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Master Thesis

Symmetries of the Standard Model

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Abstract

This thesis presents a study of the symmetries of the standard model of particle physics. More specifically how a 1+1 dimensional (confined) model can be linked to a 3+1 dimensional (asymptotic) model. Supersymmetry is included in the discussion as a mechanism to connect the internal and external symmetries, providing a remarkable mathematical framework. Supersymmetry is now almost on the edge of being excluded as a symmetry of which half of the particles are still missing, but this is not a problem in the given discussion. We will discuss a 1+1 dimensional supersymmetric model that can be connected to a 3+1 dimensional theory without supersymmetry. The 1+1 dimensional super-algebra will be examined and it will be shown that the two space-time coordinates are still independent, as in the non-supersymmetric case.

Following this is a discussion of the topic of internal symmetries, where SU(3) plays a prominent role. We will examine the substructures of SU(3) in the hope to find a way to link the strong sector SU(3) to the SU(2)×U(1) subgroup describing the electroweak sector. The goal is to construct a framework in which the strong sector is, in a sense, dual to the electroweak sector. A full description of such a framework will not be given here, since it simply does not exist (yet). Presented here is a discussion of how the symmetry group of the Standard Model could possibly be rearranged. We will "unfold" the SU(3) to extract the SU(2)×U(1) subgroup that we want to identify with the electroweak sector gauge group.

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Contents

1 Introduction

The Standard Model of particle physics is a theoretical framework describing three of the four fundamental forces (the electromagnetic, weak and the strong force), only gravity remains outside. Developed in the early 1970s, this theory has successfully explained almost all experimental results and predicted a wide scope of phenomena. For example the confirmation of the existence of the top quark, the tau neutrino and the Higgs boson have contributed to the credence of the Standard Model. Despite the success of the Standard Model as a theory describing the building blocks of the universe there are still many questions left unanswered. The Standard Model does for example not explain neutrino oscillations, the baryon asymmetry, Dark Matter or the accelerating expansion of the universe, neither does it incorporate gravity, as described by general relativity. Furthermore, it does not answer the question why there are three families and not two or four and why the three generations have a different mass scale. The Standard Model is far from being a perfect theory and much work has to be done to improve the model and make it less ad hoc.
One thing that might be useful in building a new theory is supersymmetry, which predicts a matching of bosons and fermions. The standard form of supersymmetry naturally provides a connection of the two very different classes of particles, each boson is coupled to a fermion super-partner and each fermion is coupled to a boson super-partner. The Standard Model predicts all elementary particles to be massless, which is not what experimentalists have observed. Theorists have come up with the idea of symmetry breaking and the Higgs mechanism [1, 2, 3], which requires the existence of the Higgs boson. However, there is no reason why the Higgs boson should be as light as observed. From the interactions with the Standard Model particles one expects it to be a lot heavier. Supersymmetry, although on the edge of being excluded, does give a solution for this problem. The new super-particle contributions to the interactions with the Higgs boson would cancel out the contributions of the Standard Model partners, making the Higgs boson much lighter. Another prediction of supersymmetry models is that the lightest supersymmetric particle is stable and electrically neutral, making it a perfect candidate for Dark Matter. Of course supersymmetry does not solve all our problems and experiments give a lot of constraints on how this supersymmetric theory should look like, but it is a very interesting thing to look at since the mathematical framework is so remarkable, even if does not describe nature.

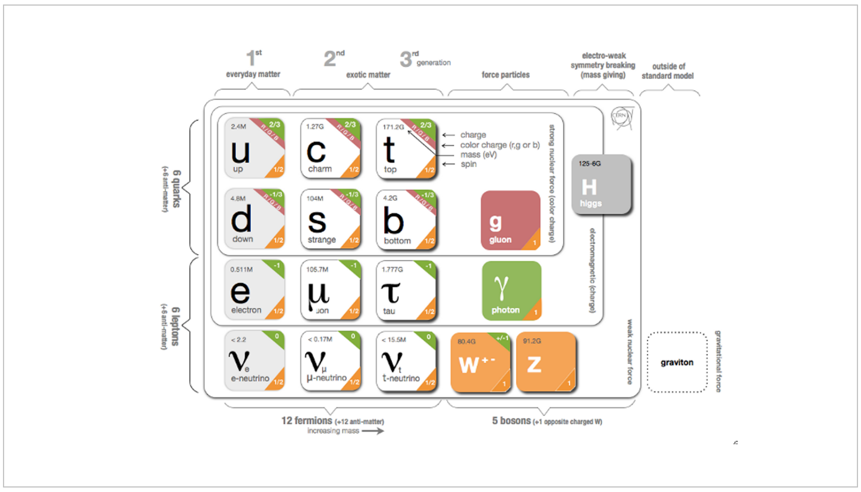

Figure 1: *Particle content of the Standard Model. Source: http://davidgalbraith.org/portfolio/ux-standard-model-of-the-standard-model/*

In this thesis a study of symmetries is presented, symmetries of the Standard Model as well as symmetries beyond the Standard Model. The gauge group of the standard model, SU(3)× SU(2)×U(1), can be viewed as the backbone of the Standard Model and thus stands at the basis of a theoretical framework that has been field leading for years. The question asked here is whether we can rearrange these gauge groups in some different way. Can we construct the Standard Model gauge group in such a way as to make the SU(3) strong sector in some sense dual to the SU(2)×U(1) electroweak sector? Thus, whether it is possible to view the electroweak sector as acting in a different (part of) space than the strong force. The main idea to make this plausible is that we have the basic constituents of the strong sector SU(3) living in a 1+1 dimensional world where the Poincaré group P(1,1) forms the natural space-time symmetry, only including one time-translation, one space translation and one boost. The confinement of the strong interactions is quite natural in 1+1 dimensional space-time. Unfortunately, we do not (yet) describe the real world with this starting point, the real world has three spatial dimensions and spin. A 3+1 dimensional theory is also needed to describe the electroweak sector, the world where the SU(2)×U(1) acts as the electroweak gauge-group. Connecting these two worlds is the key to looking at the Standard Model in this way.

The topic of this thesis originates from various discussions with my supervisor, Piet Mulders, who has come up with the idea outlined above [4]. Whether this might be a valid way to view the Standard Model will not be the topic of the thesis. I will focus on whether it is possible to rearrange symmetry groups based on this idea. I will only consider the group theory needed for this rearrangement and the field theory that has to come out in one way or another. Firstly, note that we somehow need to combine the external symmetry of the Poincaré group with the internal (gauge)symmetries of the Standard Model. This means that we also need to include supersymmetry at some point in the discussion, mainly focussing on the super-algebra. As mentioned before, this supersymmetry only resides in the 1+1 dimensional part of the theory. Hence, we aim to construct a 1+1 dimensional algebra can be incorporated in the theory. Note also that instead of the usual way of using supersymmetry in 3+1 dimensions, we do not add new particles to the Standard Model. We merely want to use supersymmetry in our 1+1 dimensional starting point in order to relate the Standard Model symmetries and look for possible rearrangements of the gauge groups.

In the first chapter of this thesis (chapter 2) we will give a review of some aspects of group theory and symmetries, followed by a review/introduction to supersymmetry. In the second chapter (chapter 3) we review some aspects of the Wess-Zumino model to couple the concept of supersymmetry to fields. After these reviews of existing literature we examine (in chapter 4) the the two-dimensional super-algebra, giving us information about the 1+1 dimensional space where the confined strong sector resides. This chapter firstly reviews a way to translate the algebra form the Weyl-spinor notation to the Dirac-spinor notation followed by a construction of the 1+1 dimensional algebra, which to our knowledge can not be found in existing literature. The last two chapters (chapter 5 and 6) present own work. Chapter 5 will discuss internal symmetries, we will examine the (sub-)structure of SU(3). First, there will be some discussion on the parametrization of the group, followed by a discussion of the subgroup SU(2)×U(1) and the orientation of this group within the larger SU(3). The last chapter will give a discussion on linking the (external) P(3,1) and P(1,1), assessing the main question whether we can link the two in such a way as to have a two dimensional space-time for the strong sector while having a four dimensional space-time for the electroweak sector. These last chapters will not include supersymmetry, this might be needed to build the theory outlined above, but it is not immediately clear how to approach the group structure in this case.

2 Symmetries

This chapter is mainly inspired on the discussions in [4, 5, 6, 7, 8]. An introduction/review will be given on the basics of (super)symmetry and the role of group theory in this discussion. Section 2.2 will give an introduction to the concept of group theory, reminding the reader of the definition of a group and discuss some examples of discrete as well as continuous groups. Section 2.3 will give a short review of gauge theory, Abelian and Non-Abelian. Finally, section 2.4 will give a short introduction to supersymmetry.

2.1 Introduction

An important distinction in the study of symmetries in physics is the one between 'external' and 'internal' symmetries. The external symmetry is coupled to the Poincaré group, these external symmetries are the symmetries of space-time. Lagrangians are in almost all cases constructed in such a way as to keep them invariant under transformations belonging to the Poincaré group. In field theory the restriction that the theory has to be Poincaré invariant gives us the scalar, vector and tensor fields for bosons and spinor fields, which are spin-$\frac{1}{2}$ representations of the Poincaré group, for fermions. The internal symmetries are symmetries that arise in the Lagrangian because fields appear in a symmetric way, e.g. a complex scalar field with the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \frac{m^2}{2}\left(\phi^*\phi\right) - \frac{\lambda}{4!}\left(\phi^*\phi\right)^2 \tag{1}$$

is invariant under the global phase shift $\phi \to e^{i\alpha}\phi$. In the group theoretical language this symmetry is described by U(1). These symmetries are internal in the sense that they do not "see" the Poincaré group, meaning that the generators of an internal group commute with all generators of the Poincaré group.

2.2 Group Theory

Group Theory has proven to be very useful in the discussion of symmetries in physics, especially in field theory. The most important symmetries that appear in field theory are the continuous symmetries, which are very effectively described using Lie groups, think about the gauge group of the standard model SU(3)×SU(2)×U(1). But also discrete symmetries play an important role in physics. For example parity, charge conjugation and time reversal are discrete symmetries, although they are part of the (continuous) Poincaré group. More recently also indications were found that the discrete group A_4 might play a role in the description of neutrino oscillations [9, 10].
First of all the definition of a group is given by the following properties:

- (1) Closed under multiplication: If $g_1, g_2 \in G$ then also $g_1 \circ g_2 \in G$
- (2) Associativity: For any three elements $g_1, g_2, g_3 \in G$, $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
- (3) Identity: There exists an element $g_0 \in G$ such that $g_0 \circ g = g$, $\forall g \in G$
- (4) Inverse: For every element $g \in G$ there exists an element $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = g_0$

The $\circ$ denotes a multiplication in the most general sense, it does not necessarily mean actual multiplication but more like the operation of the group, e.g. for the additive groups the group multiplication is addition.
For the basics of Group Theory the reader is referred to any book on Group Theory that is

written for physicists, a very useful one is the book of Howard Georgi [8]. In this section only some useful aspects of certain groups are highlighted.

2.2.1 Discrete Groups

In this subsection a few examples of discrete groups will be given, in particular the cyclic group and the permutation group will be discussed. Following this is an introduction to the alternating group A_n, especially A_4 which is believed to be an important ingredient in describing the phenomenon of neutrino oscillation.
Starting with the cyclic group, which is defined as:

$$C_n = \{e, c, c^2, ..., c^{n-1}\} = \{c \mid c^n = e\}, \tag{2}$$

in words this means that it is generated by only a single element c and that applying this element n times gives the same result as just applying the identity element. The cyclic group has the one dimensional representation $c = e^{2i\pi/n}$, i.e. it is represented by rotations in the complex plane by an angle $2\pi/n$. To be complete, it might be good to give the formal definition of a representation.

A representation of dimension n of an abstract group G is defined as a homomorphism $R:\ G \to \mathrm{GL}(n, \mathbb{C})$*, which is the group of non-singular* $n \times n$ *complex valued matrices.*

Often it is more useful to discuss representations of groups instead of the abstract group itself, since it is more intuitive to see what the action of the group is. To give a geometrical illustration for the case of the cyclic group, one can consider an n-sided polygon (n-gon), which is invariant under rotations under an angle of $2\pi/n$, or more explicitly a pentagon (5-gon) under rotations of 72 degrees.
A second example is the permutation group S_n, which is the group of permutations of n objects, it has $n!$ elements. The number of elements in this group thus increases rapidly for larger n, complicating the general discussion. A useful notation for the permutation group is called the cyclic notation. The notation (kl) means that k goes to l and l goes to k, (klm) means k goes to l, l goes to n, n goes to k. The logic of the notation continues for longer terms (kl....). As an example the elements of S_3 can be written as:

$$(), (12), (13), (23), (123), (132)$$

where () is the identity. Take a set of three objects and write them as [abc], the element (12) acts on tis set as (12)[abc]=[bac], i.e. it interchanges the first two objects. Similarly (132) acts on this set as (132)[abc]=[bca]. Going one step higher to the permutation group of four elements, S_4, and denoting the elements in the same way:

$$\begin{gathered} (),(12)(34), (13)(24), (14)(23), (123), (132), (234), (243), (341), (314), (412), (421) \\ (12), (13), (14), (23), (24), (34), (1234), (1243), (1324), (1342), (1423), (1432). \end{gathered} \tag{3}$$

Of course one can think of many ways to represent the permutation group, for example one can construct various matrix representations. When attempting this note that the permutation group can always be generated by two elements of the group. This is not immediately trivial, because this can not simply be any two elements, something that always works is to take one of the even permutations and one of the odd permutations. But let us leave the discussion of the permutation group for now and move to the case of the alternating group. Take the alternating group of four elements A_4, it consists of the elements given in the first line of 3. The group A_4 can be defined representation free as:

$$A_4 = \{S, T \mid S^2 = T^3 = (AT)^3 = \mathbb{1}\}, \tag{4}$$

where S and T are the generators. Thus also the alternating group can be generated by only two elements, now we can also see what would happen if in the case of the permutation group we choose two even permutations, in that case we would not generate S_4 but end up generating A_4. It is useful to see that the group A_4 is the symmetry group of the oriented tetrahedron. This leads to a three-dimensional representation ($R: A_4 \to \mathrm{GL}(3, \mathbb{C})$), which will also come in handy later on, given by:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

In this representation the full set of elements is given by:

$$\mathbb{1},\ T,\ TS,\ ST,\ STS = T^2ST^2,\ T^2,\ ST^2,\ TST = ST^2S,\ T^2S,\ S,\ TST^2,\ T^2ST,$$

where it is important to note that there is a C_3 subgroup of A_4 generated by only the element T: $Z_3 = \{T \mid T^3 = \mathbb{1}\}$, and in the same way there also is a C_2 subgroup generated by S.

