STUDENTS’ CONCEPT DEVELOPMENT AND UNDERSTANDING OF SINE AND COSINE FUNCTIONS

A New Theoretical and Educational Approach

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Abstract

Trigonometry is an important school subject not only for mathematics but also for some other fields. A robust understanding of trigonometric functions requires different algebraic, geometric, and graphical aspects due to the complex nature of the topic. Despite limited, the literature on trigonometry learning and teaching reveals that it is a difficult topic for students, and that students develop fragmented understanding of trigonometric functions. This study addressed two important points which appear to be problematic in research literature and educational practices: a broad understanding model of trigonometry based on conceptual connections, and a new instructional approach of trigonometric functions. The understanding model comprises different trigonometric contexts and their connections, and the new instructional approach puts a great emphasis on the connections in this model. A major difference from traditional teaching methods is that the new method starts with a focus on arcs in the unit circle together with the construction of the sine graph rather than an early introduction of radians. Based on this understanding model and the new approach, the study examined students’ concept development and understanding of the sine and cosine functions. It was conducted at a pre-university level (VWO) Dutch secondary school in Amsterdam with a mathematics class of 24 students aged 16-17. The new approach based on the implemented learning trajectory was found to be effective in terms of promoting a connected understanding of trigonometric functions, not only in terms of angles but also on the domain of real numbers. Students revealed good levels of understanding, and did not develop certain difficulties and misconceptions as found in the research literature. As a remarkable point, research findings show that students developed understanding of trigonometric graphs and function related properties. Findings have a crucial potential to shed light on neglected aspects in the literature, and to provide important educational implications to improve trigonometry instruction.
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1 INTRODUCTION

The aim of the research study reported in this thesis was to investigate students’ concept development within and understanding after the implementation of a new teaching approach to trigonometric functions.

This chapter starts with the statement of the problem that drove the study. In the second section, the significance of the study is discussed. Afterwards, the motivation of the researcher for conducting the research is presented. The chapter ends with the structure of the thesis.

1.1 Statement of the Problem

This study was based on the assumption that learning mathematics with understanding is both essential and possible as suggested by NCTM (2000). Hence, this study represents an attempt to respond to a fundamental problem in trigonometry teaching regarding learning it with understanding; as it is, students develop a fragmented understanding of trigonometric functions (e.g. Orhun, 2001; Brown, 2005; Weber, 2005; Challenger, 2009).

Trigonometry is one of the important topics in secondary school mathematics requiring integration of different algebraic, geometric and graphical reasoning. This means it is crucial to learn trigonometry with understanding based on connections among three different contexts of trigonometry:

- Triangle Trigonometry, where trigonometry is based on ratio definitions in right triangles;
- Unit Circle Trigonometry, where trigonometric functions are defined as coordinates of points on the unit circle based on rotation angles;
- Trigonometric Function Graphs, where trigonometric functions are defined in the domain of real numbers.

Developing understanding based on trigonometric connections is not easy for students, and traditional ways of teaching trigonometry do not overcome students’ difficulties. Teachers present these contexts as separate from each other based on an assumption that conceptual development of each occurs in a linear order from the first context to the last one. Such an approach promotes incomplete and disconnected understanding which is angle-measure-dominant because it generally stresses the first two contexts. Trigonometric functions in the domain of real numbers remain rather neglected. Students cannot really understand trigonometric graphs, nor do they build important connections through the radian concept between the unit circle and graphs, such as the transition from angles to real numbers through the radian concept.

This study addressed students’ learning problems in traditional methods of teaching trigonometry, which both prevent them from developing understanding based on trigonometric connections and
promote trigonometry learning based on memorization. An alternative instructional sequence was developed and tested in an attempt to address these issues.

1.2 Significance of the Study

This study is significant for two reasons. First, it has the potential to contribute to the literature on the understanding of trigonometric functions by addressing omissions noted in my rather exhaustive literature search. Second, it suggests a new method of teaching trigonometry which seems to be promising for students’ understanding based on trigonometric connections.

Research on trigonometry is sparse and quite limited. Only a few studies appear to provide significant findings about students’ understanding of trigonometric functions (e.g. Brown, 2005; Weber, 2005), although it has often been reported as a difficult topic for students. However, these studies did not provide an integrated model of trigonometric understanding addressing the three contexts of trigonometry and the connections among them. The research studies investigating students’ understanding of trigonometric functions considered only the triangle and unit circle contexts, and generally reported students’ difficulties in trigonometric functions of angles. Trigonometric functions of real numbers have not been researched together with different instructional approaches. Since the studies about students’ understanding did not include the third context (trigonometric function graphs) and the connections to this context, important aspects of trigonometric functions such as their graphs, and function concept and related properties of domain, range and periodicity were not considered in any model of trigonometric understanding. For example the function concept, either based on a formal or on a process view, was not elaborated in detail in terms of trigonometric functions.

The present study addressed such issues by investigating students’ understanding of trigonometric functions based on an integrated model of the three trigonometric contexts and connections among them. Since such a broad framework of trigonometric understanding was not found in the literature, the researcher developed his own model.

Students should be provided with appropriate learning opportunities so that they can develop a sound trigonometric understanding. The second interesting aspect of this study stems from the new teaching method of trigonometry that it suggests. This seems important because traditional methods of teaching this topic cannot help students develop a rich conceptual understanding nor overcome their learning difficulties, and the presently available literature on trigonometry does not appear to address in an adequate way other possible methods of teaching trigonometric functions.

In the literature only right triangle and unit circle methods were emphasized, but no new methods which could potentially help students develop integrated understanding were provided. The present
study addressed this point by providing a new approach to teach trigonometry as a learning trajectory for students’ concept development; this trajectory was aimed at overcoming student difficulties found in the literature. An important aspect is that the new trajectory aims to facilitate students’ knowledge construction based on taking meaningful mathematical actions within a process of mathematical investigation. It integrates the dynamic geometry software GeoGeobra in such a way that students themselves can take real advantage of using technological tools in their concept development. The trajectory was tested, and promising results were found. Therefore, this study does have potential to reveal important educational implications to improve trigonometry instruction.

1.3 Motivation for the Study

My personal motivation for the study mainly stems from my past experiences in teaching and learning trigonometry. Once a friend of mine from my bachelor program told me that I would be “a trigonometry man” because I was doing projects on trigonometry for the most important courses. I worked on creating student-centered learning activities for trigonometric concepts in order to help students develop a conceptual understanding. The activities were basically aiming at students’ construction of trigonometric knowledge of connected concepts, but were limited in scope because they were about the first two contexts of trigonometry, namely triangle trigonometry based on similarity of triangles, and unit circle trigonometry based on connections from triangle trigonometry. While I was deciding what to study for my master thesis, it was not surprising that I chose trigonometry. I was still very interested in this topic mainly because of its complex nature. However, I wanted it to be different from my previous work. So, enlarging the scope to the trigonometric functions of real numbers was what I started considering. Meanwhile, the idea of integrating some GeoGebra applets, which had been previously used for a small research project at the AMSTEL Institute, was suggested by one of my supervisors. This increased my motivation to study this topic because these applets were addressing a difficulty of students about trigonometric graphs. In addition, it was clear that they could have been effective tools in terms of teaching trigonometry of real numbers.

The second motivating factor was my interest in mathematical understanding. I wanted to do something about students’ understanding, and it should be rooted in a deep perspective. It was not interesting for me to test a new method based on statistical methods. This was the main reason for choosing qualitative research methods.

Although it may have seemed to one of my supervisors that I was reading far too much, I can say now upon reflection that my rather voracious appetite to find and study articles has resulted in a much stronger and more thorough background knowledge than I otherwise would have had. Also, the more
literature I read, the more motivated I became. Reading a lot helped me obtain a good understanding of the literature and the nature of my topic. At the beginning, my reading scope was quite large. I was reading not only certain research studies on trigonometry, but also different theoretical perspectives such as APOS theory (e.g. Dubinsky & McDonald, 2001), and Reification theory (Sfard, 1991). Even though I did not directly use most of what I read, many works certainly helped me a lot in developing my research design.

1.4 Structure of the Thesis

This thesis consists of seven chapters. After this introduction chapter, the second one presents theoretical underpinnings of the study. This starts with a brief description of constructivism, and continues with understanding. Since technology is an important aspect of the study, theoretical considerations regarding the use of technology are then provided. The chapter ends with a review of literature on trigonometry and related topics which provide a basis for the whole research study.

The third chapter presents the education and research setting of the study. This includes brief descriptions of the Dutch education system, mathematics education in the Netherlands, and trigonometry in the Dutch mathematics curriculum, as well as the school where the research was conducted and the research cohort.

In the fourth chapter, a new framework for trigonometric understanding based on three trigonometric contexts and the connections among them is presented together with the important aspects within that model which drove the design and data analysis.

The reader should note that I have chosen to present my research questions later in Chapter 5, just before the description of my research design. Given the intimate connection between the research questions and the implementation design which was based on a hypothesized learning trajectory, it was considered useful to put them together in the same chapter. This chapter also provides details of the research instruments and methods of data analysis.

The sixth chapter includes the data analysis and results regarding the Diagnostic Test which aimed at assessing students’ prior knowledge and skills, the worksheet tasks given during the lesson sequence and regarding their concept development, and the Trigonometry Test and interviews, both at the end of the lessons, for students’ understanding. These results are presented in separate sections. The reader will note that there is considerable detail in each of these sections. This was done out of consideration that some details might contribute insight into the new approach. It was difficult to condense descriptions and so on, and yet succeed to retain and keep clear, some important parts. Results leading to the answers of the research questions are summarized at the end of each respective
section, and the important results from the Diagnostic Test that affected implementation are summarized at the end of that section.

The thesis ends with the conclusion and discussion chapter. The answers to the research questions are provided in separate sections. These are followed by a discussion regarding the effectiveness of the implementation design. The chapter ends with my reflections of the study and suggestions for future research.
2 THEORETICAL BACKGROUND

This chapter provides a theoretical base for the present study. A theoretical background is presented from the most general to the most specific issues. First, constructivism which constitutes a general perspective is discussed. Then, in accordance with the aim of the research, that is, assessing students’ understanding, the meaning of (mathematical) understanding is presented. This is followed by the theoretical considerations regarding the technology use within the study. Literature on trigonometry, also considering use of technology in trigonometry instruction and related mathematical areas, is presented at the end for the purpose of providing a literature base for this subject.

2.1 Constructivism

This study is underpinned by a perspective based on constructivism regarding the fundamentals of the learning process. Constructivism can briefly be defined as the learning theory which advocates that learning is a process of active knowledge construction by the individual. According to constructivism, students construct their own knowledge, and teaching cannot be considered as a transmission of knowledge from teachers or other sources to students. This is the main stance within constructivism contrary to the behaviorist view of learning.

The definition given above is a very brief one, and there are different views under constructivism. Duffy and Cunningham (1996) described constructivism as an umbrella term for these views, and they listed two commonalities among the views regarding constructivist educational implications:

- Learning is an active process of constructing knowledge rather than acquiring knowledge.
- Instruction is a process of supporting that construction rather than communicating knowledge (p.171).

Although these statements are clear in terms of implications for education in general, the different views within the umbrella term constructivism stem basically from the question of how the construction process occurs. There are two major schools of thought on this: individual (cognitive) constructivism and social constructivism (Duffy & Cunningham, 1996).

Individual constructivism takes a stance that the mind is in the head, and hence knowledge construction primarily happens in the head of an individual. This view is based on Piaget’s account of learning and development. Piaget emphasized the role of cognitive structures in knowledge construction (as discussed in Campbell, 2002). In his account, learning occurs through assimilation and accommodation which lead to knowledge construction. Assimilation is applying an existing scheme to a new situation. If assimilation leads to failure, the existing scheme is modified in the accommodation process. Assimilation and accommodation stress the importance of prior knowledge.
Social constructivism, on the other hand, considers the mind not limited to the head but also connected to social interactions. Hence, learning has important socio-cultural aspects. This view is mainly based on Vygotsky’s ideas on learning. He emphasized the role of other people, language, objects, and society and culture in the active process of knowledge construction (Jones & Brader-Araje, 2002). The idea of the zone of proximal development (ZPD) plays a crucial role in his account. ZPD is defined as the distance between the level of actual development and the more advanced level of potential development that comes into existence in interaction between more and less capable participants (Cole & Wertsch, 1996, p. 254). Such social interactions are mediated through language and artifacts which are tools shaping and transforming mental processes.

Cole and Wertsch (1996) argued that the views of Piaget and Vygotsky can be considered as complementary as a criticism of the debates considering them contradictory. They mentioned that Piaget did not deny the role of the social world in learning, and that Vygotsky also advocated an active construction process of knowledge. They mentioned a basic difference between these two figures’ accounts, but not a contradiction: the role of cultural artifacts. For Vygotsky, mental processes are artifact-mediated, and they determine where the mind is. However, there is not a counterpart for this issue in Piaget’s account.

Two other views considered under the umbrella term constructivism are: radical constructivism and enactivism. Although they are not referred to in the literature as often as individual and social constructivism, it may be helpful to explain them briefly. Radical constructivism differs from individual constructivism in that it considers that absolute knowledge is never attainable, and true representations of the empirical and experiential worlds are not possible (Ernest, 2010). Under enactivism, on the other hand, are the importance of individuals bodily embedded in the world, and the role of bodily movements and perceptions in human understanding and communication (Ernest, 2010). Ernest mentioned that enactivism is not very different from individual and radical constructivism. He suggested that Piaget’s equilibrium, which is the balance between assimilation and accommodation, is based on a similar biological model of interacting with the environment (p. 42).

Although some different arguments can be provided within different constructivist schools, they have roots in the same fundamental idea that learning is an active construction process. It may be more beneficial for education to consider that this construction process is both individual and social, and hence individual and social views complement each other. Duffy and Cunningham (1996) presented the grounding assumptions of their view of constructivism. Among them are the following, which I consider as the fundamentals of constructivism combining individual and social views:
Chapter 2 Theoretical Background

- All knowledge is constructed; all learning is a process of construction.
- Learning should occur in contexts to which it is relevant.
- Learning is mediated by tools and signs.
- Learning is an inherently social-dialogical activity.
- Learners are multidimensional participants in a sociocultural process.
- Knowing how we know is the ultimate human accomplishment.

2.2 Understanding

The Learning Principle in the Principles and Standards for School Mathematics asserted that students must learn mathematics with understanding (NCTM, 2000), and understanding is crucially important for this study because it is about students’ understanding. In this regard, two important questions need to be answered:

1. What is understanding?
2. Why is learning with understanding important?

What is Understanding?

Skemp (1976) distinguished between two kinds of understanding: instrumental and relational. Instrumental understanding is considered as rules without meanings, while relational understanding requires conceptual connections and explaining why the rules work. Skemp discussed certain advantages of promoting one of instrumental and relational understanding over the other. Instrumental understanding can be beneficial for a short-term case within a limited context, whereas relational understanding is better for long-term learning in a broader context. In effect, Skemp placed instrumental and relational understanding as two extremes separate from each other.

Hiebert and Carpenter (1992) described understanding based on mental connections:

“A mathematical idea or procedure or a fact is understood if it is a part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of mental representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.” (p. 67)

Obviously, Hiebert and Carpenter presented a different perspective from that of Skemp. In their view, understanding applies to procedures as well as relational aspects. On the other hand, Skemp’s instrumental understanding entails procedures without connections, and hence this may not be a kind of understanding in Hiebert and Carpenter’s view because understanding of procedures requires connections in the internal network as well.

Van de Walle (2007) gave another definition of understanding based on connections. Understanding is a measure of quality and quantity of connections that an idea has with existing ideas (p. 25). The *quality* and *quantity* terms are used in a similar way to *strength* and *number* in the explanation of
Hiebert and Carpenter. Such views reveal that understanding is not of two extreme kinds of either relational or instrumental but a combining issue. Van de Walle presented understanding as a model of a continuum between relational and instrumental understandings. He distinguished two kinds of knowledge: conceptual knowledge and procedural knowledge. Conceptual knowledge consists of connections among mathematical concepts. Procedural knowledge, on the other hand, is about rules and procedures used in solving routine tasks.

Up to this point, two issues are important. First, understanding is about mental links in an internal network. Hiebert and Carpenter (1992) explained that understanding develops when a new idea, procedure, or fact is connected to an existing mental network of connections. This mental network makes clear connections to constructivism which is a similar idea to Piaget’s assimilation and accommodation. Second, understanding can apply to both conceptual and procedural knowledge. Van de Walle (2007) advised teaching procedures with links to conceptual ideas. This indicates that understanding as mental connections can entail both conceptual or relational, and procedural or instrumental aspects.

The Learning Principle (NCTM, 2000) stated that learning with understanding is essential and possible, and conceptual knowledge, factual knowledge, and procedural fluency are all desirable. Indeed, an exact separation among such aspects does not seem possible. Skemp’s extreme ends may be explained by conceptual and procedural knowledge, but understanding should link these. Rittle-Johnson, Siegler, and Alibali (2001) provided another model of understanding as a continuum, as Van de Walle did. Their model is based on the assumption that conceptual understanding and procedural skills are iterative. They tested this model, and found that conceptual knowledge predicts the gains in procedural knowledge and also that gains in procedural skills predicts improvements in conceptual knowledge. So, neither grows independently of the other.

Kilpatrick, Swafford, and Findell (2001) presented probably the most comprehensive work combining different aspects related to understanding. They defined five strands of mathematical proficiency:

1. Conceptual understanding – comprehension of mathematical concepts, operations, and relations
2. Procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
3. Strategic knowledge – ability to formulate, represent, and solve mathematics problems
4. Adaptive reasoning – capacity for logical thought, reflection, explanation, and justification
5. Productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy (p. 116).

As the most important point, Kilpatrick and colleagues mentioned that these five strands are interwoven and interdependent in the development of mathematical proficiency. Ignoring certain
inconsistencies among the various definitions of the terms understanding, conceptual understanding, and conceptual knowledge,\(^1\) for the purpose of this study understanding is considered as a combining aspect based on connections among mathematical ideas, concepts, and procedures. In the case of trigonometry, the connections among different representations are very important because deeper understanding is needed in transiting among them.

Some researchers also mentioned social aspects of understanding by advocating that understanding is not only mental (e.g., Godino, 1996; Stylianides & Stylianides, 2007). Understanding is closely related to constructivist ideas; hence, developing understanding and communicating certainly have social dimensions. However, these dimensions will not be elaborated further here.

Why is learning with understanding important?

Perhaps the greatest benefit of learning with understanding is that it helps students use their knowledge in different situations (Skemp, 1976; NCTM, 2000; Stylianides & Stylianides, 2007). NCTM’s (2000) Learning Principle stated that conceptually grounded ideas are more accessible in new situations because connections facilitate the transfer of knowledge. Another benefit is that understanding can help one to remember things more easily (Skemp, 1976); and even that there is less to remember if someone understands a mathematical idea (Van de Walle, 2007). An important issue is that understanding can help to reconstruct certain procedures, since rich internal links among mathematical ideas may not be easily forgotten because they enhance memory (Van de Walle, 2007). Van de Walle also mentioned that learning with understanding is self-generative. This means that it can create a snowball effect so that understanding something can lead to learning new things. Learning with understanding has also motivational aspects. Hiebert and Wearne (2004) spoke of understanding as leading to the idea that mathematics is fun. Similarly, Van de Walle (2007) mentioned that understanding is intrinsically rewarding and that it improves attitudes and beliefs regarding mathematics. Lastly, understanding can provide students with the idea that mathematics is useful (Hiebert & Wearne, 2004).

\(^1\)For example, Rittle-Johnson et al. used conceptual understanding and conceptual knowledge interchangeably, and Kilpatrick et al.’s conceptual understanding is similar to understanding defined by Hiebert and Carpenter.
2.3 Technology and Cognitive Tools

The present study aimed at assessing students’ understanding of sine and cosine functions after the implementation of a learning trajectory in which the use of technology plays a crucial role. It appears that technology can facilitate students’ learning and understanding, yet the effectiveness of technological tools in mathematics learning and teaching is not trivial. For instance, using a tool only for the purpose of demonstrating a process for students by a teacher may not result in better learning. In this section, a theoretical background is presented regarding technology integration within this study. A set of guidelines for effective use of technology in mathematics lessons (Garofalo, Drier, Harper, Timmerman, & Shockey, 2000), and the notion of cognitive tools (Lajoie & Derry, 1993; Van Joolingen, 1999) fit very well with informing the reader on the use of technology within this study.

A dynamic geometry software package, namely GeoGebra, was used as a cognitive tool in this study. Cognitive tool is a term elaborated by Lajoie and Derry (1993) in terms of computers as tools facilitating students’ learning. Lajoie (1993) stressed four functions of cognitive tools: “(a) support cognitive processes, such as memory and metacognitive processes; (b) share the cognitive load by providing support for lower level cognitive skills so that resources are left over for higher order thinking skills; (c) allow the learners to engage in cognitive activities that would be out of their reach otherwise; and (d) allow learners to generate and test hypotheses in the context of problem solving.” (p. 261) These functions are not mutually independent (Lajoie, 1993), and cognitive tools can be a very broad term. Van Joolingen’s (1999) definition constitutes a basis for the present study:

“... tool use by student communities can help them perform and grow their current developmental capabilities” (Vygotsky, 1978, as cited in Derry & Lajoie, 1993, p.9). This indicates that cognitive tools can be a major social component functioning in the zone of proximal development of students. A major role of a cognitive tool is to help learners by reducing the load in working memory by carrying out a part of a cognitive process for the learners. The learning process

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2 The role of technology along with other components of a learning environment is presented as a didactic model in the Research Design Chapter (Chapter 5).

3 GeoGebra is an open dynamic learning environment combining algebra and geometry. Its development was initiated by Marten Hohenwarter in 2002 in his Master Research Project, progressed in his PhD study and postdoc work. It is accessible on www.geogebra.org.
can be difficult and complex for learners, and hence cognitive tools may be very supportive. For instance, Van Joolingen (1999) gave the example that multiple processes can be carried out by cognitive tools. Indeed, such complexity can be difficult for the limited human mind, and a cognitive tool can take over a part of the cognitive process. Another role of cognitive tools can be externalizing internal representations (Jonassen, 2003).

The role of GeoGebra as a cognitive tool in this study is based on “mind extension” in that without using it, certain processes could not be performed due to limited working memory of individuals. One of these processes was dynamic relations between a moving point on the unit square, its vertical position and the travelled distance through the graph of the relevant function. The dynamic nature of GeoGebra helped students by performing these simultaneous processes for them. Another process was the transition from the unit square to the unit circle by increasing the number of sides of polygons at each step, and observing the changes on the polygon and the graph. This is a quite difficult task for students to carry out without a cognitive tool like GeoGebra. It performed this part of the process for the learners, but the important point was that the students were directing the process by increasing the number of sides with a slider. Hence, the cognitive tool performed a part of the process, which was complex for the learners, but under their control.

