The Parton Shower

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Two of the most fundamental particles have never been observed. They are however predicted by theory. We are talking about quarks and gluons, the basic building blocks for protons and neutrons. The reason they have not been observed is that they do not exist long enough to be detected. They can however exist very briefly in high energy particle collisions where high energy quarks and gluons are formed and produce so called jets. When a high energy quark moves through space it emits gluons which on their account can emit intermediate quarks and gluons. They all decay into stable observable particles. From these the nature of the jets can be determined and compared with theoretical predictions. In this thesis we discuss briefly the basic equations of the underlying theory of Quantum Chromodynamics. We use Feynman diagrams to derive the collision equations. It turns out that it is practically impossible to solve the equations analytically. They can however be used in numerical algorithms that can be calculated by computer. For this we discuss the Monte Carlo method and related veto algorithm.

Contents

1 Introduction

Particles with the color quantum numbers such as quarks and gluons that participate in LHC collision processes have a high degree of radiation.

When particles collide in high energy collisions and annihilate this can lead to a parton shower. This happens when the resulting virtual photon of the annihilation undergoes a transition into a pair of quarks. Both the quark and anti-quark emit partons i.e. additional gluons and quarks on various points allong their track. These can on their hand emit additional partons. In actuallity it is not an emission as such instead it is a type of branching where the parton splits into a pair of new partons. This braching will continue until each parton finally undergoes a transition into a hadron that can be observed. Figure 1 gives a pictorial presentation of the parton branching with the emphasis on quark-gluon emissions which has the main focus of this thesis.

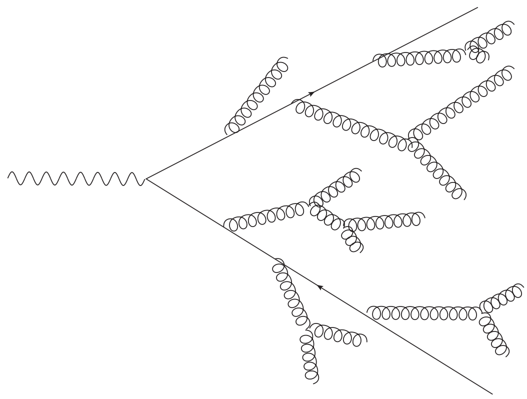

Figure 1: *Partion shower in a high energy collision*

The model of quarks and gluons is based on the theory of Quantum Electrodynamics and Chromodynamics. We will discuss the first-order approach of this model. The processes which take place in a parton shower can be understood on the basis of Feynman diagrams.

We view the creation of quark, anti-quark pairs and the corresponding phase space. We then include the gluons in the process. This gives an extension from a two- to a three-particle solution with a corresponding expansion of the phase space. The actual shower has an n-particle solution. It turns out that this is not easy to solve. The solution is found in an iterative approach based on the relation between the two- and three-particle solutions. It is further shown that the gluon emissions are subject to angular ordering which means that their path is closely related to the path of the related quark.

The final equations that make up the parton shower are not solved analytically but numerically. For this we introduce the Monte Carlo method and veto algorithm and conclude with a discussion of a simplified numerical implementation.

2 Introduction of the model

The parton shower, a stream of quarks and anti-quarks but also gluons, can best be studied in particle accelerators where highly energetic quark collisions can occur. We are mainly interested in measurable quantities as active cross-section and decay rates. These are the available process identifying characteristics.

2.1 Plane waves

We are going to investigate the inner constituents of baryon particles, to be specific quarks and gluons both members of the parton family. Quarks and gluons do not exist as free roaming particles. They might exist for a very short amount of time but eventually they will decay into more stable particles or combine to form more stable particles e.g. a proton or neutron. Free living quarks can be created in particle collisions. Just as the electron and the positron, quarks are part of the fermion family of particles and can be described by the free Dirac equation:

$$i\frac{\partial}{\partial t}\psi = (-i\hat{\alpha}\cdot\nabla - \beta m_0)\psi \tag{1}$$

with solutions in de form of

$$\psi = we^{-ip\cdot x} \tag{2}$$

where p stands for the spacetime or momentum four vector, x the spacetime coordinate and w depicts a Dirac spinor according to

$$w = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{3}$$

The spinors φ and χ are related to each other according to

$$p^0 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} m_0 \mathbb{1} & \hat{\sigma}\cdot\mathbf{p} \\ \hat{\sigma}\cdot\mathbf{p} & -m_0\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{4}$$

with $\hat{\sigma}$ the Pauli matrices. P^0 has solutions for plane waves with positive energy $p^0 = \sqrt{\mathbf{p}^2 + m_0} > 0$ as well as negative energy $p^0 = -\sqrt{\mathbf{p}^2 + m_0} < 0$.
We relate the physical representation of the positive energy solution to particles and the negative energy solutions to anti-particles.

Instead of working with the generic $\hat{\alpha}$ and β we will work with the so called covariant notation and define $\gamma^0 = \beta$ and $\gamma^i = \beta\hat{\alpha}_i$. This leads to the following free Dirac equation:

$$(i\partial\!\!\!/ - m_0)\phi = 0 \tag{5}$$

with $\partial\!\!\!/ = \gamma^\mu \partial_\mu$.

