A study of necessary and sufficient conditions for vertex transitive graphs to be Hamiltonian

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Abstract

In this thesis we study some properties of vertex transitive graphs which are necessary for graphs to be Hamiltonian. Next, we also study some sufficiency theorems. Some extra attention will be given to Kneser graphs and we prove Baranyai’s partition theorem. We study Cayley graphs and prove all Cayley graphs of abelian groups are Hamiltonian. In the end we give an overview of further known results concerning our subject.
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Definition 0.1. A Hamilton path is a path in a graph $G = (V, E)$ containing every vertex $v \in V$. A Hamilton cycle is a closed Hamilton path. We call a graph Hamiltonian if it contains a Hamilton cycle.

First studied in 1856 and named after Sir William Hamilton, the general problem of finding Hamilton paths and cycles in graphs is already quite old. Although the problem of determining if a graph is Hamiltonian is well known to be \textit{NP}-complete, there has been a lot of research to the existence of such paths and cycles in all kind of graphs. There are a lot of examples in which Hamilton paths or cycles appear. Think about the route of a truck which has to deliver his load in more than one city. Or the scheduling of more than two soccer teams which all have to play against each other team. Or even the knight’s tour on a chessboard can be translated in a Hamilton cycle in a graph. It should be clear why the problem of finding Hamilton cycles in graphs is well known as \textit{The Traveling Salesman Problem}.

![Diagram of a graph with vertices labeled AB, AC, AD, AE, BC, BD, BE, CD, CE and edges connecting them.](image)

Figure 1: The schedule of the games of more than two soccer teams, where no team can play two games in a row and each game is repeated at least once and scheduled in the same order, is not always easy to find.
The problem of finding Hamilton cycles in graphs can be studied in all kind of graphs. To make the problem less broad, in this thesis we will only take a look at highly symmetric graphs, or vertex transitive graphs.

This thesis can be divided into two parts. In the first part we will study the structure of vertex transitive graphs and see if these graphs meet the conditions for graphs to be Hamiltonian. Furthermore we will see some theorems concerning the Hamiltonicity of graphs. In the second part we will take a close look at some special classes of vertex transitive graphs; In particular, Kneser graphs are treated in Chapter 3 and we will see some Cayley graphs in Chapter 4.

We will always restrict our attention to non-directed connected simple graphs and, when not specified otherwise, to $k$-regular graphs $G = (V, E)$ of order $n \geq 3$.

For definitions in graph theory, I would like to refer to Introduction to Graph Theory [39].
Chapter 1

Vertex transitive graphs

**Definition 1.1.** A graph $G = (V, E)$ is called vertex transitive if for every pair $u, v \in V$ there exists an automorphism $f : G \to G$ that maps $u$ to $v$.

Vertex transitive graphs are very well behaved graphs and it seems unsurprising that many graphs with such a high symmetry have Hamilton cycles. A bit more surprising is that there are only five vertex transitive graphs known which do not have Hamilton cycles, and that there are no graphs known which do not contain a Hamilton path. As a consequence, László Lovász, born in 1948 in Budapest, came in 1970 with the following conjecture:

**Conjecture 1.2 (Lovász [30]).** Every finite connected vertex transitive graph contains a Hamilton path.

Another version of Lovász conjecture states that

**Conjecture 1.3.** There is only a finite number of vertex transitive graphs which do not contain a Hamilton cycle.

It is immediate that every vertex transitive graph is regular. We are going to try to find properties of vertex transitive graphs that are not shared by all regular graphs. If vertex transitive graphs are more strongly connected than regular graphs in general, it should be easier to find a Hamilton cycle. In this section I will make intensively use of *Algebraic Graph Theory* [17].

Let us first introduce some definitions and take a closer look at the structure of vertex transitive graphs. In this chapter we will see some necessary conditions for a graph to be Hamiltonian. We prove that all vertex transitive graphs are 1-tough and we will see that all vertex transitive graphs of even order have perfect matchings. Later in this thesis we will study a couple of specific classes of vertex transitive graphs more intensively.
1.1 Properties of vertex transitive graphs

**Definition 1.4.** The vertex connectivity $\kappa(G)$ is the minimum size of a vertex set $S$ such that $G - S$ is disconnected. If $S$ consists of a single vertex $v$, we call $v$ a cut-vertex. We say that a graph $G$ is $k$-connected if its connectivity is at least $k$.

**Definition 1.5.** The edge connectivity $\hat{\kappa}(G)$ is the minimum size of an edge set $F$ such that $G - F$ is disconnected. If $F$ consists of a single edge $e$, we call $e$ a bridge or a cut-edge. We say that a graph $G$ is $k$-edge-connected if its edge-connectivity is at least $k$.

**Proposition 1.6.** If $G$ is a vertex transitive graph of order $n \geq 3$, then $G$ does not contain a cut-vertex or a bridge.

**Proof.** (i) Let $G$ be vertex transitive and suppose $G$ contains a cut-vertex. Then all vertices in $G$ are cut-vertices because $G$ is vertex transitive. Let $A_1, \ldots, A_k$ be the components of $G - v$ where $v \in V$ and $k \geq 2$. We can assume without loss of generality that $A_1$ is the smallest component. Let $u \in A_1$. We know that $G - u$ falls apart in $k$ components $B_1, \ldots, B_k$ with the components $B_i$ of the same sizes of the $A_i$. Then for some $i$ it most hold that $B_i \subset A_1$ which is a contradiction of the fact that $A_1$ is the smallest component.

(i) If $G$ contains a bridge $e = \{u, v\}$, and $G$ is of order $n \geq 3$, it contains two cut-vertices $u$ and $v$ and $G$ can not be vertex transitive.

**Definition 1.7.** Let $G = (V, E)$ and $A \subseteq V$. Denote by $N_G(A)$ the neighbors of $A$ not in $A$ and by $\bar{A}$ all the vertices not in $A$ and not in $N_G(A)$: $\bar{A} = V - A - N_G(A)$. We call $A$ a fragment of $G$ if $\bar{A} \neq \emptyset$ and $|N_G(A)| = \kappa(G)$. A fragment in $G$ of minimal size is called an atom of $G$. An edge atom is a subset $S \subseteq V$ such that $d_G(S) = \hat{\kappa}(G)$ with $S$ of minimal size.

**Definition 1.8.** Let $N$ an arbitrary set. A function $f : 2^N \rightarrow \mathbb{R}$ is called a submodular function if for all $A, B \subseteq N$

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

**Theorem 1.9** (Godsil and Royle, [17]). If $G$ is a $k$-regular vertex transitive graph with $k \geq 2$, then $\hat{\kappa}(G) = k$. 

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Proof. It is immediate that \( \hat{\kappa}(G) \leq k \). So we want to prove that \( \hat{\kappa}(G) \geq k \). Let \( S \) be an edge atom of \( G \). If \( S \) consists of a single vertex then \( \hat{\kappa}(G) = d_G(S) = k \) and we are done. So assume \( |S| \geq 2 \). The graph \( G \) is vertex transitive; let \( f \) be an automorphism of \( G \) and let \( T = f(S) \). Then \( |T| = |S| \) and \( d_G(T) = d_G(S) \). We know that \( d_G(A) \) is a submodular function: for all subsets \( A, B \subseteq V \) holds that

\[
d_G(A \cup B) \leq d_G(A) + d_G(B) - d_G(A \cap B).
\]

Hence

\[
d_G(S \cup T) + d_G(S \cap T) \leq 2\hat{\kappa}(G),
\]

and because \( S \) and \( T \) are edge atoms this can only hold if either \( S = T \) or \( S \cup T = V \) or \( S \cap T = \emptyset \). Suppose that \( S \cup T = V \) and \( S \cap T \neq \emptyset \). Then \( d_G(S) = d_G(T \setminus S) \) and \( T \setminus S \) would be a smaller atom than \( S \) and \( T \). We may conclude that the edge atoms of \( G \) partition \( V \) and for all automorphisms \( f \) the following holds:

\[
f(S) = S \quad \text{or} \quad f(S) \cap S = \emptyset.
\]

Next, suppose that the subgraph of \( G \) induced by \( S \) is not regular. Let \( x, y \in S \) with \( |N_{G[S]}(x)| > |N_{G[S]}(y)| \). Let \( g \) be the automorphism that maps \( x \) to \( y \). Then

\[
g(N_{G[S]}(x)) = N_{G[S]}(y)
\]

and for at least one neighbor \( z \in N_{G[S]}(x) \) of \( x \) holds that \( g(z) \in S \) which contradicts the fact that \( g(S) = S \) or \( g(S) \cap S = \emptyset \).

Hence we know that the subgraph of \( G \) induced by \( S \) is regular of degree \( l \leq k \) and because \( G \) is connected \( l < k \). Let \( |S| \geq k \), then

\[
\hat{\kappa}(G) = d_G(S) = |S|(k - l) \geq |S| \geq k,
\]

and we are done. Assume next that \( |S| < k \). Since \( l \leq |S| - 1 \), it follows that

\[
\hat{\kappa}(G) = d_G(S) = |S|(k - l) \geq |S|(k - |S| + 1) \geq k,
\]

which proves the theorem.

\[\square\]

Theorem 1.10 (Godsil and Royle, \cite{17}). If \( G \) is a \( k \)-regular vertex transitive graph, then

\[
\frac{2}{3}(k + 1) \leq \kappa(G) \leq k.
\]

Proof. It is immediate that \( \kappa(G) \leq k \). So we want to prove that \( \kappa(G) \geq \frac{2}{3}(k + 1) \). Let \( A \) be an atom of \( G \). If \( A \) consists of a single vertex then \( \kappa(G) = |N_G(A)| = k \) and we are done. So assume \( |A| \geq 2 \). The graph \( G \) is vertex transitive; let \( f \) be an
automorphism of $G$ and let $B = f(A)$. We can check that $N_G(A)$ is a submodular function so we know
\[ N_G(A \cup B) + N_G(A \cap B) \leq 2\kappa(G), \]
and because $A$ and $B$ are atoms, this can only hold if either
\[ A = B \text{ or } A \cup B = V \text{ or } A \cap B = \emptyset, \]
and, for the same arguments we gave in the proof of Theorem 1.9, we may conclude that $V$ is partitioned by atoms of $G$.

Let $A_1$ and $A_2$ be atoms. If $A_1 \cap N_G(A_2) \neq \emptyset$, then $A_1 \subseteq N_G(A_2)$ otherwise would
\[ N_G(\tilde{A}_2 \cap A_1) \leq 2\kappa(G) - N_G(\tilde{A}_2 \cup A_1) \leq \kappa(G) \]
and $\tilde{A}_2 \cap A_1$ would be a smaller atom. This tells us that $N_G(A)$ is partitioned into atoms of $G$, so
\[ |N_G(A)| = t|A| \]
for some integer $t$. The valency of a vertex in $A$ is equal to $k$ and at most
\[ k \leq |A| - 1 + |N_G(A)| = (t + 1)|A| - 1. \]
Hence
\[ \kappa(G) = \frac{(t + 1)|A|}{t + 1} \geq \frac{t}{t + 1}(k + 1). \]

Suppose $t = 1$. Then $N_G(A)$ is an atom and $|N_G(N_G(A))| = |A|$ and since
\[ A \cap N_G(N_G(A)) \neq \emptyset, \]
it follows that $A = N_G(N_G(A))$. But then $\tilde{A} = \emptyset$ and by definition $A$ is not a fragment. So $t \geq 2$ and $\kappa(G) \geq \frac{2}{3}(k + 1)$. \hfill \square

In Figure 1.1, a graph is shown with vertex connectivity smaller than the degree of a vertex.

In the last two theorems we met a special kind of subset of $V$. 

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Definition 1.11. Let $G$ be a vertex transitive graph. A subset $S \subseteq V$ is called a block of imprimitivity of $G$ if either $f(S) = S$ or $f(S) \cap S = \emptyset$ for all automorphisms $f$ of $G$. This partitioning of $V$ is called a system of imprimitivity. A graph with no nontrivial system of imprimitivity is called primitive.

Note that a system of imprimitivity is nontrivial if there exists a block of imprimitivity $S \subset V$ with $1 < |S| < |V|$. We saw that if $A$ is an atom or an edge atom, then $A$ is a block of imprimitivity. Hence if a vertex transitive graph $G$ contains an atom or an edge atom of size bigger then one, $G$ is not primitive.

As we will see later on, the cubic graphs form a set of graphs which are the most critical to be Hamiltonian. We will pay some extra attention to cubic graphs in all sections.

**Proposition 1.12** (West, [39]). If $G$ is a cubic graph, then $\kappa(G) = \hat{\kappa}(G)$.

**Proof.** Let $G = (V, E)$ be a cubic graph and $U \subseteq V$ a vertex cut of size $\kappa(G)$. We already know that $\kappa(G) \leq \hat{\kappa}(G)$ so it suffices to find an edge cut of size $|U|$. Let $H_1, H_2$ be two components of $G - U$. Since $U$ is minimal, we know that each $v \in U$ has at least one neighbor in $H_1$ and at least one neighbor in $H_2$ and because $G$ is cubic, $v$ can not have two neighbors in both $H_1$ and $H_2$. Consider the following set of edges $S$:

$$S = \{ e \in E \mid v \in U \text{ and either } e \text{ is the only edge from } v \text{ to } H_1 \text{, or there are two edges from } v \text{ to } H_1 \text{ and } e \text{ is the edge from } v \text{ to } H_2 \}$$

Then $S$ is an edge cut of size $|U|$. \qed

With the theory of connectivity and edge-connectivity, we saw how tightly connected vertex transitive graphs are. There are two other variables concerning this subject.

**Definition 1.13.** An independent set is a set of vertices of a graph $G$ such that no two vertices in the set are adjacent. The coclique, or independency number $\alpha(G)$ is the maximum size of an independent set of vertices.

**Definition 1.14** (Chvátal, [13]). Let $c(G)$ be the number of components of $G$. A graph $G$ is said to be $t$-tough if $t \cdot c(G - S) \leq |S|$ for all $S \subseteq V$ with $c(G - S) > 1$. The largest $t$ such that $G$ is $t$-tough is called the toughness of a graph and denoted by $t(G)$. 

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Unfortunately we do not know much of the size of an independent set of vertex transitive graphs. We do have the following bound
\[ \alpha(G) \leq \frac{1}{2}(|V| + 1), \]
but this is not a real valuable bound.

We can now prove the following:

**Theorem 1.15.** All vertex transitive graphs are 1-tough.

**Proof.** Let \( G \) be a \( k \)-regular vertex transitive graph. We know by Theorem 1.9 that \( G \) has edge-connectivity \( \kappa(G) = k \). So there does not exist \( U \subseteq V \) with \( d(U) < k \). Let \( S \) be a vertex cut. Then
\[ c(G - S)k \leq d_G(S) \leq |S|k, \]
so \( c(G - S) \leq |S| \) for all \( S \) with \( c(G - S) > 1 \).

