial^2 V_t}{\partial F_t^2} dt$$

$$+ \frac{1}{2} \gamma v_t \frac{\partial^2 V_t}{\partial v_t^2} dt + \gamma \sqrt{v_t} \phi_t (\sqrt{v_t} \rho_{S, v} + \eta_d b_d(t, T) \rho_{r, d, v} - \eta_f b_f(t, T) \rho_{r, f, v}) \frac{\partial V_t}{\partial v_t} dt$$

$$= \left[ \frac{\partial V_t}{\partial t} + \frac{1}{2} v_t F_t^2 \frac{\partial^2 V_t}{\partial F_t^2} + \frac{1}{2} \eta_d^2 (b_d(t, T))^2 F_t^2 \frac{\partial^2 V_t}{\partial F_t^2} + \sqrt{v_t} \eta_d b_d(t, T) \rho_{S, r, d} F_t \frac{\partial^2 V_t}{\partial F_t^2} - \sqrt{v_t} \eta_f b_f(t, T) \rho_{S, r, f} F_t \frac{\partial^2 V_t}{\partial F_t^2}$$

$$- \eta_d b_d(t, T) \eta_f b_f(t, T) \rho_{r, d, r, f} F_t \frac{\partial^2 V_t}{\partial F_t^2} + \frac{1}{2 \gamma v_t} \frac{\partial^2 V_t}{\partial v_t^2} + \gamma v_t \phi_t \rho_{S, v} \frac{\partial^2 V_t}{\partial F_t \partial v_t}$$

$$+ \gamma \sqrt{v_t} \phi_t (\sqrt{v_t} \rho_{S, v} + \eta_d b_d(t, T) \rho_{r, d, v} - \eta_f b_f(t, T) \rho_{r, f, v}) \frac{\partial V_t}{\partial v_t} \right] dt$$

$$+ \frac{\partial V_t}{\partial F_t} dF_t + \frac{\partial V_t}{\partial v_t} dv_t$$

$$= A_V dt + \frac{\partial V_t}{\partial F_t} dF_t + \frac{\partial V_t}{\partial v_t} dv_t,$$

where $A_V$ contains all drift terms.

Note that the result is symmetric for $U$. Then $d\Pi_t$ is equal to

$$d\Pi_t = dV_t + \Delta_t dF_t + \phi_t dU_t$$

$$= (A_V + \phi_t A_U) dt + \left( \frac{\partial V_t}{\partial F_t} + \Delta_t + \phi_t \frac{\partial U_t}{\partial F_t} \right) dF_t + \left( \frac{\partial V_t}{\partial v_t} + \phi_t \frac{\partial U_t}{\partial v_t} \right) dv_t,$$

such that choosing

$$\phi_t = -\frac{\frac{\partial V_t}{\partial v_t}}{\frac{\partial U_t}{\partial v_t}}$$

$$\Delta_t = -\phi_t \frac{\partial U_t}{\partial F_t} - \frac{\partial V_t}{\partial F_t},$$

guarantees a payoff independent of stock/ volatility. Now note that $\Pi$ is defined under the forward measure meaning that it has to be a martingale under this measure. Therefore,
the drift term has to be zero:

\[ A_V + \phi_t A_U = 0 \]
\[ \Leftrightarrow \frac{\partial A_V}{\partial v_t} = \frac{\partial A_U}{\partial v_t}. \]

This implies that both sides can be written as functions of \( F_t, v_t, t \) ie \( f(F_t, v_t, t) \). As in [20] suppose \( f(F_t, v_t, t) = -\kappa(\bar{v} - v_t) \) then

\[ A_V = \kappa(\bar{v} - v_t) \frac{\partial V_t}{\partial v_t} \]
\[ \Leftrightarrow \]
\[ \frac{\partial V_t}{\partial t} + F_t^2 \left[ \frac{1}{2} \rho_s v + \frac{1}{2} \eta_d^2 (b_d(t, T))^2 + \frac{1}{2} \eta_f^2 (b_f(t, T))^2 \right] \frac{\partial^2 V_t}{\partial F_t^2} \\
+ \sqrt{\bar{v} \eta_d b_d(t, T) \rho_s r} - \sqrt{\bar{v} \eta_f b_f(t, T) \rho_s r} = 0 \]

\[ V_T(S, v, r^d, r^f, T) = g(S, v, r^d, r^f). \]

This derived PDE can be then solved using PDE methods similar to Heston’s, the only difference being that the coefficients are time-dependent. Although this means solving a different system of linear equations every iteration, the computational gain from reducing the dimensions outweighs it.
Chapter 4

Exposure of European options

This chapter focuses on European options which have one payoff at a single future date. Analytic results for a variety of European options (e.g. Binary, Vanilla, etc.) under the Black-Scholes model allow for analytic exposure evaluation as demonstrated further. Historically the majority of options and other derivatives were introduced as insurance products having positive payoffs. Hence for notation purposes the further results assume positive payoffs. In case the option of interest can result in negative payoff, most of the results can be extended with little additional effort. Exposure-wise this assumption also implies that $EPE$ is equal to $EE$.

4.1 Deterministic interest rates

In the case of deterministic discount factors several model-free results can be derived. For example, denote $S_t$ as the underlying and suppose that the option at time $T$ has a payoff $g(s)$. Then note that in a world of constant interest rates the option value at time 0 ($V(0)$) and for some future time $t > 0$ satisfy

$$V(0) = \mathbb{E}[e^{-rT}g(S_T)|\mathcal{F}_0]$$

$$= \mathbb{E}[\mathbb{E}[e^{-rT}g(S_T)|\mathcal{F}_t]|\mathcal{F}_0]$$

$$= \mathbb{E}[e^{-rt}\mathbb{E}[e^{-r(T-t)}g(S_T)|\mathcal{F}_t]|\mathcal{F}_0]$$

$$= \mathbb{E}[e^{-rt}V(t)|\mathcal{F}_0]$$

$$= e^{-rt}\mathbb{E}[V(t)|\mathcal{F}_0]$$

$$= e^{-rt}EE(t)$$

$$= EE(t),$$

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where $r$ is the relevant interest rate. Hence for such options, the exposure in current terms is constant over time and equal to existing price. In addition, this result allows to retrieve the derivatives of exposure using those of the option in question. Having obtained the derivatives one can proceed and compute the sensitivities of CVA. Now suppose that the pricing mechanism $V_t(S_t)$ is strictly increasing in spot price (e.g. call options, short put options).

$$Q(S_t \leq s|S_0 = s_0) = q \Rightarrow Q(V_t(S_t) \leq V_t(s)|S_0 = s_0) = q \Rightarrow PFE_q(t) = V_t(s),$$

where $Q$ is the risk neutral measure. This means that the $PFE$ can be retrieved by first retrieving the quantile value of the underlying and then plugging it into the pricing formula. In case of Black-Scholes the quantiles can be evaluated analytically, but then so can $PFEs$. For the majority of other models it has to be approximated numerically. Nevertheless, the big advantage of such formulation that one has to only approximate the quantile function of the underlying once and then can use it to assess the risk of a whole portfolio of derivatives. The approach here extends to weakly decreasing/increasing pricing formulas with the appropriate change of inequalities.

This result has another important practical implication. Usually to price a derivative one would calibrate a model to a given vanilla option price surface and then use the model to price other derivatives. When one is only interested in exposure evaluation one can directly use the price surface to retrieve the probability distribution which in turn yield an estimate of the quantile function. For example, in case one has obtained a call option surface (i.e. a function $C(k, t)$), he/she could retrieve the implied density function by doing the following

$$C(k, t) = e^{-rt} \int_0^\infty \max\{s - k, 0\} p(s) ds \Rightarrow e^{rt} \frac{\partial C(k, t)}{\partial k} = \int_k^\infty p(s) ds \Rightarrow e^{rt} \frac{\partial^2 C(k, t)}{\partial k^2} = -p(k),$$

using the fundamental theorem of calculus.

From analytic perspective these results are very attractive, but deterministic interest rates are rather restrictive. A practical compromise could be to assume only the foreign interest rates to be stochastic, note that the results above would still hold true. Moreover, $EE$ result would remain valid in case of derivatives with dependence on both the exchange and foreign interest rates. $PFEs$ could be directly found in case the marginal distribution of the exchange rate can be derived, but otherwise one would have to resort to more extensive numerical procedures such as FDMC which are considered further in the thesis.
4.2 Stochastic interest rates

The addition of randomness to the discount factors provides a share of complications which limit the amount of results available in analytic form. In contrast to the previous section, suppose that the interest rate market is driven by some short rate \( r^d_t \). Then

\[
\tilde{E}E(t) = \mathbb{E} \left( e^{-\int_0^t r^d_u du} V(t) | \mathcal{F}_0 \right) \\
= \mathbb{E} \left( e^{-\int_0^t r^d_u du} \mathbb{E} \left( e^{-\int_t^T r^d_u du} g(S_T) | \mathcal{F}_t \right) | \mathcal{F}_0 \right) \\
= \mathbb{E} \left( \mathbb{E} \left( e^{-\int_0^T r^d_u du} g(S_T) | \mathcal{F}_t \right) | \mathcal{F}_0 \right) \\
= \mathbb{E} \left( e^{-\int_0^T r^d_u du} g(S_T) | \mathcal{F}_0 \right) \\
= V(0),
\]

using the law of iterated expectations. Hence, for European options that do not depend on domestic interest rate the expected exposure at any future date in current terms is just the current price of the option. However, in the case of expected exposure \( EE(t) = \mathbb{E}(V_t|\mathcal{F}_0) \) the results depend on the selected model. In case of BS2HW the result would involve solving three dimensional integral or selecting some relevant numeraire to make the computation simpler. For more complex models like H2HW the measure change would result complexities which would to numerically solved.

