Master Thesis

Fine Analytic Continuation in a Fine Rhomboid Domain

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Preface

Approximation theory is an important part of complex analysis. It treats questions like:
Given a compact set $K$, which functions are uniform limits on $K$ of sequences of polynomials or rational functions?;
For such a sequence $\{f_n\}$, what properties of $f_n$ are inherited by the function $f$?;
How does this all depend on $K$?

In the early years of (complex) approximation, basic complex function theory like the Cauchy-integrals was used to prove approximation theorems, like Runge’s theorem, Weierstrass theorems and Mergelyan’s theorem. Later on, other topics in analysis, such as functional analysis, were used to give alternative proofs of the existing theorems and to prove new theorems.

One of the more recent tools of proving approximation results is the use of capacities. For a compact set $K$ that satisfies conditions involving continuous analytic capacity, Vitushkin [Vit] showed in 1967 that \( R(K) = A(K) \). However, the continuous analytic capacity of a set is not always easily computed. In 2004, Tolsa [Tol] showed that continuous analytic capacity is semi-additive and this result makes it easier to compute continuous analytic capacity of sets, because a set could be split into subsets for which we can compute the continuous analytic capacity.

In 2005, Edlund and Jöricke proved the following:
For certain compact sets $K$, every Hölder continuous function on $K$, which is analytic on the interior of $K$, admits a so called fine analytic continuation to a so called fine set.

The main goal of this thesis is to relax the condition of Hölder continuity into continuity in the result of Edlund and Jöricke.
The topology in which the work for the result by Edlund and Jöricke is done, is the *Fine Topology*; the weakest topology in which the *subharmonic functions* are continuous. [Bre, 1940]

When one is studying subharmonic functions, one is likely to encounter potential theory and, therefore, will also encounter the notion of *logarithmic capacity*. Logarithmic capacity is also used for some approximation results.

Edlund and Jörick also used the notion of holomorphic functions in the fine topology, the so called finely holomorphic functions. One of the people who studied this notion intensively is B. Fuglede. For a lemma that Lyons(1980) proved with probabilistic methods, Fuglede gave a classical proof in 1981 [Fug2]. This proof gives us an upper bound for the logarithmic capacity of certain sets. This upper bound can be used for estimates and, therefore, also for approximation results.

This thesis’ main goal is to sharpen the result by Edlund and Jöricke and therefore we have to understand some notions from Fine Topology, Subharmonic Functions, Potentials, Logarithmic Capacity and Analytic Capacities.

Because the first four are intertwined we will discuss them in the first chapter.

We will need and formulate the theorem by Vitushkin which we mentioned earlier. For better understanding of continuous analytic capacity we will also discuss *analytic capacity*. So, these two capacities will appear in the second chapter.

Our third chapter will be dedicated to the construction of the *fine rhomboid domains* in which the compact set \( K \) will be situated. We also refer to [EJ] for the construction.

After those chapters we can discuss the topics in the article by Edlund and Jöricke which are related to the result. This will be done in chapter four.

Finally, we will sharpen their result.

We will first prove that there is a fine analytic continuation for the condition of regular continuity. After that we will prove that for our compact set \( K \) we have \( R(K) = A(K) \) and thus have sharpened the result.

Note of the author:

Our second result implies our first result. However, most of the work in the process of this thesis has been done for the first result. It has been proved from scratch. The second result depends heavily on the beautiful deep work of Tolsa(2004) for which Tolsa was awarded by a European Mathematical Society award. It came only to our attention at the end of the process.
Notation and Terminology

In this thesis, we are dealing with complex function theory and therefore we introduce some notation and terminology regarding the complex plane.

Let \( B(z, r) = \{ \zeta \in \mathbb{C} : |\zeta - z| < r \} \) be the open disc in \( \mathbb{C} \) with center \( z \) and radius \( r \) and let \( \mathbb{B} \overset{\text{def}}{=} B(0, 1) \).

For a set \( S \) denote the closure by \( \overline{S} \); thus the closure of \( B(z, r) \) and \( \mathbb{B} \) will be denoted by \( B(z, r) \), respectively \( \mathbb{B} \).

The interior of a set \( S \) is denoted by \( S^\circ \), its boundary by \( \partial S \) and its complement by \( S^c \).

We also use the following notation:
- \( \mathbb{H}^+ \overset{\text{def}}{=} \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \);
- \( \mathbb{H}^- \overset{\text{def}}{=} \{ z \in \mathbb{C} : \text{Im}(z) < 0 \} \);
- \( \mathbb{C}^* = \mathbb{C} \cup \{ \infty \} \).

In the subject of function theory in the complex plane, we call a function \( f(z) \) holomorphic if it satisfies the Cauchy-Riemann equation \( \frac{\partial f}{\partial \overline{z}} = 0 \).

If a function \( f(z) \) is locally the sum of a convergent power series \( (f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n) \) we call \( f(z) \) (complex) analytic. These two definitions give rise to the same class of functions and therefore both terms are used to point out if a function is a member of this class. In this thesis we will use the term analytic.
Let us denote for a compact set $K \subset \mathbb{C}$ the function algebra with sup-norm consisting of all continuous functions on $K$ with $C(K)$.

Important subalgebras of $C(K)$ are: $P(K)$, the subset of continuous functions which can be uniformly approximated by polynomials. $R(K)$ denotes the uniform closure of rationals with poles off $K$. And $A(K)$ denotes the function algebra of all the continuous functions which are analytic on the interior of $K$ and continuous up to the boundary of $K$.

Note that $P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K)$.

If we say that a rhomb has angle $\vartheta$, then we have the following situation:

Figure 1: A Rhomb with Angle $\vartheta$

For the convenience of the reader:
We have listed most of the symbols in a List of Symbols.
All the Corollarys, Lemmas, Propositions and Theorems are listed in Corollarys, Lemmas, Propositions and Theorems.
These two lists can be found at the end of this thesis.
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Chapter 1

Fine Topology and Potential Theory

Closely related to complex function theory is potential theory, in which the class of subharmonic functions is studied. The potential defined in the section Potential Theory will be named logarithmic potential.

So, we are going to look at subharmonic functions and, therefore, we also are going to look at the weakest topology for which all subharmonic functions are continuous: the fine topology.

Important notions in this chapter: subharmonic function, Green’s function, Green potential, thin set, fine neighbourhood, finely analytic function, fine analytic continuation, balayage and logarithmic capacity.

Our main references are:

Wiegerinck [Wieg] and Helms [Hel] for the first section;
Armitage and Gardiner [Arm] and Fuglede [Fug1],[Fug2] for section 1.2;
Wermer [Wer], Ransford [Ran] and Helms [Hel] for the last section.

1.1 Subharmonic Functions

We start with the definitions of upper semi-continuity and the mean value inequality:

Definition 1.1.1 [Upper Semi-Continuity]

Let $D$ be a domain in $\mathbb{C}$. A function $v : D \to \mathbb{R} \cup \{\infty\}$ is called Upper Semi-Continuous (USC) if $\{z \in D : v(z) < M\}$ is open.

Equivalently, for every $z \in D$

$$\limsup_{w \to z} v(w) \leq v(z).$$
Definition 1.1.2 [Mean Value Inequality]
Let $D$ be a domain in $\mathbb{C}$ and $v$ be an USC-function on $D$. The function $v$ satisfies the mean value inequality if:

$$\forall z \in D \exists R = R_z : \quad v(z) \leq \bar{v}(z, r) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} v(z + re^{i\varphi})d\varphi, \quad (r < R_z).$$

Now we can define subharmonic functions:

Definition 1.1.3 [Subharmonic Functions]
Let $D$ be a domain in $\mathbb{C}$. A function $v : D \to \mathbb{R} \cup \{\infty\}$ is called subharmonic if:

- $v$ is upper semi-continuous;
- $v$ satisfies the mean value inequality.

Remark.
A function $u$ is called superharmonic if $-u$ is subharmonic.

Example 1.1.4 [Subharmonic Functions]
- $\log|f|$;
- $\sum c_i \log|f_i|, \ c_i \in \mathbb{R}_{>0}$.

Remark.
Compare the class of subharmonic functions to the class of harmonic functions:
A function $h$ is called harmonic if it is a real valued, $C^2$ and satisfies the following equation:

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 4 \frac{\partial^2 h}{\partial \bar{z}\partial z} = 0.$$

A harmonic function satisfies the mean value equality:
If $h$ is harmonic on $D$ and $B(a, R) \subset D$, then

$$h(a) = \frac{1}{2\pi} \int_{0}^{2\pi} h(a + re^{i\varphi})d\varphi, \quad 0 < r < R.$$
1.1. SUBHARMONIC FUNCTIONS

**Theorem 1.1.5**
Let \( \{v_n\} \) be a sequence of subharmonic functions pointwise monotonically decreasing to a function \( h \) that is not identically \(-\infty\), then

\[
h(z) = \lim_{n \to \infty} v_n(z)
\]
is a subharmonic function.

One frequently used property of subharmonic functions is the *maximum principle*. The principle tells us that subharmonic functions cannot attain a (local) maximum on their domain. Thus, if a subharmonic function attains a maximum on the closure of its domain, then it attains the maximum on the boundary of its domain. We will state the theorem:

**Theorem 1.1.6**
Let \( v \) be subharmonic on a domain \( D \subseteq \mathbb{C} \). Suppose that \( a \in D \) is such that \( v(a) \geq v(z) \) for all \( z \in D \). Then \( v \) is constant.

In this section we also introduce *Green’s function*, *Green potential* and *(greatest) harmonic minorant*:

**Definition 1.1.7** [Green’s function]
Let \( D \subseteq \mathbb{C}^* \) be a subdomain of \( \mathbb{C}^* \) and \( w \) a point in \( D \). A function \( G(z, w) = G_D(z, w) \) is called a *Green’s function* for \( D \) with pole at \( w \) if it satisfies:

- \( G_D(z, w) \) is harmonic on \( D \setminus \{w\} \);
- There exists a harmonic function \( h_w \) in a neighbourhood \( U \) of \( w \) such that
  \[
  G_D(z, w) = \log \frac{1}{|z - w|} + h_w(z)
  \]
on \( U \) if \( w \neq \infty \).
  