Such dynamic changes demonstrated by GeoGebra as a cognitive tool provided students with the opportunities to observe, realize, and investigate mathematical relationships. So, the use of technology went far beyond using it only as a tool for demonstrations. Garofalo et al. (2000) gave five guidelines for an appropriate use of technology in mathematics lessons which illustrate how GeoGebra as a cognitive tool is integrated in this research. These five guidelines together with some remarks regarding their use in the present study [in italics] are:

- **Introduce technology in context:** *Technology in this study was used in the context of content-based activities.*

- **Address worthwhile mathematics with appropriate pedagogy:** *Technology use did not aim to carry out pre-determined step-by-step procedures, but to help students’ investigations and promote their mathematical understanding.*

- **Take advantage of technology:** *This coincides quite well with the notion of cognitive tool. Technology helped students in an interactive and interconnected way by removing cognitive constraints.*

- **Connect mathematics topics:** *Technology was used as a tool to connect mathematical topics of functions, angle measure, periodicity, trigonometry.*

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4 All related details are explained in Chapter 5.
• Incorporate multiple representations: An important aim of technology use was to support students to understand connections among different representational systems in trigonometry. Technology was used especially to help establish connections between unit circle and graphs.

Garofalo et al. (2000) emphasized that these five guidelines are interconnected. Although I tried to provide briefly how each guideline was utilized, some italicized statements can be based on more than one guideline. For instance, using technology in a way connecting different representations or connecting different mathematics topics can also be considered under the guideline of taking advantage of technology. Nevertheless, it is clear that these guidelines provide a clear and helpful framework to incorporate technology in mathematics lessons for effective learning of students. Note also that the use of the guidelines for the use of GeoGebra as a cognitive tool is within the general perspective of constructivism.

2.4 Research Literature on Trigonometry and Related Subject

Trigonometry is an important school subject, not only in mathematics but also in other disciplines. It has many important applications in engineering, astronomy, physics, architecture, and so on. In terms of mathematics, it is one of the fundamental topics in the transition to advanced mathematics and its applications. A firm understanding of trigonometric functions is required in calculus and analysis. Hence, trigonometry has an important place in the mathematics curriculum in many countries even though its meaning may change from country to country at secondary school (e.g. Delice & Roper, 2006).

Trigonometry has a complex nature. It combines different algebraic, geometric, and graphical concepts and procedures. For instance, trigonometric functions are different from other forms of functions in that they cannot be computed directly by carrying out certain arithmetic calculations revealed by an algebraic formula. Evaluating trigonometric functions requires both geometric reasoning and algebraic reasoning. This complex nature of the topic makes it challenging for students to understand it conceptually.

Research on trigonometry is sparse (Weber, 2005; Moore, 2010), and there appear to be only a few researchers investigating students’ understanding relating to different instructional approaches, even though trigonometry has been often reported as a difficult subject for students.

This section presents a literature background for the present study. First, studies are reviewed in terms of understanding of trigonometry. This is followed by the review of the studies examining the use of technology in trigonometry teaching. Because of the narrowness of the literature, and the interrelated nature of trigonometric functions with other mathematical subjects, the section ends with
2.4.1 Understanding of Trigonometry

Despite limited research shows that students have fragmented and incomplete understanding of trigonometric functions (Weber, 2005; Brown, 2005; Challenger, 2009).

In one of a few studies which investigated students’ understanding of trigonometric functions, Weber (2005) examined college students’ understanding of trigonometric functions with an experimental design based on the notion of procept. He implemented a lesson design based on activities which are mostly hands-on, like students’ construction of the unit circle, and drawing angles and related line segments corresponding to their trigonometric values. The students encountered a learning trajectory of procedures based on their physical constructions, process based on repeated procedures, and procept based on their reflections on the processes. Weber found that experimental group of students performed much better in the posttest than did the control group who had had traditional, textbook-driven lessons. There was evidence that the experimental students demonstrated a strong understanding since they considered trigonometric functions both as processes and the results of these processes, on which they could infer the properties of these functions. However, it is also clear that the experimental students were trained in this direction within the instructional design. For instance, one of the test items examined students’ estimation of a sine value, and control group performed worse than the experimental group. When it is considered that experimental group did such tasks in the lessons, and that such activities are not generally included in traditional methods, it is not surprising to find the experimental students outperforming the control students. Nevertheless, Weber’s findings indicated that such an instructional design may help students develop a connected understanding.

Weber (2005) stated that the advantage of his unit circle trigonometry design was the process used by the students to create unit circle representations of the trigonometric functions, and that he did not think every unit circle method could lead to effective results (p. 107). This idea confirms the findings of the research conducted by Kendal and Stacey (1997), who compared two methods of teaching trigonometry. They found that teaching the ratio method first promoted a better understanding of the trigonometric functions compared to the teaching starting with the unit circle method. However, they assessed students’ understanding with tasks of solving triangles. Therefore, it is expected that ratio method would be more effective. It is important to note that the results of investigations into students’

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5 The notion of procept was defined by Gray and Tall (1994) as “the amalgam of the three components: a process, that produces a mathematical object, and a symbol which is used to represent either process or object” (as cited in Weber, 2005, p.92).
understanding depends largely on how the assessment is done. Kendal and Stacey’s study had a rather weak design in that they attempted to assess students understanding only through solving triangle tasks which are actually not very easily solvable by using unit circle trigonometry.

The studies of Weber, and of Kendal and Stacey\(^6\) reveal that students’ understanding is highly related to the aspects emphasized in the lesson design. Although the students were found to possess deep understandings in certain conditions, it can be claimed that this understanding relied on the main focus of the teaching. The latter is consistent with the research of Delice and Roper (2006). In a comparison study of English and Turkish students’ performance on trigonometry, Delice and Roper found that Turkish students were good at the algebraic aspects of trigonometry like simplifying trigonometric expressions, while English students were good at trigonometric applications in real life situations. They mentioned that differences in mathematics curricula and in teaching methods in the two countries were the reason for this performance difference. Challenger (2009) emphasized the same issue as “What you get is what you teach.”

It is difficult to find a firm framework for trigonometry understanding in the literature. Brown (2005) came up with a model of students’ understanding of trigonometric functions, but she focused on a very narrow side of trigonometry by including only the trigonometry of angles in degrees. Her model depicts that a student may consider the sine of an angle in three different ways: as a ratio, as a distance, and as a coordinate. Brown showed that students who developed the most robust understanding were able to coordinate these different components. In this regard, a student with an integrated understanding is at the center of her triangular model of these three components. If a student cannot develop such an integrated view, he or she may have problems regarding certain trigonometric aspects. For instance, Brown found that students who had a distance-dominant understanding had difficulties with the sign of a trigonometric value such as \(\sin 210^\circ\).

Although Brown’s model is useful for considering students’ trigonometric understanding, it is far from being a model which can be used for complete understanding of trigonometric functions. For instance the model is based on the connections between triangle trigonometry and the unit circle, but does not tell us anything about the relations between the unit circle and the graphs of trigonometric functions. The latter point seems crucial if students are to gain an integrated understanding of the different aspects of trigonometry. For this reason, a model which has a broader focus is proposed in the present study. The reader will find a full discussion of the model in Chapter 4.

\(^6\) These two studies were the only studies found in the literature in which different teaching methods and students’ understanding were explored in the same research study.
2.4.2 Trigonometry Instruction with Technology

Literature on technology-incorporated trigonometry teaching is also sparse and limited in scope. Although research shows positive effects of the use of technology on students’ learning and performance (Blackett & Tall, 1991; Choi-Koh, 2003; Ross, Bruce, & Sibbald, 2011; Moore, 2009; Zengin, Furkan, & Kutluca, 2012), possible effects of technology on students’ concept development remain rather unexplained. In an extensive search, no studies appeared to examine use of a technological tool thoroughly in a way that directly supported students’ concept development of trigonometric functions which could be used to underpin the present study.

Blackett and Tall (1991) demonstrated that use of a computer software package addressing triangle trigonometry had a positive effect on students’ learning. The main advantage of that software compared to the teaching without technology was that it enabled students to explore relationships between numeric and visual representations of trigonometric ratios in right triangles. Since then there have been many changes in what technology can provide for students’ learning. Kissane and Kemp (2009) exemplified the use of technology in teaching of trigonometric concepts and procedures ranging from calculators to computers. Calculators and computer programs based on computer algebra systems, and dynamic geometry software all provide students with opportunities that are not available without technology, such as investigating multiple representations of functions in a dynamic way.

Regarding the use of computer algebra systems in trigonometry instruction, Stacey and Ball (2001) noted that calculators and computer packages including computer algebra systems could be used for solving triangles, trigonometric identities, and function properties. They also mentioned that using technology in solving triangles has the advantage of calculations with exact values. This is supported by Delice and Roper (2006) who found that English students using calculators to deal with real data, unlike Turkish students, were better in word problems. Johnson and Walker (2011) also mentioned the factor of exact values, and they found that students prefer to use a hand-held calculator, when the problems are difficult, to speed the process, and to verify their solutions. They also investigated how students make sense of mathematics by using technology, and came to the conclusion that technology helps students to make sense of challenging problems.

Johnson and Walker (2011) emphasized that technology can also be used for concept development in students, which seems to be another neglected area in the literature. In 2003, on the other hand, a study conducted by Choi-Koh investigated the effect of a graphing calculator on a single student’s patterns of mathematical thinking. Choi-Koh (2003) reported that the student used the graphing calculator within three stages of mathematical thinking. First, he used it to graph the sine function, and to observe the changes in the graphs with varying coefficients in the intuitive stage. Then in the...
operative stage, the calculator helped him explain trigonometric properties. Lastly in the applicative stage, the student used it to analyze and reflect on trigonometric relations. Choi-Koh suggested that the movement between algebraic functions and their graphs was the main factor helping students proceed to the operative and applicative stages.

Along with computer algebra systems, other researchers also investigated the effects of dynamic geometry software. Zengin, Furkan, and Kutluca (2012) compared GeoGebra-integrated constructivist teaching with a solely constructivist one without GeoGebra, in a five week trigonometry course. Although they did not provide enough information about their course design in their article, they reported that constructivist teaching with technology was more effective in terms of students’ learning. They mentioned that GeoGebra is both a computer algebra system and dynamic geometry software because it includes both symbolic and visualization features related to coordinates, equations and functions, along with geometric concepts and dynamic relations.

Johari, Chan, Ramli, and Ahmat (2010) explored the effect of Geometer’s Sketchpad on students’ understanding of trigonometric functions in four categories: shape, maximum, minimum, and cycle. As a weak point of their report, they were not explicit about what they meant by these four categories. They found that students using Geometer’s Sketchpad demonstrated better understanding in the shape and cycle categories. This does not seem surprising if one considers that students probably do not need the support of a technological tool to learn the maximum and minimum values on the graph of the sine function. Johari et al.’s study also reveals that Geometer’s sketchpad students did not show understanding of drawing graphs. This finding could be directly related to design issues in their lessons, but it cannot be judged because they did not provide information about how they used technology in their design.

Using technology in lessons does not lead automatically to better results in terms of students’ learning and understanding. Of crucial importance are how technological tools are used in lessons, the kind of support students receive, and interactions between the tools and students. In this regard, Thompson (2002) mentioned the use of technological tools as didactic objects which promotes reflective mathematics discourse for knowledge construction, but that an object is not didactic on its own (as cited in Moore, 2009, p. 1481). Moore (2009) used Geometer’s Sketchpad as a didactic object enabling supportive mathematical discourse as emphasized by Thompson. Moore used two applets for angle measure and two covarying elements, namely arc length and vertical position, in a dynamic way for students to use, and found positive effects on students’ understanding based on quantitative and covariational reasoning. He mentioned that such applets could lead to understanding of the sine graph which is not easy to draw because of the concavity.
Moore’s (2009) study indicates an effective use of technology in mathematics lessons. To increase effectiveness of technology for students’ learning, technology must be integrated into classroom teaching in ways which will foster students’ concept development and understanding, and address their learning difficulties. Ross, Bruce, and Sibbald (2011) investigated the placement of computer assisted learning in the teaching sequence as a factor which may also relate to its effectiveness. They found that, in the case of transformations of the trigonometric functions, using a dynamic software package after whole class teaching of core concepts was more effective than beginning the learning unit with the software. Also at the end of the learning unit there was almost no difference between the students’ performance levels in the two groups. Hence, they suggested an integrated method of whole class teaching and technology use.

2.4.3 Related Subjects

Trigonometry is a complex subject and a deep understanding of it requires making connections and transitions among related concepts. In this regard, students’ difficulties in trigonometric functions are attributed to the lack of understanding regarding the related areas such as functions, angle measure, and graphs (e.g. Orhun, 2001; Kang, 2003; Moore, 2010). Gur (2009) mentioned that trigonometry and other related concepts are abstract and non-intuitive, and teaching based on lecturing does not help students overcome their learning difficulties.

In this section a literature review of the subjects related to trigonometry is presented. Note that this includes not only research findings specific to trigonometry learning and understanding, but also theoretical considerations which constitute a basis for covering these subjects within the present study.

Angle Measure

In general, trigonometry is first learnt as ratios in right triangles. In this phase, it is limited to the angles in degrees smaller than 90°. Then in a unit circle model this scope is expanded to any angle and also angles in radians. A transition from radians as an angle measure to real numbers as the domain of the trigonometric functions completes the picture. A rich trigonometric understanding requires being able to make transitions among these. Students’ difficulties with the angle measure can be considered as the most basic problem prohibiting them from developing a deep trigonometric understanding. Research shows that students have difficulties and incomplete understanding regarding the angle measure, both in degrees (e.g. Martinez-Sierra, 2008; Brown, 2005) and in radians (e.g. Moore, 2010, Akkoç & Akbaş Gül, 2010; Topçu, Ketil, Akkoç, Yılmaz, & Önder, 2006; Orhun, 2001).
Martinez-Sierra (2008) reported students’ difficulties with the angle measure in degrees, especially with negative angles and angles larger than 360°. Brown (2005) found that some students developed a fragile concept of rotation angles hindering their performance in finding trigonometric ratios of angles because they had problems with drawing rotation angles in the unit circle. Although students may have such difficulties with angles in degrees, research shows that students’ trigonometry conception is mostly based on angles in degrees, and this may pose some other problems such as considering trigonometric values of real numbers or radians stated without multiples of π as trigonometric values of degrees (Orhun, 2001; Topçu et al., 2006; Akkoç & Akbaş Gül, 2010). This is because students cannot develop strong understanding of radians to be able to transit to real numbers as the domain of trigonometric functions.

Along with students’ difficulties with angles in degrees, Martinez-Sierra (2008) suggested that use of radians is among the conceptual breaks in the construction of trigonometric functions. Akkoç and Akbaş Gül (2010) found that students could not define radian concept as the ratio of two lengths nor relate angle measures to the lengths of subtending arcs. Orhun (2001) also reported students’ difficulties to find the measure of arcs subtending given angles, and vice versa, noting that the calculation of angles from given arc lengths was more difficult for the students. Another student difficulty found in the literature is about recognizing real numbers as radians (Orhun, 2001; Topçu et al., 2006; Akkoç & Akbaş Gül, 2010). Orhun (2001), and Akkoç and Akbaş Gül (2010) found that students considered π radians as equal to 180, not as a number close to 3.14. Topçu et al. (2006) showed that such difficulties also exist in both pre-service and in-service teachers. They investigated concept maps of pre-service and in-service teachers and found that those with stronger understanding of radians could use the unit circle and its connections to other trigonometric concepts.

To address such problems with radians, some researchers suggested a way of teaching radians by explicating clear connections to arcs (Moore, 2010; Kang, 2003; Akkoç and Akbaş Gül, 2010). Akkoç and Akbaş Gül (2010), for instance, investigated a learning trajectory based on computer programs exploring the relations between arc lengths and angles in different circles, and defining the trigonometric functions primarily in the domain of arc lengths. Their results show that such an approach can help students having learning difficulties with the radian concept.

Moore (2010) examined a teaching method also putting an emphasis on arc lengths to set the relationships between angles and arcs. He based his study on quantitative and covariational reasoning. He suggested defining angles, both degrees and radians, as fractions of the circumference of a circle: 1 degree is the measure of an angle corresponding to the (1/360)th of any circle’s circumference while 1 radian corresponds to the (1/2π)th of any circle’s circumference. These quantitative relations indicate how to interconnect degrees and radians, as well as relating angles to arc lengths and the radius of a
circle. Moore (2010) found that such quantitative reasoning could constitute a fundamental understanding, and that investigating covariational changes of the arc lengths and vertical positions could be effective for students’ understanding of trigonometric graphs based on the fundamental concept of radian.

Considering the importance of the radian concept for students’ trigonometric understanding, the present study put a great emphasis on this concept. To overcome students’ difficulties, trigonometric functions were covered first through arc lengths. The transition from arcs to angles was expected to help students understand the connection between arcs and angles better, as well as the role of radians in transition to real numbers as the domain of trigonometric functions.

**Function Concept**

The most crucial aspect of trigonometry is perhaps related to the concept of function. Research shows that the notion of function is difficult in the context of trigonometry for students to conceptualize trigonometric concepts (sine, cosine, tangent) as functions (Weber, 2005; Challenger, 2009). Many studies on students’ understanding of trigonometric functions (e.g. Weber, 2005; Brown 2005; Kendal & Stacey, 1997) considered trigonometry based on functions of angles only. However, analytic trigonometry based on trigonometric functions of real numbers is as important for students to understand as the trigonometry of angles. The former is the part which applies to the real periodic phenomena. However, research on trigonometry appears to have ignored this part.

Grabovskij and Kotel’nikov, as long ago as 1971, emphasized the significance of the trigonometric functions of numerical arguments, and advised use of kinematic methods such as angular velocity as a way to avoid teaching trigonometric functions exclusively in terms of geometric ideas. Such a method can be effective in helping students to understand some connections to certain real life situations, but it is probably not a good way to introduce them to analytic trigonometry because of possible difficulties related to the physical concepts rather than the mathematical ones. Teaching analytic trigonometry could be done through radians as a way of connecting related algebraic and geometric underpinnings. Research shows that this is not an easy process, and students have significant difficulties in conceptualizing trigonometric functions in the domain of real numbers (Kang, 2003; Orhun, 2001). The study of Kang (2003) shows that 21% of the students claimed f(x)=sin x means the ratio of opposite to hypotenuse, and that 60% could not define y=sin x as a function. Orhun (2001) also mentioned the difficulties of students in the trigonometry of numbers. For instance, the students considered sine 30 as the sine of 30°.

For the present study, which puts a great emphasis on the function concept in the context of trigonometry, the process conception of function is of crucial importance, as it was defined by
Breidenbach, Dubinsky, Hawks and Nichols (1992), and Dubinsky and Harel (1992). Breidenbach et al. (1992), and Dubinsky and Harel (1992) defined four conceptions of function: predfunction, action, process, and object. Predfunction refers to a conception displaying no understanding. Action means a repeatable physical or mental manipulation that transforms an object. For example, plugging numbers into an algebraic expression is an action. Process, on the other hand, is an interiorized action without necessarily running all of the specific steps of the action. Lastly, object refers to the state if it is possible to perform actions on it.

Breidenbach et al. (1992) explained the difference of actions from processes as:

“Both actions and processes transform objects. The main difference between an action and a process is the need in the former for an explicit recipe or formula that describes the transformation.” (p.278)

Breidenbach and colleagues found that many students think there has to be an expression or at least the presence of variables to indicate inputs and outputs of a function. In this case, the conception of these students depicts an action view. However, when students think of a function as a total process beginning with some kind of objects (not necessarily numbers) and doing something to them (not necessarily described by an explicit formula or expression) to obtain new objects, such students have a process view.

Breidenbach et al. (1992) mentioned that understanding the function concept must include a process view. This is extremely important in terms of understanding trigonometric concepts as functions within the present study. Trigonometric functions cannot be calculated with simple arithmetic operations with respect to an algebraic formula because a trigonometric function in the form \( y = \sin x \) has no overt action to carry out. It requires a more complex process including different algebraic and geometric aspects. Hence, students need to have a process view of function to conceptualize trigonometric expressions as functions.

Tall and Bakar (1992) found that students develop prototypes for functions, and that they cannot understand the nature of the function concept although they are able to use it in practical mathematics. It is worth noting that some students consider functions as familiar formulae like \( y = x^2 \), or another parabola, or a trigonometric expression. However, before they are introduced to trigonometric functions or in the introduction, it may be too challenging for students to begin already to consider them as functions. Even (1993) found similar prototypes in prospective secondary teachers. Tall and Bakar (1992) and Even (1993) also reported that many students and prospective teachers were not able to utilize the formal definition of functions. Prospective teachers could use the vertical line test,\(^7\) but could not explain why it works. Elia, Paraooura, Gagatsis, Gravvani and Spyrou (2008) confirmed

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\(^7\) A graph represents a function if no vertical line intercepts it at more than one point; otherwise the single-valuedness would be ruined.
this in another study, and mentioned the lack of a systematic discussion of the formal definition in an explicit manner in mathematics lessons. Dubinsky and Harel (1992) also explained uniqueness in the formal definition in the process view, and found it to be one of the factors affecting students’ process conceptions. Their process notion entails a unique finishing point or output, whereas students may be confused by the idea of an injective function including also a uniqueness of the starting point or input.

Tall, McGowen and DeMarois (2000) suggested an idea of a function machine for the development of a process conception of function based on an input and output mechanism. They suggested as an advantage that such a model can apply to different function representations like formulas, graphs and Venn diagrams in a way that reflects the object-process duality of the concept.

**Graphs and Cartesian Connection**

The graphs of trigonometric functions are among the important aspects of trigonometric understanding. Research shows that students find the graphs of trigonometric functions difficult (Kutluca & Baki, 2009). Brown (2005) showed that students were unsuccessful at making even a very basic connection from the unit circle to the graph of the sine function, that is the transition between a position on the unit circle and the point that corresponds to it on the graph of \( y = \sin x \). Yet, such transitions are very important for a robust trigonometric understanding. Orhun (2001) advised teaching trigonometric functions through their graphs. Orhun’s advice can be effective in addressing students’ problems regarding the graphs of trigonometric functions and to transit to real numbers from radians. The teaching of trigonometric functions in the present study started with an emphasis on the graph of the sine function.

Graphs are important not only for trigonometric functions but also for other functions because they constitute one of the representations of a function. Connections between algebraic and graphical representations of functions are considered to be an important aspect of students’ understanding, but such connections are not easy for most students. Thus, teaching graphs in a way that is connected to algebraic representations is advised by a number of researchers (e.g., Van Dyke, 1994; Knuth, 2000).

Tall and Bakar (1992) revealed prototypes also for the graphs of functions. Their study shows that students could not distinguish the graphs representing functions from those which do not. They also found that students were looking for continuity and a regularity in graphs in order to consider them as functions. Such issues may be due to a lack of a process conception of the function as well as being unable to utilize the formal definition.