Assume that analytic expressions for some derivative are available and depend only on and are monotonic in the future spot price. In addition, suppose that the future asset distribution also has an analytic expression. Then one could invert the distribution function to get a quantile function of the underlying. This in turn can be used to fetch the required quantiles of the exchange rate and retrieve the values of the derivative price which would yield the PFEs in question. In cases where the future option price depends on more than just the future exchange rate (e.g. swap) one would have to use the analytic distribution function to simulate the distribution of the underlying and retrieve the option price for each simulated case. These simulated option prices could then be used to approximate the quantile function, so that PFEs could be retrieved.
Chapter 5

Exposure calculation using the FDMC method

Credit counterparty risk being mentioned more often than ever before, there is a pressure to be able to evaluate and price that risk. In case the options are European the guidelines from the previous chapter can be used for that purpose. For more complex derivatives, such as path dependent options, the authors in [7] propose a combined Finite Difference Monte-Carlo (FDMC) method to model the distribution of the future derivative price. The pseudo-code for the algorithm is as follows:

1. Select $\Delta t, N$, solve FD;
2. Take $t = \Delta t$;
3. Using MC to simulate $N$ realisations of the underlying;
4. For each of the simulated underlying states compute the respective derivative price using FD;
5. In case early exercise is allowed, set the derivative price to 0 for paths which have been exercised up to and including time $t$;
6. Calculate the exposure metrics using the simulated distribution of the derivative prices;
7. If needed discount the exposure metrics;
8. If $t$ is smaller than the derivative maturity take $t = t + \Delta t$ and go to step 3, else quit.

The combination of the simplicity of the MC simulations and the ability of FD methods to compute a price for a range of spot prices at once allows to compute exposure statistics in a simple and tractable manner. Next to that, the method is attractive because it is:
• Possible to include correlation between all the state variables (e.g. in H2HW exchange rate, volatility, interest rates);
• Simple to get the greeks of CVA are as shown in [8];
• Possible to correlate default probability with the underlying asset as shown in [17];
• Simple to measure exposures of path-dependent options.

In the next two sections of this chapter MC implementation is described and the solving of the PDE is reviewed in more detail.

5.1 Monte-Carlo method

To evaluate future exposure one needs to get an estimate of the future distribution of the underlying. This can be done in multiple ways:

• By an analytic expression for the distribution when it is available (e.g. BS, BS2HW);
• Distribution can be approximated using characteristic function (e.g. Levy models);
• Using the pricing PDE one can derive the respective Fokker-Planck/Kolmogorov forward equation describing the dynamics of the probability density of the underlying;
• Underlying can be simulated using some Monte-Carlo scheme.

Although others might provide specific benefits, the reason for choosing MC is that one can simulate once and then reuse those simulations for different types of options (e.g. European, American and/or Barrier). Moreover, it is straightforward to implement a general algorithm. Lastly, MC by nature is a parallel algorithm and can be moved on to distributed systems with little effort.

Regarding the case in question, interest rates and FX rate SDEs are approximated using the Euler scheme, i.e. when the process $X_t$ satisfies the PDE

$$dX_t = f(X_t)dt + \sigma dW_t,$$

it is approximated by

$$\Delta X_t = f(X_t)\Delta t + \sigma \Delta tZ.$$
where

\[ \Delta X_t = X_t - X_{t-\Delta t} \]

\[ Z \sim N(0, 1) \]

Additional attention requires the volatility process which is modeled as the CIR process

\[ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW^v_t \]

This process is restricted to being non-negative, but the long term mean \( \bar{v} \) typically is close to zero (\( \bar{v} = 0.04 \) implies 20% average annual volatility). However, Euler scheme would not implement such a restriction and some adjustments have to be made. In this work for the simulation of the volatility process QE scheme is used as proposed in [2] which significantly improves the quality of the simulation.

### 5.2 Finite Difference method

One possible method of pricing derivatives is via solving the corresponding PDE. Finite Difference (FD) method is a way to do so, but one has to give extra attention to which numerical FD scheme is stable and appropriate for the given PDE. Since any previously discussed model is a nested one under H2HW, this work concentrates on the solution of the PDE in (3.3). The solution of this 3-dimensional HHW PDE using ADI schemes was discussed in [12] and we follow their approach.

The one dimensional derivatives are approximated using the following

1st backward - \( f'(x_i) \approx \alpha_{-2}f(x_{i-2}) + \alpha_{-1}f(x_{i-1}) + \alpha_0f(x_i), \) (5.1)

1st central - \( f'(x_i) \approx \beta_{-1}f(x_{i-1}) + \beta_0f(x_i) + \beta_1f(x_{i+1}), \) (5.2)

1st forward - \( f'(x_i) \approx \gamma_0f(x_i) + \gamma_1f(x_{i+1}) + \gamma_2f(x_{i+2}), \) (5.3)

2nd central - \( f''(x_i) \approx \delta_{-1}f(x_{i-1}) + \delta_0f(x_i) + \delta_1f(x_{i+1}), \) (5.4)
with
\[ \alpha_{-2} = \frac{\Delta x_i}{\Delta x_i (\Delta x_i + \Delta x_{i-1})}, \quad \alpha_{-1} = -\frac{\Delta x_i - \Delta x_{i-1}}{\Delta x_i - \Delta x_{i-1}}, \quad \alpha_0 = \frac{\Delta x_{i-1} + 2\Delta x_i}{\Delta x_i (\Delta x_i + \Delta x_{i-1})}, \]
\[ \beta_{-1} = \frac{-\Delta x_{i+1}}{\Delta x_{i+1} (\Delta x_{i+1} + \Delta x_{i+2})}, \quad \beta_0 = \frac{\Delta x_{i+1} - \Delta x_i}{\Delta x_{i+1} - \Delta x_i}, \quad \beta_1 = \frac{\Delta x_i}{\Delta x_{i+1} (\Delta x_i + \Delta x_{i+1})}, \]
\[ \gamma_0 = \frac{-2\Delta x_{i+1} - \Delta x_{i+2}}{\Delta x_{i+1} (\Delta x_{i+1} + \Delta x_{i+2})}, \quad \gamma_1 = \frac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+1} \Delta x_{i+2}}, \quad \gamma_2 = \frac{-\Delta x_{i+1}}{\Delta x_{i+2} (\Delta x_{i+1} + \Delta x_{i+2})}, \]
\[ \delta_{-1} = \frac{2}{\Delta x_i (\Delta x_i + \Delta x_{i+1})}, \quad \delta_0 = -\frac{2}{\Delta x_i \Delta x_{i+1}}, \quad \delta_1 = \frac{2}{\Delta x_{i+1} (\Delta x_i + \Delta x_{i+1})}, \]

where \( \Delta x_i = x_i - x_{i-1} \). Equations (5.1), (5.2) and (5.3) are the backward, central and forward approximations of the first-order derivative, respectively. The formula in (5.4) is the usual central second-order derivative approximation. Since the PDEs under consideration contain mixed-terms those are approximated applying the first order approximations twice in each dimension.

Suppose one has the following meshes at hand:

- \( 0 = s_0 < ... < s_{m_S} = S_{\text{max}} \),
- \( 0 = v_0 < ... < v_{m_v} = V_{\text{max}} \),
- \( -R_{\text{max}} = r_{d0}^d < ... < r_{d_m}^d = R_{\text{max}} \),
- \( -R_{\text{max}} = r_{f0}^f < ... < r_{f_m}^f = R_{\text{max}} \),

where \( S_{\text{max}}, V_{\text{max}}, R_{\text{max}} \) are the maximum values considered in the grid and \( m_S, m_v, m_d, m_f \) are the mesh sizes in the stock, volatility, domestic and foreign interest rate domains respectively. Then define the vector \( U(t) \) to be\(^1\)

\[ U(\tau) = (u(\tau, s_0, v_0, r_{d0}^d, r_{f0}^f), u(\tau, s_0, v_0, r_{d1}^d, r_{f1}^f), ..., u(\tau, s_{m_S}, v_{m_v}, r_{d_m}^d, r_{f_m}^f))^T. \]

Using the FD approximations the PDE (3.3) can be approximated by the following system of ODEs:

\[ \frac{\partial U}{\partial \tau} = A(\tau) U + g(\tau), \]

where \( g \) contains terms related to boundary conditions. To apply ADI schemes both \( A(\tau) \)

\(^1\)The vector is formed by traversing the mesh points in the hypercube along the \( r^f \) dimension first and then along \( r^d, v, S \) respectively.
and \( g \) have to be split into

\[
A(\tau) = A_0 + A_s + A_v + A_{rd}(t) + A_{rf}(\tau) \\
g(\tau) = g_0 + g_s + g_v + g_{rd}(t) + g_{rf}(\tau),
\]

where subscript 0 means that the matrix/vector contains terms related to the mixed derivatives and in other cases denotes to which variable related terms it contains. In what follows the Hundsdorfer–Verwer (HV) scheme from [12] is applied. It iterates over the solution vector \( U \) in the following manner\(^2\):

\[
Y_0 = U_{n-1} + \Delta \tau (A(\tau_n)U_{n-1} + g(\tau_{n-1})) \\
Y_j = Y_{j-1} + \theta \Delta \tau [A_j(\tau_n)Y_j + g_j(\tau_n) - A_j(\tau_{n-1})U_{n-1} - g_j(\tau_{n-1})] ; j \in 0, s, v, r_d, r_f \\
\tilde{Y}_0 = Y_0 + \frac{1}{2} \Delta \tau [A(\tau_n)Y_{r_f} + g(\tau_n) - A(\tau_{n-1})U_{n-1} - g(\tau_{n-1})] \\
\tilde{Y}_j = Y_{j-1} + \theta \Delta \tau [A_j(\tau_n)Y_j + g_j(\tau_n) - A_j(\tau_{n-1})U_{n-1} - g_j(\tau_{n-1})] ; j \in 0, s, v, r_d, r_f \\
U_n = \tilde{Y}_{r_f},
\]

where \( \theta = \frac{1}{2} + \frac{1}{6}\sqrt{3} \) as suggested in [12].