  If \( w = \infty \):
  \[
  G_D(z, \infty) = \log |z| + h(z),
  \]
  with \( h \) harmonic in a neighbourhood of \( \infty \);

- \[
  \lim_{z \to \zeta} G_D(z, w) = 0, \quad \text{for all } \zeta \in \partial D.
  \]
CHAPTER 1. FINE TOPOLOGY AND POTENTIAL THEORY

Definition 1.1.8 [Green Potential]
Let $R$ be an open subset of $\mathbb{C}$ having a Green’s function $G$. If $\mu$ is a signed measure on $R$, then

$$G\mu(z) = \int G(z, w)d\mu(w),$$

if defined everywhere on $R$, is called the Green potential ($G$-potential) of $\mu$. If $\mu$ is a measure on $R$ and $G\mu$ is superharmonic on $R$, then $G\mu$ is called the potential of $\mu$.

Definition 1.1.9 [(Greatest) Harmonic Minorant]
If $u$ is superharmonic on the open set $R$, $h$ is harmonic on $R$, and $h \leq u$ on $R$, then $h$ is called a harmonic minorant of $u$. The function $h$ is a greatest harmonic minorant of $u$ if $h$ is a harmonic minorant and $h \geq v$ whenever $v$ is a harmonic minorant of $u$.

Theorem 1.1.10
Let $R$ be an open set having a Green’s function $G$ and let $G\mu$ be the potential of a measure $\mu$. Then the greatest harmonic minorant of $G\mu$ is zero.

Theorem 1.1.11 (Riesz Decomposition Theorem)
Let $R$ be an open subset of $\mathbb{C}$ having a Green’s function $G$ and let $u$ be superharmonic on $R$. Then there is a unique measure $\nu$ on $R$ such that if $W$ is an open subset with compact closure in $R$, then $u = G_W\nu|_W + h_W$, where $h_W$ is the greatest harmonic minorant of $u$ on $W$; if, in addition, $u \geq 0$ on $R$, then $u = G\nu + h$, where $h$ is the greatest harmonic minorant of $u$ on $R$.

Corollary 1.1.12
A non-negative superharmonic function defined on an open set $R$ having a Green’s function is a potential of a measure if and only if its greatest harmonic minorant is zero.
1.2 Fine Topology

In the usual topology it is not true in general that subharmonic functions are continuous.

**Example 1.2.1** [Discontinuous Subharmonic Function]
The following subharmonic function is defined on $B(0, \frac{1}{2})$:

$$v(z) = \sum_{n=1}^{\infty} 2^{-n} \frac{\log |z - \frac{1}{n}|}{-\log \left( \frac{1}{n} \right)}, \text{ for } z \in B(0, 1/2).$$

It is subharmonic due to the limit property of a pointwise monotonically decreasing sequence of subharmonic functions $v_m = \sum_{n=1}^{m} 2^{-n} \frac{\log |z - \frac{1}{n}|}{-\log \left( \frac{1}{n} \right)}$.

$$v(1/n) = -\infty \text{ for } n \geq 2 \text{ and } \frac{1}{n} \to 0.$$ 

However, $v(0) = \lim_{n \to \infty} -\sum_{n=1}^{m} 2^{-n} > -\infty$ and therefore the function $v(z)$ is a discontinuous subharmonic function.

In 1939, however, Brelot defined the notion of thin sets with the help of subharmonic functions. Consecutively, Cartan observed in a letter to Brelot that an equivalent definition would be embedded in the weakest topology in which all subharmonic functions were continuous. He called this topology the **fine topology**.

So, the fine topology is related to the concept of thin sets. The thinness of a set at a given point is defined as follows:

**Definition 1.2.2** [Thin Sets]
A set $S \subset \mathbb{C}$ is said to be thin at a point $p$, if one of the following holds:

- $p \notin \overline{S}$;
- $p \in \overline{S}$ and there exists a subharmonic function $\mathcal{V}$ in a neighbourhood of $p$ such that

$$\limsup_{z \in S, z \to p} \mathcal{V}(z) < \mathcal{V}(p).$$

**Remark.**
$\mathcal{V}$ can be chosen in such a way that the limit equals $-\infty$. 


For the fine topology, we are interested in sets in the topology, some of its properties, finely analytic functions and fine analytic continuation. We begin with the next lemma:

**Lemma 1.2.3**

1. A subbase for the fine topology is given by the collection of all sets of the form \( \{x : u(x) > a\} \), where \( u \) is a subharmonic function and \( a \in \mathbb{R} \).

2. The fine topology has a neighbourhood base consisting of (Euclidean) compact sets; that is, given a fine neighbourhood \( \omega \) of a point \( x \), there is a compact set \( K \subset \omega \) such that \( K \) is a fine neighbourhood of \( x \).

3. A set is finitely compact iff it is finite.

4. If \( u \) is subharmonic on an open set \( R \), then \( u \) is finely continuous on \( R \).

**Remark.**

By 2. it follows that each fine neighbourhood of \( p \) contains a fine neighbourhood of \( p \) which is closed with respect to the Euclidean Topology.

We also have the following results:

**Proposition 1.2.4**

1. The fine topology is locally connected.

2. Every usual domain is also a fine domain.

3. If \( U \) is a fine domain and \( E \) is a polar set, then \( U \setminus E \) is a fine domain, in particular it is connected.

**Remark.**

If \( U \) is a fine neighbourhood of \( p \), then \( U \setminus E \), where \( E \) is thin at \( p \), is a fine neighbourhood of \( p \).
1.3. POTENTIAL THEORY

Definition 1.2.5 [Finely Analytic functions]
A function $f : U \rightarrow \mathbb{C}$, with $U$ finely open in $\mathbb{C}$, is said to be finely analytic if every point of $U$ has a compact fine neighbourhood $V \subset U$ such that $f|_V \in R(V)$.

Thus, a rational function with poles off $K$ is finely analytic.

With this definition about finely analytic functions we can talk about fine analytic continuation:

Definition 1.2.6 [Fine Analytic Continuation]
Let $f_1$ be defined and finely analytic on a fine open set $D_1$ and let $f_2$ be defined and finely analytic on a fine open set $D_2$. We call $f_1$ the fine analytic continuation of $f_2$ if $D_1 \cap D_2 \neq \emptyset$ and there is a component $C$ of $D_1 \cap D_2$ for which $f_1|_C = f_2|_C$ holds.

If we have a function defined on a connected set $D$ with $p$ a boundary point. Then we talk about the fine analytic continuation $F$ at $p$ if the connected closed fine set is a closed fine neighbourhood of $p$.

1.3 Potential Theory

In this section we define logarithmic potentials and introduce some basic notions such as energy and logarithmic capacity.

Potential Theory has its natural setting in physics, especially within the study of conductors.

When one is dealing with conductors one will encounter notions as the state of equilibrium, energy and (logarithmic) capacity. The relation between these notions can be represented in mathematics.

We will disregard the physics and study these notions from the mathematical point of view in the coming section.
The following integral with respect to a finite Borel measure on $\mathbb{C}$ with compact support is a subharmonic function [Ran, Th.2.4.8]:

$$\text{Int}_n = \int \max(\log|z - w|, -n) d\mu(w), \quad z, w \in \mathbb{C}. $$

Thus, $\{\text{Int}_n\}_{n=1}^\infty$ is a pointwise monotonically decreasing sequence of subharmonic functions and therefore:

$$\int \log|z - w| d\mu(w), \quad z, w \in \mathbb{C}$$

is also a subharmonic function (by the limit property of pointwise monotonically decreasing sequence of subharmonic functions and the monotone convergence of integrals).

**Definition 1.3.1** [Logarithmic Potentials]
Let $\mu$ be a positive finite Borel measure on $\mathbb{C}$ with compact support. Its logarithmic potential is the function $p_\mu : \mathbb{C} \to (-\infty, \infty)$ defined by:

$$p_\mu(z) = \int \log|z - w| d\mu(w) \quad (z \in \mathbb{C}).$$

From this definition we can define Energy:

**Definition 1.3.2** [Energy]
Let $\mu$ be a positive finite Borel measure on $\mathbb{C}$ with compact support. Its energy $I(\mu)$ is given by:

$$I(\mu) \overset{\text{def}}{=} \iint \log|z - w| d\mu(z) d\mu(w) = \int p_\mu d\mu(z).$$

**Definition 1.3.3** [Polar sets]
A subset $E$ of $\mathbb{C}$ is called polar if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which supp($\mu$) is a compact subset of $E$.

Now, we will look at Borel probability measures $\mu$ on a compact set $K$ and the Borel probability measure on $K$ which maximize $I(\mu)$.

**Definition 1.3.4** [Equilibrium measure]
Let $K$ be a compact subset of $\mathbb{C}$, and denote by $\mathscr{P}(K)$ the collection of all Borel probability measures on $K$. If there exists $\nu \in \mathscr{P}(K)$ such that

$$I(\nu) = \sup_{\mu \in \mathscr{P}(K)} I(\mu),$$

then $\nu$ is called an equilibrium measure for $K$. 
1.3. POTENTIAL THEORY

The definition doesn’t tell us if there is an equilibrium measure for a compactum $K$.

For the following lemma we define weak*-convergence:

**Definition 1.3.5** [Weak*-Convergence]

Let $X$ be a compact metric space. A sequence $(\mu_n)_{n \geq 1}$ in $\mathcal{P}(X)$ is weak*-convergent to $\mu$ in $\mathcal{P}(X)$, and we write $\mu_n \stackrel{w^*}{\to} \mu$, if

$$\int_X \phi d\mu_n \to \int_X \phi d\mu, \quad \text{for each } \phi \in C(X).$$

**Lemma 1.3.6**

If $\mu_n \stackrel{w^*}{\to} \mu$ in $\mathcal{P}(K)$, then $\limsup_{n \to \infty} I(\mu_n) \leq I(\mu)$.

With the preceding lemma we can prove existence of an equilibrium measure for a compact set $K$:

**Theorem 1.3.7**

Every compact set $K \subset \mathbb{C}$ admits an equilibrium measure.