An important framework for students’ understanding regarding graphs based on connections between algebraic and graphical representations of functions is the Cartesian Connection (Schoenfeld, Smith,
& Arcavi, 1993, as cited in Brown, 2005; and Knuth, 2000). This includes two connections defined by Schoenfeld et al. (1993):

**Connection A:** The point \((x_0, y_0)\) is on the graph of the function \(y = f(x)\) if and only if the point satisfies the equation, that is \(y_0 = f(x_0)\).

**Connection B:** In the Cartesian plane, specific algebraic expressions have graphical identities. For example, \((y_2 - y_1)\) is a directed line segment with both direction and magnitude specified by mathematical convention. (as cited in Brown, 2005, p. 45-46)

Knuth (2000) investigated students’ abilities in terms of Connection A, only in the graph to equation direction in linear functions, and found that students failed to recognize or create the connections based on the graphical representations.

In the case of trigonometry, these connections may be more challenging for students because a strong trigonometric understanding does not only require making connections between algebraic and graphical representations, but also making transitions between the unit circle and graphs, both of which are defined on the Cartesian plane. Connection A is important for students to recognize points like \((x, \cos x)\) on trigonometric graphs. Connection B, on the other hand, is important for realizing coordinates and line segments as trigonometric values defined with the help of the unit circle.

**Periodicity**

Periodicity is an important aspect of trigonometric functions which provides applications in explaining some physical real life phenomena such as harmonic motion and waves. However, research on trigonometric functions has not examined periodicity although it is an important aspect of trigonometric understanding. From a mathematical point of view, the meaning of periodicity, and connections between trigonometric graphs and the unit circle in regard to periodicity are all important.

Since periodicity has not been studied in the context of trigonometry, two studies from a general mathematical stance seem important for the present study. In one of these studies, Van Dormolen and Zaslavsky (2003) discussed the issues revealed by some definitions of periodicity. Two of them:

- **Definition A:** A function \(f\) is called periodic if there exists a non-zero number \(p\), such that for every \(x\) that belongs to the domain of \(f\), the following conditions are fulfilled:
  - \((a)\) \(x \pm p\) belongs to the domain of \(f\),
  - \((b)\) \(f(x \pm p) = f(x)\).

- **Definition G:** A function is called periodic if there exists a non-trivial translation of the graph of \(f\) along the horizontal axis such that the image coincides with the original. (p.100)

Van Dormolen and Zaslavsky mentioned that these two definitions are equivalent, but definition A is analytic whereas definition G is global. They also distinguished them in that A is from a point-wise perspective while G is about the function as a whole.
Shama (1998) discussed two perspectives of periodicity: one as a process view as a dynamic collection of elements, and the other as whole unified structures in accordance with Gestalt theory. Shama’s two perspectives reflect the two definitions of Van Dormolen and Zaslavsky. Shama used these two perspectives to explain students’ errors and preferences regarding periodicity. The results of this study indicate that students made the error of considering non-periodic functions as periodic due to their process conception. For example, Shama found that students consider a repetitive pattern as an indication of periodicity even if it is periodic only in a pictorial sense. Shama explained this as a result of transferring properties of the process to its products. Regarding the Gestalt view, students’ mistakes stem from being unable to see the graphs of functions as a whole, but instead focusing on certain parts. For example, students tend to identify the part of a periodic graph between non-continuity points as a period, or they tend to view the area surrounded by a line or a curve and the x-axis together as the period.

These points of Shama (1998) may pose a difficulty for students regarding trigonometric graphs. For instance, students may think the interval \([0, \pi]\) as the period of the sine function by considering the closed curve in this interval as a whole, and by identifying this as a repeating pattern in a pictorial sense. Such difficulties can be overcome if students can attain a meaningful understanding of periodicity. In this regard, a general definition as definition G of Van Dormolen and Zaslavsky (2003) could be easier at the introductory level. In addition, connecting periodicity to both the trigonometric graphs and the unit circle can prevent students from making errors reported by Shama (1998).
3 EDUCATION & RESEARCH SETTING

The education and research setting are presented in this chapter. This study was carried out with secondary school students in Amsterdam, the Netherlands. Hence, it might be useful to look briefly at the Dutch educational system in the first section. Then the Dutch mathematics education and the trigonometry subject in the Dutch curriculum are presented because they are closely related to the scope of the research reported in this thesis. These sections are followed by the description of the school where the research took place together with the cooperating teacher. Then a description of the research cohort is provided. The chapter ends with the timeline of the research intervention.

3.1 The Dutch Education System

The Dutch educational system is organized under three main levels: primary, secondary, and tertiary. Education is compulsory for ages 5 to 16, but most children start primary education at the age of 4. Students doing a pre-vocational education have to continue a kind of compulsory partial schooling up to the age 18, and the other streams also continue at least up to age 18. Compulsory education is free of charge in public schools.

Dutch secondary education is organized under three main streams: pre-vocational secondary education (VMBO), senior general secondary education (HAVO), and pre-university education (VWO). VMBO lasts 4 years, and prepares students for secondary vocational education (MBO). HAVO lasts 5 years, and prepares students for higher professional education (HBO), while VWO lasts 6 years, and prepares students for university education (WO).

The first three years of HAVO and VWO form what is called basic secondary education. Specialization takes place in the upper levels, the last two years of HAVO and the last three years of VWO. Students can choose among four specialized subject combinations:

- Science and Technology
- Science and Health
- Economics and Society
- Culture and Society

In the last year of HAVO and VWO, students take national examinations which along with their school examinations affect their placement in tertiary education.

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8 More information can be found at http://www.govemment.nl/issues/education
3.2 Mathematics Education in the Netherlands

The Netherlands is among the high scoring countries in international studies like TIMMS and PISA. However, the mathematical skills of Dutch students are a controversial issue in the country.

Realistic mathematics education (RME) is known to be the Dutch approach to mathematics education. RME is based on Hans Freudenthal’s idea of mathematics as a human activity, and has been shaping reforms in mathematics education since the 1970’s (Van den Heuvel-Panhuizen & Wijers, 2005). Although RME can in principle provide students great opportunities to learn and understand mathematics effectively in principle, it does not seem that this is always the case in practice. Van den Heuvel-Panhuizen and Wijers (2005) stated that the national mathematics standards in the primary and lower-secondary levels are stated to be consistent with RME principles, and previous mathematics education reforms in the Netherlands have promoted the use of textbooks prepared according to the RME principles in that they provide context-based problems. The main problem seems to be that in practice RME is implemented mostly by means of using these textbooks. Researcher’s observations also confirm this. Indeed, mathematics education is highly textbook driven, and whole class teaching is not given enough importance (Van den Heuvel-Panhuizen, 2010).

The debates, apparently, start about the insufficient mathematical background of students entering secondary and even university level. Although RME-textbook driven mathematics education may promote achievement in solving context-based problems, Dutch students seem to be lacking certain algebraic skills. Van de Craats (2010) exemplified Dutch students’ entrance problems to university level mathematics in a presentation in which he stated that beginning Bachelor students very often make mistakes such as $(a+b)^2=a^2+b^2$, $1/a + 1/b = 1/(a+b)$.

Secondary students can choose one of the four mathematics packages according to their specialization and personal interests in the fourth year of HAVO and VWO: Math A, B, C, and D. Science and Technology students must take Math B; Science and Health, and Economics and Society students must take either Math A or Math B; Culture and Society students must take Math C, or one of Math A or B in VWO. Math D is, on the other hand, not a required course, but an optional one in combination with Math B in the Science and Technology profile. Hence, they can be ordered as Math D, B, A, and C, according to the level of the mathematics covered in these courses.

VWO students are in general more successful in mathematics compared to HAVO students, and the mathematics curriculum in VWO is at a more advanced level. For instance, HAVO students do not necessarily learn trigonometric functions; it is a choice of the school or mathematics teacher. At the

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9 The researcher has heard this quite often from secondary and tertiary level teachers as well as witnessing the situation in his internship in 2009. In addition, similar situations were faced in this study. For example, it was found that some students could not calculate $(-1)^2 = -1$, and some mixed up the formulae of the circumference and the area of a circle.
school where this research took place, teachers had decided not to teach trigonometric functions to HAVO students.

3.3 Trigonometry in the Dutch Mathematics Curriculum

In general, Dutch students learn trigonometry for the first time at lower levels of HAVO and VWO as trigonometric ratios in right triangles. Then in the fourth year, they learn the unit circle definitions and the graphs of sine and cosine functions in Math B. In the upper years, they continue with more advanced topics such as trigonometric equations.

The trigonometry chapter in the fourth year is a kind of introduction to trigonometric functions through the unit circle and radians. Because the teaching is mostly based on textbooks, looking at how the topic is covered in the textbook (Reichard, et al., 2007) used by students illustrates the traditional way of teaching of trigonometric functions, a method quite different to the new approach presented in this study.

The textbook approach follows the traditional order of teaching the three contexts of trigonometry: triangle, unit circle, and graphs. However, the connections among these contexts are not put clearly, which may lead to a fragmented understanding of trigonometric functions. The textbook chapter starts with the definition of the unit circle and rotation angles in degrees. Then sine and cosine are defined as the corresponding vertical and horizontal positions. The transition from triangle definitions to the unit circle is explained as in Figure 3.1.

![Figure 3.1: Unit circle definitions of trigonometric functions in the textbook](image)

Hereafter radians are defined, and it is explained how to convert degrees and radians to each other. After this, simple trigonometric equations, sin(A)=C and cos(A)= C where C = -1, 0, 1, are explained followed by exercises. Some other equations follow. Finally sine and cosine are defined as functions without an emphasis on real numbers. It is mentioned that the function is defined with numbers (getal in Dutch) instead of angles in radians. However, the transition from angles to real numbers is not covered in a clear way. Furthermore, the graph is just given, but no further information regarding how it can be drawn is included. Figure 3.2 shows these.
Although radians are defined earlier in a way that the measure of angle in radians is equal to the length of the subtending arc, this definition is too limited to help students understand the relations between arcs and angles.

The chapter of the textbook ends with transformations of the sine graph, which could be quite difficult for students who have not been given a chance to understand the sine graph itself.

### 3.4 School and Cooperating Teacher

This study was conducted at Fons Vitae Lyceum, which is a secondary school offering HAVO and VWO streams in a central location in Amsterdam. The school supplies quite nice facilities for the students such as a library, laboratories, a multimedia room, and a study hall. The school has invested a lot in technological tools. For instance, every classroom is equipped with an interactive white board.

The school was chosen by the researcher for two reasons. First, the new approach of teaching trigonometric functions was intended to be tried with general mathematics students. Hence, a public Dutch school was preferred rather than an international school. Second, the researcher already had contacts with a mathematics teacher, Nicole de Klein, of this school through the University of Amsterdam. Further, she was interested in educational research and trying out new materials, and she was already familiar with some of the ideas underlying the lesson design of the study. She is an experienced mathematics teacher, and has a reputation within the school as a very good teacher of

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10 Detailed information can be found in school’s website only in Dutch: [http://www.fonsvitae.nl](http://www.fonsvitae.nl)
mathematics. The researcher taught five lessons in one of her classes. Although she did not have a role in the research, she also helped students during their pair work.

3.5 Research Cohort

The researcher taught five lessons in a Math B class of VWO students in their 4th year. The number of students was 24, including 17 females and 7 males aged 16-17. This cohort was chosen because of the acquaintance of the researcher with their mathematics teacher as mentioned in the previous section.

The cohort was classified by their teacher as a successful group in mathematics. It was expected that they were among the highest achievement group because, generally speaking, VWO students are more capable in mathematics than are HAVO students. However, this was still a general mathematics class, Math B, not a group of exceptionally talented students like one would find in a Math D group. Hence, it can be claimed that the new teaching approach of trigonometric functions was tried with students who were at an average level of mathematics.

3.6 Timeline of the Intervention

The researcher taught five lessons. The students in the research cohort had three mathematics classes per week, each of which was 60 minutes. However, due to the school schedule, it was not possible to finish the lessons in two successive weeks. Two weeks before the lessons started, to share the latest
version of the lesson plans, the researcher met the cooperating teacher and another teacher at the same school, who wanted to implement the same lesson materials in his class.\textsuperscript{11}

Table 3.1 shows the dates of the lessons as well as the data collection.

<table>
<thead>
<tr>
<th>Intervention</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagnostic Test</td>
<td>07 February 2010</td>
</tr>
<tr>
<td>Lesson 1</td>
<td>14 February 2012</td>
</tr>
<tr>
<td>Lesson 2</td>
<td>17 February 2012</td>
</tr>
<tr>
<td>Lesson 3</td>
<td>22 February 2012</td>
</tr>
<tr>
<td>Lesson 4</td>
<td>24 February 2012</td>
</tr>
<tr>
<td>Lesson 5</td>
<td>6 March 2012</td>
</tr>
<tr>
<td>Interviews 1 – 3</td>
<td>8 March 2012</td>
</tr>
<tr>
<td>Trigonometry Test &amp; Interview 4</td>
<td>9 March 2012</td>
</tr>
</tbody>
</table>

Table 3.1: The timeline of the intervention

As it can be seen in Table 3.1, a diagnostic test was given a week before the lessons, and four interviews were done and a trigonometry test was given after the lessons. Although the initial plan was to do the interviews after the trigonometry test, it could not be realized because of the school schedule. In order to prevent the situation of doing the interviews three weeks later at the end of March, three of them had to be done one day before the test.

\textsuperscript{11} Although, he implemented the same lesson design by using the Teacher Guide (Appendix A), the research reported in this thesis is based on the data only from the researcher’s class. This is mainly because it was not possible to observe the other classes.
A new framework is presented in this chapter based on a model of trigonometric understanding. This framework drove the design of the instructional sequence and data collection instruments, as well as the analysis of the data regarding the students’ understanding of sine and cosine functions.

4.1 Theoretical Considerations for the Framework

Learning with understanding is important, and the aim of the present study is to investigate students’ understanding of sine and cosine functions. However, it is not a trivial issue how to assess understanding. In this chapter, a model of trigonometric understanding is provided based on conceptual analysis in a similar way to that defined by Thompson (2008). The model is also directly related to mathematical understanding meaning being able to make mental connections among mathematical objects and procedures.

Thompson (2008) emphasized the problem of teaching trigonometry in the two unrelated contexts of triangle and unit circle, suggesting that teaching this subject should enhance coherence among different trigonometric contexts. The model of trigonometric understanding presented in this thesis is based on developing coherent connections among three different contexts of trigonometry: (right) triangle trigonometry, unit circle trigonometry, and trigonometric function graphs. Figure 4.1 illustrates these contexts.
In the first context, trigonometric concepts are defined as ratios in right triangles. The important idea is that trigonometric ratios are useful in setting relationships between angles and side lengths, and that these ratios can be defined as trigonometric values of angles because they are constant in all triangles with the same angles because of the similarity of triangles. In this context, trigonometry is limited to angles in degrees smaller than $90^\circ$. In the second context, trigonometric concepts are defined as coordinates. This context expands trigonometry to any angles and does it both in degrees and radians, and both negative and positive angles. Unit circle definitions are important in enabling one to define trigonometric functions. The third context is derived from the second one. Trigonometric graphs illustrate trigonometric functions in the domain of real numbers as well as their periodic nature.

The mathematical connections within and among these three trigonometric contexts were explicated based on Thompson’s (2008) conceptual analysis. He defined conceptual analysis as a tool to show how mathematical ideas are connected in a coherent way. He proposed four ways that conceptual analysis can be used:

1. in building models of what students actually know at some specific time and what they comprehend in specific situations,
2. in describing ways of knowing that might be propitious for students’ mathematical learning,
3. in describing ways of knowing that might be deleterious to students’ understanding of important ideas and in describing ways of knowing that might be problematic in specific situations,
4. in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support.

(p.59)

Thompson exemplified a conceptual analysis of trigonometry based on angle measure. He suggested that a coherence of ideas in trigonometry can be gained through developing a coherent meaning of angle measure.\textsuperscript{12}

The conceptual analysis presented in this chapter differs somewhat from Thompson’s example. First of all, the fundamental idea in the new model of trigonometric understanding is a function with arcs and corresponding vertical positions for the case of sine in the unit circle prior to angle measure. It is hypothesized that this will promote a coherent meaning within which students develop their understanding. Second, my view of the conceptual analysis of trigonometry is from a broader perspective than Thompson’s example using angle measure. The model includes trigonometric functions in the domain of real numbers, which is also different from the studies reviewed in the theoretical background chapter. Lastly, it is noteworthy to mention that the conceptual analysis here is directly based on the first and fourth ways of using a conceptual analysis as described by Thompson.

\textsuperscript{12} He suggested that angle measure in degrees can cohere with radian measure by basing the idea of a degree also on arc length.
His second and third ways are more about design issues of the present study. A learning trajectory, developed by also considering students’ difficulties found in the literature, is presented in Chapter 5.

Thompson (2008) suggested that coherence is about developed meaning, and that coherent meanings are essential among and within mathematical ideas. In this regard, the trigonometric understanding model puts emphasis on both meanings within the three trigonometric contexts and among them.

### 4.2 A Model of Trigonometric Understanding

Understanding trigonometry is complex because it requires different algebraic, geometric, and graphical connections. The trigonometric understanding model presented here is fundamental to the research reported in this thesis in that it provides a framework for both the instructional design and assessment of students’ understanding. The model is based on a conceptual analysis of mathematical ideas within and among three contexts of trigonometry. Figure 4.2 shows this trigonometric understanding model.

The contexts TT, UCT, and TFG represent three contexts in which trigonometry can be partially understood, while the central point U in the model represents the desired trigonometric understanding of students. The numbered line segments represent that trigonometric understanding should entail aspects in the three trigonometric contexts and the connections among them. It is important to keep in mind that point U should not be considered static. It may have different places in between the three contexts with respect to the quality of different students’ understanding. In this regard, the areas bounded by the line segments are the spaces where students’ understanding can be attained. In

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**Figure 4.2: A model of trigonometric understanding**
addition, these areas represent the space where students’ understanding can be flexibly placed by a researcher or a teacher during students’ task solving or designing lesson materials because different tasks may require different aspects of trigonometric understanding.

For the present study, the model is utilized based on the line segments connecting understanding to the three trigonometric contexts and to their connections. The line segments numbered 1, 2 and 3 represent understanding different aspects within the three different contexts. Students’ understanding must have access through these lines to these aspects. The important point is that these aspects are not only about knowing mathematical definitions, but also about being able to elaborate on them. For example, Segment 2 not only refers to the coordinate definitions of sine and cosine, but also indicates being able to show trigonometric relationships using the unit circle, e.g., explaining why sine is an odd function, and to order trigonometric values corresponding to certain rotation angles in the positive or negative direction. Similarly, explaining relationships between angles and subtended arcs should go beyond applying the rule of converting degrees and radians. Use of proportional reasoning using the radian concept indicates a conceptual understanding. The point here is that the line segments entail more than factual knowledge.

A deeper level of understanding is represented in the model by the thicker line segments 4 and 5 which represent understanding the connections among the contexts. Note that the black dashed line segments represent the connections from TT to UCT, and from UCT to TFG. In this regard, Segment 4 represents understanding aspects of integrating triangle trigonometry to the unit circle. A task of calculating sine of an angle larger than 90°, for example, requires integrating ratio definition of sine to the unit circle to find the corresponding vertical position. Segment 5, on the other hand, represents understanding the connections between unit circle trigonometry and trigonometric graphs. Connecting algebraic, geometric, and graphical aspects of trigonometric functions indicates a deep understanding as represented by this segment, for example being able to explain why sine is a function of real numbers is represented by this segment. The explanation requires

- the coordinate definition of sine,
- the relationship between radians and arcs to transit from angles to real numbers by considering rotation angles in radians or equivalently corresponding arc lengths as the input,
- the vertical position as the output of the function, and
- combining these in the idea that each input has a unique output.

Such connections are important for trigonometric understanding by considering that trigonometric functions cannot be computed through arithmetic operations.

Various aspects were hypothesized as crucial in terms of students’ understanding in this conceptual analysis model of trigonometric understanding (Table 4.1). They inform the way in which a student’s understanding of sine and cosine functions is considered. Each aspect is coded based on the
representative line segment, and the research findings in the conclusion chapter (Chapter 7) are presented with respect to these codes.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Context or Connection</th>
<th>Code</th>
<th>Aspects</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Triangle (TT)</td>
<td>1A</td>
<td>Ratio definition and of sine/cosine in a right triangle, and applications</td>
</tr>
<tr>
<td>2</td>
<td>The unit circle (UCT)</td>
<td>2A</td>
<td>Coordinate definition of sine/cosine, and applications with directed angles or real numbers</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2B</td>
<td>Application of the coordinate definitions to evaluate sine/cosine function for certain inputs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2C</td>
<td>Trigonometric relationships that can be shown on the unit circle, e.g. sine is an odd function, i.e. sin x = -sin(-x)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2D</td>
<td>Relationship between arcs and subtended angles through the radian concept</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2E</td>
<td>Converting degrees and radians to each other by using proportions</td>
</tr>
<tr>
<td>3</td>
<td>Graphs (TFG)</td>
<td>3A</td>
<td>Reason why the graph of sine/cosine shows functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3B</td>
<td>Interpretation of the trigonometric graphs for function properties, e.g. domain and range of trigonometric functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3C</td>
<td>Periodicity and the period of trigonometric functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>Trigonometric relations that can be shown on their graphs, e.g. cos x = cos (-x)</td>
</tr>
<tr>
<td>4</td>
<td>Connections TT - UCT</td>
<td>4A</td>
<td>Trigonometric values of angles larger than 90° by integrating the ratio definitions into the unit circle through reference triangles</td>
</tr>
<tr>
<td>5</td>
<td>Connections UCT - TFG</td>
<td>5A.1</td>
<td>Drawing the graph of the sine/cosine graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5A.2</td>
<td>Interpretation of the coordinates on the sine/cosine graphs by their meanings on the unit circle</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5A.3</td>
<td>Explanations of the shape of the sine/cosine graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5A.4</td>
<td>Differences between the sine and cosine graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5A.5</td>
<td>Connection between a point given on the unit circle and its counterpart on the graph</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5B.1</td>
<td>Reason why sine/cosine are functions based on rotation angles and/or travelled arc lengths</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5B.2</td>
<td>Sine/cosine functions in the domain of real numbers</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5B.3</td>
<td>Role of radians in the transition to sine/cosine of real numbers from trigonometry of angles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5B.4</td>
<td>Meaning of the range of sine/cosine by connecting the information on the graph to the unit circle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5B.5</td>
<td>Meaning of periodicity by connecting the information on the graph to the unit circle</td>
</tr>
</tbody>
</table>

Table 4.1: Aspects crucial to students’ understanding in the model
4.3 Coherence of the Model

The understanding model of trigonometry includes the three trigonometric contexts, and the connections among them. In this study, the triangle context was considered as students’ prior knowledge, and the direct focus was their concept development of sine and cosine functions based on the second and third contexts, and the connections among all three contexts. Note that the unit circle context serves to facilitate the transition from trigonometric ratios in right triangles to the trigonometric functions of real numbers.

