Artificial Gauge Fields and optical flux lattices

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Abstract

The area of quantum simulation with ultracold atoms is a constantly evolving new area of research. Advancements are happening at rapid rate and many research lines are being currently explored. The behavior of charged particles in a magnetic field leads to many interesting physical phenomena (1). The limited tunability of the environment where such phenomena occur has lead to an interest in simulating gauge fields through the manipulation of an ultracold atomic gas in different ways (2, 3, 4). One particular method that has successfully been implemented is the accumulation of a geometrical phase (4, 5). The following literature thesis will give a brief introduction into how charged particles behave in the presence of a homogeneous magnetic field, and then continue on to discuss one of the ways this can be simulated in an ultracold quantum gas experiment through optically dressed states. Finally, this thesis will conclude with a discussion of a recent theoretical proposal for simulating a magnetic flux in ultra cold neutral atoms known as an optical flux lattice (6).
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Introduction

Ultracold neutral atoms have become an intriguing environment for the exploration of strongly correlated states of matter\(^{(7)}\). One especially interesting area is the interaction of charged particles and electromagnetic fields. This leads to phenomena such as the quantum Hall and Aharonov-Bohm effects, the Hofstadter Butterfly, and topological insulators\(^{(1, 8, 9, 10, 11, 12, 13, 14)}\). The control and tuning possible with quantum gases make them an appealing medium for conducting such experiments, but the lack of the charge on the particles makes it impossible impart a magnetic flux with real magnetic fields that will lead to the behavior similar to charged particles in such a field. This has lead to the development in recent years of methods to circumvent this issue including the use of laser beams with specific orientations, frequencies, and polarizations to manipulate the atoms \(^{(6, 8, 15)}\).

This literature thesis will discuss one of the possible methods for simulating artificial gauge fields with ultracold atoms. It will start off with a brief discussion of gauge invariance and Landau levels to provide some background information into the behavior of charged particles in a magnetic field. It will continue on to explain one way these phenomena can be simulated in ultracold atoms through the accumulation of a geometrical phase using optically dressed states. Finally, it concludes by mentioning a recent proposal for simulating a synthetic magnetic field in ultracold atoms known as an optical flux lattice\(^{(6)}\).
Gauge invariance and Landau levels

To start the discussion of charged particles interacting with magnetic fields it is important to mention gauge invariance. The root of gauge invariance comes from the idea in classical electrodynamics that the electric and magnetic fields defined by the Maxwell equations must remain invariant under a transformation of the electromagnetic potential \( \mathbf{A} \) (16). In quantum dynamics this leads to the wave function, or field, acquiring a phase in order for the wave function to remain unchanged under such a transformation. Gauge invariance is commonly accepted as the the basis of the modern theory of electroweak and strong interactions between elementary particles, which is a fundamental idea in the standard model (16).

For considering static magnetic fields, as the ones in this thesis, the postulate that there exists no magnetic monopoles and that the field is divergence free is used (8, 16, 17),

\[
\nabla \cdot \mathbf{B} = 0. \tag{2.1}
\]

We can also view this as the magnetic flux of \( \mathbf{B} \) over a closed surface being zero (8). Applying this limitation allows the magnetic field to be written as (8, 16, 17)

\[
\mathbf{B} = \nabla \times \mathbf{A}(r), \tag{2.2}
\]
where \( \mathbf{A} \) is a vector potential. The wave functions of stationary states of a particle in a magnetic field are degenerate. This is because the choice of a different vector potential could lead to the same \( \mathbf{B} \) field and have the same physical effects \((8, 17)\). According to gauge invariance these two vector potentials of the same field are connected through the gradient of an arbitrary scalar function \( \chi(r) \) \((8, 16, 17)\),

\[
\mathbf{A}'(r) = \mathbf{A} + \nabla \chi(r). \tag{2.3}
\]

We can look at a uniform magnetic field along the z-axis, \( \mathbf{B} = e_z B \), where \( e_j \) is the unit vector of axis \( j \), and \( B \) is the magnitude of the magnetic field, to better understand this. The Hamiltonian of a charged particle confined to the \( e_x - e_y \) plane moving in this field can be written as \((8, 17)\)

\[
\hat{H} = \frac{(\hat{p} - q \mathbf{A}(r))^2}{2M}, \tag{2.4}
\]

where \( q \) is the charge of the particle. This equation comes from classical electromagnetism and can be extended to quantum mechanics through the traditional manner by replacing the classical momentum with the momentum operator, \( \hat{p} = -i\hbar \nabla \) \((17)\). For a wave function \( \psi(r,t) \) the Schrödinger equation is then \((17)\)

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(r,t)}{\partial t} = \frac{(-i\hbar \nabla - q \mathbf{A}(r))^2}{2M} \psi(r,t). \tag{2.5}
\]

The transformation in Eqn. \((2.3)\) cannot change the eigenfunctions of the operator Eqn. \((2.4)\). Specifically, the modulus squared, \( |\psi|^2 \), must remain unchanged \((17)\). To rectify this issue the substitution \((8, 16, 17)\)

\[
\psi'(r,t) = \psi(r,t)e^{i\frac{qB}{\hbar}\chi(r)}, \tag{2.6}
\]

can be made. If \( \psi \) is a solution to the Schrödinger equation with vector potential \( \mathbf{A} \), then the wave function \( \psi' \) is the solution of a Schrödinger equation with potential \( \mathbf{A}' \) \((8, 17)\). By adding the gradient of a scalar field to the potential \( \mathbf{A} \), the phase of the wave function changes proportionally as well, which is the definition of a gauge invariant quantity in quantum dynamics \((16)\). This also means to determine the eigenstates for a specific situation we must choose the specific gauge. Of the unlimited possible choices, two commonly used vector fields in the \( xy \) planes are the Landau gauge \( (A_{\text{Land}}) \) and the symmetric gauge \( (A_{\text{sym}}) \) \((1, 7, 8, 17)\),
The two gauges are proportional to each other with a function \( \chi \) existing that allows for the transformation between them. In this case, \( \chi \) would be proportional to \( Bxy \) (8).

Let’s continue by looking at the cyclotron motion of a particle in 2D caused by a homogeneous \( B \) field perpendicular to the 2D plane of the particle. The Hamiltonian of Eqn. (2.4) can be rewritten based on the kinetic momentum \( \Pi \) (7, 8),

\[
\hat{H} = \frac{\hat{p}_x^2}{2M} + \frac{(\hat{p}_y - qB\hat{x})^2}{2M},
\]

(2.8)

where \( \Pi = \hat{p} - qA(\hat{r}) \). The kinetic momentum operator satisfies the commutation relation \([\Pi_x, \Pi_y] = i\hbar B\), which shows us the operator algebra is formally equivalent to that of a harmonic oscillator (7, 8, 17).

