Description of pattern formation in semi-arid landscapes by weakly non-linear stability analysis

Marieke Jesse

Thesis
under supervision of dr. G.M. Hek

Universiteit van Amsterdam, Faculty of Science
Korteweg-de Vries Institute for Mathematics
Plantage Muidergracht 24, 1018 TV Amsterdam
The Netherlands

November 2006
## Contents

1 Introduction ........................................ 4  
   1.1 Introduction ........................................ 4  
   1.2 Model ........................................ 5  
   1.3 Dimensional Analysis and Scaling .................... 5  

2 Linear stability analysis ............................. 8  
   2.1 Homogeneous model ................................... 8  
   2.2 Model without diffusion of plant biomass .......... 10  
   2.3 Model with diffusion of plant biomass .............. 13  

3 Weakly non-linear stability analysis ............... 18  
   3.1 Perturbation of the equilibrium ..................... 18  
   3.2 Turing-Hopf matrix .................................. 21  
   3.3 Solve the unknowns of the form $X_{ij}$ and $Y_{ij}$ . 22  
   3.4 The Complex Ginzburg-Landau equation ............... 23  

4 Dynamics of the Ginzburg-Landau equation .......... 25  
   4.1 Scaling the Ginzburg-Landau equation ............... 25  
   4.2 Stability Analysis .................................. 27  
   4.3 Solutions ........................................ 29
Chapter 1

Introduction

1.1 Introduction

In semi-arid areas, which are defined as having a precipitation of 250-500 millimeter per year, vegetation cover can be patterned such that distinct bands of vegetation are visible on hillsides, whereas the vegetation is distributed as mosaics in flat areas. These patterns have been described from e.g. Africa, Australia and Mexico. Due to their scale it is difficult to see these patterns from the ground; hence they have only been discovered via aerial photography. The hills on which these patterns occur generally have slopes of approximately 0.25%. The vegetation is concentrated in bands of 100 to 250 meter in width and are separated by gaps, in which vegetation is absent or sparse. These gaps can have widths ranging from 200 meters to 1 kilometer. There is discussion on the exact cause for these banded patterns. However there is a broad consensus that the key factor is competition for water, Sherrat [7]. The hypothesis about how these patterns arise on hillsides can be described as follows.

The water is flowing downhill, it does not infiltrate the bare areas, to a region with vegetation where it is absorbed and supports plant growth. The water is exhausted at the downhillside of the vegetation stripe causing the next gap to occur and the band to be maintained uphill of the stripe. The soil just uphill of the vegetation band is moist and is able to support vegetation. As a result, bands are reported to gradually migrate uphill. The hypothesis has not been completely tested yet. Although Klausmeier [3] proposed a model from which pattern formation in semi-arid areas can be studied, this model has been studied with only linear stability analysis by Sherrat [7]. A non-linear stability analysis has not been applied yet, this will be done in this thesis.

In the following of this chapter I will use the techniques of non-dimensionalizing and scaling to simplify Klausmeiers model to the model I am going to work with. Linear stability analysis will then be applied in the second chapter. The stability of the equilibria will be determined in the case that a) only advection but no diffusion is present and b) in the case that both advection and diffusion are present. Using linear stability analysis, parameter values can be be determined for which pattern formation is possible. However, we do not know what these patterns will look like.

In Chapter 3 I apply a weakly non-linear stability analysis. With this analysis we will derive the Complex Ginzburg-Landau equation which describes the pattern by an amplitude
1.2 Model

The model proposed by Klausmeier [3] to describe plant growth on semi-arid hills is given by two partial differential equations:

\[ \frac{\partial U}{\partial T} = \underbrace{\text{plant growth}}_{k_1 U^2 W} - \underbrace{\text{plant loss}}_{k_2 U} + k_3 \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) \]  
\( (1.1a) \)

\[ \frac{\partial W}{\partial T} = k_4 - k_5 W - k_6 U^2 W + k_7 \frac{\partial W}{\partial X} \]  
\( (1.1b) \)

Here \( W(X, Y, T) \) represents the water and \( U(X, Y, T) \) the plant biomass in a two-dimensional area with dimensions \( X \) and \( Y \). The direction of \( X \) is taken in the direction of the slope with uphill being the positive direction and \( Y \) is taken perpendicular to the slope of the hill. Note that the model includes an advection and a diffusion term. The advection term describes the water flow along the hill and the diffusion term the spreading of the plant biomass. The time is given by \( T \) and the parameters \( k_1, \ldots, k_7 \) are constants, which are defined as follows

- \( k_1 \) is a factor representing the plant’s growth
- \( k_2 \) is the coefficient which describes the loss of plant biomass
- \( k_3 \) is the coefficient for the dispersal, which models the expansion of the population through seeds or spores
- \( k_4 \) models the rainfall, which is assumed to be constant and continuous
- \( k_5 \) defines the water lost due to evaporation
- \( k_6 \) describes the rate at which plants take up water
- \( k_7 \) is the speed at which the water flows downhill (i.e. in the negative \( X \) direction).

In this model a lot of parameters occur. In order to simplify the model a transformation is applied to the variables and parameters. Following Sherrat [7], we use the techniques of non-dimensionalizing and scaling to accomplish this.

1.3 Dimensional Analysis and Scaling

To start we apply a transformation on both the variables and the parameters to non-dimensionalize the model. This is done by writing each variable as a product of a scale
and a dimensionless variable. Take $U = [u] \tilde{u}$, where $[u]$ is the chosen scale and $\tilde{u}$ is the corresponding dimensionless variable. Write in similar way

$W = [w] \tilde{w}$

$T = [t] \tilde{t}$

$X = [x] \tilde{x}$

$Y = [y] \tilde{y}$

Now substitute these into the model (1.1). Then,

$$\frac{[w]}{[t]} \frac{\partial \tilde{w}}{\partial t} = k_4 - k_5 [w] \tilde{w} - k_6 [u]^2 \tilde{u}^2 [w] \tilde{w} + k_7 \frac{[w]}{[x]} \frac{\partial \tilde{w}}{\partial x}$$

$$\implies \frac{\partial \tilde{w}}{\partial t} = \frac{[t]}{[w]} k_4 - k_5 [t] \tilde{w} - k_6 [t] [u]^2 \tilde{u}^2 \tilde{w} + k_7 \frac{[t]}{[x]} \frac{\partial \tilde{w}}{\partial x}$$

(1.2)

and

$$\frac{\partial \tilde{u}}{\partial t} = [u] [t] k_1 \tilde{u}^2 \tilde{w} - k_2 [t] \tilde{u} + \frac{k_3 [t]}{[x]^2} \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{k_3 [t]}{[y]^2} \frac{\partial^2 \tilde{u}}{\partial y^2}$$

(1.3)

The model is now in a dimensionless form, but we still have to choose the values for the scales $[w]$, $[u]$, $[t]$, $[x]$ and $[y]$. This is done by scaling, which aims to reduce the number of parameters in the model. When the scales are cleverly chosen, some coefficients in (1.2) and (1.3) can become one.

Start with $k_5 [t]$, this scales to 1 if $[t] = \frac{1}{k_5}$. Notice that in (1.3) the parameter $k_2 [t]$ will not be equal to one, once $[t] = \frac{1}{k_5}$ is chosen. Now use the scale chosen for $[t]$ to determine the other scalings by imposing:

$$\frac{k_3}{k_5 [x]^2} = 1 \implies \frac{[x]}{\sqrt{k_3}} = \frac{\sqrt{k_5}}{\sqrt{k_6}}$$

$$\frac{k_3}{k_5 [y]^2} = 1 \implies \frac{[y]}{\sqrt{k_3}} = \frac{\sqrt{k_5}}{\sqrt{k_6}}$$

$$\frac{k_6 [u]^2}{k_5} = 1 \implies \frac{[u]}{\sqrt{k_5}} = \frac{\sqrt{k_5}}{\sqrt{k_6}}$$

and finally

$$\frac{k_1 \sqrt{k_5}}{k_5 \sqrt{k_6}} [w] = 1 \implies \frac{[w]}{\sqrt{k_5 k_6}} = \frac{\sqrt{k_5 k_6}}{k_1}.$$  

We have now determined the following scalings:

$$\tilde{u} = \frac{\sqrt{k_6}}{\sqrt{k_5}} U,$$

$$\tilde{w} = \frac{k_1}{\sqrt{k_5 k_6}} W,$$

$$\tilde{t} = k_5 T,$$

$$\tilde{x} = \frac{\sqrt{k_3}}{\sqrt{k_5}} X,$$

$$\tilde{y} = \frac{\sqrt{k_3}}{\sqrt{k_5}} Y.$$
1.3 Dimensional Analysis and Scaling

With the appropriate scaling, model (1.1) is thus simplified to

\[ \frac{\partial \tilde{u}}{\partial \tilde{t}} = \tilde{w} \tilde{u}^2 - \frac{k_2}{k_5} \tilde{u} + \frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \]
\[ \frac{\partial \tilde{w}}{\partial \tilde{t}} = \frac{k_1 k_4}{k_5 \sqrt{k_3 k_6}} - \tilde{w} - \tilde{w} \tilde{u}^2 + \frac{k_7}{\sqrt{k_3 k_5}} \frac{\partial \tilde{w}}{\partial \tilde{x}} \]

Drop the tildes and define \( A = \frac{k_1 k_4}{k_5 \sqrt{k_3 k_6}} \), \( B = \frac{k_2}{k_5} \) and \( \nu = \frac{k_7}{\sqrt{k_3 k_5}} \):

\[ \frac{\partial u}{\partial t} = u^2 w - B u + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \] \hfill (1.5a)
\[ \frac{\partial w}{\partial t} = A - w - u^2 w + \nu \frac{\partial w}{\partial x} \] \hfill (1.5b)

In this model (1.5) only 3 parameters remain:

- \( A \), which controls the water input
- \( B \), which measures plant loss
- \( \nu \), which controls the rate at which water flows downhill.