Thompson (2008) defined coherence that is revealed by conceptual analysis as the developed meaning. Coherence, then, can be considered as the glue connecting different mathematical ideas, or helping students develop conceptual understanding based on making connections among these ideas. Thompson emphasized the importance of angle measure for coherence; but in the model presented in this thesis, the coherence is established through arc lengths rather than angle measure.

In the work reported here, it should be noted pointed out that arc lengths are emphasized before angle measure. In the conceptual analysis underlying the trigonometric understanding in this study, arcs were aimed to serve as the glue between the second and third contexts through taking meaningful actions. For example, the metaphor of travelling along the unit circle, referring to the arcs as travelled distances, was used to help students develop coherent meanings based on arcs. Angle measure was addressed after almost all connections between the unit circle and trigonometric graph contexts were set.
5 RESEARCH QUESTIONS & RESEARCH DESIGN

The research questions and design issues are presented in this chapter. Considering that the research questions are connected to the understanding model and the hypothetical learning trajectory underlying the instructional design, it was decided to include them between the understanding model (Chapter 4) and the learning trajectory (Section 5.2). After Section 5.2, instructional materials are described in the next section. Note that a full description of lesson plans are given in Appendix A including all designed worksheets with answer keys. This is followed by two other sections, the implementation design and the changes in the design which occurred during the implementation. Afterwards, data collection instruments are presented in Section 5.6. It was important to include these instruments in the same chapter as the instructional materials because worksheets were also a means of data collection. The chapter ends with a description of the data analysis methods.

5.1 Research Questions

The research study reported in this thesis is a single case study. Gay, Mills, and Airasian (2009) defined case study research as qualitative research in which the researcher focuses on a unit of study known as a bounded system (p. 426). The case as the bounded system in the present study was a single mathematics classroom in a Dutch secondary school (Chapter 3) where the researcher tried the new teaching approach of trigonometric functions.

Gay et al. (2009) emphasized that case study is appropriate when the researcher is looking for answers to a descriptive (e.g., what happened?) or an explanatory (e.g., how or why did something happen?) questions. The research was conducted to answer these two descriptive research questions:

1. What task-related difficulties do students face in their concept development within the designed lesson sequence based on a hypothetical learning trajectory of trigonometric functions?

2. What characteristics relating to students’ understanding of sine and cosine can be found in the data resulting from the intervention based on a model of trigonometric understanding?

The aim of the first research question is to expose the kind of difficulties and challenges students have while working on the instructional tasks in pairs. Since the designed lesson sequence was based on a new theoretical approach, which may be difficult for high school students, it was of great importance to answer this question. Moreover, it was expected that it would provide suggestions about alterations to the original designed tasks, which would then have a possible value of increasing the effectiveness of the intervention if others wish to try it.

13 This new approach is presented in Section 5.2.
The second question, on the other hand, aims to describe the understanding of students which they were expected to develop in the lesson sequence. Students’ understanding was explored in terms of the conceptual aspects of the trigonometric understanding as presented earlier (Chapter 4). The expectation was that the answer to this question in addressing the main issues found in the literature in the field (Chapter 2), would provide insight into the process of teaching and learning trigonometry.

5.2 Research Intervention: A New Approach to Teaching Trigonometry

A lesson sequence was designed to overcome some perceived problems in traditional ways of teaching trigonometric functions (see Section 3.3 for the textbook approach). The theoretical considerations (Chapter 2) and the model of trigonometric understanding (Chapter 4) together form the basis of the teaching approach formulated and tested in this study.

In this section design issues are discussed first to stress the main differences of the new approach to teaching trigonometry. This is followed by the hypothesized learning trajectory which explains the development of the targeted mathematical content within the lesson design.

5.2.1 Design Issues of the Instructional Sequence

The main focuses of the lesson sequence are sine and cosine as functions in the domain of real numbers, and the connections among the three trigonometry contexts: triangle trigonometry, unit circle trigonometry, and trigonometric function graphs. The way in which these aspects are addressed in the lesson sequence is what makes the approach underlying the lesson design different from traditional ways of teaching this topic.

First Focus: Trigonometric functions in the domain of real numbers

The starting point was arc lengths and vertical positions to define sine and cosine functions, instead of an early introduction to radians based on rotation angles in the unit circle. It was expected that students would understand the connections between arcs and subtending angles better when the transition is in the direction from arcs to angles, and that they would be able to conceptualize the sine and cosine functions of real numbers through arc lengths. The function concept was to be discussed through the graph which is formed by plotting the vertical position against the travelled distance by a moving point on the unit circle which corresponds to arc lengths. After students investigated the relationship between arcs and subtending angles by engaging in proportional reasoning, the graph would then be connected to the sine of rotation angles subtending the arc lengths used to draw the graph. A formal introduction to radian measure completed the picture by serving as a transition from angles to real numbers as the domain of trigonometric functions.
There are two important issues in this way of covering trigonometric functions of real numbers. First, the lessons were designed so that students could construct their knowledge through meaningful mathematical actions. For instance, instead of providing them with the sine graph, they were expected to construct it as a tool with which they could investigate mathematics further. However, it is not easy to draw the graph showing the vertical position corresponding to the arc lengths on the unit circle. Hence, the lesson sequence was designed to start with a unit square for which such a graph is easier to draw because it has a linear structure instead of concavity as in the case of the unit circle. The unit square was defined as the square in which the center is at the origin of the Cartesian plane and sides are perpendicular to the axes, each side being of 2 unit-lengths. Then students were meant to realize the changes in the graph as the number of the sides of a polygon starting with the unit square increased to very large numbers. Mathematically speaking, as the number of sides approaches infinity, the polygon will approach the unit circle. This transition was designed to be done with the dynamic geometry software GeoGebra. Since the effects of this change on the polygon and the graph were difficult to imagine, GeoGebra would serve as a cognitive tool for students. (See Section 2.3 for cognitive tools.)

The second issue is the use of a journey metaphor with a moving point along the unit square, the unit circle, or along any polygon inbetween. Instead of using pure mathematical talk including, for example, vertical positions corresponding to certain arcs, it was assumed that students would find it easier to think and speak of the distances travelled by the moving point. It was thought that this approach would help students to make their mental connections, as explained by Presmeg (2007): “... a metaphor introduced by the teacher becomes part and parcel of the fabric of classroom discourse, and thus helps many students in their building of connections for mathematical concepts” (p. 169). It was also assumed that actions based on such a metaphor would promote in students a process view of trigonometric functions.

Second Focus: Teaching trigonometry in a connected manner

Traditional, textbook driven methods of teaching trigonometry provide a linear strategy for transitions in the order “Triangle Trigonometry → Unit Circle Trigonometry → Trigonometric Graphs”. The way in which these connections were addressed in this study was different as a result of the assumption that such a linear development does not always fit with students’ concept development. In the lesson design, the graph was covered first together with the connections to the unit circle through the journey metaphor of a moving point. Note that this included a number of transitions between the two contexts, unit circle and graph. Students were expected to develop a connected understanding beginning in very early phases. The function was defined first through its graph in the domain of travelled arc lengths on the unit circle. Then it was connected to the coordinate definitions through the
relationships between arcs and angles. Finally, the first context was connected by integrating ratio definitions into the unit circle in order to calculate the trigonometric values of any angle by using the unit circle.

5.2.2 The Hypothetical Learning Trajectory

Clements and Sarama (2004) defined the construct of a hypothetical learning trajectory as a cognitive tool grounded in constructivism which reveals a description of students’ thinking and learning in a mathematical area, and at the same time a route through which they can develop mental processes and actions for the learning goals in the mathematical domain.

My hypothetical learning trajectory is presented as mathematical steps within seven stages, starting from the unit square and ending with some elaborations on the three trigonometric contexts. The trajectory describes all the mathematics required to learn trigonometry in the new approach. Note that formulating a learning trajectory is also related to a conceptual analysis (Clements & Sarama, 2004). In this regard, the learning trajectory presented in this section can be considered as the detailed steps underlying the conceptual analysis presented in Chapter 4. Instructional activities are an important aspect of a learning trajectory. However, in this section only the mathematical ideas that students were expected to understand are described. The new approach is not only different in its pedagogical aspects, but also the development of the mathematical content is considered from a different perspective. Lesson materials based on this learning trajectory are discussed later on (Section 5.3).

The hypothetical learning trajectory is described in terms of seven stages. The first stage includes actions on the unit square. The unit square is defined as the square having its center at the origin of the Cartesian Coordinate system, and its corners at the points (1, 1), (-1, 1), (-1, -1), and (1, -1). The learning trajectory starts with drawing the graph illustrating the vertical position versus the travelled distance of a moving point along the unit square. Later, there is a transition to the unit circle. On the unit square, relatively easy compared to the unit circle, but meaningful actions were hypothesized as promoting a good conceptual development.
Stage 1: The Unit Square

1a. The following graph illustrates the vertical position versus the travelled distance of a moving point P starting from (1, 0) in the counter clockwise direction along the unit square.

1b. A point on the graph, for instance (2, 1), has the meaning that when point P has moved 2 units on the square, its vertical position is 1. Hence it corresponds to the position (0, 1) on the unit square.

1c. When P moves 8 units along the unit square it comes back to the starting point for the first time. Further, when it moves any positive integer multiple of 8, it always passes through the starting position.

1d. The positions of P when it is passing through the starting point correspond to the points (8k, 0) on the graph where k is a positive integer.

1e. 1c & 1d illustrate the shortest repeating interval on the graph, which is called periodicity, and hence the period of the graph is 8, and this corresponds to the perimeter of the unit square.

1f. The graph can be extended in the direction of the negative x-axis by following the same repeating pattern in the opposite direction.

1g. The extended part illustrates the movement of point P in the clockwise direction. For instance, the point (-5, 1) is interpreted as when P moves 5 units along the unit square in the clockwise direction, its vertical position is 1.

1h. The graph illustrates a function whose input is travelled distance and output is vertical position. It is a function because for every input, there only one output.

The second stage includes the first steps on the unit circle to define a function on the arc lengths.

Stage 2: Transition to the Unit Circle

2a. If the number of sides of the polygon starting with the unit square is increased infinitely many times, the polygon will converge to the unit circle.

2b. When the polygon approaches the unit circle, the graph in Stage 1 will approach the graph of the sine function. In this step it is defined in the domain of travelled arc lengths.

2c. When the graph approaches the one of the sine function, its period will approach 2π. For instance for the octagon (8-gon) and dodecagon (12-gon), it is approximately 6.6 and 6.4, respectively.

2d. For the unit circle, the period of the graph is 2π because its circumference is 2π.

2f. The graph represents a function, say s, in the domain of travelled arc lengths.

In the previous stage, the function has not yet been called the sine function. It is in the third stage that the function is connected to the sine of rotation angles in degrees. Through the relationship between arcs and angles, the formula of the function is obtained.

Stage 3: Naming the Graph as Sine

3a. When point P moves a certain arc length along the unit circle, its corresponding coordinates can be found by considering the proportion of the arc length to the circumference of the unit circle. For instance, when it moves, π/2 its coordinates will be (0, 1) because this corresponds to a quarter of the unit circle. Such reasoning connects arcs and coordinates.

3b. A movement of point P for a certain arc length along the unit circle can also be described by the rotation of a segment by the angle subtended by that arc. For instance, a movement of an arc length, π/4 can be described by a rotation of 45°.

3c. The vertical position of P at any position can be written as sine of the corresponding rotation angle. This is to say that sine of an angle is defined as the corresponding vertical position. This can be verified for the situation that P moves an arc length of π/4 corresponding to the rotation angle 45° by using the ratio definition of sine.

3d. Using the relationship between rotation angles and subtended arcs, and the coordinate definition of sine, the formula of the function s can be found by proportional reasoning:

\[
s(x) = \sin \left( \frac{\pi}{180°} x \right)
\]

\[
\Rightarrow \quad \alpha = \frac{180° x}{\pi} \quad \Rightarrow \quad s(x) = \sin \left( \frac{180° x}{\pi} \right)
\]
In the fourth stage there is a transition from degrees to radians, through which the sine function is defined in the domain of real numbers.

**Stage 4: Closing the Gap with Radians**

4a. Angles can be measured in radians or degrees. The measure of an angle in radians is defined as the ratio of the arc length subtending that angle to the radius of the circle. Radian measure is needed to define sine function in the domain of real numbers.

4b. The definition of radian indicates that the measure of an angle is equivalent to the subtending arc length for the unit circle because its radius is 1.

4c. Using the relationship between angles and arcs, angles in degrees and radians can be converted to each other by proportions, for example, 2π radians corresponds to 360°.

4d. Converting 180°/π into radians, the formula \( s(x) = \sin\left(\frac{180°}{\pi}x\right) \) becomes \( s(x) = \sin (x \text{ rad}) \).

4e. Considering that \( x \) radians is equivalent to the subtended arc length, the unit of radian can be omitted. Hence the sine function can be defined in real numbers, i.e. \( \sin (x \text{ rad})=\sin x \).

4f. The Sine function has the following properties:
   - Its domain is real numbers.
   - Its range is \([-1, 1]\).
   - Its period is \(2\pi\).

The previous stages are about the trigonometric contexts of the unit circle and the graph, i.e., the second and third contexts. In the fifth stage the first context, triangle trigonometry, is connected to the second one, unit circle trigonometry, by integrating the ratio definition of sine to the unit circle trigonometry.

**Stage 5: Integrating Triangle Trigonometry to Unit Circle Trigonometry**

5a. The sine value of any angle can be found by the corresponding vertical coordinate of the rotation about the origin with that angle.

5b. This requires calculating the sine value of an acute angle by using its sine value in a right angle triangle which is called a reference triangle. For instance, \( \sin 210° = -\sin 30° \)

In the sixth stage, the cosine function is introduced through similar steps to those of the sine function, beginning with the unit square.

**Stage 6: Introduction to the Cosine Function**

6a. The following graph illustrates the horizontal position versus the travelled distance of a moving point P starting from \((1, 0)\) in the counter clockwise direction.

6b. If the number of sides of a polygon starting with the unit square increases, the cosine graph will be obtained.

6c. All steps in Stages 1-5 also apply to the case of the horizontal position. Hence Cosine of an angle is defined as the corresponding horizontal coordinate.

   The formula of the function represented by this graph, say \( c(x) = \cos \left(\frac{180°}{\pi}x\right) \).

   Using the radian and the relationship of the angles and arcs, it is \( c(x) = \cos (x) \).

   It has the same domain, range, and period as the sine function.
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This last stage includes certain trigonometric applications in a connected way regarding the three trigonometric contexts.

**Stage 7: Elaborating on Three Contexts of Sine and Cosine**

7a. Trigonometric contexts of triangle, the unit circle, and the graphs are useful to show trigonometric relationships. Certain equalities can be shown for all of the three, whereas some others can be shown only for the second and third contexts. Such actions could be useful for making connections among the three contexts.

7b. Simple trigonometric equations of the form sin x = 0, sin x = 1 or sin x = -1, or sin x = sin y or cos x = cos y can be solved using the either unit circle or the graph. Trigonometric equations have infinitely many solutions.

### 5.3 Instructional Materials: Worksheets & GeoGebra Applets

The hypothetical learning trajectory just described was utilized to create the instructional materials which consisted of worksheets and GeoGebra Applets. Table 5.1 shows which worksheets and GeoGebra applets address which steps of the learning trajectory.

<table>
<thead>
<tr>
<th>WORKSHEETS</th>
<th>GEOGEBRA APPLETS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worksheet 1a</td>
<td>GeoGebra 1</td>
</tr>
<tr>
<td>Worksheet 1b</td>
<td>1a, 1b, 1e, 1f</td>
</tr>
<tr>
<td>Worksheet 2</td>
<td>GeoGebra 2</td>
</tr>
<tr>
<td>Worksheet 3</td>
<td>1e, 1f, 2a-2f</td>
</tr>
<tr>
<td>Worksheet 4</td>
<td>GeoGebra 3</td>
</tr>
<tr>
<td>Worksheet 5</td>
<td>2a-2f</td>
</tr>
<tr>
<td>Worksheet 6</td>
<td>GeoGebra 4</td>
</tr>
<tr>
<td>Worksheet 7</td>
<td>3a, 3b, 3c, 3d</td>
</tr>
<tr>
<td>Worksheet 8</td>
<td>4a-4f, 5a, 5b</td>
</tr>
<tr>
<td>Worksheet 9</td>
<td>6a-6c</td>
</tr>
<tr>
<td>Worksheet 10</td>
<td>7a, 7b</td>
</tr>
<tr>
<td>Worksheet 11</td>
<td>1a, 1b, 1e, 1f</td>
</tr>
</tbody>
</table>

The bottom row shows the distinguished steps of the learning trajectory in Section 5.2.

Table 5.1: Steps of the learning trajectory addressed in the instructional materials

Eleven (1a, 1b – 10) worksheets were prepared as tools to guide students in their mathematical investigations with the help of the four GeoGebra applets. Worksheets can be found in Appendix A within a teacher guide explaining the lesson plans in detail. Brief descriptions of the four GeoGebra applets are provided here.

**GeoGebra Applet 1** illustrates the graph of the vertical position of the moving point against the travelled distance along the unit square. It presents the dynamic relations between the moving point P, the point Q on the x-axis corresponding to the travelled distance, and the point S on the graph connecting the information of travelled distance and vertical position. A screenshot from this applet is presented in Figure 5.1.
This applet allows students to observe the relative changes in these points by dragging the point Q, i.e., by altering the travelled distance. Dragging the point Q from a position towards the origin is assumed to help students extend the graph in the negative x direction, which will correspond to the negative angles later on.

**GeoGebra Applet 2** demonstrates the changes in the graph and the polygon as a result of the increase of the number of sides of the polygon starting with the unit square. Figures 5.2 and 5.3 illustrate the situation for pentagon and decagon.

This applet serves as a cognitive tool to observe the fact that as $n \to \infty$, polygon $\to$ unit circle and to notice the changes in the periodicity. For instance, the period for $n=8$ is approximately 6.6, and 6.4 for $n=12$, and it approaches $2\pi$ as the polygon approaches the unit circle.

**GeoGebra Applet 3** demonstrates dynamic relations regarding the unit circle only. It illustrates the movement or rotation along the unit circle, and the graph which is the sine graph. Figure 5.4 shows a screenshot from this applet illustrating the position on the graph when the point P moves $\frac{3\pi}{2}$. 

Figure 5.1: A Screenshot from GeoGebra Applet 1

Figure 5.2: The graph $s$ for $n=5$

Figure 5.3: The graph $s$ for $n=5$

Figure 5.4: The graph $s$ when the point P moves $\frac{3\pi}{2}$. 
GeoGebra Applet 4 enables students to visualize the relationships between angles in degrees and radians, and arc lengths subtending these angles. Figure 5.5 shows a screenshot from this applet.

5.4 Description of Implementation Design with a Didactic Model

Five lessons were designed based on the hypothetical learning trajectory and with the instructional materials. Table 5.2 shows the materials according to the lesson planning.

<table>
<thead>
<tr>
<th>Lesson Number</th>
<th>Worksheets</th>
<th>GeoGebra Applet</th>
<th>Homework</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson 1</td>
<td>1a, 1b, 2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lesson 2</td>
<td>3, 4, 5</td>
<td>2, 3</td>
<td>2</td>
</tr>
<tr>
<td>Lesson 3</td>
<td>6, 7</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Lesson 4</td>
<td>8, 9</td>
<td>-</td>
<td>4</td>
</tr>
<tr>
<td>Lesson 5</td>
<td>10</td>
<td>-</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 5.2: Lesson materials as designed

The lessons were implemented in a student-centered approach by creating a learning environment in which students were encouraged to construct their knowledge through interactions with peers, the researcher and the regular classroom teacher. In this, GeoGebra was used as a cognitive tool with
some tasks in the worksheets. This student-centered implementation can be described as a didactic model (Figure 5.6).\textsuperscript{14}

![Diagram of the didactic model](image)

**Figure 5.6: Didactic model underlying the implementation design**

The didactic model developed in this research study shows how the characteristics of the implementation design can be described based on a constructivist learning environment. This didactical model puts the student at the center of the learning environment. Then, it shows the interactions among the four elements embedded in a context: student, teacher, technology, and peers. These interactions are illustrated with two sided arrows in the figure because the interactions occur bidirectionally in the knowledge construction. The arrows have different colors: the blue ones represent the interactions with an individual student, while the green ones show other possible interactions. The important point is that all of these elements are interconnected, and learning occurs through the all interactions within a particular context.

The context had three important characteristics within the present study. First, it was related to the targeted mathematics content, i.e. trigonometry. Second, the context was developed and presented to the students through learning activities which determined the basis for the interactions supplied with the tasks on the worksheets. Third, it addressed the engagement in a particular kind of mathematical activity, mathematical investigations.

The interactions with technology and with peers are important aspects of this model. In the present study, technology was considered to be the GeoGebra applets which were used as cognitive tools.

\textsuperscript{14} Tall (1986) defined a didactic tetrahedron including different elements of a learning environment. Although the didactic model used here is different from Tall’s, his model provided insight to the researcher so that he could develop his own didactic model (Figure 5.6).
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(details in Chapter 2). The peer factor was important to be considered because students did not only learn on their own by interacting with the researcher and the GeoGebra applets, they also learned by interacting with their peers. Pair-work and follow-up classroom discussions were of crucial importance. Students worked in pairs (or in groups in certain cases) on each worksheet, during which the researcher (and the cooperating teacher sometimes) helped and gave feedback to particular pairs. After the pairs completed each worksheet, a classroom discussion took place, and the researcher paid attention to emphasizing and summarizing the important points. In this way, students could understand the important steps to take in order to attain the aims of the lesson design.

At the end of every lesson, students were given a homework assignment either to review previous lessons, or to get ready for the next lesson. However, most students did not deliver their homework assignments as requested; hence, the researcher could not obtain enough information from homework papers about their progress.

5.5 Changes in the Instructional Design

This was an exploratory research study in a master program, and the lesson design could not be tested ahead of time. Given this situation, the implemented lessons were a little different from the original design.

As revealed by the analysis of the Diagnostic Test (Section 6.1), the students did not know the formal definition of the function concept, nor could distinguish graphs belonging to functions from those which do not represent a function. Hence, the definition of a function using single-valuedness and an input-output mechanism through two graphs, only one showing a function, were included as an introduction to Lesson 1. The teacher guidelines in Appendix A include this change in the lesson design.