To show a specific example we can briefly consider the Landau gauge where \( A(\mathbf{r}) = Bx \mathbf{e}_y \). The Hamiltonian takes the form (1, 8, 17)

\[
\hat{H} = \frac{\hat{p}_x^2}{2M} + \frac{(\hat{p}_y - qB\hat{x})^2}{2M}.
\]

(2.9)

The position operator \( \hat{y} \) is absent due to the choice of gauge. This means \( \hat{p}_y \) will commute with this Hamiltonian which can also be seen as the momentum in \( y \) being conserved (17). Because of this we can look for a solution in the form of plane waves along \( y \) axis (8),

\[
\Psi = \psi(x)e^{iky}.
\]

(2.10)

By considering a sample of a finite size \( L_y \) along the \( y \) axis, and taking periodic boundary conditions into consideration, the quantum number \( k \) becomes quantized such that \( k = 2\pi n_y/L_y \). We can then solve the Schrödinger equation for this wave function and Hamiltonian (1, 8, 17),

\[
\frac{-\hbar^2}{2M} \psi''_k(x) + \left(k - \frac{qBx}{\hbar}\right)^2 \psi_k(x) = E\psi(x).
\]

(2.11)
Making the substitution of \( \omega_c = |q|B/M \) allows for Eqn. (2.11) to be written as (8, 17),

\[
\frac{-\hbar^2}{2M} \psi''_k(x) + \frac{1}{2} M \omega_c^2 (x - x_k)^2 \psi_k(x) = E \psi(x),
\]

(2.12)

where \( x_k = \hbar k/qB = k \ell_{mag}^2 \). An eigenvalue equation of a harmonic oscillator centered at \( x_k \) with oscillation frequency \( \omega_k \) can be seen for each wave vector \( k \). The term \( \omega_c \) is known as the cyclotron frequency, and \( \ell_{mag} \) is called the magnetic length, which can be understood as the minimum size of a cyclotron orbit based on the Heisenberg inequality: \( \Delta r \Delta v \geq \hbar/(2M) \) (8). For each \( k \) vector, highly degenerate energy levels emerge known as Landau levels (8, 17, 18),

\[
E_n = (n + \frac{1}{2}) \hbar \omega_c, \quad n \in \mathbb{N}.
\]

(2.13)

The states belonging to the ground level of this harmonic oscillator are referred to as the lowest Landau levels (LLLs). The Landau levels are the quantized cyclotron orbits of charged particles in the presence of a magnetic field (17).

By taking a linear combination of the different oscillators in the \( y \) plane, we can find the LLL ground state wave functions. Each one of these functions is proportional to a Gaussian centered at \( x_k \) with width \( \ell_{mag} \) (8, 17):

\[
\Psi_k(r) \propto e^{-\frac{(x-x_k)^2}{2\ell_{mag}^2}} e^{iky}.
\]

(2.14)

These wave function’s can be considered to be orthogonal due to their different variations in \( y \). This leads to the relation of (8)

\[
\int \Psi_{k}'(r) \Psi_{k'}(r)dy \propto \int e^{-iky} e^{ik'y}dy = L_y \delta_{n_y,n_y'},
\]

(2.15)

where \( n_y \) and \( n_y' \) are quantized integers characterizing the wave numbers \( k \) and \( k' \), and \( \delta_{n_y,n_y'} \) is the Kronecker delta. The periodicity in \( k \) leads to the separation of two consecutive \( k \) values by \( 2\pi/L_y \). This causes the values of the corresponding oscillators to also be closely clumped together (8):

\[
x_k - x_{k'} = \frac{2\pi \ell_{mag}^2}{L_y} \iff k - k' = \frac{2\pi}{L_y}.
\]

(2.16)

Since the value of \( \ell_{mag} \) is much smaller than the area \( L_y \), the distance between oscillator centers is very small.
We can now determine the degeneracy of Landau levels for a given area $L_x \times L_y$ by considering only the eigenstates $\Psi_k$ whose center $x_k$ are located within this rectangle. This imposes a limitation on the wave number $k$ based on the size of the width of the level $L_x$ (8):

$$0 \leq x_k \leq L_x \Rightarrow 0 \leq k \leq \frac{L_x}{\ell_{mag}^2}. \quad (2.17)$$

This brings us to the number of degenerate $\Psi_k$ states per area, $A = L_x L_y$ (1, 7, 8):

$$\mathcal{N} \approx \frac{L_x L_y}{2\pi \ell_{mag}^2} = \frac{A}{2\pi \ell_{mag}^2}. \quad (2.18)$$

We can see this as each independent eigenstate in a given Landau level occupying an area of $2\pi \ell_{mag}^2$. Using the relationship of the magnetic length to the $B$-field strength allows for the degeneracy to be written as (1, 7, 8)

$$\mathcal{N} = \frac{A B q}{\hbar} = \frac{\Phi}{\Phi_0}, \quad (2.19)$$

where the flux quanta, $\Phi_0 = h/q$, and the flux of the magnetic field, $\Phi = AB$ have been introduced. The flux quanta can be considered as the magnetic flux per area $2\pi \ell_{mag}^2$ and plays a crucial role in the quantum phenomena that occur when a magnetic field is present (1). At first glance it might seem weird that we went from a system with two degrees of freedom, a particle moving in a plane, to one represented by a single harmonic oscillator, but this is what leads to the wild degeneracy that makes the Landau levels so interesting (1).

The Landau levels play an important role in many different physical phenomena. They are the base for understanding the quantized interactions between particles and magnetic fields, and can be seen in many different cold atom simulation experiments (7, 19, 20, 21, 22, 23, 24). The Landau levels form a band like structure similar to particles in a periodic lattice with each of the levels having a Chern number of 1 (7, 21). Because of this an analogy can also be made between a lowest band energy band with Chern number 1 and a filled Lowest Landau Level (6, 18, 19, 22, 25). At integer fillings of the Landau levels we see the emergence of the integer quantum Hall effect (11, 26). The fractional quantum Hall effect is also explained through the fractional filling of Landau levels with quasi-particles (1, 12). Finally we can also consider the tight-binding scenario with charged particles confined to a periodic potential. An
interesting phenomena occurs when such a system is exposed to a magnetic field that is a rational fraction number of flux quanta $\Phi_0$. If the magnetic flux is parameterized such that $\Phi/\Phi_0 = p/q$, where $p$ and $q$ and co-prime integers, it will cause each Landau level to split into $p$ subbands (10, 27). This leads to the beautiful fractal pattern known as the Harper Hofstadter Butterfly (10, 19, 27). All of these interesting systems are linked back to an understanding of the Landau levels. The highly controllable and tunable quantum systems available in ultracold atom experiments make them an appealing choice for simulating these types of phenomena.
3

Geometric phases and dressed states

Frequently in quantum mechanics we find a state that evolves in time in such a way that it returns to its initial state after time $T$. As this state evolves, it is able to acquire a memory of the trajectory it has taken in the form of a phase added to its wave function (28). Specifically, it was discovered by Sir Michael Berry, that as this occurs the phase can include a purely geometric component (5). This additional component contains a memory of its motion after it returns to its initial physical state and is known as the Berry phase (5, 28).

An anholonomic system is one described by a set of parameters that evolves in such a way that the system returns to the same initial parameters, but not the same original state (5, 28, 29). A commonly used example is the Foucault pendulum, which describes the behavior of a pendulum oscillating on the earth’s surface as the earth rotates. If the pendulum is positioned directly at the equator, it will return to the same position and same oscillation plane after 24 hours. However, if the pendulum is located anywhere else, the oscillation plane of the pendulum will be rotated by an angle related to the distance from the equator (8, 29, 30).

For looking at the Berry’s phase we must consider an anholonomic system like this in order to generate a geometric phase. The Berry’s phase is gauge invariant, geometrical, and closely linked to gauge field theories and differential geometries (31).
This makes it a powerful unifying concept in many different areas of study such as in quantum Hall physics (1). In simple terms the Berry’s phase can be seen to generate Aharonov-Bohm like phases even for non-charged particles (5, 8, 9, 31).