Concerning the toughness of a graph, note that we can write
\[ t(G) = \min_{S \subseteq V, c(G - S) > 1} \frac{|S|}{c(G - S)}. \]
It follows that
\[ t(G) \leq \frac{\kappa(G)}{2}, \quad \text{and} \quad t(G) \geq \frac{\kappa(G)}{\alpha(G)}, \]
and \( \kappa(G) \geq 2 \) if \( G \) is vertex transitive.

There are more variants we can look at concerning vertex transitive graphs. The following theorem shows the relation between vertex transitive graphs and matchings.

**Theorem 1.16** (Godsil and Royle, [17]). Let \( G \) be a vertex transitive graph. Then \( G \) has a matching that misses at most one vertex.

**Proof.** Let \( G \) be a vertex transitive graph. A vertex \( u \) in \( G \) is called critical if every maximum matching covers \( u \). It should be clear that if \( G \) contains a critical vertex, then all vertices in \( G \) are critical because \( G \) is vertex transitive. It follows that \( G \) has a perfect matching if \( G \) contains a critical vertex. So assume \( G \) contains no critical vertices. Then every vertex in \( G \) is missed by at least one matching. Let \( M_v \) be a matching of maximum size that misses a vertex \( v \). Suppose there exists a matching \( N \) of maximum size that misses the two vertices \( u, v \in V \). Let \( P \) be the path between \( u \) and \( v \), this path exists because \( G \) is connected. We will prove that \( N \) does not exist by induction on the length of \( P \).
If $u$ and $v$ are connected by an edge, then we could add the edge $\{u, v\}$ to $N$, contradicting the fact that $N$ is maximum. Let the length of $P$ be at least two. Let $x$ be a vertex on $P$ different from $u$ and $v$. By the induction hypothesis there exists no maximal matching in $G$ that misses both $u$ and $x$ or both $v$ and $x$. Hence we know that $N$ covers $x$, and that $u$ and $v$ are covered by $M_x$.

Consider the symmetric difference $N \Delta M_x$. We know that $N \Delta M_x$ has maximum valency two and each component is either an alternating path or an alternating cycle, relative to both $N$ and $M_x$. The vertices $u, v, x$ have degree one in $N \Delta M_x$.

Let $P_u$ be the path with one endpoint in $u$ and $P_v$ the path with one endpoint in $v$. If $P_u = P_v$, the path is an $N$-augmenting path with endpoints $u$ and $v$, contradicting the fact that $N$ is maximal. If the second endpoint of both $P_u$ and $P_v$ is equal to $x$ we must conclude that $u = v$. This concludes the proof that $G$ cannot have a maximal matching that misses two of its vertices.

1.2 Non-Hamiltonian graphs

There are only five vertex transitive graphs known which do not have Hamilton cycles. We are very interested in these particular graphs because they can lead us to some 'rule' in which cases vertex transitive graphs do not contain Hamilton cycles. The first non-Hamiltonian vertex transitive graph is not interesting at all: it is $K_2$, the graph on two vertices and one edge. The second and third are the Petersen graph on 10 vertices, see Figure 1.2, and the Coxeter graph on 28 vertices depicted in Figure 1.4. We will see both of these graphs in this section. The remaining two non-Hamiltonian vertex transitive graphs are obtained from the Petersen and Coxeter graphs by replacing each vertex by a triangle, for picturing this see Figure 1.5. This way you get two 3-regular graphs on 30 and 84 vertices respectively. If you repeat the blowing up process a second time the graphs will not longer stay vertex transitive, although they are still 3-regular and non-Hamiltonian.

Ignoring $K_2$, these four non-Hamiltonian graphs are all 3-regular, also called cubic. For this reason we will pay some extra attention to cubic graphs during this thesis.

The Petersen graph

The Petersen graph (see Figure 1.2) is the cubic graph on 10 vertices and 15 edges and the complement of the line graph of $K_5$. Because the Petersen graph is a small graph, it appeared to be a useful example or counterexample for many problems in graph theory. It is the smallest bridgeless cubic graph with no Hamiltonian cycle and one of the five known non-Hamiltonian vertex transitive graphs.

**Theorem 1.17.** The Petersen graph does not contain a Hamilton cycle.

**Proof.** The Petersen graph is constructed from circulants based on $\mathbb{Z}_5$ (see Chapter 4 for detailed information of graphs based on circulants). Start with the circulants
Figure 1.2: The Petersen graph and a cycle plus perfect matching. The first is three edge-colorable, the second can be colored with two colors.

\[ G(\mathbb{Z}_5, \{1\}) \text{ and } G(\mathbb{Z}_5, \{2\}) \text{ and then add five more edges, joining each one of the same elements of } \mathbb{Z}_5 \text{ in the two circulants. We first show that the Petersen graph is not three edge-colorable. Suppose you can color the edges of the graph with colors } \{1, 2, 3\}. \text{ Without loss of generality we can assume that there are two edges in } G(\mathbb{Z}_5, \{1\}) \text{ with color 1, two edges with color 2 and one edge with color 3. Now the colors for the edges that connect } G(\mathbb{Z}_5, \{1\}) \text{ and } G(\mathbb{Z}_5, \{2\}) \text{ are set. Next we can not give the edges of } G(\mathbb{Z}_5, \{2\}) \text{ a proper coloring and the Petersen graph is not three edge-colorable.} \]

Suppose the Petersen graph does have a Hamilton cycle. Then because the graph is three-regular, it is equal to a 10-cycle plus a perfect matching. Color the 10-cycle with color 1 and 2 in alternating order, and the edges of the perfect matching with color 3. Then the graph would be three edge-colorable contradicting the fact that it was not. \(\square\)

More proofs of the fact that the Petersen graph is non-Hamiltonian are known and we will refer to this graph a couple of times in this thesis.

The Coxeter graph

The Coxeter graph is the cubic graph on 28 vertices and 42 edges as shown in Figure 1.4 and another vertex transitive graph which is not Hamiltonian. It is

Figure 1.3: The Fano plane: The antiflags of the Fano plane are the vertices of the Coxeter graph.
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Figure 1.4: The Coxeter graph

constructed from the Fano plane, depicted in Figure 1.3, by drawing a vertex for each antiflag in the Fano plane. An antiflag is an ordered pair \((p, l)\), where \(p\) is a point and \(l\) is a line such that \(p \notin l\). Two such antiflags \((p, l)\) and \((q, m)\) are adjacent in the graph if the set \(\{p, q\} \cup l \cup m\) contains all points of the Fano plane. Another way to see this is to make a vertex for each three points which do not lie on a line. Two triplets are adjacent if they are disjoint. This makes the Coxeter graph a subgraph of the Kneser graph \(K(7, 3)\) which we will discuss in Chapter 3.

**Theorem 1.18** (Godsil e.a. [17]). The Coxeter graph does not contain a Hamilton cycle.

Different than the non-Hamiltonicity of the Petersen graph, the non existence of a Hamilton cycle in the Coxeter graph is hard to prove. A direct proof is found by Tutte [37]. Later in this thesis we will see another way for approaching Hamiltonicity, for which we need the following:

**Definition 1.19.** A graph \(G = (V, E)\) is called \(k\)-edge regular if for every pair \(p, q\) of paths of length \(k\) there exists an automorphism that maps \(p\) to \(q\).

**Theorem 1.20.** The Coxeter graph is \(3\)-edge regular.

A proof of this theorem is given by Godsil and Royle [17].

**The line graph of the subdivision graph**

The non-Hamiltonicity of the two remaining non-Hamiltonian vertex transitive graphs is not hard to see.

**Definition 1.21.** The subdivision graph \(S(G)\) of a graph \(G\) is obtained by putting one new vertex in the middle of each edge of \(G\). The vertex set of \(S\) is equal to \(V \cup E\), where two vertices \(v, e\) are adjacent if \(v \in e\) in \(G\).
Proposition 1.22. Let $G$ be a cubic graph. Then $G$ is Hamiltonian if and only if the line graph of the subdivision graph $L(S(G))$ is Hamiltonian.

Proof. The line graph of the subdivision graph of a cubic graph $G$ is equal to $G$ with each vertex replaced by a triangle, and is itself a cubic graph. The proposition follows directly from the fact that if a cubic graph on $n$ vertices has a Hamilton cycle, then it is equal to an $n$-cycle (plus a perfect matching) and it is easily seen that a cycle with each vertex replaced by a triangle is still a cycle. \hfill \square

Since we have proved that the Petersen graph and Coxeter graph have no Hamilton cycles, it follows that the line graphs of the subdivision graphs of these two are not Hamiltonian either. These are the two remaining non-Hamiltonian vertex transitive graphs we mentioned before. We can repeat the blowing up process infinitely many times (by taking the line graphs of the subdivision graphs again) and we will get an infinite sequence of non-Hamiltonian graphs. However, these graphs will no longer be vertex transitive.
Chapter 2

Hamiltonicity

There are different ways to find out if a graph contains a Hamilton cycle. We already saw some necessary conditions for a graph to be Hamiltonian, which vertex transitive graphs meet. Think about not having bridges or cut-vertices, or for cubic graphs to have perfect matchings. We will take a close look at some other conditions for a graph to be Hamiltonian.

2.1 Matchings and high degrees

In this section we will see that the relation between matchings and cycles is not only reserved for cubic graphs. Häggkvist conjectured the following theorem which was proved by Berman [8].

**Theorem 2.1** (Häggkvist). Let $G$ be a $k$-regular graph of order $n \geq 3$ with $k \geq \frac{1}{2}(n+1)$. Then every set of independent edges of $G$ lies in a cycle.

The following proposition follows almost directly from this theorem.

**Proposition 2.2.** Let $G$ be a vertex transitive $k$-regular graph of order $n$ and $k \geq \frac{1}{2}(n+1)$. Let $M$ be a matching that misses at most one vertex. Then $G$ contains a Hamilton cycle $C$ such that the edges of $M$ are contained in $C$.

**Proof.** Let $G$ be a vertex transitive graph. First suppose $G$ has an even number of vertices and let $M$ be a matching of size $\frac{1}{2}n$. We saw in Theorem 1.16 that this matching exists for vertex transitive graphs. Let the degree of the vertices of the graph be at least $\frac{1}{2}(n+1)$. Then it follows directly from Theorem 2.1 that $M$ is contained in a cycle of size $n$ which is a Hamilton cycle.

Suppose next that $G$ has an odd number of vertices and $M$ is a matching of size $\frac{1}{2}(n-1)$. Then it follows from Theorem 2.1 that $M$ is contained in a cycle $C$ of length at least $n-1$. If $|V(C)| = n$, $C$ clearly is a Hamilton cycle and we are done. So assume $|V(C)| = n-1$ and let $v$ be the vertex not in $C$. Note that $C$ consists of edges of $M$ and edges of $E(G)\setminus M$ in alternating order. Assume there
are no two vertices \( x, y \in V(C) \cap N_G(v) \) such that \( \{x, y\} \in E(C) \setminus M \). Then

\[
d_G(v) \leq \frac{1}{2} |M| < \frac{1}{2} (n + 1)
\]

which is a contradiction. So there are \( x, y \in V(C) \cap N_G(v) \) such that \( \{x, y\} \in E(C) \setminus M \) and \( C \setminus \{x, y\} \cup \{\{x, v\}, \{v, y\}\} \) gives the desired Hamilton cycle. \( \Box \)

Dirac slightly improved this bound by

**Theorem 2.3** (Dirac, [39]). Let \( G \) be a \( k \)-regular graph of order \( n \geq 3 \) such that \( 2k \geq n \). Then \( G \) is Hamiltonian.

**Proof.** Let \( G \) be a \( k \)-regular graph with \( k \geq \frac{n}{2} \). Assume \( G \) contains no Hamilton cycle. Let \( G' \) be the graph obtained from \( G \) by adding a maximum number of edges without creating a Hamilton cycle. This means that adding any edge to \( G' \) will create a Hamilton cycle.

Let \( u, v \in V \) be two non-adjacent vertices in \( G' \) and let \( P \) be the Hamilton path

\[
P = (u = v_1, v_2, \ldots, v_n = v).
\]

Define

\[
S = \{i : v_{i+1} \in N(u)\} \quad \text{and} \quad T = \{i : v_i \in N(v)\}.
\]

Then \( S \cap T \neq \emptyset \) otherwise would \( S \cup T = n \) while \( n \notin S \cup T \).

Let \( l \in S \cap T \). Then

\[
(u, v_2, \ldots, v_l, v, v_{n-1}, \ldots, v_{l+1}, u)
\]

is a Hamilton cycle in \( G' \), a contradiction. So we can not extend \( G \) to a maximal graph \( G' \) and we may conclude that \( G \) contains a Hamilton cycle itself. \( \Box \)

Jackson [22] at his turn improved this bound by

**Theorem 2.4** (Jackson). A \( k \)-regular 2-connected graph on \( n \) vertices is Hamiltonian if \( n \leq 3k \).

We end this section with the following conjecture:

**Conjecture 2.5.** Let \( G \) be a cubic graph. Then there exists a perfect matching \( M \) in \( G \) such that \( G - M \) consists of either one cycle, and \( G \) is Hamiltonian, or of two non-connected cycles.

We know from Theorem 1.16 that all vertex transitive cubic graphs have perfect matchings. The only known non-Hamiltonian cubic vertex transitive graphs \( G \) have perfect matchings \( M \) such that \( G - M \) consists of two non-connected cycles. If Conjecture 2.5 is true, it is easy to find a Hamilton path and Conjecture 1.2 would be true for all cubic graphs.
2.2 Connectivity and toughness

In the previous chapter we studied the connectivity of vertex transitive graphs because we expect some relation between the connectivity and the Hamiltonicity of graphs. There exists a theorem stated by Chvátal and Erdős [14] concerning this relation. For this we first need the following theorem of Dirac [12].

**Theorem 2.6.** Let $G$ be a graph with $\kappa(G) \geq 2$ and let $X$ be a set of $\kappa(G)$ vertices of $G$. Then there exists a cycle in $G$ containing every vertex of $X$.

With this theorem we can prove the following:

**Theorem 2.7** (Chvátal and Erdős). Let $G$ be a graph of order $n \geq 3$. If $\kappa(G) \geq \alpha(G)$ then $G$ is Hamiltonian.