From here on we will focus on this simplified model (1.5), with \( t \in \mathbb{R}_{\geq 0}, (x, y) \in \mathbb{R}^2 \) and \( u, w : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \). In order to restrict the analysis to biologically meaningful parameter values, we impose \( A, B \geq 0 \). For \( \nu \) we impose \( \nu > 0 \), as \( \nu = 0 \) indicates a flat surface. Klausmeier [3] already showed that for this specific case, pattern formation is impossible to occur for biological meaningful parameters.
Chapter 2

Linear stability analysis

In this chapter we are going to analyze the linear stability of the model (1.5). With this analysis we will find that equilibria in this model are either linearly stable or linearly unstable. Since we are interested in pattern formation, we will look at the equilibrium from which a pattern can grow. This equilibrium has to be stable without the advection and/or diffusion term, but it has to become unstable to small perturbations when (one of) these terms are present.

First the stability of the homogeneous model will be analyzed to find a stable equilibrium. Then the stability of the equilibrium will be studied when a) only advection but no diffusion is present and b) both advection and diffusion are present.

2.1 Homogeneous model

So the first step to analyze this model is to look at the spatially homogeneous equilibria. This means that the spatial derivatives are equal to zero:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= w u^2 - B u \\
\frac{\partial w}{\partial t} &= A - w - w u^2
\end{align*}
\]

This model will first be analyzed for arbitrary \( A \) and \( B \), afterwards we will restrict ourselves to the condition \( A, B \geq 0 \).

For the model (2.1) we have to compute the equilibria and determine their stability. The equilibria \((u, w)\) are solutions of

\[
\begin{align*}
w u^2 - B u &= 0 \\
A - w - w u^2 &= 0
\end{align*}
\]

Determine the roots of the first equation:

\[
\frac{\partial u}{\partial t} = 0 \iff (w u - B) = 0 \iff u = 0 \text{ or } u = \frac{B}{w}
\]

and use the values just found for \( u \) to determine the roots of the second equation:

\[
\frac{\partial w}{\partial t} = 0 \text{ and } u = 0 \implies w = A
\]
2.1 Homogeneous model

Figure 2.1: Bifurcation diagram for the homogeneous equilibria. The solid curves denotes a stable equilibrium, the dashed one an unstable equilibrium.

\[ \frac{\partial w}{\partial t} = 0 \quad \text{and} \quad u = \frac{B}{w} \implies w = \frac{1}{2} (A \pm \sqrt{A^2 - 4B^2}) \]

Thus there are three equilibria:

- \((u_1, w_1) = (0, A)\)
- \((u_2, w_2) = \left( \frac{2B}{A - \sqrt{A^2 - 4B^2}}, \frac{1}{2}(A - \sqrt{A^2 - 4B^2}) \right)\)
- \((u_3, w_3) = \left( \frac{2B}{A + \sqrt{A^2 - 4B^2}}, \frac{1}{2}(A + \sqrt{A^2 - 4B^2}) \right)\)

In the first equilibrium, no plants are present. The last two equilibria are equal if \(A = \pm 2B\). When \(A < 2B\), the equilibria \((u_2, w_2)\) and \((u_3, w_3)\) are complex and since there is no biological meaning for such values these equilibria will be ignored.

The linear stability of these three points can be determined by calculating the eigenvalues of the Jacobi matrix in these points. The Jacobi matrix for this model is given by

\[ J = \begin{pmatrix} 2uw - B & u^2 \\ -2uw & -1 - u^2 \end{pmatrix} \quad (2.2) \]

The eigenvalues of the Jacobi matrix in the equilibrium \((u_1, w_1)\) are \(\lambda_1 = -1\) and \(\lambda_2 = -B\). This equilibrium is a stable node for \(B > 0\), otherwise it is unstable.

When \(A = \pm 2B\) one of the eigenvalues of the Jacobi matrix in the equilibria \((u_2, w_2)\) and \((u_3, w_3)\) equals zero. The equilibrium \((u_2, w_2)\) has two negative eigenvalues and is thus stable for \(A > 2B\). The equilibrium \((u_3, w_3)\) is unstable for \(A > 2B\), since it has one positive and one negative eigenvalue.

So, summarizing, if

\begin{align*}
A < 2B & : \quad 1 \text{ equilibrium, stable} \\
A = 2B & : \quad 2 \text{ equilibria, one stable and one nonhyperbolic} \\
A > 2B & : \quad 3 \text{ equilibria, two stable and one unstable}
\end{align*}

We are looking for a stable equilibrium in the homogeneous state and \((u_2, w_2) = \left( \frac{2B}{A - \sqrt{A^2 - 4B^2}}, \frac{1}{2}(A - \sqrt{A^2 - 4B^2}) \right)\) satisfies this condition under the biologically relevant conditions \(A, B \geq 0\) and \(A \geq 2B\).
Linear stability analysis

In Figure 2.1 a bifurcation diagram of this model is shown. As is visible, a transcritical bifurcation occurs for $B = 0$. The equilibrium $(u_1, w_1)$ is unstable for $B < 0$. At the origin there is a bifurcation point at which the two equilibria $(u_1, w_1)$ and $(u_3, w_3)$ coincide, this is a nonhyperbolic equilibrium. For $B > 0$, the equilibrium $(u_1, w_1)$ now becomes stable and $(u_3, w_3)$ becomes unstable. At the points $B = \pm \frac{1}{2}A$, a fold bifurcation occurs, the equilibria $(u_2, w_2)$ and $(u_3, w_3)$ collide and disappear.

2.2 Model without diffusion of plant biomass

In the absence of diffusion the model reads

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u^2 w - Bu \\
\frac{\partial w}{\partial t} &= A - w - u^2 w + \nu \frac{\partial w}{\partial x}
\end{align*}
\] (2.3a, 2.3b)

From now on we will only consider positive parameter values for $A$ and $B$.

In the previous section the stable equilibrium in the absence of spatial variation was found. For $A, B > 0$ and $A \geq 2B$, this stable equilibrium is $(u_s, w_s)$, where

\[
\begin{align*}
u
u u_s &= \frac{2B}{A - \sqrt{A^2 - 4B^2}} \\
\nu w_s &= \frac{1}{2}(A - \sqrt{A^2 - 4B^2}).
\end{align*}
\]

We shall investigate the stability of this equilibrium with the advection term, but without diffusion. Therefore the model (2.3) will be linearized around the steady state $(u_s, w_s)$, because this allows us to determine the local stability of this equilibrium.

Write $\tilde{u} = u - u_s$ and $\tilde{w} = w - w_s$ and substitute this into model (2.3). Then the linear terms of (2.3) give the new system

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} &= a \tilde{u} + b \tilde{w} \\
\frac{\partial \tilde{w}}{\partial t} &= c \tilde{u} + d \tilde{w} + \nu \frac{\partial \tilde{w}}{\partial x}
\end{align*}
\] (2.4a, 2.4b)

where the coefficients are given by

\[
\begin{align*}
u a &= B \\
\nu b &= \frac{4B^2}{(A - \sqrt{A^2 - 4B^2})^2} \\
\nu c &= -2B \\
\nu d &= \frac{-2A}{A - \sqrt{A^2 - 4B^2}}
\end{align*}
\]

(2.5a, 2.5b, 2.5c, 2.5d)

To determine the linear stability of the homogeneous equilibrium $(u_s, w_s)$ under inhomogeneous perturbations, we look for solutions proportional to $e^{ikx + \lambda t}$, so set

\[
\begin{pmatrix}
\tilde{u} \\
\tilde{w}
\end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} e^{ikx + \lambda t},
\] (2.6)
where \( k \in \mathbb{R} \) is the wavenumber, \( \lambda \in \mathbb{C} \) is the growth rate and \( f, g \in \mathbb{C} \). The wavenumber \( k \) represents the \( x \) dependence of the perturbation at \((u_s, w_s)\). The wavenumber is associated to the wavelength \( \sigma \) by \( \sigma = \frac{2\pi}{k} \).