To see how this geometric Berry’s phase emerges, we consider a state based on a parameter $\lambda$ that evolves slowly through time. The state evolves in such a way that $\lambda$ is the same at time $t = 0$ and $t = T$,

$$\lambda(0) \rightarrow \lambda(t) \rightarrow \lambda(T) = \lambda(0),$$

and that the adiabatic theorem is valid. This means we can assume the wave function will be found in the same eigenstate initially and after time $T$ (1, 8, 29, 32, 33). The parameter $\lambda$ is controlled by an external operator such that for each value of $\lambda$, the wave function $|\psi_n(\lambda)\rangle$ and energy $E_n(\lambda)$ are the eigenstates and energies of the Hamiltonian $\hat{H}(\lambda)$ (5, 8, 29, 31),

$$\hat{H}(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle.$$  

(3.2)

We can also assume that the set of all possible wave functions, $\{ |\psi_n(\lambda)\rangle \}$, form an orthonormal basis in Hilbert space for each value of $\lambda$ (8).

The state vector’s evolution is then calculated to examine how the state is changing as $\lambda$ slowly evolves (8). We can determine the state vector at time $T$ using the time dependent wave function (8)

$$|\psi_n(t)\rangle = \sum_n c_n(t)|\psi_n(\lambda(t))\rangle.$$  

(3.3)

For simplification, we can examine a system that initially resides in a single eigenstate $|\psi_{\ell}\rangle$,

$$c_{\ell}(0) = 1, \quad c_n(0) = 0 \quad \text{if} \quad n \neq \ell.$$  

(3.4)

The system is proportional to $|\psi_{\ell}(\lambda(t))\rangle$ at any time $t$ (8) since we are considering a system in which the adiabatic theorem is valid. This allows for us to use the Schrödinger equation to determine the time evolution operator (8),
\[
\frac{i\hbar}{\partial t} \frac{\partial c_\ell}{\partial t} = \left[ E_\ell - i\hbar \frac{\partial}{\partial t} \langle \psi_\ell | \nabla \psi_\ell \rangle \right] c_\ell, \quad (3.5)
\]

and to introduce the vector known as the Berry’s connection, \( A_\ell(\lambda) \),

\[
A_\ell(\lambda) = i\hbar \langle \psi_\ell | \nabla \psi_\ell \rangle. \quad (3.6)
\]

We can perform the integration of Eqn. (3.5) over the period \( t = 0 \to t = T \) by considering a situation in which the parameter \( \lambda \) follows a closed contour \( C \), such that the condition of Eqn. (3.1) is fulfilled (8),

\[
c_\ell(T) = \frac{-i}{\hbar} \int_0^T \left[ E_\ell(t) - \frac{\partial}{\partial t} A_\ell(\lambda) \right] c_\ell(t) \, dt = e^{i\Phi_{dy}(T)} e^{i\Phi_{geo}(T)} c_\ell(0). \quad (3.7)
\]

Two quantities have been introduced. The first is the dynamical phase \( \Phi_{dy} \) (4, 8),

\[
\Phi_{dy}(T) = \int_0^T E_\ell(t) dt.
\quad (3.8)
\]

We can think of this as the part of the phase that tells us how long the trip took. This is because as the time increases so does the dynamical phase (29). The second term, \( \Phi_{geo} \), is the geometrical phase, which tells us where the system has been,

\[
\Phi_{geo} = \int_0^T \frac{\partial}{\partial t} A_\ell dt = \oint_C A_\ell \cdot d\lambda. \quad (3.9)
\]

This phase, also referred to as the Berry’s Phase, no longer relies on the time it took to travel the path. Instead, it only relies on the trajectory of \( \lambda \) during the evolution of the system (5, 8, 29). These quantities can be seen as gauge invariant since they will remain unchanged if the wave function \( |\psi_n(\lambda(t))\rangle \) is multiplied by an arbitrary phase (8, 29, 31). This also means both the exponential terms of the dynamic and geometric phases can be thought of as physical quantities.

By thinking of \( \lambda \) as a parameter describing either the position of a particle or its quasi-momentum, we can examine a new real, gauge invariant, vector field known as the Berry’s curvature (4, 5, 29),

\[
\mathcal{B}_\ell = \nabla \times A_\ell. \quad (3.10)
\]
This can be thought of as an artificial magnetic field (8). The Berry’s phase collected by a particle as the parameter $\lambda$ travels along the closed contour $C$ becomes (5, 8, 29, 31)
\[
\Phi_{geo} = \oint_S \mathbf{B}_\ell \cdot d\mathbf{a},
\] (3.11)
where $a$ is the area of the surface $S$ that is enclosed by the contour of $C$. In the last step Stokes’s theorem was used to rewrite the geometric phase. This theorem states; the surface integral of the curl of a vector field, $\mathbf{F}$, over a surface, $S$, in 3D Euclidean space is equivalent to the contour integral of the vector field over the boundary of the surface $\partial S$ (29, 31, 34),
\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{F} \cdot dr.
\] (3.12)

This leads us to an anholonomic phase that can be seen to be analogous to the Aharonov-Bohm phase (8, 9, 29). The Berry’s curvature and phase are gauge invariant quantities. This is because a different choice of gauge can lead to a different Berry’s connection but the same Berry’s curvature (8, 35).

The adiabatic following of a dressed state can lead to the accumulation of a Berry’s phase for ultracold atoms in a laser field (4, 8, 15, 35). To define the Hamiltonian under the rotating wave approximation (RWA) we use the center of mass motion of the atom based on the position operator $\hat{r}$ and momentum $\hat{p}$, and the movement of the atom through internal states based on interactions with the light field (4, 6, 8, 18, 21, 25, 35);
\[
\hat{H} = \frac{\hat{p}^2}{2M} \hat{1} + \hat{H}_{int}(\hat{r}),
\] (3.13)
where $\hat{1}$ is the identity operator in the internal Hilbert space, and $\hat{H}_{int}$ is the time independent Hamiltonian describing the coupling of the atom’s internal degrees of freedom with the light field (4, 6, 8, 21, 22). We can treat $\hat{r}$ as an external parameter and define the dressed states as the eigenstates of $\hat{H}_{int}$ (4, 8),
\[
\hat{H}_{int}(r)|\psi_n(r)\rangle = E_n(r)|\psi_n(r)\rangle,
\] (3.14)
with $\{|\psi_n(r)\rangle\}$ forming a complete basis in Hilbert space at any point $r$. The total wave function can then be written as the linear combination of all possible quantum states(8);
\[
\Psi(r, t) = \sum_n \phi_n(r, t)|\psi_n(r)\rangle.
\] (3.15)
Using the same logic as above in Eqn. (3.3), we can assume the atoms will remain in their initial state $\ell$ and that the states $n \neq \ell$ can be neglected. This allows for the Schrödinger equation (4, 8),

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( \frac{-\hbar^2}{2M} \Delta + \mathbf{H}_{int}(r) \right) \Psi(r,t), \quad (3.16)$$

to be written in terms of the probability amplitude $\phi_{\ell}(4, 8)$:

$$i\hbar \frac{\partial \phi_{\ell}}{\partial t} = \left[ \left( \frac{\mathbf{p} - A_\ell(r)}{2M} \right)^2 + E_{\ell}(r) + V_{\ell}(r) \right] \phi_{\ell}(r,t). \quad (3.17)$$

Eqn. (3.17) has the same structure of the scalar Schrödinger equation presented earlier in Eqn. (3.6), for a charged particle moving in a magnetic field created by the vector potential $A_\ell$, only with an additional potential (4, 8). This potential is the sum of two terms: the energy of the occupied dressed state $E_{\ell}(r)$, also known as the adiabatic scalar energy arising from the spatial variations in the potential, and the geometric scalar potential $V_{\ell}(r) (4, 8, 35)$,

$$V_{\ell}(r) = \frac{\hbar^2}{2M} \sum_{n \neq \ell} |\langle n | \nabla | \ell \rangle|^2. \quad (3.18)$$

This scalar potential can be thought of as the energy created by the micro-motion of the atom as it makes virtual transitions between other dressed states $\psi_{n \neq \ell}$ and the initial dressed state $\psi_{\ell} (4, 8, 35, 36)$.