**Proof.** Let $G$ be a graph with at least three vertices and let $\kappa(G) = s$ and $\alpha(G) \leq s$. Let $S = \{x_1, \ldots, x_s\}$ be a disconnecting set of order $s$. According to theorem 2.6, every subset $U \subseteq V$ with $|U| = s$ is contained in a cycle in $G$. Let $C$ be the longest cycle containing the vertices of $S$. Assume $C$ is not a Hamilton cycle and let $v \notin C$. According to Menger’s theorem there exist pairwise disjoint paths $p_i$, such that $p_i$ goes from $v$ to $x_i$, where $i = 1, \ldots, s$. Fix an orientation for $C$ and let $y_i$ be the predecessor of $x_i$ in $C$.

i) Assume there are $i, j \in \{1, \ldots, s\}$ with $x_i = y_j$, then we could replace the edge $\{y_j, x_j\}$ by the path $(p_i, p_j)$ and obtain a longer cycle, which is a contradiction. So $x_i \neq y_j$, $i, j = 1, \ldots, s$.

ii) Assume there is an $i \in \{1, \ldots, s\}$ with $y_i$ adjacent to $x$, then we could replace the edge $\{y_i, v\}$ by the path $(y_i, v, p_i)$ and obtain a longer cycle, which is a contradiction. So $y_i$ is not adjacent to $v$ for all $i \in \{1, \ldots, s\}$.

iii) There are $a, b \in \{1, \ldots, s\}$ for which the edge $\{y_a, y_b\}$ exists in $G$. If not, $\{y_1, \ldots, y_s, v\}$ form an independent set of size $s + 1$, which is a contradiction.

Now we could delete the edges $\{y_a, x_a\}$ and $\{y_b, x_b\}$ from $C$ and replace them by

![Figure 2.1: A cycle including the vertices of an independent set](image-url)
\( \{y_a, y_b\} \) and \((p_b, p_a)\) and obtain a longer cycle (see Figure 2.1), which is a contradiction. We may conclude that \( C \) is a Hamilton cycle.

As a result of this theorem we give the following proposition without proof.

**Proposition 2.8** (Chvátal and Erdös [14]). Let \( G \) be a graph of order \( n \geq 3 \). If \( \kappa(G) + 1 \geq \alpha(G) \) then \( G \) has a Hamilton path.

We already know some bounds for \( \kappa(G) \) for \( k \)-regular vertex transitive graphs:

\[
\frac{2}{3}(k + 1) \leq \kappa(G) \leq k.
\]

If we can say something about the bounds of \( \alpha(G) \), we know how useful Theorem 2.7 is. Without any further knowledge of \( G \), the only thing we can say about vertex transitive graphs on \( n \) vertices is that \( \alpha(G) \leq \frac{1}{2}(n + 1) \). So if

\[
\frac{1}{2}(n + 1) \leq \frac{2}{3}(k + 1),
\]

then \( G \) is Hamiltonian, which really is not a good bound: We have seen better bounds by different theorems before.

Another necessary condition for a graph to be Hamiltonian, is to be 1-tough. We already saw that vertex transitive graphs meet this constraint. It is an interesting question if there exists a \( t \) such that every \( t \)-tough (vertex transitive) graph is Hamiltonian. Chvátal [13] stated the following conjecture.

**Conjecture 2.9.** There exists \( t_0 \) such that every \( t_0 \)-tough graph is Hamiltonian.

This conjecture is valid for planar graphs and \( t_0 > 3/2 \): Every \( t \)-tough graph with \( t > \frac{3}{2} \) is 4-connected and by Tutte’s theorem [38], every 4-connected planar graph is Hamiltonian. Unfortunately this is not relevant in order of our study of vertex transitive graphs because most vertex transitive graphs will not be planar. The toughness of the Petersen graph is equal to \( \frac{4}{3} \) and the toughness of the Coxeter graph is equal to \( \frac{3}{2} \). In fact, there are no counterexamples known of the following:

**Conjecture 2.10.** Every 2-tough graph is Hamiltonian.
2.3 Graphs and eigenspaces

Let \( G = (V, E) \) be a graph and \( A_G \) be the \( V \times V \) matrix with entries \( a_{ij} = 1 \) if \( \{i, j\} \in E \) and 0 otherwise. We call \( A_G \) the adjacency matrix of \( G \). Further let \( L_G = D_G - A_G \) be the Laplacian matrix of \( G \), where \( D_G \) is the diagonal matrix with the degrees of the vertices of \( G \) on its diagonal. In this thesis we only look at vertex transitive graphs so we can write the Laplacian as \( L_G = kI - A_G \), where \( k \) is the degree of the vertices of \( G \) and \( I \) is the identity matrix of appropriate dimension. There has been a lot of study on the eigenvalues \( \lambda_i \) of a graph, \( i = 1 \ldots n \). For a survey of results, see for instance Mohar [35]. By the eigenvalues of a graph we mean the eigenvalues of the adjacency matrix \( A_G \). We will order the eigenvalues such that \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). The set of all eigenvalues of a graph \( G \) is called the spectrum \( \text{spec}(G) \) of a graph \( G \).

Of course you can also take a look at the eigenvalues of the Laplacian matrix \( L_G \). To avoid misunderstanding we shall write \( \lambda_i(A_G) \) and \( \lambda_i(L_G) \) for the \( i \)-th eigenvalue of the adjacency matrix and the Laplacian matrix respectively. Note that the eigenvalues of the Laplacian are ordered such that \( \lambda_1(L_G) \) is the largest eigenvalue and:

\[
\lambda_i(L_G) = k - \lambda_i(A_G)
\]  

In this section we will see that we have a lower bound for the Laplacian eigenvalues of a graph \( G \), for \( G \) to be Hamiltonian (Heuvel [19]). We first choose an orientation on \( G \) by choosing for every edge \( e \) of \( G \) one of its end vertices as the initial vertex and the other as the terminal vertex. Define the oriented incidence matrix of \( G \) with respect to the chosen orientation by the \( V \times E \) matrix \( M_G \) with entries

\[
(M(G))_{ve} = \begin{cases} 
1, & \text{if } v \text{ is the terminal vertex of } e, \\
-1, & \text{if } v \text{ is the initial vertex of } e, \\
0, & \text{if } v \text{ and } e \text{ are not incident}.
\end{cases}
\]

Let \( M_G^T \) denotes the transpose of \( M_G \). It holds that \( L_G = M_G \cdot M_G^T \).

Both \( M_G \cdot M_G^T \) and \( M_H \cdot M_H^T \) are symmetric matrices with only nonnegative eigenvalues and, apart from the multiplicity of the eigenvalue 0, their spectra coincide. So

\[
\text{spec}(M_G^T \cdot M_G) = \{\text{spec}(M_G \cdot M_G^T), 0_{(m-n)}\}
\]

where \( n \) is the number of vertices of \( G \) and \( m \) is the number of edges of \( G \). Let \( H \) be a subgraph of \( G \) obtained by deleting an edge of \( G \). Then the matrix \( M_H^T \cdot M_H \) is obtained from \( M_G^T \cdot M_G \) by deleting the corresponding row and column of the deleted edge (\( M_G^T \cdot M_G \) is an \( E \times E \) matrix). By the Interlacing Property of symmetric matrices (van Lint [29]) we now have

\[
\lambda_i(M_G^T \cdot M_G) \geq \lambda_i(M_H^T \cdot M_H) \geq \lambda_{i+1}(M_G^T \cdot M_G), \quad i = 1 \ldots n - 1.
\]
It follows that
\[ \lambda_i(L_G) \geq \lambda_i(L_H) \geq \lambda_{i+1}(L_G), \quad i = 1 \ldots n - 1. \]

If you delete \( m' \) edges from \( G \) this results in
\[ \lambda_i(L_G) \geq \lambda_i(L_H) \geq \lambda_{i+m'}(L_G), \quad i + m' \leq n. \]

**Theorem 2.11** (Heuvel, [19]). Let \( G \) be a graph on \( n \) vertices and \( \lambda_i(L_G), i = 1 \ldots n \) its Laplacian eigenvalues. If \( G \) contains a Hamilton cycle then
\[ \lambda_i(L_G) \geq \lambda_i(L_{C_n}) \geq \begin{cases} \lambda_{m-n+i}(L_G) & \text{if } 1 \leq i \leq 2n - m \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** If \( G \) contains a Hamilton cycle \( C_n \), you can delete \( m' = m - n \) edges from \( G \) to contain a subgraph \( H \) equal to \( C_n \). Now we have
\[ \lambda_i(L_G) \geq \lambda_i(L_H) \geq \lambda_{i+m-n}(L_G), \quad i + m - n \leq n. \]

It follows directly from Theorem 2.11 and (2.1) that for \( k \)-regular graphs on \( n \) vertices it holds that
\[ \lambda_i(A_G) \leq \lambda_i(A_{C_{10}}) + (k - 2). \]

We see for the Petersen graph \( P \) with spectrum \( \text{spec}\{-2[4], 1[5], 3[1]\} \) that
\[ \lambda_5(A_P) = 1 > 2\cos(3\pi/5) + 1 = \lambda_5(A_{C_{10}}) + 1 \]
which makes the Petersen graph non-Hamiltonian.

We now take a closer look at the Coxeter graph \( C \). Suppose \( C \) contains a Hamilton cycle \( H \). As a result of Theorem 1.20, we can choose any path of length three which should be contained in \( H \). Lets say we choose the path
\[ P = (v_i, v_{i+1}, v_{i+2}, v_{i+3}) \subset H. \]

Each vertex \( v_j \) of \( C \) is contained in three edges; let \( e_j \) be the edge with \( v_j \in e_j \) and \( e_j \notin H \). Let
\[ C' = C \setminus \{e_{i+1}, e_{i+2}\}. \quad (2.2) \]

Then it should hold that \( C' \) still contains the cycle \( H \). In Table 2.1, the Laplacian eigenvalues of \( C_{28} \), the Coxeter graph and the Coxeter graph minus the two non-adjacent edges as in (2.2) are shown. Unfortunately, the Interlacing Property still holds for \( C' \), and we can not exclude the Coxeter graph to be Hamiltonian. Nevertheless, it is a good example of deleting edges from graphs and comparing their eigenvalues.
### Table 2.1: \( \lambda \) values for the Laplacian eigenvalues of \( C_{28} \), the Coxeter graph \( C \) and the Coxeter graph minus the two edges \( C' \) as in (2.2).

<table>
<thead>
<tr>
<th>( \lambda_i )</th>
<th>( C_{28} )</th>
<th>( C )</th>
<th>( C' )</th>
<th>( \lambda_i )</th>
<th>( C_{28} )</th>
<th>( C )</th>
<th>( C' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>4</td>
<td>5.414</td>
<td>5.414</td>
<td>( \lambda_{15} )</td>
<td>2</td>
<td>2.586</td>
<td>2.586</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>3.950</td>
<td>5.414</td>
<td>5.414</td>
<td>( \lambda_{16} )</td>
<td>1.555</td>
<td>2.586</td>
<td>2.586</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>3.950</td>
<td>5.414</td>
<td>5.414</td>
<td>( \lambda_{17} )</td>
<td>1.555</td>
<td>2.586</td>
<td>2.586</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>3.802</td>
<td>5.414</td>
<td>5.414</td>
<td>( \lambda_{18} )</td>
<td>1.132</td>
<td>2.586</td>
<td>2.215</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>3.802</td>
<td>5.414</td>
<td>5.228</td>
<td>( \lambda_{19} )</td>
<td>1.132</td>
<td>2.586</td>
<td>1.864</td>
</tr>
<tr>
<td>( \lambda_6 )</td>
<td>3.564</td>
<td>5.414</td>
<td>4.751</td>
<td>( \lambda_{20} )</td>
<td>0.753</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_7 )</td>
<td>3.564</td>
<td>4</td>
<td>4</td>
<td>( \lambda_{21} )</td>
<td>0.753</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_8 )</td>
<td>3.247</td>
<td>4</td>
<td>4</td>
<td>( \lambda_{22} )</td>
<td>0.436</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_9 )</td>
<td>2.247</td>
<td>4</td>
<td>4</td>
<td>( \lambda_{23} )</td>
<td>0.436</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_{10} )</td>
<td>2.868</td>
<td>4</td>
<td>4</td>
<td>( \lambda_{24} )</td>
<td>0.198</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_{11} )</td>
<td>2.868</td>
<td>4</td>
<td>4</td>
<td>( \lambda_{25} )</td>
<td>0.198</td>
<td>1</td>
<td>1</td>
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<tr>
<td>( \lambda_{12} )</td>
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<td>4</td>
<td>3.464</td>
<td>( \lambda_{26} )</td>
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<td>0.716</td>
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</tr>
<tr>
<td>( \lambda_{13} )</td>
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<td>4</td>
<td>3.318</td>
<td>( \lambda_{27} )</td>
<td>0.050</td>
<td>1</td>
<td>0.444</td>
</tr>
<tr>
<td>( \lambda_{14} )</td>
<td>2</td>
<td>2.586</td>
<td>2.586</td>
<td>( \lambda_{28} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Chapter 3

Kneser graphs

**Definition 3.1.** A \( k \)-subset of a set \( N \) is a subset of \( N \) with cardinality \( k \). The Kneser graph \( K(n,k) \) has as vertices the \( k \)-subsets of \( \{1,2,\ldots,n\} \) where two vertices are adjacent if the \( k \)-subsets are disjoint.

Kneser graphs are named after Martin Kneser, who first investigated them in 1955. The graph in the introduction of this thesis is the Kneser graph \( K(5,2) \). This graph is also equal to the Petersen graph, so we know that there is no solution for the soccer problem described. Though for most pairs \( (n,k) \) the use of Kneser graphs is a good way to find a schedule for the matches to play, in fact for all \( n \geq 6 \) there exist solutions of this scheduling problem as we will prove in this section. Of course you can find many other applications comparable to this soccer problem.

**Theorem 3.2.** All Kneser graphs are vertex transitive.

*Proof.* Let \( f \) be some permutation of \( \{1,2,\ldots,n\} \). Then \( f \) induces an automorphism of \( K(n,k) \): Let \( S = \{a_1,\ldots,a_k\} \) and \( T = \{b_1,\ldots,b_k\} \) be vertices of \( K(n,k) \). Then, for example,

\[
(a_1b_1)(a_2b_2)\cdots(a_kb_k)
\]

(3.1)

is a permutation of \( \{1,2,\ldots,n\} \) that sends \( S \) to \( T \) and the neighbors of \( S \) to the neighbors of \( T \). So (3.1) is an automorphism of \( K(n,k) \) and all Kneser graphs are vertex transitive. Note that this is only an example of the many permutations that map \( S \) onto \( T \). \( \square \)

As a result of Conjecture 1.3 we now have the following.

**Conjecture 3.3.** All Kneser graphs are Hamiltonian, except for \( K(5,2) \), which is the Petersen graph.

It is already shown for a lot of Kneser graphs that they are Hamiltonian. In particular, \( K(n,k) \) has a Hamilton cycle for \( k = 1 \) and \( n \geq 3 \), which is the complete graph \( K_n \). Heinrich and Wallis [18] showed the Hamiltonicity for \( k = 2 \) and \( n \geq 6 \) and for \( k = 3 \) and \( n \geq 7 \). Chen [10] constructed a Hamilton cycle for
n \geq 3k and even improved this bound later on by \( n \geq 2,62k + 1 \). Note that for a graph to be connected it is necessary that \( n \geq 2k + 1 \) which still gives us a gap for Kneser graphs to be Hamiltonian when \( 2k + 1 \leq n \leq 2,62k + 1 \).

In this section we will study some of the results known. We prove an important theorem of Baranyai, we introduce Gray codes, and prove they exist.