Using (2.6), calculate the expressions
\[
\frac{\partial \tilde{u}}{\partial t} = f \lambda e^{ikx + \lambda t} \\
\frac{\partial \tilde{w}}{\partial t} = g \lambda e^{ikx + \lambda t} \\
\frac{\partial \tilde{w}}{\partial x} = g i k e^{ikx + \lambda t}
\]
and substitute these into the linearized system (2.4):
\[
f \lambda e^{ikx + \lambda t} = (af + bg)e^{ikx + \lambda t} \\
g \lambda e^{ikx + \lambda t} = (cf + dg + i k g \nu)e^{ikx + \lambda t}
\]
which equals
\[
\begin{pmatrix} f \\ g \end{pmatrix} \lambda e^{ikx + \lambda t} = \begin{pmatrix} a & b \\ c & d + ik \nu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} e^{ikx + \lambda t}.
\]
Dividing by \( e^{ikx + \lambda t} \) results in the eigenvalue problem
\[
\begin{pmatrix} f \\ g \end{pmatrix} \lambda = \begin{pmatrix} a & b \\ c & d + ik \nu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.7}
\]
with characteristic polynomial
\[
\lambda^2 + \lambda(-a - i k \nu - d) + a(i k \nu + d) - bc = 0. \tag{2.8}
\]
The equilibrium is stable when the real parts of both eigenvalues are negative and becomes unstable when one of the real parts is positive.

The characteristic polynomial (2.8) gives the eigenvalues
\[
\lambda_{1,2} = \frac{1}{2} \left[ (a + d + ik \nu) \pm \sqrt{\alpha + i \beta} \right] \tag{2.9}
\]
where the eigenvalues \( \lambda_{1,2} \) are functions of the wavenumber \( k \) and \( \alpha = (a + d)^2 - k^2 \nu^2 - 4(ad - bc) \) and \( \beta = 2k \nu(d - a) \).

From Calculus we know that
\[
\sqrt{\alpha + i \beta} = \pm \sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2} + \alpha)} + \text{sign}(\beta)i \sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2} - \alpha)},
\]
where \( \text{sign}(\beta) = 1 \) when \( \beta \geq 0 \) and \( \text{sign}(\beta) = -1 \) when \( \beta < 0 \), so we can write the real and imaginary part of \( \lambda \) as follows:
\[
\text{Re}(\lambda) = \frac{1}{2} \left[ (a + d) \pm \sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2} + \alpha)} \right] \tag{2.10}
\]
\[
\text{Im}(\lambda) = \frac{1}{2} \left[ k \nu \pm \text{sign}(\beta)\sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2} - \alpha)} \right] \tag{2.11}
\]
Now we have to find the real part of that eigenvalue that is likely to become positive for some \( k \). In the absence of advection, the equilibrium is stable, so \( \text{Re}(\lambda_{1,2}) < 0 \). This is equivalent to the following two conditions

\[
 a + d < 0 \quad \text{and} \quad ad - bc > 0.
\]

In words: if the trace, which is the sum of the two eigenvalues, is negative and the determinant, which is the product of the two eigenvalues, is positive, then the real parts of both eigenvalues are negative. Since \( a + d < 0 \), \( \lambda_1(k) \) with

\[
 \text{Re}(\lambda_1) = \frac{1}{2} \left[ (a + d) + \sqrt{\frac{1}{2}(\alpha^2 + \beta^2 + \alpha)} \right]
\]

is the eigenvalue with real part that can become positive for some \( k \). The other eigenvalue \( \lambda_2(k) \) stays negative for all \( k \). We are interested in that \( k \) for which \( \text{Re}(\lambda_1(k)) = 0 \), because when \( \text{Re}(\lambda_1(k)) > 0 \) the equilibrium becomes unstable and a pattern can grow.

As shown in Figure 2.2 for different values of the velocity \( \nu \) and given \( A \) and \( B \), there is a different wavenumber \( k \) such that \( \text{Re}(\lambda_1(k)) = 0 \). This wavenumber is called the critical wavenumber and denoted by \( k_c \). In fact, such critical wavenumber \( k_c \) exists for any \( \nu > 0 \).

So if \( |k| > |k_c| \) the homogeneous equilibrium \( (u_s, w_s) \) becomes unstable, which means that without diffusion the solution \( (u_s, w_s) \) is always unstable under perturbations with large wavenumber \( |k| \). The occurring patterns are a result of multiple waves with wavenumber \( k > k_c \) and will therefore apparently look chaotic.

To find the critical wavenumber, we write \( \lambda = \mu + i \omega \) and substitute this into the characteristic polynomial (2.8), this results in

\[
 \mu^2 - \omega^2 + 2i \mu \omega + (\mu + i \omega)(-a - i k \nu - d) + a(i k \nu + d) - bc = 0. \quad (2.12)
\]

Set \( \mu = 0 \), since we want to have \( \text{Re}(\lambda(k_c)) = 0 \). Then there are two equations to solve, one for the imaginary part of the characteristic polynomial and one for the real part:

\[
 -\omega^2 - k \nu \omega + ad - bc = 0
\]

\[
 a k \nu - d \omega - a \omega = 0
\]
From the second equation we derive the imaginary part of the eigenvalue $\omega = \frac{ak\nu}{\nu^2}$. With $\omega$ we can solve the first equation for $k$ and find that the critical wavenumber $k_c$ is given by

$$k_c = \pm \left( \frac{a + d}{\nu} \sqrt{\frac{bc - ad}{ad}} \right). \quad (2.13)$$

This expression (2.13) corresponds to Figure 2.2, the critical wavenumber $k_c$ is large for small $\nu$.

When $\nu$ is small, the water flows downhill more slowly as a result of a smaller slope of the hill, resulting in patterns caused by small wavelengths. A possible biological explanation for this is that the water is absorbed faster into the soil as compared to a steeper hill and thus allows plant growth in more places.

### 2.3 Model with diffusion of plant biomass

Now we will look at the stability of the homogeneous equilibrium in the presence of both advection and diffusion in model (1.5). For simplicity only the $x$-direction is taken into consideration, this means that we set $u_{yy} = 0$. The biological meaning of this choice is, that we only consider patterns that are homogeneous in the $y$-direction. The model is thus given by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u^2 w - Bu + \frac{\partial^2 u}{\partial x^2} \quad (2.14a) \\
\frac{\partial w}{\partial t} &= A - w - u^2 w + \nu \frac{\partial w}{\partial x} \quad (2.14b)
\end{align*}
\]

As in the previous section, first we linearize around the equilibrium $(u_s, w_s)$ and then look for a solution proportional to $e^{i k x + \lambda t}$. The whole procedure is the same, except that now there is the extra term $\frac{\partial^2 u}{\partial x^2}$. Which means that the eigenvalue problem comes down to

\[
\begin{pmatrix}
 f \\
 g
\end{pmatrix}
\lambda
\begin{pmatrix}
 f \\
 g
\end{pmatrix}
\]

(2.15)

Notice that when the $y$-direction was also taken into account, a second wavenumber associated to $y$ would be involved. Then the solution would be proportional to $e^{i(kx + ly) + \lambda t}$.

The characteristic polynomial of (2.15) is

$$\lambda^2 + \lambda(k^2 - a - i k \nu - d) + (a - k^2)(i k \nu + d) - bc = 0 \quad (2.16)$$

and we can write the real and imaginary part of $\lambda$ as follows:

\[
\begin{align*}
\text{Re}(\lambda) &= \frac{1}{2} \left[ (a + d - k^2) \pm \sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2 + \alpha})} \right] \quad (2.17) \\
\text{Im}(\lambda) &= \frac{1}{2} \left[ k \nu \pm \text{sign}(\beta) \sqrt{\frac{1}{2}(\sqrt{\alpha^2 + \beta^2 - \alpha})} \right] \quad (2.18)
\end{align*}
\]

where now

\[
\begin{align*}
\alpha &= (a + d - k^2)^2 - k^2 \nu^2 - 4(ad - bc - dk^2) \quad (2.19a) \\
\beta &= 2k \nu(k^2 - a + d) \quad (2.19b)
\end{align*}
\]
Linear stability analysis

\[ \alpha = (a + d - k^2)^2 - k^2\nu^2 - 4(ad - bc - dk^2) \quad \text{and} \quad \beta = 2k\nu(k^2 - a + d). \]

In the previous section, we have seen that the homogeneous equilibrium \((u_s, w_s)\) is unstable under perturbations with a wavenumber bigger than \(|k_c|\) in the absence of diffusion. We will show below, that when diffusion is present, there is a critical velocity \(\nu_c\) such that for any \(\nu < \nu_c\) the homogeneous equilibrium is always stable, see Figure 2.3(c). As soon as \(\nu > \nu_c\) the homogeneous equilibrium \((u_s, w_s)\) is unstable under perturbations with wavenumbers around \(k_c\). Here the occurring patterns are a result of waves with only a small band of possible wavenumbers around \(k_c\).