The density ρ and current density $\mathbf{j}$ combined satisfy the continuity equation $\delta_t \rho + \nabla\cdot\mathbf{j} = 0$ from which we obtain $\rho = w^\dagger w\,|u(x,t)|^2$ and $\mathbf{j} = w^\dagger \hat{\alpha} w\,|u(x,t)|^2$

Normalization of the wave function is chosen such that a unit volume satisfies $\int d^3x\rho = 2\omega$ with ω the energy in $\omega = \sqrt{\mathbf{p}^2 + m^2}$.

We now construct the spinors $u(p,s)$ and $v(p,s)$ for positive and negative energy respectively in such a way that they represent energy, momentum and spin projection by defining

$$u(p,s) = \sqrt{\omega + m}\begin{pmatrix} \varphi^s \\ \frac{\hat{\sigma}\cdot\mathbf{p}}{\omega+m}\varphi^s \end{pmatrix} \tag{6}$$

and

$$u(p,s) = \sqrt{\omega + m}\begin{pmatrix} \frac{\hat{\sigma}\cdot\mathbf{p}}{\omega+m}\chi^s \\ \chi^s \end{pmatrix} \tag{7}$$

with s = 1,2 representing spin up and spin down.

From this we obtain the normalized fermion wave functions for positive and negative energy respectively:

$$\psi(f) = \frac{1}{\sqrt{V}}u(p,s)e^{-ip\cdot x}, \tag{8}$$

$$\psi(\bar{f}) = \frac{1}{\sqrt{V}}v(p,s)e^{-ip\cdot x} \tag{9}$$

with V the normalized volume element.

To evaluate transitions that occur in high energy interactions we are interested in the transition amplitude. We rely on first order perturbation theory where the perturbation is formed by the electromagnetic potential. This leads to the transition matrix

$$-f\int d^4x j^\mu A_\mu \tag{10}$$

with $j^\mu = (-e)\bar{\psi}_f\gamma^\mu\psi_i$ the fermion current density and A^μ the electro magnetic potential. Combined with (8) we can write the current density as

$$j^\mu = \frac{-e}{V}\bar{u}_f(p',s')\gamma^\mu u_i(p,s)e^{i(p'-p)\cdot x} \tag{11}$$

This gives us a very powerful instrument for evaluating the scattering process and in particular the parton shower by means of Feynman diagrams.

We now have an expression for the transition amplitude

$$S_{fi} = \frac{-ie}{V}\int d^4x\bar{u}_f(p',s')\gamma^\mu u_i(p,s)e^{i(p'-p)\cdot x}A_\mu \tag{12}$$

The electromagnetic field potential A_μ is the interaction potential. It is in the case of a scattering process related to the interacting particle. To learn more about the interaction amplitude we turn our focus on the interaction potential. From electrodynamics we know that A_μ obeys the Maxwell equation

$$\Box A^\mu - \delta^m u(\delta_\nu A^\nu) = j^\mu \tag{13}$$

Furthermore we can specify a gauge, the Lorentz gauge, such that $\Box A^\mu = j^\mu$. By definition of the d'Alembertian we find that $\Box^{-1}\Box = \mathbb{1}$ which we can use to find an expression for A^μ.
By operating on a plane wave we have $\Box^{-1}\Box e^{-iq\cdot x} = \Box^{-1} - q^2e^{-iq\cdot x}$ from which we can deduce that $\Box^{-1}e^{-iq\cdot x} = \frac{1}{-q^2}e^{-iq\cdot x}$ and that $A^\mu = \Box j^\mu = \frac{-1}{q^2}j^\mu$. This leads to the transition amplitude matrix

$$S_{fi} = \frac{-ie^2}{V^2}\int d^4x\bar{u}_f(p',s')\gamma^\mu u_i(p,s)e^{i(p'-p)\cdot x} \times \frac{1}{q^2}\bar{v}_f(k',s')\gamma^\mu v_i(k,s)e^{i(k'-k)\cdot x} \tag{14}$$

with q^μ the transfered momentum $p' - p = k' - k$.

The transition probability P_{fi} is the transition amplitude squared integrated over time and the phase space of the final state particles.

$$P_{fi} = \left| \frac{ie^2}{V^2} \int d^4x \bar{u}_f \gamma^\mu u_i \frac{1}{q^2} \bar{v}_f \gamma^\nu v_i e^{p'+p-k'-k} \right|^2 \frac{V_f d^3p}{(2\pi)^3} \frac{V_f d^3p}{(2\pi)^3} \tag{15}$$

From the transition probability it is one step closer to the differential cross section $d\sigma$. For this we look at the flux of the incomming particles and the density of the target. Both the flux and target are normalized to 2ω particles per unit volume V. For the flux this leads to a current density $|j| = |v_{rel}| \frac{2\omega_b}{V}$ with v_{rel} the relative speed between the beam and target particles. To obtain the density of the target we integrate the density over a unit volume and obtain $\rho_t = \frac{2\omega_t}{V}$. We now have the luminosity by combining the current and target density:

$$L = |v_{rel}| \frac{2\omega_b}{V} \frac{2\omega_t}{V} \tag{16}$$

From this the differential cross section is readily obtained:

$$\begin{aligned} d\sigma &= \frac{1}{L} \frac{dP_{fi}}{dt} && (17) \\ &= \frac{V^2}{2\omega_b \omega_t |v_{rel}|} \left| \frac{ie^2}{V^2} \int d^4x \bar{u}_f \gamma^\mu u_i \frac{1}{q^2} \bar{v}_f \gamma^\nu v_i e^{p'+p-k'-k} \right|^2 \\ &\quad \times \frac{V_f d^3p}{(2\pi)^3} \frac{V_f d^3p}{(2\pi)^3} && (18) \end{aligned}$$