There were some other changes in the lesson design during the implementation due to unexpected factors. One of them was that the researcher could not estimate in advance the exact amount of time that could effectively be used in every lesson. Although 60 minutes were allocated to each lessons, the real time for implementation was much less than this because of the regular teacher’s announcements to the students and the time lost in starting computers. Also some students had problems in starting the GeoGebra applets. Another reason was that the researcher could not estimate exactly how long students would need to complete each worksheet. The students also had more difficulties to carry out the worksheet tasks than expected. Their major problems were: English was difficult as an instructional language for most of the students, the designed tasks were very unfamiliar to them, and mathematical investigations which they were expected to engage in were difficult because they were unaccustomed to this way of learning mathematics.
The implemented lessons were changed somewhat from the original design (Table 5.2) so that these problems could be adequately addressed. The implementation of lesson design is shown in Table 5.3. Some particulars are worth noting. Worksheets 5 and 9 were assigned first as homework, then discussed in the lessons. Worksheet 4 was, on the other hand, removed from the schedule. A similar task to relate arc lengths to the coordinates was done in the first part of Worksheet 5. Additionally, Worksheet 7 was shortened, and Worksheet 8 was both shortened and divided into two parts. Excluded parts of the worksheets were done as whole class discussions.

<table>
<thead>
<tr>
<th>Lesson Number</th>
<th>Worksheets</th>
<th>GeoGebra Applet</th>
<th>Homework</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson 1</td>
<td>1a, 1b</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>Lesson 2</td>
<td>2, 3</td>
<td>1,2</td>
<td>2, Worksheet 5</td>
</tr>
<tr>
<td>Lesson 3</td>
<td>5, 6</td>
<td>3</td>
<td>3, Applet 4</td>
</tr>
<tr>
<td>Lesson 4</td>
<td>7, 8a</td>
<td>-</td>
<td>4, Worksheet 9, 8b</td>
</tr>
<tr>
<td>Lesson 5</td>
<td>10</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.3: Instructional design as implemented

The biggest change was in Lesson 5. Because of the delays and because of the difficulties revealed by the students, trigonometric equations were excluded from the design. Considering that they were included mainly upon the request of the cooperating teacher, this change did not affect the design to an important extent. Instead of trigonometric equations, a new worksheet was prepared. The tasks of sine and cos 210° and showing the equality cos x = cos -x were given in the changed version of Worksheet 10. Additionally, a review of the previous lessons was done, and homework questions and some important points were discussed. The initial idea to show the students’ the sine graph in two versions, one with integers and one with multiples of π, was abandoned because the researcher decided to ask this in the interviews to see whether students could conceptualize the sine as a function of real numbers.

The answer keys of every worksheet and homework assignment were uploaded to the e-learning website of the school after they were covered in the lessons. In addition, the GeoGebra applets were put to this website so that the students could use them online in the lessons, and they could reach them from home as well. Note that the designed worksheets can be found in Appendix A as a teacher guide. However, the changed versions of the worksheets (Worksheets 7, 8 and 10) together with the others were also used as a means of data collection. So they are also attached in Appendix B.

5.6 Data Collection Instruments

The data were collected from 24 secondary school students of a mathematics classroom before, during, and after the implementation of five lessons which were taught by the researcher. The data were collected through a diagnostic test, a trigonometry test, worksheets, audio recordings, and
interviews. Table 5.4 shows these the data collection instruments developed to collect data to answer the particular research questions.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Research Question</th>
<th>Time (5 lessons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagnostic Test</td>
<td>-</td>
<td>Before</td>
</tr>
<tr>
<td>Worksheets</td>
<td>1</td>
<td>During</td>
</tr>
<tr>
<td>Audio Recordings</td>
<td>1</td>
<td>During</td>
</tr>
<tr>
<td>Trigonometry Test</td>
<td>2</td>
<td>After</td>
</tr>
<tr>
<td>Interviews</td>
<td>2</td>
<td>After</td>
</tr>
</tbody>
</table>

Table 5.4: Data Collection Instruments according to the Research Questions

5.6.1 Diagnostic Test

The aim of the Diagnostic Test (Appendix C), was to evaluate students’ prior knowledge and skills so as to have an idea of their readiness for the lessons. The half hour test was given to the students one week before the lessons started.

Trigonometry is a complex subject combining different algebraic and geometric concepts and representations. Accordingly, the diagnostic test was prepared considering different algebraic and geometric skills and knowledge that were necessary for effective learning. The target subjects aimed to evaluate with this test were:

- Trigonometric ratios in a right angled triangle
- The function concept
- Cartesian connection and analyzing graphs
- Circumference of a circle and arc lengths corresponding to certain central angles

The test includes nine open ended questions addressing the subjects listed above. Table 5.5 presents information regarding the aim of each test question.

<table>
<thead>
<tr>
<th>Question</th>
<th>Target concept/subject</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Right angle triangle definitions of sine and cosine</td>
<td>Calculating cosine of an angle whose sine is given</td>
</tr>
<tr>
<td>2</td>
<td>Function</td>
<td>Defining what a function is</td>
</tr>
<tr>
<td>3</td>
<td>Function</td>
<td>Deciding if a given graph shows a function</td>
</tr>
<tr>
<td>4</td>
<td>Function</td>
<td>Evaluating a function for given inputs and outputs, and determining its domain and range</td>
</tr>
<tr>
<td>5</td>
<td>Graph analysis</td>
<td>Calculating the y coordinate on a graph using its equation</td>
</tr>
<tr>
<td>6</td>
<td>Function and graph analysis</td>
<td>Deciding if a given graph is function, evaluating the function using its graph</td>
</tr>
<tr>
<td>7</td>
<td>Circle and proportions</td>
<td>Calculating the circumference of a circle and some arc lengths corresponding to given central angles using proportions</td>
</tr>
</tbody>
</table>

Table 5.5: Addressed Concepts and Tasks in the Diagnostic Test
For the tasks related to the function concept, the ideas and research of Tall and Bakar (1992), and for the ones related to analyzing graphs, the Cartesian Connection of Schoenfeld, Smith, and Arcavi, (1993, as cited in Brown, 2005), and Knuth (2000) were utilized to create the test items.

### 5.6.2 Worksheets and Audio Recordings

Data were collected through audio recordings of students’ discussions on worksheet tasks to answer the first research question about students’ concept development.

The worksheets that were designed by the researcher were also used as a means of data collection. All worksheets which were used to collect data can be found in Appendix B. The pair work of some students was also audio recorded to collect supporting data regarding their discussions of the worksheet tasks. In every lesson, the discussions of four pairs were recorded. In the first lesson, the pairs were decided according to the advice of the cooperating teacher. However, there were some changes in the other lessons according to students’ performances.

### 5.6.3 Trigonometry Test

The Trigonometry Test (Appendix D) designed to assess students’ understanding was prepared considering the framework aspects provided in Chapter 4. It was given to the students 3 days after the classes finished. They were asked to complete it in 50 minutes. The Trigonometry Test included nine open-ended questions, some of which have subparts. Table 5.6 below presents the context addressed by the task in each test item.

<table>
<thead>
<tr>
<th>Question</th>
<th>Content</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The graph of cosine function</td>
<td>Drawing and interpreting the graph of the cosine function, and marking the point on the graph corresponding to ( \cos(3) )</td>
</tr>
<tr>
<td>2</td>
<td>Sine function and its domain</td>
<td>Stating what ( x ) is in ( y = \sin x )</td>
</tr>
<tr>
<td>3</td>
<td>Sine function and properties: periodicity, domain and range</td>
<td>Explaining why sine is a function, stating its periodicity with its meaning, and stating its domain and range.</td>
</tr>
<tr>
<td>4</td>
<td>Sine function</td>
<td>Evaluating ( f(x) = \sin x ) at ( x = -\frac{\pi}{2} ) and stating what ( \pi ) stands for in the function.</td>
</tr>
<tr>
<td>5</td>
<td>Integrating triangle trigonometry to the unit circle</td>
<td>Computing ( \sin(225^\circ) )</td>
</tr>
<tr>
<td>6</td>
<td>Cosine of angles on the unit circle</td>
<td>Showing the relationship between ( \cos(340^\circ) ) and ( \cos(200^\circ) )</td>
</tr>
<tr>
<td>7</td>
<td>Trigonometric relationships</td>
<td>Showing ( \sin(-x) = -\sin x ) on the unit circle or on the graph of sine function</td>
</tr>
<tr>
<td>8</td>
<td>Relationship between arcs and subtended angles</td>
<td>Calculating the angle subtended by an arc of ( \frac{\pi}{3} ) on the unit circle, and vice versa</td>
</tr>
<tr>
<td>9</td>
<td>Sine corresponding to rotations, connections from the unit circle to the sine graph.</td>
<td>Ordering the sine values corresponding to certain rotations on the unit circle, and marking the point on the sine graph corresponding to one of these positions</td>
</tr>
</tbody>
</table>

**Table 5.6: Content and Tasks in the Trigonometry Test**
Some of the test items were taken from previous studies because they address important aspects of trigonometric understanding. The second question was taken from Kang (2003). The sixth and the last part of the ninth question was taken from Brown (2005). Although it is a very common kind of task, the eighth question was taken and asked in the same way as Orhun (2001) did because she found interesting results.

5.6.4 Semi-structured Interviews

Four semi-structured interviews were done to collect data regarding students’ understanding revealed by their responses to the interview tasks. The initial plan was to choose the interviewees according to their test performance. However, this could not be actualized because of the school schedule. Three of the interviews had to be done before the test. Thus, the interviewees were chosen according to performance of the students in the lessons and to the advice of the cooperating teacher. Two of the interviewees can be categorized as above-average while the other two can be categorized as average students according to their mathematical ability level advised by the cooperating teacher.

The interview tasks were designed by the researcher considering the key mathematical points underlying the lesson sequence. Interview tasks can be found in Appendix E. The following are the main points of the interview tasks:

- The graph of the sine and cosine functions and the connection to the unit circle
- The reason why sine is a function
- Characteristics of the sine function (domain, range, periodicity) together with connections between the graph and the unit circle
- Relationship between arcs and angles
- Transition from sine of angles to the sine function of real numbers
- Connections among three trigonometric contexts

Although it was not related to the research questions, the interviewees were also asked about their ideas and advice regarding the lessons. It was considered that these could be important in terms of the limitations of the study and indications for future research.

5.7 Methods of Data Analysis

The data were analyzed qualitatively to answer the two research questions about students’ concept development and understanding related to the sine and cosine functions.

The diagnostic test was analyzed question by question to gain an idea about students’ prior knowledge and skills. Similarly, the Trigonometry Test was also analyzed question by question. However, assuming that the analysis can be reported in a better way with respect to the theoretical underpinnings of the research, it is presented according to the main mathematical points of trigonometry understanding. Interviews, on the other hand, were analyzed interviewee by interviewee,
but it is also presented with respect to the trigonometry understanding. The longer versions which are more descriptive are included in the supplementary DVD to this thesis.

The basis of the data analysis is the issues of trigonometry understanding as discussed in Chapter 4. Because the Trigonometry Test was relatively long, the researcher transferred all student responses with their characteristics regarding the understanding framework to spreadsheets. Although this was a primitive way, it helped the researcher quite a lot to analyze students’ responses more deeply by allowing him to make comparisons and to detect certain commonalities of the student responses.

The audio recordings of the interviews were transcribed. Then they were analyzed according to the key aspects of trigonometric understanding to come up with the results to answer the research question about students’ understanding.

Analyzing audio recordings of group discussions, on the other hand, was more difficult for the researcher. He could not transcribe himself the recordings because the students discussed the worksheet tasks in Dutch. There were twenty recordings in total each of which was quite long. Hence, it was not possible to transcribe all of the recordings. The researcher surveyed the students’ written responses on the worksheets. The recordings related to interesting ones were translated by keeping the research question about students’ difficulties with the tasks in mind.
6 DATA ANALYSIS & RESULTS

The analysis of data regarding the two research questions about students’ concept development and understanding of the sine and cosine functions is presented in this chapter together with the analysis of the diagnostic test indicating students’ initial knowledge and skills. The analysis of the diagnostic test is followed by the analysis of students’ responses to worksheet tasks and the audio recordings of their discussions in the second section. The last two sections provide the analysis of trigonometry test and interviews for the purpose of answering the second research question.

The students were given numbers between 1 and 24 to report the data. This way was preferred to giving them pseudonyms because the data reported here includes 24 students. Students with numbers can make it easier for the reader to follow responses of the same students to different tasks or in different instruments.

6.1 Analysis of the Diagnostic Test

The diagnostic test was given to the students one week before the lessons started in a class hour. Twenty-three of the twenty-four students in the cohort took the test. The students were asked to complete the test in half an hour. They were observed to be tired before the test because in the previous lesson they had had a physics test. They were informed at the beginning of the test that the test would not be graded.

In this part, the results of the diagnostic test are presented. After a question-by-question analysis, a summary as an indication to how ready the students were for the lessons is provided. Analysis of the questions on the test was done qualitatively and was based on the mathematical validity of students’ responses so that the latter could provide insight into where students were before the lessons in this study. The examples of student responses were given in the case that either they represented the biggest portion of the sample, or that they were somehow extreme or interesting in the way of students’ thinking. The complete diagnostic test can be found in Appendix C.

**Question 1:**

Results of the first question, which asked students to calculate the cosine value of an angle whose sine is given as 3/5, generally revealed that students possessed basic knowledge about sine and cosine as ratios in right angle triangles. This conclusion is derived from the fact that despite some missing procedures, all students but two could relate the task to a right triangle.

Seven students (out of twenty-three) could answer the question correctly while only three were found to be totally wrong. Two of the latter did not provide any response while the other student mixed up
the definitions of sine and cosine. The remaining thirteen students’ answers were found to be partially correct. An overwhelming number of these thirteen students gave correct definitions of trigonometric ratios, but did not apply or were confused about finding the hypotenuse with Pythagoras’s Theorem. This shows a lack of procedural knowledge in such calculations.

The most interesting response to this question was given by Student 2 who provided a partially correct answer to the first question. She could relate the given information to a right triangle correctly, and she was right in her definitions. However, she tried a much more difficult way compared to Pythagoras’s Theorem as shown in Figure 6.1.

It is actually impressive that the student was trying to calculate the angle \( \alpha \) by using the inverse of sine. Although she could not provide the correct response, her answer made it clear that she was thinking at a higher level and that she already knew something about inverse trigonometric functions.

In short, although not every student could give a fully correct answer for Question 1, mostly due to some procedural problems, it can be concluded that students had a sufficient level of knowledge of sine and cosine ratios. Therefore, they were considered ready to proceed to upper levels with trigonometric ratios in the designed lesson-sequence.

**Question 2:**

The second question, which asked for an explanation of the meaning of a function, was found to be difficult for students. However, this was not surprising because in Dutch secondary education, students are not taught a formal definition of function. First, at lower levels, they study graphs as functions and then in higher levels, they define them as an input-output mechanism.
Of the twenty-three students, eighteen referred to the existence of a formula to explain what a function is. Seven students also mentioned graphs. Only five students referred to an input and output mechanism. However, some of the other students might have meant a similar mechanism by talking about formulas and computations. This may be a cue that students possessed a process understanding of the function concept. Finally, eight students referred only to lines and parabolas as functions, probably because these were the function types to which they had been exposed.

Example student responses:

- **Student 3:** “A function is a formula where you can fill in x to get y. If you draw a function in a coordinate system, you will get a line.”
- **Student 4:** “I am not sure but in the past we had a subject that was about filling in a function for x and then get something for the y.”
- **Student 6:** “A function is another symbol for the thing you want to compute. It is mostly for a graphic. f(x) = 2x + 5.”
- **Student 9:** “A function is a name for a formula where you want to find x. The points that come out as your answer form a line or something like it.”
- **Student 15:** “The formula of regular line.”
- **Student 16:** “A function is another word for a formula.”
- **Student 21:** “A function is a formula that shows what something does. Like y=3 + 5x is the height here.”

In these statements, it can be seen that many students associated function with a formula. The extreme case is Student 16 who claimed that a function was a formula.

As a conclusion about the second question, it can be claimed that students had ideas in accordance with their past experiences about the function concept. Also, the students who gave only specific examples such as lines or parabolas showed that not every student had a process view of function.

**Question 3:**

Question 3, which aimed to examine if the students could determine if a given graph belongs to a function, was found to be another difficult task in this test. This was not surprising because the students had previously not learnt about a formal definition, and instead had worked on certain graphs of functions without having any particular reason for why they represent functions.

The first graph, which was a polynomial, was marked by all students as a function. The only non-function graph among those given was stated to be a function by fifteen students. Another problematic issue with this question seemed to be about linear functions. Five students claimed that the graph y=x is not a function while twelve students said that the graph y=2 is not a function. Hence the biggest misconception found here is that students thought that a constant function is not a function. The sine graph, on the other hand, was considered as a function by sixteen students whereas seven students thought it was not a function.
Two students provided correct answers to all parts of this question. Student 3 stated that the last graph belonged to the sine function, and Student 9 mentioned that “It is a function where you want to find x by using sin.”

As an explanation to their response for the last graph (sine), some students provided the following:

- Student 4: “I think this is a function because when you zoom out to a parallelogram, I will be going down/up again. So this will create a wave kind of figure.”
- Student 12: “All functions make a graph. The graph will depend on the numbers given.”
- Student 13: “It is a repeating parabola and a parabola represents a function.”
- Student 14: “It goes up and down and I do not know a formula that goes up and down like that.”
- Student 15: “It is ongoing. You cannot make a formula for it.”
- Student 19: “I cannot see how much waves there are, there seems to be unendless [sic] much of waves. And that is not a function.”

Student 12 was a student who thought that every function had a graphical representation, but she made a logical mistake by considering that this also means that every graph belongs to a function. A remarkable result here was that some students made comments about “ongoing, repeating, wavy” structure of the graph, but they concluded with different answers. For instance, the wavy figure going up and down is a function for Student 4, but not for Students 14, 15, and 19. They also differed in their explanations why it is not a function: for Student 14 a function cannot go up and down like that graph, for Student 15 no formula can represent such a graph, and for Student 19 an infinite number of waves is the reason. Student 13 had a very different way of thinking. He claimed that the graph is of a function because it is a collection of parabolas.

As a short conclusion, the students did not really know how to decide if a given graph belongs to a function. Furthermore, the constant function was considered as a non-function by many of the students. Finally, only two students could give a sufficient answer for the reason why the sine graph is a function because of their familiarity with the sine function. On the other hand, it may be a reasonable finding that sixteen students considered the sine graph as a function although they did not specify a reason.

**Question 4:**

The students had varying success in the fourth question which was about domain and range of the given function, and evaluating the function for the given values of x and y.

Six students answered all parts of the question correctly. Nine students gave only the correct domain while eleven gave only the correct range. Sixteen students in total appeared to know what domain and range are by definition. Two students mixed up domain and range.
Evaluating the function for \( x = 4 \) was carried out correctly by all students but one who made a calculation mistake. Finding the inputs where the function has the value 30 was a bit problematic because fourteen students found only one solution for the respective quadratic equation. Despite this common procedural mistake, it was a good point that the students had almost no problem with function evaluation.

Student 16 gave this answer:

\[
\text{Domain}= < \leftarrow, \rightarrow > \ / \ \text{Range}= \{ 5, \rightarrow \} \ / \ f(4)=21 \ / \ f(z)=30 \Rightarrow z=5 \quad ( \text{arrow means } \infty )
\]

She was one of the student who just missed the negative root (-5) for the last part.

Student 4 gave this answer:

\[
\text{Domain:} \quad \text{“I know it is the most wide coordinates on the x axis but I forgot how to calculate.”} \\
\text{Range:} \quad \text{“I know it is the most up/down coordinates on the y axis but also forgot how to calculate.”} \\
f(4)=21 \ \text{and} \ f(z)=30 \Rightarrow z = 5 \ \text{or} \ -5.
\]

He was one of the students who could evaluate the function correctly for the given values. He could not calculate the domain and range, but could provide correct definitions.

In short, the students were quite good at this question in spite of some procedural mistakes, and a few students who were confused with or could not remember domain and range concepts.

**Question 5:**

Question 5, which asked the students to calculate the y value corresponding to an x value on the given graph by using its formula was found to be the easiest question for the students. Fourteen students gave the correct answer while only three students were found with an insufficient answer. Two of these three did not provide any response, and the other student made a mistake by thinking that the formula \( y=(x - 2)^3 + 2 \) has the y-intersection at the point (0, 2) although the task was not about y-intersection. Additionally, he could not distinguish the given function from the ones of the form \( y = x^n + 2 \). He wrote:

\[
\text{“The +2 in formula is the y coordinate from the graph in point A. So the y coordinate is 2.”}
\]

Six students’ answers were classified as partially correct because of calculation mistakes. One of them, Student 11, distributed the parentheses \( (x - 2)(x - 2)(x - 2) \), and made a mistake. Another student stated the connection, but did not complete the calculation. As the most common mistake, three students calculated the cube of negative 1 as positive 1, and hence came up with a wrong answer.
Chapter 6  Data Analysis & Results

As a conclusion, the students as a whole were found to be quite successful at integrating the information given by the formula of a function with its graph to find an unknown coordinate, (which is referred to as Cartesian Connection) despite some procedural mistakes.

**Question 6:**

Question 6 asked students in the first part if the given graph of a piecewise function represents a function, and to analyze it for some given x and y values at the last three parts. Students mostly did well on the last three parts while they had problems with the first part asking if the graph represents a function. This was as expected according to the student responses in Question 3. The piecewise linear structure in the given graph was the main issue for the students.

Only four students stated that this given graph can represent a function. Two of them did not provide any explanation; the other two students said:

- Student 4: “Yes it can. Lines are also derived from linear functions.”
- Student 6: “Yes when you fill a letter in for x you can get y.”

It is worth noting that these two students were among the ones who considered lines as functions in Question 3. An interesting finding was that one of the four students who stated that it is a function, Student 2, stated in the third question that the linear graphs are not functions. This may mean that the same issue may bring along different responses from the same student when it is asked in different forms.

Another interesting answer came from Student 8:

“At the schuine [Dutch word for inclined] points it is a function. Not at the straight points because y is then always the same.”

Here it was clear that she was considering different parts of the graph, and that she was not considering a constant function as a function.

Some other students were also paying attention like Student 8 to the different parts of the graph, but they considered this piecewise structure as an obstacle to it being a function. Some of them:

- Student 13: “No it has straight lines in it and there are different pieces.”
- Student 17: “No because it is not a regular function. He has pieces where the functions are different.
- Student 19: “No, it is an irregular graph. There is not a system in it.”
- Student 21: “No there is no constant change but random differences.”

These students were looking for regularity or some kind of system in the graph. Because of the piecewise structure, they considered it as a non-function graph.
For the last three parts of Question 6, almost every student gave the correct answer. Student 23 left the whole question empty. Student 8 could not answer part b because interestingly she was trying to find a formula for the function, and Student 14 mixed x coordinates with y coordinates.

In short, students were rather successful on Question 6. Part a was found to be the most challenging, as expected, due to students’ lack of a function definition. Another remarkable result was that almost all of the students were able to analyze the given graph correctly.