**Proof**

Put $M = \sup_{\mu \in \mathcal{P}(K)} I(\mu)$, and choose a sequence $(\mu_n)$ in $\mathcal{P}(K)$ such that $I(\mu_n) \to M$ as $n \to \infty$. There is a subsequence $(\mu_{n_k})$ which is weak*-convergent to some $\nu \in \mathcal{P}(K)$ [Ran, Appendix A.4]. By the previous lemma

$$I(\nu) \geq \limsup_{k \to \infty} I(\mu_{n_k}) = M.$$

Because $\nu \in \mathcal{P}(K)$ we have now that $\nu$ is an equilibrium measure for $K$.

\[ \square \]

At last we can discuss the logarithmic capacity of a set.
Definition 1.3.8 [Logarithmic Capacity]
The logarithmic capacity of a subset $E$ of $\mathbb{C}$ is given by
\[
\text{Cap}(E) \overset{\text{def}}{=} \sup_{\mu} e^{I(\mu)},
\]
where the supremum is taken over all Borel probability measures $\mu$ on $\mathbb{C}$ whose support is a compact subset of $E$. In particular, if $K$ is a compact set with equilibrium measure $\nu$, then
\[
\text{Cap}(K) \overset{\text{def}}{=} e^{I(\nu)}.
\]

Example 1.3.9 Let $K$ be a singleton $\{a\}$, $a \in \mathbb{C}$. The only measure on $K$ is the Dirac-measure $\delta_a$ of $\{a\}$ and therefore, this measure is the equilibrium measure. Then the energy is:
\[
I(\delta_a) = \int \int \log|z - w|d\delta_a(w)d\delta_a(z)
= \int \log|z - a|d\delta_a(z)
= \lim_{r \to 0} \int_{B(a,r)} \log|z - a|d\delta_a(z)
= -\infty.
\]
And therefore the capacity of a singleton:
\[
\text{Cap}(\{a\}) = e^{I(\delta_a)} = 0.
\]

The only basic property of logarithmic capacity that we will need, is the following:
If $E \subseteq F$, then $\text{Cap}(E) \leq \text{Cap}(F)$.

The statement is true, because: if $E \subseteq F$, then every element $\mu_E$ of $\mathcal{P}(E)$ is also an element $\mu_F$ of $\mathcal{P}(F)$. Thus, $\sup_{\mu_E \in \mathcal{P}(E)} e^{I(\mu_E)} \leq \sup_{\mu_F \in \mathcal{P}(F)} e^{I(\mu_F)}$. In other words: $\text{Cap}(E) \leq \text{Cap}(F)$.

For other basic and more advanced properties we refer to Ransford [Ran, chapter 5].

However, there is a theorem which helps us to easily compute some capacities.

Theorem 1.3.10
Let $K \subset \mathbb{C}$ be a compact non-polar set, and let $\Omega(K)$ be the component of $\mathbb{C}^* \setminus K$ which contains $\infty$. Then
\[
G_{\Omega(K)}(z, \infty) = \log|z| - \log(\text{Cap}(K)) + o(1) \quad \text{as} \quad z \to \infty.
\]
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Proof
Let $\nu$ be the equilibrium measure for $K$. We have by Ransford [Ran, Th.4.4.2]:

$$G_{\Omega(K)} = p_{\nu}(z) - I(\nu) = p_{\nu}(z) - \log(\text{Cap}(K)) \quad (z \in \Omega(K) \setminus \{\infty\}).$$

By [Ran, Th.3.1.2]:

$$p_{\nu}(z) = \log|z| + o(1) \quad \text{as} \quad z \to \infty.$$

Thus,

$$G_{\Omega(K)}(z, \infty) = \log|z| - \log(\text{Cap}(K)) + o(1) \quad \text{as} \quad z \to \infty.$$  \hfill \Box

Example 1.3.11
With this theorem we have that $\text{Cap}(B(w, r)) = r$, because

$$G_{\Omega(B(w, r))} = \log \frac{|z - w|}{r} \overset{r \to \infty}{\Longrightarrow} \log|z| - \log r + o(1).$$

With the expression in the previous theorem, we conclude that $\text{Cap}(K) = r$.

At this point, we also introduce the notions of reduced functions, balayage and some of the features of balayage.

Let $R$ be an open set in $\mathbb{C}$ having a Green’s function $G_R$ and let $u$ be a non-negative superharmonic function on $R$.

If $E$ is a subset of $R$, define:

$$\Phi_u^E = \{v : v \geq 0 \text{ superharmonic on } R, \ v \geq u \text{ on } E\}$$

and let

$$R_u^E = \inf\{v : v \in \Phi_u^E\}.$$

Definition 1.3.12 [Reduced Function]
$R_u^E$ is called the reduced function of $u$ relative to $E$ in $R$.

The reduced function need not to be superharmonic, because it could fail the Lower Semi-Continuity [Hel, pg.134]. Let us look at the lower regularization $\hat{R}_u^E$ of $R_u^E$.

$$\hat{R}_u^E(z) = \liminf_{w \to z, \ w, z \in R} R_u^E(w).$$
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Lemma 1.3.13
\( \hat{R}_u^E \) is superharmonic on \( R \).

Definition 1.3.14 [Balayage]
\( \hat{R}_u^E \) is called the balayage (or regularized reduced function) of \( u \) relative to \( E \) in \( R \).

Theorem 1.3.15
Let \( u \) be a bounded non-negative superharmonic function on \( B \). Then for \( K \) a relative compact subset of \( B \) we have that \( \hat{R}_u^E \) is a potential.

For the proof we refer to Helms[Hel, Theorem 7.12(v), pg.135].
The theorem is proved there for \( u \) a non-negative superharmonic function, an arbitrary open set \( R \) and \( K \) a compact subset of \( R \). However, we are interested in \( u \equiv 1 \), \( R = B \) and \( K \) a relative compact subset of \( B \). And with these conditions, we can use the proof in Helms.

We will state one more theorem about balayage. This theorem is found in Armitage and Gardiner[Arm].

Theorem 1.3.16
Let \( R \) be an open set with a Green’s function, let \( E \subseteq R \) and \( z \in R \), and let \( u \) be a non-negative superharmonic function.
If \( E \) is thin at \( z \) and \( u \) is bounded from above on a neighbourhood of \( z \), then
\[
\lim_{r \downarrow 0} \hat{R}_u^{B(z,r) \setminus E}(z) = 0.
\]

For our first result we are going to use the following lemma, due to Lyons [Lyo], about a relation between logarithmic capacity and fine sets. The lemma and its proof are extracted from [Fug2] and slightly adapted to our benefits.

However, we first have to define a certain set.
Let \( W_n(z) = \{ w \in \mathbb{C} : |Re(z) - Re(w)|, |Im(z) - Im(w)| \leq 2^{-n} \} \) and
\( W_n^\circ(z) = \{ w \in \mathbb{C} : |Re(z) - Re(w)|, |Im(z) - Im(w)| < 2^{-n} \} \).
Define:
\[
C_n(z) = W_n(z) - W_{n+1}^\circ(z).
\]
(1.1)

\( C_n(z) \) is a (closed square) collar.
1.3. POTENTIAL THEORY

Lemma 1.3.17
Let $z_0 \in U \subset \mathbb{C}$, where $U$ is a fine open set, and let $E \supset U^C$, be an open set which is thin at $z_0$. For every real number $\alpha > 0$ there exists a closed fine neighbourhood $U_\alpha$ of $z_0$ in $U$ and a number $N_\alpha$ such that:

$$\text{Cap}(C_n(z) \cap E) \leq 2^{-n\alpha}$$

for every $z \in U_\alpha$ and every $n \geq N_\alpha$.

Proof
We look at the case that $z_0 = 0 (\in U)$. Thus the complement of $U$, $U^C$, is thin at 0 and there exists an open set $E$ which is thin at 0 and which contains $U^C$. The Green’s function for the unit disc $\mathbb{B}$ will be written as:

$$G(z, w) = \log \frac{|z - w^*|}{|z - w||w^*|}, \quad z, w \in \mathbb{B},$$

here we have that $w^* = 1/\overline{w}$ and define $G(z, 0) = -\log|z| = \lim_{w \to 0} G(z, w)$.

From now on we assume

$$|z| < \frac{1}{2}, \quad |w| < \frac{1}{2},$$

then we have $|w^*| > 2, \frac{3}{4}|w^*| < |z - w^*| < \frac{5}{4}|w^*|$ and thus we have

$$\log \frac{3/4}{|z - w|} < G(z, w) < \log \frac{5/4}{|z - w|}.$$

Now, the $G$-capacity ($G\text{-Cap}(\cdot)$) of a Borel set $A \subset \mathbb{B}$ is defined by

$$\frac{1}{G\text{-Cap}(A)} \overset{\text{def}}{=} \inf \iint G(z, w)d\mu(z)d\mu(w),$$

where the infimum is taken over all Borel probability measures $\mu$ with support on $A$. If we have that $A \subset B(0, 1/2)$, then we have

$$\log \frac{3}{4} \leq \iint \left( G(z, w) - \log \frac{1}{|z - w|} \right) d\mu(z)d\mu(w) \leq \log \frac{5}{4},$$

therefore we have the following inequalities with the $G$-capacity and the logarithmic capacity:

$$\log \frac{3}{4} \leq \frac{1}{G\text{-Cap}(A)} - \log \frac{1}{\text{Cap}(A)} \leq \log \frac{5}{4}, \quad A \subset B(0, 1/2).$$
Let’s apply this to $A = C_n(z) \cap E$ for $n \geq 3$, an open set $E$ which is thin at 0 and $|z| < \frac{1}{4}$. The equilibrium $G$-potential for $C_n(z) \cap E$ of $z$, related to the balayage relative to $B(0, 1)$, can be expressed by the associated equilibrium measure $\mu$ of $C_n(z) \cap E$. We have:

$$
\widehat{R}_{1}^{C_n(z) \cap E}(z) = \int_{\mathbb{B}} G(z, w) d\mu(w) \geq \int_{C_n(z)} \log \frac{3/4}{|z - w|} d\mu(w)
$$

$$
\geq \log \left( \frac{3}{4} \cdot 2^n \right) \cdot G-Cap(C_n(z) \cap E)
$$

$$
\geq \log \left( \frac{3}{4} \cdot 2^n \right) \cdot \left( \log \frac{1}{Cap(C_n(z) \cap E)} + \log \frac{5}{4} \right)^{-1}.
$$