The transition probability is sharply peaked where the initial momentum equals the final momentum leading to a delta function hence

$$d\sigma = \frac{V^2}{2\omega_b \omega_t |v_{rel}|} \left| \frac{ie^2}{V^2} \int d^4x \bar{u}_f \gamma^\mu u_i \frac{1}{q^2} \bar{v}_f \gamma^\nu v_i \right|^2 \delta(p + k - p' - k') \frac{V_f d^3p}{(2\pi)^3} \frac{V_f d^3p}{(2\pi)^3} \tag{19}$$

2.2 Gluon emission

To study the parton shower process it can be reduced to the creation of a quark, an anti-quark and a gluon where the latter is emitted from either the quark or the anti-quark. When we apply this process iteratively we create the shower.

To determine the invariant amplitude we make use of a Feynman diagram in which the annihilation of a electron, anti-electron pair will form the starting point for the shower.

Instead of the gluon creation at the quark line it can just as well be created at the anti-quark line. This is depicted in the following Feynman diagram and is also part of the invariant amplitude.

These diagrams lead to the following expression with

$$\begin{aligned} &\bar{v}_\alpha(k_2, s_2)(-ie\gamma^\mu)_{\alpha\beta} u_\beta(k_1, s_1) \frac{i\eta_{\mu\nu}}{(k_1 + k_2)^2} \\ &\qquad \times \bar{u}_\gamma(p_1, s_1')(-ig_s t_a \not{\epsilon}_{\gamma\delta}(p, \lambda)) \frac{-i(\not{p}_1 + \not{p})_{\delta\eta}}{(p_1 + p)^2} (-ie\gamma^\nu)_{\eta\kappa} v_\kappa(p_2, s_2') \end{aligned} \tag{20}$$

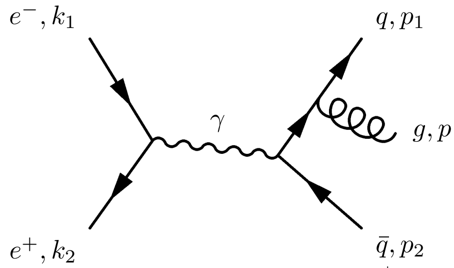

Figure 2: *Feynman diagram for $e^-e^+ -> qg\bar{q}$*

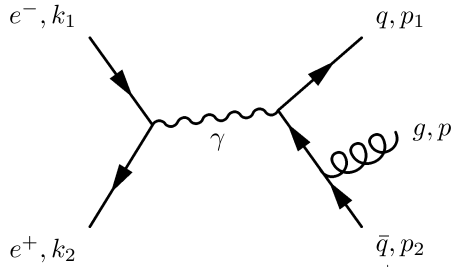

Figure 3: *Feynman diagram for $e^-e^+ -> q\bar{q}g$*

for $e^-e^+ -> qg\bar{q}$ (Figure 2) and

$$\begin{aligned}
&\bar{v}_\alpha(k_2, s_2)(-ie\gamma^\mu)_{\alpha\beta}u_\beta(k_1, s_1)\frac{i\eta_{\mu\nu}}{(k_1+k_2)^2} \\
&\qquad \times \bar{u}_\gamma(p_1, s_1')(-ie\gamma^\nu)_{\gamma\eta}\frac{-i(\not{p}_1+\not{p})_{\eta\delta}}{(p_1+p)^2}(-ig_s t_a \not{\epsilon}_{\delta\kappa}(p, \lambda))v_\kappa(p_2, s_2') \qquad (21)
\end{aligned}$$

for $e^-e^+ -> q\bar{q}g$ (Figure 3), leading to a total invariant amplitude:

$$M = \frac{ie^2 g_s}{2k_1 \cdot k_2}(\bar{v}_2\gamma^\mu u_1)\bar{u}_{1'}\{\not{\epsilon}\frac{\not{p}_1+\not{p}}{2p_1 \cdot p}\gamma_\mu + \gamma_\mu\frac{\not{p}_2+\not{p}}{2p_2 \cdot p}\not{\epsilon}\}v_{2'} \qquad (22)$$

2.3 Tracing the matrix

As we have seen in the previous chapter, the invariant amplitude for the creation of a quark, anti-quark and gluon triplet from the annihilation of a electron with a positron evaluates to two Feynman diagrams each representing the emission of a gluon in the respective quark line. Here we focus mainly on the creation process and make use of trace techniques to evaluate the resulting matrix. The trace gives the eigenvalues of the matrix.