This value \(k_c\) is determined as follows: when we draw a graph of \(\Re(\lambda)\) versus \(k\), we want this graph to have a local maximum \(\Re(\lambda) = 0\) for \(\nu = \nu_c\) at some point \(k\), which we will call \(k_c\), as is shown in Figure 2.3(b).

Clearly at \(\Re(\lambda) = 0\) a bifurcation occurs. This bifurcation is called pure Hopf when \(\lambda_{1,2} = \pm i\omega\), so their real parts equal zero, at \(k = 0\). A pure Turing bifurcation takes place when \(\lambda = 0\) and \(\frac{\partial \lambda}{\partial k} = 0\) at \(k_c \neq 0\).

In our model (2.14) a combination of both occurs: a Turing-Hopf bifurcation. This kind of bifurcation occurs when \(\Re(\lambda) = 0\), \(\Im(\lambda) \neq 0\) and \(\frac{\partial \Re(\lambda)}{\partial k} = 0\) at \(k_c \neq 0\).

So when diffusion is involved, two conditions have to be satisfied for a Turing-Hopf bifurcation:

\[ \Re(\lambda) = 0 \quad \text{and} \quad \frac{\partial \Re(\lambda)}{\partial k} = 0 \quad (2.20) \]

These conditions give a critical wavenumber \(k_c\) and also a critical velocity \(\nu_c\). For \(\nu > \nu_c\) the solution will grow exponentially in time.

To determine \(k_c\) and \(\nu_c\), write \(\lambda = \mu + i\omega\) and substitute this into the characteristic polynomial (2.16),

\[ \mu^2 - \omega^2 + 2i\mu\omega + (\mu + i\omega)(k^2 - a - i\nu - d) + (a - k^2)(i\nu + d) - bc = 0 \quad (2.21) \]

the conditions (2.20) now correspond with \(\mu = 0\) and \(\frac{\partial \mu}{\partial k} = 0\). The characteristic polynomial (2.21) can be split in a real and an imaginary part

\[ \begin{align*}
\mu^2 + \mu(k_2 - a - d) - bc + a d - dk^2 + k\nu\omega - \omega^2 &= 0 \quad (2.22) \\
2\mu\omega - k\nu\mu + a\nu - k^3\nu - a\omega - d\omega + k^2\omega &= 0 \quad (2.23)
\end{align*} \]
The derivative of (2.22) together with the condition $\mu = \frac{\partial \mu}{\partial k} = 0$ gives

$$-2d k + \nu \omega + k \nu v - 2 \omega v = 0$$  \hspace{1cm} (2.24)

where $v = \frac{\partial \omega}{\partial k}$. From (2.23) we can solve the imaginary part of the eigenvalue, $\omega$, using the conditions (2.20), which results in $\omega = \frac{ak\nu - k^3v}{a + d - k^2}$.

Now solve (2.24) to determine $\nu$, this results in

$$\nu = \sqrt{\frac{(a + d - k^2)^3}{a^2 - 2dk^2 + a(d - k^2)}}$$  \hspace{1cm} (2.25)

Substitute this value for $\nu$ in (2.22), use the condition $\mu = 0$ and solve for $k$:

$$k^2 = d(i\sqrt{3} - 1) \psi_1 + (2ad - 2d^2) \psi_2 + (-1 - i\sqrt{3}) \psi_2^2$$  \hspace{1cm} (2.26)

where

$$\psi_1 = -3abc + 4a^2d - 6bcd + 4ad^2 + d^3$$

$$\psi_2 = \left[\sqrt{(d^3 - \psi_1^2 + d\phi^2 - d^2 \phi)}\right]^\frac{1}{3}$$

with

$$\phi = 8a^3d - 9bcd^2 + d^4 + a^2(-9bc + 12d^2) + a(-9bcd + 6d^3).$$

Note that although $k$ looks complex, $k \in \mathbb{R}$.

For the model (2.14) we have thus found a critical velocity $\nu_c$, given by (2.25), and a critical wavenumber $k_c$, given by (2.26), at which the homogeneous equilibrium becomes unstable and therefore any perturbation with wavenumber $k$ close to $k_c$ will grow exponentially in time for $\nu > \nu_c$ according to the linear analysis. In the next chapter we will apply a weakly non-linear analysis to see if we are able to draw some conclusions about the real behaviour of these perturbations that grow, since in reality they probably will not keep growing exponentially.

Since the expression for $k_c$ is rather involved, we use the values of the parameters given by Klausmeier [3] to determine some $k_c$ and $\nu_c$ explicitly. Klausmeier gives estimates for two different types of vegetation, grass and trees. The estimates by Klausmeier for trees and grass are $A = 0.077$ up to 0.23 with $B = 0.045$ and $A = 0.94$ up to 2.81 with $B = 0.45$, respectively. Klausmeier also estimated a value for $\nu$, which is for a typical hill with vegetation stripes 182.5.

As shown in Figure 2.4(a), the critical velocity $\nu_c$ is an increasing function in $A$. Therefore in Table 2.3 the critical wavenumber and critical velocity for fixed $B$ are calculated for the lowest and highest possible value for $A$ determined by Klausmeier. However, since $A > 2B$ we use $A = 0.091$ instead of $A = 0.077$ as lowest value for $A$ at $B = 0.045$. This gives that for trees $\nu_c$ varies from 16 up to 277 and for grass from 6 up to 133. For the value $\nu = 182.5$ taken by Sherrat [7] the equilibrium is always unstable to perturbations with wavenumber around $k_c$ for grass and for trees it can be either stable or unstable. Therefore patterns can
Figure 2.4: The variation in velocity for different values of $A$ and $B$, with $A \geq 2B$. The dashed line is the approximation of Sherrat [7].

![Graph showing variation in velocity](image)

Table 2.1: Table of values for $\nu_c$ and $k_c$ for different $A$ and $B$

<table>
<thead>
<tr>
<th></th>
<th>$B = 0.045$</th>
<th>$B = 0.45$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.091</td>
<td>0.94</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>18.88</td>
<td>6.23</td>
</tr>
<tr>
<td>$k_c$</td>
<td>$\pm 0.11$</td>
<td>$\pm 0.36$</td>
</tr>
</tbody>
</table>

always occur on a typical hill with $\nu = 182.5$ when the vegetation is grass, but for trees it depends on the amount of precipitation.

The graph in Figure 2.4 shows the critical velocity against the rainfall $A$ and the plant loss $B$: the critical velocity increases with $A$ and decreases with $B$. This makes intuitive sense, since the water will flow faster downhill when there is more rain and it will flow slower downhill when there is a lot of vegetation which holds the water. In the same graph the approximation of Sherrat [7] is drawn for his values for $\nu = \nu_c$ and $k = k_c$. Sherrat estimated the critical velocity $\nu_c = 2\sqrt{-bc(a-d)}a$ by expanding the eigenvalues as a powerseries in $1/\sqrt{\nu}$, since $\nu$ is, compared to the other parameter values $a, b, c$ and $d$, much larger and he assumed that the wavenumber $k$ is of order $1/\sqrt{\nu}$.

In Figure 2.5 the critical wavelength is drawn as a function of $A$ respectively $B$, while the other parameter is fixed. The dashed line is again the approximation of Sherrat [7], with $k_c = (-bc(a-d))^{1/2}/\sqrt{\nu_c}$. According to the exact solution, the critical wavenumber $k_c$ becomes zero for $A = BA$. Clearly, Sherrat’s approximation fails when $B$ approaches $2A$. For varying $A$ it is not possible to draw the graph with Sherrat’s approximation, since the critical wavenumber $k_c$ then only depends on $B$. In his article Sherrat draws a graph of the wavelength for fixed $\nu = 182.5$, thus where the equilibrium is unstable. The wavelength then increases with $B$ and decreases with $A$. 
Figure 2.5: The variation in wavelength, defined as $\frac{2\pi}{k_c}$, for different values of $A$ and $B$, with $A \geq 2B$. When $B \rightarrow 2A$ the wavelength goes to infinity due to the fact that for $B = 2A$ the critical wavenumber $k_c$ equals zero. The dashed line is the approximation of Sherrat [7].
Chapter 3

Weakly non-linear stability analysis

The linear stability analysis showed that the homogeneous equilibrium of model (2.14) is stable as long as \( \nu < \nu_c \), where \( \nu_c = \nu_c(A, B) \). As soon as \( \nu > \nu_c \), it becomes linearly unstable against perturbations with wavenumbers close to \( k = \pm k_c \).