### 3.1 Baranyai’s Partition Theorem

**Definition 3.4.** A hypergraph \( H \) is a pair \((V, E)\) where \( V \) is a set of vertices and \( E \) is a set of subsets of \( V \). The \( k \)-uniform hypergraph \( K^k_n \) has \( n \) vertices and all the \( k \)-subsets of \( \{1, 2, \ldots, n\} \) as edges.

The degree \( d(v) \) of a vertex \( v \) of a hypergraph is defined as the number of edges \( E \) with \( v \in E \). We call a hypergraph almost regular if \( |d(u) - d(v)| \leq 1 \) for all pairs \( u, v \in V \). A famous theorem on partitioning the edges of a hypergraph is stated by Baranyai.

**Theorem 3.5** (Baranyai’s partition theorem). Let \( a_1, \ldots, a_t \) be positive integers with \( \sum_{i=1}^{t} a_i = \binom{n}{k} \). Then the edges of \( K^k_n \) can be partitioned into almost regular hypergraphs \( H_i = (V, A_i) \) with vertex set \( V = \{1, \ldots, n\} \) and edge set \( A_i = \{E_{i1}^{\alpha_i}, \ldots, E_{i\alpha_i}^{\alpha_i}\} \), for \( i = 1 \ldots t \).

Schrijver and Brouwer [29] proved this theorem for the case that all \( a_i \) are equal and \( n = sk \) for \( s \in \mathbb{N} \). We will extend this proof to the general case. In our proof we will use the following theorem:

**Theorem 3.6** (Integer flow theorem). Let \( D = (V, A) \) be a directed graph and let \( s, t \in V \). Let \( f : A \to \mathbb{R}_+ \) be an \( s-t \) flow of value \( \beta \). Then there exists an \( s-t \) flow \( f_{\text{int}} : A \to \mathbb{Z}_+ \) of value \( \lfloor \beta \rfloor \) or of value \( \lceil \beta \rceil \) such that

\[
\lfloor f(a) \rfloor \leq f_{\text{int}}(a) \leq \lceil f(a) \rceil
\]

for each arc \( a \in A \).

**Proof.** We are going to prove by induction a somewhat stronger statement. Let \( a_1, \ldots, a_t \) be positive integers with \( \sum_{i=1}^{t} a_i = \binom{n}{k} \). We assert that for any \( l = 0, \ldots, n \), there exist families of subsets of \( \{1, \ldots, l\} \)

\[
A_1, A_2, \ldots, A_t
\]

such that \( |A_i| = a_i \) and each subset \( S \subseteq \{1, \ldots, l\} \) is contained in exactly \( \binom{n-l}{k-|S|} \) of the \( A_i \). To be sure that exactly \( a_i \) subsets are contained in every \( A_i \), we allow subsets to occur with a multiplicity in every \( A_i \). Notice that

\[
\binom{n-l}{k-|S|} = 0 \quad \text{if} \quad |S| > k.
\]
3.1. BARANYAI’S PARTITION THEOREM

If we take \( l = 0 \), these families indeed exist; each \( A_i \) will consist of \( a_i \) copies of \( \emptyset \).

Assume for some \( l < n \) families with the desired properties exist. We now can construct a bipartite graph \( G \) with vertex sets

\[
U = \{ A_1, A_2, \ldots, A_t \}, \quad W = 2^{\{1, \ldots, l\}},
\]

so \( W \) consists of all subsets of \( \{1, \ldots, l\} \). A vertex \( A_i \) is connected to a vertex \( S \in W \) if \( S \in A_i \). If \( S \) occurs in \( A_i \) with multiplicity \( a \), then we make \( a \) parallel edges between \( A_i \) and \( S \). Then \( G \) has the following degrees:

\[
d_G(A_i) = a_i, \quad d_G(S) = \left( \frac{n - l}{k - |S|} \right).
\]

Direct every edge from \( U \) to \( W \). Let \( \sigma \) be a source and connect \( \sigma \) to the vertices of \( U \). Let \( \tau \) be a sink and connect \( \tau \) to the vertices of \( W \). See Figure 3.1 for the construction of this graph. Let \( f \) be the flow from \( \sigma \) to \( \tau \) where the flow value of the edges leaving \( \sigma \), the flow value of the edges from \( A_i \) to each of its members \( S \) and the flow value of the edges from a subset \( S \) to \( \tau \) are respectively equal to

\[
f(\sigma, A_i) = \frac{ka_i - \sum_{S \in A_i} |S|}{n - l}, \quad f(A_i, S) = \frac{k - |S|}{n - l}, \quad f(S, \tau) = \left( \frac{n - l - 1}{k - |S| - 1} \right).
\]

This indeed is a flow:

\[
\sum_{a \in \mathcal{A}^{out}(A_i)} f(a) = \sum_{S \in A_i} \frac{k - |S|}{n - l} = \frac{1}{n - l} \left( a_i k - \sum_{S \in A_i} |S| \right) = \sum_{a \in \mathcal{A}^{out}(A_i)} f(a), \quad \text{and}
\]
\[
\sum_{a \in d^\text{in}(S)} f(a) = d^\text{in}(S) \frac{k - |S|}{n - l} = \frac{k - |S|}{n - l} \left( n - l \right) = \left( n - l - 1 \right) \quad (3.2)
\]

\[
= \sum_{a \in d^\text{out}(S)} f(a).
\]

It follows because of the Integer flow theorem that there also exists an integer flow \( f_{\text{int}} \) of the same value for this network with

\[
|f(a)| \leq f_{\text{int}}(a) \leq [f(a)]
\]

for all arcs \( a \) in \( G \). Let \( h_l = a; k - \sum_{S \in A_i} |S| \). All edges leaving \( \sigma \) will get some flow value

\[
k_i = f_{\text{int}}(\sigma, A_i) \in \left\{ \left\lfloor \frac{h_l}{n - l} \right\rfloor, \left\lceil \frac{h_l}{n - l} \right\rceil \right\}
\]

under \( f_{\text{int}} \). Furthermore

\[
f_{\text{int}}(A_i, S) \in \left\{ \left\lfloor \frac{k - |S|}{n - l} \right\rfloor, \left\lceil \frac{k - |S|}{n - l} \right\rceil \right\},
\]

and the outflow of a vertex \( S \in W \) remains the same under \( f_{\text{int}} \).

We are now going to add the extra element \( l + 1 \). Denote by \( S_{ij} \) those members of \( A_i \) with

\[
f_{\text{int}}(A_i, S_{ij}) = \left\lceil \frac{k - |S_{ij}|}{n - l} \right\rceil,
\]

where \( j = 1, \ldots, m \) with \( m \leq k_i \). Note that it is possible that there are sets \( S_{ij} \) equal to each other. Next let

\[
T_{ij} = S_{ij} \cup \{l + 1\}. \quad (3.3)
\]

We first prove by induction to \( l \) that it follows from this construction that

\[
\frac{k - |S|}{n - l} \leq 1 \quad \text{for all } l \text{ and for all } S \subseteq \{1, \ldots, l\}. \quad (3.4)
\]

For \( l = 0 \) this follows from the fact that \( k \leq n \). Assume (3.4) holds for some \( l \leq n \). If it even holds that

\[
k - |S| < n - l \quad \text{for all } S \subseteq \{1, \ldots, l\},
\]

then it is immediate that also

\[
k - |T| \leq n - (l + 1) \quad \text{for all } T \subseteq \{1, \ldots, l + 1\}.
\]

Suppose for some \( S \subseteq \{1, \ldots, l\} \) equality holds in (3.4). Then

\[
\left\lfloor \frac{k - |S|}{n - l} \right\rfloor = \frac{k - |S|}{n - l} = 1
\]
3.1. BARANYAI’S PARTITION THEOREM

and according to the construction in (3.3), the element \( l + 1 \) is added to the subset \( S \) and it still holds that

\[
\frac{k - (|S| + 1)}{n - (l + 1)} = 1
\]

and we have proved (3.4).

We may conclude that \( f_{\text{int}} \) assigns the value 1 to \( k_i \) of the edges leaving \( A_i \) and 0 to the others. It follows from (3.2) that

\[
\sum_{a \in \partial^\text{in}(S)} f_{\text{int}}(a) = \sum_{i: f_{\text{int}}(A_i, S) = 1} 1 = \binom{n - l - 1}{k - |S| - 1}, \text{ for all } S,
\]

so for each \( S \) the set \( S \cup \{l + 1\} \) will be covered by \( \binom{n - l - 1}{k - |S| - 1} \) of the \( A_i \), including multiplicity.

Define

\[
B_i = A_i \setminus \{S_{ij}\}_{j=1}^{k_i} \cup \{T_{ij}\}_{j=1}^{k_i}.
\]

Let \( S \subseteq \{1, \ldots, l\} \) and \( T \subseteq \{1, \ldots, l + 1\} \). We distinguish two cases: \( l + 1 \not\in T \) and \( T \) is contained in

\[
\binom{n - l}{k - |S|} - \binom{n - l - 1}{k - |S| - 1} = \binom{n - l - 1}{k - |T|}
\]

of the \( B_i \), or \( l + 1 \in T \) and \( T \) is contained in

\[
\binom{n - l - 1}{k - |S| - 1} = \binom{n - l - 1}{k - |T|}
\]

of the \( B_i \). We see that it holds that there are \( a_i \) subsets \( T \subseteq \{1, \ldots, l + 1\} \) in each \( B_i \) and each subset \( T \) is contained in exactly \( \binom{n - l - 1}{k - |T|} \) of the \( B_i \). We can conclude that \( B_1, B_2, \ldots, B_t \) are families of subsets of \( \{1, \ldots, l + 1\} \) with the desired properties.

If we take \( l = n \) we have

\[
\binom{0}{k - |S|} = \begin{cases} 1 & \text{if } |S| = k, \\ 0 & \text{otherwise}. \end{cases}
\]

so all subsets \( S \) are \( k \)-subsets and we have found a partition with \( a_i \) \( k \)-subsets in each \( A_i \) and because \( \sum_{i=1}^{t} a_i = \binom{n}{k} \), all \( k \)-subsets of \( \{1, \ldots, n\} \) will meet exactly one hypergraph \( A_i \). Next, we only have to prove that this partition gives us almost regular hypergraphs \( H_i = (V, A_i) \). We saw in our induction step that any new element \( l \) is covered by \( A_i \) exactly \( k_i \) times with

\[
k_i \in \left\{ \left\lfloor \frac{h_l}{n-l} \right\rfloor, \left\lceil \frac{h_l}{n-l} \right\rceil \right\}
\]
and $h_l = a_i k - \sum_{S \in A_i} |S|$. If we can prove that $k_i \in \{ \lfloor \frac{a_i k}{n} \rfloor, \lceil \frac{a_i k}{n} \rceil \}$ for all $l$, then we have proved that the number of times every element of $\{1, \ldots, n\}$ appears in $A_i$ differs at most one.

Proof by induction to $l$. For $l = 0$ this is easy to see:

$$k_i \in \{ \frac{h_0}{n}, \frac{h_0}{n} \} = \{ \lfloor \frac{a_i k}{n} \rfloor, \lceil \frac{a_i k}{n} \rceil \}.$$

Let for some $l < n$, $k_i \in \{ \lfloor \frac{a_i k}{n} \rfloor, \lceil \frac{a_i k}{n} \rceil \}$. We know that for $l + 1$ holds that $k_i \in \{ \lfloor \frac{h_{l+1}}{n-l-1} \rfloor, \lceil \frac{h_{l+1}}{n-l-1} \rceil \}$. It is not difficult to see that

$$h_{l+1} \in \{ \lfloor \frac{h_l(n-l-1)}{n-l} \rfloor, \lceil \frac{h_l(n-l-1)}{n-l} \rceil \}.$$

It follows that

$$h_{l+1} \geq \frac{(n-l-1)(n-l)\lfloor \frac{a_i k}{n} \rfloor}{n-l} \quad \text{and} \quad h_{l+1} \leq \frac{(n-l-1)(n-l)\lceil \frac{a_i k}{n} \rceil}{n-l}.$$

Hence, for all $l$ $k_i \in \{ \lfloor \frac{a_i k}{n} \rfloor, \lceil \frac{a_i k}{n} \rceil \}$. □

### 3.2 Gray Codes

The term combinatorial Gray code was introduced in 1980 to refer to any listing of combinatorial objects so that successive objects differ in some pre-specified small way. This notion generalizes the classical reflected binary code for listing $n$-bit binary numbers such that successive numbers differ in exactly one bit position. The reflected binary code was introduced by Frank Gray in 1947. For now, we will work with the following definition:

**Definition 3.7.** Let $m = \binom{a}{b}$. Let us denote a set $\{1, \ldots, a\}$ by $[a]$ and the set of all $b$-subsets of $a$ by $\binom{[a]}{b}$. A Gray code $\mathcal{G}(a, b)$ is a list $(D_1, \ldots, D_m)$ of all the members of $\binom{[a]}{b}$ such that $|D_i \cap D_{i+1}| = |D_m \cap D_1| = b - 1$ for $i = 1 \ldots m$.

Gray codes can be used for error correction in digital communications as for many listings in combinatorics, such as listing permutations or combinations. Gray codes are also used in many proofs involving Hamilton cycles.

**Theorem 3.8.** Gray Codes exist for all $a \geq b \geq 1$. 
3.2. GRAY CODES

Figure 3.2: The Gray code $G(n+1,b)$

**Proof.** We are going to prove the slightly stronger statement that for all $a \geq b \geq 1$ and for any pair of $b$-sets $u,v$ with $u \cap v = b - 1$, there exists a Gray code $(D_1, \ldots, D_m)$ with $u = D_i, v = D_{i+1}$. Proof by induction to $a$. It is easy to check that for $a = 1, 2, 3$ and all $1 \leq b \leq a$ and for any pair of $b$-sets, Gray codes exist with the desired properties. Assume for $a = n \geq 3$ and all $1 \leq b \leq a$ and for any pair of $b$-sets there exists a Gray code $G(a,b)$ with the desired properties. Take $a = n + 1$. The members of $\binom{[n+1]}{b}$ can be written as

$$\binom{[n+1]}{b} = \binom{[n]}{b} \cup \left( \binom{[n]}{b-1} \oplus \{n+1\} \right),$$

where $A \oplus \{e\}$ means that the element $e$ is added to every element of the set $A$. Let

$$v \in \binom{[n]}{b-1} \oplus \{n+1\} \text{ and } u = v \setminus \{n+1\} \cup \{z\}, \quad z \in [n].$$

We are going to prove that there exists a Gray code $G(n+1,b)$ such that $\{u,v\}$ is subject of $G(n+1,b)$.