We evaluate the diagram by 'walking' the quark fermion lines against the charge flow to obtain the trace expression for the matrix. Note that we evaluate in the massless limit hence $(p_i + p_j)^2 = p_i^2 + 2p_i \cdot p_j + p_j^2 = 2p_i \cdot p_j$. In our approximation of the parton shower we are not interested in the individual spin interactions. We therefore take the sum over the spin momenta to obtain the invariant amplitude:

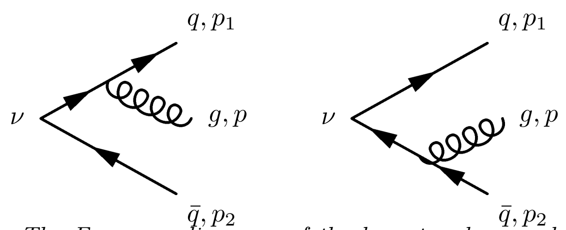

Figure 4: *The Feynman diagrams of the lowest order quark, anti-quark and gluon approximation*

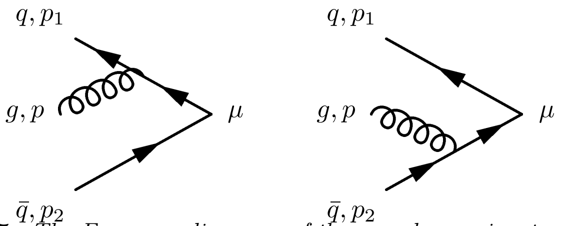

Figure 5: *The Feynman diagrams of the complex conjugate of the lowest order quark, anti-quark and gluon approximation*

$$\begin{aligned}
\sum_{spins} |M_{qg\bar{q}} + M_{q\bar{q}g}|^2 &= \sum_{spins} \bar{u}(p_1)\left[\gamma^\alpha \frac{\not{p}_1 + \not{p}}{2p_1 \cdot p}\gamma^\nu + \gamma^\nu \frac{-\not{p}_2 - \not{p}}{2p_2 \cdot p}\gamma^\alpha\right] v(p_2) \\
&\times \bar{v}(p_2)\left[\gamma^\mu \frac{\not{p}_1 + \not{p}}{2p_1 \cdot p}\gamma^\alpha + \gamma^\alpha \frac{-\not{p}_2 - \not{p}}{2p_2 \cdot p}\gamma^\mu\right] u(p_1) \\
&= tr\left\{\not{p_1}\left[\gamma^\alpha \frac{\not{p}_1 + \not{p}}{2p_1 \cdot p}\gamma^\nu + \gamma^\nu \frac{-\not{p}_2 - \not{p}}{2p_2 \cdot p}\gamma^\alpha\right]\right\} \\
&\times tr\left\{\not{p_2}\left[\gamma^\mu \frac{\not{p}_1 + \not{p}}{2p_1 \cdot p}\gamma^\alpha + \gamma^\alpha \frac{-\not{p}_2 - \not{p}}{2p_2 \cdot p}\gamma^\mu\right]\right\}
\end{aligned} \tag{23}$$

Taking into account the annihilation of the electron positron pair that leads the process we get the following expression for the invariant amplitude:

$$|M|^2 \propto \frac{(k_1 \cdot p_1)^2 + (k_1 \cdot p_2)^2 + (k_2 \cdot p_1)^2 + (k_2 \cdot p_2)^2}{k_1 \cdot k_2 \; p_1 \cdot p \; p_2 \cdot p} \tag{24}$$

3 The parton shower

In the previous chapters we have covered the basics for high energy scattering and decay processes. We have also made a first step toward the parton shower by introducing quark gluon emission. When we develop the one gluon emission into a full parton shower the complexity for the analytical solution grows with each additional gluon. We seek a solution in the approximation of the two particle creation process. For this we discuss the soft gluon and the collinear limit of(24) and compare it to the invariant amplitude of the two particle solution which we discuss first.

3.1 Two particle solution

The two particle solution of the annihilation/creation process has the following Feynman diagram

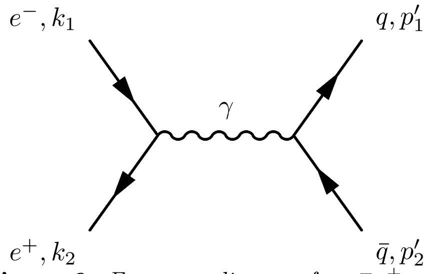

Figure 6: *Feynman diagram for* $e^-e^+- > q\bar{q}$

To obtain an expression for the invariant amplitude we evaluate the electron and quark line just as we did for the quark-gluon creation. Again we take the sum over the spin momenta and evaluate in the massless limit.

$$\begin{aligned} |M_B|^2 &\propto \sum_{spins} \frac{\bar{v}(k_2)\gamma^{mu}u(k_1)\bar{u}(k_1)\gamma^{nu}v(k_2)}{(k_1+k_2)^2}\bar{u}(p_1')\gamma^{\nu}v(p_2')\bar{v}(p_2')\gamma^{\mu}u(p_1') \\ &= \frac{tr\ \{\not k_2\gamma^{\mu}\not k_1\}}{(k_1+k_2)^2} tr\ \{\not p_1'\gamma^{\nu}\not p_2'\gamma^{\mu}\} \end{aligned} \tag{25}$$