To look at a simple example we consider a two level atom. An excited atomic state with a longer lifetime than the relevant time scale of an experiment is chosen so that spontaneous emissions can be ignored. An example of this level scheme can be seen in Fig. 3.1(a) (8). We can find this type of level scheme in strontium as well as other atoms (8, 37). A $\Lambda$ style Raman transition scheme as in Fig. 3.1 (b), in which two internal states are linked through an additional excited state, is another possible level scheme (4, 8, 15, 35). With the proper choice of basis, this system reduces to a two level like system (38). By looking at a two level system, we are able to describe the atom-laser coupling purely based on the Rabi frequency $\kappa$ and the detuning $\Delta$.

The interaction Hamiltonian can be written as (8)

$$\mathbf{H}_{int} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}; \quad (3.19)$$
Figure 3.1: An example of two toy models that can be used for the description of a two level system used in the text. For (b) when the detuning of $\Delta_e$ is large enough, the excited state acts only as a relay between the two ground states so can be neglected allowing for the system to be considered as a two level system in the subspace $\{|g_1\rangle, |g_2\rangle\}$ with two parameters the Raman detuning $\Delta$ and the two photon Rabi frequency $\kappa = \frac{a_+ a_-}{2\Delta_e}$ (8). Taken from (8).

using the basis of $\{|g\rangle, |e\rangle\}$ or $\{|g_1\rangle, |g_2\rangle\}$. The Rabi frequency $\kappa$ is associated with the coupling strength between the light and internal states. Both $\kappa$ and the light-induced detuning $\Delta$ depend on the center of mass motion of the atom (4, 35). We can transform this Hamiltonian to one defined by the mixing angle $\theta$, phase angle $\phi$, and generalized Rabi frequency $\Omega$ using (4, 8)

$$\Omega = \sqrt{\Delta^2 + |\kappa|^2}, \quad \cos \theta = \frac{\Delta}{\Omega}, \quad \sin \theta = \frac{|\kappa|}{\Omega}, \quad \kappa = |\kappa| e^{i\phi}. \quad (3.20)$$

This allows the atom light coupling to be written as (4, 8)

$$\hat{H}_{\text{int}} = \frac{\hbar \Omega^2}{2} \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}. \quad (3.21)$$

The corresponding eigenstates of this matrix are (4, 8)

$$|\psi_+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} -e^{-i\phi} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}, \quad (3.22)$$

with eigenvalues of $\pm \hbar \Omega/2$. By assuming the atom’s state adiabatically follows one of these dressed states we can write the Berry’s connection and curvature for each state using Eqn. (3.6) and Eqn. (3.10) (4, 8, 35),

$$\mathcal{A}_\pm = \pm \frac{\hbar}{2} (\cos \theta - 1) \nabla \phi, \quad \& \quad \mathcal{B}_\pm = \pm \frac{\hbar}{2} \nabla (\cos \theta) \times \nabla \phi. \quad (3.23)$$
We can then also calculate the strength of the geometric scalar potential $V_\pm$ for each state (8);

$$V_\pm(r) = \frac{\hbar^2}{8M} \left[ (\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2 \right]. \quad (3.24)$$

Looking at the definition of the Berry’s curvature in Eqn. (3.23), we see for the simulation of a non zero magnetic flux to occur we must have both a gradient of the mixing and phase angles, $\theta$ and $\phi$ respectively (8). A gradient of the mixing angle can be obtained through either a detuning or intensity gradient of the coupling beams (8).

The three expressions above $V$, $B$, and $A$ are all calculated for a system with only two internal states. Only a change to the prefactor of these equations is required to make an extension to a system with $N$ internal dressed states $|m_F\rangle \equiv |m_F = -f, -f+1, ..., f\rangle \equiv |m_F(r)\rangle$, where $f = \frac{N-1}{2}$ is the total angular momentum quantum number (35). These internal states do not even need to necessarily represent true spin states as long as the interaction Hamiltonian is defined using the proper angular momentum algebra (35). The state dependent Berry connection and curvature both change from $\hbar$ to $\hbar m_F$, and the scalar potential becomes (35)

$$V_\pm(r) = \frac{\hbar^2 g_{f,m_F}}{4M} \left[ (\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2 \right], \quad (3.25)$$

where $g_{f,m_F} = f(f + 1) - m_F^2$. Although the scalar potential relies on both the total momentum $f$ as well as the projection $m_F$, we can see that it reduces to the proper expression for the 2 state case. For further details on this see Ref. (35).

For the Hamiltonian used in Eqn. (3.17) to be valid we must be in the regime where the angular velocity of the eigenstate $|\psi_\ell\rangle$ is much smaller than the generalized Rabi frequency, $\Omega$, of the two level system (8). Looking at the expressions of the dressed states in Eqn. (3.22), we see the states vary on a length scale of $k^{-1} = \lambda/2\pi$. This allows for us to estimate the angular velocity of a dressed state as $\sim kv$ for an atom moving at velocity $v$. Using the recoil velocity acquired when an atom absorbs a photon, $\hbar k/M$, as the approximate velocity with which the atoms are traveling we come to the condition;

$$E_r \ll \hbar \Omega, \quad (3.26)$$
where the recoil energy $E_r = \hbar^2 k^2 / 2M$ has been introduced. This condition must be fulfilled for the adiabatic approximation to be considered applicable in Eqn. (3.17) (8).

Creating a synthetic magnetic field through the coupling of light fields with the internal atomic states was demonstrated in 2009 by the group of I. Spielman in a Bose Einstein Condensate (BEC) of $^{87}$Rb (15). This was done through the use of a detuning gradient (15). In Fig. 3.2 an overview of this experiment can be seen. This includes the basic experimental geometry in (A) consisting of a pair of Raman laser beams with momentum difference along $\hat{x}$, a level diagram for the states used in the Raman coupling of the three $m_F = 0$ and $\pm 1$ in (B), and the tunable energy–momentum dispersion relations in (C) (15). The graph in (D) shows how the detuning gradient created by an external magnetic field gradient generates a spatial gradient, $A_x^*$ leading to an approximately uniform synthetic $B^*$ (15). Finally, the images in (E, F) show absorption images of the atomic clouds after a 25.1 ms time-of-flight (TOF) expansion where both the momentum and spin states can be seen. When a gradient of detuning is added to the system, (f), vortices emerge in all three of the $m_F$ states. The presence of these vortices in a BEC cloud are a clear indication a synthetic magnetic field has been created (8, 15). The number of vortices can also be to increase proportionally to the size of the detuning gradient and the length of time that the BEC was exposed to the gradient (15). This provides one example of how an adiabatically dressed state has been used experimentally to create an artificial magnetic flux.
Figure 3.2: A summary of the scheme implemented in Ref. (15). (A) The experimental setup used in which the BEC is suspended by a crossed dipole trap. There is a magnetic field $B = (B_0 - b' y) e_y$ where $b'$ is the gradient of the magnetic field that causes a gradient of the laser detuning (15). The two Raman beams are travelling along $e_y = e_x$ and linearly polarized along $e_y = e_x$ and have frequencies $\omega_L$ and $\omega_L + \Delta \omega_L$. (B) The Raman coupling scheme within the $F=1$ manifold. $\omega_z$ and $\varepsilon$ are the linear and quadratic Zeeman shifts respectively, $\delta$ is the Raman detuning. (C) The energy–momentum dispersion relations where the grey lines represent the states before Raman coupling, and the three colored lines represent the dressed states. An arrow also indicates where the minimum is. (D) A plot of the vector potential versus the Raman detuning where a shift in the minimum can be seen with respect to the detuning. Finally, two absorption images taken after a 25.1 ms TOF can be seen: (E) one image without any detuning gradient, (F) one image with a detuning gradient in which vortices can be seen. The different momentum states $m_F = 0, \pm 1$ are separated along the $e_y$ axis due to the Stern-Gerlach effect (15). Taken from Ref. (15).
Optical flux Lattices