2.2.2 Continuous- /Lie- Groups

Lie groups are defined as groups with elements g_i labelled by continuous parameters, in the contrary to discrete groups where the parameters are discrete, as the name already implies. A Lie group has an infinite number of elements, in other words a continuous spectrum of elements, and the multiplication law of a Lie group depends smoothly on the continuous parameters, whereas a discrete group is completely defined by a finite set of elements. To make things more clear and also to clarify the language and notation used, we now consider a few examples.
The Unitary Group U(N) is the group of $N \times N$ matrices, hence if U is a matrix in this group, $UU^\dagger = \mathbb{1}$ and $|\det(U)| = 1$. The identity element of this group is simply the identity matrix, a representation of this group can be given in exponential form as

$$U = e^{i\theta T}, \tag{6}$$

where the T is called the generator of the group U(N). Note that the generator T is an hermitian matrix since

$$UU^\dagger = e^{i\theta T} e^{-i\theta T^\dagger} = 1 \ \to \ T = T^\dagger.$$

A second example is the Orthogonal Group O(N), the group of orthogonal $N \times N$ matrices, hence if O is a matrix in this group, $O^T O = \mathbb{1}$ and $\det(O) = \pm 1$. The generators Q of O(N) are anti-symmetric:

$$O^T O = e^{\phi Q^T} e^{\phi Q} \ \to \ Q^T = -Q.$$

From the Orthogonal group we can define the Special Orthogonal Group SO(N) as the group with only elements O for which $\det(O) = 1$. Actually the group O(N) is the "double cover" of SO(N), there exists a two-to-one map between O(N) and SO(N). The elements of SO(N) correspond to all "proper" rotations in N-dimensional space, where O(N) also includes "improper" (orientation changing) rotations for which $\det(O) = -1$. When a system is invariant under SO(N) transformations the system is said to be rotationally invariant (i.e. it is just a mathematical way of discussing rotations and rotational invariance).
The same thing can be done for the Unitary Group: the Special Unitary Group SU(N) is defined as the group of $N \times N$ matrices for which (when $U \in SU(N)$) $\det(U) = 1$. Note that SU(N) is generated by a set of traceless hermitian matrices T, since now:

$$\det(U) = \det(e^{i\theta T}) = e^{i\theta \mathrm{tr}(T)} = 1 \ \to \ \mathrm{tr}(T) = 0.$$

The careful reader might have noticed that formally one has to take into account the fact that SU(N) (as well as U(N),O(N) and SO(N)) has more than one generator. To be specific one needs d generators, where d is de dimension of a group, defined as the number of parameters needed to describe any group element. To verify that everything done above still holds when there is more than one generator, write a general element of SU(N) as

$$U = \Pi_{a=1}^{d} e^{i\theta_a T_a} \tag{7}$$

and observe that for the elements of SU(N) we have:

$$UU^\dagger = \Pi_{a=1}^{d}\Pi_{b=1}^{d} e^{i\theta_a T_a} e^{-i\theta_b T_b^\dagger} = e^{i\Sigma_{a=1}^{d}\theta_a T_a} e^{-i\Sigma_{b=1}^{d}\theta_b T_b^\dagger} = e^{i\Sigma_{a=1}^{d}\theta_a (T_a - T_a^\dagger)} = 1 \;\rightarrow\; T_a = T_a^\dagger \;\forall a$$
$$\det(U) = \det(e^{i\Sigma_{a=1}^{d}\theta_a T_a}) = e^{i\Sigma_{a=1}^{d}\theta_a \mathrm{tr}(T_a)} = 1 \;\rightarrow\; \mathrm{tr}(T_a) = 0 \;\forall a$$

where the fact was used that all generators are independent of each other and that $\theta_a \neq 0 \;\forall a$. This is still very sloppy, since SU(N) is a non-Abelian group we can not simply combine the two exponents after the third equal sign in the upper equation. To be thorough, we have to invoke the Baker-Campbell-Hausdorff formula and check that everything works out. However one will arrive at the same conclusion since all the sums in the Baker-Campbell-Hausdorff formula are over all possible commutation relations. This concludes the discussion of continuous groups for now, in the next section these continuous groups are used to transform fields and discuss the behaviour of a theory under these transformations. In chapter 5 we will return to the discussion of the special unitary group and give the parametrizations of SU(2) and SU(3).

2.3 Gauge Symmetry

Different configurations of unobservable fields often result in the same measurable quantities, such as energy, charge and mass. A transformation from a certain field configuration to another field configuration is called a gauge transformation, since the measurable quantities do not change under such a transformation, there is a gauge invariance. And when there is an invariance, there is something called a symmetry, which leads to the concept of gauge symmetry. First we will consider Abelian gauge transformations, and later generalize to the case of non-Abelian gauge transformations.

When discussing gauge transformations we can make a distinction between global and local gauge transformations. An example of a global gauge transformation is a (global) phase shift of the scalar field (which is a U(1) transformation under an angle α): $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ as was also mentioned in section 2.1 where the Lagrangian of equation 1 is invariant under this transformation. The other type of transformation, the "local" gauge transformation, depend explicitly on the space-time point(s) x and is given by $\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)$ (this is essentially a local U(1) transformation). But now the transformation spoils the gauge invariance of the complex scalar Lagrangian, since:

$$\phi(x) \rightarrow e^{i\alpha(x)}\phi(x) \tag{8a}$$
$$\phi^*(x) \rightarrow e^{-i\alpha(x)}\phi^*(x) \tag{8b}$$
$$\partial_\mu\phi(x) \rightarrow e^{i\alpha(x)}\partial_\mu\phi(x) + i(\partial_\mu\alpha(x))e^{i\alpha(x)}\phi(x). \tag{8c}$$

A solution to this problem is to introduce something called a covariant derivative defined as:

$$D_\mu\phi(x) \equiv (\partial_\mu + iA_\mu(x))\phi(x), \tag{9}$$

where a field vector is introduced. Performing the local gauge transformation on the covariant derivative of a field gives:

$$\begin{aligned}
D_\mu\phi(x) &\to (\partial_\mu + iA_\mu(x))e^{i\alpha(x)}\phi(x) \\
&= e^{i\alpha(x)}\partial_\mu\phi(x) + i(\partial_\mu\alpha(x))e^{i\alpha(x)}\phi(x) + iA_\mu e^{i\alpha(x)}\phi(x) \\
&\equiv e^{i\alpha}D_\mu\phi(x),
\end{aligned} \tag{10}$$

hence now we see that to obtain the correct invariance in the final Lagrangian, the introduced vector field should transform as:

$$A_\mu \to A_\mu - \partial_\mu\alpha(x). \tag{11}$$

The Lagrangian of equation 1 can be made invariant under the local U(1) transformations if the normal derivatives are replaced by covariant derivatives and an extra term is added:

$$\mathcal{L} = \frac{1}{2}D_\mu\phi^* D^\mu\phi - \frac{m^2}{2}(\phi^*\phi) - \frac{\lambda}{4!}(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{12}$$

where the extra term was introduced to account for the variation of the new vector field. The symbol $F_{\mu\nu}$ is often called the field strength tensor and is given in terms of the vector field as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{13}$$

The theory developed above is called scalar QED, which is a simplified version of QED. (Normal QED includes fermions and thus spinor fields, making the discussion somwhat more complicated due to the gamma's. But except for being more mathematically challenging the ideas are the same.) This can be found in almost every book on quantum field theory, but it is mentioned here to be complete in our current discussion of gauge theory.
The formalism can be generalized to include transformations belonging to non-Abelian groups and hence obtain a non-Abelian gauge theory, note that the symmetry group is now a Lie-Group G generated by generators T_a with the following algebra:

$$[T_a, T_b] = if_{abc}T_c, \tag{14}$$

where the f_{abc} are the structure constants. In an Abelian group the structure constants are zero, and for a compact Lie-group they are anti-symmetric in the three indices. For a field transforming under a non-Abelian Lie group:

$$\phi(x) \to e^{i\alpha^n L_n}\phi(x) \overset{\text{inf}}{=} (1 + i\alpha^n L_n)\phi(x) \tag{15}$$

The L_n are matrix representations of the generators corresponding to the representations of the fields. Consider, as an example, a three-component field transforming under SO(3) transformations with generators in matrix representation,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

(Or similarly for SU(2) with as generators the Pauli matrices.) The field then transforms as:

$$\vec{\phi} \to e^{i\alpha^n L_n}\vec{\phi} \overset{\text{inf}}{=} (1 + i\alpha^n L_n)\vec{\phi} = \vec{\phi} + i\alpha_1 \begin{pmatrix} 0 \\ -i\phi_3 \\ i\phi_2 \end{pmatrix} + i\alpha_2 \begin{pmatrix} i\phi_3 \\ 0 \\ -i\phi_1 \end{pmatrix} + i\alpha_3 \begin{pmatrix} -i\phi_2 \\ i\phi_1 \\ 0 \end{pmatrix} = \vec{\phi} - \vec{\alpha} \times \vec{\phi}. \tag{17}$$

Again the process becomes more difficult when one considers local gauge transformations $\Lambda = e^{i\alpha^n(x)L_n}$, the fields transform under local transformations as:

$$\begin{aligned}\phi(x) &\to \Lambda\phi(x) \\ \partial_\mu\phi(x) &\to (\Lambda\partial_\mu + \partial_\mu\Lambda)\phi(x).\end{aligned} \tag{18}$$

The fields $\phi(x)$ may in general have many components, say equal to the dimension d, and the Λ is thus a d×d matrix. Introducing the covariant derivative as was done for the Abelian case (note that the covariant derivative is also a matrix now, as it should also be, in the same representation as the fields):

$$D_\mu\phi(x) \equiv (\mathbb{1}\partial_\mu - igW_\mu)\phi(x), \tag{19}$$

where the introduced fields $W_\mu = W_\mu^a L_a$ are matrix valued. The transformation of the covariant derivative now becomes:

$$D_\mu\phi(x) \to (\Lambda\partial_\mu + \partial_\mu\Lambda - igW_\mu\Lambda)\phi(x). \tag{20}$$

Requiring that the covariant derivative transforms in such a way that $D_\mu\phi(x) \to \Lambda D_\mu\phi(x)$, hence that $D_\mu \to \Lambda D_\mu \Lambda^{-1}$, which is the same as in the Abelian case. The W_μ have to transforms as:

$$W_\mu \to \Lambda W_\mu \Lambda^{-1} - \frac{i}{g}(\partial_\mu\Lambda)\Lambda^{-1}. \tag{21}$$

To make the theory (i.e. the Lagrangian) again invariant under this local gauge transformation, new vector fields (W_μ) are introduced into the Lagrangian. Similar to the approach for Abelian gauge theory a term like $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ has to be introduced. For this to work out, consider the generalized field strength tensor:

$$G_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = D_\mu W_\nu - D_\nu W_\mu - ig[W_\mu, W_\nu], \tag{22}$$

or when explicitly writing the field indices:

$$G^a_{\mu\nu} = D_\mu W^a_\nu - D_\nu W^a_\mu + gf^a_{\ bc}W^b_\mu W^c_\nu. \tag{23}$$

Note that the last term indeed disappears in the Abelian case, since the structure constants are all zero, or in other words all generators commute with each other.
The complex scalar Lagrangian for the non-Abelian gauge theory now obtains the form:

$$\mathcal{L} = \frac{1}{2}D_\mu\phi^* D^\mu\phi - \frac{m^2}{2}\left(\phi^*\phi\right) - \frac{\lambda}{4!}\left(\phi^*\phi\right)^2 - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a, \tag{24}$$

where the last term can also be written as the trace $G^a_{\mu\nu}G^{\mu\nu}_a = 2\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})$, where the factor of 2 is convention. This concludes the discussion of gauge symmetries.

2.4 Supersymmetry

In this section we will give a review of some concepts of supersymmetry, mostly following [5]. The focus will be on the structure of the algebra. Note that there is no unique way to implement supersymmetry, in general there are many different ways to introduce supersymmetry in a physical system, also depending on the situation. Here we mainly focus on the usual way supersymmetry is used in particle physics. The study of supersymmetry is very interesting, whether it arises in the real world or not, since it gives a remarkable mathematical structure that is worth studying on its own. Besides this one might also want to study it because it can be used as a mechanism to gain a better understanding of quantum field theory in general. For the aim of this thesis we are mainly concerned with the mathematical structure of the 1+1 dimensional variant which we will introduce in chapter 4, hence we will not use the "usual" supersymmetry where new particles are introduced.

As mentioned above, an important distinction is made between external symmetries (the ones related to space-time transformations, like the Poincaré group, parity, charge conjugation and time reversal) and internal symmetries (which arise by combining several particles, like the gauge groups U(1) of electromagnetism, SU(2) for the weak interactions and SU(3) for the strong interactions). By definition, the internal symmetry has generators that commute with the generators of the Poincaré group, specifically the generators $[T_a, T_b] = i f_{ab}{}^c T_c$ commute with the Casimir operators of the Poincaré group, $[T_a, P^2] = 0$ and $[T_a, W^2] = 0$ (where W_μ is the "Pauli-Lubanski pseudovector"). This means that particle states related to each other by an internal symmetry, have the same mass and spin. This is an important thing to remember.

The question now arises whether it makes sense to combine the internal- and external symmetry in some non-trivial way. This leads to the discussion two remarkable theorems that have to be studied before continuing.

Coleman-Mandula Theorem [11]

Sidney Coleman and Jeffrey Mandula published an article in 1967 answering this question negative. They showed it is not possible to mix internal and external symmetries, this statement is often also called the Coleman-Mandula no-go theorem.

In their paper they start out with 4 basic assumptions and one (ugly) technical assumption:

- Lorentz-invariance: This assumption basically boils down to the fact that the total symmetry group of a theory should have a subgroup locally isomorphic to the Poincaré group, which makes sense since all of field theory is assumed to be Poincaré invariant.
- Particle-finiteness: This means that all particle types in a theory should correspond to positive-energy representations of the Poincaré group and that for some finite mass M there is only a finite amount of particle types with mass less than M.
- Weak elastic analyticity: Elastic-scattering amplitudes are assumed to be analytic functions of the center-of-mass energy and the invariant momentum transfer, at least in some neighbourhood of the physical region. This is a somewhat strange assumption at first sight, but it is something that is assumed by most people. Note that (and this is something that Coleman and Mandula also state in their paper themselves) that this theorem is note true if this assumption is left out. There exist groups that are not direct products, however theories based on these group structures do not allow scattering, except in the forward and backward directions. They cite T.F. Jordan for this comment [12].
- Occurrence of Scattering: Two plane waves scatter at (almost) all energies.
- The technical assumption: The generators of the symmetry group of the particular theory, can be considered as operators in momentum space and should have distributions of their kernels. (No detail here.)

Using these assumptions and some technical statements following these assumptions, they argue that an infinitesimal generator of a symmetry group of the S-matrix is the sum of an infinitesimal translation, an infinitesimal Lorentz-transformation and some infinitesimal internal symmetry transformation. This is the same a stating that every symmetry group of the S-matrix is a direct product of the Poincaré-group and an internal symmetry group.