Because $\log \frac{3}{4} \geq -\frac{1}{2} \log 2 \geq -\frac{1}{2} \log(2^n)$ and

$$
\log \frac{1}{Cap(C_n(z) \cap E)} \geq \log \frac{1}{Cap(C_n(z))} \geq \log \left( \frac{2^n}{\sqrt{2}} \right) \geq \log \frac{5}{4},
$$

we arrive at (for $n \geq 2$, $|z| < \frac{1}{4}$):

$$
\widehat{R}_{1}^{C_n(z) \cap E}(z) \geq \frac{(n/4) \log 2}{-\log Cap(C_n(z) \cap E)}.
$$

(1.2)

Now by Theorem 1.3.16, for a given number $\alpha > 0$ there is an $r > 0$ ($r < 1/4$) such that

$$
\widehat{R}_{1}^{B(0, 2r) \cap E}(0) < \frac{1}{4\alpha}.
$$

Choose $N_{\alpha}$ and a fine neighbourhood $U_{\alpha} \subset B(0, r)$ of 0 such that

$$
\sqrt{2} \cdot 2^{-N_{\alpha}} < r, \quad \widehat{R}_{1}^{B(0, 2r) \cap E}(z) < \frac{1}{4\alpha} \quad \forall z \in U_{\alpha}.
$$

Because $C_n(z) \subset B(0, 2r)$ for $z \in U_{\alpha}$ ($\subset B(0, 1/4)$) and $n \geq N_{\alpha}$, we get

$$
\widehat{R}_{1}^{C_n(z) \cap E}(z) < \frac{1}{4\alpha} \quad z \in U_{\alpha}, \quad n \geq N_{\alpha}.
$$

Combining this with (1.2) the required inequality follows

$$
Cap(C_n(z) \cap E) \leq 2^{-na},
$$

(1.3)

for $z \in U_{\alpha}$ and $n \geq N_{\alpha}$. □
Chapter 2

Analytic Capacities

In the previous chapter we discussed logarithmic capacity which is related to subharmonic functions. In this chapter we will introduce other kinds of capacity which are related to analytic functions. These analytic capacities are more natural in the study of rational approximation. So, we are going to look at (bounded) analytic capacity and continuous analytic capacity.

The main reference for this chapter is Gamelin[Gam].

2.1 (Bounded) Analytic Capacity

We start this section with a part about admissible functions:

For a compact set $K \subset \mathbb{C}$ write $\Omega(K)$ for the unbounded component of $K^C$ together with the point at $\infty$. Note that if $K \subset \mathbb{C}$ is compact and connected, then $\Omega(K)$ is simply connected with more than one boundary point.

**Definition 2.1.1** [Admissible functions]

Let $K$ be compact in $\mathbb{C}$.

A function $f$ is called admissible for $K$ if:

- $f$ is defined and analytic on $\Omega(K);
- \|f\|_{\Omega(K)} \leq 1;
- f(\infty) = 0.$

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For any arbitrary set $E$ a function is called admissible if it is admissible for a compact subset of $E$. The set of admissible functions for $E$ is denoted by $\mathcal{A}(E)$.

Because an admissible function $f$ is analytic at $\infty$ and equal to 0 in $\infty$, $f$ can be represented as Laurent series as follows:

$$f = \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

With a local change of coordinates it follows that:

$$a_1 = \lim_{z \to \infty} z(f(z) - f(\infty)) = f'(\infty).$$

**Definition 2.1.2** [(Bounded) Analytic Capacity]

The (bounded) analytic capacity of a set $E$ is denoted by:

$$\gamma(E) = \sup \{|f'(\infty)| : f \in \mathcal{A}(E)\}.$$  

Remark.

For this capacity we use the term analytic capacity rather than bounded analytic capacity.

It is easy to see that: $E \subseteq F \Rightarrow \gamma(E) \leq \gamma(F)$.

For every compact set $K_E \subset E$, we have that $K_E \subseteq K_F$ for a compact set $K_F \subset F$. Then $\Omega(K_F) \subseteq \Omega(K_E)$ and therefore if $f \in \mathcal{A}(E)$, then $f \in \mathcal{A}(F)$. Analytic capacity is also translation invariant.

For a compact set $K$, Ahlfors showed that there is always a unique admissible function $g$ for $K$ satisfying $g'(\infty) = \gamma(K)$. The proof is like a proof for the Riemann Mapping Theorem: existence is showed due to the fact that $\mathcal{A}(K)$ is a normal family. The function $g$ is called the Ahlfors function for $K$.

At this moment, we want to remark that $\gamma(K) \leq \text{Cap}(K)$ for a compact non-polar $K$.

If we have the Ahlfors function $g(z)$ for $K$, then we have that $\log|g(z)|$ is subharmonic and $\leq 0$ on $\Omega(K)$. $g(z)$ is admissible and therefore of the form $g(z) = \frac{\gamma(K)}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots$. 
Thus, log\(|g(z)|\) will behave as \(-\log|z| + \log(\gamma(K))\) for \(z \to \infty\).

We also have that \(G_{\Omega(K)}(z, \infty)\) is subharmonic on \(\Omega(K)\) and
\[
\lim_{z \to \zeta} G_{\Omega(K)}(z, \infty) = 0 \quad \text{for} \quad \zeta \in \partial\Omega(K).
\]

Thus, \(\log|g(z)| + G_{\Omega(K)}(z, \infty)\) is a subharmonic function on \(\Omega(K)\) and
\[
\lim_{z \to \zeta} \log|g(z)| + G_{\Omega(K)}(z, \infty) \leq 0 \quad \text{for} \quad \zeta \in \partial\Omega(K).
\]

With the maximum principle of subharmonic functions, we conclude that
\[
\log|g(z)| + G_{\Omega(K)}(z, \infty) \leq 0 \quad \text{on the whole of} \quad \Omega(K).
\]

Let \(z \to \infty\), then we have that \(-\log|z| + \log(\gamma(K)) + \log|z| - \log(\text{Cap}(\Omega(K))) \leq 0\).

It follows that \(\gamma(K) \leq \text{Cap}(\Omega(K))\).
Example 2.1.4
Let \( L \in \mathbb{R} \) and \( K = [-\frac{L}{2}; \frac{L}{2}] \). Consider the following function:

\[
z = (w + \frac{1}{w}) \frac{L}{4}.
\]

This function maps \( \mathbb{B} \) conformally onto \( \mathbb{C}^* \setminus [-\frac{L}{2}; \frac{L}{2}] \equiv \Omega(K) \), where \( z \in \Omega(K) \) and \( w \in \mathbb{B} \). Thus, we can express \( w \) in terms of \( z \).

Now we compute: \( \gamma(K) = \lim_{z \to \infty} z \cdot w(z) \), substituting our starting function we obtain: \( \gamma(K) = \lim_{z \to \infty} (w(z)^2 + 1) \frac{L}{4} = \frac{L}{4} \).

Because analytic capacity is invariant under translation, \( \gamma(K) = \frac{L}{4} \) for every straight line segment with length \( L \).

Now, we discuss the analytic capacity of a compact subset of a line segment:

**Theorem 2.1.5 (Capacity of a compact subset of a line.)**

Let \( K \) be a compact subset of a line, and let \( L \) be the length of \( K \). Then

\[
\frac{L}{4} \leq \gamma(K) \leq \frac{L}{\pi}.
\]

**Proof**

We can assume that \( K \) lies on the real axis. Let

\[
f(z) = \frac{1}{2} \int_K \frac{ds}{z - s}.
\]

If \( z = x + iy \), where \( y \neq 0 \), then

\[
|\text{Im}(f(z))| = \frac{|y|}{2} \int_K \frac{ds}{(x - s)^2 + y^2} \leq \frac{|y|}{2} \int_{-\infty}^{\infty} \frac{ds}{s^2 + y^2} = \frac{\pi}{2}.
\]

Consequently the range of \( f \) lies in the horizontal strip \(-\frac{\pi}{2} < \text{Im}(w) < \frac{\pi}{2}\).
2.2. CONTINUOUS ANALYTIC CAPACITY

If we compose $f$ with the conformal map $w \rightarrow \frac{w-1}{e^w+1}$ of this horizontal strip onto the interior of the unit disc, we obtain a function

$$h(z) = \frac{e^{f(z)} - 1}{e^{f(z)} + 1}$$

which satisfies $|h| \leq 1$. Also, $h(\infty) = 0$, so $h$ is admissible. Since $f'(\infty) = \frac{L}{4}$, we obtain $h'(\infty) = \frac{L}{4}$. Consequently:

$$\gamma(K) \geq \frac{L}{4}.$$  (2.1)

For the other inequality, let $g$ be the Ahlfors function of $K$. Given $\epsilon > 0$, there exists a finite union $\Gamma$ of smooth closed Jordan curves in $\Omega(K)$, such that $\Gamma$ surrounds $K$ in the usual sense of contour integration, and such that the length of $\Gamma$ does not exceed $2L + \epsilon$. Then

$$\gamma(K) = g'(\infty) = \frac{1}{2\pi i} \int_{\Gamma} g(z)dz \leq \frac{2L + \epsilon}{2\pi}.$$  

Letting $\epsilon$ tend to zero, we obtain $\gamma(K) \leq \frac{L}{\pi}$, as was required.

\[ \square \]

Remark.

Pommerenke showed that the capacity is equal to $\frac{L}{\pi}$.

2.2 Continuous Analytic Capacity

Let us look at the space of admissible functions for $E \subset \mathbb{C}$ which are defined and continuous on the whole Riemann sphere $S^2$: the space $A(E)$.

Definition 2.2.1

For a $E \subset \mathbb{C}$ a function $f$ belongs to $A(E)$ if:

- $f$ is analytic off some compact subset of $E$;
- $f$ is continuous on $S^2$;
- $\|f\|_{S^2} \leq 1$;
- $f(\infty) = 0$.
Remark.
It is clear that $\mathcal{AC}(E) \subset \mathcal{A}(E)$.