In the massless limit this leads to:

$$\begin{aligned} |M_B|^2 &\propto \frac{k_1\cdot p_1'\ k_2\cdot p_2' + k_1\cdot p_2'\ k_2\cdot p_1'}{(k_1\cdot k_2)^2} \\ &\propto \frac{t^2+u^2}{s^2} \end{aligned} \tag{26}$$

with t,u and s the Mandelstam variables also in the massless limit:

$$\begin{aligned} \frac{s}{2} &= k_1\cdot k_2 = p_1'\cdot p_2' \\ \frac{-t}{2} &= k_1\cdot p_1' = k_2\cdot p_2' \\ \frac{-u}{2} &= k_1\cdot p_2' = k_2\cdot p_1' \end{aligned}$$

3.2 Soft gluon limit

In the soft gluon limit we evaluate (24) in the limit $p- > 0$ and compare it to (26). In case $p- > 0$ we see that $p_1 - > p_1'$ and $p_2 - > p_2'$.

When we also take into account the relations according to the Mandelstam variables we can write:

$$|M|^2 \propto |M_B|^2 \times \frac{p_1\cdot p_2}{(p_1\cdot p)(p_2\cdot p)} \tag{27}$$

The proportionality factor that relates the three particle solution to the two particle solution in the soft gluon limit is known as the eikonal factor. We will use this to learn more about the angular dependency between the gluon and the quark.

3.3 Angular ordering

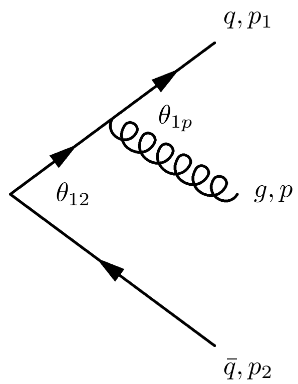

Figure 7: *The angular ordering of the gluon*

We want to evaluate the proportionality factor in the soft gluon limit for a gluon emission. We therefore designate the quark and the anti-quark with index i and j respectively. If we look at the inner product of the two quark four vectors p_i and p_j where we take $p_i = (E_i, 0, 0, k_i)$ and $p_j = (E_j, 0, k_j \sin\theta_{ij}, k_j \cos\theta_{ij})$ and make use of the fact that $v_n = \frac{k_n}{E_n}$ we find:

$$p_i \cdot p_j = E_i E_j (1 - v_i v_j \cos\theta_{ij}) \tag{28}$$

We can do a similar substitution for the inner product between a quark line and the gluon where we take the gluon four vector as $p = \omega(1, 0, \sin\theta_{ip}, \cos\theta_i)$ and find:

$$p_i \cdot p = E_i \omega (1 - v_i \cos\theta_{ip}) \tag{29}$$

We can combine these with the proportionality factor found in the soft gluon limit (27) and find:

$$W_{ij} = \frac{\omega^2 (p_i \cdot p_j)}{(p_i \cdot p)(p_j \cdot p)} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i \cos\theta_{ip})\ (1 - v_j \cos\theta_{jp})} \tag{30}$$

In the massless limit $v_i = v_j = 1$. We can make a smart substitution [3] and rewrite W_{ij} as the sum

$$W_{ij} = W_{ij}^{[i]} + W_{ij}^{[j]} \tag{31}$$

with

$$W_{ij}^{[i]} = \frac{1}{2}\left(W_{ij} + \frac{1}{1 - \cos\theta_{ip}} + \frac{1}{1 - \cos\theta_{jp}}\right) \tag{32}$$

We can now integrate e.g. $W_{ij}^{[i]}$ in the azimuthal plane with respect to the quark line to obtain the average azimuthal gluon emission. We find that this leads to an interesting aspect with regard to emission of gluons. It turns out that the angle between the quark and gluon is limited by the angle between the quark and the anti-quark i.e.

$$\int_0^{2\pi} \frac{d\phi_{ip}}{2\pi} W_{ij}^{[i]} = \frac{1}{1 - \cos\theta_{ip}}$$

when $\theta_{ip} < \theta_{ij}$ and zero otherwise. The physical explanation for this phenomenon is that when the angle between the gluon and the quark is larger than the angle between the quark and the anti-quark, the gluon can no longer distinguish the separate quark charges within a single wavelength and therefore no longer sees a charge. Without charge there is no interaction.

This fact gives credence to the assumption that the gluon's axis is in line with the quark. This is investigated in the next section, the collinear limit.

3.4 Collinear limit

The soft gluon limit derivation does not distinguish between the two possible gluon diagrams where the gluon originates from either the quark or the anti-quark. This in contrast to the collinear limit derivation. Here we examine the situation where the gluon is closely related to the quark. The solution found however is the same as when the gluon originates from the anti-quark. In the collinear limit we treat the gluon/quark as a single 'particle' with four momentum P thus obtaining in the limit a two particle solution. We relate the gluon to this four momentum as $p = (1-z)P$ and the quark as $p_1 = zP$. In addition we assume that P^2 is very small. Using this and the relations obtained from the Mandelstam variables we rewrite (24) and obtain:

$$\begin{aligned} |M|^2 &\propto \frac{z^2(k_1 \cdot P)^2 + (k_1 \cdot p_2)^2 + z^2(k_2 \cdot P)^2 + (k_2 \cdot p_2)^2}{k_1 \cdot k_2 \; z(z-1)P \cdot P \; (1-z)p_2.P} \\ &= \left(\frac{1+z^2}{1-z}\right) \frac{1}{z(z-1)P^2} \, |M_B|^2 \end{aligned} \tag{33}$$

where we have to make the observation that P takes the function of p_1' in (26).