Haag-Łopuszański-Sohnius Theorem [13]

The Coleman-Mandula theorem was realized to contain a hidden assumption, a loophole. This assumption is that all symmetries concerned are assumed to be Lie-algebraic in nature, which means that one can in principle consider spinorial symmetries, whose generators would have half-integer spin and hence be by definition not Lie-algebraic. These generators would be fermionic and have anti-commutation relations instead of commutation relations, defying the Lie-algebraic nature. Including this spinorial symmetry and adding it to the Poincaré group is also known as supersymmetry. In the paper of Haag, Łopuszański and Sohnius, published in 1975, they narrow down the possibilities to use spinorial representations to only the spin-$\frac{1}{2}$ generators.
They start of by making the assumptions that a generator of the supersymmetry of the S-matrix is any operator in the Hilbert space that has the properties that (1) it commutes with the S-matrix, (2) it acts additively on states of several incoming particles and (3) it connects only particle types which have the same mass. In the end they end up with a consistent algebra for supersymmetry, but we will postpone this result till after giving a small introduction to the language and formalism of supersymmetry.

These two papers are basically the starting point of the new theoretical framework, based on this fermionic symmetry, known as supersymmetry. Supersymmetry can be defined as the symmetry obtained when one adds anti-commuting spin-$\frac{1}{2}$ generators to the Poincaré group. To discuss the structure of this additional symmetry the approach of [5] and some aspects in [7] will be used here.
Write the fermionic generators as Weyl spinors Q_α, taking a parity-invariant theory and hence considering also their conjugates $\bar{Q}_{\dot{\alpha}}$ they fill up a 4-component Dirac spinor:

$$Q_D = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}. \tag{25}$$

These Q's are called the "supercharges", in general there is a label on these charges Q^A_B, where the index A indicates different fermionic operators, which leads to the discussion of extended-supersymmetries which will be discussed later on. Consider for now the minimal case, where there is only one fermionic operator, called $\mathcal{N} = 1$ supersymmetry.
As is usual, introduce the vectors:

$$\sigma^\mu \equiv (\mathbb{1}, \sigma^i) \qquad \bar{\sigma}^\mu \equiv (\mathbb{1}, -\sigma^i), \tag{26}$$

where field indices are chosen such that the contraction of the Dirac spinor with the γ^μ works out. Adding these field indices explicitly to the vectors of equation 26 gives:

$$\sigma^\mu_{\alpha\dot{\beta}} \qquad \bar{\sigma}^{\mu\dot{\alpha}\beta}. \tag{27}$$

One now usually introduces the following objects,

$$\begin{aligned}
(\sigma^{\mu\nu})_\alpha^{\ \beta} &= \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\bar\sigma^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\bar\sigma^{\mu\dot\gamma\beta}\right) \\
(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} &= \frac{i}{4}\left(\bar\sigma^{\mu\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - \bar\sigma^{\nu\dot\alpha\gamma}\sigma^\mu_{\gamma\dot\beta}\right) \qquad (28)
\end{aligned}$$

which makes writing the full set of commutation relations for the $\mathcal{N}=1$ supersymmetry more compact and neat.
Remember that the Poincaré algebra can be written as:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, J_{\rho\sigma}] &= i\eta_{\mu\rho}R_\sigma - i\eta_{\mu\sigma}P_\rho \\
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho}), \qquad (29)
\end{aligned}$$

where the rotations and boosts are combined into the tensor $J_{\mu\nu}$ defined as $J_{ij} = -J_{ji} = \epsilon_{ijk}J_k$, $J_{i0} = -J_{0i} = -K_i$. In like manner the full set of commutation relations of the $\mathcal{N}=1$ super-Poincaré algebra become:

$$\begin{aligned}
[P_\mu, P_\nu] = [P_\mu, Q_\alpha] = [P_\mu, \bar Q_{\dot\alpha}] &= \{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \\
[P_\mu, J_{\rho\sigma}] &= i\eta_{\mu\rho}R_\sigma - i\eta_{\mu\sigma}P_\rho \\
[Q_\alpha, J_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^{\ \beta}Q_\beta \\
[\bar Q_{\dot\alpha}, J_{\mu\nu}] &= -\bar Q_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\beta}_{\ \dot\alpha} \\
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho}) \\
\{Q_\alpha, \bar Q_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha}P_\mu. \qquad (30)
\end{aligned}$$

The concept of fermionic generators also allows for a new possibility in supersymmetry, namely that the Q's can be charged under some operator of an internal symmetry group which is generated by some element R. This brings up the topic of "R-symmetry". R-Symmetry is the symmetry transforming the supercharges into each other, in the case where $\mathcal{N}=1$ this symmetry is locally isomorphic to a U(1) (a somewhat hand-waving argument for this is that when writing down anti-commutation relations between Q's, one has to take into account that the anti-commutation relations are symmetric in the field-indices, e.g. α and β, and there is no symmetric Lorentz invariant object to place on the other side. Thus, the anti-commutator has to be trivial), but in extended supersymmetries this can become some non-Abelian group. The commutation relations that need no be added to equation's 30 to incorporate this R-symmetry are given by,

$$\begin{aligned}
&[R, R] = [R, P_\mu] = [R, J_{\mu\nu}] = 0 \\
&[Q_\alpha, R] = Q_\alpha \qquad [\bar Q_{\dot\alpha}, R] = \bar Q_{\dot\alpha}. \qquad (31)
\end{aligned}$$

The first line is just the statement that the internal symmetry group should commute with the Pointcaré group. Note here that this new group generated by R is indeed an internal group with respect to the Poincaré group, but not (necessarily) with respect to the super-Poincaré group. Also note that the R-symmetry assigns opposite charges to the left- and right-handed supercharges. Written in exponential form this gives:

$$Q_\alpha \to e^{-i\rho}Q_\alpha \qquad \bar Q_{\dot\alpha} \to e^{i\rho}\bar Q_{\dot\alpha}, \qquad (32)$$

where the generator takes the form $R = \mathrm{diag}(-1, 1)$.
This whole procedure is easily generalised to $\mathcal{N} > 1$ supersymmetry by introducing an index

$A = 1, 2, ..., \mathcal{N}$ to label the different supercharges. Also introducing an anti-symmetric matrix $\mathcal{Z}^{AB}$ this generalizes to:

$$\begin{aligned}
\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB} \\
\{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta}\mathcal{Z}^{AB} \\
\{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}}(\mathcal{Z}^*)^{AB},
\end{aligned} \tag{33}$$

where the first line is similar to the $\mathcal{N} = 1$ case, and in the second and third line the anti-commutation relations are no longer trivial since there is a symmetric Lorentz invariant object now, namely the combination of the two anti-symmetric tensors $\epsilon_{\alpha\beta}\mathcal{Z}^{AB}$ which can be placed on the right side of the equation. One may wonder what kind of object the $\mathcal{Z}^{AB}$ actually is, it turns out that this can only be an element of the center group, which is one of the results of [13].

We will conclude this section by stating the generalization of the R-Parity, this is also the same as stating the result from the Haag-Łopuszański-Sohnius Theorem and it also summarizes anything we need to know about the super algebra.

The complete set of commutation relations for a supersymmetic theory including internal symmetry generators B_l are summarized below.

$$\begin{aligned}
[P_\mu, P_\nu] &= [P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0 \\
[P_\mu, J_{\rho\sigma}] &= i\eta_{\mu\rho}R_\sigma - i\eta_{\mu\sigma}P_\rho \\
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho}) \\
[Q^A_\alpha, J_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^{\ \beta} Q^A_\beta \\
[\bar{Q}^A_{\dot{\alpha}}, J_{\mu\nu}] &= -\bar{Q}^A_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \\
\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB} \\
\{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta}\mathcal{Z}^{AB} \\
\{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}}(\mathcal{Z}^*)^{AB} \\
[B_l, B_m] &= i f_{lmn} B_n \\
[Q^A_\alpha, B_l] &= (s_l)^A_{\ B} Q^B_\alpha \\
[\bar{Q}^A_{\dot{\alpha}}, B_l] &= (\bar{s}_l)^A_{\ B} \bar{Q}^B_{\dot{\alpha}}
\end{aligned} \tag{34}$$

Here the f_{lmn} are the structure constants of the internal symmetry group (a compact Lie group), the $(s_l)^A_{\ B}$ are Hermitian ($s_l^{AB} = \bar{s}_l^{BA}$) representation matrices of the generators of this compact Lie group in a ν-dimensional representation. The $\mathcal{Z}^{AB}$ are elements of the center group, meaning that they commute with all elements of the group.

$$[\mathcal{Z}^{AB}, G] = 0 \tag{35}$$

Here G is any element of the complete group (the super-Poincaré group and the internal compact Lie group).

3 The Wess-Zumino Model

This chapter is used to examine the Wess-Zumino model [14] in a few forms. First we will start of with the Wess-Zumino model including the auxiliary fields and linear term, in section 3.1 we will clean up and argue that the linear term can be left, out as always, and then integrate out the auxiliary fields. In section 3.2 we will give the Wess-Zumino Lagrangian for Weyl fermions and the most compact form of the Lagrangian.
The discussion of the Wess-Zumino model presents a way to couple the algebra discussed in the previous chapter to the concept of fields. This is very important to eventually formulate the theory described in the introduction. However, one should note that the Wess-Zumino model is a supersymmetric model for 3+1 dimensions. Eventually one should aim to find a 1+1 dimensional equivalent, but for this it is also important to review the (existing) case of the Wess-Zimino model. Of course this is not the only model one could examine, but is is a good staring point at the very least.

3.1 Cleaning up the mess

Starting with the Wess-Zumino Lagrangian for an $\mathcal{N} = 1$ supersymmetry from [14], using the signature $(+ - - -)$, a construction will be given to obtain a useful form of the Lagrangian and it's equations of motion. The Lagrangian with all terms consistent with the superalgebra is given by:

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}\left(\partial_\mu\phi_s\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_p\right)^2 + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ &+ m\left(F\phi_s + G\phi_p - \frac{1}{2}\bar{\psi}\psi\right) \\ &+ g\left[F\left(\phi_s^2 - \phi_p^2\right) + 2G\phi_s\phi_p - \bar{\psi}\left(\phi_s + i\gamma_5\phi_p\right)\psi\right] \\ &+ \lambda F \end{aligned} \tag{36}$$

where ϕ_s and ϕ_p are respectively a scalar-field and a pseudoscalar-field, ψ is a Majorana spinor and F and G are two auxiliary fields. Note that we can get rid of the term proportional to λ by a shift of the scalar-field ϕ_s (which is quite general since linear terms can always be left out of the theory).
Consider only the following terms in the Lagrangian which depend on the scalar field:

$$\mathscr{L}' = mF\phi_s + gF\phi_s^2 + 2gG\phi_s\phi_p - \bar{\psi}\phi_s\psi, \tag{37}$$

now apply a shift to the scalar-field $\phi_s \to \phi_s + \alpha$, the change in the Lagrangian due to this shift is

$$\delta\mathscr{L}' = mF\alpha + gF\alpha^2 + 2g\alpha F\phi_s + 2g\alpha G\phi_p - \alpha\bar{\psi}\psi. \tag{38}$$

By redefining $m \to m + 2\alpha g$, the last three terms can be absorbed, the first two terms are left and they should cancel against λF leading to:

$$m\alpha + g\alpha^2 + \lambda = 0. \tag{39}$$

Which obviously has solutions: $\alpha = \left(-m \pm \sqrt{m^2 - 4\lambda g}\right)/g^2$, hence the last term of equation 36 drops out by shifting the scalar field by α and redefining the mass.

$$
\begin{aligned}
\mathscr{L} = & \frac{1}{2}\left(\partial_\mu \phi_s\right)^2 + \frac{1}{2}\left(\partial_\mu \phi_p\right)^2 + \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\
& + m\left(F\phi_s + G\phi_p - \frac{1}{2}\bar{\psi}\psi\right) \\
& + g\left[F\left(\phi_s^2 - \phi_p^2\right) + 2G\phi_s\phi_p - \bar{\psi}\left(\phi_s + i\gamma_5\phi_p\right)\psi\right]
\end{aligned}
\tag{40}
$$

The Lagrangian with auxiliary fields is unusual, a more conventional form is obtained by integrating out the auxiliary fields, this can be done by inserting the equations of motion to remove them from the Lagrangian. For this to work it is essential that the auxiliary fields F and G appear algebraically (without derivative terms), otherwise non-local terms will appear in the action. The equations of motion for F and G are given by:

$$
\frac{\delta \mathscr{L}}{\delta A} - \partial_\mu \frac{\delta \mathscr{L}}{\delta \partial_\mu A} = 0 \tag{41}
$$

$$
F = -m\phi_s - g\left(\phi_s^2 - \phi_p^2\right) \qquad G = -m\phi_p - 2g\phi_s\phi_p \tag{42}
$$

Substituting this back into the Lagrangian gives

$$
\begin{aligned}
\frac{1}{2}\left[F^2 + G^2\right] &= \frac{1}{2}\left[\left(-m\phi_s - g\left(\phi_s^2 - \phi_p^2\right)\right)^2 + \left(-m\phi_p - 2g\phi_s\phi_p\right)^2\right] \\
&= \frac{1}{2}\left[m^2\phi_s^2 + 2mg\phi_s\left(\phi_s^2 - \phi_p^2\right) + g^2\left(\phi_s^2 - \phi_p^2\right)^2 + m^2\phi_p^2 + 4mg\phi_s\phi_p^2 + 4g^2\phi_s^2\phi_p^2\right] \\
&= \frac{1}{2}\left[m^2\left(\phi_s^2 + \phi_p^2\right) + 2mg\phi_s\left(\phi_s^2 + \phi_p^2\right) + g^2\left(\phi_s^2 + \phi_p^2\right)^2\right] \\
m\left(F\phi_s + G\phi_p\right) &= m\left[-m\phi_s^2 - g\phi_s\left(\phi_s^2 - \phi_p^2\right) - m\phi_p^2 - 2g\phi_s\phi_p^2\right] \\
&= -m^2\left(\phi_s^2 + \phi_p^2\right) - mg\phi_s\left(\phi_s^2 + \phi_p^2\right) \\
g\left[F\left(\phi_s^2 - \phi_p^2\right) + 2G\phi_s\phi_p\right] &= g\left[-m\phi_s\left(\phi_s^2 - \phi_p^2\right) - g\left(\phi_s^2 - \phi_p^2\right)^2 - 2m\phi_s\phi_p^2 - 4g\phi_s^2\phi_p 2\right] \\
&= -mg\phi_s\left(\phi_s^2 + \phi_p^2\right) - g^2\left(\phi_s^2 + \phi_p^2\right)^2 .
\end{aligned}
$$

Hence the Wess-Zumino Lagrangian can be written as

$$
\begin{aligned}
\mathscr{L} = & \frac{1}{2}\left(\partial_\mu \phi_s\right)^2 + \frac{1}{2}\left(\partial_\mu \phi_p\right)^2 + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{m}{2}\bar{\psi}\psi - g\bar{\psi}\left(\phi_s + i\gamma_5\phi_p\right)\psi \\
& - \frac{1}{2}m^2\left(\phi_s^2 + \phi_p^2\right) - mg\phi_s\left(\phi_s^2 + \phi_p^2\right) - \frac{1}{2}g^2\left(\phi_s^2 + \phi_p^2\right)^2 .
\end{aligned}
\tag{43}
$$