We now define the continuous analytic capacity of a set $E$:

**Definition 2.2.2** [Continuous Analytic Capacity]
The continuous analytic capacity of the set $E$ is

$$\alpha(E) = \sup\{|f'(\infty)| : f \in \mathcal{AC}(E)\}.$$ 

Remark.
We know that $\mathcal{AC}(E) \subset \mathcal{A}(E)$ and therefore we have that $\alpha(E) \leq \gamma(E)$. For further relations between the analytic capacity and the continuous one we refer to Gamelin [Gam].

Remark.
It is clear that $\alpha(E) \leq \alpha(F)$ for $E \subseteq F$, because $\mathcal{AC}(E) \subseteq \mathcal{A}(E) \subseteq \mathcal{AC}(F)$.

The continuous analytic capacity of a line segment $L$ is zero, because the analytic function on $\mathbb{C} \cup \{\infty\} \setminus L$ can be extended to an analytic function on $\mathbb{C} \cup \{\infty\}$. Then we have a bounded entire analytic function and with Liouville’s Theorem the function is constant.

Every compact set of 2-dimensional Lebesgue measure 0 has continuous analytic capacity zero.

The continuous analytic capacity has an interesting semi-additive property. More precisely, we have the following theorem of Tolsa [Tol]:

**Theorem 2.2.3** (Semi-Additivity of Continuous Analytic Capacity)

*Let $E_i$, $i \geq 1$, be Borel sets in $\mathbb{C}$. Then,

$$\alpha\left(\bigcup_{i=1}^{\infty} E_i\right) \leq C \sum_{i=1}^{\infty} \alpha(E_i),$$

(2.2)

where $C$ is an absolute constant.*

The proof is too complex to comprehend for the author at the moment. Nevertheless, we will use this theorem for our second result.
For our second result we will also need the following theorem due to Vitushkin.

**Theorem 2.2.4 (Vitushkin’s Theorem)**

The following are equivalent for a compact set $K$:

1. $R(K) = A(K)$.
2. For every bounded open set $D$ we have: $\alpha(D \setminus K) = \alpha(D \setminus K^o)$.
3. For each complex $z$, each $r > 1$, and each $\delta > 0$:
   \[ \alpha \left( \overline{B(z, \delta)} \setminus K^o \right) \leq \alpha \left( \overline{B(z, r\delta)} \setminus K \right). \]
4. There exist $r \geq 1$ and $c > 0$ such that for all complex $z$ and all $\delta > 0$:
   \[ \alpha \left( \overline{B(z, \delta)} \setminus K^o \right) \leq c \cdot \alpha \left( \overline{B(z, r\delta)} \setminus K \right). \]
5. For each $z \in \partial K$, there exists $r \geq 1$:
   \[ \limsup_{\delta \to 0} \frac{\alpha \left( \overline{B(z, \delta)} \setminus K^o \right)}{\alpha \left( \overline{B(z, r\delta)} \setminus K \right)} < \infty. \]

Remark.
We are going to use that (1)$\Leftrightarrow$(5) and thus, if we have (5) then we have proven that $R(K) = A(K)$. 
CHAPTER 2. ANALYTIC CAPACITIES
Chapter 3

Fine Rhomboid Domains

We start this chapter with the definition of Pseudocontinuation:

**Definition 3.0.5** [Pseudocontinuation]

Let $D$ be a domain with rectifiable boundary, let $\mathcal{E}_D \subset \partial D$ be a set of positive measure and let $D_e \subset \{z \in \mathbb{C} : z \notin \overline{D}\}$ be a domain.

A function $f \in A(D)$ is said to have pseudocontinuation from $D$ across the set $\mathcal{E}_D$ to $D_e$ if for all $z \in \mathcal{E}_D$ the domain $D_e$ contains a truncated non-tangential cone with vertices at $z$, and there exists an analytic function $\tilde{f}$ in $D_e$, such that the non-tangential limits of $f$ and $\tilde{f}$ at $z$ are the same. If this is the case, we call $\tilde{f}$ the pseudocontinuation of $f$.

In our situation $D = \mathbb{B}$, $D_e = B_e \subset \{z \in \mathbb{C} : |z| > 1\}$ and $\mathcal{E}_D = \mathcal{E}_\mathbb{B}$ which we denote by $\mathcal{E}$.

With the definition of pseudocontinuation at hand we can construct the fine rhomboid domains we will work on.

**Remark.**

We speak of fine rhomboid domains. However, they are not really a domain, because they will not be necessarily open in the fine topology.

Our fine rhomboid domains will have the form $\overline{B}_i \cup \overline{B}_e$, where $B_i$ and $B_e$ are domains. $B_i \cup B_e$ is not connected. But if there are points on $\overline{B}_i \cap \overline{B}_e$ at which $\mathbb{C} \setminus (\overline{B}_i \cup \overline{B}_e)$ is thin, then there is at least one fine open set $W$, which is connected, and for which $B_i \cup B_e \subset W \subset \overline{B}_i \cup \overline{B}_e$ holds.
Let us consider the situation that $\mathcal{E}$ is closed and that the angles and the diameters of the truncated non-tangential cones in $B_e$ are uniformly bounded from below by positive constants.

We may assume that all the cones have the same angle and that $B_e$ is a bounded domain, otherwise we could shrink $B_e$. We may also assume that the pseudocontinuation is continuous in $B_e$.

Let $B_i \subset \mathbb{B}$ be symmetric to $B_e$ in a neighbourhood of $\mathcal{E}$ and therefore we have that all the bounded components of the complement of $B_i \cup B_e$ are similar open rhombs $\Diamond_l$ (because the angles are the same), which contains the arcs of $\partial \mathbb{B} \setminus \mathcal{E}$.

We can assume that the unbounded component of $\mathbb{C} \setminus (B_i \cup B_e)$ intersects $\partial \mathbb{B}$ along a connected arc.

We call $B_i \cup B_e$ our fine rhomboid domain.

We will display a visual sketch of such a fine rhomboid domain:

![Figure 3.1: A Fine Rhomboid Domain](image)
Chapter 4

The Result of Edlund and Jöricke

4.1 Proposition

In this chapter we present the result of Edlund and Jöricke and its proof and therefore our main reference for this chapter is [EJ].

Proposition 4.1.1

Let $B_i$ and $B_e$ be as in chapter 3 and let $f \in A(B_i \cup B_e)$. Put $U = \mathbb{C} \setminus (B_i \cup B_e)$. If $U$ is thin at a point $p \in \mathcal{E}$ and $f$ is Hölder continuous of order $\alpha \in (0,1]$, then $f|_{B_i}$ has fine analytic continuation $f|_{V_p}$ at $p$ for a fine neighbourhood $V_p \subset B_i \cup B_e$ of $p$.

Recall the definition of Hölder continuity:

Definition 4.1.2 [Hölder Continuity]

A function $f \in C(D)$ on a (bounded) domain $D$ is called Hölder Continuous of order $\alpha$, $0 < \alpha \leq 1$, if:

$\exists$ constant $H$: $|f(z_1) - f(z_2)| \leq H|z_1 - z_2|^\alpha$, $z_1, z_2 \in \overline{D}$. 

4.2 \( \Diamond_l \) and Thinnes

Let us study \( U = \mathbb{C} \setminus (\overline{B} \cup \overline{B_e}) \). \( U \) is by construction of our fine rhomboid domain a union of similar open rhombs \( \Diamond_l \). We have that \( U \) is thin at a point \( p \). Are there any properties of \( U \) for being thin?

Well, yes. And we are going to study some of those properties in this section.

Observe that \( \mathcal{E} = \overline{B} \cap \overline{B_e} \).

We start this section with the following proposition:

**Proposition 4.2.1**

Let \( E \subset \mathbb{C} \) be thin at the point \( 0 \in \overline{E} \). Let \( T : \mathbb{C} \to \mathbb{C} \) be a mapping which satisfies a Lipschitz condition (\( |T(z_1) - T(z_2)| \leq C|z_1 - z_2| \) for \( z_1, z_2 \in \mathbb{C} \)) and such that \( T(0) = 0 \) and \( |T(z)| \geq c|z| \) for \( z \in \mathbb{C} \). (\( C \) and \( c \) are positive constants). Then the set \( T(E) \) is thin at 0.

With this proposition we can prove the following lemma:

**Lemma 4.2.2**

Let \( \bigcup_l \Diamond_l \) be a union of open rhombs, with endpoints of their symmetry axes on \( \partial B \), which is thin at \( p \in \partial B \). Then \( \bigcup_l \overline{\Diamond_l} \) is thin at \( p \).

**Proof**

\( \bigcup_l \Diamond_l \) is thin at \( p \). This means that \( \bigcup_l I_l \), the union of open pairwise disjoint arcs on \( \partial B \) associated to \( \bigcup_l \Diamond_l \), is thin at \( p \in \partial B \). Because \( \bigcup_l I_l \) is thin at \( p \) so is \( \bigcup_l \overline{T_l} \) due to the fact that they differ only by a countable (and therefore thin) set. Denote the union of the boundary of the rhombs by \( \Lambda \). We can split \( \Lambda \) in two parts: the part outside the unit disc, \( \Lambda_e \), and the part inside the unit disc, \( \Lambda_i \). These parts can be represented as graphs over \( \bigcup_l I_l \):

\[
\Lambda_e = \left\{ r_e e^{i\phi} \mid r_e = r_e(\phi), e^{i\phi} \in \bigcup_l \overline{T_l} \right\} \quad \text{and} \quad \Lambda_i = \left\{ r_i e^{i\phi} \mid r_i = r_i(\phi), e^{i\phi} \in \bigcup_l \overline{T_l} \right\}.
\]

The mapping \( T(e^{i\phi}) = r_e e^{i\phi} \) where \( e^{i\phi} \in \bigcup_l \overline{T_l} \) can be extended to the whole plane as a Lipschitz continuous mapping which satisfies the conditions of the preceding proposition with 0 replaced by \( p \). We also do this for \( \Lambda_i \). Thinness is invariant under translation and in view of Proposition 4.2.1 it is clear that \( \Lambda_e \) and \( \Lambda_i \) are thin at \( p \). Therefore, \( \Lambda \) is thin at \( p \). Thus, we have that the union of the boundaries of the closed rhombs is thin at \( p \).
Because \( p \in \partial B \) and \( \bigcup_l \overline{\delta}_l \cap \partial B \) is also thin at \( p \), we conclude with the maximum principle that the union of the closed rhombs is thin at \( p \).

\[
(\sup_{z \in \delta_l} \mathcal{V}(z) < \sup_{z \in \partial \delta_l} \mathcal{V}(z) < \mathcal{V}(p))
\]

\[\square\]

We will make three more observations of the rhombs in relation with thinness and only prove the third observation:

1. If the \( \bigcup_l \overline{\delta}_l \) is thin at \( p \), then there are arbitrarily small numbers \( r > 0 \) for which \( \partial B(p, r) \cap \bigcup_l \overline{\delta}_l = \emptyset \).