A different type of laser lattice configuration for trapping ultracold atoms was proposed by Nigel Cooper (6). A traditional optical lattice periodically confines atoms in real space, whereas this new lattice configuration periodically confines atoms in momentum space (25). This is known as an "optical flux lattice" (6). Not only can an optical flux lattice impart the usual periodic potential of a conventional optical lattice, it can also produce a non-zero periodic magnetic flux density with non-zero mean and large magnitude of $\Phi \sim 1/\lambda^2$ (6). An important distinction between previous proposals that take place in the tight binding regime and this new one is the continuous function of the flux density with relation to position. This is because the flux lattice is not restricted to the tight binding regime in real space (6). We will also see that optical flux lattices can cause energy bands with non zero Chern numbers to be visible, and lowest energy bands that are topologically equivalent to the lowest Landau level (6, 18, 21, 22, 35, 39).

Let’s start from Eqn. (3.13) to examine how an optical flux lattice can be implemented. We can redefine the interaction Hamiltonian as (6, 22, 35)

$$\hat{H}_{int}(r) = \hbar m_F \hat{v}(r) = \hbar m_F \hat{\sigma} \cdot \hat{\Omega} = \hbar m_F \begin{pmatrix} \Omega_z & \Omega_x - i\Omega_y \\ \Omega_x + i\Omega_y & -\Omega_z \end{pmatrix},$$

(4.1)

$\hat{\sigma}$ is the vector of Pauli matrices and $\hat{\Omega} = \hat{\Omega}(r) = \{\Omega_x(r), \Omega_y(r), \Omega_z(r)\} \equiv \{\Omega_x, \Omega_y, \Omega_z\}$ is the vector describing the spatially dependent coupling between the atomic internal states. The coupling vector can be parameterized to the spherical coordinates of the earlier example through (35)

$$\tan \phi = \frac{\Omega_y}{\Omega_x}, \quad \cos \theta = \frac{\Omega_z}{\Omega}, \quad \text{and} \quad \Omega = \sqrt{\Omega_x^2 + \Omega_y^2 + \Omega_z^2},$$

(4.2)
The matrix element $\Omega_z$ can be seen to represent the light induced detuning of the transition, and the element $\Omega_x \pm i\Omega_y$ can be seen to depict the transition matrix element coupling the states together (35). This is then also consistent with definitions outlined in Eqn. (3.19). The transformation does not change the functions describing the Berry’s connection and curvature in Eqn. (3.23) (6, 35).

We can then consider the coupling vector $\hat{\Omega}$ to be spatially periodic in the $e_x - e_y$ plane (35),

$$\hat{\Omega}(r) = \hat{\Omega}(r + r_{n,m}), \quad \text{with} \quad r_{n,m} = na_1 + ma_2.$$  \hspace{1cm} (4.3)

The primitive vectors $a_1$ and $a_2$ define the 2-D lattice spacing in $e_x - e_y$. Because of this periodicity, the collected phase must be a multiple of $2\pi$ causing the total magnetic flux over an elementary cell to be zero (6, 8, 35),

$$\hbar \Phi = \oint_{\text{cell}} A \cdot dr = \int_{\text{cell}} B \cdot da = 0,$$  \hspace{1cm} (4.4)

where the substitution $B = B(r) + B_{AB}$ has been made to the Berry’s curvature in Eqn. (3.23). The total phase of zero is a combination of a continuous background magnetic field $B(r)$, as well as possible Aharonov Bohm style gauge dependent singular fluxes $B_{AB}(r)$ (4, 6, 35). Together these AB style singularities carry a non zero number of Dirac flux quanta, and can cause a nonzero flux to be experienced over the cell (35). These “Dirac strings” coincide with the points where the $\nabla \phi$ contribution of Berry’s connection may be singular. This can occur if the $(\cos \theta - 1)$ term (for our specific gauge choice) does not properly compensate the singularity by going to zero as well in Eqn. (3.23) (35). This happens at $\cos \theta \to -1$ in this gauge choice (35), and the Berry’s connection reduces to $A \to -2\hbar m_F \nabla \phi$ (35). The physically observable flux can be seen as the remaining flux after these non-measurable singularities have been removed (35),

$$\hbar \Phi' = \int_{\text{cell}} B \cdot da = -\oint B_{AB}(r) \cdot da.$$  \hspace{1cm} (4.5)

We can also calculate this remaining flux through the summation of the singular points in the vector potential around which the integration is carried out (35)

$$\hbar \Phi' = -\sum_{\text{sing}} \oint A \cdot dr.$$  \hspace{1cm} (4.6)
4.1 Square optical flux lattice

At each one of the points the vector potential contains a Dirac string piercing the $e_x - e_y$ plane, and imposes the requirement that for a non-zero flux $\Phi'$, the sum of singular contributions must be nonzero \((6, 22, 35)\). An optical flux lattice contains an array of gauge dependent Dirac strings and a non-staggered background magnetic field of the opposite sign. Since the Dirac strings are non-measurable they must be removed from any physical consideration, which causes a non-zero magnetic flux to be possible over an optical flux lattice \((6, 35)\).

There are three main requirements for the generation of an optical flux lattice. The first is that the coupling laser \((\Omega_{x,y})\) must generate optical vortices \((6)\). The degeneracy of dressed states at the core of these vortices must be eliminated, which leads to the second requirement that the species dependent potential \((\Omega_z)\) at these points must be nonzero \((6)\). The final requirement is that the species dependent potential varies in such a way that it produces an unequal number of vortices and antivortices per unit cell. This is because total flux through the unit cell can be seen to be \(N_\phi = N_{v+}^+ - N_{av}^+\) where \(N_{v/av}^+\) is the number of vortices/antivortices with a positive \(\Omega_z\) \((6)\). There are multiple different ways that the above requirements can be fulfilled to allow for an optical flux lattice to be generated.