To complete the discussion of the Wess-Zumino Lagrangian one should obtain the equations of motion for the various fields, at $g = 0$ the scalar fields should reduce to the Klein-Gordon equation and for the spinors one should obtain the Dirac equation.

$$\frac{\delta \mathscr{L}}{\delta \phi_s} = -g\bar{\psi}\psi - m^2\phi_s - mg\left[3\phi_s^2 + \phi_p^2\right] - 2g^2\phi_s\left(\phi_s^2 + \phi_p^2\right)$$
$$\Rightarrow \left[\Box + m^2\right]\phi_s = -g\left[\bar{\psi}\psi + m\left[3\phi_s^2 + \phi_p^2\right] + 2g\phi_s\left(\phi_s^2 + \phi_p^2\right)\right] \tag{44a}$$

$$\frac{\delta \mathscr{L}}{\delta \phi_p} = -ig\bar{\psi}\gamma_5\psi - m^2\phi_p - 2mg\phi_s\phi_p - 2g^2\phi_p\left(\phi_s^2 + \phi_p^2\right)$$
$$\Rightarrow \left[\Box + m^2\right]\phi_p = -g\left[i\bar{\psi}\gamma_5\psi + 2m\phi_s\phi_p + 2g\phi_p\left(\phi_s^2 + \phi_p^2\right)\right] \tag{44b}$$

$$\frac{\delta \mathscr{L}}{\delta \bar{\psi}} = \frac{i}{2}\partial\!\!\!/\psi - \frac{m}{2}\psi - g\left(\phi_s + i\gamma_5\phi_p\right)\psi$$
$$\Rightarrow \left(i\partial\!\!\!/ - m\right)\psi = 2g\left(\phi_s + i\gamma_5\phi_p\right)\psi \tag{44c}$$

$$\frac{\delta \mathscr{L}}{\delta \psi} = -\frac{m}{2}\bar{\psi} - g\bar{\psi}\left(\phi_s + i\gamma_5\phi_p\right) \qquad \partial_\mu\frac{\delta \mathscr{L}}{\delta \partial_\mu \psi} = \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu$$
$$\Rightarrow i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 2g\bar{\psi}\left(\phi_s + i\gamma_5\phi_p\right) \tag{44d}$$

Which for $g = 0$ indeed satisfies the necessary equations. The above equations thus give the dynamics of a minimal supersymmetric model with one Dirac spinor, a scalar and a pseudoscalar field.

3.2 Rewriting The Wess-Zumino Lagrangian

Now it might be interesting to write the whole Lagrangian in terms of Left- and Right-handed fields, where we use the chiral representation of the gamma matrices given in appendix A. Writing the scalar and pseudoscalar fields in terms of Left- and Right-handed gives,

$$\phi_s = \frac{1}{\sqrt{2}}\left(\phi_R + \phi_L\right) \qquad \phi_p = \frac{1}{\sqrt{2}}\left(\phi_R - \phi_L\right) \tag{45}$$

and writing the fermion field in terms of the Left- and Right-handed components,

$$\psi = \begin{pmatrix}\chi_L \\ \chi_R\end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix}\chi_R^\dagger & \chi_L^\dagger\end{pmatrix}. \tag{46}$$

Using this, we can write the Lagrangian as:

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}\left(\partial_\mu\phi_R\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_L\right)^2 + \frac{i}{2}\left[\chi_R^\dagger\sigma^\mu\partial_\mu\chi_R + \chi_L^\dagger\bar{\sigma}^\mu\partial_\mu\chi_L\right] \\ &- \frac{m}{2}\left[\chi_R^\dagger\chi_L + \chi_L^\dagger\chi_R\right] - g'\left[\chi_R^\dagger\left[(1-i)\phi_L + (1+i)\phi_R\right]\chi_L + \chi_L^\dagger\left[(1+i)\phi_L + (1-i)\phi_R\right]\chi_R\right] \\ &- \frac{1}{2}m^2\left(\phi_R^2 + \phi_L^2\right) - mg'\left(\phi_R + \phi_L\right)\left(\phi_R^2 + \phi_L^2\right) - (g')^2\left(\phi_R^2 + \phi_L^2\right)^2,\end{aligned} \tag{47}$$

where $g' = g/\sqrt{2}$. This equation is quite long due to the introduced left and right handed scalar field. A more attractive form to write the Wess-Zumino Lagrangian is by using the complex scalar field and just leaving the Weyl spinors inside the Dirac spinor.

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}}\left(\phi_s + i\phi_p\right) & \phi_s &= \frac{1}{\sqrt{2}}\left(\phi^* + \phi\right) \\ \phi^* &= \frac{1}{\sqrt{2}}\left(\phi_s - i\phi_p\right) & \phi_p &= \frac{i}{\sqrt{2}}\left(\phi^* - \phi\right)\end{aligned} \tag{48}$$

which gives

$$\begin{aligned}\mathscr{L} = \partial_\mu\phi\partial^\mu\phi^* + \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - \frac{m}{2}\bar{\psi}\psi - \frac{g''}{2}\bar{\psi}\left[\phi(1+\gamma_5) + \phi^*(1-\gamma_5)\right]\psi \\ - m^2\phi\phi^* - mg''\left[\phi^2\phi^* + \phi(\phi^*)^2\right] - (g'')^2\phi^2(\phi^*)^2,\end{aligned} \tag{49}$$

where $g'' = \sqrt{2}g$. This can be written even more compact using projection operators,

$$\begin{aligned}\mathscr{L} = \partial_\mu\phi\partial^\mu\phi^* + \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - \frac{m}{2}\bar{\psi}\psi - g''\bar{\psi}\left[\phi P_R + \phi^* P_L\right]\psi \\ - m^2\phi\phi^* - mg''\left[\phi^2\phi^* + \phi(\phi^*)^2\right] - (g'')^2\phi^2(\phi^*)^2,\end{aligned} \tag{50}$$

where the left and right projection operators are introduced.

$$P_R = \frac{1+\gamma_5}{2} \qquad P_L = \frac{1-\gamma_5}{2} \tag{51}$$

This last form also removes the factor i in the interaction term of the fermion with the scalar, one can now of course continue and give the equations of motion for the complex scalar field, but we will not do that here.

4 The Super-Algebra for 1+1 dimensional space-time

In this chapter we present a review of how to rewrite the fermionic part of the super-algebra. The super-algebra, as presented in chapter 2, gives the fermionic part in terms of Weyl spinors, but this is not a dimensional invariant formulations as Weyl spinors do not necessarily exist in higher or lower dimensional cases. The Dirac-spinor formulation can be defined in arbitrary dimensions. We will thus translate the algebra to Dirac-spinors using the same approach as in the QFT book by Weinberg [15]. The application and discussion of this algebra in the 1+1 dimensional case can (to our knowledge) not be found in existing literature and will be presented here as a result of the study.

The Poincaré algebra in 1+1 dimensions is a lot more easy than the 3+1 dimensional case since there is only one space dimension we only have one space and one time translation and only one boost. The Hamiltonian H can be seen as generating time translations, the momentum operator P generates the spatial translations and the Lorentz group $SO(1,1)$ is generated by one element K. Defining for the translations

$$P_{\pm} = H \pm P \tag{52}$$

the full algebra of $P(1,1)$ can be summarized as shown below.

$$[P_+, P_-] = 0 \qquad [P_{\pm}, K] = \pm i P_{\pm} \tag{53}$$

For the fermionic operators, the super-charges, it is less clear what to do. In 2 dimensional space-time the concept of spin does not exist, so the question arises: What do we mean by fermions in 2 dimensional space-time? The only thing that we know is that the generators of these “fermions” should be Grassmannian generators, but their nature is not immediately clear. To solve this conceptual difficulty, we go back to the discussion of the 4 dimensional super algebra first. In the four dimensional case we took the supersymmetry generators to be Weyl spinors, which does not straightforwardly translate to the 2 dimensional case. One thing that can be attempted is to translate the algebra to Dirac-spinors, which can be defined in any space-time.

For simplicity we ignore the possibility of an internal symmetry and the existence of central charges. Remember that we had two types of super-charges, Q_α which has to be in the $(1/2,0)$-representation of the Lorentz-group and its conjugate $Q^\dagger_{\dot\alpha}$ in the $(0,1/2)$-representation.[1] The (anti-)commutation relations satisfy $\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$ where we for now only take one generator. We can write the two Weyl spinors into one Dirac spinor (or in fact a Majorana spinor).

$$Q_M = \begin{pmatrix} Q \\ -\epsilon Q^* \end{pmatrix} \qquad \bar{Q}_M = \begin{pmatrix} Q^T\epsilon & Q^\dagger \end{pmatrix} \tag{54}$$

The charge conjugation matrix is given by[2]

$$\mathcal{C} = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \qquad \epsilon = i\sigma_2 \tag{55}$$

and the Majorana condition is indeed satisfied:

$$Q_M^c = \mathcal{C}\bar{Q}_M^T = \mathcal{C}\gamma^0 Q_M^* = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} Q^* \\ -\epsilon Q \end{pmatrix} = \begin{pmatrix} Q \\ -\epsilon Q^* \end{pmatrix} = Q_M \tag{56}$$

[1] Note that we use a † for the conjugate Weyl spinor instead of the bar that we used in chapter 2. We do this so that we can reserve the bar for the conjugate dirac spinor which has a factor of γ^0 in front.

[2] This definition differs from the one in appendix A by an overall minus sign. This is not very important since the definition gives us the freedom to introduce a minus sign and most authors seem to use this charge conjugation matrix, so we might as well do the same.

Using the above information we can write the anti-commutation relations for Dirac super-charges.

$$\{Q_M, \bar{Q}_M\} = \begin{pmatrix} \{Q, Q^T\}\epsilon & \{Q, Q^\dagger\} \\ -\epsilon\{Q^*, Q^T\}\epsilon & -\epsilon\{Q^*, Q^\dagger\} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2\sigma^\mu P_\mu \\ -\epsilon\left[2\sigma^\mu P_\mu\right]^T \epsilon & 0 \end{pmatrix} = 2\gamma^\mu P_\mu \tag{57}$$

The Dirac representation can in principle be generalized to any number of dimensions, for the present purpose this wll be 2 dimensions. In 2D the dirac spinor will be a two component vector-like object.[3] This can be written down as,

$$Q_D = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \qquad \bar{Q}_D = \begin{pmatrix} Q_2^* & Q_1^* \end{pmatrix} \tag{58}$$

and the commutation relation as

$$\{Q_D, \bar{Q}_D\} = 2\gamma^\mu P_\mu = 2\begin{pmatrix} 0 & H+P \\ H-P & 0 \end{pmatrix}, \tag{59}$$

where

$$P_0 = H \qquad P_1 = P. \tag{60}$$

Hence we obtain for the supercharges of the 1+1 dimensional algebra

$$\{Q_1, Q_1^*\} = 2(H+P) = 2P_+$$
$$\{Q_2, Q_2^*\} = 2(H-P) = 2P_- \tag{61}$$

Equations 53 and 61 together give the complete super-algebra for the 1+1 dimensional case, without including central charges.
If we now include the central charges, i.e. allow for an internal symmetry, the discussion becomes somewhat more complicated. In the first place the anti-commutation relations, with all indices explicitly written, are given in equation 33 and the Dirac super-charges are given by, writing explicitly the indices on the epsilon,

$$Q^A = \begin{pmatrix} Q_\alpha^A \\ -\epsilon_{\dot{\alpha}\dot{\gamma}}(Q^*)^A_{\dot{\gamma}} \end{pmatrix} \qquad \bar{Q}^A = \begin{pmatrix} (Q^T)^A_\gamma \epsilon_{\gamma\alpha} & (Q^\dagger)^A_{\dot{\alpha}} \end{pmatrix} \tag{62}$$

where there is a sum over the repeated indices. And being very careful with the indices, the anti-commutation relation for the Dirac super-charges becomes,

$$\{Q^A, \bar{Q}^B\} = \begin{pmatrix} \{Q^A_\alpha, (Q^T)^B_\gamma \epsilon_{\gamma\beta}\} & \{Q^A_\alpha, (Q^\dagger)^B_{\dot{\beta}}\} \\ \{-\epsilon_{\dot{\alpha}\dot{\gamma}}(Q^*)^A_{\dot{\gamma}}, (Q^T)^B_\gamma \epsilon_{\gamma\beta}\} & \{-\epsilon_{\dot{\alpha}\dot{\gamma}}(Q^*)^A_{\dot{\gamma}}, (Q^\dagger)^B_{\dot{\beta}}\} \end{pmatrix}$$
$$= \begin{pmatrix} \epsilon_{\alpha\gamma}\epsilon_{\gamma\beta}\mathcal{Z}^{AB} & 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{AB} \\ -\epsilon_{\dot{\alpha}\dot{\gamma}}\left[2\sigma^\mu_{\dot{\gamma}\gamma} P_\mu \delta^{AB}\right]^T \epsilon_{\gamma\beta} & -\epsilon_{\dot{\alpha}\dot{\gamma}}\left[-\epsilon_{\dot{\gamma}\dot{\beta}}(\mathcal{Z}^*)^{AB}\right]^T \end{pmatrix}$$
$$= \begin{pmatrix} \delta_{\alpha\beta}\mathcal{Z}^{AB} & 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{AB} \\ 2\bar{\sigma}^\mu_{\dot{\alpha}\beta} P_\mu \delta^{AB} & \delta_{\dot{\alpha}\dot{\beta}}(\mathcal{Z}^*)^{BA} \end{pmatrix}$$
$$= 2\gamma^\mu P_\mu \delta^{AB} + \frac{1-\gamma_5}{2}\mathcal{Z}^{AB} + \frac{1+\gamma_5}{2}(\mathcal{Z}^*)^{BA}. \tag{63}$$

[3]Using the 2 dimensional gamma matrices from appendix A.

For the 1+1D case, take again equation 58 as the dirac super-charge and the 2D chiral gamma matrices.

$$\{Q^A, \bar{Q}^B\} = 2\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} H + \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} P\right]\delta^{AB} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathcal{Z}^{AB} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}(\mathcal{Z}^*)^{BA}$$
$$= \begin{pmatrix}\mathcal{Z}^{AB} & 2(H+P)\delta^{AB}\\ 2(H-P)\delta^{AB} & (\mathcal{Z}^*)^{BA}\end{pmatrix} \tag{64}$$

Thus the anti-commutation relations of the 1+1 dimensional super-Poincaré algebra are given by,

$$\begin{aligned}&\{Q_1^A, (Q_1^*)^B\} = 2(H+P)\delta^{AB} \qquad &&\{Q_1^A, (Q_2^*)^B\} = \mathcal{Z}^{AB}\\ &\{Q_2^A, (Q_2^*)^B\} = 2(H-P)\delta^{AB} \qquad &&\{Q_2^A, (Q_1^*)^B\} = (\mathcal{Z}^*)^{BA},\end{aligned} \tag{65}$$

where the two on the right can be written as

$$\{Q_a^A, (Q_b^*)^B\} = \epsilon_{ab}\mathcal{Z}^{AB} \tag{66}$$

or its complex conjugate.