2. We may assume that the \( \overline{\delta}_l \)'s are pairwise disjoint.

3. If \( \bigcup_l \overline{\delta}_l \) is thin at \( p \) then there exists another sequence of similar (open) rhombs \( \delta_l' \) related to disjoint open arcs \( I_l' \) of \( \partial B \), such that \( \bigcup_j \delta_l' \) is thin at \( p \) and \( \bigcup_l \overline{\delta}_l \subset \bigcup_j \delta_l' \).

**Proof**

Because \( \bigcup_l \overline{\delta}_l \) is thin at \( p \), there exists a subharmonic function \( \mathcal{V} \) and \( a \in \mathbb{R} \) such that \( \limsup_{z \in \bigcup_l \overline{\delta}_l, z \to p} \mathcal{V}(z) < a < \mathcal{V}(p) \). Then for each \( \overline{T}_l \) contained in a small neighbourhood \( V_p \) of \( p \): \( \sup_{\overline{T}_l} \mathcal{V} < a \). Hence \( \sup_{\overline{\delta}_l} \mathcal{V} < a \) for some open arc \( \overline{\delta}_l \supset \overline{T}_l \). Take also for the other arcs \( \overline{T}_l \) suitable open arcs \( \overline{\delta}_l \) containing them. The set \( \bigcup_l \overline{\delta}_l \) is thin at \( p \). Let \( I_l' \) be the connected components of this last union and associate rhombs \( \delta_l' \) to them.

\[\square\]

### 4.3 Proof of the Result

We want fine analytic continuation for a fine neighbourhood \( V_p \). Suppose we have \( F(z) = \mathcal{G}(z) + \mathcal{C}(z) \) on \( V_p \) with \( \mathcal{G} \in A(\overline{B(p, r)}) \) and \( \mathcal{C} \) is the Cauchy-type integral of a finite Borel measure \( \mu \) concentrated on the union of open rhombs such that for an increasing sequence of compacta \( \kappa_n \subset \bigcup_l \overline{\delta}_l \) the functions

\[
\mathcal{C}_n(z) = -\frac{1}{\pi} \int_{\kappa_n} \frac{d\mu(\xi)}{\xi - z}, \quad z \notin \kappa_n,
\]

converge uniformly to \( \mathcal{C}(z) \) on \( V_p \).
Let us write $F_n(z) = \mathcal{G}(z) + C_n(z)$, then $F_n(z) \in R(V_p)$ and $F_n(z)$ converges uniformly $F(z)$. Thus $F(z)$ is finely analytic. If $F(z)|_{\mathcal{V}_{\rho} \cap \mathbb{B}} = f(z)$, then $F(z)$ is a fine analytic continuation of $f(z)$ and this will prove the proposition.

For the convenience of the reader we will state Privalov’s Theorem, found in [Wen], which will be used in the proof:

**Theorem 4.3.1 (Privalov’s Theorem)**

Let $M_1, M_2, M_3, M_4 \in \mathbb{R}$ be constants.

Suppose $\Phi(z) = u(z) + iv(z)$ is analytic on $\mathbb{B}$, and $|v(0)| \leq M_1 < \infty$.

Furthermore, suppose $u(z)$ is continuous on $\mathbb{B}$, with $C_\alpha[u(z), \Gamma] \leq M_2 < \infty$; or equivalently, if for $|z| = 1$:

$$|u(z)| \leq M_2,$$

$$|u(e^{i\theta_1}) - u(e^{i\theta_2})| \leq M_2 |e^{i\theta_1} - e^{i\theta_2}|^\alpha, \quad 0 < \alpha < 1,$$

then in $\mathbb{B}$ we have:

$$|\Phi'(z)| \leq M_2 M_3 (1 - |z|)^{\alpha-1},$$

and $\Phi(z)$ is continuous on $\overline{\mathbb{B}}$ with $C_\alpha[\Phi(z), \mathbb{B}] \leq M_2 M_4$; or equivalently:

$$|\Phi(z)| \leq M_2 M_4, \quad |\Phi(z_1) - \Phi(z_2)| \leq M_2 M_4 |z_1 - z_2|^\alpha,$$

where $M_3 = M_3(\alpha)$ and $M_4 = M_4(\alpha, M_1, M_2)$.

In other words: $\Phi$ has a Hölder continuous extension to the whole of $\mathbb{B}$.

**Proof of the result by Edlund and Jörick**

We will assume that the $\diamond_l$ are pairwise disjoint (shrinking $B_i$ and $B_e$ otherwise) and prove fine analytic continuation to $\overline{B(p, r) \setminus \bigcup_j \Diamond_j'}$ for a suitable small $r > 0$. We may assume that $r > 0$ is chosen so that $\partial B(p, r)$ does not meet $\bigcup_j \Diamond_j'$. Keep notation $\overline{\Diamond_l}$ for only those of the original rhombs which are contained in $B(p, r)$ and $\Diamond_j'$ for those of the enlarged rhombs which are contained there.

The function $f$ is analytic in each of the domains $B_i' \overset{\text{def}}{=} B(p, r) \cap \mathbb{B} \setminus \bigcup_l \overline{\Diamond_l}$ and $B_e' \overset{\text{def}}{=} B(p, r) \cap (\mathbb{C} \setminus \mathbb{B}) \setminus \bigcup_l \overline{\Diamond_l}$ and Hölder continuous in the union of the closures.
Both domains have rectifiable boundary, hence by Cauchy’s formula
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B'} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\partial B_e} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B'_i \cup B'_e
\]

The contours of integration are always oriented as boundaries of relatively compact domains. Note that one of the integrals in the sum above will be equal to zero. Using that \( \partial B \cap B(p, r) \setminus \bigcup_{l} \delta_l = \emptyset \cap B(p, r) = \partial B'_i \cap \partial B'_e \) and integration over this set is carried out twice in opposite direction we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B(p, r)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \sum_{l} \int_{\partial \delta_l} \frac{f(\zeta)}{\zeta - z} d\zeta \]
def \( J(z) = \sum_{l} J_l(z), \quad z \in B'_i \cup B'_e \)

By Privalov’s Theorem, \( J(z) \) extends to a Hölder continuous function of order \( \alpha \) in \( \overline{B(p, r)} \) if \( \alpha < 1 \) and of any order less than 1 if \( \alpha = 1 \). The measure \( f(\zeta) d\zeta \) on \( \bigcup \partial \delta_l \) is a finite Borel measure concentrated on a subset of \( \bigcup \delta_j \).

To prove the proposition let:
\( \kappa_n = \bigcup_{l=1}^n \delta_l \) and \( F_n(z) = J(p + (1 - \frac{1}{n})(z - p)) - \sum_{l=1}^n J_l(z), \quad z \in \overline{B(p, r)} \setminus \kappa_n \).
We have to check that the \( F_n \) converge uniformly to \( f \) on compacta in \( B'_i \cup B'_e \), that is, on compact sets of the form \( \overline{B(p, r)} \setminus \bigcup \delta_j \). To obtain a uniform estimate of \( J_l \) on \( B(p, r) \setminus \bigcup \delta_l \) we use that for \( z \notin \delta_l \) the Cauchy type integral with pole at \( z \) of the constant function \( f(z) \) along \( \partial \delta_l \) vanishes. We get for \( z \in \overline{B(p, r)} \setminus \delta_l \)
\[
|J_l(z)| = \left| \frac{1}{2\pi i} \int_{\partial \delta_l} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq C \int_{\partial \delta_l} \frac{|\zeta - z|^\alpha}{|\zeta - z|} |d\zeta|.
\]
Let \( N > n \) be a natural number and let \( z \in \overline{B(p, r)} \setminus \bigcup_{l=n}^N \delta_l \). Then
\[
\sum_{n \leq l \leq N} |J_l(z)| \leq C \int_{\bigcup_{n \leq l \leq N} \partial \delta_l} \frac{|\zeta - z|^\alpha}{|\zeta - z|} |d\zeta| \leq C \int_{\bigcup_{l \geq n} \partial \delta_l} \frac{|\zeta - z|^\alpha}{|\zeta - z|} |d\zeta|.
\]
Represent the contour of integration on the right hand side as the union of its part \( \Lambda_k^- \) contained in \( \overline{\mathbb{B}} \) and its part \( \Lambda_k^+ \) contained in \( \mathbb{C} \setminus \overline{\mathbb{B}} \). Each of the parts is the graph of a Lipschitz continuous function over a subset of \( \partial \mathbb{B} \cap \overline{B(p, r)} \) with uniform estimate for the Lipschitz constant (which depends on the angle of the truncated non-tangential cones contributing to \( B_i \) and \( B_e \)).
For a point $\xi \neq 0$ we denote by $\xi'$ its radial projection to the circle, $\xi' = \frac{\xi}{|\xi|}$. Using the inequality $|\zeta - z| \geq c|\zeta' - z'|$ and estimating the arc-length on $\Lambda_k^+$ and on $\Lambda_k^-$ by the arc-length of the radial projection we obtain

$$\sum_{n \leq l \leq N} |J_l(z)| \leq C \int_{\partial B \cap \bigcup_{l \geq n} \overline{\Omega}_l} |\zeta' - z'|^{\alpha - 1} |d\zeta'|$$

$$\leq C' \int_{\gamma_n} |e^{i\phi} - 1|^{\alpha - 1} |d\phi|, \quad z \in \overline{B(p, r)} \setminus \bigcup_{l = n}^{N} \overline{\Omega}_l,$$

where $\gamma_n$ is the arc of the circle which is symmetric around the point 1 and has length $\text{mes}_1(\partial B \cap \bigcup_{l \geq n} \overline{\Omega}_l)$. Since $\alpha > 0$ the right hand side converges to zero for $n \to \infty$. This proves the proposition.