Of course we would like to take a closer look at the pattern that can occur near this bifurcation point. However, linear stability analysis does not suffice to determine the longterm behaviour of these patterns. Therefore we are going to apply a weakly non-linear stability analysis, in order to describe the patterns by amplitude functions and to determine if these perturbations grow or decay due to the effects of the non-linear terms. Since it is a weakly non-linear stability analysis, we can only describe what happens if perturbations close to the bifurcation values occur. We take \( \nu \) as the main bifurcation parameter, so we assume \( A, B \) fixed.

3.1 Perturbation of the equilibrium

As is common in weakly non-linear stability analysis, we take \( \nu \) close to \( \nu_c \); \( \nu = \nu_c + r \epsilon^2 \), where \( 0 < \epsilon << 1 \), since the analysis is applied close to the bifurcation. The parameter \( r \) determines whether we are, according to the linear theory, in the stable \((r < 0)\) or unstable \((r > 0)\) area.

Assume \((u, w)\) is a solution of (2.14) that is a small perturbation of \((u_s, w_s)\) written as

\[
\begin{pmatrix}
u(x,t) \\
w(x,t)
\end{pmatrix} = \begin{pmatrix}
u_s \\
w_s
\end{pmatrix} + \epsilon \begin{pmatrix}
A_1(\zeta, \tau) \\
A_2(\zeta, \tau)
\end{pmatrix} e^{ik_c x + \omega t} + \text{h.o.t. + c.c.},
\]

where h.o.t. means higher order terms and c.c. complex conjugated. Here \( A_1(\zeta, \tau), A_2(\zeta, \tau) : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \), describe the amplitude of the periodic function \( e^{ik_c x + \omega t} \) and \( \omega \) is the imaginary part of \( \lambda_1 \) at criticality, which we find after evaluating \( \lambda_1 \) in \( k_c \) and \( \nu_\text{c}, \text{i.e.} \; \omega = \frac{a k_c \nu_c - b \nu_c^2}{a + d - k_c^2} \).

We will denote this \( \omega_c \) and this is the critical frequency. The perturbation is chosen to have \( e^{ik_c x + \omega_c t} \) as basic form, since according to our linear analysis this is the most unstable wave for \( \nu \) near \( \nu_c \). Since the eigenvalue \( \lambda_1 \) is complex, the periodic function \( e^{ik_c x + \omega_c t} \) is time-dependent, which means that the wave described by (3.1) is a nonstationary travelling wave, Doelman [2]. Therefore there is a group velocity \( v \) which determines the velocity of the amplitude oscillations and is given by \( v = \frac{\partial \omega}{\partial k} \) evaluated in \( \omega_c \).
For \( \nu = \nu_c + r \varepsilon^2 \), with \( r > 0 \), there is an interval with center \( k_c \) for which \( \text{Re}(\lambda(k, \nu)) > 0 \). The width of this interval is described by Doelman [2] to be of order \( \varepsilon \). Doelman [2] shows that the linear theory predicts the amplitude of the critical wave \( e^{i(k_c x + \omega t)} \) to be a function of \( \varepsilon(x - vt) \) and \( \varepsilon^2 t \), therefore we assume that the non-linear amplitude \( A_{1,2}(\zeta, \tau) \) is of the same spatial and temporal scale. This leads to the two new variables already introduced

\[
\zeta = \varepsilon (x - vt) \quad \text{and} \quad \tau = \varepsilon^2 t
\]  

(3.2)

For the particular \((u, w)\) of (3.1), we calculate the expressions \( \frac{\partial u}{\partial t}, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 u}{\partial x^2} \) and \( u^2 w \) of (2.14) using (3.1):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= i \varepsilon A_1 \omega_c e^{i(k_c x + \omega t)} - \nu \varepsilon^2 \frac{\partial A_1}{\partial \zeta} e^{i(k_c x + \omega t)} + \varepsilon^3 \frac{\partial A_1}{\partial \tau} e^{i(k_c x + \omega t)} + c.c. \quad (3.3a) \\
\frac{\partial w}{\partial t} &= i \varepsilon A_2 \omega_c e^{i(k_c x + \omega t)} - \nu \varepsilon^2 \frac{\partial A_2}{\partial \zeta} e^{i(k_c x + \omega t)} + \varepsilon^3 \frac{\partial A_2}{\partial \tau} e^{i(k_c x + \omega t)} + c.c. \quad (3.3b) \\
\frac{\partial w}{\partial x} &= i \varepsilon k_c A_2 e^{i(k_c x + \omega t)} + \varepsilon^2 \frac{\partial A_2}{\partial \zeta} e^{i(k_c x + \omega t)}x + c.c. \quad (3.3c) \\
\frac{\partial^2 u}{\partial x^2} &= -\varepsilon^2 k_c^2 A_1 e^{i(k_c x + \omega t)} + 2i \varepsilon^2 k_c \frac{\partial A_1}{\partial \zeta} e^{i(k_c x + \omega t)} + \varepsilon^3 \frac{\partial^2 A_1}{\partial \zeta^2} e^{i(k_c x + \omega t)} + c.c. \quad (3.3d) \\
u^2 w &= u_s^2 w_s + \varepsilon^2 w_s A_2^2 e^{2i(k_c x + \omega t)} + 2\nu \varepsilon w_s A_1 e^{i(k_c x + \omega t)} + \varepsilon u_s^2 A_2 e^{i(k_c x + \omega t)} + \varepsilon^3 A_2^2 e^{3i(k_c x + \omega t)} + 2\varepsilon^2 u_s A_1 e^{2i(k_c x + \omega t)} + c.c. \quad (3.3e)
\end{align*}
\]

As we see higher order perturbative terms of the structure \( \varepsilon e^{i(k_c x + \omega t)}, \varepsilon^2 e^{i(k_c x + \omega t)}, \varepsilon^3 e^{i(k_c x + \omega t)}, \varepsilon^2 e^{2i(k_c x + \omega t)} \) and \( \varepsilon^3 e^{3i(k_c x + \omega t)} \) appear in the equations. Therefore we have to take them into consideration in the perturbation of \((u_s, w_s)\) as well. So write \( u \) as follows

\[
u(x, t) = u_s + e^{i(k_c x + \omega t)} [e \ A_1 + \varepsilon^2 X_{12} + \varepsilon^3 X_{13} + O(\varepsilon^4)] + e^{-i(k_c x + \omega t)} [e \ A_1 + \varepsilon^2 X_{13} + \varepsilon^3 X_{13} + O(\varepsilon^4)] + e^{2i(k_c x + \omega t)} [\varepsilon^2 X_{22} + \varepsilon^3 X_{23} + O(\varepsilon^4)] + c.c. \\
+ e^{3i(k_c x + \omega t)} [\varepsilon^3 X_{33} + O(\varepsilon^4)] + c.c. + \text{h.o.t.} \quad (3.4)
\]

and

\[
w(x, t) = w_s + e^{i(k_c x + \omega t)} [e A_2 + \varepsilon^2 Y_{12} + \varepsilon^3 Y_{13} + O(\varepsilon^4)] + e^{-i(k_c x + \omega t)} [e A_2 + \varepsilon^2 Y_{13} + \varepsilon^3 Y_{13} + O(\varepsilon^4)] + e^{2i(k_c x + \omega t)} [\varepsilon^2 Y_{22} + \varepsilon^3 Y_{23} + O(\varepsilon^4)] + c.c. + e^{3i(k_c x + \omega t)} [\varepsilon^3 Y_{33} + O(\varepsilon^4)] + c.c. + \text{h.o.t.} \quad (3.5)
\]

Here \( X_{ij} \) and \( Y_{ij} \) are complex functions of the parameters \( \zeta \) and \( \tau \) describing the higher order perturbations. Now calculate the expressions of the model again, but with (3.4) and
We use the notation $v = \frac{\partial \omega}{\partial \tau}$ evaluated in $\omega_c$ as above.