Having skipped the boundary and mesh construction specifics, we return to properly define this in the following subsections.

**Boundary conditions**

Solving a partial differential equation for an initial value problem requires selecting appropriate boundary conditions. Usually academic papers focus on a specific derivative and fine-tune the boundary conditions (e.g. [12]) per derivative. However, this approach limits the generality of the algorithm and in a worst-case scenario be the root of numerical instability (e.g. oscillations arising in case Neumann boundary conditions are used in the volatility direction for Heston PDE). Hence thorough this work analysis is done by taking the payoff as the boundary condition. In other words, for every mesh point \((S, v, r_d, r_f)\) belonging to the boundary of \([0, S_{max}][0, V_{max}] \times [-R_{max}, R_{max}] \times [-R_{max}, R_{max}]\) the following holds

\[
U(\tau, S, v, r_d, r_f) = g(S, v, r_d, r_f).
\]

One might argue that for any specific payoff there are better boundary conditions, however, as long as the boundary condition does not cause instability in the solving scheme the

\(^2\)When \( j \) is 0, s, v, rd, rf, \( j - 1 \) refers to the previous element in the list 0, s, v, rd, rf
impact of a "bad" boundary condition can be diminished by choosing large enough hyper-
cube. Therefore, a better boundary condition would improve mostly the solutions close to
the boundary which by construction are extreme cases.

To insure against a possible "wrong" boundary condition one has to choose a far away
boundary point. If one is using a uniform grid this would entail either a large number of
grid points or big gaps between the grid points implying a poor solution approximation. To
deal with this, the next section describes a non uniform grid which effectively assigns grid
points.

5.2.1 Spatial grid construction

This thesis concentrates on rather simple derivatives (European and American puts,
Up-and-out calls) which have in common high curvature around the strike price. Using this
property, the FD grid is constructed following the guidelines presented in [12]. Recall that
the PDE is defined on $\mathbb{R}^3 \times \mathbb{R}^+$. To solve it numerically the domain has to be truncated to
$[0, S_{max}] \times [-R^d_{max}, R^d_{max}] \times [-R^f_{max}, R^f_{max}] \times [0, V_{max}]$. In what follows a restriction $R_{max} = R^d_{max} = R^f_{max}$ is applied, since the differences in interest rate grids are relatively small when
large endpoints are chosen.

At the starting point grids for all four dimensions have to be chosen:

- Due the fact that call/put option payoff is not differentiable at the strike $K$, in the
  $s$-direction the grid is chosen to be dense and uniform in $[S_{left}, S_{right}] \subset [0, S_{max}]$
  where $K \subset [S_{left}, S_{right}]$. Using some $m_s \geq 1$, $d_s > 0$ and equidistant set of points
  $\xi_{min} = \xi_0 < \xi_1 < ... < \xi_{m_s} = \xi_{max}$ given by

\[
\xi_{min} = \sinh^{-1}\left(\frac{-S_{left}}{d_s}\right)
\]
\[
\xi_{int} = \frac{S_{right} - S_{left}}{d_s}
\]
\[
\xi_{max} = \xi_{int} + \sinh^{-1}\left(\frac{S_{max} - S_{right}}{d_s}\right),
\]

the mesh in $0 = s_0 < s_1 < ... < s_{m_s} = S_{max}$ is defined by

\[
s_i = \phi(\xi_i), i = 0, ..., m_s
\]
where

\[
\phi(\xi) = \begin{cases} 
S_{\text{left}} + d_S \sinh(\xi), & \xi_{\text{min}} \leq \xi \leq 0 \\
S_{\text{left}} + d_S \xi, & 0 \leq \xi \leq \xi_{\text{int}} \\
S_{\text{right}} + d_S \sinh(\xi - \xi_{\text{int}}), & \xi_{\text{int}} < \xi \leq \xi_{\text{max}}
\end{cases}
\]

In this mesh parametrisation the parameter \(d_S\) controls the amount of points in the interval \([S_{\text{left}}, S_{\text{right}}]\).

- For the \(v\)-direction parameters \(m_v, d_v, V_{\text{max}}\) have to be selected. At first the following is computed

\[
\Delta \eta = \frac{1}{m_v} \sinh^{-1} \left( \frac{V_{\text{max}}}{d_v} \right),
\]

and the intermediate mesh is generated by \(\eta_i = i \Delta \eta, i = 0, ..., m_v\). Then the required mesh for the volatility is calculated by

\[
v_i = d_v \sinh(\eta_i), i = 0, ..., m_v.
\]

The parameter \(d_v\) regulates the amount of points close to 0.

- The domestic and foreign interest rate grids have the same formulations with different parameters. Hence, only general scheme is presented here. In construction of the mesh the parameters \(m_r, d_r, c_r\) and \(R_{\text{max}}\) are required. First one computes

\[
\Delta \zeta = \frac{1}{m_r} \left[ \sinh^{-1} \left( \frac{R_{\text{max}} - c_r}{d_r} \right) - \sinh^{-1} \left( \frac{-R_{\text{max}} - c_r}{d_r} \right) \right],
\]

which is then used for the intermediate grid defined by

\[
\zeta_i = \sinh^{-1} \left( \frac{-R_{\text{max}} - c_r}{d_r} \right) + i \Delta \zeta, i = 0, ..., m^r.
\]

The latter is then used to generate the \(r\)-grid by

\[
r_i = c_r + d_r \sinh(\zeta_i), i = 0, ..., m^r.
\]

The parameters \(c_r\) sets the center of the grid and \(d_r\) governs the grid density around it. As shown in [12] the meshes are smooth in all three directions.

In this work the following grid parameters are used:

- \(S_{\text{max}} = 25K, S_{\text{left}} = \max\{0.5, e^{-0.25T}\}\), \(m_s = 4m, d_S = K/20\)
\begin{itemize}
  \item $V_{\text{max}} = 15, m_d = 2m, d_v = V_{\text{max}}/2000$
  \item $R_{\text{max}} = 1, m_r = m, d_r = R_{\text{max}}/400$
  \item $\Delta t = T/n$.
\end{itemize}

where $m, n$ are chosen to get results with a specified level of precision. For the put option $S_{\text{right}} = K$ and $S_{\text{max}} = 25K$, whereas in the case of a up-and-out call option $S_{\text{right}} = S_{\text{max}} = B$.

It is noteworthy to mention that although [12] provided suggestions for the required parameters, during the experimental stage of this work it was observed that further fine-tuning is required. One has to be particularly cautious about the volatility dimension. For some of conducted experiments the initial variance was between 0 and first grid point leading to poor price estimates. To counter this the grid in $v$ dimension was constructed more dense around zero by choosing smaller value for $d_v$.

Also, it was noticed that the number of grid points can be reduced by choosing a smaller number of grid points in the interest rate dimensions. This can be explained by the fact that HW processes imply rather simple dynamics on the resulting derivative, so that the solution is flatter when compared to the curvature of solution in the variance dimension.

For long maturities MC simulations can explore scenarios outside the hypercube that FD is solved on. In situations like this prices are set equal to payoff as if the maturity was today. Extrapolations were avoided since close to the boundary there are few points so extrapolations are highly unreliable.

Implementation of high dimensional PDE solvers can become tedious increasing the code bug rate and, therefore, development time. To ease this stage for those interested, Appendix A contains notes on how high-dimensional PDE problems can be addressed using the Kronecker products. Such an implementation combines these ideas with sparse matrices and LU decomposition yielding a well-structured and efficient script.

5.3 Extensions of the FD solution scheme

5.3.1 Pricing a portfolio of European options

The discussions with my supervisors yielded an observation that the exposure for a complete portfolio of European derivatives can be computed with only one sweep in FD method. The computation at its core is similar to pricing a swap option and the latter has been explored in [6].
The first step requires focusing on cash flows instead of payoffs. Suppose one has $M$ options with maturities $T_i$ and payoffs $g_i(S_{T_i}, v_{T_i}, r^d_{T_i}, r^f_{T_i})$. Then define the cash-flow function $g(t, s)$ as

$$g(t, s, v^d, r^f) = \sum_{i=1}^{N} 1_{\{t = T_i\}} g_i(s, v^d, r^d, r^f).$$

To price this stream of payoffs a mild adjustment has to be made to the FD scheme. The solution at the initial point has to be set to $U_0 = g(T - \tau_0, \vec{S}, \vec{v}, \vec{r}^d, \vec{r}^f)$ with vectors $\vec{S}, \vec{v}, \vec{r}^d, \vec{r}^f$ defined as in Appendix A.1.2. Also, (5.9) has to be changed to

$$U_n = \tilde{Y}_{r^f} + g(T - \tau_n, \vec{S}, \vec{v}, \vec{r}^d, \vec{r}^f).$$

(5.10)

This allows one can price a portfolio of derivatives with payoffs depending on a combination of exchange rate, volatility, domestic and foreign rates. The only drawback of this approach is the need to choose a relevant spatial discretisation which would be more dense close to the heavily curved parts of the payoffs.