5 Parametrizations

This chapter is a discussion of a few possible parametrizations for SU(3), starting with a parametrization of SU(2) which will be discussed in full detail. The complete mathematically profound discussion can be found in books on Group Theory, the book that helped to write the chapter is the book of Francis D. Murnaghan [16]. The conventions used are the ones that are used in most literature, see for an example J.W.F. Valle [17]. Section 5.3 gives a view on the substructures of SU(3), based on the idea described in the introduction.

5.1 Parametrizations of SU(2)

In this section the parametrization of SU(2) will be discussed in as much detail as possible. A few options for parametrization will be discussed, but it starts with general arguments.
Starting with a general 2×2 matrix with complex entries,

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{67}$$

and implementing the two SU(2) conditions (1. Unitarity) $U^\dagger U = \mathbb{1}$ and (2. "Special") $\det(U) = 1$ leads to the following conditions for a matrix representation of SU(2):

$$(1):\ \bar{a}a + \bar{c}c = 1,\ \bar{b}b + \bar{d}d = 1 \text{ and } \bar{a}b + \bar{c}d = 0,$$
$$(2):\ ad - bc = 1.$$

Now first assume that $b = 0$, in that case $ad = \bar{d}d = 1 \to d = \bar{a}$ ($|a| = 1$) and $\bar{c}d = 0 \to c = 0$ which gives: $U = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$. A second option is to take $b \neq 0$ and $a = 0$, in this case $-bc = \bar{c}c = 1 \to c = -\bar{b}$ ($|b| = 1$) and $\bar{c}d = 0 \to d = 0$, leading to $U = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}$. For the general case, where $b \neq 0$ and $a \neq 0$, after some manipulations of the conditions one finds $d = \bar{a}$ and $c = -\bar{b}$. Taking this all into consideration gives the general form of an SU(2) matrix as:

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{68}$$

Reconsidering condition (1): $\bar{a}a + \bar{b}b = 1$ shows that a choice can be made to write the complex parameters a and b in terms of two phases and an angle as $a = e^{i\alpha}\cos\theta$ and $b = e^{i\beta}\sin\theta$ (note that this choice is not unique):

$$U = \begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix}. \tag{69}$$

This can immediately be rewritten in a more convenient form as,

$$U(\theta_{12}, \chi, \delta_{12}) = D(\chi, -\chi)U_{12}(\theta_{12}, \delta_{12}), \tag{70}$$

where the $D(\chi, -\chi) = \text{diag}(e^{i\chi},\ e^{-i\chi})$ and the (12) index indicates rotations about the 3-axis, which will be a more intuitive notation when we move to the discussion of the SU(3) parametrization. The notation will be implemented here for consistency. Explicitly writing the matrix form as:

$$U = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & e^{-i\delta_{12}}\sin\theta_{12} \\ -e^{i\delta_{12}}\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} = \begin{pmatrix} e^{i\chi}\cos\theta_{12} & e^{i(\chi-\delta_{12})}\sin\theta_{12} \\ -e^{-i(\chi-\delta_{12})}\sin\theta_{12} & e^{-i\chi}\cos\theta_{12} \end{pmatrix}, \tag{71}$$

shows the connection to the choices of phases, the last equal sign gives the relation between the parameters: $\chi = \alpha$ and $\delta_{12} = \alpha - \beta$.
Now since $U(-\theta_{12}, \chi, \delta_{12}) = U(\theta_{12}, \chi, \delta_{12} + \pi)$ and $U(\theta_{12} + \pi, \chi, \delta_{12}) = U(\theta_{12}, \chi + \pi, \delta_{12})$ it is sufficient to take $0 \leq \theta_{12} < 2\pi$, $0 \leq \delta_{12} \leq \pi$ and $0 \leq \chi \leq \pi$, noting that in the interval $0 \leq \theta_{12} < 2\pi$ the end-points are identified. Other choices can be made for these intervals, the options are listed below.

θ_{12}	δ_{12}	χ
$[0, 2\pi)$	$[0, \pi]$	$[0, \pi]$
$[0, \pi]$	$[0, 2\pi)$	$[0, \pi]$
$[0, \pi]$	$[0, \pi]$	$[0, 2\pi)$
$[0, \pi/2]$	$[0, 2\pi)$	$[0, 2\pi)$

Table 1: *This table lists the possibilities for choosing the intervals of the three parameters. The intervals could also have been chosen as for example $-\pi \leq \theta_{12} < \pi$, $-\pi/2 \leq \delta_{12} \leq \pi/2$ and $-\pi/2 \leq \chi \leq \pi/2$ but this is exactly the same as the first option in the table shifted with $-\pi$. These are the only (obvious) unique choices for the ranges of these parameters. The first three arise form the realisation that setting $\theta_{12} \to \pi - \theta_{12}$ is the same as setting $\delta_{12} \to \delta_{12} + \pi$ and $\chi \to \chi + \pi$ and the last one can be understood by comparing this parametrization with the one in terms of the Euler-angles for $SU(2)$ (see below).*

Rewriting equation 71 in terms of the Pauli matrices gives:

$$\begin{aligned} U_{12}(\theta_{12}, \delta_{12}) &= \begin{pmatrix} \cos\theta_{12} & e^{-i\delta_{12}} \sin\theta_{12} \\ -e^{i\delta_{12}} \sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\delta_{12}} & 0 \\ 0 & e^{i\delta_{12}} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} e^{i\delta_{12}} & 0 \\ 0 & e^{-i\delta_{12}} \end{pmatrix} = e^{-i\delta_{12}\tau_3/2} e^{i\theta_{12}\tau_2} e^{i\delta_{12}\tau_3/2} \end{aligned} \tag{72}$$

and one can thus write,

$$U(\theta_{12}, \chi, \delta_{12}) = e^{i\chi\tau_3} e^{-i\delta_{12}\tau_3/2} e^{i\theta_{12}\tau_2} e^{i\delta_{12}\tau_3/2} \tag{73}$$

where the τ_i are the familiar Pauli matrices.
Another very useful parametrization that could have been chosen is the one in terms of the Euler-angles. An explanation for the use of Euler-angles for $SU(2)$ is given in chapter 3 of [18]. The representation of the $SU(2)$ in terms of the Euler-Angles ($0 \leq \rho \leq \pi$, $0 \leq \theta \leq \pi/2$ and $0 \leq \psi \leq 2\pi$) would be:

$$U(\rho, \theta, \psi) = e^{i\rho\tau_3} e^{i\theta\tau_2} e^{i\psi\tau_3} = \begin{pmatrix} e^{i(\rho+\psi)} \cos(\theta) & e^{i(\rho-\psi)} \sin(\theta) \\ e^{i(\psi-\rho)} \sin(\theta) & e^{-i(\rho+\psi)} \cos(\theta) \end{pmatrix}, \tag{74}$$

where one can easily identify $\theta = \theta_{12}$, $\delta_{12} = 2\psi$ and $\chi = 2(\rho + \psi)$. From this set of phases it can intuitively be seen that one can also take the parameter ranges $0 \leq \delta_{12} < 2\pi$, $0 \leq \chi < 2\pi$ and $0 \leq \theta_{12} < \pi/2$ in the standard parametrization mentioned above, this option is already listed in the last line of table 1.
It can be instructive to make a link to a third, more familiar, parametrization. This third parametrization is the one in terms of a rotation angle $0 \leq \phi \leq$ and azimuthal angles $\hat{n}(\vartheta, \varphi)$ (with $0 \leq \vartheta \leq \pi$ and $0 \leq \varphi < \pi$),

$$\begin{aligned} U(\phi, \hat{n}(\vartheta, \varphi)) &= \mathbb{1} \cos(\phi/2) + i\vec{\tau} \cdot \hat{n} \sin(\phi/2) \\ &= \begin{pmatrix} \cos(\phi/2) \ + i \sin(\phi/2) \cos(\vartheta) & i \sin(\phi/2) \sin(\vartheta) e^{-i\varphi} \\ i \sin(\phi/2) \sin(\vartheta) e^{i\varphi} & \cos(\phi/2) \ - i \sin(\phi/2) \cos(\vartheta) \end{pmatrix}. \end{aligned} \tag{75}$$

This parametrization can easily be related to the one in terms of χ and θ_{12}: (denoting an entry of the matrix as u_{ab}), $u_{11}+u_{22} = 2\cos(\theta_{12})\cos(\chi) = 2\cos(\phi/2)$, $u_{11}-u_{22} = 2i\cos(\theta_{12})\cos(\chi) = 2i\sin(\phi/2)\cos(\vartheta)$ and $u_{12}\cdot u_{21} = -\sin^2(\theta_{12}) = -\sin^2(\phi/2)\sin^2(\vartheta)$. Combining the statements above leads to the relations between the various parameters, which are given by:

$$\tan(\chi) = \tan(\phi/2)\cos(\vartheta),\ \sin(\theta_{12}) = \sin(\phi/2)\sin(\vartheta) \text{ and } \phi - \pi/2 = \delta_{12} - \chi. \tag{76}$$

Linking this parametrization intervals to the standard parametrization,

$$\begin{aligned} &\tan(\chi)[\vartheta = 0] = \tan(\phi/2) && \sin(\theta_{12})[\vartheta = 0] = 0 \\ &\tan(\chi)[\vartheta = \pi/2] = 0 && \sin(\theta_{12})[\vartheta = \pi/2] = \sin(\phi/2) \\ &\tan(\chi)[\vartheta = \pi] = -\tan(\phi/2) && \sin(\theta_{12})[\vartheta = \pi] = 0, \end{aligned}$$

shows that $0 \leq \chi \leq \pi$, $0 \leq \theta_{12} \leq \pi$ and $0 \leq \delta_{12} < 2\pi$, which is the second set of ranges in table 1.

5.2 Parametrization of SU(3)

The parametrization of SU(3) equivalent to the one above for SU(2), referred to as the standard parametrization, would involve 3 different unimodular 3×3 matrices which can be chosen to be:

$$U_{12}(\theta_{12},\delta_{12}) = \begin{pmatrix} \cos\theta_{12} & e^{-i\delta_{12}}\sin\theta_{12} & 0 \\ -e^{i\delta_{12}}\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-i\delta_{12}\lambda_3/2}e^{i\theta_{12}\lambda_2}e^{i\delta_{12}\lambda_3/2} \tag{77}$$

$$U_{13}(\theta_{13},\delta_{13}) = \begin{pmatrix} \cos\theta_{13} & 0 & e^{-i\delta_{13}}\sin\theta_{13} \\ 0 & 1 & 0 \\ -e^{i\delta_{13}}\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} = e^{-i\delta_{13}\lambda_V/2}e^{i\theta_{13}\lambda_5}e^{i\delta_{13}\lambda_V/2} \tag{78}$$

$$U_{23}(\theta_{23},\delta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & e^{-i\delta_{23}}\sin\theta_{23} \\ 0 & -e^{i\delta_{23}}\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} = e^{-i\delta_{23}\lambda_U/2}e^{i\theta_{23}\lambda_7}e^{i\delta_{23}\lambda_U/2} \tag{79}$$

and a diagonal unimodular matrix D as in equation 70,

$$U = D(\xi_1,\xi_2,\xi_3)U_{23}(\theta_{23},\delta_{23})U_{13}(\theta_{13},\delta_{13})U_{12}(\theta_{12},\delta_{12}) \tag{80}$$

where $\xi_3 = -(\xi_1+\xi_2)$ because of the $\det(D) = 1$ condition. Here we can note that the number of parameters used in the equation above is equal to the number of generators in SU(3). This choice of parametrization is mathematically justified in [16], but we do not need the mathematical rigour here.

The transposition matrices $T_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ can be used to relate the U_{ab} matrices to each other as follows:

$$\begin{aligned} U_{13}(\theta_{13},\delta_{13}) &= T_{23}U_{12}(\theta_{13},\delta_{13})T_{23} \\ U_{23}(\theta_{23},\delta_{23}) &= T_{12}T_{23}U_{12}(\theta_{23},\delta_{23})T_{23}T_{12}. \end{aligned} \tag{81}$$

Hence, U can be rewritten as:

$$U = D(\xi_1,\xi_2,\xi_3)T_{12}T_{23}U_{12}(\theta_{23},\delta_{23})T_{23}T_{12}T_{23}U_{12}(\theta_{13},\delta_{13})T_{23}U_{12}(\theta_{12},\delta_{12}) \tag{82}$$

When defining

$$U_{31}(\theta_{31}, \delta_{31}) = U_{13}(\theta_{13} = -\theta_{31}, \delta_{13} = -\delta_{31}), \tag{83}$$

and redefining equation 80 as

$$U = D(\xi_1, \xi_2, \xi_3)U_{23}(\theta_{23}, \delta_{23})U_{31}(\theta_{31}, \delta_{31})U_{12}(\theta_{12}, \delta_{12}) \tag{84}$$

we can use the A_4 (Z_3) generator

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{85}$$

to write:

$$\begin{aligned} U_{31}(\theta_{31}, \delta_{31}) &= TU_{12}(\theta_{31}, \delta_{31})T^2 \\ U_{23}(\theta_{23}, \delta_{23}) &= T^2U_{12}(\theta_{23}, \delta_{23})T. \end{aligned} \tag{86}$$

Equation 84 can now be written as:

$$U = D(\xi_1, \xi_2, \xi_3)T^2U_{12}(\theta_{23}, \delta_{23})T^2U_{12}(\theta_{31}, \delta_{31})T^2U_{12}(\theta_{12}, \delta_{12}). \tag{87}$$

An important thing to note here is that to transform between these two definitions, one has to find a matrix that takes $U_{13}(\theta_{13}, \delta_{13})$ to $U_{31}(\theta_{13}, \delta_{13})$, we can do this using the matrix

$$T' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{88}$$

This gives $U_{31}(\theta_{13}, \delta_{13}) = T'U_{13}(\theta_{13}, \delta_{13})T' = U_{13}^T(\theta_{13}, \delta_{13})$. However T' is not an element of A_4, but T' does by itself generate a Z_2 algebra and T generates a Z_3 algebra. So, to be able to make this redefinition we need an extra symmetry group, namely Z_2. To rewrite equation 80 into equation 87, one uses the generators of both Z_2 and Z_3, but in a representation where the generator of Z_2 does not commute with the generator of Z_3. It can thus be verified that the group generated by these generators is $S_3 \cong D_3 \cong C_2 \rtimes C_3$.
Returning to the relations between the U_{ab} using transposition matrices, it can be seen that T' can also be identified as the T_{13} transposition matrix and T can be identified by $T_{23}T_{12}$. But this matrix is already in the symmetry group created by T_{12} and T_{23}, disguised as $T_{12}T_{23}T_{12} = T_{13}$. Looking more closely, one can see that $T_{12}T_{23} = T_{123}$ and $T_{23}T_{12} = T_{321}$ and hence the whole S_3 symmetry is generated. Which leads to the conclusion that equation 87 is also equivalent to equation 82 and the symmetry needed to write the SU(3) representation as stated in these equations is a discrete S_3 symmetry.

5.3 Subgroups of SU(3)

Now it is time to clarify why the above discussion was included in this thesis. This is because it leads up to the main point of the presented study. As should be obvious by now, SU(3) had an SU(2) subgroup. The most intuitive way to see this is to look at the generators of SU(3).