Let $x \in v$ and

$$\tilde{v} = v \setminus \{x\} \cup \{y\}, \quad \text{and} \quad \tilde{u} = u \setminus \{x\} \cup \{y\}, \quad \text{for some } y \notin v.$$

We know, because of the induction hypothesis, that there exist Gray codes

$$(D_1, \ldots, \tilde{u}, u, \ldots, D_m)$$

of $\binom{[n]}{b}$ and

$$(E_1, \ldots, \tilde{v} \setminus \{n+1\}, v \setminus \{n+1\}, \ldots, E_l)$$

of $\binom{[n]}{b-1}$, where $m = \binom{n}{b}$ and $l = \binom{n}{b-1}$. Define $\tilde{E}_i = E_i \oplus \{n+1\}$ for $i = 1, \ldots, l$. Notice that

$$(\tilde{E}_1, \tilde{v}, v, \ldots, \tilde{E}_l)$$
is a Gray code of $\left( \binom{n}{b} \right) \oplus \{n + 1\}$. Then, because $\tilde{u} = \tilde{v} \setminus \{n + 1\} \cup \{z\}$,

$$(v = \hat{E}_j, \ldots, \hat{E}_t, \hat{E}_1, \ldots, \hat{E}_{j-1} = \tilde{v}, \tilde{u} = D_{i-1}, D_{i-2}, \ldots, D_1, D_{m}, \ldots, D_i = u)$$

is a Gray code of $\left( \binom{n+1}{b} \right)$. For picturing this, see Figure 3.2.

Notice that if we would do the same for $|D_i \cap D_{i+1}| = |D_m \cap D_1| = b - k$, it doesn’t work because we get for $k = 2, a = 5$ and $b = 2, 3$ a system inducing the Petersen graph, and for $k > 2$ we can not construct a list such as in Figure 3.2.

We can now prove the following theorem of Chen and Füredi [9].

**Theorem 3.9** (Chen, [9]). The Kneser graph $K(n, k)$ is Hamiltonian for $n = sk$, where $3 \leq s \in \mathbb{N}$.

**Proof.** Let $n = sk$, $3 \leq s \in \mathbb{N}$ and $m = \binom{n-1}{k-1}$. Denote a $k$-set of $[n]$ by $E^i_j$. By Baranyai’s partition theorem there exists a partition

$$\binom{n}{k} = \bigcup_{i=1}^{m}\{E^i_1, \ldots, E^i_s\},$$

such that

$$E^i_j \cap E^i_k = \emptyset, \text{ and } \bigcup_{j=1}^{s} E^i_j = [n].$$

Without loss of generality we may assume that $\{n\} \in E^i_1$ for all $i$. Let $D_i = E^i_1 \setminus \{n\}$. Note that

$$\bigcup_{i=1}^{m} D_i = \binom{n-1}{k-1}.$$ 

Then (permute if necessary) $\{D_i\}_{i=1}^{m}$ forms a Gray code $G(n - 1, k - 1)$. Let $z_i = D_{i+1} \setminus D_i$ and $z_m = D_1 \setminus D_m$. We can assume that (permute if necessary) $z_i \notin E^i_s$. Note that it is important that $s \geq 3$. Then $E^i_{i+1} \subseteq E^i_1 \cup \{z_i\}$ and $E^i_{i+1} \cap E^i_s = \emptyset$. Then $E^i_{i+1} \subseteq E^i_1 \cup \{z_i\}$ and $E^i_{i+1} \cap E^i_s = \emptyset$. Now

$$E^1_1, \ldots, E^1_s, E^2_1, \ldots, E^2_s, \ldots, E^m_1, \ldots, E^m_s$$

form a Hamilton cycle through $K(n, k)$. See Figure 3.3.
3.3 Proof of Conjecture 3.3 for the case \( n > 3k \)

We prove Conjecture 3.3 for the case \( n > 3k \) by induction on \( k \). The proof is due to Chen [10].

For \( k = 1 \) the Kneser graph \( K(n,1) \) is equal to the complete graph and obviously has a Hamilton cycle for \( n > 3 \). \textit{Induction Hypothesis:} The Kneser graph \( K(n,k-1) \) has a Hamilton cycle for all \( n > 3(k-1) \).

Let \( n > 3k \). We are going to prove that \( K(n,k) \) has a Hamilton cycle. We will do this by building a Hamilton cycle through the vertices

\[
V = \left( \begin{array}{c} \frac{n}{k} \end{array} \right) = V_0 \cup V_1 \cup V_2 \cup V_3, \text{ with }
\]

\[
V_0 = \left( \begin{array}{c} \frac{n-2}{k} \end{array} \right), \quad V_1 = \left( \begin{array}{c} \frac{n-2}{k-1} \end{array} \right) \oplus \{n-1\},
\]

\[
V_2 = \left( \begin{array}{c} \frac{n-2}{k-1} \end{array} \right) \oplus \{n\}, \text{ and } V_3 = \left( \begin{array}{c} \frac{n-2}{k-2} \end{array} \right) \oplus \{n-1, n\},
\]

where the \( V_i \) are disjoint.

Let us construct a system where we can use Baranyai's partition theorem. Let \( n-1 = sk+r \), where \( 3 \leq s \in \mathbb{N} \) and \( 0 \leq r \leq k-1 \). Further, let

\[
\left( \begin{array}{c} \frac{n-1}{k} \end{array} \right) = s(t-1) + q, \text{ where } 1 \leq q \leq s \text{ and let } m = \left( \begin{array}{c} \frac{n-2}{k-1} \end{array} \right).
\]

Note that

\[
s(t-1) + q = \left( \begin{array}{c} \frac{n-1}{k} \end{array} \right) = \left( \begin{array}{c} \frac{sk+r}{k} \end{array} \right) \geq s \left( \begin{array}{c} \frac{sk+r-1}{k-1} \end{array} \right) = s \left( \begin{array}{c} \frac{n-2}{k-1} \end{array} \right) = sm, \quad (3.5)
\]
whence \( t \geq m \). Define
\[
\begin{align*}
    a_i &= s \quad \text{for } i \leq t - 2 \\
    a_{t-1} &= s, \quad a_t = q \quad \text{if } q \geq 2 \\
    a_{t-1} &= s - 1, \quad a_t = 2 \quad \text{if } q = 1.
\end{align*}
\]
(3.6)

Then
\[
\sum_{i=1}^{t} a_i = \binom{n-1}{k} = |E(K_{n-1}^k)|,
\]
and by Baranyai’s theorem the edge set \( E(K_{n-1}^k) \) of \( K_{n-1}^k \) can be partitioned into \( t \) almost regular hypergraphs \( H_i = ([n-1], \{E_1^i, \ldots, E_{a_i}^i\}) \), \( i = 1 \ldots t \). Recall that the \( E_j^i \) are \( k \)-subsets of \([n-1]\).

Every vertex occurs in exactly \( m \) of the edges \( E_j^i \) and because \( H_i \) is almost regular and \( m \leq t \) it follows that every vertex of \( (H_i) \) has degree 1 or 0 in \( H_i \) for all \( i \), and the edges \( \{E_1^i, \ldots, E_{a_i}^i\} \) are disjoint for all \( i \).

The vertex \( \{n-1\} \in [n-1] \) occurs in exactly \( m \) of the edges \( E_j^i \) and because the edges of \( H_i \) are disjoint for all \( i \) we can assume without loss of generality that \( \{n-1\} \in E_j^i \) for \( 1 \leq i \leq m \). Define \( D_i = E_j^i \setminus \{n-1\} \). Hence \( |D_i| = k - 1 \) and \( \{D_i\}_{i=1}^{m} = \binom{n-2}{k-1} \).

Let us first look at the case where \( m = t \). Then \( \{n-1\} \in E_j^i \) for all \( i \). It follows from (3.5) and (3.6) that \( q = s \) and \( a_i \geq 3 \) for all \( i \). Define
\[
R_i = \{E_2^i, \ldots, E_{a_i}^i\},
\]
(3.7)
then
\[
\{R_i\}_{i=1}^{t} = \binom{n-1}{k} - \left( \binom{n-2}{k-1} \oplus \{n-1\} \right) = \binom{n-2}{k}.
\]

Notice that the set of edges of \( \{R_i\} \) is equal to the set of vertices \( V_0 \).

We assumed that \( n > 3k \) whence also \( n-2 \geq 3(k-1) \). By the induction hypothesis we know that there exists a Hamilton cycle \( C_1 \) through the Kneser graph \( K(n-2, k-1) \). Notice that the vertices of \( K(n-2, k-1) \) are equal to \( \{D_i, \ i = 1, \ldots, m\} \).

Without loss of generality we can assume that
\[
C_1 = (D_1, D_2, \ldots, D_m),
\]
and
\[
C_2 = (D_1 \cup \{n-1\}, R_{11}, \ldots, R_{1(s-1)}, D_1 \cup \{n\}, D_2 \cup \{n-1\}, \ldots,
D_m \cup \{n-1\}, R_{m1}, \ldots, R_{m(s-1)}, D_m \cup \{n\}),
\]
is a cycle through all the vertices of \( V_0 \cup V_1 \cup V_2 \) in \( K(n, k) \), where all the vertices \( R_{ij}, j = 1, \ldots, c \) of \( R_i \) are passed in arbitrary order.
3.3. PROOF OF CONJECTURE 3.3 FOR THE CASE \( N > 3K \)

Next, we only have to add the vertices of \( V_3 = \binom{n-2}{k-2} \oplus \{n-1, n\} \) to obtain a Hamilton cycle through \( K(n, k) \).

Define a bipartite graph \( B \) with partite sets \( U = \binom{n-2}{k-2} \) and \( W = \binom{n-2}{k-1} \) respectively. Let a vertex \( U \in U \) be adjacent to a vertex \( W \in W \) if and only if \( U \subset W \).

It follows that \( d_B(U) = (n-2) - (k-2) > k - 1 = d_B(W) \).

Assume there exists \( S \subseteq U \) with \( |S| > |N(S)| \). Then

\[
|S|(n-k) > |N(S)|d_B(W) = d_B(N(S)) \geq d_B(S) = |S|(n-k),
\]

a contradiction, and so by the theorem of Hall, there exists a matching \( M \) in \( B \) of size \( U \). Hence for all \( U \in U \) there exists a \( D_i = W \in W \) such that \( \{U, W\} \in M \) for some \( i \). Then the following cycle is a Hamilton cycle in \( K(n, k) \):

\[
C_3 = \{D_1 \cup \{n-1\}, R_{11}, U_1 \cup \{n-1, n\}, R_{12}, \ldots, R_{1c}, D_1 \cup \{n\}, D_2 \cup \{n-1\}, R_{21}, U_2 \cup \{n-1, n\}, R_{22}, \ldots, R_{2c}, D_2 \cup \{n\}, \ldots, D_m \cup \{n-1\}, R_{m1}, U_m \cup \{n-1, n\}, R_{m2}, \ldots, R_{mc}, D_m \cup \{n\}\},
\]

where \( U_i \) is skipped if it doesn’t exists (\( |U| < m \)). Notice that it is important that \( |R_i| \geq 2 \) for all \( i \), as we have defined in (3.6) and (3.7). In Figure 3.4 the cycle \( C_3 \) is shown.

Now we only have to consider the case where \( m < t \). Let

\[
R_i = \begin{cases} 
\{E_2^i, \ldots, E_{a_i}^i\} & \text{for } 1 \leq i \leq m \\
\{E_1^i, \ldots, E_{a_i}^i\} & \text{for } m+1 \leq i \leq t.
\end{cases}
\]

For a cycle through all the vertices of \( K(n, k) \) we have to add those \( R_i \) not in \( C_3 \). That is, those \( R_i \) with \( i > m \). We may assume that the element \( n-1 \) is not covered in the hypergraph \( H_t \) (otherwise we would have taken another element).
We will consider the special case where \( q = 1, t - 1 = m \) and \( s = 3 \) later on and ignore it for now. Then it still holds that \( |R_i| \geq 2 \).

Define a bipartite graph \( \tilde{B} \) with partite sets \( X = [m] \) and \( Y = [t]\setminus[m] \) respectively. Let a vertex \( x \in X \) be adjacent to a vertex \( y \in Y \) if and only if \( D_x \subset E^y_j \), for some \( E^y_j \subset R_y \). Note that \( D_x \) is a \((k - 1)\)-subset of \([n - 2]\) and \( E^y_j \) is a \( k \)-subset of \([n - 1]\) missing the vertex \( \{n - 1\} \), whence each \( k \)-subset \( E^y_j \) contains \( k \) distinct \((k - 1)\)-subsets \( D_x \). It follows that

\[
d_{\tilde{B}}(y) = a_y \cdot k \geq \begin{cases} \text{sk} & \text{for } y \leq t - 2 \\ (s - 1)k & \text{if } y = t - 1 \\ 2k & \text{if } y = t. \end{cases}
\]

\[
d_{\tilde{B}}(x) \leq (n - 2) - (k - 1) = (s - 1)k + r \leq sk - 1.
\]

Assume there exists \( S \subseteq Y \) with \(|S| > |N_{\tilde{B}}(S)|\). Then

\[
(|S| - 1)(sk - 1) > |N_{\tilde{B}}(S)| \max_{x \in N_{\tilde{B}}(S)} d_{\tilde{B}}(x)
\]

\[
\geq \sum_{y \in S} d_{\tilde{B}}(y) \geq sk(|S| - 2) + k(s - 1) + 2k.
\]

This implies that \(|S| \leq 1 - k\), which is not possible, and by the theorem of Hall, there exists a matching \( \tilde{M} \) in \( \tilde{B} \) of size \( Y \). Hence for all \( y \in Y \) there exists a different \( x \in X \) such that \( \{x, y\} \in \tilde{M} \). In other words, for all \( y \in [t]\setminus[m] \) there exists \( x \in [m] \) such that \( D_x \subset E^y_j \in R_y \) for some \( j \leq a_y \). Then because the number of edges of \( R_y \) is greater or equal than two and \( \{n - 1\} \notin R_y \), we know that there exists a \( j' \) such that \( E^y_{j'} \cap (D_x \cup \{n - 1\}) = \emptyset \). Furthermore let \( E^y_{j'} \setminus D_x = \{z_i\} \in [n - 2] \). We know that there exists a \( j' \) such that \( z_i \notin E^y_{j'} \). Now replace

\[
D_x \cup \{n - 1\}, \ R_{x1}, \ U_x \cup \{n - 1, n\}, \ R_{x2}, \ldots, R_{xc}, \ D_x \cup \{n\}
\]

in \( C_3 \) by

\[
D_x \cup \{n - 1\}, \ E^y_{j'}, \ R_y, \ E^y_{j'}, \ E^x_{j'}, \ U_x \cup \{n - 1, n\}, \ R_x, \ D_x \cup \{n\},
\]

then you’ve got a Hamilton cycle through \( K(n, k) \). See Figure 3.5.

Now we only have to consider the case where \( q = 1, t - 1 = m \) and \( s = 3 \). We
already saw that if \( m < t \) then \( r \geq 1 \). The sketched situation can never happen because in that case would \( n - 1 = 3k + r \) and

\[
\frac{n - 1}{k} \binom{n - 2}{k - 1} = \binom{n - 1}{k} = 3 \binom{n - 2}{k - 1} + 1.
\]

Combining these two gives \((3k + r)m = (3m + 1)k\) resulting in \( rm = k \) which can never happen for \( k > 1 \).