### 5.3.2 Pricing American and Barrier options

FD methods are valued because of their ability to price several early-exercise options. Here the idea of the previous subsection is generalised to price American and Barrier options. Generalize the step (5.10) to

$$U_n = \tilde{g}(\tilde{Y}_{r^f}, T - \tau_n, \vec{S}, \vec{v}, \vec{r}^d, \vec{r}^f).$$

where $\tilde{g}$ is some user selected function. Then a $T$-maturity up-and-out call option with barrier $B$ and strike $K$ can be priced by selecting:

$$\tilde{g}(y, t, s, v^d, r^f) = 1_{\{t < T\}} 1_{\{s < B\}} y + 1_{\{t = T\}} 1_{\{s < B\}} (s - K)^+$$

To retrieve the price of an American call option the function to be chosen is:

$$\tilde{g}(y, t, s, v^d, r^f) = \max\{y, (s - K)^+\}$$

This formulation has little impact on the underlying mathematics apart from notation, but has helped in achieving implementation efficiency and code quality. Moreover, it allowed

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3In this section a single argument function of a vector should be understood as vector containing element-wise functions of the vector. For multi-argument function the result would be a vector with function applied to respective vector elements.
to adapt the regular FD solver to more complex options resulting in shorter development time.
Chapter 6

Case study

The previous chapter examined the pricing differences under several FX models. Although it gave inference for an arbitrary parameter set, the market conditions can be vastly different. To address these issues in this chapter, real data is used to calibrate parameters and draw conclusions in actual market scenarios.

6.1 Market conventions

The calibration of the models is executed by fitting market prices with model prices. However, the volatilities in FX markets is defined in terms of Delta levels, so that conversion to prices has to take place before calibration. The conversion procedure outlined here follows [19].

In exchange markets the quotes are given in risk reversals (\(RR\)) and butterflies (\(FLY\)). For \(x\) Delta level and fixed maturity the definitions of the two are:

\[
RR(x) = \sigma_C(x) - \sigma_P(x)
\]
\[
FLY(x) = \frac{\sigma_C(x) + \sigma_P(x)}{2} - \sigma_{ATM},
\]

where \(\sigma_C(x)\) and \(\sigma_P(x)\) are the volatilities of the call and put options which both have Delta equal to \(x\) and \(\sigma_{ATM}\) is the volatility of an at-the-money option. Note that at-the-money level for FX options is the where call and put have the same Delta, but different sign.

System (6.1) can be solved for the call and put option volatilities yielding

\[
\sigma_C(x) = FLY(x) + \frac{RR(x)}{2} + \sigma_{ATM}
\]
\[
\sigma_P(x) = FLY(x) - \frac{RR(x)}{2} + \sigma_{ATM}.
\]
Having obtained the volatilities one proceeds with the calculation of the strike, which for EURUSD means using the equation:

\[
K(w, \sigma, \Delta) = F(t, T) \exp \left[ -w\sigma \sqrt{T - t} \Phi^{-1} \left( \frac{\Delta}{P(t, T)} \right) + \frac{\sigma^2(T - t)}{2} \right],
\]

with

\[
F(t, T) = S_t \frac{P(t, T)}{P(t, T)}.
\]

and \( w = 1 \) in case of a call and \( w = -1 \) otherwise.

### 6.2 Data

The focus of this case study is options on EURUSD exchange rate. The dataset\(^1\) includes European option volatilities, USD and EUR yield curve data as measured on 2014-04-11 and which is available for inquiry in Appendix tables C.1, C.2 and C.3 respectively. Note that in this case USD is the domestic currency (ccy) and EUR is the foreign one.

Figures 6.1 and 6.2 presents the volatility and surfaces computed using the risk reversal

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\(^1\)The author wishes to express gratitude to ING for providing the data.
and butterfly data. A quick look at the chart reveals that the smile has a substantial skew.

The shape of the skew in Figure 6.1 imply that out of the money puts are going to be more expensive than if they were priced with at-the-money volatility levels. Therefore, the implied chances of the EURUSD decreasing are higher than they would be in simple Black-Scholes.

### 6.3 Calibration

The pricing of derivatives under H2HW require estimating 14 parameters: 2 per HW component, 4 for volatility process and 6 correlations. Although technically it is possible to calibrate all the parameters to one surface, it would entail a different parameter set per product calibrated. For example, in case parameter fitting is performed for EURUSD and GBPUSD options separately, it would lead to having differing parameters for the USD interest rate component. Therefore, calibrating all the components to a single option surface introduces ambiguity in the choice of parameters for pricing other options.

The calibration scheme used contains:

1. Calibrating HW components to swaptions;
2. Calibrating Heston to the price surface;
3. Retrieving observable correlations from historical data (\(\rho_{S,r_d}, \rho_{S,r_f}, \rho_{r_d,r_f}\));

4. Calibrating the remaining correlations to the price surface.

Note that before calibrating H2HW, Heston is calibrated. This is done in order to reduce the computational complexity and simplify the optimization problem. Moreover, efficient calibration procedures for Heston are available.

Since pure HW model is of little interest in the context of this work the estimates for the interest rate processes were provided by ING:

\[
\begin{align*}
\lambda_d &= 0.01, \eta_d = 0.00917940269182; \\
\lambda_f &= 0.01, \eta_f = 0.00754794484330; \\
\rho_{S,r_d} &= -0.01144108, \rho_{S,r_f} = -0.3124035, \rho_{r_d,r_f} = 0.6336561.
\end{align*}
\]

For calibration purposes FD methods are too time consuming since per run they provide prices of option with single strike under different spot prices. On the other hand, the goal of price fitting is to match the prices of options with multiple strike prices and one spot price. Having this in mind, calibration is implemented by using the COS method as described in [9]. The target to minimize was the squared error, i.e.

\[
\sum_{i=1}^{N_T} \sum_{j=1}^{N_K} (C_{Mkt}(T_i, K_{i,j}) - C_{Hes}(T_i, K_{i,j}, \phi))^2
\]

where

- \(N_T\) is the number of different maturities;
- \(T_i\) denotes the \(i\)-th maturity;
- \(N_K\) is defined as the number of different strikes per maturity;
- \(K_{i,j}\) means the \(j\)-th strike of \(i\)-th maturity;
- \(C_{Mkt}(t, k)\) gives the observed market price of an European call with maturity \(t\) and strike \(k\);
- \(C_{Hes}(t, k, \phi)\) stands for the price of an European call with maturity \(t\) and strike \(k\) under the Heston model with parameters \(\phi\).

Solving for optimal parameters under Matlab’s \textit{fminunc} yielded the following parameters:

\[
\begin{align*}
v_0 &= 0.0037, \kappa = 0.6054, \bar{v} = 0.0116, \gamma = 0.1811, \rho_{S,v} = -0.24.
\end{align*}
\]
Figure 6.3 presents calibration errors. It can be observed that in absolute terms the fit worsens with increasing maturity. Moreover it seems to lack the flexibility to capture the curvature around the strike price as the error in most maturities is highest in that region. Analysing the Heston implied prices in a relative sense reveals that the percentage difference is greatest for out of the money calls. The relative difference is substantial in the very short term options (e.g. overnight, 1 week, 2 weeks) which is a known feature of Heston model. On the other hand those options have very small values and a small error is large relative to the price.

The last step of the calibration procedure involves fitting the full H2HW model. Using the parameters obtained in HW and Heston calibration only two parameters of correlation need to be fitted $\rho_{v,r}^d$ and $\rho_{v,r}^f$. Note that these parameters have to be restricted so that it results in a proper covariance matrix between the Brownian motions. These parameters were optimised using the grid search method over the space of valid correlation values. The algorithm yielded $\rho_{v,r}^d = -0.96$ and $\rho_{v,r}^f = -0.45$ as the optimal values.

### 6.4 Calculation of default probabilities using CDS premiums

The FX models define the prices and discount factors per parameter set. To be able to compute the CVAs also survival rate probabilities are required. Since these are not quoted explicitly in the market, the information has to be inferred using the credit default swaps.

A credit default swap (CDS) is a financial product invented for the bond market by Blythe Masters of JP Morgan in 1994. Using this contract the buyer receives a predefined amount of money (aka. notional) in case the party under consideration defaults. In return the buyer pays regularly some percentage of the notional to the seller until the party defaults or the contract reaches maturity.

By itself the CDS premium provides no explicit information on probabilities, but coupled with a model one can extract them. In this work the model for this purpose is chosen following the work in [8]. It starts by defining:

$$q(t) = e^{-\lambda_{haz}t}, \lambda_{haz} > 0$$

where $q(t)$ is the survival probability and $\lambda_{haz}$ is the hazard rate parameter setting the general likelihood of party’s default.

On the other hand, as shown in [1] the CDS premium $S$ can be computed using the
Figure 6.3: Difference between Heston implied and market prices.
formula:

\[ S = \frac{(1 - R) \sum_{i=1}^{N} D(t_i) (q(t_{i-1}) - q(t_i))}{\sum_{i=1}^{N} D(t_i) q(t_i) \Delta t + \sum_{i=1}^{N} D(t_i) (q(t_{i-1}) - q(t_i)) \frac{\Delta t}{2}} \]

where

- \( \Delta t \) - time between buyer’s payments;
- \( t_i = i \Delta t, i = 1, ..., N \) - timepoints when CDS buyer pays to the insurer;
- \( R \in [0, 1] \) - recovery rate given as proportion of capital recovered given counterparty’s default;
- \( D(\cdot) > 0 \) - discount curve.

In Appendix B it is show that there is a one-to-one correspondence between the hazard rate \( \lambda_{haz} \) and CDS premium \( S \). The conversion can be done using the formula:

\[ \lambda_{haz} = \frac{1}{\Delta t} \ln \left( \frac{1 - R + S \frac{\Delta t}{2}}{1 - R - S \frac{\Delta t}{2}} \right). \]

(a) CDS premium and equivalent hazard rate for different recovery and hazard rates with quarterly payments. (b) Survival probabilities for different recovery and hazard rates over time (years) with quarterly payments.