4.1 Square optical flux lattice

Let us consider a simple model for a square optical flux lattice created by five laser beams intersecting at right angles: one pair of counterpropagating beams along $e_x$, one pair of counterpropagating beams along $e_y$, and a single beam propagating along the $e_z$ axis \((6, 35)\). We can define dimensionless coordinates based on a coupling period of $2a$ along $e_x$ and $e_y$. The spatially periodic coupling vector $\hat{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ can then be represented as \((6, 35)\)

\[
\Omega_x = \Omega_\perp \cos(\bar{x}), \quad \Omega_y = \Omega_\perp \cos(\bar{y}), \quad \Omega_z = \Omega_\parallel \sin(\bar{x}) \sin(\bar{y}),
\]

with \((\bar{x}, \bar{y}, \bar{z}) = \frac{a}{2}(x, y, z)\). In the symmetric case $\Omega_\perp = \Omega_\parallel$ the following reduces down to the scheme described in Ref. \((6)\), and follows directly the procedure outlined in Ref. \((35)\). The total Rabi frequency for the coupling vector can be written as \((35)\)

\[
\Omega = \sqrt{\Omega_\parallel^2 + (\Omega_\perp^2 - \Omega_\parallel^2)(f_x^2 + f_y^2) + \Omega_\parallel f_x^2 f_y^2},
\]
where the substitution of \( f_u = \cos(u) \) has been made. The adiabatic energies \( E_{mF} = \hbar m_F \Omega \) have a periodicity, which is half that of the atom-light coupling (35).

We can consider the case where \( m_F < 0 \), and the coupling vector strength fulfills \( \Omega_\perp^2 > \Omega_\parallel^2/2 \). The minima of the adiabatic scalar potential created by the beams are found at \( \bar{x}_n = \pi n \) and \( \bar{y}_m = \pi m \) where the value of \( \Omega_z \) goes to zero. The maxima are found at the points where \( \Omega_x + i\Omega_y \) goes to zero: \( \bar{x}_{n,max} = \pi(n + 1/2) \) and \( \bar{y}_{m,max} = \pi(m + 1/2) \) (35). By solving for the total Rabi frequency Eqn. (4.8), the adiabatic scalar energies for the minima and maxima of the potential can be seen to be \( E_{min} = \hbar m_F \sqrt{2} \Omega_\perp \) and \( E_{max} = \hbar m_F \Omega_\parallel \) (35).

The value of the coupling vector components in Eqn. (4.8) can be estimated around these maxima energy points to analyze the possibility of creating a flux over this lattice (35):

\[
\Omega_x \approx -\Omega_\perp (\bar{x} - \bar{x}_{n,max})(-1)^n, \quad \Omega_y \approx -\Omega_\perp (\bar{y} - \bar{y}_{m,max})(-1)^m, \quad \Omega_z \approx \Omega_\parallel (-1)^{n+m}.
\]

(4.9)

When the value of \( (n + m) \) is even(odd) the sign of \( \Omega_z \) is negative(positive), while the sign of the angle \( \phi \) in Eqn. (4.2) is the opposite. This can also be viewed as \( \phi \) rotating anticlockwise(clockwise) for even(odd) values of \( (n + m) \) (35).

To better understand how this will ensure that non zero singularities will occur in the integration of Eqn. (4.11), we can look at the Berry’s curvature represented in terms of a unit vector \( \hat{N} = \hat{\Omega}/\Omega \) (35)

\[
\mathcal{B}(r) = -\hbar m_F \frac{\nabla N_x \times \nabla N_y}{N_z}.
\]

(4.10)

From this we can see that if the sign of \( \Omega_z \) alternates where \( \Omega_{x,y} = 0 \), there is a possibility for it to compensate the alternating sign of \( \nabla N_x \times \nabla N_y \) at the integration points leading to a nonzero magnetic flux (35).

We can now consider the magnetic flux passing the elementary cell defined by the boundaries \( \bar{x} \in [0, 2\pi) \) and \( \bar{y} \in [0, 2\pi) \) based on the singularities created from the Dirac strings piercing the \( e_x - e_y \) plane. For the vector potential \( A \) these points appear at the odd values of \( n + m \) where \( \Omega_z = -\Omega_\parallel \) (6, 35). There are two sites that fulfill this
4.1 Square optical flux lattice

Figure 4.1: A square optical flux lattice calculated for $\beta = 1$. (A) A plot of the position dependent Bloch vector $\Omega$. The vectors show the $e_x - e_y$ components and the color shows the value of the $z$ component. The $z$-component is zero in the white region, positive in the red region, and negative in the blue region increasing to $+1,-1$ respectively as the regions get darker. (B) The arrows show the value of $A(r)$ and the color depicts the strength of the geometrical potential $V(r)$. (C) The magnitude of the effective magnetic field $B$ along the $e_z$ axis. It is worth noting that the maxima of the geometric potential $V(r)$ and the largest $B$ fall in the same regions (35). The plots in (B) and (C) both correspond to the color scale on the right of the images. Taken from Ref. (35).

for the unit cell: $n = 1, m = 0$ and $n = 0, = 1$. Each site provides $2m_F$ flux quanta yielding a total background flux of (35)

$$\Phi' = -\frac{1}{\hbar} \sum_{\text{sing}} \oint A \cdot d\mathbf{r} = -8\pi m_F. \quad (4.11)$$

In the case of a spin 1/2 particle as described in Ref. (6) this yields two flux quanta per unit cell. The components $\Omega$ in the $e_x - e_y$ plane form two vortices and antivortices in the unit cell. However, the Bloch vector in this plane varies smoothly with the sign of the $\Omega_z = \pm 1$ at the vortex cores. This causes all of these areas to wrap the Bloch sphere in the same way resulting in each vortex having the same contribution to the overall flux for the unit cell (6). An example of this can be seen in Fig. 4.1(a) where the local description of the Bloch vector over an elementary cell is provided. The color in the plot depicts the $e_z$ component, and the arrows show the $e_x - e_y$ components (35).

Using Eqn. (4.10) we can solve for an explicit equation for the magnetic flux density (35)

$$B(r) = \hbar m_F \left(\frac{\pi}{a}\right)^2 \frac{\beta(f_x^2 f_y^2 - 1)}{[f_x^2 + f_y^2 + \beta g_x^2 g_y^2]^{3/2}} e_z, \quad (4.12)$$
Figure 4.2: A setup proposed in Ref. (35) for implementing a square optical flux lattice. Two pairs of counterpropagating beams are circularly polarized and traveling $e_x - e_y$ plane. A fifth beam is traveling along the $e_z$ plane with linear polarization. Taken from Ref. (35).

where $g_u = \sin(\tilde{u})$ and $\beta = \Omega_y/\Omega_{\perp}$. A plot of the strength of the flux density $B(r)$ can be seen in Fig. 4.1(c). The strength of the geometric scalar potentials can also be calculated, as in Ref. (35), and shown to correspond to the strength of the magnetic flux present in the cell as seen in Fig. 4.1.