□
Chapter 5

Refined Propositions

In this chapter we will formulate two refinements of the proposition and give the proofs of the new results.

For our first result, we will use a few notions from analytic capacity and logarithmic capacity.

We are going to use Vitushkin’s Theorem and moreover the result by Tolsa for our second result. Therefore, we also use some notions of continuous analytic capacity.

5.1 Propositions

Proposition 5.1.1 (Weak Refined Proposition)
Let $B_i$ and $B_e$ be as in chapter 3 and let $f \in A(B_i \cup B_e)$.
Put $U = \mathbb{C} \setminus (B_i \cup B_e)$. If $U$ is thin at a point $p \in \mathcal{E}$ and $f$ is continuous, then $f|_{B_i}$ has fine analytic continuation $f|_{V_p}$ at $p$ for a fine neighbourhood $V_p \subset B_i \cup B_e$ of $p$.

Proposition 5.1.2 (Strong Refined Proposition)
Let $B_i' \cup B_e'$ be as in the proof by Edlund and Jöricke (section 4.3). Then: $R(K) = A(K)$, with $K = B_i' \cup B_e'$.

Remark.
If the strong refined proposition is fullfilled, then the weak refined proposition is fullfilled, because then $V_p = B_i' \cup B_e'$. So, if we have proven Proposition 5.1.2 we also have proven Proposition 5.1.1. However, as mentioned earlier, most of the work for this thesis is done in view of the weak refined proposition and therefore we would also like to present the weak refined proposition and its proof.
CHAPTER 5. REFINED PROPOSITIONS

5.2 Proof of the Weak Refined Proposition

The proof of the proposition of Edlund and Jöricke uses the fact that there are arbitrarily small numbers $r > 0$ such that $\partial B(p, r) \cap \bigcup_l \tilde{\vartheta}_l = \emptyset$ if $\bigcup_l \tilde{\vartheta}_l$ is thin at $p$.

We are going to use that fact for our problem.

Proof

We have arbitrarily small numbers $r > 0$ for which $\partial B(p, r) \cap \bigcup_l \tilde{\vartheta}_l = \emptyset$, because $\bigcup_l \tilde{\vartheta}_l$ is thin at $p$.

(The following is with respect to such $B(p, r)$). Because $r > 0$ is arbitrarily small and in view of Proposition 4.2.1, we can transform our problem to a problem on the real axis, where $p = 0$ and the rhombs maintain their form. And thus, we assume that the union of the rhombs is thin at 0 (thinness is invariant under translation).

And, with this point of view, we have that $\mathcal{E} = \mathbb{R} \setminus \bigcup_l \vartheta_l$.

Let $\vartheta'_j$ denote the enlarged rhombs described in the observations before the proof of the result of Edlund and Jöricke.

(We will keep the following notation: $\vartheta_l$ for the original rhombs in $B(0, r)$ and $\vartheta'_j$ for the enlarged ones.)

We have, just as in the proof of Edlund and Jöricke, that the function $f$ is analytic in the domains $B'_{i} \overset{\text{def}}{=} B(0, r) \cap \mathbb{H}^- \setminus \bigcup_l \tilde{\vartheta}_l$ and $B'_{e} \overset{\text{def}}{=} B(0, r) \cap \mathbb{H}^+ \setminus \bigcup_l \tilde{\vartheta}_l$ and continuous in the union of the closures.

Both domains have rectifiable boundary, hence by Cauchy’s formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B'_i} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\partial B'_e} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B'_i \cup B'_e$$

The contours of integration are always oriented as boundaries of relatively compact domains. Note that one of the integrals in the sum above is always zero, because the integrand in that integral is analytic on the domain of integration.
Using that \( \mathbb{R} \cap B(0, r) \setminus \bigcup_i \partial \Omega_i \subseteq \mathcal{E} \cap B(0, r) = \partial B'_i \cap \partial B'_c \) and integration over this set is carried out twice in opposite direction and therefore cancels out, we obtain:

\[
f(z) = \frac{1}{2\pi i} \int_{\partial B(0, r)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \sum_l \int_{\partial \Omega_l} \frac{f(\zeta)}{\zeta - z} d\zeta
def \mathcal{G}(z) - \mathcal{C}(z), \quad z \in B'_i \cup B'_c
\]

We have that \( \mathcal{G}(z) \) is an analytic function on the disc \( B(0, r) \) and that the measure \( f(\zeta) d\zeta \) on \( \bigcup_i \partial \Omega_i \) is a finite Borel measure concentrated on a subset of \( \bigcup_j \partial \Omega'_j \).

To prove the weak refined proposition, we have to show that

\[
f(z) = \mathcal{G}(z) - \frac{1}{2\pi i} \lim_{n \to \infty} \sum_{l=1}^n \int_{\partial \Omega_l} \frac{f(\zeta)}{\zeta - z} d\zeta
\]

converges uniformly on a closed fine neighbourhood of \( p \).

In fact, we only have to show that

\[
\mathcal{C}(z) = \frac{1}{2\pi i} \lim_{n \to \infty} \sum_{l=1}^n \int_{\partial \Omega_l} \frac{f(\zeta)}{\zeta - z} d\zeta
\]

converges uniformly on that neighbourhood of \( p \).

(REmark that \( \frac{1}{2\pi i} \sum_l \int_{\partial \Omega_l} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\bigcup_l \partial \Omega_l} \frac{f(\zeta)}{\zeta - z} d\zeta \).)

We have to make some preparations.

Let \( S \) be some union of the open rhombs \( \Omega_i \), then:

- \(|f|\) will assume a maximum on \( \partial S \);
- \(|\zeta - z| \geq d(z, S)\) for \( \zeta \in \partial S \) and \( z \notin S \) not a boundary point.
  \( d(z, S) \) stands for the distance between \( z \) and \( S \).

And therefore:

\[
\int_{\partial S} \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta| \leq \frac{\max_{\zeta \in \partial S} |f(\zeta)|}{d(z, S)} \int_{\partial S} |d\zeta|.
\]
This also holds when the contour of the contour integral is split into pieces and we regard the sum of the integrals over those pieces.

We will look at $\bigcup_l \partial \Omega_l$ which has a nice boundary. The last integral will, therefore, be the circumference of $\bigcup_l \partial \Omega_l$ ($|\bigcup_l \partial \Omega_l|$) which is the same as the circumference of $\bigcup_l \partial \Omega_l$. The circumference of the (closed) rhombs are in relation with their symmetry-axes $I_l$, which are disjoint.

However, we are also going to look at the collars $C_n(z)$, as defined in equation (1.1), for $z$ an element in a closed fine neighbourhood of $p$.

Let us look at the following:

$$\int_{C_n(z)\cap \bigcup_l \partial \Omega_l} \frac{1}{|\zeta - z|} |d\zeta| \leq \frac{1}{d(z, C_n(z))} \int_{C_n(z)\cap \bigcup_l \partial \Omega_l} |d\zeta|.$$

For $C_n(z) \cap \bigcup_l \partial \Omega_l$ we have to regard three situations:

1. $0 \leq d(z, \mathbb{R}) < 2^{-n-1}$;
2. $2^{-n-1} \leq d(z, \mathbb{R}) < 2^{-n}$;
3. $d(z, \mathbb{R}) \geq 2^{-n}$.

Remarks.
Due to symmetry, we only have to look at $z$ on or above the real line. Note that $\vartheta$ is the angle of the rhombi (as seen in Figure 1).
5.2. \textit{Proof of the Weak Refined Proposition}\hfill 43

We would like to estimate $|C_n(z) \cap \bigcup_l \partial \omega_l|$ with a constant times the analytic capacity of $C_n(z) \cap \bigcup_l \overline{\omega}_l$.

For this goal, we are going to use compact subsets of line segments.

In the first situation we have to choose between two line segments:

1.

Let us look at the following sets:

$S_1 = \{ w \in \mathbb{R} \cap C_n(z) \cap \bigcup_l \overline{\omega}_l \}$ and

$S_2 = \{ w \in C_n(z) : |Re(z) - Re(w)| \leq 2^{-n-1}, Im(z) - Im(w) = 2^{-n-1}, w \in \bigcup_l \overline{\omega}_l \}$.

Let $L_n$ be the one of these two sets such that $|L_n| = \max(|S_1|, |S_2|)$.

Then,

$$\int_{C_n(z) \cap \bigcup_l \partial \omega_l} |d\zeta| = \left| C_n(z) \cap \bigcup_l \partial \omega_l \right| \leq 2 \left( \frac{|L_n|}{\cos(\vartheta)} \right) \leq \frac{4}{\cos(\vartheta)} \left( 4 \cdot \gamma(L_n) \right).$$

Let $c = \frac{16}{\cos(\vartheta)}$. With $(E \subseteq F \Rightarrow \gamma(E) \leq \gamma(F))$ we have:

$$\int_{C_n(z) \cap \bigcup_l \partial \omega_l} |d\zeta| \leq c \cdot \gamma \left( C_n(z) \cap \bigcup_l \overline{\omega}_l \right).$$

Thus,

$$\int_{C_n(z) \cap \bigcup_l \partial \omega_l} \left| \frac{f(\zeta)}{\zeta - z} \right| |d\zeta| \leq \max_{\zeta \in \bigcup_l \partial \omega_l} |f(\zeta)| \cdot c \cdot \frac{\gamma(C_n(z) \cap \bigcup_l \overline{\omega}_l)}{d(z, C_n(z))}.$$

For the second situation, we only have to look at $\{ w \in \mathbb{R} \cap C_n(z) \cap \bigcup_l \overline{\omega}_l \}$.

By doing this, we obtain a constant which is different by the constant of situation 1. by a factor $\frac{1}{2}$. Therefore, we can use the same constant.