Figure 6.4a shows the CDS level versus the resulting hazard rates. One can observe that in case of \( R = 0 \) and an approximation \( \lambda_{haz} = S \) can provide sufficiently accurate results. Figure 6.4b plots the survival probabilities over time for multiple recovery rates and CDS premiums. It can be noted that in case CDS premium is small the impact of recovery rate is negligible. This conforms with the financial reasoning since low enough CDS implies close to
0 default probability and in this case one just does not expect the company to default and any capital recovery is unnecessary. Lastly, the change of payment frequency had insignificant impact on hazard rates and survival probabilities.

6.5 The impact of stochastic interest rates and volatility on derivative exposures

The focus of this section is to quantify the impact of adding stochastic volatility and interest rate risk on the prices and exposure profiles for a couple of derivative contracts. The analysis starts by extending the results of [7] about American put option and then moves on to up-and-out call option.

The measures tracked in this work include:

- Simple price/ CVA differences to assess the impact in monetary terms;
- Price/ CVA differences expressed as percentage of BS2HW. This metric provides a relative measure allowing to compare differences across strikes and maturities;
- An additional run of BS2HW is executed with volatility increased by 0.01 and using the resulting difference between the two BS2HW runs gives the sensitivity of the Price/ CVA to 1% change in volatility. Then Price/ CVA differences between models are expressed in terms of 1% BS2HW vega. Options are usually quoted in terms of their BS volatilities. Hence this provides an estimate how much the BS volatility has to be adjusted to price in interest rate risk and volatility skew.

This section uses the calibrated parameter set which in complete form are:

\[
\begin{align*}
S_0 &= 1.3886 \\
\lambda_d &= 0.01, \eta_d = 0.00917940269182; \\
\lambda_f &= 0.01, \eta_f = 0.00754794484330; \\
v_0 &= 0.0037, \bar{\nu} = 0.0116, \kappa = 0.6054, \gamma = 0.1811;
\end{align*}
\]

\[
\begin{pmatrix}
1 & \rho_{S,v} & \rho_{S,r}^d & \rho_{S,r}^f \\
... & 1 & \rho_{\nu,r}^d & \rho_{\nu,r}^f \\
... & ... & 1 & \rho_{r,r}^d, r_f \\
... & ... & ... & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & -0.24 & -0.01144108 & -0.3124035 \\
... & 1 & -0.96 & -0.45 \\
... & ... & 1 & 0.6336561 \\
... & ... & ... & 1
\end{pmatrix}.
\]

It has to be mentioned that HW model is reduced to Vasicek. This is done for clarity as in case of yields curves the effect of yields would be entangled with the impact of stochastic...
volatility possibly resulting in mixed/ wrong conclusions about skew impact. Ergo, IR parameters are

\[
\begin{align*}
r_0^d &= 0.001714, \theta_d(t) = \theta_d = 0.0042286 \\
r_0^f &= 0.0012175, \theta_f(t) = \theta_f = 0.0055060
\end{align*}
\]

(6.3)

which are chosen similar to ones prevailing in the market.

### 6.5.1 American put

The starting point of early exercise options on EURUSD is the American put option. American option is a generalisation of European version since it allows for exercise anytime until maturity. Although American call option under stochastic interest rates is not equal to its European counterpart, the results are computed for American put options such that they could be compared to other research.

#### Error analysis

The application of the FD scheme requires grid parameter selection. In order to confirm the validity of the results an error analysis is performed. In the case considered the FD grid depends on two parameters:

- **m** - regulating the density of points in the spatial dimensions. Recall that \( m_S = 4m, m_v = 2m, m_{r_d} = m_{r_f} = m; \)

- **n** - controlling the number of points in the time dimension.

The limiting factor is the computational time required to solve the H2HW PDE since already for \( m = 16 \) implies solving a system of 524288 equations at every time step. The largest system that fit into the memory of a late 2014 Macbook Pro 13’ was that with \( m = 32 \) and is taken as the reference price. Then other \( m, n \) combinations were explored and the errors are presented in Table 6.1. It can be observed that for \( m = 16 \) for all the models the errors are below 1% of the reference price. Also, it takes less than half the computation time of \( m = 32 \) and, therefore, \( m = 16 \) is chosen for the discretisation in space. Regarding time dimension \( n = 64 \) provides errors less than 0.40% which provides a good accuracy/ speed trade-off for all three FX models.

An observation has to be made that BS2HW is a special case of H2HW when \( \gamma = 0, \eta_0 = \bar{\eta} \) and BS is a special case of BS2HW when \( \theta_d = r_0^d, \eta_d = 0, \theta_f = r_0^f, \eta_f = 0. \) However, the solvers in each case are different. This is used to check for solver consistency which is achieved by restricting the more general models to align with the nested ones. The results are presented in Table 6.2 showing that the models are consistent up to 0.25%.
(a) H2HW, reference price 0.118867.

<table>
<thead>
<tr>
<th>m \ n</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
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<td>6.31%</td>
<td>6.87%</td>
<td>7.06%</td>
<td>7.14%</td>
<td>7.17%</td>
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<tr>
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<td>2.12%</td>
<td>1.64%</td>
<td>1.46%</td>
<td>1.37%</td>
<td>1.33%</td>
</tr>
<tr>
<td>16</td>
<td>1.19%</td>
<td>0.58%</td>
<td>0.48%</td>
<td>0.40%</td>
<td>0.36%</td>
</tr>
<tr>
<td>32</td>
<td>0.91%</td>
<td>0.22%</td>
<td>0.12%</td>
<td>0.04%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

(b) BS2HW, reference price 0.14534.

<table>
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<th>16</th>
<th>32</th>
<th>64</th>
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<td>0.43%</td>
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<td>0.10%</td>
<td>0.03%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

(c) BS, reference price 0.133819.

Table 6.1: Absolute errors of a 5 year at-the-money put option as percentage of the reference price

Table 6.2: Relative errors in the limiting cases of nested models for the American put option.
Sensitivity analysis

For the purposes of better understanding of the underlying dynamics this section deals with the sensitivities of an at-the-money American put option to the model parameters. Parameter impact is conducted comparing the price given under the parameters described in (6.2) and (6.3) to the cases when one of the parameters is changed. Apart from correlations the changed parameters are twice the size of the original parameters. However, correlations are chosen such that either they are zero or some other value that does satisfy the constraint of positive semi-definite covariance matrix of the Wiener processes. Having obtained the price under the modified parameter set the derivative with respect to the parameter is approximated using the finite difference formula:

\[
\frac{\partial p}{\partial x} \approx \frac{\hat{p} - \tilde{p}}{\Delta x}
\]

where \( p \) is the option price, \( \hat{p} \) ( \( \tilde{p} \) ) is its approximation under the regular ( modified ) parameter set, \( x \) is the parameter in consideration and \( \Delta x \) is the size of the change in \( x \) between the two sets.

Table 6.3 presents the resulting prices and derivative approximations which imply the following:

- The doubling of the vol-of-vol yielded the largest decrease in the price;
- Increased initial volatility had a large impact on the shortest maturity option, but the impact decayed with increasing maturity;
- Similarly, an increase in mean reversion affected the short term options the most;
- An increase in the long term volatility had a significant impact over all maturities, but the impact increased with longer maturities;
- Changes in the volatilities of the interest rate components had a minor impact for the shorter maturity options, but a significant one for the 10 year option;
- Derivative-wise interest rate volatilities affected long maturity options the most.

An at-the-money option was insensitive to the changes in the correlation matrix, mean reversion of the foreign interest rate. This implies that during calibration these values can be excluded from the optimization and be replaced by some estimates from the market experts with minimal impact on the resulting fit.
(a) Prices in domestic currency (ccy) for each marginal parameter case.

Table 6.3: American put option price sensitivity to parameters under H2HW. The reference case is defined by parameters in (6.2) and (6.3) and in other cases parameters are changed one at a time.

(b) Finite difference approximations of the derivatives.
Prices

Table 6.4 presents the price comparisons under the three models for several maturity and strike combinations. For exact prices of the options please consult the Table D.1 in the Appendix. The following can be observed:

- The exclusion of interest rate risk leads to Black-Scholes model underpricing the options. 10 year at-the-money option price difference reaches 18% of BS2HW and as the option moves out-of-money the relative difference increases. However, volatility adjustment needed for BS2HW to match BS increases as the option moves in-the-money;
- For all strikes the underpricing of BS tends to worsen when maturity increases;
- The relative adjustment of prices for the volatility skew has an increasing effect as options go out-of-money;
- The adjustment of volatility needed for BS2HW to match H2HW is not monotonous in strike. The adjustment is substantial for far out-of-money options and is most significant at the strike level, but for the rest it is up to one 1% BS2HW Vega;
- Skew impact diminishes for longer maturity options;
- Stochastic volatility dynamics changes the pricing of the derivatives the most at short maturities where at at-the-money strike the price adjustment is equal to 37% of BS2HW price;

From these observations it can concluded that the interest rate risk has to be included when pricing options with longer maturities. It is of crucial importance for the in-the-money options. On the other hand, volatility skew affects mostly short-dated out-of-money options. Additional care should be taken for at-the-money Vega adjustments.

The impact of the model for pricing derivatives is of crucial importance while the contract is being signed. However, having the derivative in the portfolio future value dynamics are more important in assessing the risks involved. To address this question in the next section an overview of the risk dynamics of the value of such a contract in the future is assessed through the exposure profiling.

Exposure profiles

Although the information on spot price differences between models for different options can provide valuable insights on the model relevance, the assessment of derivative exposure requires analysis of the distribution of the future options prices. It is noteworthy to mention
Table 6.4: BS and H2HW prices relative to BS2HW prices for multiple maturities (T) and strikes (K).