When \( n < 3k \), the Kneser graph \( K(n, k) \) has no triangles. Chen [10] used Baranyai’s partition theorem again and found that all Kneser graphs are Hamiltonian for

\[
n \geq \frac{1}{2}(3k + 1 + \sqrt{5k^2 - 2k + 1})
\]

which is smaller than \( 2.62k + 1 \).

### 3.4 The Chvátal-Erdös condition

As we have seen in Theorem 2.7, a condition for a graph \( G \) to be Hamiltonian is that the independency number \( \alpha(G) \) does not exceed the connectivity \( \kappa(G) \). The independency number of the Kneser graph \( K(n, k) \) is equal to

\[
\alpha(K(n, k)) = \binom{n - 1}{k - 1}.
\]

To see this think of \( \alpha \) as the size of a set \( A \) containing all \( k \)-subsets of \( n \) with one specific element \( a \in \{1, \ldots, n\} \). You get \( \binom{n - 1}{k - 1} \) subsets which aren’t adjacent for all pairs. This is the size of the independent set.

We will see in section 3.6 that \( \kappa(K(n, k)) = \binom{n - k}{k} \). So now we have: If

\[
\binom{n - 1}{k - 1} \leq \binom{n - k}{k},
\]

then \( K(n, k) \) is Hamiltonian.

Chen and Lih [12] defined the following:

\[
a(n, k) = \frac{\binom{n - 1}{k - 1}}{\binom{n - k}{k}} \quad \text{and} \quad N_0(k) = \min \{ n \mid a(n, k) \leq 1 \}.
\]

**Proposition 3.10.** If \( a(n, k) \leq 1 \) then \( a(n + 1, k) \leq 1 \).

**Proof.** Let \( a(n, k) \leq 1 \). Then

\[
a(n + 1, k) = \frac{a(n, k)n(n + 1 - 2k)}{(n - k + 1)^2} \leq \frac{n(n + 1 - 2k)}{(n - k + 1)^2} \leq 1.
\]
Now we’ve got the following theorem:

**Theorem 3.11.** For fixed $k$ the graph $K(n, k)$ is Hamiltonian if $n \geq N_0(k)$.

The results for $N_0(k)$ are shown in Table 3.1 for $k = 1, \ldots, 20$. Unfortunately there are no results here for $n < 3k$. Note that $N_0(k)$ always exists because for sufficiently large $n$

\[
\binom{n-1}{k-1} \sim n^{k-1} \quad \text{and} \quad \binom{n-k}{k} \sim n^k.
\]

### Table 3.1: Hamiltonian Kneser Graphs

<table>
<thead>
<tr>
<th>$k$</th>
<th>$K(n,k)$</th>
<th>$N_0(k)$</th>
<th>$n = ak + b$</th>
<th>$k$</th>
<th>$K(n,k)$</th>
<th>$N_0(k)$</th>
<th>$n = ak + b$</th>
</tr>
</thead>
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<td>1</td>
<td>$K(n,1)$</td>
<td>3</td>
<td>$3k$</td>
<td>11</td>
<td>$K(n,11)$</td>
<td>75</td>
<td>$6k+9$</td>
</tr>
<tr>
<td>2</td>
<td>$K(n,2)$</td>
<td>6</td>
<td>$3k$</td>
<td>12</td>
<td>$K(n,12)$</td>
<td>86</td>
<td>$7k+2$</td>
</tr>
<tr>
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<td>$9k+16$</td>
</tr>
</tbody>
</table>

3.5 The Odd Graph

**Definition 3.12.** The Odd graph $O_{k+1}$ is the graph $K(2k+1,k)$

The Odd graph is the Kneser graph $K(n, k)$ where $n$ is minimal under the condition that the graph is connected. There has been done some research to Odd graphs but we are nowhere near a proof of the conjecture that almost all Odd graphs have Hamilton cycles (Note that $O_2$ is equal to the Petersen graph). The proofs we gave so far for Hamiltonian Kneser graphs do not concern Odd graphs, except for the triangle $K(3,1)$. So far the Hamiltonicity of the Odd graph has been proven for $k$ starting at 3 up to 7: Balaban [5] exhibited Hamilton cycles for $k = 3$ and $k = 4$, Meredith and Lloyd [32] established Hamilton cycles for $k = 5$ and $k = 6$ and Mather [31] showed a Hamilton cycle for $k = 7$. These proofs are basically constructions of Hamilton cycles and does not provide a method for proving that all Odd graphs have Hamilton cycles. We do have the much stronger conjecture of Meredith and Lloyd [33] about $k$-ply Hamiltonian Odd graphs.

**Definition 3.13.** A graph is called $m$-ply Hamiltonian if it contains $m$-edge disjoint Hamilton cycles.
Conjecture 3.14 (Meredith and Lloyd). The graphs $O_{2k}$ and $O_{2k+1}$ are $k$-ply Hamiltonian for all $k \geq 2$.

### 3.6 The Uniform Subset Graph

**Definition 3.15.** The uniform subset graph $G(n, k, t)$ has as vertices the $k$-subsets of $\{1, 2, \ldots, n\}$ where two vertices are adjacent if the $k$-subsets have $t$ elements in common.

**Definition 3.16.** A graph $G = (V, E)$ is edge transitive if for all $e, f \in E$, there exists an automorphism of $G$ that maps the endpoints of $e$ to the endpoints of $f$.

**Theorem 3.17** (Chen and Lih [12]). $G(n, k, t)$ is edge transitive.

**Proof.** Let $\{A, B\}$ be an edge of $G(n, k, t)$. Note that the sets $A \cap B$, $A \setminus B$ and $B \setminus A$ are pairwise disjoint. Let $\{C, D\}$ be another edge of $G(n, k, t)$. There exists a permutation $f$ of $\{1, \ldots, n\}$ that maps $A \cap B$ onto $C \cap D$, $A \setminus B$ onto $C \setminus D$ and $B \setminus A$ onto $D \setminus C$. Hence, $f(A) = C$, $f(B) = D$ and $\{f(A), f(B)\} \in E$ and $G(n, k, t)$ is edge transitive. \hfill $\square$

**Proposition 3.18.** The connectivity $\kappa$ of $G(n, k, t)$ is equal to $\binom{k}{t} \binom{n-k}{k-t}$, where $t < k$.

We know that the degree of a vertex $v$ in $G(n, k, t)$ is equal to

$$d_G(v) = \binom{k}{t} \binom{n-k}{k-t},$$

so we want to prove that $\kappa(G) = d_G(v)$. Chen and Lih [12] prove this is true by use of a theorem of Lovász [30] that states that all connected graphs with an edge transitive automorphism group and with all degrees at least $r$, is $r$-connected.

### 3.6.1 The Johnson graph

**Definition 3.19.** The Johnson graph $J(n, k)$ is the graph $G(n, k, k-1)$.

The Johnson graph is derived from the Johnson distance, which is the number of elements between two $k$-subsets, the $k$-subsets differ.

**Theorem 3.20.** All Johnson graphs are Hamiltonian.

Notice that finding a Hamilton cycle in a Johnson graph is the same as finding a Gray code as we did in Section 3.2. So we already proved the theorem above. In the following, we want to give another proof as well and for this let us first prove the following induction theorem of Chen and Lih [12]:

---

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CHAPTER 3. KNESEER GRAPHS

Theorem 3.21. If both $G(n, k, t)$ and $G(n, k + 1, t + 1)$ are Hamiltonian, so is $G(n + 1, k + 1, t + 1)$

Proof. Assume $G(n, k, t)$ and $G(n, k + 1, t + 1)$ are Hamiltonian. Hence there exists a cycle $C_1$ through all the vertices of $G(n, k, t)$ and there exists a cycle $C_2$ through all the vertices of $G(n, k + 1, t + 1)$. Notice that if you add the element $\{n + 1\}$ to all the $k$-subsets of $n$, then $C_1$ is isomorphic to a cycle of length $\binom{n}{k}$ through all the vertices containing the element $\{n + 1\}$ in $G(n + 1, k + 1, t + 1)$. Furthermore, $C_2$ is a cycle of length $\binom{n}{k+1}$ in $G(n + 1, k + 1, t + 1)$ through all the vertices without the element $\{n + 1\}$.

Let $\{a, b\}$ be an edge of $C_2$ and let $a_0 \in a \cap b$, $a_1 \in a \setminus b$ and $b_1 \in b \setminus a$. Consider the vertices

$$x = \{n + 1, a_1\} \cup b \setminus \{a_0, b_1\} \quad \text{and} \quad y = \{n + 1, b_1\} \cup a \setminus \{a_0, a_1\}.$$ 

Then $x$ and $y$ are connected and because the uniform subset graph is edge transitive, we can assume that the edge $\{x, y\}$ is subject of the cycle $C_1$. Then the cycle

$$C_1 \setminus \{x, y\} \cup \{x, a\} \cup C_2 \setminus \{a, b\} \cup \{b, y\}$$

is a Hamilton cycle (of length $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$) in $G(n + 1, k + 1, t + 1)$, see Figure 3.6.

Theorem 3.22. Let $k$ be fixed and

$$n_0 = \min\{n | G(m, k, 0) \text{ is Hamiltonian } \forall m \geq n\}.$$ 

If $G(n_0, k + r, r)$ is Hamiltonian for $r = 0, \ldots, n_0 - 2k - 1$, then $G(n, k + r, r)$ is Hamiltonian for $r = 0, \ldots, n - 2k$ and $n \geq n_0$.
3.6. THE UNIFORM SUBSET GRAPH

Proof. Fix \( k \) and \( n_0 \) such that \( G(n, k, 0) \) is Hamiltonian for all \( n \geq n_0 \). Notice that \( n_0 \) exists because of the results on Kneser graphs (i.e. Theorem 3.9). Assume \( G(n_0, k + r, r) \) is Hamiltonian for \( r = 0, \ldots, n_0 - 2k - 1 \). Notice that the graphs \( G(n, k, t) \) and \( G(n, n - k, n - 2k + t) \) are isomorphic because of the complementary subsets. So we know that \( G(n, n - k, n - 2k) \) is Hamiltonian for all \( n \geq n_0 \). It follows directly by Theorem 3.21 that \( G(n, k + r, r) \) is Hamiltonian for \( r = 0, \ldots, n - 2k \). For picturing this, see Figure 3.7.

\[
\begin{array}{cccccc}
(n_0, k, 0) & (n_0, k + 1, 1) & \ldots & (n_0, n_0 - k, n_0 - 2k) \\
(n_0 + 1, k, 0) & (n_0 + 1, k + 1, 1) & \ldots & (n_0 + 1, n_0 - k, n_0 - 2k) & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
(n, k, 0) & (n, k + 1, 1) & \ldots & (n, n_0 - k, n_0 - 2k) & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
(n, n_0 + 1, k, n_0 + 1 - k, n_0 + 1 - 2k) & \ldots & (n, n - k, n - 2k)
\end{array}
\]

Figure 3.7: \( G(n, k + r, r) \) is Hamiltonian for all \( n \geq n_0 \) and \( r = 0, \ldots, n - 2k \).

Proof of Theorem 3.20. We want to prove that \( G(n, k, k - 1) \) is Hamiltonian for all admissible \( (n, k, k - 1) \). We know that \( G(n, 1, 0) \) is Hamiltonian for all \( n \geq 3 \). Furthermore, \( G(3, 2, 1) \) is a triangle and obviously Hamiltonian. So by Theorem 3.22 we now know that \( G(n, 1 + r, r) \) is Hamiltonian for \( r = 0, 1, \ldots, n - 2 \). In other words, \( G(n, k, k - 1) \) is Hamiltonian for all admissible \( (n, k, k - 1) \) and \( n \geq 3 \).

3.6.2 Other Hamiltonian Subset Graphs

Let \( N_1(k) = \min \{ n \mid \binom{n}{k} \leq 3k \binom{n-k}{k-1} \} \). Chen and Lih [12] show by direct computations that \( \binom{n}{k} \leq 3k \binom{n-k}{k-1} \) when \( n = k^2 - k \) and \( k \geq 3 \), so \( N_1(k) \) exists for \( k \geq 3 \). By Theorem 2.4 we now have the following theorem:

**Theorem 3.23.** The graph \( G(n, k, 1) \) is Hamiltonian for all \( n \geq N_1(k) \).

Proof. First note that for \( k = 2 \) it easily follows from Theorem 3.21 that \( G(n, 2, 1) \) is Hamiltonian for \( n \geq 3 \). Let \( k \geq 3 \), \( n \geq 2k - 1 \) and \( N_1(k) = \min \{ n \mid \binom{n}{k} \leq 3k \binom{n-k}{k-1} \} \). The results for \( N_1(k) \) are shown in Table 3.2 for \( k = 2, \ldots, 20 \). We
If we look at the results in Table 3.1, we see that $N_1(k)$ increases faster than $k$ and $N_1(k) \leq (k-1)k+1 \leq k^2$ for all $k$. If $n \leq k^2$, some regular calculations show us that if $n \geq N_1(k)$, $G(n,k,1)$ is Hamiltonian.

If we look at the results in Table 3.1, we see that $N_0(k)$ increases faster than $k$ as well and $N_0(k) \leq k^2 + 2$ for all $k$. So we have $N_0(k-1) \leq (k-1)^2 + 2 \leq k^2$ and it follows from Section 3.4 that $G(k^2, k-1, 0)$ is Hamiltonian. Together with the above, Theorem 3.21 tells us that $G(n,k,1)$ is Hamiltonian for all $n \geq k^2$.

Hence now we have that $G(n,k,1)$ is Hamiltonian for all $n \geq N_1(k)$ and the theorem is proved.

If we make this method more general we have the following:
Let $k \geq t + 1$ and $N_t(k) = \min\{n| \binom{n}{k} \leq 3\binom{k}{t}\binom{n-k}{t-1}\}$ and let $f_t(k)$ be an upper bound for $N_t(k)$. If $N_t(k)$ exists, $f_t(k) \leq k^2/t$ and $f_{t-1}(k) \leq k^2/t$ then $G(n,k,t)$ is Hamiltonian for $n \geq N_t(k)$. Unfortunately we do not have an expression for $f_t(k)$ so along this way, there is no easy way to check the Hamiltonicity for $G(n,k,t)$.

If we go back to Theorem 3.22 we also see that:

i) $G(n,2,0)$ is Hamiltonian for all $n \geq 6$. Furthermore, $G(6,3,1)$ is Hamiltonian and $G(6,4,2)$ is not connected so no admissible pair. Hence $G(n,2+r,r)$ is Hamiltonian for $r = 0, 1, \ldots, n-4$ and $n \geq 6$. In other words, $G(n,k,k-2)$ is Hamiltonian for all admissible $(n,k,k-2)$ with $n \geq 6$.

ii) $G(n,3,0)$ is Hamiltonian for all $n \geq 7$, see Heinrich and Wallis [18]. Furthermore, $G(7,3,0)$ and $G(7,4,1)$ are Hamiltonian (in fact, they are isomorphic). Hence $G(n,3+r,r)$ is Hamiltonian for $r = 0, 1, \ldots, n-6$ and $n \geq 7$. In other words, $G(n,k,k-3)$ is Hamiltonian for all admissible $(n,k,k-3)$ with $n \geq 7$.