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \epsilon i \omega_c A_1 e^{i(k_c x + \omega_c t)} \\
&\quad + \epsilon^2 \left[ (i \omega_c X_{12} - v \frac{\partial A_1}{\partial \zeta}) e^{i(k_c x + \omega_c t)} + 2i \omega_c X_{22} e^{2i(k_c x + \omega_c t)} \right] \\
&\quad + \epsilon^3 \left[ \frac{\partial X_{02}}{\partial \zeta} + \left( \frac{\partial A_1}{\partial \tau} + i \omega_c X_{13} + v \frac{\partial X_{12}}{\partial \zeta} \right) e^{i(k_c x + \omega_c t)} \right] \\
&\quad + \left( 2i \omega_c X_{23} + v \frac{\partial X_{22}}{\partial \tau} \right) e^{2i(k_c x + \omega_c t)} + 3i \omega_c X_{33} e^{3i(k_c x + \omega_c t)} \\
&\quad + \text{h.o.t.} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w}{\partial t} &= \epsilon i \omega_c A_2 e^{i(k_c x + \omega_c t)} \\
&\quad + \epsilon^2 \left[ e^{i(k_c x + \omega_c t)} (i \omega_c Y_{12} - v \frac{\partial A_2}{\partial \zeta}) + 2i \omega_c Y_{22} e^{2i(k_c x + \omega_c t)} \right] \\
&\quad + \epsilon^3 \left[ \frac{\partial Y_{02}}{\partial \zeta} + e^{i(k_c x + \omega_c t)} \left( \frac{\partial A_2}{\partial \tau} + i \omega_c Y_{13} + v \frac{\partial Y_{12}}{\partial \zeta} \right) \right] \\
&\quad + 2i \omega_c Y_{23} + v \frac{\partial Y_{22}}{\partial \tau} \right) e^{3i(k_c x + \omega_c t)} + 3i \omega_c Y_{33} \\
&\quad + \text{h.o.t.} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w}{\partial x} &= \epsilon A_2 i k_c e^{i(k_c x + \omega_c t)} \\
&\quad + \epsilon^2 \left[ e^{i(k_c x + \omega_c t)} \left( \frac{\partial A_2}{\partial \zeta} + i k_c Y_{12} \right) + 2i k_c Y_{22} e^{2i(k_c x + \omega_c t)} \right] \\
&\quad + \epsilon^3 \left[ \frac{\partial Y_{02}}{\partial \zeta} + e^{i(k_c x + \omega_c t)} \left( i k_c Y_{13} + \frac{\partial Y_{12}}{\partial \zeta} \right) + 2i k_c Y_{23} e^{2i(k_c x + \omega_c t)} + 3i k_c Y_{33} e^{3i(k_c x + \omega_c t)} \right] \\
&\quad + \text{h.o.t.} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= -\epsilon k_c^2 A_1 e^{i(k_c x + \omega_c t)} \\
&\quad + \epsilon^2 \left[ e^{i(k_c x + \omega_c t)} \left( 2i k_c \frac{\partial A_1}{\partial \zeta} - k_c^2 X_{12} \right) - 4 k_c^2 X_{22} e^{2i(k_c x + \omega_c t)} \right] \\
&\quad + \epsilon^3 \left[ e^{i(k_c x + \omega_c t)} \left( \frac{\partial^2 A_1}{\partial \zeta^2} - k_c^2 X_{13} + i k_c \frac{\partial X_{12}}{\partial \zeta} \right) \right] \\
&\quad + 2i k_c \frac{\partial Y_{22}}{\partial \tau} \right) e^{2i(k_c x + \omega_c t)} - 4 k_c^2 X_{23} - 9 k_c^2 X_{33} e^{3i(k_c x + \omega_c t)} \right] \\
&\quad + \text{h.o.t.} \\
\end{align*}
\]
3.2 Turing-Hopf matrix

In order to find a pattern of the form (3.4) and (3.5) that is a solution of our model (2.14), we are going to solve the unknown functions $X_{ij}$ and $Y_{ij}$. The following matrix will be encountered a few times:

$$T = \begin{pmatrix} a - k_c^2 - i\omega_c & b \\ c & d + i(k_c\nu_c - \omega_c) \end{pmatrix} \tag{3.11}$$

We will call this matrix the Turing-Hopf matrix $T$. Matrix $T$ has by construction one eigenvalue $\lambda_1$, which equals zero, with eigenvector $p = (p_1, p_2)$ and an eigenvalue $\lambda_2$,

$$\lambda_2 = \frac{1}{2} \left[ a - k_c^2 + d - i(k_c\nu_c - 2\omega_c) - \sqrt{\alpha + i\beta} \right] \tag{3.12}$$

with $\alpha$ and $\beta$ defined as in (2.19), with eigenvector $q = (q_1, q_2)$.

The first eigenvector $p$ is a solution of the two equations

$$\begin{align*}
(a - k_c^2 - i\omega_c) p_1 + b p_2 &= 0 \\
(c p_1 + (d + i(k_c\nu_c - \omega_c)) p_2 &= 0
\end{align*}$$

Since both equations are equal modulo a multiplication factor, either can be used to describe the eigenvector $p$. We choose

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -b \\ a - k_c^2 - i\omega_c \end{pmatrix} \tag{3.13}$$
The second eigenvector $q$ is a solution of the two equations
\[
\begin{align*}
(a - k_c^2 - i\omega_c - \lambda_2) p_1 + b p_2 &= 0 \\
c p_1 + (d + i(k_c \nu_c - \omega_c) - \lambda_2) p_2 &= 0
\end{align*}
\]

Here we choose
\[
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} =
\begin{pmatrix}
\lambda_2 - (d + i(k_c \nu_c - \omega_c)) \\
c
\end{pmatrix}
\]

(3.14)

One of the eigenvalues equals zero, hence the range of this matrix $T$ is spanned by $q$. The equation $T x = r$, with $r = (r_1, r_2)$, has a solution $x$ if $r \in \langle q \rangle$. This implies that there are only solutions if $r$ satisfies
\[
q_1 r_2 - q_2 r_1 = 0.
\]

(3.15)

This relation (3.15) is called the solvability condition and will be used later on to solve for the Ginzburg-Landau equation.

### 3.3 Solve the unknowns of the form $X_{ij}$ and $Y_{ij}$

In order to find the Ginzburg-Landau equation that describes the dynamics of the amplitudes $A_1$ and $A_2$, we first have to solve the unknowns of the form $X_{ij}$ and $Y_{ij}$. These unknowns can be found by gathering the terms of the same order in $\epsilon$ and $e^{i(k_c x + \omega_c t)}$.

However, before we are going to solve the unknowns, we consider the leading terms of order $O(\epsilon)$
\[
\begin{pmatrix}
a - k_c^2 - i\omega_c \\
c
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

On the left the Turing-matrix $T$ is stated, therefore $A_1$ and $A_2$ are related to each other by the eigenvector corresponding to its zero eigenvalue: $-b A_1 = (a - k_c^2 - i\omega) A_2$. Hence a single amplitude function can be introduced
\[
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} = \begin{pmatrix}
-b \\
(a - k_c^2 - i\omega)
\end{pmatrix} A,
\]

(3.16)

where $A = A(\zeta, \tau) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the new amplitude function.

The first unknowns encountered are $X_{02}$ and $Y_{02}$ of order $O(\epsilon^2)$
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
X_{02} \\
Y_{02}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(3.17)

Solving this system gives two possible solutions: $X_{02} = Y_{02} = 0$ or $a d - b c = 0$. The latter implies that $A = \pm B$. We will not consider this special case in which $(u_s, w_s)$ undergoes a fold bifurcation.
Now $X_{12}$ and $Y_{12}$

$$
\begin{pmatrix}
(a - k_c^2 - i\omega_c & b \\
\frac{c}{d + i(k_c \nu_c - \omega_c)}
\end{pmatrix}
\begin{pmatrix}
X_{12} \\
Y_{12}
\end{pmatrix}
= \begin{pmatrix}
(v - 2i k_c) \frac{\partial A_1}{\partial \zeta} \\
(v - \nu) \frac{\partial A_2}{\partial \zeta}
\end{pmatrix}
= \begin{pmatrix}
\frac{(2ik_c - v)b}{(v - \nu)(a - k_c^2 - i\omega_c)} \\
(\nu - \nu)(a - k_c^2 - i\omega_c)
\end{pmatrix} \frac{\partial A}{\partial \zeta}.
$$

Again on the left the Turing matrix $T$ can be seen. This equation only has a solution if the vector on the right hand side is in the range of $T$. This is indeed the case, since this vector is a multiple of the non-zero eigenvector $q$, hence the solvability condition (3.15) is fulfilled. Therefore a particular solution is given by

$$
\begin{pmatrix}
X_{12} \\
Y_{12}
\end{pmatrix}
= \frac{1}{\lambda_2} \begin{pmatrix}
\frac{(2ik_c - v)b}{(v - \nu)(a - k_c^2 - i\omega_c)} \\
\frac{(v - \nu)(a - k_c^2 - i\omega_c)}
\end{pmatrix} \frac{\partial A}{\partial \zeta}.
$$

For $X_{22}$ and $Y_{22}$ we have to solve

$$
\begin{pmatrix}
a - 4k_c^2 - 2i\omega_c \\
\frac{c}{d + 2i(k_c \nu_c - \omega_c)}
\end{pmatrix}
\begin{pmatrix}
X_{22} \\
Y_{22}
\end{pmatrix}
= \begin{pmatrix}
w_s A_1^2 + 2u_s A_1 A_2 & 1 \\
-1 & -1
\end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} A^2
$$

Note that the matrix on the left is not the Turing matrix $T$; this equation can be solved by inverting the matrix, which results in

$$
\begin{pmatrix}
X_{22} \\
Y_{22}
\end{pmatrix}
= \frac{b(-2a u_s + 2k_c^2 u_s + b w_s + 2i u_s \omega_c) A^2}{-b c + (a - 4k_c^2)(d + 2i k_c \nu_c) - 2i(a + d - 4k_c^2 + 2i k_c \nu_c) \omega_c - 4\omega_c^2}
\begin{pmatrix}
b + d + 2i k_c \nu_c - 2i \omega_c \\
-(a + c - 4k_c^2 - 2i \omega_c)
\end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$

(3.18)

Since the expressions for $X_{22}$ and $Y_{22}$ are needed in the next section, they will be abbreviated to

$$
\begin{pmatrix}
X_{22} \\
Y_{22}
\end{pmatrix}
= \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} A^2
$$

3.4 The Complex Ginzburg-Landau equation

The functions determined in the previous section are necessary to derive the Ginzburg-Landau equation. This equation is derived from the terms of order $\epsilon^3 e^{i(k_c x + \omega t)}$. Take $X_{02} = Y_{02} = 0$, which means that we assume that $A > 2B$ and that we are in the generic situation according to (3.17). Also we assumed at the beginning of this chapter $\nu = \nu_c + r \epsilon^2$. The new parameter $r$ appears at order $\epsilon^3 e^{i(k_c x + \omega_c t)}$, as we see below.