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{89}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The first three generators listed (λ_1,λ_2,λ_3) are just the Pauli matrices with an extra column and row of zero's and thus generate an $SU(2)$. But this is not the only possibility for finding and $SU(2)$ subgroup of $SU(3)$. By introducing the concept of I-spin, U-spin and V-spin in the following way:

$$\lambda_I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_V = (\sqrt{3}\lambda_8 + \lambda_3)/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \lambda_U = (\sqrt{3}\lambda_8 - \lambda_3)/2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we can see that aside from the combination λ_1,λ_2,λ_I there are two other SU(2) subgroups generated by λ_4,λ_5,λ_V and by λ_6,λ_7,λ_U. The relation between these orientations of the SU(2) subgroup is precisely the group Z_3 with the matrix representation of the generator T as discussed before. To compare with the previous section, the unimodular matrices U_{12}, U_{13} and U_{23} correspond respectively to the I-, V- and U-spin orientations, so by building up the SU(3) parametrization we already used this concept. More specifically, the Cartan subalgebra of $SU(3)$ generated by $I_3 = \frac{1}{2}\lambda_3$ and $Y_I = \sqrt{3}\lambda_8$ (corresponding to the isospin and hypercharge quantum numbers), which gives the weights and roots of the chosen representation, resides in the $SU(2) \times U(1)$ subgroup.
However, one can argue that existence of the U-spin and V-spin orientations is nothing more than just choosing another representation for the set of generators λ_i, and thus it is actually part of something more general. Consider the SO(3) subgroup, generated by for example λ_2,λ_5,λ_7, this can be used to rotate the basis of SU(3) and hence "rotate" the SU(2) subgroup inside SU(3).
Now, to go one step further, note that one can include for example the λ_8 as generator of an U(1), and combine this with the with the I-spin $SU(2)$ of the previous statement. Leading to the conclusion that SU(3) has an $SU(2) \times U(1)$ subgroup. This indicates that there is a substructure inside of SU(3) that can be described as an $SU(2) \times U(1)$ with an infinite amount of orientations, with an SO(3) group switching between these orientations. Now considering all of this, one can write this more formally as:

$$SU(3) \supset SO(3) \circ [SU(2) \times U(1)]\,, \tag{90}$$

where the $\circ$ operation is not yet well-defined operation. It is understood as the operation that lets the SO(3) rotate between the different orientations of the $SU(2) \times U(1)$ subgroup.
With this structure defined, we want to interpret the SO(3) subgroup as the spatial rotations, fixing the meaning of a part of the SU(3) group. The SU(3) group itself is identified with the strong interactions. The $SU(2) \times U(1)$ subgroup can accordingly be identified as the electroweak gauge group. So in other words, if we take the SU(3) strong sector to live in a 1+1 dimensional space-time, this could imply that the $SU(2) \times U(1)$ comes out of the strong sector and leaves behind spatial rotations. These spatial rotations can then be absorbed into the 1+1 dimensional Poincaré group. Which we are tempted to write down as,

$$P(1,1) \times SU(3) \sim P(3,1) \circ [SU(2) \times U(1)]\,. \tag{91}$$

This is actually the structure we wanted to describe, although it is not a well defined structure. We should check if this SO(3) group can be absorbed in the P(1,1) group, if that works out in a satisfactory way, we have a well defined structure. We will attempt to solve this problem in the next chapter.
For now, we want to return to the special role of the SU(2)×U(1) orientations linked by the center group Z_3. Because they are linked by the center group, we can more or less have these orientations simultaneously, meaning that they are inside the SU(3) in such a way that they can

be discussed in context of the basis chosen above. While the other, by SO(3) linked, orientations can not be "seen" in a single examination of the basis. To emphasise this special role we can include it explicitly in the notation,

$$SU(3) \supset SO(3) \circ (Z_3 \circ [SU(2) \times U(1)]) . \tag{92}$$

But this would mean that we have taken, as to say, three times the subgroup we were after. The point here is that we have an $SU(2) \times U(1)$ that has three special orientations inside $SU(3)$, but these three orientations can be chosen freely due to the $SO(3)$ rotational freedom. The explicit inclusion is part of some speculative remarks. One thing is that this could be related to the concept of the three different families we have in the Standard Model, the three different spin orientations inside this $SU(3)$ could be related to the three families, in some way. How (an of) this will work is not immediately clear, but since Z_3 is the center group these three orientations play a special role, and this special role may just be the special role of the families. The second thing is that it is not even strange to write the Z_3 down explicitly. The QCD SU(3) is actually not the full SU(3), often we use $\mathrm{SU}(3)_{QCD} = \mathrm{SU}(3)/Z_3$. So keeping in mind that we are doing physics and that we only need the QCD SU(3) to describe the strong sector interactions, we can write down the Z_3 explicitly without introducing more symmetry.
Let us make a further note on equation 91, the left hand side of the equation gives rise to the covariant derivative,

$$E(1,1): \qquad iD_\mu \phi^i = i\partial_\mu \phi^i + g \sum_{a=1,\ldots,8} A^a_\mu (T_a)^i{}_j \phi^j . \tag{93}$$

Similarly for the right hand side,

$$E(3,1): \qquad iD_\mu \phi^i = i\partial_\mu \phi^i + g \sum_{a=1,2,3,8} A^a_\mu (T_a)^i{}_j \phi^j , \tag{94}$$

which can further be written down as,

$$\begin{aligned} E(3,1): \qquad iD_\mu &= i\partial_\mu + \frac{g}{2} \left(\sum_{a=1,2,3} W^a_\mu \lambda_a + B_\mu \lambda_8 \right) \\ &= i\partial_\mu + g \sum_{a=1,2,3} W^a_\mu I_a + \frac{g}{2\sqrt{3}} B_\mu Y_I . \end{aligned} \tag{95}$$

Comparing this to the standard formalism, this means that the usual g' is linked to g as $g = \sqrt{3} g'$, which is actually a good zeroth order result. The zeroth order approximation of the weak angle, $\sin(\theta_W) = \frac{1}{2}$, gives that $\frac{g}{g'} = \tan(\theta_W) = \sqrt{3}$. Hence there is agreement with the electroweak sector of standard model, on the other hand we need a consistent 1+1 dimensional QCD theory. This whole discussion is of course highly speculative, since the group theory behind this idea is not well understood. A few remaining questions are: What is the operation $\circ$? Is this even a well-defined operation? If this operation does exist, then what mechanism can allow us to combine the group of rotations with the 1+1 dimensional space-time symmetries? Or maybe even more urgent, can the electroweak sector be seen as he asymptotic limit of the strong sector? Can the two sectors be dual?
These questions will not be answered here, for we do not know the answers. In the next chapter we will focus on the question whether it is possible to combine the P(1,1) and the SO(3) into a P(3,1).

6 The Poincaré Group

The aim of this chapter is to write the Poincaré group in 3+1 dimensions as some product of the Poincaré group in 1+1 dimensions and the special orthogonal group for $N = 3$. First the systematics of the Poincaré group will be examined, especially the well known decomposition $P(3,1) \cong R^4 \rtimes SO(3,1)$. Then, another decomposition will be studied namely the candidate $P(1,1) \bowtie SO(3)$. It will be discussed whether this is a good candidate or not. The last remarks in this chapter are on the commutator group.

6.1 $P(3,1) \cong R^4 \rtimes SO(3,1)$

The Poincaré group is well studied throughout physics and the Poincaré transformations are well known to be $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + r^\mu$ with $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$. In matrix notation this is: $x \to x' = \Lambda x + r$, with $\Lambda^T \eta \Lambda = \eta$. Where the Λ denotes the Lorentz transformations (rotations and boosts) and r denotes the translations. In analogy with this, we denote the elements of $R^4 \ni r$ and $SO(3,1) \ni \Lambda$, then $(r, \Lambda) \in P(3,1)$. The multiplication rule for the Poincaré algebra is

$$(r_2, \Lambda_2)(r_1, \Lambda_1) = (r_2 + \Lambda_2 r_1, \Lambda_2 \Lambda_1), \tag{96}$$

which is exactly the same as for the semi-direct product $R^4 \rtimes SO(3,1)$, when the homomorphism is $\phi_\Lambda(r) = \Lambda r$, which is a homomorphism since it transforms R^4 to itself preserving the group structure. The inverse element is $(-\Lambda^{-1} r, \Lambda^{-1})$:

$$(-\Lambda^{-1} r, \Lambda^{-1})(r, \Lambda) = (-\Lambda^{-1} r + \Lambda^{-1} r, \Lambda^{-1} \Lambda) = (0, \mathbb{1}).$$

A change of notation can be made, instead of $(r, \Lambda) \in P(3,1)$, one can implement the simpler notation: $r\Lambda \in P(3,1)$. When using this notation, one should be careful and always remember that the group product of R^4 and $SO(3,1)$ are different, namely addition and multiplication respectively, hence

$$r_2 \Lambda_2 (r_1 \Lambda_1) = r_2 + \Lambda_2 (r_1 \Lambda_1) = r_2 + \Lambda_2 r_1 + \Lambda_2 \Lambda_1. \tag{97}$$

This can be made clear by acting with the group elements on some object, call this object x:

$$\begin{aligned} (r\Lambda)x &= r + \Lambda x \\ (r_2 \Lambda_2)(r_1 \Lambda_1)x = (r_2 \Lambda_2)(r_1 + \Lambda_1 x) &= r_2 + \Lambda_2 r_1 + \Lambda_2 \Lambda_1 x \end{aligned} \tag{98}$$

completely in agreement with the Poincaré transformation rule mentioned above.
Note that this discussion is not restricted to $SO(3,1)$, everything mentioned above is also true for the more general $P(N,1) \cong R^N \rtimes SO(N,1)$ for $N \geq 1$.

6.2 The Idea

First observe that it is not strange to assume that $P(3,1)$ can be build up using the lower dimensional $P(1,1)$ and the rotation group $SO(3)$. The group $P(1,1) \cong R^2 \rtimes SO(1,1)$ represents (in physics) time-translations, translations in one spatial direction, say x, and one boost-matrix: a_t, a_x and K_x. The group $SO(3)$ can be represented by rotations about three orthogonal axis, the generators are R_x, R_y and R_z. Observe that if we do an x-translation followed by a rotation about the y-axis, we get a translation in the xz-plane, and if we do a x-translation followed by a rotation about the z-axis we get a translation in the xy-plane. In other words, the generator

a_x combined with R_y produces the generator $a_z \in R^4$, and the generator a_x combined with R_z produces a generator a_y.

$$R_z a_x \sim a_y \qquad\qquad R_y a_x \sim a_z \tag{99}$$

Following the same approach for the boost generator K_x, first boosting in xt-plane and then rotating about the z-axis generates a boost in the xty-subspace and boosting in the xt-plane and then rotating about the y-axis generates a boost in th xtz-subspace.

$$R_z K_x \sim K_y \qquad\qquad R_y K_x \sim K_z \tag{100}$$

Now all generators of $P(3,1)$ are accounted for, the only thing that is left is to check for redundancies. For this, consider all other configurations of the generators a_t, a_x, K_x, R_x, R_y and R_z.

$$\begin{aligned} R_x a_x &\sim a_x & R_x a_t &\sim a_t \\ R_x K_x &\sim K_x & R_y a_t &\sim a_t \\ & & R_z a_t &\sim a_t \end{aligned} \tag{101}$$

From the above considerations one can conclude that no redundancies occur, so it is expected that $P(3,1)$ can be written as some combination of $P(1,1)$ and $SO(3)$. Since neither of the two groups are normal in $P(3,1)$, but both are subgroups of $P(3,1)$ and the intersection of the two groups is identity $P(1,1) \cap SO(3) \cong \{e\}$ it seems that it is possible to write $P(3,1) \cong P(1,1) \bowtie SO(3)$. We explore this possibility in the next section.

6.3 $P(1,1) \bowtie SO(3)$ and $P(1,1) \rtimes SO(3)$

In this section we will explore the possibilities mentioned in the section title and we will conclude that the 3+1 dimensional Poincaré group can not be constructed in such a way. Using the definitions of the two products we cannot choose the homomorphisms such that they give the operations we want in the Poincaré group.
Write $p \in P(1,1)$, $a \in R^2$, $K \in SO(1,1)$ and $R \in SO(3)$ (remember $r \in SO(3,1)$, $\Lambda \in SO(3,1)$ and $(r,\Lambda) \in P(3,1)$). If $P(3,1) \cong P(1,1) \bowtie SO(3)$, the elements of $P(3,1)$ can be written $(p,R) \in P(3,1)$.
Writing the multiplication rule in $P(1,1) \bowtie SO(3)$ explicitly, using equation 130. Note that the usual operation of rotations inside the Poincaré group is matrix multiplication, for this reason the we use the homomorphism $(P(1,1) \times SO(3) \to P(1,1))$ $\alpha_R(p) = Rp$ and the anti-homomorphism $(P(1,1) \times SO(3) \to SO(3))$ $\beta^p(R) = R^p$.

$$\begin{aligned} (p_2,R_2)\circ(p_1,R_1) &= (p_2\alpha_{R_2}(p_1), \beta^{p_1}(R_2)R_1) = (p_2R_2p_1, R_2^{p_1}R_1) \\ &= ((a_2,K_2)R_2(a_1,K_1), R_2^{(a_1,K_1)}R_1) \\ &= ((a_2,K_2)(R_2a_1,R_2K_1), (R_2a_1+R_2K_1)R_1) \\ &= ((a_2+K_2R_2a_1, K_2R_2K_1), R_2a_1+R_2K_1R_1) \end{aligned} \tag{102}$$

Equation 102 gives an incorrect multiplication, since the product of two elements is not clearly an element of the group. The first element in the last line $(a_2 + K_2R_2a_1, K_2R_2K_1)$ is not an element of P(1,1) since it involves an element of SO(3) which is not in there. Similarly the second element $R_2a_1 + R_2K_1R_1$ is no longer an element of SO(3) due to the boos which transforms the

element outside the range of SO(3).
The simpler case of the semi-direct product $P(1,1) \rtimes SO(3)$ can be excluded even easier.

$$(p_2, R_2) \circ (p_1, R_1) = (p_2 \phi_{R_2}(p_1), R_2 R_1) \tag{103}$$

The homomorphism here, $\phi_{R_2}(p_1)$, translates to $R_2 p_1 R_2^{-1}$. If we assume this transformation returns a P(1,1) element, thus assuming the P(1,1) is normal in $P(1,1) \rtimes SO(3)$, the SO(3) acts on a different space than the P(1,1) boost, which is not the case in the Poincaré group.

6.4 Commutator Group

One option still remains, it gives at least some relation between P(1,1) and P(3,1), this is the commutator group. The commutator group of two groups can be defined as follows:

Consider two groups G and H, with elements $g \in G$ and $H \in H$, then the commutator group is defined as the group with elements $[g, h] = g^{-1}h^{-1}gh$. Thus, if both G and H have distinct algebra's that act in the same space, the commutator subgroup is the group that completes the algebra. This group is denoted as $[G, H]$.