<table>
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<th>$k$</th>
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<th>$N_1(k)$</th>
<th>$n=ak+b$</th>
<th>$k$</th>
<th>$G(n,k,1)$</th>
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<td>$G(n,20,1)$</td>
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</tr>
</tbody>
</table>

Table 3.2: Hamiltonian Subset Graphs
Chapter 4

Cayley graphs

The Cayley graph is named after Arthur Cayley (1821-1895) and occurs today frequently as a model of interconnection networks.

**Definition 4.1.** Let $G$ be a finite group and $S$ be a minimal generating set of $G$. A Cayley graph $\text{Cay}(G, S)$ is a graph with as vertices the elements $g \in G$. Two vertices $g_1$ and $g_2$ are adjacent if $g_2 = g_1 \cdot s$, where $s \in S$.

Note that a generating set $S$ is minimal if $S$ generates $G$ but no proper subset of $S$ does.

**Theorem 4.2** (Gosil and Royle, [17]). All Cayley graphs are vertex transitive.

*Proof.* Let $G$ be a group and $X = (V, E)$ be the Cayley graph of some set of generators $S$ of $G$. So $V = G$ and $\{g_1, g_2\} \in E$ if $g_2 = g_1 \cdot s$, where $s \in S$. Let $f$ be the automorphism of $G$ given by

$$f(g) := xg \text{ for some } x \in G.$$ 

Let $\{g, h\} \in E$. We are going to prove that $\{f(g), f(h)\} \in E$. Without loss of generality we can assume that $h = gs$ for some $s \in S$. It follows that

$$f(h) = xh = xgs = f(g)s,$$

and $\{f(g), f(h)\} \in E$ under $f$ and $f$ is an automorphism.

If you want to particularly map some element $g$ to $g'$, take $x = g' \cdot g^{-1}$ which makes each Cayley graph vertex transitive. 

Ignoring $K_2$, the known non-Hamiltonian vertex transitive graphs are not Cayley graphs. This leads to the following conjecture:

**Conjecture 4.3.** All Cayley graphs are Hamiltonian.
The advantage of this conjecture above Conjecture 1.2 is that Cayley graphs correspond to a finite group \( G \) and a generating set \( S \). Thus one can ask for which \( G \) and \( S \) the conjecture holds rather than attack it in full generality.

In this chapter we prove the Hamiltonicity of all Cayley graphs based on any abelian group. We also see some results of Cayley graphs of a dihedral or symmetric group.

\section{Abelian groups}

As you will expect, we start to look at the Cayley graphs based on some set of generators of an abelian group. Proofs that these graphs have Hamilton paths can be found in [41], [42], and [21].

\subsection{Cycle graphs}

The cycle graph \( C_n \) is a graph on \( n \) vertices containing a single cycle through all vertices. All Cayley graphs based on a generating set of a single element are isomorphic to a cycle graph. Groups that can be generated by a single element (the cyclic groups) are of course abelian.

The Cartesian Product \( C_{n_1} \times \cdots \times C_{n_k} \) of \( k \) cycles can be seen as an \( n_1 \times \cdots \times n_k \) grid where the vertex \((i_1, \ldots, i_k)\) is adjacent only to the vertices

\[(i_1 \pm 1 \mod n_1, i_2, \ldots, i_k), \ldots, (i_1, i_2, \ldots, i_k \pm 1 \mod n_k).
\]

\begin{proposition}
The cartesian product \( C_{n_1} \times \cdots \times C_{n_k} \) of \( k \) cycles is Hamiltonian.
\end{proposition}

\begin{proof}
Proof by induction to \( k \). If \( k = 1 \) the lemma is trivial. Assume the lemma holds for \( k = m \). Let \( X = C_{n_1} \times \cdots \times C_{n_{m+1}} \) and assume first at least one \( a \in \{n_1, \ldots, n_{m+1}\} \) is even. Without loss of generality we can assume \( n_{m+1} \) is even. Let \( P = (v_1, \ldots, v_l) \) with \( v_1 = v_l \), the Hamilton cycle through \( C_{n_1} \times \cdots \times C_{n_m} \). Then

\[P' = ((v_1, 0), \ldots, (v_{l-1}, 0), (v_{l-1}, 1), \ldots, (v_1, 1), \ldots, (v_1, n_{m+1} - 1))\]

is a Hamilton cycle through \( X \), see Figure 4.2.

\end{proof}
4.1. ABELIAN GROUPS

Figure 4.2: Hamilton cycles through $C_{n_1} \times \cdots \times C_{n_k}$.

Let now all $a \in \{n_1, \ldots, n_{m+1}\}$ be odd and again let $P = (v_1, \ldots, v_l)$ with $v_1 = v_l$, be the Hamilton cycle through $C_{n_1} \times \cdots \times C_{n_m}$. Then

$$P' = ((v_1, 0), \ldots, (v_1, n_{m+1} - 1), (v_2, n_{m+1} - 1), \ldots, (v_2, 1), \ldots, (v_l - 2, 1), (v_l - 1, n_{m+1} - 1), (v_l - 1, 0), \ldots, (v_2, 0))$$

is a Hamilton cycle through $X$, see Figure 4.2.

We can conclude that if $X$ is a Cayley graph and $X \cong \prod_{k=1}^{k_i} C_k$ then $X$ is Hamiltonian. Thus, if $G$ is a group equal to the product of cyclic groups, then $G$ has at least one Hamiltonian Cayley graph.

4.1.2 General abelian groups

It leaves us to prove that all Cayley graphs of all abelian groups are Hamiltonian. We first give an idea of the proof.

Let $G$ be an abelian group and $S$ a minimal set of generators. If $S$ consists of one element, the group $G$ is cyclic and thus Hamiltonian. Every new generator gives us cycles in the Cayley graph; the cycles are of length of the order of the generator.

We can 'build' a grid in the same way as we did before by using the generators as coordinates. We stop in each direction if we meet an element we met before by use of the preceding generators.

**Theorem 4.5.** Let $G$ be an abelian group and $S = \{s_1, \ldots, s_r\}$ a minimal set of generators. Let, for $i = 2, \ldots, r$, $v_i$ be the smallest integer such that

$$v_i s_i = \sum_{j=1}^{i-1} a_j s_j$$

for some numbers $a_j$, and let $v_1$ be equal to the order of $s_1$. Then the Hamilton cycle in the grid

$$A = C_{v_1} \times C_{v_2} \times \cdots \times C_{v_r} \quad (4.1)$$

generates a Hamilton cycle in $\text{Cay}(G, S)$ as well.
Proof. It is not hard to see that if \( v_i \) is equal to the order of \( s_i \) for all \( i \), then \( \text{Cay}(G, S) \) is isomorphic to the product of cycles \( C_{v_i} \) with \( v_1v_2 \cdots v_r = n \). If this is not the case we still want to prove that every element of \( \text{Cay}(G, S) \) is contained in the grid (4.1) exactly once. We prove this by induction to the number of generators. It is immediate that if \( r = 1 \) no element \( x \in G \) is contained in \( A \) more than once. Assume for \( r = k - 1 \) no element is contained in \( A \) more than once. Let \( r = k \) and suppose there is an element \( x \) of \( G \) that is contained in the grid \( A \) more than once. Then there are \( c_i, d_i < v_i \) such that

\[
x = \sum_{i=1}^{k} c_is_i = \sum_{i=1}^{k} d_is_i,
\]

where at least for some \( i \) holds that \( c_i \neq d_i \). Recall that \( v_k \) is the minimal integer such that

\[
v_ks_k = \sum_{i=1}^{k-1} a_is_i.
\]

We consider three different cases:

\( i) \) If \( c_k = d_k \), then

\[
y = x - c_ks_k = x - d_ks_k,
\]

and \( y \) is an element that is contained in \( C_{v_1} \times C_{v_2} \times \cdots \times C_{v_{k-1}} \) more than once, contradicting our induction hypothesis.

\( ii) \) If \( c_k < d_k \), then

\[
(d_k - c_k)s_k = \sum_{i=1}^{k-1} (c_i - d_i)s_i,
\]

but then \( d_k - c_k < d_k < v_k \) contradicting the fact that \( v_k \) was minimal.

\( iii) \) If \( c_k > d_k \) the proof goes the same as in \( ii) \).

Hence we have proved that no element of \( G \) is contained in the grid \( A \) more than once. It remains to prove that every element of \( G \) is contained in the grid \( A \) at least once. Suppose there is an element \( x \in G \) which does not lay in our grid. So

\[
x = a_{11}s_1 + a_{12}s_2 + \ldots + a_{1r}s_r
\]

with \( a_{1i} \geq v_i \) for at least one \( i \). Let \( j \) be the largest index for which \( a_{1j} \geq v_j \), then we can write

\[
x = a_{11}s_1 + \ldots + v_js_j + a_{2j}s_j + \ldots + a_{1r}s_r
\]

\[
= a_{11}s_1 + \ldots + \sum_{i=1}^{j-1} a_{2i}s_i + a_{2j}s_j + \ldots + a_{1r}s_r
\]

\[
= (a_{11} + a_{21})s_1 + \ldots + (a_{1(j-1)}a_{2(j-1)})s_{j-1} + a_{2j}s_j + \ldots + a_{1r}s_r.
\]
4.2. THE PRODUCT OF HAMILTONIAN CAYLEY GRAPHS

We can continue this process until all coefficients \( b_i = a_{i1} + \ldots + a_{ki} \) are smaller than \( v_i \), contradicting our assumption that \( x \) lay outside our grid \( A \).

A final note must be made by the grid we use. In the previous section we saw the Hamilton cycle through such a grid, with the only difference that for each \( C_{v_i} \), the vertex \( v_i - 1 \) is connected to the vertex 0. With the 'cutting of' process we used in this proof, this is not longer the case except at least for the first generator. Fortunately, if we take a look at the Hamilton cycles in Figure 4.2, this is enough.

We can conclude that every Cayley graph of any abelian group is Hamiltonian.

An example of a Cayley graph based on an abelian group is the Circulant graph. The Circulant graph \( \text{Circ}(n, s_1, \ldots, s_r) \) is a graph on \( n \) vertices in which the \( i \)-th vertex is adjacent to the \((i + j)\)-th and \((i - j)\)-th vertices for each \( j \) in a certain list \( L = \{s_1, \ldots, s_r\} \). The Circulant graph on \( n \) vertices is equal to the Cayley graph based on the group \( \mathbb{Z}_n \) with generators \( s_1, \ldots, s_r \). Theorem 4.5 gives us a Hamilton cycle through \( \text{Circ}(n, s_1, \ldots, s_r) \).

4.2 The product of Hamiltonian Cayley graphs

We now take a look at the product of Cayley graphs. So far, we only saw the cartesian product of graphs: In Section 4.1.1 we saw that if \( X \) en \( Y \) are graphs with vertices \( x_i \in X, y_i \in Y \), then two vertices \((x_1, y_1), (x_2, y_2) \in X \times Y \) are adjacent if \( x_1 \) is adjacent to \( x_2 \) in \( X \) or if \( y_1 \) is adjacent to \( y_2 \) in \( Y \). If we want to say something about some Cayley graph of the product of groups \( G \times H \), we need another definition of products of graphs.

**Definition 4.6.** The conjunction product \( \text{Cay}(G, S) \cdot \text{Cay}(H, T) \) of two Cayley graphs is the graph \( \text{Cay}(G \times H, S \times T) \).

We see that if \( X \) en \( Y \) are graphs with vertices \( x_i \in X, y_i \in Y \), then two vertices \((x_1, y_1), (x_2, y_2) \in X \cdot Y \) are adjacent if \( x_1 \) is adjacent to \( x_2 \) in \( X \) and if \( y_1 \) is adjacent to \( y_2 \) in \( Y \).

![Figure 4.3: \( \mathbb{Z}_6 \) and \( \mathbb{Z}_{12} \) both with generators 2 and 3](image)
Figure 4.4: The Cayley graph of the quaternion group with generating set \{i, j\}. All Cayley graphs of the quaternion group are isomorphic and it is easy to find a Hamilton cycle.

**Conjecture 4.7** (Keating, [25]). If there is a Hamilton cycle in each of \(\text{Cay}(G, S)\) and \(\text{Cay}(H, T)\), then there is a Hamilton cycle in the conjunction

\[
\text{Cay}(G, S) \cdot \text{Cay}(H, T),
\]

unless it is not connected.

Keating proved this conjecture is true for all cases except the case where \(\text{Cay}(G, S) = C_2\), which is still open. We also see that if \(\text{Cay}(G, S) = C_2\), \(H\) is of odd order, and \((v_0, \ldots, v_k = v_0)\) is a Hamilton cycle through \(\text{Cay}(H, T)\), then

\[
((0, v_0), (1, v_1), (0, v_2), \ldots, (1, v_k), (0, v_1), \ldots, (0, v_k))
\]

is a Hamilton cycle through \(\text{Cay}(G, S) \cdot \text{Cay}(H, T)\). Hence, the only open case of Conjecture 4.7 is the case where \(\text{Cay}(G, S) = C_2\) and \(H\) is of even order.

An example of a Cayley graph based on the product of groups is the Cayley graph of a Hamiltonian group. A Hamiltonian group is a non-abelian group for which every subgroup is normal. Every Hamiltonian group \(G\) is a direct product of the form

\[
G = Q_8 \times B \times A,
\]

where \(Q_8\) is the quaternion group, \(B\) is a number of copies of \(Z_2\) and \(A\) is an odd order abelian group (Alspach and Qin [1]). It is easy to show that \(\text{Cay}(Q_8, S)\) is Hamiltonian for all generating sets \(S\) (see Figure 4.2). Furthermore, we know that \(B \times A\) is an abelian group and we proved in the previous section that all Cayley graphs of this group are Hamiltonian. Then, by the results of Conjecture 4.7, all Cayley graphs of \(Q_8 \times (B \times A)\) are also Hamiltonian.

### 4.3 The Dihedral group

The dihedral group \(D_n\) is the group of symmetries of a regular polygon with \(n\) sides, including both rotations and reflections. A regular polygon with \(n\) sides has
2n different symmetries: n rotational symmetries and n reflection symmetries. In general, $D_n$ has elements

$$R_0, \ldots, R_{n-1} \text{ and } S_0, \ldots, S_{n-1},$$

with $I = R_0 = R_n$, $I = S_i^2$ for all $i$ and $S_i R_j S_i = R_{-j}$ for all $i, j$. Furthermore, composition is given by the following:

$$R_i R_j = R_{i+j}, \quad R_i S_j = S_{i+j}, \quad S_i R_j = S_{i-j}, \quad S_i S_j = R_{i-j}.$$  

It should be clear that the dihedral group is not abelian and a generating set should contain at least one reflection symmetry.