We get the following equation at this order:

$$
\begin{pmatrix}
a - k_c^2 - i\omega_c \\
\frac{c}{d + ik_c \nu_c - i\omega_c}
\end{pmatrix}
\begin{pmatrix} X_{13} \\ Y_{13} \end{pmatrix}
$$
In the next chapter the dynamics of the Ginzburg-Landau equation (3.21) will be studied. Now apply the transformation with 
\[ q = \frac{q_1}{c} \]
This leaves us with the Complex Ginzburg-Landau equation
\[ \kappa \frac{\partial A}{\partial \tau} = (c_1 + ic_2) \frac{\partial^2 A}{\partial \xi^2} + (c_3 + ic_4) \tau A - (c_5 + ic_6) |A|^2 A \]  
with
\[
\kappa = b + q_1 (a - k_c^2 - i \omega_c) \\
c_1 + ic_2 = \frac{1}{\lambda_2} (v - i k_c) (2i k_c - v) b + b - \frac{1}{\lambda_2} (v - \nu_c)^2 (a - k_c^2 - i \omega_c) q_1 \\
c_3 + ic_4 = i k_c q_1 (a - k_c^2 - i \omega_c) \\
c_5 + ic_6 = -(q_1 + 1) [b^2 (-3a + 3k_c^2 + \omega_c) + 2bw_s \gamma_1 + 2bu_s \gamma_2 - 2as (a - k_c^2 + i \omega_c) \gamma_1] .
\]
Here \( \kappa \in \mathbb{C} \), \( c_i \in \mathbb{R} \) and \( q_1 \in \mathbb{C} \). Now apply the transformation \( A = e^{-i c_1 \tau} \hat{A} \), see Van Hecke et al [8], and divide all three coefficients by \( \kappa \), then the resulting equation is
\[ \frac{\partial \hat{A}}{\partial \tau} = (\hat{c}_1 + i\hat{c}_2) \frac{\partial^2 \hat{A}}{\partial \xi^2} + \hat{c}_3 \tau \hat{A} - (\hat{c}_5 + i\hat{c}_6) |\hat{A}|^2 \hat{A} \]  
In the next chapter the dynamics of the Ginzburg-Landau equation (3.21) will be studied.
Chapter 4

Dynamics of the Ginzburg-Landau equation

In the previous chapter we have found the Ginzburg-Landau equation which describes the behaviour of our equilibrium close to the bifurcation. In this chapter we will analyze the stability of this equation, since stable solutions are able to produce stable patterns in the model (1.5).

4.1 Scaling the Ginzburg-Landau equation

In the literature there is a so-called standard form to write the Ginzburg-Landau equation in, see for instance Van Saarloos [6]. By scaling we will write the equation (3.21) in this standard form

\[ \frac{\partial A}{\partial \tau} = (1 + i \eta_1) \frac{\partial^2 A}{\partial \zeta^2} + rA - (1 + i \eta_2) |A|^2 A \tag{4.1} \]

The procedure is the same as in Chapter 1, however we will perform this in two steps. The first step is to get rid of the coefficient \( \tilde{c}_3 \) by scaling \( \tau = \frac{1}{\tilde{c}_3} \tilde{\tau} \). Substituting this in (3.21) results in

\[ \frac{\partial \hat{A}}{\partial \tilde{\tau}} = c_1 + ic_2 \frac{\partial^2 \hat{A}}{\partial \zeta^2} + r\hat{A} - \frac{c_5 + ic_6}{c_3} |\hat{A}|^2 \hat{A} \tag{4.2} \]

Note that the \( c_i \) are now the same as in (3.20).

Before we scale this equation to the standard form, we have to check the sign of \( \frac{c_1}{c_3} \) and \( \frac{c_5}{c_3} \). The coefficient \( \frac{c_1}{c_3} \) should be positive and \( \frac{c_5}{c_3} \) determines whether a sub- or a supercritical bifurcation occurs at \( r = 0 \). If \( \frac{c_5}{c_3} > 0 \) a supercritical bifurcation occurs, but if the sign is negative a subcritical bifurcation would happen. A supercritical bifurcation means that the branch that arises at this point is stable.

In Figure 4.1 the graph is shown of the coefficients \( \frac{c_1}{c_3} \) and \( \frac{c_5}{c_3} \) for \( B = 0.045 \). For this value \( B \) the sign of the coefficients is clearly positive, the same holds for \( B = 0.45 \). However, for relatively large \( B \) the value \( \frac{c_5}{c_3} \) becomes negative for some \( A \), for example Figure 4.2. This change of sign is interesting, since it means that if patterns are observed they are not necessarily small amplitude patterns. If under subcritical parameter conditions a pattern is
Dynamics of the Ginzburg-Landau equation

Figure 4.1: The solid line is the graph of the coefficient $\frac{c_1}{c_3}$ and the dashed line the one of $\frac{c_5}{c_3}$ for $B = 0.045$. 

Figure 4.2: The graph of the coefficient $\frac{c_5}{c_3}$ for $B = 5$. Clearly, the sign of the coefficient becomes negative for some $A$.

observed for $\nu > \nu_c$ it is in general not of small amplitude type and therefore not described by a Ginzburg-Landau equation. They are in fact not Turing/Hopf patterns, although this is often what people suggest in literature. In the graph in Figure 4.2 the coefficient $\frac{c_5}{c_3}$ against $A$ for $B = 5$ is drawn.

So numerical computation showed that for $A > 2B$ and $B = 0.045$ or $B = 0.45$, the coefficients $\frac{c_1}{c_3}$ and $\frac{c_5}{c_3}$ are positive, therefore a supercritical bifurcation will occur at $r = 0$.

Now we can scale the equation to the standard form (4.1), write

$$\hat{A} = [a] \tilde{A}$$
$$\zeta = [x] \tilde{\zeta}$$

and substitute them into the (3.21). Then,

$$\frac{\partial \hat{A}}{\partial \tau} = \frac{c_1 + ic_2}{c_3} \frac{1}{|x|^2} \frac{\partial^2 \hat{A}}{\partial \zeta^2} + r \hat{A} - \frac{c_5 + ic_6}{c_3} [a]^2 |\hat{A}|^2 \hat{A}. \quad (4.3)$$

Choose the scalings by imposing

$$\frac{c_1 + ic_2}{c_3 |x|^2} = 1 + i \eta_1 \implies |x| = \sqrt{\frac{c_1}{c_3}}$$
$$\frac{(c_5 + ic_6)|a|^2}{c_3} = 1 + i \eta_2 \implies |a| = \sqrt{\frac{c_3}{c_5}}.$$

The Ginzburg-Landau equation (3.20) we derived is now, with the appropriate scaling,
written in the standard form (4.1) with
\[ \eta_1 = \frac{c_2}{c_1} \]  
\[ \eta_2 = \frac{c_6}{c_5} \]  
(4.4a)  
(4.4b)

Remember that \( r \) determines only whether we are in the stable \( (r < 0) \) or unstable \( (r > 0) \) area and when \( r = 0 \) we are precisely in the bifurcation point.

### 4.2 Stability Analysis

The Ginzburg-Landau equation for the amplitude of the perturbation is thus given by
\[ \frac{\partial \tilde{A}}{\partial \tilde{\tau}} = (1 + i \eta_1) \frac{\partial^2 \tilde{A}}{\partial \tilde{\zeta}^2} + r \tilde{A} - (1 + i \eta_2) |\tilde{A}|^2 \tilde{A} \]  
(4.5)

For this equation different solutions exist, the simplest ones, Van Hecke et al [8], being the periodic solutions of the form
\[ \tilde{A}(\zeta, \tau) = A_0 e^{i(K \zeta - \Omega \tau)} \]  
(4.6)

with \( A_0, K, \Omega \in \mathbb{R} \). These solutions are called phase winding solutions and they describe the steady state periodic patterns with a wavenumber that is somewhat bigger \( (K > 0) \) or somewhat smaller \( (K < 0) \) than the critical wavenumber \( k_c \), see Van Saarloos [5]. In the same way, \( \Omega \) describes the difference between the critical frequency \( \omega_c \) and the frequency of the pattern.