If we choose G to be the 1+1 dimensional Poincaté group living on some xt-plane and choose H to be the SO(3) group acting on the xyz-space, where they share the x-axis. The two groups influence each other, making the commutator group non-trivial. One can easily verify that this indeed produces the 3+1 dimensional Poincaré group. This establishes that,

$$P(3,1) \cong [P(1,1), SO(3)] \, . \tag{104}$$

The only problem with this structure is that it is not very easy to use. The two groups used to construct the P(3,1) are locked in a structure that does not allow us to pick out the SO(3) and use it to build up an SU(3) group as mentioned in section 5.3.

7 Discussion and Conclusions

In this last chapter we summarize the most important remarks made in this thesis and discuss the results. In the first part (sections 2-4) we gave an elaborate review of group theory and super(symmetry) followed by the construction of the 2D super-algebra. The algebra constructed in chapter 4 shows that we only have one Dirac spinor generating the fermionic part of the algebra and that the two components of this spinor are independent of each other. In addition, equation 59 shows that when describing the 1+1 dimensional space-time with P_+ and P_- instead of H and P, the two scalar supercharges decouple. This is an effect that is very similar to the behaviour we see in the Einstein equations for 1+1 dimensional space-time, discussed in appendix C, where the "time" and "space" coordinate decouple. In two dimensional space-time there is no mixing between space and time coordinates. In my opinion this might be the same effect we see in equations 61 where it seems that Q_1 only couples to P_+ and Q_2 only to P_-. Of course this effect partly disappears when internal symmetries are allowed, since Q_1 and Q_2 are then coupled by an element of the center group of the Poincaré group and the internal group. But here we should note that when discussing the Einstein equations in pure GR, one also ignores any internal symmetries.

In chapter 5 we have shown how the group structure of $SU(3)$ has an $SU(2) \times U(1)$ subgroup with three special orientations (the I-, U- and V-spin orientations). That the $SU(3)$ has an $SU(2) \times U(1)$ subgroup is of course well known, but the given discussion of the orientations is new. These orientations are special in the sense that they have a Z_3 center group linking the three orientations, which could speculatively be linked to the three-family structure seen in the electroweak-sector of the standard model. The precise structure behind this is not well understood, since the three orientations linked by this Z_3 are not the only ones, in general one can rotate the whole $SU(3)$ basis with an $SO(3)$. Allowing for infinite orientations of this subgroup. However, independent of the basis there will always be this trio of I-, U- and V-spin orientations recognisable. This strongly indicates that we can consistently work with this structure, the question of "how" still remains.

Speculating beyond what was mentioned in this chapter, one could even say that including the space-time symmetries of the Poincaré groups (two dimensional or four dimensional) this Z_3 symmetry might grow to an A_4 symmetry. The A_4 symmetry plays a role in the lepton family mixing and is important for neutrino oscillations as described in [9, 10], where it is an ingredient in the mass mixing. During the time studying the possibilities, it has not become clear to me how or why this structure resides in there. We should also note that there are many models for describing neutrino oscillation, but when this A_4 resides in this theory there might be a link to these studies.

In the last chapter we have seen that there appears to be no group-product mechanism that allows us to connect the $P(3,1)$ to the $P(1,1)$, this does make the idea stated in chapter 5, summarized by equation 91, somewhat unlikely. The $\circ$ operation in this equation cannot be a group product, therefore it is not clear what kind of operation this is in the full space. This question remains unanswered.

A solution to this problem can be to use the commutator group to link $P(3,1)$ and $P(1,1)$. However, this becomes problematic since the $SO(3)$ is "locked" inside this commutator group making it impossible to take it out and absorb it into the $SU(3)$. And it would also violate the Coleman-Mandula no-go theorem, when combining the internal and external symmetries.

The story is not finished here, we only hinted that some paths to solving this problem are dead ends. We would like to see the group structure from chapter 5 studied in more detail, to see whether an A_4 symmetry does emerge when including the space-time symmetries. It would also be interesting to look at the super algebra more closely, to see how this can be connected to the

discussion in chapter 6.
As a final statement we would like to mention that it still seems possible to "rearrange" the gauge group of the Standard Model using the space-time symmetries and supersymmetry, even though it seems unlikely that this would lead to many new insights. In the best case we find an A_4 symmetry inside the rearrangement, but aside from that, no new mechanism is expected to be found. The most exiting result that can be obtained is that this rearrangement of the Standard Model gauge group can lead to an understanding why the number 3 seems to appear everywhere in the Standard Model, in the amount of colours, space-directions and families. The model described in this thesis gives a way to link these three 3's. The three space-directions would be linked to the three colours, since the $SO(3)$ comes directly out of the $SU(3)$ and also the three families (Z_3) could be directly linked to the three colours. The story is definitely not finished here, it only appears to be more difficult than initially thought.

Appendices

A Gamma Matrices and Spinors in 1+1 Dimensions

A.1 Gamma matrices

Standard Representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad C = -i\gamma^2\gamma^0 = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \tag{105}$$

C is the charge conjugation matrix and $\epsilon = i\sigma^j$.

Chiral Representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad C = -i\gamma^2\gamma^0 = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \tag{106}$$

Majorana Representation

The Majorana Representation is defined such that the γ^μ are real.

$$\gamma^0 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$
$$\gamma_5 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad C = -i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{107}$$

Another option is to take $\gamma_2^M = \gamma_5^{st}$ with respect to the standard representation, hence $\gamma_5^M = -\gamma_2^{st}$ and the rest of the matrices equal to those in the standard representation. The charge conjugation matrix is $C = -i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ in that case.

1+1 Dimensional Gamma Matrices

Now this is actually very simple since there are only two gamma matrices and $\eta_{\mu\nu} = \mathrm{diag}(1, -1)$. The gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}$ and $\gamma_\mu^T = \gamma_0\gamma_\mu\gamma_0$, the charge conjugation operator satisfies $C\gamma_\mu C^{-1} = -\gamma_\mu^T$. Hence,

$$\begin{aligned}
&\gamma_0\gamma_0 = \mathbb{1} \qquad \gamma_1\gamma_1 = -\mathbb{1} \qquad \gamma_0\gamma_1 = -\gamma_1\gamma_0 \\
&\gamma_1^T = \gamma_0\gamma_1\gamma_0 = -\gamma_1 \qquad \gamma_0^T = \gamma_0 \\
&C\gamma_0 C^{-1} = -\gamma_0 \to C\gamma_0 = -\gamma_0 C \\
&C\gamma_1 C^{-1} = -\gamma_1^T = \gamma_1 \to C\gamma_1 = \gamma_1 C \\
&\to C = \pm\gamma_1.
\end{aligned}$$

The standard representation can be written as,

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = -\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

And the chiral representation as,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = -\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

both qualify to be Majorana representations since all the gamma matrices are now real.

A.2 Spinor fields

The chiral represention is used her since it is the most convenient representation. Consider for this section the Lagrangian,

$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \tag{108}$$

3+1 Dimensions
The γ^μ can conveniently be written as,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{109}$$

where $\sigma^\mu = \{\mathbb{1}, \sigma^j\}$ and $\bar{\sigma^\mu} = \{\mathbb{1}, -\sigma^j\}$ and the four component spinor field as,

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}. \tag{110}$$

The Lagrangian can now be written in terms of these Weyl-spinors as,

$$\mathscr{L} = i\left[\eta^\dagger\sigma^\mu\partial_\mu\eta + \xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi\right] + m\left[\eta^\dagger\xi + \xi^\dagger\eta\right]. \tag{111}$$

When one considers a Majorana field, one must set the conjugated spinor equal to the normal spinor,

$$\psi^c = C\bar{\psi}^T = C\gamma^0\psi^* = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} -\epsilon\eta^* \\ \epsilon\xi^* \end{pmatrix} \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{112}$$

So the majorana spinor can be written as:

$$\psi = \begin{pmatrix} \xi \\ \epsilon\xi^* \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} -\xi^T\epsilon & \xi^\dagger \end{pmatrix}. \tag{113}$$

Now writing the Lagrangian for the Majorana field, we obtain:

$$\mathscr{L} = i\left[\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - \xi^T\epsilon\sigma^\mu\partial_\mu\epsilon\xi^*\right] + m\left[\xi^T\epsilon\xi - \xi^\dagger\epsilon\xi^*\right]. \tag{114}$$

1+1 Dimensions
Using a similar notation for the γ^μ:

$$\gamma^\mu = \begin{pmatrix} 0 & n^\mu \\ \bar{n}^\mu & 0 \end{pmatrix} \tag{115}$$

where $\bar{n}^\mu = \{1, -1\}$ and $n^\mu = \{1, 1\}$. Also writing the spinors as for the 3+1 dimensional case, when doing so note that the ξ and η are now scalar Grassmann fields and not Weyl-spinors.

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} \eta^* & \xi^* \end{pmatrix} \tag{116}$$

The Lagrangian can now be written as

$$\mathscr{L} = i\left[\eta^* n^\mu\partial_\mu\eta + \xi^*\bar{n}^\mu\partial_\mu\xi\right] - m\left[\eta^*\xi + \xi^*\eta\right], \tag{117}$$

or when adopting the notation $\bar{n}^\mu\partial_\mu = \partial_0 - \partial_1 = \partial_-$ and $n^\mu\partial_\mu = \partial_0 + \partial_1 = \partial_+$ as,

$$\mathscr{L} = i\left[\eta^*\partial_+\eta + \xi^*\partial_-\xi\right] - m\left[\eta^*\xi + \xi^*\eta\right]. \tag{118}$$

When applying the Majorana condition on 2-dimensional spinors something peculiar happens:

$$\psi^c = C\bar{\psi}^T = C\gamma^0\psi^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} -\xi^* \\ \eta^* \end{pmatrix} \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{119}$$

hence ξ is real and η pure imaginary.

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} \eta & -\xi \end{pmatrix}. \tag{120}$$

The Lagrangian for a 1+1 dimensional Majorana field can thus be written as

$$\begin{aligned} \mathscr{L} &= i\left[\xi\partial_-\xi - \eta\partial_+\eta\right] - m\left[\eta\xi - \xi\eta\right] \\ &= i\left[\xi\partial_-\xi - \eta\partial_+\eta\right] - 2m\left[\eta\xi\right] \end{aligned} \tag{121}$$

where in the last line the fact that ξ and η are Grassmann was used. For this case it is interesting to note that, because the components anti-commute,

$$\bar{\psi}\gamma_5\psi = \begin{pmatrix} \eta & -\xi \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = -\xi\eta - \eta\xi = 0 \tag{122}$$

which means that in 1+1 dimensions Majorana fermions do not interact with pseudo-scalars.

B Definitions of Group Products

This appendix introduces the different ways of combining groups into a larger group structure. Various types of group products will be discussed and will, in some cases, be clarified with a few examples, in all cases we will first give the formal mathematical definition.

B.1 Direct Product

The direct product of two groups $G = A \times B$ is defined by the following statements:

1. The underlying set is the Cartesian product $A \times B$, where the group elements are ordered pairs $g = (a, b)$, with $g \in G$, $a \in A$ and $b \in B$
2. The binary operation on $A \times B$ is defined as: $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$

An alternative (and possibly more plain) definition:

1. The intersection of A and B is trivial, i.e. $A \cap B = \{e\}$
2. Every element in G can be expressed as the product of an element in A and an element in B
3. Both A and B are normal in G
   (a) all elements of A commute with those of B
   (b) every element in G can be written as $g = (a, b)$, where $g \in G$, $a \in A$ and $b \in B$ with the multiplication rule $g_1 \circ g_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$

An Example

Take the direct product $G = C_2 \times C_n$ where $C_2 = \{e, a\}$ and $C_n = \{e, b, b^2, \dots, b^{n-1}\}$. Now the direct product of these groups is the group

$$\begin{aligned} G &= \{(e,e), (e,b), (e,b^2), \dots, (e,b^{n-1}), (a,e), (a,b), (a,b^2), \dots, (a,b^{n-1}\} \\ &= \{E, u, u^2, \dots, u^{n-1}, v, vu, vu^2, \dots, vu^{n-1}\} \end{aligned} \tag{123}$$

where $E = (e, e)$ is the identity element, $u = (e, b)$ and $v = (a, e)$. G is a group with $2n$ elements that satisfy $u^n = v^2 = e$, but since $uvu = (e, b)(a, e)(e, b) = (a, b^2) \neq v^{-1}$ this is not the D_n group. The resulting group is a combination of two separate cyclic groups that do not influence each other. Notice that for the case where we have $n = 2$: $C_2 \times C_2 \cong D_2$, since then $a^2 = b^2 = e$ and $uvu = (a, e) = v = v^{-1}$.

Another Example

Sometimes it is easier to use the concept of algebra's instead of groups, for this example consider Lie-algebras instead of groups, the Lie-algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (where the direct sum $\oplus$ is the direct product equivalent for combining algebras) is well known and can be verified fairly easily. First note that the $\mathfrak{so}(4)$ commutation relations are:

$$[T_{ab}, T_{cd}] = -i(\delta_{bc}T_{ad} + \delta_{ad}T_{bc} - \delta_{bd}T_{ac} - \delta_{ac}T_{db}), \tag{124}$$

rewriting these commutation relations using the $\mathfrak{so}(3)$ generators $J_x = T_{23}$, $J_y = T_{13}$, $J_z = T_{12}$ and writing the other generators as $K_i = T_{i4}$. Now the (rewritten) commutation relations are:

$$[J_i, J_j] = i\epsilon_{ijk}J_k,\ [J_i, K_j] = i\epsilon_{ijk}K_k \text{ and } [K_i, K_j] = i\epsilon_{ijk}J_k. \tag{125}$$

At first glance it seems that this is very different from the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ commutation relations, but applying a change of basis: $J_{\pm i} = J_i \pm K_i$ gives

$$[J_{\pm i}, J_{\pm j}] = i\epsilon_{ijk}J_{\pm k} \text{ and } [J_{\pm i}, J_{\mp j}] = 0, \tag{126}$$

which is exactly the set commutation relations of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ which has generators $(\sigma_i, 0)$ and $(0, \sigma_i)$ for $i = x, y, z$, where σ_i are the Pauli-matrices. Hence the isomorphism of the two algebras is evident.

Note: the group equivalent of this isomorphism is the quotient group $SO(4) \cong (SU(2) \times SU(2))/Z_2$ which means that $SU(2) \times SU(2)$ is a double cover of $SO(4)$.

Note 2: $SO(4)$ is the only non-simple special orthogonal group, all groups $SO(N)$ for $N \leq 3$ and $N > 4$ are simple groups.

B.2 Semi-Direct Product

The semi-direct product $G = N \rtimes H$ with N normal in G is defined as follows:

1. $N \cap H = \{e\}$
2. For every $g \in G$ there is a unique $n \in N$ and $h \in H$ such that $g = hn$
3. For every $g \in G$ there is a unique $n \in N$ and $h \in H$ such that $g = nh$
4. The composition $\pi \circ i$, where $i : H \to G$ and $\pi : G \to G/N$ is an isomorphism (one-to-one) between H and the quotient group G/N.