**Question 7:**

In Question 7 students were asked to calculate the circumference of a circle and some arc lengths corresponding to certain central angles. It was another one with which most students had no important difficulties. Five students could not connect arc lengths to the angles by using the circumference of the circle and the corresponding proportions, and three of these five left the whole question empty.

Ten students could not remember the correct formula for the circumference of a circle. However, they were successful in the conceptual underpinning of the question, in other words in the arc-angle relations. The most common problem was using the formula of area for the circumference. Two students provided interesting and uncommon answers. Student 8 thought that when the ant walks along the whole circle, the distance will be the same amount as the radius:

“The distance he has walked is the same as the radius because it is a circle and because circle has no corners the length from the middle of the circle to the line is everywhere the same.”

However, for the other parts she could engage in proportional reasoning, and provide sufficient responses, although he used a wrong formula for the circumference.

Student 1, on the other hand, could not reach a sufficient answer because she was thinking in quite a different way. She was trying to find the distances in parts b and c by considering corresponding circle parts as right triangles. Her work can be seen in the following figure.

![Figure 6.2: The response of Student 1 trying to find the arc length with a triangle](image-url)
Although many students used an incorrect formula, it is a very remarkable finding that most of them could achieve to relate arc lengths to angles by using proportional reasoning. Many of them could also explain this relationship in words very well.

**Conclusion**

Trigonometry is a complex subject combining different algebraic and geometric concepts and representations. Accordingly, the diagnostic test was prepared considering different algebraic and geometric skills and knowledge that were necessary for the students’ effective learning. The aim was to see how ready the students were for the lesson sequence within this research. The target subjects aimed to assess with this test were:

- Trigonometric ratios in a right angled triangle
- The function concept
- Cartesian connection and analyzing graphs
- Circumference of a circle, and arc lengths corresponding certain central angles

Analysis of the diagnostic test results, which was provided here, revealed the students were generally ready for the designed lesson sequence despite some problematic issues. Many student mistakes were found to be due to lack of procedural fluency, or difficulties in remembering certain formulas. However, conceptual validity of most responses should be given more importance.

Table 6.1 below summarized the students’ responses to the diagnostic test questions.

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Aim of the Question</th>
<th>Results (N = 23)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>To assess if the students can calculate cosine of an angle whose sine is given by using the ratio definitions of sine and cosine</td>
<td>7 correct, 3 wrong/no reply, and 13 partial Despite procedural mistakes like those with the Pythagoras Theorem, the students knew the ratio definitions and related the given information to a right triangle.</td>
</tr>
<tr>
<td>2</td>
<td>To examine the students’ ideas about the function concept through their definitions</td>
<td>No one provided a formal definition. Formula and graph were often mentioned. Some had a process view of function.</td>
</tr>
<tr>
<td>3</td>
<td>To assess if the students can determine if a given graph represents a function.</td>
<td>Everyone stated the graph of a polynomial to be a function. 15 students marked the non-function graph as function. Linear and constant function graphs were stated by many to be non-functions. 16 students mentioned that the graph of sine belongs to a function.</td>
</tr>
<tr>
<td>4</td>
<td>To see if students know about domain and range, and how to evaluate a function for given values as input or output</td>
<td>Although they made calculation mistakes, 16 students knew what domain and range are. Only 2 mixed up the two. All students but one could evaluate the function correctly for the given input. To find the input for the given output, the common mistake was that many students did not calculate the negative root of the respective quadratic equation.</td>
</tr>
<tr>
<td>5</td>
<td>To assess if the students can analyze a given graph and connect the information between the graph and formula of a function (Cartesian Connection)</td>
<td>14 students provided the correct answer whereas 3 were wrong or did not provide an answer. The common mistake of the rest was that the cube of -1 was calculated as +1.</td>
</tr>
<tr>
<td>6</td>
<td>To assess if the students analyze the graph of a piecewise function and consider it as a function.</td>
<td>Only 4 students mentioned it as a function. The rest mentioned the different pieces of the graph as an irregularity for the reason why it is not a function.</td>
</tr>
<tr>
<td>7</td>
<td>To assess if the students can calculate the circumference of a circle, and some arc on the circle by using a proportional reasoning with the given central angles.</td>
<td>Only 5 students could not relate the arcs to the given angles. 10 students could not remember the formula of the circumference, and some used the formula of area for the circumference.</td>
</tr>
</tbody>
</table>

Table 6.1: Summary of student responses to the items in the Diagnostic Test
The students were generally found to be successful in trigonometric ratios, Cartesian connection and analyzing graphs, and finding arc lengths by using proportions of angles and arcs. Despite some procedural mistakes, the conceptual connections of the students, which were more important for this research, can be considered to be at quite a good level according to the results of the diagnostic test. The most problematic issue was the function concept. As expected considering the structure of Dutch mathematics education, students were not able to provide a formal definition of the function concept, and most were unable to differentiate function graphs from non-function ones. As consistent with the findings of Tall and Bakar (1992), students tended to mark the graphs that were not familiar to them like the piecewise one as a non-function, and they emphasized a familiar formula or the existence of a formula to define a function. Some students mentioned that because the graph is not regular, it could not be a function. Another consistent finding with Tall and Bakar is that many students did not consider the constant function as a function. Nevertheless, they were mostly very good at evaluating functions, and domain and range concepts. Their problems about the function concept were most likely to be because they had not yet learned about functions in a systematic and formal way.

In conclusion, the students were considered ready for the lesson sequence because of their prior knowledge and skills as measured by the diagnostic test. However, it was still needed to utilize the test results for the sake of more effective lessons. After the test, the students’ test papers were given back to them with the key to the test. Since the function concept was their main problem in the test, it was decided to make an introduction to the first lesson using a process definition of function (input-output), domain and range concepts, and how to decide if a given graph represents a function by using the formal definition. In addition, it was also decided that it would be beneficial in the lessons to remind the students of sine and cosine definitions as ratios, and of the circumference formula of a circle.

### 6.2 Analysis of Worksheets

The analysis of worksheet tasks are provided in this section for the purpose of finding students’ difficulties with the hypothesized learning trajectory (See Chapter 5). This analysis is supported by the audio recordings of the students’ group discussions. The results are reported here worksheet by worksheet for every lesson. All worksheets can be found in Appendix B. There are some issues which might be useful for the reader to keep in mind while reading this section. As mentioned in Section 5.5, there were changes in the worksheets because of a shortage of time during the implementation. The results presented here are according to the tasks in the changed versions of the Worksheets 7-10. The worksheets in Appendix B include these changes. In addition, Worksheets 5, 8, and 9 were given as homework to students. Considering that the tasks included some important steps of the learning trajectory, the analysis of student responses to these homework assignments is also reported here.
However, very few students delivered their homework sheets; hence, the analysis of them is quite limited.

A constraint occurred in the analysis of audio recordings. The students discussed the tasks in Dutch. Hence the researcher could not understand the discussions in the recordings, and it was not possible to listen to and transcribe all of them. The researcher determined the points on students’ worksheets which might be interesting, and then received help from a friend to understand the respective parts in the recordings. However, it was not possible in most cases to get a lot from the recordings because the students discussions were not as detailed as expected. This was most probably because the kind of tasks requiring mathematical investigation was very different for students, and they were not used to doing group work in classes.

### 6.2.1 Analysis of Student Responses to Worksheet Tasks

The analysis of worksheet tasks is presented in this part according to the separate worksheets in each lesson. Although the students were told to work with the same partners in pairs, this was not always the case. In the second and fifth lessons there were twelve groups, in the fourth one ten, and in the others there were eleven groups. The student responses are reported here in a given group. When a result is provided regarding, for instance, Students 14 and 15, this means that they worked together on the corresponding task, and their responses belong to their particular group.

#### Lesson 1

**Worksheet 1a**

It was a difficult task for the students to draw the graph illustrating the vertical position of a moving point P versus the distance it travels along the unit square. However, eight of the eleven groups came up with the correct graph in the end. Since the students did not understand the task well enough to carry it out easily, the help of the researcher and the teacher was crucial for students to make progress. For instance, Students 14 and 15 were first trying to find a formula for the function to draw the graph. Similarly, Students 5 and 9 mentioned that the task is difficult because “*There is not even a formula*”. The reason for this is probably that students were only familiar with drawing graphs of functions whose formulas were given; hence, this was a different kind of task for them.

However, the researcher and the teacher were able to help the students understand the task so that they could draw the graph. This was mostly in the form of explaining some coordinates for the students. It was good that the students could proceed and complete the task. The following excerpt shows how one group reacted to and continued after the teachers’ help.
[T: teacher, S: student]

...  
T: 1, 2, 3 [referring to the travelled parts of the unit square], then the vertical position is 1.  
S 15: Yes, so 1.  
T: Thus (3, 1) etc. And that way you complete it.  
S 15: That one is at 0 and the other at 1. You have not moved. At 1 you still have travelled 0  
[referring to the starting point (1, 0)]. At 2 you travelled that. At 4, it is no longer 0, but  
minus, right? Then it is minus and then again it is again at 0.  
S 14: I do not understand.  
...

S 15: Here [at (1, 0)] the distance travelled is 0. So, it is (0, 0). After this, 1 step is travelled, so  
it has the vertical position 1.

It can be seen here after the teacher gave hints to the students, Student 15 understood the task, and  
could explain it to his peer. He could proceed from the teachers’ explanations to finding the  
coordinates of the starting point by reasoning that at the beginning no distance had been travelled.

Students 5 and 9, one of the three groups which could not draw a correct graph, plotted incorrect  
coordinates and did not connect them to obtain the graph. Although the excerpt below from their  
group discussion shows that they could interpret the specific points on the square correctly, they could  
not plot them correctly to draw the graph.  

“If you have not travelled, you are here [at (1, 0)]. Then one above, then the travelled distance is  
1. Then it stays here at 1 and you have the vertical position 1. Then you have one step to the side,  
and then you have travelled 2. And if you keep moving, then the y-coordinate becomes 0. And then  
you go to -1, and then you go again to the top.”

The other two groups of Students 2 and 4; and Students 6, 10 and 11 each made a mistake about the  
linear structure of the required graph by drawing some parts of the graph as curves as it can be seen in  
Figure 6.3.

![Figure 6.3: The graph of Students 6, 10 and 11](image)

This was actually a big issue not only for these two groups, but the majority of the students  
considered this in their work. The researcher explained why the coordinates had to be connected by  
lines with a constant rate of change, as there was the same amount of change in the travelled distance  
and in the vertical position for the intervals where the vertical position changed.
Worksheet 1b:

Only three groups out of eleven could not provide the correct answers on the first task. They wrote (10, 1), (14, -1) and (5, -1) respectively as the positions on the unit square corresponding to the given points A(2, 1), B(6, -1), and C(13, -1) on the graph. The following excerpt from the discussion of Students 14 and 15 shows that they considered travelled lengths corresponding to the same positions as the given coordinates on the graph. In other words, they determined some other points on the graph corresponding to the same positions on the unit square.

S 15: You must have a point where they meet.
S 14: (14, -1).

Therefore, their mistake was because they did not understand the task in English, rather than because of an important mathematical mistake. They investigated the periodicity in a way. They realized their mistake before the worksheets were collected by stating “We were completely wrong. The problem is that we incorrectly translated “correspond”.”

On the second task, all of the groups gave the correct answers for the travelled distances when the point P returns to its starting position for the first, second, and third times, and they could write the coordinates on the graph corresponding to these positions. However, only three groups stated that x coordinates are the integer multiples of 8, but many groups mentioned that the y coordinates of all points are 0.

As expected, the students provided responses such as “It repeats itself, it is a function, its domain is [0, \( \rightarrow \• \])” However, two groups wrote that the range was (-1, 1) indicating that they did not know the mathematical difference between parentheses and brackets. Two groups of Students 2 and 4, and 14 and 15 wrote that the y coordinates were -1, 0, 1. Although they could not state it correctly, these students probably referred to the range. Students 1 and 12, on the other hand, provided an argument which was about periodicity. “If you divide the value of y by 8, you will get the amount of times you went around the square.” Considering that they might have come to this point as a result of the previous task, they probably meant the x-coordinate but wrote y by mistake.

Lesson 2

Worksheet 2

The students generally performed well regarding recognition of the relationships among the points S, Q and P. Students 8 and 19 explained the fact that the point S on the graph combines the information from the points P and Q as:

“If you get x of Q and y of P then you get the coordinates of S.”
Students 13 and 17 wrote the following:

"S is the point that is the same as P on the square but then in the graph. P is the high [sic] of S and Q is the travelled distance of P."

Although the language is not very clear here, it can be inferred that the students realized that the point S gives information regarding the vertical position corresponding to travelled lengths as shown by the point Q.

The students could also draw the graph on the negative side of the x-axis although many of them could not understand the task because of the word “extend” at the beginning. Once they understood, they could give explanations of the extended part of the graph. Some pairs explained that the extended part shows the movement of P in the clockwise direction while others also gave the corresponding point on the unit square to (-5, 1) on the graph. For instance, the group of Students 7 and 12 stated that “-5 \( \rightarrow \) go 5 units back, with the vertical position 1.” And the group of Students 2 and 4 stated, “The point (-5, 1) is equal to (-1, 1) on the unit square.” These students obviously realized that the movement is in the reverse direction, which shows that they could relate the graph to the unit square quite well.

Students 13 and 17, on the other hand, tried to explain the point (-5, 1) by showing the positions of the points S, Q and P. Although they could position S and Q correctly as S is the point (-5, 1) on the graph and Q is the travelled distance of 5 units on the negative side, they marked P at (-1, 0) instead of (-1, 1). It is interesting that they could not determine the position of P correctly although they had mentioned “P has the same height as S” in the previous question. They most probably counted 5 units on the unit square incorrectly. Figure 6.4 shows their graph.

![Figure 6.4: Extension of the graph and related points by Students 13 and 17](image)

On the last part of Worksheet 2, every group stated that the graph shows a function. Some groups mentioned the formal definition “There is only one y for every x.” as discussed in the first lesson, while others combined this definition with an input-output mechanism instead of x and y. However, some groups did not give the formal definition correctly by using “one y” instead of “only one y.” There is evidence that these students could not distinguish “one y” from “only one y” by the
concerned mathematical language rather than a mistake. This can be seen in the response of the group of Students 10 and 24:

“It is a function because if you have one input then there is only one output. For every x there is one y.”

As it can be seen in these statements, while the students mentioned “only one output” in the first sentence, they wrote “one y” in the second one. This seems to indicate that they could not distinguish them.

Worksheet 3

Students 20 and 21 left the worksheet unanswered. Except for the group of Students 14 and 15, all other groups were able to find the period of the graphs for the 8-gon and 12-gon correctly, which are approximately 6.6 and 6.4 respectively. This indicates that the students had learnt periodicity well in the previous lesson. Nevertheless, six groups gave the written response that the period was (6.6, 0) for n=8, and (6.4, 0) for n=12. It is not clear there if these students gained an idea that the period is a point instead of an interval, or they confused with the coordinates because they were focusing on the changes in the coordinates defining the period. On the other hand, Students 14 and 15 stated the periods as [0, 6.7] and [0, 6.4]. Their audio reveals that they made the same mistake as the ones with (6.6, 0) and (6.4, 0), but in their case they placed x and y coordinates incorrectly in brackets as well as not being able to read the first x-coordinate correctly in the applet.

For the question what happens to the polygon and the graph when n gets larger, three groups provided unusual responses. Students 5 and 9 stated “The line becomes a wave instead of straight lines.” This was unusual among the student responses because they could put the change in the graph with nice words reflecting a basic characteristic of the sine curve. The groups of Students 1 and 16, and Students 14 and 15 mentioned that the period gets smaller, which was a correct observation.

Probably the most important question in this worksheet concerned verifying that the period approaches $2\pi$ for very large values of $n$ because students were required to connect the concept of periodicity to the unit circle. Nine of the eleven groups could explain this connection very well by stating that it corresponded to the circumference of the circle with the radius 1.\textsuperscript{15} Students 3 and 18, on the other hand, wrote “It becomes a circle with diameter [of] 2 [units], therefore the area becomes $2\pi$.” It is quite possible that they used the word “area” for “circumference”. However, this is not clear because they did not write the formula they used.

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\textsuperscript{15} Note that it had not been called the unit circle yet at this point.
Lesson 3

Worksheet 5 (Homework)

Due to a shortage of time in the lessons, Worksheet 5 was given as homework to the students, then discussed in lesson 3. Only sixteen students out of twenty-four delivered their homework. Six did it in three pairs, but two pairs and one student did the first task only.\(^{16}\)

Of sixteen students, fourteen could find the rotation angles and the coordinates of the moving point corresponding to a movement of \(\frac{3\pi}{2}\) along the unit circle. Only Student 15 made a mistake with his claim that the movement corresponds to a rotation of 90°. This was wrong because the direction was given as counter clockwise in the task. Student 17 wrote the coordinates as (-1, 0) instead of (0, -1). In addition, all students who completed the homework wrote down the correct central angles in degrees corresponding to the given arc lengths. Student 15 did not repeat his mistake in the previous task here. These results show that relating angles and arcs to each other was not difficult for the students who completed the task.

For the task of giving the smallest positive value of \(x\) for which \(s(x) = -1\), two students did not give an answer. Some students mentioned that the input of the function is arc lengths. Student 8, who mentioned that \(x\) represents the travelled distance as the input of the function, was the only student who gave the wrong answer of 0 for \(x\).

For the given claim \(s(x) = \sin(\alpha)\), three students did not give any response while the other thirteen stated that \(\alpha\) stands for an angle subtended by the arc length \(x\). However none of them could verify the claim for \(x = \frac{\pi}{4}\) although some calculated \(\sin 45^\circ\). Except for two students, the other fourteen answered all parts correctly concerning the relationship between angles and arcs for given \(x\) values.

Although seven students wrote the correct formula for the function as \(s(x) = \sin\left(\frac{180^\circ x}{\pi}\right)\), only two of them explained their work. Student 7 explained her work by using the proportions with the variables \(x\) and \(\alpha\) as required.

Although the data from Worksheet 5 does not reveal students’ difficulties to a great extent, the most plausible explanation can be that the students were able to relate arcs to angles from travelled lengths to rotations, and could connect information to the graph, but it was difficult for them to verify the claim that the function \(s\) is equal to the sine of the angles corresponding to the arcs, and come up with

\(^{16}\) Although the data from this worksheet may be of limited value because the students did not do the tasks in class, it is still useful to look at their work because the problems given was based on the transition from arc lengths to rotations angles.
the formula of the function. Their difficulty with the formula was also observed in the lesson. Because they were not clear about how the formula was obtained by proportional reasoning when the researcher explained it, they asked their regular teacher to explain it in Dutch.

**Worksheet 6**

It was very difficult for the students to understand the first task of elaborating the given definition of radians to the unit circle and finding the angle in radians corresponding to the given arc length. The following dialogue shows not only their difficulty but also their resistance to this kind of task:

[R: Researcher]
S 16: We do not know what radian is?
R: Here is the definition. You can use this definition to answer the question.
S 16: Yeah but we never learnt it before.

It shows that the students were not familiar with the type of task asking them to elaborate on a given claim or definition, infer results, and do this as a mathematical investigation.

As another indication that the students did not understand the task or did not read it carefully, four of the eleven groups stated that the angle is 45° although the question was to give the angle in radians. Only two groups, Students 20 and 21, and Students 7 and 12 could state that the angle in radians is equal to the corresponding arc length.

Although every group could state the angles in degrees and radians corresponding to the arc length of 2π, only five groups clearly explained how to convert degrees and radians to each other.

The fourth part of the worksheet which was rewriting the formula \( s(x) = \sin(\frac{180\degree}{\pi}x) \) by converting \( \frac{180\degree}{\pi} \) to radians and explaining what \( x \) can denote was extremely difficult for the students. Only four groups could convert \( \frac{180\degree}{\pi} \) to radians correctly. Two of them, Students 20 and 21, and Students 14 and 15, stated that after the conversion to radians the formula becomes \( s(x) = \sin(x) \), and Students 10, 11 and 24 stated that it becomes \( s(x) = \sin(x \text{ radians}) \). Only one group, Students 20 and 21, could state that \( x \) can denote arc lengths or, equivalently, angles in radians while the other two mentioned only arc lengths.

Not every group provided correct responses to all three parts about the characteristics of the sine function. This was most probably because there was not enough time for most groups. The last part about some characteristics of the sine function was also difficult for the students. Only four groups could state the domain, range and period correctly. Although the others provided more correct answers for the domain and range, they gave wrong answers for periodicity, or left it unanswered. Interestingly, the groups of Students 13 and 17, Students 8 and 19, and Students 1 and 16 stated that
the period is 8. They probably could not remember the last version of the graph for the unit circle, or they just confused it with the graph for the unit square.

Lesson 4

Worksheet 7

It was in general an easy task for the students to find the sine of a given acute angle $\alpha$ and connect it to a right triangle in first part. Seven of the ten groups clearly showed on their worksheets that sine of the given angle, which is the corresponding vertical position, can be found in the right triangle by dividing the opposite side to the hypotenuse, which is 1 because it is equal to the radius of the unit circle. Among the others, Students 7 and 12 wrote that the sine value is equal to the opposite side, but they did not state their explanation. The audio recording of their discussions reveals that they determined the correct right triangle and mentioned that sine value is equal to the opposite side of this right triangle. However, it is not clear that they came to this by considering the ratio definition of sine.

Students 10, 11 and 24 drew the correct triangle, and stated that the hypotenuse is 1. However, they could not connect this triangle to the vertical position as the sine of the given angle. They could not remember that the sine value can be found with the vertical position without the triangle, and hence they could not connect the vertical position to the opposite side of the triangle. They had the same problem in the second part of the worksheet, which was connecting $\sin 120^\circ$ to a triangle. After the researcher reminded them that they could first find the sine value with the corresponding vertical position, and then they could try to connect it to a right triangle, the rest was easy for them.

The other nine groups could connect $\sin 120^\circ$ to a right triangle. All of them stated that $\sin 120^\circ = \sin 60^\circ$ because in the right triangle the opposite side to $60^\circ$ is equal to the vertical position which is $\sin 120^\circ$ and the radius is 1. It was also observed that this second part was easier for the students after they did the first part with the help of the researcher for some groups. In addition, as a very good point regarding the students’ understanding to integrate triangle trigonometry to the unit circle, two groups, Students 1 and 16, and Students 13 and 17 mentioned that $\sin 120^\circ$ can also be calculated by using $\cos 30^\circ$ referring to a different right triangle in the second quadrant.

Worksheet 8 – Part 1

After a discussion about the graph showing the horizontal position versus the travelled distance along the unit square, and about the change of the graph when the unit square approached the unit circle, the students were given the first part of Worksheet 8 as an introduction to the cosine function.
Students were found to be quite successful on writing the coordinates of the points given on the graph of horizontal position versus travelled arc length along the unit circle. Only the group of Students 1 and 16 gave completely incorrect coordinates. For example they wrote $A(0, 1)$ instead of $A\left(-\frac{3\pi}{2}, 0\right)$. Their audio reveals that they made this mistake because they were talking about the coordinates of the points on the unit circle corresponding to the given points on the graph. They did not understand the task correctly, and hence they did something else. Nevertheless, even if they did not do the correct task, their responses show their capability to relate the information from the graph to the unit circle.