For the third situation we only have to look at the set $\{ w \in C_n(z) : Im(z) - Im(w) = 2^{-n}, w \in \bigcup_l \overline{\omega}_l \}$. The constant we obtain in this situation will differ with the first constant by a factor $\frac{1}{4}$ and therefore the first constant will also be sufficient in the last situation.

So, define $c = \frac{16}{\cos(\vartheta)}$. 

2.

Let \( L_n = \{ w \in \mathbb{R} \cap C_n(z) \cap \bigcup_l \delta_l \} \).

Then,
\[
\int_{C_n(z) \cap \bigcup_l \delta_l} |d\zeta| = \left| C_n(z) \cap \bigcup_l \partial \delta_l \right| \leq \frac{2 \cdot |L_n|}{\cos(\vartheta)}
\]
\[
\leq \frac{2}{\cos(\vartheta)} \left( 4 \cdot \gamma(L_n) \right)
\]
\[
\leq \frac{8}{\cos(\vartheta)} \cdot \gamma \left( C_n(z) \cap \bigcup_l \delta_l \right).
\]

We have:
\[
\int_{C_n(z) \cap \bigcup_l \delta_l} \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta| \leq \max_{\zeta \in \bigcup_l \delta_l} |f(\zeta)| \cdot \frac{8}{\cos(\vartheta)} \cdot \frac{\gamma(C_n(z) \cap \bigcup_l \delta_l)}{d(z, C_n(z))}
\]
\[
\leq \max_{\zeta \in \bigcup_l \delta_l} |f(\zeta)| \cdot c \cdot \frac{\gamma(C_n(z) \cap \bigcup_l \delta_l)}{d(z, C_n(z))}.
\]

3.

Let \( L_n = \{ w \in C_n(z) : \text{Im}(z) - \text{Im}(w) = 2^{-n}, w \in \bigcup_l \delta_l \} \).

Then,
\[
\int_{C_n(z) \cap \bigcup_l \delta_l} |d\zeta| = \left| C_n(z) \cap \bigcup_l \partial \delta_l \right| \leq \frac{|L_n|}{\cos(\vartheta)}
\]
\[
\leq \frac{1}{\cos(\vartheta)} \left( 4 \cdot \gamma(L_n) \right)
\]
\[
\leq \frac{4}{\cos(\vartheta)} \cdot \gamma \left( C_n(z) \cap \bigcup_l \delta_l \right).
\]
5.2. PROOF OF THE WEAK REFINED PROPOSITION

We have:

\[
\int_{C_n(z) \cap U \setminus \partial \Omega} \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta| \leq \max_{\zeta \in U \setminus \partial \Omega} |f(\zeta)| \cdot \frac{4}{\cos(\vartheta)} \cdot \frac{\gamma(C_n(z) \cap U \setminus \partial \Omega)}{d(z, C_n(z))} \\
\leq \max_{\zeta \in U \setminus \partial \Omega} |f(\zeta)| \cdot c \cdot \frac{\gamma(C_n(z) \cap U \setminus \partial \Omega)}{d(z, C_n(z))}.
\]

So, for all three situations we have:

\[
\int_{C_n(z) \cap U \setminus \partial \Omega} \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta| \leq \max_{\zeta \in U \setminus \partial \Omega} |f(\zeta)| \cdot c \cdot \frac{\gamma(C_n(z) \cap U \setminus \partial \Omega)}{d(z, C_n(z))} \tag{5.1}
\]

Our preparations are finished.

Before we estimate the integral, let us recall Lemma 1.3.17:

Let \( z_0 \in U \subset \mathbb{C} \), where \( U \) is a fine open set, and let \( E \supset U^C \), be an open set which is thin at \( z_0 \). For every real number \( \alpha > 0 \) there exists a closed fine neighbourhood \( U_\alpha \) of \( z_0 \) in \( U \) and a number \( N_\alpha \) such that:

\[
\text{Cap}(C_n(z) \cap E) \leq 2^{-n\alpha} \tag{1.3}
\]

for every \( z \in U_\alpha \) and every \( n \geq N_\alpha \).

Regard \( \mathbb{B} \setminus \bigcup_j \partial l_j \) as \( U \).

Let \( z_0 = 0 \) and \( E = \bigcup_j \partial l_j' \).

For every \( \alpha \) there is a closed fine neighbourhood \( U_\alpha \) for which (1.3) holds. Then \( U_\alpha^0 = U_\alpha \cap B(0, r) \) is also a closed fine neighbourhood for which (1.3) holds.
Let us estimate the integral for $z \in U_0^\alpha$:

$$|\mathcal{C}(z)| = \left| \frac{1}{2\pi i} \int_{\partial U_0} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\bigcup_n (C_n(z) \cap \partial U_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \sum_n \int_{C_n(z) \cap \partial U_0} \left| \frac{f(\zeta)}{\zeta - z} \right| \, d\zeta$$

$$(5.1) \leq \frac{c}{2\pi} \max_{\zeta \in \partial U_0} |f(\zeta)| \sum_n \frac{\gamma(C_n(z) \cap \bigcup_l \bar{\mathcal{O}}_l)}{d(z, C_n(z))}$$

$$\leq \frac{c}{2\pi} \max_{\zeta \in \partial U_0} |f(\zeta)| \sum_n \frac{\text{Cap}(C_n(z) \cap \bigcup_j \mathcal{O}_j)}{d(z, C_n(z))}$$

$$\leq \frac{c}{2\pi} \max_{\zeta \in \partial U_0} |f(\zeta)| \left( \sum_{n=1}^{N-1} \frac{\text{Cap}(C_n(z) \cap \bigcup_j \mathcal{O}_j)}{d(z, C_n(z))} + \sum_{n \geq N} \frac{\text{Cap}(C_n(z) \cap \bigcup_j \mathcal{O}_j)}{d(z, C_n(z))} \right)$$

$$(1.3) \leq \frac{c}{2\pi} \max_{\zeta \in \partial U_0} |f(\zeta)| \left( C + \sum_{n \geq N} \frac{2^{-n\alpha}}{2^{-n-1}} \right)$$

$$\leq \frac{c}{2\pi} \max_{\zeta \in \partial U_0} |f(\zeta)| \left( C + 2 \sum_{n \geq N} \frac{2^n}{2^m} \right)$$

We are going to choose our $\alpha \geq 2$.

To show that $\mathcal{C}(z)$ converges uniformly for the closed fine neighbourhood $U_0^\alpha$ we will show that $\{ \mathcal{C}_m(z) \}_{m=1}^\infty$ is a uniform Cauchy sequence in $U_0^\alpha$, where

$$\mathcal{C}_m(z) = \frac{1}{2\pi i} \int_{\bigcup_{n=1}^m (C_n(z) \cap \partial U_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Let $N_\alpha \leq m_1 \leq m_2$. 

5.3. PROOF OF THE STRONG REFINED PROPOSITION

We have that for all \( z \in U_0^\alpha \):

\[
|\mathcal{C}_{m_1}(z) - \mathcal{C}_{m_2}(z)| = \left| -\frac{1}{2\pi i} \int_{\bigcup_{n=m_1+1}^{m_2} (C_n(z) \cap \partial_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \\
\leq \frac{c}{\pi} \max_{\zeta \in \bigcup_0 \partial_0} |f(\zeta)| \sum_{n=m_1+1}^{m_2} \frac{2^n}{2^{m_2}}.
\]

With the last expression we see that \( \{ \mathcal{C}_m(z) \}_{m=1}^\infty \) is indeed a uniform Cauchy sequence in \( U_0^\alpha \) and therefore converges uniformly to \( \mathcal{C}(z) \) for \( z \in U_0^\alpha \).
Thus, there is a closed fine neighbourhood for which \( \mathcal{C}(z) \) converges uniformly and we have proven the weak refined proposition.

\[\square\]

5.3 Proof of the Strong Refined Proposition

We also wanted to know if \( R(K) = A(K) \) for our specific \( K = \overline{B_i} \cup \overline{B_e} \).
We will show that this is indeed true by showing that 
(5)(of Vitushkin’s Theorem) holds:

\[
\limsup_{\delta \to 0} \frac{\alpha\left(\overline{B(z, \delta)} \setminus K^\circ\right)}{\alpha\left(\overline{B(z, r\delta)} \setminus K\right)} < \infty.
\]

We will show this by using some properties of continuous analytic capacity and, most important, the result by Tolsa (Theorem 2.2.3).

Our first tool will be the result by Tolsa.
Then we use \( E \subseteq F \Rightarrow \alpha(E) \leq \alpha(F) \).
Finally, we use that some of the continuous analytic capacities are zero.
Let us start:

\[
\limsup_{\delta \to 0} \frac{\alpha \left( \overline{B(z, \delta)} \setminus K^c \right)}{\alpha \left( \overline{B(z, r\delta)} \setminus K^c \right)} = \limsup_{\delta \to 0} \frac{\alpha \left( \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right) \cup \left( \mathbb{R} \setminus \bigcup_l \overline{B(z, \delta)} \right) \right)}{\alpha \left( \overline{B(z, r\delta)} \cap \bigcup_j \partial_j \right)}
\]

\[
= \limsup_{\delta \to 0} \frac{\alpha \left( \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right) \cup \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right) \cup \left( \mathbb{R} \setminus \bigcup_l \overline{B(z, \delta)} \right) \right)}{\alpha \left( \overline{B(z, \delta)} \cap \bigcup_j \partial_j \right)}
\]

\[
\leq \limsup_{\delta \to 0} \frac{C \left( \alpha \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right) \right) + \alpha \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right) + \alpha \left( \mathbb{R} \setminus \bigcup_l \overline{B(z, \delta)} \right)}{\alpha \left( \overline{B(z, \delta)} \cap \bigcup_j \partial_j \right)}
\]

\[
\leq \limsup_{\delta \to 0} \frac{C \cdot \alpha \left( \overline{B(z, \delta)} \cap \bigcup_l \partial_l \right)}{\alpha \left( \overline{B(z, \delta)} \cap \bigcup_j \partial_j \right)} \leq \limsup_{\delta \to 0} C = C < \infty
\]

\[
\square
\]

We have now proven that \( R(K) = A(K) \) for \( K = \overline{B_i} \cup \overline{B_c} \) and thus have proven the strong refined proposition.
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