5. There exists a homomorphism $G \to H$ that acts as the identity on H and has kernel N

Another way to define the semi-direct product is to relax the definition of the direct product, namely by relaxing the third condition such that only one of the two subgroups A or B is required to be normal. The resulting product still consists of ordered pairs $g = (a, b)$, but with a slightly more complicated multiplication rule.
The multiplication rule for $G = N \rtimes H$ is:

$$(n, h) \cdot (n', h') = (n\phi_h(n'), hh'), \tag{127}$$

where ϕ_h is a group homomorphism $\phi : H \to Aut(N)$ and the group $Aut(N)$ is the group of automorphisms of N. Since N is normal in G there exists an obvious automorphism of N, that is $n \to hnh^{-1}$ for each $h \in H$. The mapping $H \to Aut(N)$ that is given by $h \to (n \to hnh^{-1})$ is a group automorphism.
Hence the group homomorphism mentioned above is : $\phi_h(n) = hnh^{-1}$. To prove that this is indeed a homomorphism one has to show that $\phi_{th} = \phi_t \circ \phi_h$:

$$\phi_{th}(n) = (th)n(th)^{-1} = t(hnh^{-1})t^{-1} = \phi_t(hnh^{-1}) = \phi_t(\phi_h(n)) = \phi_t \circ \phi_h(n).$$

Example

Take $N = Z_n$ and $H = Z_2$, as group homomorphism take the map $\phi_h(n) : n \to n^{-1}$, which is a good homomorphism since N is Abelian and the homomorphism has order 2. Now let $a \in Z_2 \mid a^2 = 1$ and $b \in Z_n \mid b^n = 1$, then $Z_n \rtimes Z_2$ has elements:

$$v = (e_n, a), u = (b, e_h) \mid u^n = v^2 = 1 \text{ and } vuv = u^{-1}, \tag{128}$$

the last statement can be shown explicitly:

$$\begin{aligned} vuv = vuv^{-1} &= (e_n, a) \cdot (b, e_h) \cdot (e_n, a) = (e_n, a) \cdot (b \cdot \phi_{e_h}(e_n), e_h \cdot a) = (e_n, a) \cdot (b, a) \\ &= (e_n \cdot \phi_a(b), a^2) = (b^{-1}, e_h) = u^{-1}. \end{aligned} \tag{129}$$

Hence $Z_n \rtimes Z_2 \cong D_n$. To empathise the difference of the semi-direct product and the direct product one can compare this example to the first example of section B.1. Note that where the direct product generates a symmetry group with two sub-symmetries which are not influenced by each other, the semi-direct product generates something larger with a more complicated structure. It is now evident that the semi-direct product is a generalization of the direct product, the direct product is just the semi-direct product with as homomorphism the identity map.
A definition similar to the one given above and some more examples can be found in [19].

B.3 Zappa-Szép Product

For the Zappa-Szép product [20, 21], let G be a group with identity element e, and let H and K be subgroups of G (Note: NOT normal subgroups, just subgroups).

1. $G = HK$ and $H \cap K = \{e\}$
2. for each $g \in G$ there exists a unique $h \in H$ and $k \in K$ such that $g = hk$

If either (and hence both) of these statements hold, then G is said to be the internal Zappa-Szép product of H and K and is denoted: $G = H \bowtie K$.

As for the semi-direct product, the Zappa-Szép product is obtained by relaxing the third condition of the definition of the direct product further, to the point that neither A nor B is required to be a normal subgroup of G.
The multiplication rule for the Zappa-Szép product is a bit more difficult than the one for the semi-direct product, the elements of this product group can be written as $(h,k) \in G = H \bowtie K$, which was also done for the other products. The multiplication rule now is given by:

$$(h_2, k_2) \circ (h_1, k_2) = (h_2 \alpha_{k_2}(h_1), \beta^{h_1}(k_2) k_1). \tag{130}$$

Now there are two mappings in the multiplication rule, the first mapping α_k is homomorphism $\alpha : HK \to H$, similar to what was mentioned for the semi-direct product, hence α satisfies $\alpha_{kk'}(h) = kk'(h) = k(k'h) = k\alpha_{k'}(h) = \alpha_k \circ \alpha_{k'}(h)$. The second mapping β^h is something called an anti-homomorphism $\beta : HK \to K$ and satisfies $\beta^{hh'}(k) = k^{hh'} = (k^h)^{h'} = (\beta^h(k))^{h'} = \beta^{h'} \circ \beta^h(k)$, where one should notice the reversed order of the h and h', hence the "anti". A clarification of the notation: k^h indicates some form of right-multiplication of k with h, the systematics of this notation should be clear form the multiplication rule.

B.4 Tensor Product for Groups

The tensor product for groups $M \otimes N$ is the group generated by elements $m \otimes n$, with

1. $mm' \otimes n = (mm'm^{-1} \otimes {}^m n)(m \otimes n)$
2. $m \otimes nn' = (m \otimes n)({}^n m \otimes nn'n^{-1})$

for all $m, m' \in M$ and $n, n' \in N$.
The groups M, N are equipped with and action of M on N denoted ${}^m n$ and similarly with an action of N on M denoted ${}^n m$ for all elements $m \in M$ and $n \in N$.
This is the definition given by Ronald Brown and Jean-Louis Loday in [22] on page 314.

Note on the tensor product for representations (Vector-spaces)

Normally the tensor product is only defined on vector spaces, take for example the vector space tensor product $V \otimes W$ of two representations of the group G, which is itself a representation of G. An element $g \in G$ then acts on a basis element of the vector space $v \otimes w$ as: $g(v \otimes w) = gv \otimes gw$.
If V_1 is a representation of G_1 and V_2 a representation of G_2, then $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, this is called the external tensor product.
If V is a representation of the group G and W a representation of the group H, then the vector space $V \otimes W$ is a representation of the group $G \times H$, an element (g,h) of $G \times H$ acts on a basis element $v \otimes w$ as: $(g,h)(v \otimes w) = gv \otimes hw$.

C Einstein Equations in 1+1 Dimensions

In this appendix we will give a short derivation of the Einstein equations from the Einstein-Hilbert action and give the equations for general 1+1 dimensional metric. We will show that the space and time components of the 1+1 dimensional space-time get decoupled leading to the absence of interesting dynamics in 1+1 dimensional space time.

C.1 Derivation of Einstein Equations form Einstein-Hilbert action

This sections follows the computation given in chapter 4 of Sean Carroll's "Spacetime and Geometry" [23]. We start by stating the Einstein-Hilbert Action.

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(\frac{c^4}{16\pi G} R + \mathscr{L}_M \right) \tag{131}$$

In this equation the $\mathscr{L}_M$ describes the matter fields appearing in the theory and $g = \det(g_{\mu\nu})$. Now treating the metric as a field, the variation of the action is given by ($K = c^4/16\pi G$):

$$\begin{aligned} \delta\mathcal{S} &= \int d^4x \left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} (KR + \mathscr{L}_M) + \sqrt{-g} \left(K \frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta \mathscr{L}_M}{\delta g^{\mu\nu}} \right) \right) \delta g^{\mu\nu} \\ &= \int \sqrt{-g}\, d^4x \left(\frac{KR}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + K \frac{\delta R}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}\mathscr{L}_M}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}. \end{aligned} \tag{132}$$

There are three derivatives to determine, we start with the most cumbersome one, the variation of the Ricci scalar with respect to the metric

$$\frac{\delta R}{\delta g^{\mu\nu}} = \frac{\delta \left(g^{\rho\sigma} R_{\rho\sigma} \right)}{\delta g^{\mu\nu}} = \frac{\delta g^{\rho\sigma}}{\delta g^{\mu\nu}} R_{\rho\sigma} + g^{\rho\sigma} \frac{\delta R_{\rho\sigma}}{\delta g^{\mu\nu}} = R_{\mu\nu} + g^{\rho\sigma} \frac{\delta R_{\rho\sigma}}{\delta g^{\mu\nu}}. \tag{133}$$

Nothing was done in this step, the variation of the Ricci tensor with respect to the metric has still to be computed. Remember that the Ricci tensor is just a contraction of the Riemann tensor, $R_{\rho\sigma} = R^{\mu}{}_{\rho\mu\sigma}$. The definition of the Riemann tensor and it's variation are given below.

$$R^{\mu}{}_{\rho\nu\sigma} \equiv \partial_\nu \Gamma^{\mu}{}_{\sigma\rho} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\sigma\rho} - \Gamma^{\mu}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\nu\rho} \tag{134}$$

$$\delta R^{\mu}{}_{\rho\nu\sigma} = \partial_\nu \delta\Gamma^{\mu}{}_{\sigma\rho} - \partial_\sigma \delta\Gamma^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\sigma\rho} + \Gamma^{\mu}{}_{\nu\lambda} \delta\Gamma^{\lambda}{}_{\sigma\rho} - \delta\Gamma^{\mu}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\mu}{}_{\sigma\lambda} \delta\Gamma^{\lambda}{}_{\nu\rho} \tag{135}$$

Using,

$$\nabla_\nu (\delta\Gamma^{\mu}{}_{\sigma\rho}) = \partial_\nu \delta\Gamma^{\mu}{}_{\sigma\rho} - \Gamma^{\alpha}{}_{\nu\sigma} \delta\Gamma^{\mu}{}_{\alpha\rho} - \Gamma^{\alpha}{}_{\nu\rho} \delta\Gamma^{\mu}{}_{\sigma\alpha} + \Gamma^{\mu}{}_{\nu\alpha} \delta\Gamma^{\alpha}{}_{\sigma\rho} \tag{136}$$

we can rewrite the variation of the Riemann tensor as:

$$\delta R^{\mu}{}_{\rho\nu\sigma} = \nabla_\nu (\delta\Gamma^{\mu}{}_{\sigma\rho}) - \nabla_\sigma (\delta\Gamma^{\mu}{}_{\nu\rho}). \tag{137}$$

The variation of the Ricci tensor is

$$\delta R_{\rho\sigma} = \delta R^{\mu}{}_{\rho\mu\sigma} = \nabla_\mu (\delta\Gamma^{\mu}{}_{\sigma\rho}) - \nabla_\sigma (\delta\Gamma^{\mu}{}_{\mu\rho}). \tag{138}$$

Contracting with the metric gives

$$g^{\rho\sigma} \delta R_{\rho\sigma} = \nabla_\lambda (g^{\rho\sigma} \delta\Gamma^{\lambda}{}_{\sigma\rho} - g^{\rho\lambda} \delta\Gamma^{\mu}{}_{\mu\rho}), \tag{139}$$

which is a total derivative. Hence,

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + \text{Total Derivative}. \tag{140}$$

Now the second term that we should rewrite is the term with $\sqrt{-g}$, which reduces to the term given below.[4]

$$\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{R}{\sqrt{-g}} \left(\frac{-1}{2\sqrt{-g}} \frac{\delta g}{\delta g^{\mu\nu}} \right) = \frac{R}{\sqrt{-g}} \left(\frac{g g_{\rho\sigma}}{2\sqrt{-g}} \frac{\delta g^{\rho\sigma}}{\delta g^{\mu\nu}} \right) = -\frac{R}{2} g_{\mu\nu} \tag{141}$$

[4] Here we used $\delta \ln(\det M) = \delta \mathrm{Tr}(\ln M) = \mathrm{Tr}(\delta \ln M)$, which translates to $\frac{1}{\det M} \delta \det M = \mathrm{Tr}(M^{-1} \delta M)$. Applying this to g gives $\delta = -g g^{\mu\nu} g_{\mu\nu}$.

Defining the Hilbert-Stress-Energy Tensor[5]:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta\sqrt{-g}\mathscr{L}_M}{\delta g^{\mu\nu}}, \tag{142}$$

the variation of the action in equation 132 can be written as,

$$\delta S = \int \sqrt{-g}\, d^4x \left(\frac{-KR}{2} g_{\mu\nu} + K R_{\mu\nu} - \frac{1}{2} T_{\mu\nu} \right) \delta g^{\mu\nu}. \tag{143}$$

Setting this equal to zero gives the Einstein equations.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{144}$$

C.2 Einstein Equations in 1+1 Dimensions

In this section we will derive the Einstein equations for 1+1 dimensional space-time. In general the 2 dimensional metric can be taken diagonal by a convenient choice of the coordinate system.

$$g_{\mu\nu} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad g_{11} = A \qquad g_{22} = B \qquad g^{11} = \frac{1}{A} \qquad g^{22} = \frac{1}{B} \tag{145}$$

Using the definition of the Riemann tensor given in equation 134, and the general properties of the Riemann tensor derived from that definition, we can (for a 2 dimensional space-time) notice that:

$$R^{\alpha}{}_{\beta 11} = R^{\alpha}{}_{\beta 22} = R^{1}{}_{1\alpha\beta} = R^{2}{}_{2\alpha\beta} = 0 \tag{146}$$

for $\alpha, \beta = 1, 2$. And the other components are also linked to each other, leaving us with only one independent component.

$$R^{1}{}_{212} = R^{1}{}_{221} = -\frac{B}{A} R^{2}{}_{112} = \frac{B}{A} R^{1}{}_{121} \tag{147}$$

For the interested reader in terms of A and B the full expression is

$$R^{1}{}_{212} = -\frac{\partial_1^2 B + \partial_2^2 A}{2A} + \frac{\partial_1 B \partial_1 A + \partial_2 A \partial_2 A}{4A^2} + \frac{\partial_2 A \partial_2 B + \partial_1 B \partial_1 B}{4AB}, \tag{148}$$

where the ∂_i represent derivatives to the respective coordinates. To obtain this equation one should first compute the Christoffel symbols and then do some tedious computations, but this is not relevant for the discussion since we do not need an explicit term to get the result we aim to get.
Determining the components of the Ricci tensor has now become trivial:

$$R_{12} = R_{21} = 0 \qquad R_{11} = \frac{A}{B} R_{22}. \tag{149}$$

The off-diagonal components of the Einstein tensor ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$) are zero ($G_{12} = G_{21} = 0$) and the diagonal components are given by,

$$\begin{aligned} G_{11} &= R_{11} - \frac{1}{2}(g^{11} R_{11} + g^{22} R_{22}) g_{11} = \frac{1}{2} R_{11} - \frac{1}{2}\frac{A}{B} R_{22} = 0 \\ G_{22} &= R_{22} - \frac{1}{2}(g^{11} R_{11} + g^{22} R_{22}) g_{22} = \frac{1}{2} R_{22} - \frac{1}{2}\frac{B}{A} R_{11} = 0, \end{aligned} \tag{150}$$

[5] Showing that this action is similar to the Noether Energy-Momentum tensor provides a challenge.

hence all components of the Einstein tensor are zero. No dynamics is left in this space-time unless we add the cosmological constant to the Einstein equations, in that case:

$$T_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \qquad \rightarrow \qquad \begin{pmatrix} \rho & 0 \\ 0 & p \end{pmatrix} = \frac{\Lambda}{8\pi G} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{151}$$

In words, the Energy-Momentum tensor is proportional to the metric. Furthermore, the result gives end two independent equations, one for the "space" and one for the "time" component of the 2 dimensional space-time. This means that "space" and "time" are decoupled in a two dimensional world.

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