(a) Difference (model price - BS2HW price).

(b) Difference as percentage of BS2HW price.

(c) Difference as multiple of BS2HW 1% vega.
that for early exercise options two interdependent factors contribute to their exposure profiles: the early exercise conditions and the distribution of the underlying. Figure 6.5 provides exposure profiles of three options with varying maturities. Moreover, Figure 6.6 presents the exercise dynamics for the 10 year version of the American put. The inquiry into the charts reveals that:

- The difference in expected exposures of the 1 year American put option is dominated by the pricing mechanisms as the volatility levels are not sufficient to result in a meaningful amount of early exercises of such option;

- The difference between skew-adjusted model and the rest in 97.5% PFE is close to constant during the lifetime of 1 year option;

- The inclusion of interest rate risk is insignificant in the case of 1 year option;

- As the maturity increases the expected exposure of H2HW and BS2HW converge;

- In the case of 5 and 10 year options the PFE 97.5% of H2HW exceeds that of BS2HW close to maturity having been the opposite immediately after the issue of the contract. This can be explained by the fact that H2HW starts with a small initial volatility;

- The decrease in exposure metrics is related to the early exercise of the options as observed by [7]. The impact increases as maturity grows and significant differences can be observed between the three models as displayed in Figure 6.6. There are around three times more options exercised just before the maturity in models excluding the skew than in the H2HW;

- Similarly to pricing the impact of skew is smaller than that of stochastic interest rates when considering the 5 year and 10 year options.

**CVAs**

The considerations about exposure profiles here were made by analysing the risk in nominal future terms. Several of these factors come into play only close to maturity for long dated options. However, due to compound interest the discount factor decreases exponentially with time making far in the future prices negligible in current terms. To address these issues the next step is to evaluate and compare resulting CVAs.

The effect of CDS premium on resulting CVA for two maturities is presented in Figure 6.7. Since there were little differences between BS and BS2HW for the option of 1 year maturity there is negligible difference between the resulting CVA under all CDS values. On the other
Figure 6.5: At-the-money American put maturity exposure profiles for several maturities under different models.
Figure 6.6: At the money American put with 10 year maturity percentage of exercised options over time.

Figure 6.7: At the money American put option CVA under different models against multiple CDS levels.

hand, the CVA for that option is smaller when the underlying is H2HW the difference being up to 36% of BS2HW price in the case when CDS premium is 4%. Also, over the whole CDS range for the 1 year option the relationship between CDS and CVA is close to linear. Differences arise when the maturity grows longer, for example in the 10 year option. In all cases the increase of CDS premium amplified the differences between models. The CVAs for a range of maturities, strikes and CDS can be found the in the Appendix Table D.2 and the comparison in Table 6.5. From the latter one can note the very similar relative differences across CDS premiums supporting the claim that CVA as a function of CDS premium is close to linear. A more close inquiry into the table 6.5 reveals that

- Given a fixed maturity, the exclusion of stochastic interest rates relatively affect the options further from the spot the most. Vega-wise the adjustment is largest for in the money options;

- Recall that under BS a significant share of the options is exercised before maturity.
It results in a decrease of expected exposure which in itself decreases the CVA. Since for longer maturities more options are exercised the decrease in CVA is larger. This is clearly visible in the relative differences;

- When adjusting for skew the relative differences are largest at shorter maturities and out-of-money and decrease with increasing maturities and strikes;

Apart from these specific insights a general pattern is visible that out-of-money options CVA relative difference is almost exactly the same as that of the prices. However, as options go into money and beyond, the relation fades away. This is a result of the early exercise features. The more in-the-money an option is, the more likely it is to be exercised before maturity decreasing the expected exposure and CVA as a result. On the other hand, the exercise point is highly dependent on the model as shown in figure 6.6. To conclude this section, one can say that for out-of-the money options the underlying model influences the levels of exposure through pricing, but for in-the-money options the difference in CVA arises due the early exercise conditions governed by the model.
Table 6.5: American put option CVA comparison for several maturities and strikes.

(a) Simple difference.

<table>
<thead>
<tr>
<th>Simple difference</th>
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<th>( T = 2 )</th>
<th>( T = 5 )</th>
<th>( T = 10 )</th>
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(b) In percentages of BS2HW prices

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(c) In 1% Vega of BS2HW.
6.5.2 Up and out call option

The American Put option extended European put which added the option to exercise prematurely. Another extension of the European options are the barrier options. However, since the American option is more flexible its price is no less than that of its European counterpart. In specific scenarios for an individual investor/trader the price of both of them might be unreasonably high. A way to get a similar payoff to that of an European option, but for less the amount, is by making use of the barrier options. Barrier options are such options that provide similar payoffs at maturity, but become worthless if the underlying asset value crosses some barrier during the lifetime of the option. These options are especially attractive for technical traders who track support and resistance levels of the underlying.

This section focuses on the up-and-out call option which provides the call payoff at maturity unless the underlying touches of goes above the barrier level at any time up to and including maturity. From the definition it also follows that the barrier price has to be higher than both spot and strike price for the option value to be non-zero.

Error analysis

Table 6.6: Absolute errors of a 5 year up-and-out at-the-money option with barrier 1.8052 as percentage of the reference price.
Table 6.7: Relative errors in the limiting cases of nested models in the case of up-and-out call option.

Up-and-out call option extends European options by allowing for early termination in case the asset touches the barrier level. Accounting for such path dependency is straightforward to implement with the FD method. On the other hand, the selected option has a non-continuous payoff at the barrier level which can be a challenge for finite difference schemes to handle.

Table 6.6 presents the errors in the estimates of the barrier option prices for multiple time and spatial grid configurations. Although the grid is adjusted to better fit the barrier option payoff, the difference between simplest and most dense grid is significantly larger than in the American put. It is also evident that the error is more sensitive to the choice of time discretisation. Furthermore, there are notable differences in errors between models per fixed grid setting. Black-Scholes error varies considerably between different temporal grids, whereas BS2HW and H2HW errors are more stable per fixed spatial grid.

Similarly as in the American put case the system of solvers are checked for internal consistency with results presented in Table 6.7. The models agree up to 0.2% of the reference which stands in favour of the internal consistency.

Taking these results into account the choice for the grid dimensions remains the same as in the American put option which is expected to provide results with up to 1% error.

**Sensitivity analysis**

It has been observed that changes to the volatilities of processes had the most profound impact on the American put option pricing. The price estimates of an up-and-out at-the-money call option changes are presented in Table 6.8. It has to be noted that the major drivers of the option price in this case are terms related to stochastic volatility. The discount factor volatility has a significant contribution which increases with time. As expected the influence of initial spot, volatility and short rates decays over time.

**Price**

Pricing differences between models across strikes, maturities and barrier levels contribute significantly to all stages of derivative risk assessment from issue to maturity. A complete list
(a) Prices in domestic currency (ccy) under different parameters. (b) Finite difference approximations of the derivatives.

Table 6.8: H2HW parameter sensitivity data

of prices is contained in the Appendix Figure D.3. In addition, Appendix Figure D.4 shows the differences between H2HW/BS and BS2HW. Here we concentrate on Figure 6.9 which expresses those differences in relative and BS2HW 1% Vega terms.

Close examination of the figures leads to noticing that:

- Despite several exceptions the volatility adjustment required for BS2HW to match BS is up to one BS2HW 1% Vega different;

- The exclusion of random components in the stochastic interest rates affects shorter maturity out-of-money and longer maturity in-the-money options the most;

- Black-Scholes prices overestimate (underestimate) the option prices up to 15% (5.3%) compared to BS2HW price;

- Skew adjusted model has a much bigger relative and Vega-wise impact in the barrier case than in the previously analysed American put;

- The adjustment for skew resulted in some prices being more than three times that of BS2HW and some were reduced by up to 80%;
Table 6.9: Up-and-out call option with barrier $B$, strike $K$ and maturity $T$ BS and H2HW price differences with BS2HW.

(a) Differences as percentage of BS2HW price.

(b) Differences as multiples of 1% BS2HW Vega.
• The differences between H2HW and BS2HW are most prominent for the options with the smallest barrier level;

• In the case of the smallest barrier level the impact of added volatility dynamics are not monotonous in time, i.e. The differences for 2 and 5 year options are greater than those of 1 and 10 year options.

Similarly to American put option, the stochastic interest rates effect is largest at the long term options. However, in all cases their prices are no further a BS2HW Vega away. On the other hand, the skew shifts prices significantly for all the option parameters considered. Observing under H2HW that options with smallest barrier are more expensive and options with larger barrier levels are cheaper one can deduce that these changes occur not because the option expiry dynamics change, but because the asset is less likely to terminate above the strike price.

**Exposure**

To compare the exposure dynamics implied by the three models four extreme cases of at-the-money options are analysed:

1. Short maturity \((T = 1)\) with barrier \((B = 1.5275)\) close to strike \((S_0 = 1.3886)\);

2. Short maturity \((T = 1)\) with barrier \((B = 2.0829)\) far from strike;

3. Short maturity \((T = 10)\) with barrier close to strike;

4. Short maturity \((T = 10)\) with barrier far from strike;

Figure 6.8 presents exposure profiles of these 4 cases. It conforms to the observations made in [8] about the exposure growth shape for the Barrier options. As observed previously, in case of European options the exposure \((PFE_{97.5\%})\) is acquired quickly just after the issue, but the growth slows down considerably as the option reaches maturity. Since for far enough barrier the options start to mimic their European counterparts, their exposure profiles tend to be similar as well. However, for sufficiently large maturities a substantial share of these options are terminated due stock reaching the barrier yielding a decrease in exposure.