A setup was proposed for a square flux lattice in Ref. (35) for an atom with an $F = 1$ ground state as found in some commonly used alkali atoms. The setup (Fig. 4.2) contains two pairs of counter propagating beams with circular polarization in the $e_x - e_y$ and one vertical beam with linear polarization and an additional frequency offset $\delta \omega$ in comparison to the horizontal beams, and are used to induce Raman transitions (35). The total electric field of this setup is

$$E_\omega = E_{x^+} + E_{x^-} + E_{y^+} + E_{y^-}, \quad \text{and} \quad E_{\omega^+} = E_{z},$$

(4.13)

where $E_\omega$ is the electric field in the $e_x - e_y$ plane and $E_{\omega^+}$ is the field with angular frequency $\omega + \delta \omega$ traveling along $e_z$. The individual beam fields can be written as (35)
4.1 Square optical flux lattice

\[ E_{x,z} = E_{xy} \left( e^{i\phi/2} \cos \theta_p e_z \pm e^{-i\phi/2} \sin \theta_p e_y \right) e^{\pm i k_R x}, \]
\[ E_{y,z} = E_{xy} \left( e^{-i\phi/2} \cos \theta_p e_z \mp e^{i\phi/2} \sin \theta_p e_x \right) e^{\pm i k_R y}, \]
\[ E_z = \frac{E_{xy}}{\sqrt{2}} (e_x + e_y) e^{ik_r z}. \]

The four horizontal beams are assumed to have the same electric field amplitude of \( E_{x,y} \), the vertical beam has electric field amplitude \( E_z \), and both have the wave vector \( k_R = 2\pi/\lambda \). The angle \( \phi \) describes the ellipticity of the lasers travelling in the \( e_x - e_y \) plane with \( \theta_p \) defining the angle from vertical that the major axes are tipped.

If \( \phi = \pi/2 \) all four of the beams have right handed circular polarization. The relative phase difference between the two pairs of counter propagating beams for each horizontal axis are described by \( \delta \phi_x (\delta \phi_y) \) for the \( e_x (e_y) \) axis. The phase difference \( \delta \phi_{xy} \) is the overall phase difference between all the beams in the \( e_x - e_y \) plane (35). There are also similar phases that can be considered with the \( e_z \) axis, but this only leads to slight displacements along \( e_z \), and active stabilization of the phase between the \( e_{x,y} \) beams and the \( e_z \) beam will not be required (35).

Using the field above to define the scalar potential and properly expressing the Hamiltonian allowing for the coupling vector and magnetic flux of the system to be found. To see the sensitivity of the system to certain perturbations we can examine the solutions to this more complex example. When the proper steps are taken the magnetic flux density becomes (35)

\[ \mathcal{B}(r) = \hbar m_F \left( \frac{\pi}{\alpha} \right)^2 \frac{\beta (f_x^2 f_y^2 - 1 - \tilde{\delta} g_x g_y) \sin(-2\varphi_-)}{f_x^2 + f_y^2 + 2 f_x f_y \cos(2\varphi_-) + \beta(\delta + g_x g_y)^2)^{3/2}} e_z. \quad (4.14) \]

The definitions of \( f_u \) and \( g_u \) remain the same as above, but now \( \bar{x} = k_R \bar{x} - \delta \phi_x/2 \) and \( \bar{y} = k_R \bar{y} - \delta \phi_y/2 \), and without loss of generality \( \bar{z} = \pi/4 \) can be used. The term \( \beta = \Omega_\parallel/\Omega_\perp \) remains unchanged, however, the terms for the coupling strengths become \( \Omega_\parallel = 4u_v E_{xy}^2 \sin(\varphi_\parallel) \sin^2 \theta_p/\hbar \) and \( \Omega_\perp = 2u_v E_{xy} E_z \cos \theta_p/\hbar \), where \( u_v \) determines the vector light shift (35). The substitution of \( \phi_\pm = (\delta \phi_{xy} \pm \phi/2) \) has also been used in these expressions, and \( \tilde{\delta} = \delta/\hbar \Omega_\parallel \). Because the expression for \( \mathcal{B}(r) \) relies on so many parameters let us look at the first order sensitivity of this example to perturbations \( \delta = \Delta \delta \) and \( \phi_- = -\pi/4 + \Delta \varphi_- \). Perturbations to \( \phi_+ \) can be neglected since they
only appear in a quadratic term of $\Omega_\parallel$, and the angle $\theta_p$ can be thought of as fixed, which allows for imperfections in the setting to be compensated through adjusting the intensity of the Raman beams (35). We can then look at the RWA effective coupling vector for this system:

$$\hat{\Omega} = \Omega_\perp [\cos(x) + \Delta \varphi \cos(y')]e_x + \Omega_\parallel [-\Delta \varphi \cos(x) - \cos(y)]e_y + \Omega_\parallel [\sin(x) \sin(y) + \Delta \tilde{\delta}]e_z.$$  

(4.15)

Looking at this simplified coupling we can see that the field should be unaltered by fluctuations in the detuning and phase to the first order. The coupling strength $\Omega_\parallel$ must be large in comparison to the fluctuations $\Delta \tilde{\delta}$, which mainly come from noise in the external magnetic field (35).

Finally to end our discussion of square flux lattices we can look at the band structure and Chern numbers characterized for this setup in Ref. (35). Figure 4.3 shows the band structure computed from the Hamiltonian Eqn. (3.13) with the properly defined terms (see Ref. (35) for details). The magnetic field might not be important to the lowest bands if the adiabatic scalar potential does not properly compensate the geometric scalar potential $V(r)$ created by the atom-light interactions. The Chern numbers of the bands are extracted because non zero Chern number bands are analogous to the band structure of charged particles in a magnetic field (6, 35). The Hamiltonian is first modified through a spatially dependent rotation that halves the size of the unit cell and doubles the area of the Brillouin zone (6, 22, 35). The band gaps for the optimum parameters can be seen in Fig. 4.3(a) where a gap between the ground and first excited band of $\Delta E_{01} = 0.107E_L$ is found for the optimum coupling parameters, where $E_L = 2E_R = \hbar^2 k_R^2 / M$ is twice the recoil energy (35). We can see that the lowest Chern bands do have non zero Chern numbers over a wide range of values, with the red lines in Fig. 4.3 (b) showing the values of the coupling strengths where the band gap $\Delta E_{01}$ is non-negligible (35). These results support that a square lattice could be used to create an optical flux lattice capable of creating an artificial magnetic flux (35).
4.2 Triangular flux Lattice

Figure 4.3: (A) The band structure of the lowest four bands including their Chern numbers, and the energy gap of $\Delta E_{01}$ between the first two energy bands. The computation was done for $\Omega_\perp = 1.905E_L$, $\Omega_\parallel = 51E_L$, $\delta = 0$, and $\phi_- = -\pi/4$. The scalar light shift created by the coupling beams was also incorporated into these calculations at $U_\perp = -195E_L$ and $U_\parallel = 0$. See Ref. (35) for further details. (B) This plot shows an evaluation of the ground band Chern number based on different coupling parameters. The scalar light shifts were left constant and values of $\Omega_\perp$ and $\Omega_\parallel$. The red contours shows the parameters at which the gap $\Delta E_{01}$ appears. The blue cross indicates where the band gap is maximum, and where the band structure found in (A) was calculated (35). Taken from Ref. (35).

4.2 Triangular flux Lattice

A second example of a possible setup for implementing an optical flux lattice is described in Ref. (22). This paper shows a triangular optical flux lattice generated for atoms with a ground state of $J_g = 1/2$ and excited state $J_e = 1/2$. This lattice uses three beams travelling in the $e_x - e_y$ at a frequency $\omega_L$ and linear polarization of angle $\theta_p$ from the vertical axis. A fourth $\sigma_-(m = -1)$ polarized beam propagates along the $e_z$ axis at a frequency of $\omega_L + \delta$. Fig. 4.4 provides an illustration of this setup along with the coupling of states that occurs from these different beams (22). A magnetic field is assumed to be present in the $z$ direction separating the ground states $g_{\pm}$ by $\delta$. The $\sim \pi (m = 0)$ polarized horizontal beams and $\sigma_-$ combine to give the proper Raman coupling for the system (22).