Substitute the expression (4.6) in (4.5) and divide by \( e^{i(K \zeta - \Omega \tau)} \) then
\[ -i A_0 \Omega = -K^2 A_0 - i \eta_1 K^2 A_0 + r A_0 - A_0^3 - i \eta_2 A_0^3. \]  
(4.7)

From the above equation it follows that either
\[ \Omega = \eta_1 K^2 + \eta_2 A_0^2 \text{ and } r = K^2 + A_0^2 \]  
(4.8)

holds, or
\[ A_0 = 0 \rightarrow \tilde{A}(\zeta, \tau) = 0 \]  
(4.9)

implying that \( A \) has a bounded amplitude \( A_0 \). For both solutions we are interested in its stability. First we will investigate the non-trivial solution \( \tilde{A} \neq 0 \) and then the trivial one \( A = 0 \).

Perturb the solution (4.6) a little
\[ \tilde{A}(\zeta, \tau) = [A_0 + \rho(\zeta, \tau)] e^{i(K \zeta - \Omega \tau + \theta(\zeta, \tau))}. \]  
(4.10)

Substitute this in (4.5), linearize the equations and use the relations found in (4.8). This gives two equations, one for the real and one for the imaginary part:
\[ \rho_\tau = -2 A_0^2 \rho - 2 \eta_1 K \rho_\zeta + \rho_\zeta - 2 K A_0 \theta_\zeta - A_0 \eta_1 \theta_\zeta \]  
(4.11a)
\[ A_0 \theta_\tau = -2 A_0^2 \eta_2 \rho + 2 K \rho_\zeta + \eta_1 \rho_\zeta - 2 K A_0 \eta_1 \theta_\zeta + A_0 \theta_\zeta \]  
(4.11b)
Since (4.11) are linear equations in $\rho$ and $\theta$ with constant coefficients it is possible to write the solutions $\rho$ and $\theta$ as linear combinations of solutions:

$$\rho(\zeta, \tau) = X(\tau) e^{ik\zeta}$$

and

$$A_0 \theta(\zeta, \tau) = Y(\tau) e^{ik\zeta}.$$  

Substitute these in (4.11) and divide by $e^{ik\zeta}$

$$(X_{\tau}) = \left( \begin{array}{c} -2i k K \eta_1 - 2A_0^2 - k^2 \\ 2i k K - k^2 \eta_1 - 2A_0^2 \eta_2 \\ -2i k K \eta_1 - k^2 \end{array} \right) (X_{\tau})$$

To analyze the stability of the solution we have to determine the characteristic polynomial of the above matrix:

$$\lambda^2 + \lambda(2k^2 + 2A_0^2 + 4i k K \eta_1) + k^2(2A_0^2 + 4 K^2 \eta_1^2 - 2A_0^2 \eta_1 \eta_2 - 4K^4) + k^4(1 + \eta_1^2) + 4i k K A_0^2(\eta_1 - \eta_2) = 0$$

(4.13)

substitute $\tilde{\lambda} = \lambda + 2i k K \eta_1$ in (4.13) resulting in

$$\tilde{\lambda}^2 + \tilde{\lambda}(2k^2 + 2A_0^2) + k^2(1 + \eta_1^2) + k^2(2A_0^2 - 4 K^4 - 2A_0^2 \eta_1 \eta_2) + 4i k K A_0^2(\eta_1 - \eta_2) = 0$$

(4.14)

For stability the real parts of the eigenvalues $\tilde{\lambda}$ have to be negative for all $k$. For small $k$ the condition is, Van Saarloos [6],

$$A_0^2(1 + \eta_1 \eta_2) - 2K^2(1 + \eta_2^2) > 0$$

(4.15)

using the condition (4.8), this results in

$$K^2 < \frac{r(1 + \eta_1 \eta_2)}{3 + \eta_1 \eta_2 + 2\eta_2^2}$$

(4.16)

from which follows that

$$1 + \eta_1 \eta_2 > 0.$$  

(4.17)

This is called the Newell condition and when this condition is violated, i.e. $1 + \eta_1 \eta_2 < 0$, all phasewinding solutions are linearly unstable, see Van Saarloos [6].

This Newell condition is satisfied for the values for $A > 2B$ and $B$ estimated by Klausmeier [3] for trees as well as grass.

Now back to the other possible solution of (4.7), $A_0 = 0$ which implied $\tilde{A}(\zeta, \tau) = 0$. Also for this we perturb the solution a little

$$\tilde{A}(\zeta, \tau) = (\alpha + i \beta) e^{i k\zeta + w \tau}$$

(4.18)

Substitute this in (4.5) and linearize around $\alpha$ and $\beta$, this gives

$$w(\begin{array}{c} \alpha \\ \beta \end{array}) = \left( \begin{array}{cc} r - k^2 & k^2 \eta_1 \\ -k^2 \eta_1 & r - k^2 \end{array} \right) (\begin{array}{c} \alpha \\ \beta \end{array})$$

(4.19)

The eigenvalues of this matrix are $w_{1,2} = r - k^2 \pm i k^2 \eta_1$. The real part of these eigenvalues is positive when $r > 0$, which means we are in the unstable area according to the linear theory, and $k^2 < r$. And indeed, when $r < 0$ the eigenvalues are negative and thus the equilibrium is stable.
4.3 Solutions

We assumed that the solution \((u, w)\) is a small perturbation of the equilibrium \((u_s, w_s)\) written as

\[
\begin{pmatrix}
    u(x, t) \\
    w(x, t)
\end{pmatrix} = \begin{pmatrix}
    u_s \\
    w_s
\end{pmatrix} + \epsilon \left( a - k_c^2 - i \omega_c \right) A e^{i(k_c x + \omega_c t)} + \text{h.o.t. + c.c.},
\]

In the previous sections we have found an expression for the amplitude function \(A\). However, we scaled some variables, but to determine the amplitude function \(A\) we have to substitute these scalings back. The scalings were

\[
\begin{align*}
    \tau &= \frac{1}{c_3} \tilde{\tau} \quad \text{→} \quad \tilde{\tau} = \frac{c_3}{\kappa} \tau \\
    \zeta &= \sqrt{\frac{c_1}{c_3}} \tilde{\zeta} \quad \text{→} \quad \tilde{\zeta} = \sqrt{\frac{c_3}{c_1}} \zeta \\
    A &= \sqrt{\frac{c_3}{c_5}} \tilde{A} \quad \text{→} \quad \tilde{A} = \sqrt{\frac{c_5}{c_3}} A
\end{align*}
\]

and since

\[A = e^{-i c_4 \tau} \tilde{A} \quad \text{→} \quad \tilde{A} = e^{i c_4 \tau} A\]

it follows that \(\tilde{A} = \sqrt{\frac{c_5}{c_3}} e^{i c_4 \tau} A\).

The results found for the standard form of the Ginzburg-Landau equation are now brought back to our Ginzburg-Landau equation (3.20), then

\[
A(\zeta, \tau) = A_0 e^{i(K \zeta + \Omega \tau)}
\]

with

\[
A_0^2 = \frac{c_3}{c_5} e^{-2i c_4 \tau (r - \frac{c_3}{c_1} K^2)}
\]

and

\[
\Omega = -\frac{c_2 c_3}{c_1^2} K^2 - \frac{c_3 c_6}{c_5^2} e^{-2i c_4 \tau (r - \frac{c_2}{c_1} K^2)}
\]

is stable when \(1 + \frac{c_2 c_6}{c_1^2 c_5^2} > 0\).

The solution \((u, w)\) of our model (2.14) that is a small perturbation of the equilibrium \((u_s, w_s)\) can thus be described by

\[
\begin{pmatrix}
    u(x, t) \\
    w(x, t)
\end{pmatrix} = \begin{pmatrix}
    u_s \\
    w_s
\end{pmatrix} + \epsilon \left( a - k_c^2 - i \omega_c \right) A_0 e^{i[(k_c + \Omega) (x - v t) + (\omega_c - \Omega)e^2 t]} + \text{h.o.t. + c.c.}
\]

Since the Newell condition is always satisfied for the values estimated by Klausmeier [3] for trees and grass, the solution (4.24) describes traveling waves with wavenumber around \(k_c\) which are stable. So patterns with wavenumber around \(k_c\) occur on hills and are in these parameter regions really of small amplitude type and thus described by the Ginzburg-Landau equation.
Bibliography