The situation is completely different for the cases where barrier level is close. In those cases the price is low because of high probability of crossing the barrier, thus limiting the exposure. However, as the option approaches maturity the crossing of barrier either already happened or is unlikely to happen, so that the possibility of receiving positive payoff drives the price and, therefore, the exposure up. The combination of these properties provide a unique opportunity to tune exposure profiles as desired by shifting barrier level up or down.
Figure 6.8: Exposure profiles of at the money up and out call option under different models and several barrier, strike levels and maturity combinations.
The changes in dynamics by using different models are as follows:

- The differences between BS and its IR risk adjusted version are negligible;

- The adjustment for volatility smile yields a smaller probability of crossing the barrier which in turn yields to higher prices and exposure for the option with barrier close to spot. The impact is largest for the 1 year option;

- The low initial volatility results in lower barrier crossing probability in the shortest term, but for a larger barrier the probability of higher is lower thus drawing the price and exposure down;

- In the case of high barrier 10 year option the PFE 97.5% peaks later and decreases slower than compared to BS and BS2HW;

- The latter effect is in alignment with observation that volatility skew has a different impact of different maturity options which is not necessarily monotonic.

It can be deduced that up-and-out call option price is mostly driven by the probability of crossing the barrier and the role of the payoff becomes significant only if the probability of the cross is sufficiently small. This leads to the lower importance of the discount factor and interest rate component altogether, so that the inclusion of interest rate risk has only a mild effect on the exposures.

Recall that in the chosen parameters the skew was mildly negative. Distribution-wise this entails a "fat" left tail, but sharply decaying right one. This has a profound effect on the barrier options because of the significantly lower barrier crossing probability.

CVA

In this section the combination of the price together with exposure is assessed by looking at the resulting CVA. Appendix Figure D.1 displays the dependence between the CDS premium and resulting CVA. The conclusions are in alignment with the ones in the American put case. Short dated CVA increases close to linearly in terms of CDS premium and there are signs of slight non-linearity in the 10 year maturity option. The linearity might disappear when accounting for wrong-way risk, but the current setting excludes this possibility due computational limitations.

For a fixed model CVA of a barrier option is a function of CDS premium, maturity, barrier and strike levels. Having observed the close to linear relationship between CDS and CVA in this section CDS is restricted to have a premium of 2%. The resulting CVA is contained in Appendix Table D.5. The comparison of the models is presented in Table 6.9.
Examining figure 6.9 reveals that

- The exclusion of interest rate risk affects more long dated options and out-of-money options;
- The difference between BS and BS2HW in BS2HW Vegas is considerably higher for the CVA that it was for the prices;
- The relative differences between H2HW and BS2HW are smaller in absolute value than it was for prices;
- Skew adjusted CVA’s relative difference increases with time;
- Vega-wise BS difference is greatest for out of money options;
- Vega adjustment required for BS2HW to match price of H2HW is greatest for at-the-money options.

The exclusion of random element in the interest rate had a much smaller impact on the exposure profiles than the inclusion of stochasticity in the volatility. Still, addition of a short rate model affected the most longer maturity options. Contrary to the prices the CVA difference can reach substantial levels for long maturity options in both relative and Vega terms which is likely to be due the correlation structure between the exchange rate and IRs.

The addition of stochastic volatility impacts the CVAs of all the options studied. Similar to all the previous cases H2HW changes the dynamics of the short term exchange rate substantially shifting prices and CVAs. The price changes are largest for options with barrier close to the spot and for further out of money options.
### Figure 6.9: Up-and-out call option with maturity (T), strike (K) and barrier (B) CVA under H2HW, BS differences with BS2HW expressed in multiple metrics.

#### (a) Difference as % of BS2HW price.

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<th>Relative B / K</th>
<th>T = 1</th>
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<th>T = 5</th>
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<td>-0.8%</td>
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<td>-70%</td>
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<td>21%</td>
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</tr>
<tr>
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<td>-70%</td>
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<td>-4.9%</td>
<td>-3.4%</td>
<td>-0.6%</td>
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#### (b) Difference as multiple of 1% BS2HW Vega.

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<td>-11.61</td>
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<td>-0.09</td>
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<td>-0.86</td>
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<td>-0.09</td>
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<td>-0.09</td>
<td>-0.18</td>
<td>-0.41</td>
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</tbody>
</table>
Chapter 7

Discussion

Regulators and financial institutions are constantly working on researching and pricing all the risks included in the derivative contracts. The global crisis of 2008 shifted the focus on the assessment of the risks due to the counterparty’s default. Although collateral eliminates the need for any credit risk adjustment, it is hard to implement in the less liquid over-the-counter derivative markets. For these cases, financial engineers have to revert to the estimation of the credit value adjustment (CVA).

The research conducted during the writing of this thesis concentrated on applying the FDMC method first suggested by [7] to evaluate CVA. Monte Carlo is used to provide the distribution of the underlying asset, which can include various early termination conditions. In the next step the resulting underlying distribution coupled with the finite difference scheme’s ability to price an option for a whole range of spot prices allows for a robust, yet general exposure estimation algorithm. However, there are several caveats that one has to be careful about. It is observed that FD performance varies significantly per parameter set, hence some parameter dependent grids could be a topic for future research. Using more dense grids could be an option, but in the case of H2HW the dimensions of the equation systems become too large to solve on commodity hardware even for rather small sizes.

The object of study is the FX market because of their high derivative volume and a multitude of risks involved such as interest rate risk and volatility skew. The analysis is built up by comparing the regular Black-Scholes model with its extensions obtained by the adding of stochastic components in the interest rates and volatility. The impact is measured for American put options and up-and-out call options. The former is chosen due its high trading volume, whereas the latter is used as a proxy for more tail-sensitive options.

As expected it is observed that the interest rate risk contributed the most for long maturity options. The difference is larger for the American put option than the barrier one. Addition of stochastic volatility is mostly prominent in out of money options with shorter
maturities. In contrast to the stochastic IRs, stochastic volatility not only changed the distribution of the option value per fixed time point, but also had a significant impact on the dynamics of that distribution. The selection of initial and long term volatility allows to calibrate the rate of exposure acquisition over time.

The following decision tree could be used as a guide for pricing options in the current FX markets¹:

1. For short dated American in-the-money options ignoring skew and interest rate volatility has minor relative impact, so BS is sufficient;

2. For short dated American out-of-money options the exclusion of stochastic interest rates is allowed as long as the prices are corrected for volatility skew. Hence, Heston could be an option;

3. With longer maturities the impact of the skew on the American options diminishes, but that of stochastic interest rates increases, so that one should consider model no simpler than BS2HW;

4. In the close to zero interest rate world, the chances of crossing barriers are governed mostly by the volatility skew dynamics. Therefore, pricing of barrier options with Heston could be sufficient.

A separate note is required for the up-and-out call option. This class of options seems to be less sensitive to interest rate risk, but highly dependent on the volatility skew. In the case of at-the-money up-and-out call option the price can vary up to 40% of the BS price. Also, the addition of a barrier to an European payoff not only reduces the price, but also allows to fine-tune the exposure over time to fit specific needs. This way the barrier option becomes a very attractive addition to a portfolio to meet certain risk requirements.

The comparison of the American put and a barrier option reveals another important aspect of the exposure measurement. In case one is interested in evaluating the exposures of a large portfolio of assets case by case analysis is too labour intensive. Therefore, a single metric has to be chosen reflecting the whole profile. Our example shows that metrics like maximum exposure could highly overestimate the exposure. Therefore, a portfolio which is rather risk-less during most of its lifetime, but at a single point in time acquires a substantial amount of exposure would be flagged as very risky. This is of prime importance when barrier ( or other early exercise ) options are considered.

¹Note that the terms short dated and long dated are functions of model parameters. In the current context and parameter setting terms short dated refers to options with maturities of around one year and long dated to options with maturities as long as a decade.
Chapter 8

Bibliography


Appendices
Appendix A

Notes on the implementation of the PDE solver

A.1 Spatial discretization

A.1.1 One dimensional case

Suppose one is interested in solving the following time-dependent PDE:

\[
\frac{\partial u}{\partial t} = \sum_{i=0}^{N} f_i(x, u) \frac{\partial^i u}{\partial x^i},
\]

on a bounded interval \([x_{\text{min}}, x_{\text{max}}]\). Following the lines of Finite Difference method (FD) the \(m\) points \(x_i, i = 1, ..., m\) belonging to the interval are selected. Define \(\vec{x} = (x_1, ..., x_m)^T\) and \(\vec{u} = u(\vec{x})\) with \(x_{\text{min}} = x_1, x_i < x_j, i < j\) and \(x_{\text{max}} = x_m\). Then approximations of the spatial derivatives take the form

\[
\frac{\partial^i u}{\partial x^i}(\vec{x}) \approx A_{ix} \vec{u} + g_{ix}
\]

where \(A_{ix}\) and \(g_{ix}\) depend on the selections of FD schemes and boundary conditions. Hence the approximate ODEs become\(^1\)

\[
\frac{\partial u}{\partial t}(\vec{x}) = \sum_{i=0}^{N} ((f_i(\vec{x}, \vec{u}^i) \cdot A_{ix}) \vec{u} + \sum_{i=0}^{N} f_i(\vec{x}, \vec{u}) \cdot g_{ix} = A \vec{u} + g
\]

where \(\cdot\) defines the element-wise (Hadamard) product and \(i_m\) is the \(m\)-dimensional column unit vector. Time integration of this system of ODEs yields the solution.