Using the proper angular momentum algebra for the $\{|g_+\},|g_-\rangle\}$ basis the interaction Hamiltonian becomes

$$\hat{H}_{int} = \frac{\hbar \kappa_0^2}{3\Delta} \hat{I} + \frac{\hbar}{3\Delta} \left( \frac{|\kappa_-|^2 - |\kappa_+|^2}{E\kappa_0^2} |\kappa_-|^2 \kappa_0^2 |\kappa_+|^2 \right),$$

(4.16)
Figure 4.4: (A) The level system that is considered in Ref. (22) with a ground state of $J_g = 1/2$ and excited state $J_e = 1/2$. The Zeeman splitting between the ground states is $\delta$, and the ground and excited state are coupled through two lasers, one with frequency $\omega_L$ and one with frequency $\omega_L + \delta$. (B) A drawing of the proposed setup with three beams in the $xy$-plane propagating with wave vectors at an angle of $2\pi/3$ from each other. Each beam has a linear polarization at an angle of $\theta$ from the $z$-axis (22). A fourth beam of circularly polarized beam propagates along the $z$-axis at a frequency of $\omega_L + \delta$ (22). Taken from Ref. (22)


4.2 Triangular flux Lattice

where $\kappa_m, m = 0, \pm 1$, are the Rabi frequencies characterizing the light field, and $\hbar m$ is the angular momentum gained along $e_z$ gained by an atom when it absorbs a photon and $\kappa_{tot}^2 = \sum_m |\kappa_m|^2$. $\Delta \omega_L - \omega_A$ gives the detuning of lasers from atomic resonance $\omega_A$, and the condition $|\Delta| \gg |\delta| \gg |\kappa_m|$ has been imposed (22). The term $E$ is a uniform and adjustable coupling characterizing the field of the $\omega_L + \delta$ laser, which is assumed to be a plane wave propagating along $z$ (22). The ac Stark shift induced by this beam has also been incorporated into $\delta$. The superposition of the three plane waves at $\omega_L$ with triangular symmetry intersect each other at angle of $2\pi/3$, with $k_1 = -k/2(\sqrt{3}, 1, 0), k_2 = -k(\sqrt{3}, -1, 0)$, and $k_3 = -k/2(0, 1, 0)$. This leads to a total coupling vector of (22).

$$\kappa = \kappa \sum_{i=1}^{3} e^{ik_i \cdot r} [\cos \theta \hat{e}_z + \sin \theta (\hat{e}_z \times \hat{k}_i)],$$

where $\kappa$ is the Rabi frequency of a single beam. For interpreting the results of the optimization of this Hamiltonian the relative amplitude of the $\omega + \delta$ with respect to the $\omega_L$ field $\epsilon = E/\kappa$ will be used, and the energy associated with the atom-light interaction will be represented as $E = \hbar \kappa^2 / (3\Delta)$. The band structure was computed and optimized focusing on three conditions; a ground state band should have a non-trivial topology characterized by a non-zero Chern number similar to that of the lowest Landau levels, the band should be very narrow in units of recoil energy to mimic the flatness of a Landau level and should be separated from the first excited state by a large gap (8, 22). Fig. 4.5 shows the results of this process for a cut through the first Brillouin zone, and for a few different parameter choices. The dotted blue line represents the situation where $\epsilon = \theta_\rho = 0$. This choice of parameters causes the atoms to only experience a scalar potential $\propto |\kappa_0|^2$ (22). Three special points can be seen inside the first Brillouin zone due to the state dependent gauge transformation that was performed (again to maximize the size of the Brillouin zone): a Dirac point for $g_+$, a Dirac point for $g_-$, an additional Dirac point that is coincident for both the $g_\pm$ ground states (22). These points can be seen circled in blue on Fig. 4.5 (A). As the coupling parameters are modified gaps in the spectrum will open up at these Dirac points, and when both $\epsilon$ and $\theta$ are non zero, the optical dressing leads to a net flux through the unit cell, which indicates time reversal symmetry breaking. At this point the lowest bands can acquire a non zero Chern number (22). In this regime where $E \leq E_r$ a tight binding model is not
4.2 Triangular flux Lattice

Figure 4.5: (A) The band structure for an \( F = 1/2 \) atom experiencing energy associated with the atom light coupling of \( \mathcal{E} = 1.8E_R \) along a path \( \mathbf{k} = -k(2k_2 - 2k_1) - k_3/2 \) can be seen in the left panel. The blue dotted line represents the case of \( \epsilon = \theta = 0 \), where the decoupled \( m_F = \pm 1/2 \) states each have two Dirac points indicated by the blue circles (22). The green line shows the case of \( \epsilon = \theta = 0.1 \) where the time-reversal symmetry breaks and a splitting of the energy bands at the Dirac points can be seen. In this regime the lower two bands have a net Chern number of 1 (22). The solid black line gives an example of intermediate coupling where \( \epsilon = 0.4 \), and \( \theta = 0.3 \), which leads to a lowest energy band of Chern number 1 (22). In the right panel, the density of states for the intermediate coupling can be seen. The lowest band has width of \( \Delta E \approx 0.1E_R \) (22). (B) The density of states calculated for \( \mathcal{E} = 2E_R, \ \theta = \pi/4, \ \text{and} \ \epsilon = 1.3 \). The lowest band has a width of \( \Delta E \approx 0.001 \) and is separated from the next band by about \( 0.4E_R \) (22). This figure is taken from Ref. (22).

The inclusion of next-nearest-tunneling and beyond allows for the atoms to perform loops that enclose flux and breaks time-reversal symmetry (22). The bands represented by the green dashed lines in Fig. 4.5 specifically split in such a way that the lowest two bands have a net Chern number of one (22). The two lowest bands will continue to have a net Chern number of one as long as the ratio \( \theta^2/\epsilon \) is small enough. For \( \mathcal{E} = 1.8E_R \) this limit is \( \simeq 0.19 \) after which there is a transition back to a topologically trivial case where the bands have a total Chern number of zero (22). As the couplings \( \epsilon \) and \( \theta \) continue to increase past the pertubative limit the lowest band becomes a narrow band with Chern number of 1 as seen in the the solid black lines of Fig. 4.5 (A). Figure 4.5 (B) shows an example where
the parameters have been tuned for a narrow band and large gap to the next level. A gap between the lowest band and the other bands of about $0.4E_R$ is visible and the width of the band is $\Delta E \simeq 0.01E_R$ after this optimization (22). The parameters for this final example is for $\mathcal{E} = 2E_R$, $\theta = \pi/4$, and $\epsilon = 1.3$ (22).

This provides a second example for the possible implementation of an optical flux lattice. Optical flux lattices require very few lasers for implementation and use well establish techniques making them an appealing option for future experiments (22). Finally, optical flux lattices should allow for the exploration of strongly correlated phenomena with ultracold atoms (22).
Conclusion

This literature thesis has provided some background into the behavior of charged particles in the presence of an external magnetic field and some of the phenomena that can appear in such situations to provide some motivation for the simulation of artificial gauge fields with neutral atoms. We went on to discuss the accumulation of a geometric phase and how this Berry’s phase can be created through an optically dressed state. Finally, we ended the discussion with a brief exploration into one concept for implementing a synthetic magnetic flux known as an optical flux lattice. The study of artificial gauge fields is an extremely fast evolving area of research with new breakthroughs happening all the time. This thesis provides a quick introduction to only one of the many possible schemes that have been proposed and implemented.
References


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