Three groups could provide the correct coordinates only for all points except point $G$ which was $(2\pi, 1)$. Two of them did not write it while the group of Students 6 and 22 wrote it as $G(2\pi, 0)$. Their work clearly revealed that they considered sine of $2\pi$ instead of cosine, and hence they came up with 0 instead of 1.

It can be concluded that the previous tasks for sine helped students to make the connections between the unit circle and the graph through arc lengths because they did not have significant problems with the first part of Worksheet 8.

**Worksheet 8 – Part 2 and Worksheet 9 (Homework)**

Due to the time limitations, the second part of Worksheet 8 and all of Worksheet 9 were assigned as homework as a means of preparation for the last lesson. Eight students out of twenty-four delivered Worksheet 8 – part 2. The students generally provided correct answers for the domain, range and period of the cosine function, but two of them could not give answers for the requested cosine values. On the other hand, only three students delivered Worksheet 9, and only one of them, Student 1 could show the given trigonometric equality by using a right triangle and the figure of sine and cosine graphs. Although it is not possible to know the exact reason why they did not deliver homework papers, this may be an indication that proof tasks were difficult for them.\(^{17}\)

**Lesson 5**

**Worksheet 10**

As mentioned in Section 5.5, the lesson plan was changed in Lesson 5. At the beginning, the previous lessons and the cosine function were discussed. The researcher explained how cosine can be defined as a function of real numbers through the equation $c(x) = \cos\left(\frac{180^\circ x}{\pi}\right)$. Although this was almost the same as it was for sine, it was still confusing for the students as observed in the lesson.

\(^{17}\) The analysis of trigonometry test (Section 6.3) also confirms this.
Nine groups out of twelve were able to show the equality \( \cos(x) = \cos(-x) \), but two of them, Students 8 and 19, and Students 1 and 16 showed it for \( 90^\circ \) only. However, they could explain a generalization in words. Students 1 and 16 stated that “If you [move] clockwise or counter clockwise with the same arc length, your horizontal position will be equal.” This clearly shows that they were aware that the negative sign was only about the direction, and the horizontal position is always the same, not only for \( 90^\circ \). Additionally, they showed the equality by using the graph of cosine as well as the unit circle.

Students 22 and 6 explained that \( \cos\left(\frac{180^\circ x}{\pi}\right) = \cos x \) by using the radians in this part of the worksheet. Although this was not needed for this task, it shows that at least some students understood this equation of the cosine function, which was observed to be difficult in general for the students.

Two groups of Students 10 and 11, and Students 5 and 9 could not show the equality. On the other hand, Students 7 and 12 did not provide a clear response for this task. They marked only 4 points, each of which were on different quadrants such that they were symmetric two by two with respect to the x axis. It is clear in their audio recording that they realized the equality on their correct figure, but they could not figure out how to explain it as they stated “How am I supposed to write this?”

On the other hand, the second task “Calculate \( \cos 210^\circ \) and \( \sin 210^\circ \)” was found to be more challenging for the students. Only two groups, Students 1 and 16, and Students 8 and 19, could provide the correct answers for the two calculations. The group of Students 13, 17, and 24 succeeded with the help of the researcher. Students 7 and 12 could give the correct answers, but they did not provide any explanations, and they gave the rounded answer for \( \cos 210^\circ \). Thus, they probably used calculators.

Three other groups showed the correct position on the unit circle and a correct triangle to use for the calculation, but they could not finish their work. Interestingly one of them, Students 14 and 15, first tried to convert the degrees to radians. It can be heard in their audio recordings that after the researcher told them that they did not need radians, they came up with the conclusion that they could find \( \sin 210^\circ \) by \( \sin 30^\circ \) by referring to the correct position on the unit circle, but they could not finish it because there was no time left. The last three groups did not provide any answer for this task. In the class discussion, the researcher realized that the students did not remember a correct combination of the side lengths of a \((30^\circ, 60^\circ, 90^\circ)\) triangle. It might have been a common difficulty of the students on this task.
6.2.2 Summary Remarks

The analysis of student responses to worksheet tasks reveals the steps with which the students had difficulties. Note that these worksheet tasks address the steps of the learning trajectory as explained in Chapter 5.

The first task which was found to be difficult for the students was drawing the graph illustrating the vertical position of moving point along the unit square versus the distance it travels. The reason for this difficulty was basically that the students did not understand the task at the beginning. However, after the researcher and cooperating teacher explained the task to the groups, the task was easy for them. Apart from the difficulty with understanding the task, it was unclear to many students whether they needed to draw lines or curves to connect the points they marked in order to obtain the graph.

The second difficulty was with the task of obtaining the formula of the graph by using the relationship between the arcs and subtended angles in the formula \( s(x) = \sin(\alpha) \). Although students did not have difficulties with the relationship of such angles and arcs, most of them were not able to come up with the formula. Since this task was a homework assignment, the data may not be reliable. However, with this said, students’ difficulties on this were also confirmed in the classroom. Understanding the proportional operation to get the formula was not easy for the students. A possible explanation is that students could not set the mathematical relationship between the function \( s \) and the sine of an angle as well as applying the proportional calculations. In addition, students also had difficulties in obtaining the same formula for the cosine function.

Another difficulty was the conversion of \( \frac{180^\circ}{\pi} \) to 1 radian, and using this in the formula of the function to transit to the sine of real numbers. Although these were found to be difficult in the group work of the students, much of the difficulty was cleared up by the end of the classroom discussions.

The last difficulty of the students was with calculating sine and cosine values of 210°. Although students were found to be good at illustrating the corresponding rotation in a figure, and pointing to the correct position, only a few groups could come up the correct answers. It was found that students did not know the side lengths of a right triangle of 30° and 60° to calculate the sine and cosine ratios.

These were the main mathematical difficulties of the students in their group work. In addition, it was found that students had also problems in explaining their reasoning and their responses. This was because they were not used to engaging in such kinds of mathematical tasks. Furthermore, there were some problems with mathematical language and notations in their written responses. For example, some students did not know the difference between \([-1, 1]\) and \((-1, 1)\), and some did not distinguish
“one y” from “only one y” in their function definition. The researcher was not able to detect such issues during the implementation.

### 6.3 Analysis of the Trigonometry Test

The trigonometry test, which aimed to assess students’ knowledge and understanding of the key points of the lesson-sequence, was given to the students three days after the intervention finished. All students in the research cohort, twenty-four in total, took the test. They were asked to complete the test in 50 minutes, and were told that their marks would count towards their mathematics grade. In regard to their difficulties with English in the lessons, only the phrase “arrange in ascending order” was explained at the beginning of the test because the other terms had already been covered in the lessons.

The first part of this section provides the analysis of the student responses to the test items. The whole test can be found in Appendix D. The section ends with some important remarks regarding the analysis of the test.

### 6.3.1 Analysis of Student Responses to Test Items

The results of the trigonometry test were analyzed qualitatively here by paying attention to the key mathematical ideas underlying the test items and the lesson-sequence. In this part, first the number of responses to each question, and a summary of student responses are provided in tables. Then the analysis of student responses to the test items is presented regarding the common key aspects related to different test questions. They are:

- Drawing and interpreting the graph of cosine,
- The notion of function and related properties,
- Finding sine or cosine values on the unit circle,
- Showing trigonometric relationships on the unit circle or on the graphs,
- Relationships between arc lengths or angles in radians and angles in degrees.

Twenty-four students took the test, but it is not the case that every student provided a response to every question. Therefore, it might be useful first to present the number of students who answered each question (Table 6.2).

<table>
<thead>
<tr>
<th>Question</th>
<th>1a</th>
<th>1b</th>
<th>1c</th>
<th>2</th>
<th>3a</th>
<th>3b</th>
<th>3c</th>
<th>4a</th>
<th>4b</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8a</th>
<th>8b</th>
<th>9a</th>
<th>9b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of responses (out of 24)</td>
<td>23</td>
<td>16</td>
<td>13</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24 *</td>
<td>23</td>
<td>23</td>
<td>24</td>
<td>24</td>
<td>23</td>
<td>23</td>
<td>24</td>
<td>21</td>
<td>24</td>
</tr>
</tbody>
</table>

*Although everyone responded to Question 3c, not everyone stated both domain, range and periodicity as requested.

Table 6.2: Number of responses to each question
Before reporting the results, Trigonometry Test questions are given in Figure 6.5.

1. a. Draw the graph of \( y = \cos x \) on the figure below. Write down the coordinates of some special points you use on the graph.
   
b. Write down the characteristics that you can derive from the graph.
   
c. Mark the point corresponding to \( \cos (3) \) on the graph.

2. Which of the choices below does \( x \) in the function \( y = \sin x \) represent? Explain your choice.
   i. An angle in degrees
   ii. An angle in radians
   iii. A real number

3. a. Why is sine a function?
   
b. What is the period of the sine function? What does periodicity actually mean?
   
c. What is the domain and the range of the sine function?

4. For the function \( f(x) = \sin x \),
   a. Calculate \( f(-\frac{\pi}{2}) \). Explain your answer.
   
b. What does \( \pi \) stand for in part a?

5. What is the value of \( \sin (225^\circ) \)? Explain (Show) how you got to your answer.

6. Are \( \cos 340^\circ \) and \( \cos 200^\circ \) equal, opposite, or neither? Use the unit circle below to show your answer.

7. Show that \( \sin(-x) = -\sin x \) on the unit circle or on the graph of sine function.

8. a. What is the arc length corresponding to a central angle \( 60^\circ \) in the unit circle?
   
b. How many degrees is the central angle that corresponds to an arc of length \( \frac{\pi}{3} \) ?

9. A, B, C and D are points on the unit circle as shown in the figure below. They correspond to some rotations about the origin in the counter-clockwise direction starting from \((1, 0)\).

   a. Arrange in increasing order the sine values of the angles corresponding to these rotations: \( \sin A, \sin B, \sin C \) and \( \sin D \).
   
b. Mark the point that corresponds to \( B \) on the graph of the sine function below.

Figure 6.5: Trigonometry Test Questions
A summary of the Trigonometry Test results is provided in Table 6.3 below.

<table>
<thead>
<tr>
<th>Question</th>
<th>Aim of the Question</th>
<th>Results (n=24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>To examine if students can draw and interpret the graph of the cosine function, and mark the point on the graph corresponding to cos 3.</td>
<td>Only 11 students could draw the correct curve, but only 9 of them could write the correct coordinates. 8 students attempted to draw the graph of sine, only one of whom could draw it correctly with correct coordinates. 8 students drew graphs wrongly, and one student left this part unanswered. For the characteristics derived from the graph, 8 students did not write anything. The others were found in general to be good at stating domain, range and periodicity. 3 students showed a good understanding in marking the point cos 3 while 11 did not answer this part. Students problems in drawing the graph made it more difficult for them to plot cos 3.</td>
</tr>
<tr>
<td>2</td>
<td>To assess if the students consider the x as a real number in y=sin x because the domain of the function is real numbers.</td>
<td>9 students chose the correct option mostly with sufficient explanations, a real number. 5 students, on the other hand, chose the option of an angle in degrees while 9 others chose an angle in radians. 1 student chose both options of an angle in radians and degrees stating that they are equivalent. Some students who chose the option of an angle in radians indicated a connection to arc lengths, so it is hard to see if they considered real numbers as well.</td>
</tr>
<tr>
<td>3</td>
<td>To examine if the students can explain why sine is a function, state its period with its meaning, and state its domain and range.</td>
<td>Students mostly provided sufficient arguments for why sine is a function. 15 students referred to a formal definition. Of them, 6 did not distinguish &quot;a&quot;, or one y &quot;from &quot;only one y&quot;; 3 combined the formal definition with an input-output mechanism. 6 students mentioned only a process conception based on an input-output mechanism. Only 3 students were unsuccessful. The students were good at periodicity: 17 could explain what it meant correctly by one of the unit circle or graph or both. Since only 11 students stated that the period is 2π, it can be assumed that some did not understand the whole question. 13 students stated both the domain and range of sine correctly. 4 students stated only one of these correctly while other 2 students mixed them up; the notation using brackets or parentheses for the range was confusing for 3 students.</td>
</tr>
<tr>
<td>4</td>
<td>To assess if students can evaluate the function f(x)=sin x at x=π/2 and state what x stands for in the function.</td>
<td>Only 6 students managed to provide a fully correct answer for the first part. 3 students made the direction mistake regarding the rotation along the unit circle. Interestingly, 4 students, in total, claimed that f(x)=π/2, and 90°. The function notation might have been difficult for the students here. Regarding the second part about π, 13 students referred to arc-lengths. Some of them connected it to the circumference of the circle, and 2 students mentioned 3.14. 3 students referred to radians which were difficult to interpret while 5 students mentioned 180°.</td>
</tr>
<tr>
<td>5</td>
<td>To assess if students can compute sin (225°).</td>
<td>10 students were found to be successful at this task although 19 students were good at indicating the correct position on the unit circle. A similar situation to Question 4 was found that 6 students claimed sin 225° = 5√3/4.</td>
</tr>
<tr>
<td>6</td>
<td>To evaluate if students can choose the correct relationship and show it on the unit circle.</td>
<td>14 students chose the correct option and most of them could provide a good explanation. 3 students indicated the vertical position as a mistake. One student referred only the arc lengths on the unit circle. This can be a similar mistake to the ones in the fourth and fifth questions about considering sine as equal to an angle or arc length.</td>
</tr>
<tr>
<td>7</td>
<td>To evaluate if the students can show a given relationship using the unit circle or graph.</td>
<td>Only 6 students could explain the relationship sufficiently, only one of them used the graph of sine, unlike the others who used the unit circle. Most of the others provided some explanations but could not show it clearly. Most students attempted to show the given relationship for only certain values indicating that they did not know enough about proof.</td>
</tr>
<tr>
<td>8</td>
<td>To assess if the students convert radians and degrees to each other when asked for the relationship between an arc and the corresponding central angle.</td>
<td>15 students provided the correct answer for the measure the arc corresponding to the central angle 80° while 20 students provided the correct answer for the measure of the angle to the arc length n/3.</td>
</tr>
<tr>
<td>9</td>
<td>To evaluate if the students can order the sine values corresponding to certain rotations on the unit circle, and mark the point on the sine graph corresponding to one of these positions.</td>
<td>Only 6 students could give the correct arrangement. Apparently, some students did not pay attention to the negative or positive signs. And 1 student clearly arranged the angles instead of the corresponding vertical positions. 4 students tried to estimate the angles corresponding to the rotations. Apparently some students did not understand the task. As in Questions 1 and 6, some students were not careful if the question was about sine or cosine. For the second part of the question19 students were successful.</td>
</tr>
</tbody>
</table>

Table 6.3: Summary of student responses on the Trigonometry Test
**Drawing and Interpreting the Cosine Graph**

Among the mathematical aspects underlying the Trigonometry Test were drawing and interpreting the graph of the cosine function. Because only eleven students could draw the correct curve in Question 1, it can be claimed that the students had problems with drawing the graph of the cosine function. It should be noted that two of the students who drew the graph correctly wrote the coordinates in degrees. This may be considered somehow problematic from the mathematical point that the task was about cosine as a function of real numbers. In addition, there is evidence that some students drew the graph from memory, but they had problems with remembering the correct coordinates as seen in the answers of two students who made mistakes with the coordinates although their curves were correct as it is exemplified in Figure 6.6.

![Figure 6.6: The graph sketched by Student 11](image)

It is also interesting that six students attempted to draw the graph of the sine function instead of the one of cosine. Five students sketched completely incorrect graphs that were of neither sine nor cosine. Interestingly, Students 4 and 20 drew the reflected version of the cosine function with respect to the x-axis. Although it is difficult to infer why these students made such a mistake, it might be another consequence of students’ unsuccessful efforts to memorize the curve.

In the case of Student 15, who was one of the interviewees, there was an exceptional situation for the first question. He drew the correct graph with correct coordinates, but after seeing the last part of the first question that was about marking the point corresponding to \( \cos(3) \), he got confused and changed his graph to the one illustrating the moving point P along the unit square instead of the unit circle. \( \cos(3) \) confused him because integer 3 was easy to count with the case of the unit square. On the other hand, it was clear from his paper that he was able to draw the cosine graph at the first phase correctly.

About interpreting the graph, the students provided expected types of responses probably because of the similar task done in the class, but it is not that every student mentioned all of the concepts function, domain, range, and periodicity. Only two students mentioned that the graph shows a periodic function with the domain of real numbers and the range \([-1, 1]\), and with the period \(2\pi\).
others mentioned that it is a function and about the period, but few students wrote about the domain and range.\(^{18}\)

Students 1 and 8, who sketched the graph with angles in degrees as the x values, stated that the period was \(2\pi\). This may show that students drawing the graph with degrees may still be aware of the connection to real numbers or radians. On the other hand, two students mixed up the domain and range; whereas Student 16 revealed confusion about the definition of a function:

“It is a function... one x for every y.”

**Function Concept & Related Properties**

Another important aspect of the test was to assess the students’ knowledge regarding the concept of function. Among the important issues was how students explained the reason why sine is a function in Question 3. Students generally provided sufficient arguments which can be categorized in three clusters. The first one consists of ten students stating a formal definition as “For every x, there is only one y.”. The second cluster includes five students who provided an input-output mechanism such as “When you put an x, you get a y.”. Lastly, the third cluster is of five students who provided a combined response of the formal and input-output definitions such as “For every input, there is only one output.”. The responses of the students in the third cluster can be considered as revealing a better level of understanding because they considered and combined two different ways of conceptualizing the notion of function.

Six of the fifteen students who referred to a formal definition in the first and third clusters did not distinguish “a, or one” from “only one”. It is difficult to interpret what those students were actually considering by “a, or one” y. Although they were considered as providing a formal definition, they may have considered only an input-output mechanism, but a more likely possibility is that either or both the mathematical language or English was difficult for them for the distinction of “only one y” from “a, or one y.”.

Five students were found with insufficient answers: two of them mixed the places of x and y by stating “there is one x for every y.”, one of the others stated that it is a function because it is a number while one mentioned that “you want to know y.” The last student left this part unanswered.

As one of the main focuses of the lesson-sequence, the students’ conceptions of sine and cosine as the functions of real numbers were assessed in the test as were reflected by their responses to the last part of Question 1, Question 2, second part of Question 3, and the last part of Question 4. The last part of the first question, “Mark the point corresponding to cos (3) on the graph.” was found to be very

\(^{18}\) Although the same task was done in the first lesson for the sine graph, it appeared in the interviews that students had still problems with the word *characteristics*. This may have also been the case in the test.
difficult for students. Only three students showed a good understanding for this part, and eleven students left this part unanswered. One of the three students with better responses marked only the point \( x=3 \) just before the \( x \) coordinate she had plotted as \( \pi \). Although she did not show the point on the graph as requested, she was still able to consider the domain of the function as real numbers. One of the other two students marked the correct point as corresponding to \( \cos(3) \), but she also marked the point \( \cos(-3) \) as \( \cos(3) \) strangely. Student 11, on the other hand, marked the point on the graph just before the point whose \( x \)-coordinate is \( \pi \). However, she was mistaken due to the wrong coordinates of her graph in part a. Her work can be seen in Figure 6.5. It can be claimed that students’ difficulties with drawing the graph made this part more difficult for them.

When the students were asked in the second part of Question 4, “What does \( \pi \) stand for in part b (\( f(-\frac{\pi}{2}) \))?”, thirteen mentioned that \( \pi \) was for the arc length, two of whom approximated its value as 3.14.

The following are example responses:

Student 1: “With \( \pi \) you can explain arc lengths. The whole circle has a circumference of \( 2\pi \). With \( \pi/2 \) you can explain a quarter of the circle.”

Student 11: “\( \pi =3,14 \). It is a negative \( \pi \) that means it goes the same way as a clock.”

On the other hand, three students mentioned that \( \pi \) stands for radians. Considering that the sine was given in the function form, the explanations related to arc lengths as real numbers were more acceptable. It is interesting that although the students were not really unsuccessful on this item, they revealed a much worse performance on marking the point \( \cos(3) \) on the graph.

Although thirteen students referred to arc lengths for \( \pi \) in Question 4, only nine students chose the option that \( x \) stands for a real number in Question 2. Two of these nine students did not provide any explanation while the others gave the following explanations:

Student 2: “It is the vertical position the representing \( y \) from the function.”

Student 3: “Because you can fill in every real number \( x \).”

Student 4: “\( x \) represents the horizontal position down the graph which can be only a real number.”

Student 6: “Because for every \( y \) is a \( x \), for every number is a number.”

Student 12: “You want to know what the vertical position is. That is calculated in the case in real numbers.”

Student 15: “For every \( x \) is one \( y \), i [the option angle in degrees] is not possible, ii [the option angle in radians] not possible then you get \( \sin \pi \) or something and cannot calculate.”

Student 22: “Because you do sine of a number (\( x \)), you get another number that equals to \( y \).”

Among these, Students 3, 4, 6, and 22 provided better arguments than the others. Students 3, 6, and 22 revealed a clear connection to the domain and range by probably considering the function as an input-output mechanism, and Student 6 had confusion with the definition of a function.

On the other hand, nine students chose the option radians, and three of them mentioned a connection to the arc lengths. Hence, those students might have considered the relationship that the angle in radians is equal to the corresponding arc length in the unit circle, as Student19 stated that
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"[After choosing the option radians] Because in the graph you can see how much the point P has walked [sic] around the unit circle."

Moreover, sixteen students stated the domain of the sine function correctly in the third question, but the number of students who chose the correct option in the second question was only nine. The correct option was that x represents a real number in \( y = \sin x \). This could be an indication that some students were not able to connect the second question and also \( \pi \) in the fourth question to the domain of the function. Therefore, it can be claimed that although the students were good at conceptualizing sine or cosine as a function of real numbers by connecting it through arc lengths, they could not state it explicitly when it is asked in the function form of \( y = \sin x \). This may be because they had problems with the meaning of domain.

Regarding the range and the periodicity of the sine function, students were able to state the range of the sine function. However, 3 students had problems with the bracket notation as they mentioned \((-1, 1)\) as the range instead of \([-1, 1]\). It can be inferred that students revealed a better performance about the periodicity because only four students did not give an explanation for it. On the other hand, only fourteen stated the period, three of whom were wrong. This might be because the students did not realize that the task was both explaining the meaning of periodicity and stating the period of the sine function.