**Proposition 4.8.** If a set $A = \{a_1, \ldots, a_k\}$ is a minimal generating set of $\mathbb{Z}_n$, then

$$\hat{A} = \{R_{a_1}, R_{a_2}, \ldots, R_{a_k}, S_{a_k}\}$$  

(4.2)

is a minimal generating set of $D_n$.

**Proof.** Let $A$ be a minimal generating set of $\mathbb{Z}_n$. Then every element $g \in \mathbb{Z}_n$ can be written as $g = n_1a_1 + \ldots + n_k a_k$ for some numbers $n_i$. It follows that for every $R_g, S_g \in D_n$,

$$R_g = R_{n_1a_1+\ldots+n_k a_k} = R_{n_1a_1} \cdots R_{n_k a_k} = R_{a_1}^{n_1} \cdots R_{a_k}^{n_k},$$  

and

$$S_g = S_{n_1a_1+\ldots+n_k a_k} = R_{n_1a_1} \cdots R_{n_{k-1}a_{k-1}} S_{n_k a_k} = R_{a_1}^{n_1} \cdots R_{a_{k-1}}^{n_{k-1}} S_{a_k},$$

and $\hat{A}$ is a generating set. Every rotation symmetry $R_g$ can be written as a combination of the rotation symmetries of $\hat{A}$ because

$$S_{a_k} R_{m_i a_i} S_{a_k} = R_{-m_i a_i} = R_{a_i}^{-m_i}.$$  

To prove that $\hat{A}$ is minimal, we suppose that it is not. Then there exists $i \in \{1, \ldots, k\}$ and there are $m_j$ with

$$R_{a_i} = R_{m_1 a_1} \cdots R_{m_{i-1} a_{i-1}} \cdots R_{m_{i+1} a_{i+1}} \cdots R_{m_k a_k} = R_{m_1 a_1 + \ldots + m_{i-1} a_{i-1} + m_{i+1} a_{i+1} + \ldots + m_k a_k}$$

and $a_i = m_1 a_1 + \ldots + m_{i-1} a_{i-1} + m_{i+1} a_{i+1} + \ldots + m_k a_k$, contradicting the fact that $A$ is minimal. \qed

It follows directly from the fact that all Cayley graphs $\text{Cay}(G, S)$ with $G$ abelian are Hamiltonian, that if $A$ is of the form (4.2) then $\text{Cay}(G, S)$ is Hamiltonian.

**Theorem 4.9.** Let $\text{Cay}(D_n, A)$ be a Cayley graph of a dihedral group $D_n$ where $A$ contains at least one rotation. Then $\text{Cay}(D_n, A)$ is Hamiltonian.
Proof. Let $A$ be a minimal generating set of the dihedral group $D_n$ containing at least one rotation. Let $G$ be the subgraph of the Cay$(D_n, A)$ with vertex set $D_n$ and edge set

$$A' = \{ (x, y) \mid yx^{-1} = R_a, \ x, y \in D_n, \ R_a \in A \}.$$ 

Then we know that $G$ consists of components $G_1, \ldots, G_m$, which are all isomorphic to a circulant graph $\mathbb{Z}_k$. Note that $k$ is equal to the number of vertices generated by the rotations in $A$ and $m = 2n/k$. We saw in section 4.1.2 that these components contain a cycle $C_k$. Let $\{S_{a_1}, \ldots, S_{a_l}\}$ be the reflections in $A$ and let

$$\tilde{G} = G \cup \{ (u, v) \mid vu^{-1} = S_{a_1}, \ u, v \in D_n \}.$$ 

Let a vertex $u_1$ in a component $G_p$ be joined with a vertex $v_1$ in a component $G_q$. Then there is for all $x \in G_p$, a vertex $y \in G_q$ with $S_{a_1}x = y$. More specifically, because $S_{a_1}R_a = R_a^{-1}$, it follows that if $v_1'$ is a neighbor of $u_1$ in $G_p$, then $S_{a_1}v_1' = v_1'$ and $v_1'$ is a neighbor of $v_1$. Hence by adding edges generated by $S_{a_1}$ to $\tilde{G}$, we get $m/2$ components with each components containing a cycle $C_{2k}$.

Go on by adding the edges generated by $S_{a_2}$. Let $\{u_2, u_2'\}$ be an edge of a component $\tilde{G}_p$ of $\tilde{G}$ other than $\{u_1, u_1'\}$ with $u_2'u_2^{-1} = R_a$ for some $R_a \in A$. We again see that $u_2$ and $u_2'$ have neighbors $v_2 = S_{a_2}u_2$ and $v_2' = S_{a_2}u_2'$ respectively in another component $\tilde{G}_q$ with $v_2'^{-1} = R_a^{-1}$.

We can go on with this procedure by adding edges generated by $S_{a_3}, \ldots, S_{a_l}$ one at a time. Start each time with an edge generated by a rotation symmetry. After adding the edges of each reflection, the components become larger but remain contain cycles through the entire component. We need all reflections for making the graph connected because $A$ is minimal and we end up with a cycle through all vertices of $D_n$.

Hence now we know that all Cayley graphs of a dihedral group are Hamiltonian if the generating set contains at least one rotation. Our further considerations can be restricted to the case that the generating set contains only reflections.

If $S = \{S_i, S_j\}$ is a generating set for $D_n$, then

$$C = (I, S_i, S_jS_i, \ldots, S_j(S_jS_i)^{n-1}, (S_jS_i)^n = I)$$

is a Hamilton cycle and Cay$(D_n, S)$ is Hamiltonian for all generating sets containing two elements. As a result we have

**Proposition 4.10** (Holsztyński and Strube [21]). All Cayley graphs of the dihedral group $D_p$, where $p$ is a prime, are Hamiltonian.

**Proof.** Let $p$ be a prime number. The only minimal generating sets of $D_p$ are either of the form $A_1 = \{R_{a_1}, S_{a_1}\}$ or $A_2 = \{S_{a_1}, S_{a_2}\}$. We have already shown that Cayley graphs of the dihedral group with these forms of generating sets are Hamiltonian. \[\square\]
In fact, Witte [43] showed that every Cayley graph on a group of prime-power order greater than two is Hamiltonian.

Alspach and Zhang [2] have proved that $\text{Cay}(D_n, S)$ is Hamiltonian for $S = \{S_i, S_j, S_k\}$ and hereby proved that all cubic Cayley graphs based on a dihedral group are Hamiltonian. Unfortunately there are no results known for the case where the generating set consist of more than three reflections. The problem lies, of course, in the non-abelian property of the reflection symmetries.

4.4 The Symmetric group

The symmetric group $S_n$ is the group of all the permutations of $\{1, \ldots, n\}$. The group $S_n$ has order $n!$ and is not abelian for $n > 2$. We already saw $D_n$, which is a subgroup of $S_n$. The symmetric group occurs in change ringing: Ringing all $n$ bells in a bell tower in one of the $n!$ possible ways is called a change. Ringing all $n!$ different changes is called an extent. Change ringing is the art of ringing all $n!$ possible changes such that no change differs in its order of ringing by more than two positions from one change to the next. Two bells can only be swapped in their orders and the last ringing change must be equal to the first. Change ringing emerged in England in the 17th century and is, particularly in England, still a popular occupation. The group theoretical underpinnings of change ringing have been pursued by mathematicians who have done a lot of study on change ringing throughout the years since. White [40] has shown that the constraints give rise to a set of transpositions in $S_n$ and that an extent exists if and only if the Cayley graph on $S_n$ generated by these transpositions is Hamiltonian.

**Definition 4.11.** An involution is a group element of order two.

**Definition 4.12.** A cycle is a permutation $f$ for which there exists an element $x$ in $\{1, \ldots, n\}$ such that $x, f(x), \ldots, f^k(x) = x$ are the only elements moved by $f$. $k$ is called the length of the cycle and is equal to its order. Cycles of length two are called transpositions.

Note that this definition of a cycle has nothing to do with a cycle in a graph and that an involution need not always be a transposition.

**Theorem 4.13** (Pak and Radoičić, [36]). If $S_n$ is generated by three involutions $\alpha, \beta, \gamma$ such that two of them commute, then the Cayley graph $\text{Cay}(S_n, \{\alpha, \beta, \gamma\})$ is Hamiltonian.

**Proof.** Let $\alpha, \beta, \gamma$ be three involutions that generate $S_n$ and let $\alpha \beta = \beta \alpha$. Define

$$d_y(X_i) = \{g \in S_n - X_i \mid g = xy, \ x \in X_i\},$$
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Figure 4.5: A spanning cycle through \( X_{i+1} \). Note that it is important that \( \alpha\beta = \beta\alpha \).

where \( X_1 \) is equal to the dihedral group generated by \( \beta \) and \( \gamma \),

\[
X_{i+1} = X_i \cup y_i D_n \quad \text{and} \quad y_i \in d_{\alpha}(X_i) \subset S_n - X_i.
\]

We are going to prove by induction that there exists a spanning cycle of \( X_i \) for all \( i \). The proof is based on the same idea as the proof we gave for Theorem 4.9.

We already saw in (4.3) a spanning cycle through \( X_1 \). Note that, by definition of a group, \( d_{\beta}(X_1) = d_{\gamma}(X_1) = \emptyset \) and at this point, new vertices can only be reached by the use of \( \alpha \). Assume there exists a spanning cycle of \( X_i \). It still holds that \( d_{\beta}(X_i) = d_{\gamma}(X_i) = \emptyset \) and if \( d_{\alpha}(X_i) = \emptyset \) (4.4) as well, then it follows that \( X_i = S_n \) and we have found a Hamilton cycle through \( \text{Cay}(S_n, \{\alpha, \beta, \gamma\}) \). Assume \( d_{\alpha}(X_i) \neq \emptyset \) and let \( y_i \in d_{\alpha}(X_i) \). Then \( X_i \cap y_i D_n = \emptyset \), otherwise would \( y_i h = x \in X_i \) for some \( h \in D_n \). This implies that \( y_i \in X_i \) which is a contradiction. Hence, only ‘new’ vertices are added to \( X_i \) to obtain \( X_{i+1} = X_i \cup y_i D_n \). By the induction hypothesis, there exists a spanning cycle \( C_1 \) through \( X_i \) and by (4.3) there exists a spanning cycle \( C_2 \) through \( y_i D_n \). We know that \( y_i \alpha = x \alpha^2 = x \) for some \( x \in X_i \) and because \( y_i = x \alpha \notin X_i \), \( x \) is on \( C_1 \) and connected to \( x\beta \) and \( x\gamma \). On the other hand \( y_i \) is in \( X_{i+1} \) so on \( C_2 \) and connected to \( y_i\beta \) and \( y_i\gamma \). Then because \( \alpha\beta = \beta\alpha \)

\[
(C_1 \setminus \{x, x\beta\} \cup C_2 \setminus \{y_i, y_i\beta\})
\]

is a spanning cycle through \( X_{i+1} \). If we continue this process un till (4.4) holds, we will find our Hamilton cycle through \( S_n \). \[\square\]

Another known result is the following:

**Theorem 4.14** (Kompel’makher and Liskovets, [27]). The Cayley graph \( \text{Cay}(S_n, S) \) is Hamiltonian if \( S \) consists only of transpositions.

Furthermore, it is proved that \( G \) is Hamiltonian if \( S \) consists of either \( n-1 \) right rotations: \( \{(12), (123), \ldots, (12 \cdots n)\} \) or \( n-1 \) left rotations: \( \{(21), (321), \ldots, (n \ n-1 \cdots 1)\} \) (Jiang and Ruskey [23]).
A problem which remained open for a long time is the one where $S$ contains one cycle of length $n$ and one transposition. Compton and Williamson [15] were able to construct a Hamilton cycle in this graph using Gray Codes, as we did for some particular Kneser graphs.

Although the Hamiltonicity of $\text{Cay}(S_n, S')$ is proved for many $S$, there are still many more open cases in order of this problem.
Chapter 5

Prime graphs

There are lots of results known about the Hamiltonicity of vertex transitive graphs where the order of the graph is some function of a prime number $p$. For instance, Alspach [3] proved that with exception of the Petersen graph, all connected vertex transitive graphs of order $2p$ are Hamiltonian. And Kutnar and Marušić [28] proved that with exception of the Coxeter graph, all connected vertex transitive graphs of order $4p$ are Hamiltonian.

Furthermore, if we restrict ourself to Cayley graph, we have for instance the following results:

**Theorem 5.1** (Witte, [43]). Every Cayley graph on a group of prime-power order greater than 2, and any generating set is Hamiltonian.

Witte uses the following theorem about quotient groups:

**Theorem 5.2.** Let $S$ be a generating set for a group $G$ and $N$ a normal subgroup of $G$. Let $\bar{S}$ be the image of $S$ in the quotient group $G/N$. If there is a Hamilton cycle $C = (a_1, a_2, \ldots, a_n)$ in $Cay(G/N, \bar{S})$ such that $a_1a_2\cdots a_n$ generates $N$, then $|N|$ copies of $C$ gives a Hamilton cycle in $Cay(G, S)$.

The following theorem of Jungreis, [24] ensures the Hamiltonicity of many Cayley graphs.

**Theorem 5.3.** Let $p$ and $q$ be prime numbers. Every Cayley graph on a group of order $pq$, $4q$ ($q > 3$), $p^2q$ ($2 < p < q$), $2p^2$, $2pq$, $8p$ or $4p^2$, and any generating set is Hamiltonian.

We already saw in Proposition 4.10 an example of such a theorem.
Epilogue

In this thesis I studied some properties of vertex transitive graphs with the purpose to discuss the existence of Hamilton cycles in such graphs. In particular, I studied the structure of Kneser graphs in more detail and found out that the only $n$ for which it is not yet proved that Kneser graphs $K(n, k)$ are Hamiltonian is when $2k + 1 \leq n \leq 2, 62k + 1$. During this study, I proved Baranyai's partition theorem, the existence of Gray codes and some results about uniform subset graphs.

In the last part of my thesis, I did some research on Cayley graphs and the existence of Hamilton cycles in such graphs. I started by proving that all Cayley graphs of any abelian group have Hamilton cycles. From here on, I found out about the results known of Hamiltonicity concerning the product of groups, Hamiltonian groups, dihedral groups and symmetric groups.

When I started my master’s thesis, I, of course, had to pick a subject first. I always loved graph theory and combinatorics so that choice was already made. There are a lot of open problems in any kind of mathematics. Graph theory is no different. I chose for a problem with some algebraic aspects and the conjecture of Lovász seemed a suitable problem.

On the one side the problem appeared to be very broad and I had to decide which way I wanted to go in my research. On the other hand the problem was already fully investigated by lots of people and it seemed really hard to make a contribution. Nevertheless, I found my way to contribute to the problem; I worked out some results known and found some holes to close.

Thanks goes to my supervisor Dion, who is always online, and to my friend Bart, who always likes to think with me.
Bibliography