3.5 Phase space

We have seen that the emission of a single gluon introduces more complexity with regard to the invariant scattering amplitude but that we can make an approximation based on the two particle solution. If we can make a similar approximation for the phase space we can consequently work towards a solution for the parton shower as an iterative process of gluon emissions.

When emitting the gluon the momentum of the incomming quark is divided between the gluon and the ongoing quark. A fraction of the energy is transfered to the gluon whereas the outgoing quark holds on to the remainder of the energy. We define the energy fractions as

$$z = \frac{\omega}{E_1} = 1 - \frac{E_2}{E_1} \tag{34}$$

with ω the energy of the gluon and E_1 and E_2 the energy of the incomming and outgoing quark respectively. When we take into account the transverse conservation of momentum we can also make an observation of the opening angle. If we define the opening angle $\theta = \theta_g + \theta_2$ where the latter two are taken with respect to the direction of the incomming quark we find the following relation:

$$\theta = \frac{\theta_g}{1-z} = \frac{\theta_2}{z} \tag{35}$$

We now compare the phase space of the two particle system with the phase space of the three particle system. The phase space of the two particle system is:

$$d\Phi_n = \frac{d^3\mathbf{p'}_1}{2(2\pi)^3 E'_1}\frac{d^3\mathbf{p'}_2}{2(2\pi)^3 E'_2} \tag{36}$$

For the three particle system the phase space has an additional term to reflect the third particle, the gluon:

$$d\Phi_{n+1} = \frac{d^3\mathbf{p'}_1}{2(2\pi)^3 E'_1}\frac{d^3\mathbf{p}_2}{2(2\pi)^3 E_2}\frac{d^3\mathbf{p}_g}{2(2\pi)^3 E_g} \tag{37}$$

The modification of the calculation of the phase space with regard to the two particle process is that the phase space of the incomming quark designated with index $2'$ is replaced with the phase space of the gluon and of the outgoing quark on the condition that we are evaluating the process at a fixed four momentum for the gluon i.e. $p'_2 = p_2 + p_g$. In this case the differential volume of the incomming quark $d^3\mathbf{p'}_2$ equals the differential volume of the outgoing quark $d^3\mathbf{p}_2$. This leads to an expression of the three particle phase space based upon the two particle phase space:

$$d\Phi_{n+1} = d\Phi_n \frac{E_1}{E_2}\frac{d^3\mathbf{p}_g}{2(2\pi)^3\omega} \tag{38}$$

In the small angle approximation we can make use of the relations (34) and (35) to rewrite the phase space relation (38) as a function of the fraction z and the four momentum transfer t:

$$\begin{aligned} d\Phi_{n+1} &= d\Phi_n \frac{1}{2(2\pi)^3}\int \omega d\omega \theta_g d\theta_g d\phi dp_1^2 \frac{dz}{1-z}\delta(p_1^2 - \omega E_2 \theta^2)\delta(z - \frac{\omega_g}{E_1}) \\ &= d\Phi_n \frac{1}{4(2\pi)^3} dt dz d\phi \end{aligned} \tag{39}$$

This ultimately leads to the following expression for the differential cross section:

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t}\frac{d\Phi}{2\pi}\frac{\alpha_s}{2\pi} CF \tag{40}$$

with α_s the coupling constant $\frac{g^2}{4\pi}$, C the colour factor and F the polarization dependent z-distribution. We can relate the latter two to the parton splitting function $\hat{P}(z)$ found earlier by integrating (40) over the azimuthal angle ϕ and find

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi}\hat{P}(z) \tag{41}$$

We now have a function that we can use to describe the parton shower as an iterative process.

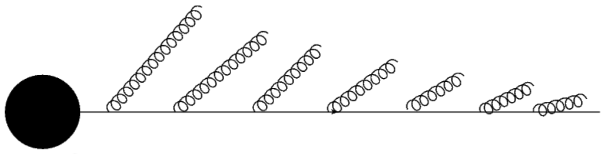

Figure 8: *Parton shower evolution*

4 Parton shower evolution

By evaluating the parton shower as an iterative process of gluon emissions we can view it as a radioactive process with the gluon as the radiated particle. At any point in time it is possible that branching takes place of the quark into a quark/gluon. For this we want to derive an evolution equation $f(x,t)$ that gives the change in parton distribution as function of the parton momentum fractions depicted by δx and δt. The evolution equation will give the possibility of change in momentum fraction [3].

$$\delta f(x,t) = \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(z/x,t) \tag{42}$$

To be able to use (42) in a numerical approach the Sudakov factor is introduced:

$$\Delta(t) \equiv e^{-\int_{t_0}^{t} \frac{dt'}{t'} \int dz \frac{\alpha_s}{2\pi} P(z)} \tag{43}$$

which enables us to rewrite (42) as

$$\frac{\delta f}{\Delta} = \frac{dt}{t} \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(x/z,t) \tag{44}$$