\(^1\text{For notational simplicity, } f(\vec{x}) = (f(x_1), ..., f(x_m))^T \text{ and } f(\vec{x}, \vec{y}) = (f(x_1, y_1), ..., f(x_m, y_m))^T.\)
\section*{A.1.2 K-dimensional case}

Implementation of a PDE solver with several variables can become tedious and prone to errors. Here we will explain our approach which provides a simplistic structure. Suppose the PDE of interest is

\[
\frac{\partial u}{\partial t} = \sum_{k=1}^{K} f_k(u, x^{(1)}, \ldots, x^{(K)}) \frac{\partial u}{\partial x^{(k)}} + \sum_{k=1}^{K} \sum_{l=k}^{K} f_{k,l}(u, x^{(1)}, \ldots, x^{(K)}) \frac{\partial^2 u}{\partial x^{(k)} \partial x^{(l)}},
\]

on a hypercube \([x^{(1)}_{\min}, x^{(1)}_{\max}] \times \ldots \times [x^{(K)}_{\min}, x^{(K)}_{\max}]\). First discretize domain of each variable separately, yielding \(\vec{x}^{(1)}, \ldots, \vec{x}^{(K)}\) with \(\vec{x}^{(k)}\) having the dimension \(m_k\). Then define

\[
\vec{X}^{(k)} = i^{m_{k+1}} \otimes \vec{x}^{(k)} \otimes i^{m_k},
\]

where \(\otimes\) is the Kronecker product and \(i_m\) is the \(m\)-dimensional unit column vector. Take \(\vec{U} = u(\vec{X}^{(1)}, \ldots, \vec{X}^{(K)})\). Having this structure one can derive the non-mixed derivative approximations to be

\[
\frac{\partial u}{\partial x^{(k)}}(\vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) \approx I^{m_{k+1}} \otimes A_k \otimes I^{m_k} \vec{U} + i^{m_{k+1}} \otimes g_k \otimes i^{m_k},
\]

\[
\frac{\partial^2 u}{\partial (x^{(k)})^2}(\vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) \approx I^{m_{k+1}} \otimes A_{kk} \otimes I^{m_k} \vec{U} + i^{m_{k+1}} \otimes g_{kk} \otimes i^{m_k},
\]

where \(A_k, A_{kk}\) and \(g_k, g_{kk}\) are determined by the selected FD scheme and boundary conditions in the \(k\)-dimension. Then the mixed derivatives can be approximated applying the same logic twice i.e.

\[
\frac{\partial^2 u}{\partial x^{(k)} \partial x^{(l)}}(\vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) \approx A_k A_l \vec{U} + G_{kl} = A_{kl} \vec{U} + G_{kl},
\]

then the approximating system of ODEs becomes

\[
\frac{\partial u}{\partial t}(\vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) = \sum_{k=1}^{K} f_k(\vec{U}, \vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) \cdot (A_k \vec{U} + G_k)
\]

\[
+ \sum_{k=1}^{K} \sum_{l=k}^{K} f_{k,l}(\vec{U}, \vec{X}^{(1)}, \ldots, \vec{X}^{(K)}) \cdot (A_{kl} \vec{U} + G_{kl}).
\]

\footnote{We limit ourselves to 2nd order PDEs, but the idea can be generalized to higher order equations.}
Appendix B

From CDS premiums to hazard rates

Following the work done in [8] assume the following model:

\[ q(t) = e^{-\lambda_{haz} t}, \lambda_{haz} > 0, \]  

(B.1)

where \( q(t) \) is the survival probability and \( \lambda_{haz} \) is the hazard rate parameter setting the general likelihood of party’s default.

On the other hand, as shown in [1] the CDS premium \( S \) can be computed using the formula:

\[ S = \frac{(1 - R) \sum_{i=1}^{N} D(t_i)(q(t_{i-1}) - q(t_i))}{\sum_{i=1}^{N} D(t_i)q(t_i)\Delta t + \sum_{i=1}^{N} D(t_i)(q(t_{i-1}) - q(t_i))\frac{\Delta t}{2}}, \]  

(B.2)

where

- \( \Delta t \) - time between buyer’s payments;
- \( t_i = i\Delta t, i = 1,N \) - timepoints when CDS buyer pays to the insurer;
- \( R \in [0,1] \) - recovery rate given as proportion of capital recovered given counterparty’s default;
- \( D(\cdot) > 0 \) - discount curve.

Substitution of the default model (B.1) into CDS premium calculation formula (B.2) the following yields:

\[ S = \frac{(1 - R) \sum_{i=1}^{N} D(t_i)(e^{-\lambda_{haz} t_{i-1}} - e^{-\lambda_{haz} t_i})}{\sum_{i=1}^{N} D(t_i)e^{-\lambda_{haz} t_i}\Delta t + \sum_{i=1}^{N} D(t_i)(e^{-\lambda_{haz} t_{i-1}} - e^{-\lambda_{haz} t_i})\frac{\Delta t}{2}}. \]
Solving for $\lambda_{\text{haz}}$ yields

$$S = \frac{(1 - R) \sum_{i=1}^{N} D(t_i) e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}}}{\sum_{i=1}^{N} D(t_i) e^{-\lambda_{\text{haz}} t_i} \Delta t + \sum_{i=1}^{N} D(t_i) (e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}} \Delta t^2)}$$

$$\Leftrightarrow S \left( \sum_{i=1}^{N} D(t_i) e^{-\lambda_{\text{haz}} t_i} \Delta t + \sum_{i=1}^{N} D(t_i) (e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}} \Delta t^2) \right) = (1 - R) \sum_{i=1}^{N} D(t_i) (e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}})$$

$$\Leftrightarrow S \sum_{i=1}^{N} D(t_i) (e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}} \Delta t) + \frac{N}{2} \sum_{i=1}^{N} D(t_i) (e^{-\lambda_{\text{haz}} t_{i-1} - e^{-\lambda_{\text{haz}} t_i}}) = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} D(t_i) \left( e^{-\lambda_{\text{haz}} t_{i-1}} \left( \frac{S \Delta t}{2} + (R - 1) \right) + e^{-\lambda_{\text{haz}} t_{i-1}} \left( \frac{S \Delta t}{2} + (1 - R) \right) \right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} D(t_i) e^{-\lambda_{\text{haz}} t_{i-1}} \left( \left( \frac{S \Delta t}{2} + (R - 1) \right) + e^{-\lambda_{\text{haz}} \Delta t} \left( \frac{S \Delta t}{2} + (1 - R) \right) \right) = 0$$

$$\Leftrightarrow \left( \left( \frac{S \Delta t}{2} + (R - 1) \right) + e^{-\lambda_{\text{haz}} \Delta t} \left( \frac{S \Delta t}{2} + (1 - R) \right) \right) \sum_{i=1}^{N} D(t_i) e^{-\lambda_{\text{haz}} t_{i-1}} = 0.$$
real number. Therefore, if CDS premium satisfies

\[ 1 - R - S \frac{d}{2} > 0, \]

the equivalent hazard rate is equal to

\[ \lambda_{haz} = \frac{1}{\Delta t} \ln \left( \frac{1 - R + S \frac{\Delta t}{2}}{1 - R - S \frac{\Delta t}{2}} \right), \]

which completes the derivation.
Appendix C

Market data used in the study

Table C.1: At the money, 10 and 25 delta risk reversals and butterfly volatilities as measured at 11/04/14

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<th>25RR</th>
<th>10RR</th>
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<td>0.55</td>
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<td>0.57</td>
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<td>0.57</td>
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<td>6 months</td>
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<tr>
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<td>0.25</td>
<td>0.97</td>
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<tr>
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<td>-1.86</td>
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<td>0.97</td>
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<td>-2.06</td>
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<td>0.98</td>
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<td>-3.70</td>
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<td>1.34</td>
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Table C.2: Yield curve data for the American Dollar as measured at 11/04/14

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<th>Discount Factor</th>
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<td>0.9996</td>
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<td>11/07/14</td>
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<td>0.2301</td>
<td>0.9994</td>
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<tr>
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<td>0.2310</td>
<td>0.9992</td>
</tr>
<tr>
<td>16/09/14</td>
<td>157</td>
<td>0.2312</td>
<td>0.9990</td>
</tr>
<tr>
<td>15/10/14</td>
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<td>0.2317</td>
<td>0.9988</td>
</tr>
<tr>
<td>17/12/14</td>
<td>250</td>
<td>0.2376</td>
<td>0.9984</td>
</tr>
<tr>
<td>18/03/15</td>
<td>341</td>
<td>0.2496</td>
<td>0.9977</td>
</tr>
<tr>
<td>17/06/15</td>
<td>432</td>
<td>0.2752</td>
<td>0.9967</td>
</tr>
<tr>
<td>16/09/15</td>
<td>523</td>
<td>0.3196</td>
<td>0.9954</td>
</tr>
<tr>
<td>16/12/15</td>
<td>614</td>
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<tr>
<td>16/03/16</td>
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<td>18/04/17</td>
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<td>0.9728</td>
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<td>2.8471</td>
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<td>16/04/29</td>
<td>5484</td>
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<td>0.6054</td>
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Appendix D

Case study additional tables

D.1 American Put

D.1.1 Price

Table D.1: American put prices for multiple strikes and maturities under different models
Table D.2: American put option CVA under different models for several maturities and strikes

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(c) BS

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## D.1.2 CVA

### D.2 Up and out call

#### D.2.1 Price

Table D.3: Up and out call prices for multiple maturities ($T$), barrier levels ($B$), strikes ($K$) and asset models

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Table D.3: Up and out call prices for multiple maturities ($T$), barrier levels ($B$), strikes ($K$) and asset models
Table D.4: Up and out call price differences under with BS2HW

D.2.2 CVA

Figure D.1: The observed relation between CDS premium and resulting CVA for multiple up and out at the money call options of several maturities
Table D.5: Up and out call options for multiple barrier (B), strike (K) levels and three models for CDS of 2%