We can integrate both sides with respect to t and the initial distribution at t_0 to obtain f(x,t):

$$\int_{t_0}^{t} \frac{\delta}{\delta t'} \left(\frac{f}{\Delta}(t') \right) dt' = \int_{t_0}^{t} \frac{dt'}{t'\Delta(t')} \int \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(x/z,t)$$

$$\frac{f}{\Delta}(t) - \frac{f}{\Delta}(t_0) = \int_{t_0}^{t} \frac{dt'}{t'\Delta(t')} \int \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(x/z,t)$$

$$f(x,t) = \Delta(t) f(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(x/z,t) \tag{45}$$

Here we have used that fact that $\Delta(t_0) = 1$. From this we learn that the physical interpretation of the Sudakov form factor is the probability that no branching takes place between t_0 and t. Furthermore the Sudakov form factor can be used in describing the parton shower evolution to derive the probability R that no branching takes place between two subsequent values of t i.e.

$$R = \frac{\Delta_{t_a}}{\Delta_{t_b}} \tag{46}$$

The parton shower is a natural process which means that at any point in time a gluon can be emitted. This gluon will have a certain minimum momentum but this is never more then the total momentum of the emitting quark. To find this momentum we compare the differential cross section of this partial branching with the total differential cross section possible based upon the parton splitting function and a randomly chosen fraction R' such that

$$\int_{z_{min}}^{z} dz \frac{\alpha_s}{2\pi} P(z) = R' \int_{z_{min}}^{z_{max}} dz \frac{\alpha_s}{2\pi} P(z) \tag{47}$$

where z_{min} and z_{max} the lower and upper bounds for the gluon's momentum fraction.

The equations (46) and (47) form the basis for a Monte Carlo approach of the numerical evaluation of the parton shower.

4.1 Monte Carlo and the veto algorithm

To find a solution for the equations found in section 4 we take the stochastic approach. The parton shower evolves in such a way that the future state is completely determined by the present state. The transition to a new state is determined by a probabilistic process, the parton splitting function $P(z)$. This type of statistic process is known as a Markov process. It can be simulated by Monte Carlo methods where the basis is a sum over the average values of a uniform distribution based on random sampling.

$$I = \int_{x_1}^{x_2} f(x)dx = (x_2 - x_1)\langle f(x) \rangle \approx \frac{(x_2 - x_1)}{N} \sum_{i=1}^{N} f(x_i) \tag{48}$$

with x_i taken randomly in the domain $[x_1, x_2]$.

The accuracy of this solution increases with N according to the central limit theorem

$$error \propto \frac{1}{\sqrt(N)} \tag{49}$$

The Monte Carlo method does not take into account the 'radioactive process' of the parton shower. For this we introduce the veto algorithm [4]. If we express the probability that nothing has happened until time t by $N(t)$ and the probability that something happens at time t by $P(t)$ we can combine these to get a basic expression for a radioactive type of process i.e.

$$P(t) = f(t)N(t) \tag{50}$$

If we take $N(0) = 1$ we find that this (50) has the following solution:

$$P(t) = f(t)e^{-\int_0^t f(t')dt'} \tag{51}$$

We can relate this to the parton shower evolution equations by means of the Sudakov factor. This corresponds with the exponential factor in (51), the probability that nothing has happened yet at time t.

At this point we can introduce the actual algorithm based on these expressions that gives us the development of the parton shower in a timely fashion. The basis is a Monte Carlo technique where we relate the function $f(t)$ to another function $g(t)$ for which the primitive is known on the condition that $f(t) < g(t)$ for all $t \geqq 0$. The difference between Monte Carlo and the veto algorithm is that in the former we take t according to a random distribution whereas here we start with $i = 0, t = 0$ and demand that $t_i > t_{i-1}$. The first step in the algorithm is to select a value for t such that this corresponds with a random event i.e.

$$\int_0^t P(t')dt' = N(0) - N(t) = 1 - e^{-\int_0^t f(t')dt'} = 1 - R \tag{52}$$

where R is randomly choosen number in the range $[0, 1]$ according to a uniform distribution. If $f(t)$ where a easily solvable function the solution would be

$$t = F^{-1}(F(0) - lnR) \tag{53}$$

In case $f(t)$ is not easily solvable it can be compared to a second function $g(t)$ that is easily solvable on the premise that $f(g) \leq g(t)$ for all $t \geq 0$. The reason we can to this is that the probability that $f(x)/g(x) > R$ is proportional to $f(x)$. Instead of using F to find t we now use the primitive of $g(t)$ and obtain

$$t = G^{-1}(G(0) - lnR) \tag{54}$$

which brings use to the actual next step where we compare the ratio $f(t)/g(t)$ with a new random number R'. If the ratio is greater we have found our match otherwise we discard t and try again from this point unless we have found a maximum for t. This signals the end of the shower. We keep repeating these steps whereby each time a match is found the maximum is decreased by this amount. Summarizing we get the following for the veto algorithm:

1	start with $i = 0$ and $t_0 = 0$;
2	increase i by 1 and select $t_i = G^{-1}(G(t_{i-1}) - lnR)$ on the condition that $t_{i-1} < t_i \leq t_{max}$;
3	stop if $t_i = t_{max}$ otherwise continue at step 4;
4	if $\frac{f(t_i)}{g(t_i)} > R'$ t_i is our answer and stands for a gluon emission otherwise return to step 2;
5	decrease t_{max} with t_i and return to step 1.

4.2 A one dimensional shower

As a first approach the parton shower can be evaluated by means of a simple one dimensional model in which the Sudakov factor is defined by:

$$\Delta = e^{-a \int_1^{x_2} dz \frac{Q(z)}{z}} \tag{55}$$

We start by defining a lower bound on the energy after branching and call this x_0. The maximum energy available to the particle is x_M. It decreases each time an emission takes place. We pick a random number R_1 in the domain $[0, 1]$ and solve for z the equation

$$e^{-a \int_z^{x_M} \frac{dz'}{z'}} \equiv (\frac{z}{x_M})^a = R_1 \tag{56}$$

We then pick a second random number R_2. When this is less then Q(z) the branching is rejected and we return to the step where we choose R_1 and redo

(56) for a new value of R_1 and with $x_M = z$ from the previous iteration. On the other hand when Q(z) is less then R_2 we have an emission with energy fraction z if this is greater then our lower bound x_0. Again we return to the selection for a new value of R_1 and repeat the procedure with $x_M = z$ and keep repeating the whole process until we have found an emission with x less then x_0. This signals the end of the shower.

The results of the simulation are given in the following table

event	particle	energy fraction
	1	1.00
1	2	2.72E-002
	3	5.16E-013
	1	1.00
2	2	2.72E-007
	3	2.67E-014
	4	6.32E-015
	1	1.00
3	2	4.95E-005
	3	3.81E-009
	1	1.00
	2	5.23E-006
4	3	3.00E-006
	4	2.11E-006
	5	7.14E-008
	6	8.51E-010
	1	1.00
	2	4.76E-004
	3	1.57E-006
5	4	2.25E-007
	5	1.41E-009
	6	1.28E-010
	7	8.38E-011

Table 1: *Parton shower simulation results for a one dimensional case*

5 Discussion

We want to conclude this thesis with the following observations. We can conclude that an analytical solution for the parton shower evolution is nearly impossible to find. Even a three particle solution is not confined to one fixed solution instead it has a solution space. The solution to the parton shower evolution is found in a stochastic numeric approach where the exact solution is not determined upfront yet it is choosen from an infinite set of possible paths. The only thing that is certain is that the shower will end. We see the same behavior in nature from the fact that even here the future is not set at every and each point.

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6 Appendices

6.1 Trace technique

$$
\begin{aligned}
tr\{AB\} &= tr\{BA\} \\
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu}\mathbb{1} \\
tr\{\mathbb{1}\} &= 4 \\
tr\{\gamma^\mu\} &= 0 \\
tr\{\gamma^\mu, \gamma^\nu\} &= tr\{2\eta^{\mu\nu}\mathbb{1}\} = 8\eta^{\mu\nu} \\
tr\{\gamma^\mu\gamma^\nu\} &= \frac{1}{2}tr\{\gamma^\mu\gamma^\nu\} + \frac{1}{2}tr\{\gamma^\mu\gamma^\nu\} \\
&= \frac{1}{2}\{\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu\} + \frac{1}{2}tr\{\gamma^\mu\gamma^\nu\} \\
&= \frac{1}{2}(8\eta^{\mu\nu} - tr\{\gamma^\nu\gamma^\mu\}) + \frac{1}{2}tr\{\gamma^\mu\gamma^\nu\} \\
&= 4\eta^{\mu\nu} \\
tr\{\gamma^\mu\gamma^\nu\gamma^\epsilon\gamma^\sigma\} &= tr\{(\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu)\gamma^\epsilon\gamma^\sigma\} \\
&= 2\eta^{\mu\nu}tr\{\gamma^\epsilon\gamma^\sigma\} - tr\{\gamma^\nu\gamma^\mu\gamma^\epsilon\gamma^\sigma\} \\
&= 8\eta^{\mu\nu}\eta^{\epsilon\sigma} - 2\eta^{\mu\epsilon}tr\{\gamma^\nu\gamma^\sigma\} + tr\{\gamma^\nu\gamma^\epsilon\gamma^\mu\gamma^\sigma\} \\
&= 8\eta^{\mu\nu}\eta^{\epsilon\sigma} - 8\eta^{\mu\epsilon}\eta^{\nu\sigma} + 8\eta^{\nu\epsilon}\eta^{\mu\sigma} - tr\{\gamma^\nu\gamma^\epsilon\gamma^\sigma\gamma^\mu\} \\
&= 8\eta^{\mu\nu}\eta^{\epsilon\sigma} - 8\eta^{\mu\epsilon}\eta^{\nu\sigma} + 8\eta^{\nu\epsilon}\eta^{\mu\sigma} - tr\{\gamma^\mu\gamma^\nu\gamma^\epsilon\gamma^\sigma\} \\
tr\{\gamma^\mu\gamma^\nu\gamma^\epsilon\gamma^\sigma\} &= 4\eta^{\mu\nu}\eta^{\epsilon\sigma} - 4\eta^{\mu\epsilon}\eta^{\nu\sigma} + 4\eta^{\nu\epsilon}\eta^{\mu\sigma}
\end{aligned}
$$