The last issue regarding the function concept in the context of trigonometry was evaluating the function \( f(x) = \sin x \) for \( x = -\frac{\pi}{2} \). Although it was a standard type of task, only six students could provide the correct answer with a sufficient explanation and drawing. On the other hand, seven students indicated the correct direction due to the negative sign, and the position on the unit circle although they could not come up with the final result. In addition, 6 students mentioned \( f(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) \) or \( f(-\frac{\pi}{2}) = \sin(-90^\circ) \) without giving the final result of -1. A different answer from the others was given by Student 11 revealing also a different level of understanding. She drew a correct figure and mentioned that \( f(x) = \sin(-90^\circ) = \sin(270^\circ) \) in a way to connect positive and negative directions. These responses may mean that although the students were good at finding the value of the function using the vertical position on the unit circle, the function notation given in the form \( f(x) = \sin x \) in the question confused them, and they could not give the final answer.
In addition, three students interestingly stated that the answer is \( f(x) = -90^\circ \). These students indicated the correct direction and the correct point corresponding to the requested sine value. The work of Student 10 illustrating this situation is given in Figure 6.7, and one student mentioned \( f(x) = 90^\circ \) by making also a mistake with the direction as well as giving the function value as an angle.

**Unit Circle Definitions of Sine and Cosine**

Another aspect of the students’ understanding assessed in the Trigonometry Test was if the students were able to consider the sine or cosine of angles with the corresponding vertical or horizontal positions on the unit circle in different types of tasks as in Questions 5, 6, and 9. Note that Question 5 also requires integrating ratio definition of sine to the unit circle.

In Question 5, “What is the value of \( \sin (225^\circ) \)? Explain (Show) how you got to your answer.”, nineteen students could point to the correct position on the unit circle, but only ten students could provide a better level of understanding related to integrating triangle trigonometry to the unit circle definition. Among them was Student 13 who drew a correct figure and illustrated the situation well, mentioning that the needed coordinate for \( \sin (225^\circ) \) can be found by \( \cos (45^\circ) \). This was different from the answers of others who calculated it with \( \sin (45^\circ) \). However, she did not or could not calculate the value of \( \cos (45^\circ) \) as the final step.

An interesting finding is that some students converted \( 225^\circ \) to radians although they did not need to for this task. Six students could provide a correct drawing and could convert \( 225^\circ \) into radians, but wrote that \( \sin (225^\circ) = \frac{5\pi}{4} \). Four of these students were among those who equated the required sine value to the angle \( 90^\circ \) or \(-90^\circ\) in the fourth question. Nevertheless, one of the exceptional two students was one of those who could provide the correct answer for the fourth question. Hence, it is difficult to claim an exact interpretation for this common mistake. These students might have had problems in mathematical writing\(^\text{19}\) or they may have really developed a conceptual flaw in their thinking that sine is equal to an arc length or angle rather than the corresponding vertical position. Student 10’s response is given in Figure 6.8 as an example of this kind of mistake.

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\(^{19}\) Note that it was a common problem of many students not to write their answers in a proper and systematic way.
In Question 6, all students but two could draw the correct rotation angles, however not all of them pointed to the corresponding horizontal positions. Three students considered the vertical positions for cosine by mistake, where one revealed that he had been considering the corresponding arc lengths, instead of the horizontal positions. A similar mistake was made in Question 9 by three students who considered the horizontal positions corresponding to the given rotations although the task was about sine values. Such mistakes were most probably due to carelessness during the test.

In Question 9, the students could not perform well enough to arrange the sine values corresponding to the given sine values. Only six students could arrange correctly the corresponding sine values in increasing order. Student 19 provided the correct answer with the best explanation. Her work is presented in Figure 6.9 below.

On the other hand, there is some evidence that the task was not really clear for some students. As exemplified in Figure 6.10, five students tried to estimate the corresponding angles and arrange them instead of their sine. In addition, a student put a question mark, and another student wrote “I do not understand.” for this question. So, it appears that these particular seven students did not understand what they were expected to do. Once again another possibility could be some students considered that sine is equal to the corresponding angle or arc length. However, the same individuals did not repeat this mistake. So, it cannot be given as a misconception of students.
On the other hand, the majority of the students, eighteen in total, could mark the correct point on the sine graph corresponding to one of the given rotations. Only five students could not do it, and one student made a mistake with the direction of the rotation. Thus, the students were found to be quite good at connecting the information from the unit circle to the graph of the sine function.

**Showing Trigonometric Relationships**

Questions 6 and 7 aimed to assess students’ skills in showing the given trigonometric relationships using the unit circle, and the unit circle or the graph, respectively. As a group, students were found to be more successful on Question 6 than on Question 7. Possibly this was because the proof work in Question 7 with a variable \( x \) was confusing to students. Fourteen students chose the correct option *opposite* for the relationship between \( \cos(200^\circ) \) and \( \cos(340^\circ) \). Students’ mistakes of drawing the correct angles on the unit circle, considering sine instead of cosine, or considering only arc lengths were the reasons prohibiting their success in showing that \( \cos(200^\circ) \) and \( \cos(340^\circ) \) are opposite.

In Question 7, only six students were able to explain the given equality sufficiently. Fourteen of the other twenty-three students could not give a clear and sufficient explanation although eight of them revealed a kind of understanding by being able to explain the equality in words, but they could not or did not relate their explanations to a drawing with the unit circle or graph, or they made partial mistakes with the figures. Student 22, who was one of the few students trying to show the equality on the graph, provided an explanation, but did not show her work on the graph clearly, as can be seen in Figure 6.11.

![Figure 6.10: Ordering the given angles as the order of sine values by Student 10](image-url)
Interestingly, all of the students who could show the equality used specific values for $x$. This may be because the students were not capable of doing mathematical proofs, and they did not know the difference between verifying a given equality for a certain value and proving it for a variable. Student 8, for instance, showed the equality for $x = 45^\circ$:

Last but not least, only six students preferred to show the equality on the graph. Although the others used the unit circle, they could not perform as well as they could on Question 6. This is perhaps another indication that students had difficulties with proving trigonometric relationships given with a variable. Another possibility could be the minus signs in the equality were confusing.

**Arc – Angle Relationship through Radians**

The last aspect in this test was evaluating the students’ knowledge regarding the arc lengths and subtended angles in Question 8. The students were generally successful on this task. Fifteen students provided the correct answer $\frac{\pi}{3}$ for the arc length corresponding to a central angle of $60^\circ$. As a good
point, almost all of them explained their work with proportional reasoning. On the other hand, twenty students gave the correct answer $60^\circ$ for the angle corresponding to an arc length $\frac{\pi}{3}$. It was interesting that more students could give the correct answer when the same task was asked from the reverse direction, in other words, from arc length to angle. This might be because some students might have interchanged $\pi$ with $180^\circ$ in the second part of the question unlike the students who used a proportional reasoning and could solve the both parts of the question.

6.3.2 Summary Remarks

The analysis of the student responses in the trigonometry test was presented above. Students were found to be good at their understanding regarding certain tasks whereas they provided problematic answers for other tasks.

Generally speaking, it can be concluded from these results that the students revealed a good level of performance on Questions 3, 6, 8, and part b of Question 9; a moderate level of performance on Questions 1, 2, and 5; and a more limited level of performance on Questions 4, 7, and part a of Question 9 compared to the other questions. This is shown in Table 6.4 below. This means that students performed better on the test items about the concepts of domain, range, periodicity and the reason(s) why sine is a function; showing a trigonometric relationship given with degrees on the unit circle; and connecting the information given on the unit circle to the graph of sine. On the other hand, they performed moderately better on the items regarding drawing and interpreting the graph of the cosine function, stating that $x$ represents a real number in the expression $y = \sin x$, and computing $\sin 225^\circ$. However, fewer students could succeed in the tasks regarding $\sin \left( -\frac{\pi}{2} \right)$, showing a given algebraic relationship on the graph or unit circle, and ordering sine values corresponding to certain rotations on the unit circle.

<table>
<thead>
<tr>
<th>Level</th>
<th>Question Number</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>3, 6, 8, 9b</td>
<td>Function concept including domain, range, and periodicity, showing trigonometric relations on the unit circle, connecting the information from the unit circle to the graph of sine.</td>
</tr>
<tr>
<td>Moderate</td>
<td>1, 2, 5</td>
<td>Drawing the graph of cosine, stating that $x$ represents a real number in $y = \sin x$, computing $\sin 225^\circ$.</td>
</tr>
<tr>
<td>Limited</td>
<td>4, 7, 9a</td>
<td>$\sin \left( -\frac{\pi}{2} \right)$, proving a trigonometric relationship for any $x$, ordering the sine values corresponding to given rotations</td>
</tr>
</tbody>
</table>

Table 6.4: A general classification of the students’ performance on the test items

Nonetheless, such comments on students’ performance are of limited value for two reasons. First, they are very general in that they cannot reflect the deep understanding of the individual students. For instance, it is stated that students’ performance was limited on calculating $\sin \left( -\frac{\pi}{2} \right)$, but it cannot be
claimed that their understanding of it was completely poor because they knew the definition of sine, and the proper direction and the corresponding arc. Their mistakes could be due to the use of the function notation which was not familiar to them. Secondly and more importantly, such a test from which these comments were gathered can only provide restricted information about the understanding of the students although it is useful to see a general picture illustrating where the students had reached by the end of the lessons. This is mostly because it is almost impossible to be sure about how students think while responding to a test question, and the students’ responses were not always clear enough to interpret. In this regard, the researcher faced some limitations in the analysis. For example, it was difficult to make certain claims regarding the student responses evaluating the sine function for certain inputs by considering the angles in degrees although they mentioned \( \pi \) was for an arc length or real number. Similarly, it was difficult to interpret student responses in the case that Student 8 drew the graph of cosine function with angles in degrees, although she mentioned that \( x \) stands for a real number in the second question. Another example was interpreting students’ understanding when they considered cosine but the question was about sine, or vice versa. Moreover, it was not clear if some students meant “only one \( y \)” by “\( a, \ or \ one \ y \)”. Finally, some of the test questions were not clear for the students as understood by some student comments, as for the ninth question.

There are some specific points worth mentioning here. First of all, when the students’ responses to the second question are compared with the ones to the third question, it can be claimed that some students did not know about the meaning of the domain of a function or they could not consider it in a connected manner in the context of sine function. Although most of the students could state the domain of the sine function correctly, few of them chose that \( x \) stands for a real number in \( y = \sin x \). Another issue is that there is an indication that students may have developed a misconception like sine is equal to an angle or an arc because some students stated such claims in Questions 4, 5, and 9. However, when the responses of the individuals were compared for these questions, it is clear that such a misconception is not systematic, but only Student 10 provided such claims for all three questions. Hence, it is more likely that she alone developed such a misconception. The last point is that integrating triangle trigonometry to the unit circle to compute \( \sin 225^\circ \) was grasped by most students, but although they were able to illustrate the situation on the unit circle and to mark the corresponding point, they could not succeed to calculate \( \sin 45^\circ \) or \( \cos 45^\circ \).

Despite all these limitations and issues, it can be claimed that in general the students revealed somehow a kind of conceptual understanding in varying levels. However, some lack of procedural connections or a low ability to report their answers clearly seem to be the main problem.
6.4 Analysis of the Interviews

Four semi-structured interviews were carried out to elucidate students’ understanding of the sine and cosine functions regarding the important aspects of trigonometry understanding as described in the trigonometric understanding framework (Chapter 4).

It had been planned that the interviews would be carried out a few days after the Trigonometry Test with students to be chosen according to their performances on the test. However, due to the school schedule, three of the interviews had to be conducted before the test. Otherwise, the interviews would have had to put off for two weeks. This was not preferred by the researcher, so the last interview was done immediately after the test. In addition, the first interview had to be done in the lunch break of the students, and this meant there was a time limitation in the first interview, so some of the planned questions were removed from that interview.

Table 6.5 shows interview times as well as some information regarding the four interviewees.

<table>
<thead>
<tr>
<th>Interview Number</th>
<th>Interviewee No.</th>
<th>Interview done (Trigonometry Test)</th>
<th>Performance on the Trigonometry Test</th>
<th>Gender</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S 16</td>
<td>1 day before</td>
<td>Average</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>S 11</td>
<td>1 day before</td>
<td>Above Average</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>S 17</td>
<td>1 day before</td>
<td>Below Average</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>S 15</td>
<td>Immediately after the test</td>
<td>Above Average</td>
<td>M</td>
</tr>
</tbody>
</table>

In this section, analysis of student responses to the interview tasks considering common aspects of trigonometry understanding (Chapter 4) is presented after the description of interviewees’ portraits and some remarks about the interviews. The section ends with summary remarks regarding the results of the analysis of the interviews.

6.4.1 Short Portraits of the Interviewees

Three girls and one boy were interviewed. They were chosen according to their performance in the lessons and with the advice of the regular teacher. In Table 6.5, the performance of these students on the Trigonometry Test is also mentioned as above average, average, and below average. The interviewees, luckily, revealed different performance levels in the test. (The intention of the researcher was to choose the interviewees according to the test results, but this was not possible.)

The first interviewee, Student 16, was observed to be a good student in the lessons with her interest and contributions to the group and class discussions. She said once to the researcher that she was not happy with the speed of the lessons because they were too slow. Although she did quite well on most of the test questions, most of her mistakes were probably due to her confusion or carelessness, either
of which may well have prevented her from being an above average student. The following were her weaknesses in the test:

- She drew the graph of the sine function instead of the cosine function.
- She mixed up the terms *domain* and *range* as well as showing confusion with the definition of a function: “There is one x for every y.”
- She chose the option that x stands for an angle in degrees in the expression $y = \sin x$, and she mentioned that it also corresponds to an angle in radians.
- She could not calculate the angle corresponding to a given arc and vice versa.
- She considered the cosine values related to given rotations for the last question although the question was about the sine values.

The second interviewee, Student 11, was observed to be quiet in the classes, although she looked interested in the given tasks done in pairs. Her test performance was much better than most of the students. She was, therefore, an above average student. Among her missing points in the test were the following:

- She wrote some incorrect coordinates in the graph of cosine, but she could draw the correct curve.
- She wrote only “When x = 0, y=1” for the characteristics that could be derived from the graph.
- She chose the options that x is for an angle in degrees and radians in the expression $y=\sin x$.
- She did not give the full answer for $\sin (-90^\circ)$, but just wrote $f(x) = \sin (-90^\circ) = \sin (270^\circ)$.
- She just wrote that she did not understand the question about ordering the sine values corresponding to the given rotations.

The third interviewee, Student 17, was generally quiet in the lessons, and a couple of times she gave the impression that she could not really understand what was to be done in the worksheets. She was interviewed also the day before the Trigonometry Test, in which she performed below average. Her main difference from Student 16, (average student on the test) was that she could not show the given trigonometric equalities on the unit circle or on the graph for the sixth and seventh questions. For the sixth, she considered the y values although the question was about cosine. Her other missing points were:

- She could draw the graph correctly, but her coordinates were wrong. She probably drew the curve from memory, and could not remember the coordinates correctly.
- She chose the option of an angle in radians for the x in $y = \sin x$.
- She wrote the range of the sine function as $(1, -1)$.
- She could not convert radians to degrees or vice versa when the question was given as an arc and the corresponding central angle.
- She could not arrange the sine values corresponding to the given rotations in the correct order, and she could not mark the point on the graph corresponding to one of these rotations.

The last interviewee, Student 15, was among those students who paid a lot of attention to the tasks in the worksheets. He was very active both in the group work and in the class discussions. His performance on the test was one of the highest in the cohort. As mentioned in the analysis of the test (Section 6.3), he was confused about the graph of cosine. After seeing the question to mark the point
corresponding to \( \cos(3) \), he changed his graph which was correct to one showing the horizontal position of a moving point along the unit square. This was his biggest mistake. On the other hand, it is worth mentioning that he was the only interviewee who chose the option of a real number for \( x \) in the expression \( y = \sin x \). He was also the only one among the interviewees who could arrange the sine values corresponding to the given rotations in the correct order. Apart from the graph, he only made small mistakes with the signs of the sine values in questions of \( \sin\left(-\frac{\pi}{2}\right) \) and \( \sin(225^\circ) \).

### 6.4.2 Some Remarks Related to the Interviews

After the short portraits of the interviewers, I would like to emphasize my viewpoint regarding the interviews. I did not have high expectations beforehand regarding performances of the interviewees in connection with my impressions in the lessons. However, interviewees’ performances were much better than expected.

The main aim of the interviews was to illuminate students’ understanding of the sine and cosine functions as reflected by their task solutions and their discussions. The dialogues with me as the interviewer and researcher were also of crucial importance, and in this regard, I intended to facilitate students’ talk and only guide them throughout the interviews, not instruct them directly. However, in fact in some cases I gave hints to the interviewees to help them continue elaborating further. More importantly, I often needed to confirm what the students were mentioning or referring to because not all of their arguments were clear. To help me analyze the audio recordings regarding some unclear expressions such as “from here to here”, I simply confirmed by repetition, that is by stating the proper coordinates. Similarly, because the students had particular problems with the terminology, I needed to confirm what they meant by using the proper vocabulary, or even remind them of the correct words in some cases.

The students revealed many uncertainties during the interviews, either because of their difficulties to make themselves clear in English or because explaining things was something different or difficult for them. In such cases, it was important for the continuation of the interviews to tell the students more about the related tasks and give further explanations. For instance, the first and third interviewees reacted very often to the tasks like “I don’t know what you are asking.”, “I do not know.”, or “No idea.”. However, after an additional explanation or a follow up question, it was seen that they actually knew enough about the tasks to provide answers. Similarly, the interviewees had problems with the word “characteristics” in interpreting the sine graph. Hence, I needed to ask them more specific questions. Another instance was that Student 16 mixed up the words \( \text{horizontal} \) and \( \text{vertical} \), and the terms \( \text{domain} \) and \( \text{range} \). In such cases, I needed either to ask further questions to help them correct themselves or to correct their verbal mistakes.
6.4.3 Analysis of Student Responses to the Interview Tasks

Analysis of students’ responses to the interview tasks are provided in this section under subtitles with respect to the key mathematical aspects underlying trigonometric understanding. They are drawing and interpreting the sine graph, function concept and related properties, radians, calculating sine of various angles and explaining their relationship. Connections among concepts and the three trigonometry contexts are an important aspect of the analysis presented here. Although it is not directly related to the research questions, students ideas about the lesson sequence are also reported followed by the exceptional situation of Student 15 about his cosine graph in the test, which was also discussed in his interview.

Table 6.6 shows the summary of the students’ responses in the interview according to these important aspects of trigonometric understanding.
<table>
<thead>
<tr>
<th>Topic</th>
<th>Student 16 (interviewee 1)</th>
<th>Student 11 (interviewee 2)</th>
<th>Student 17 (interviewee 3)</th>
<th>Student 15 (interviewee 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawing the graph of the sine function</td>
<td>first drew the curve, then wrote down the coordinates, and explained them using the arc lengths as travelled distance along the unit circle and vertical position</td>
<td>Same as Student 16</td>
<td>needed the help of the researcher. She first calculated some coordinates by using arc lengths and vertical positions, then drew the curve</td>
<td>first drew the curve and then wrote down the coordinates; was trying to explain why sine is the vertical position by the triangle definition; later on could also connect to the arc lengths.</td>
</tr>
<tr>
<td>The concave down shape of the graph between (0, 0) and ((\pi/2, 1))</td>
<td>could explain why it is not linear by arguing about the unit square; mentioned that all the coordinates create such a concave up curve; also referred to GeoGebra Applet.</td>
<td>could explain why it is not linear by arguing about the unit square; also referred to GeoGebra Applet.</td>
<td>Question was not asked.*</td>
<td>first explained why it is not linear by arguing about the unit square; explained why it is not concave down by the characteristics of the movement along the unit circle.</td>
</tr>
<tr>
<td>Function concept</td>
<td>mentioned that it is a function because for every (x) there is only one (y).</td>
<td>mentioned both an input-output mechanism and the formal definition through a sample graph which is not a function</td>
<td>mentioned an input-output mechanism.</td>
<td>gave the formal definition.</td>
</tr>
<tr>
<td>Domain and Range</td>
<td>mixed up the domain and range, but could explain well on the (x) and (y) axes.</td>
<td>stated domain and range correctly and explained the meaning of range on the unit circle.</td>
<td>stated the domain and range correctly, could explain range after the follow up question why (2) and (-2) are not in the range.</td>
<td>stated the domain and range of the sine function correctly, and explained what range means on the unit circle.</td>
</tr>
<tr>
<td>Periodicity</td>
<td>did not remember and because of time limitations the researcher preferred not to focus on it.</td>
<td>first said that she did not remember; after the researcher reminded her of the definition, could state that it is (2\pi), and could explain its meaning on the unit circle.</td>
<td>could explain the periodicity very well both on the graph and on the unit circle.</td>
<td>explained it in his own words using a copy-paste idea on the graph, and could explain it on the unit circle as well.</td>
</tr>
<tr>
<td>Difference between the graph with (\pi) values and with integers</td>
<td>Question was not asked.*</td>
<td>mentioned they are the same because (\pi) is a number; also showed the point on the graph with integers corresponding to (x%\pi).</td>
<td>mentioned they are the same, and she showed the point on the graph with integers corresponding to (x=2\pi).</td>
<td>mentioned they are the same because (\pi=3.14), and showed it on the graph with integers.</td>
</tr>
<tr>
<td>Radian</td>
<td>mentioned radians as angles corresponding to an arc, and that radians are equal to the corresponding arc length, but could not explain the reason; could also convert to degrees.</td>
<td>stated that (\pi) stands both for an angle and an arc length; knew that angles in radians and arc lengths are equivalent.</td>
<td>stated that angles in radians are equal to corresponding arc lengths, but could not explain the reason.</td>
<td>stated that angles in radians are equal to the corresponding arc lengths.</td>
</tr>
<tr>
<td>(\sin(\frac{\pi}{2}))</td>
<td>pointed to the correct position and direction on the unit circle, but could not give the answer; could find the answer on the graph.</td>
<td>got confused by considering ((\pi/2)) as (\pi) on the unit circle, but corrected herself after checking the graph.</td>
<td>Despite some uncertainties, could give the correct answer with the correct position and direction on the unit circle.</td>
<td>could answer it using the unit circle with the correct position and direction.</td>
</tr>
<tr>
<td>(\sin(\frac{3\pi}{2}) = \sin(\frac{-\pi}{2}))</td>
<td>could find the answer for (\sin(3\pi/2)) on the unit circle, and explained the equality both with the position on the unit circle and the periodicity.</td>
<td>could find the answer for (\sin(3\pi/2)) on the unit circle, and explained the equality both with the position on the unit circle and the periodicity.</td>
<td>could give the correct answer for (\sin(3\pi/2)).</td>
<td>could explain the equality by periodicity.</td>
</tr>
<tr>
<td>(\sin(210^\circ))</td>
<td>pointed to the correct position on the unit circle, and calculated it with sin 30°, but could not remember the sides of the right triangle.</td>
<td>pointed to the correct position on the unit circle, and calculated it with (\cos60^\circ), but could not remember the sides of the right triangle.</td>
<td>pointed to the correct position on the unit circle, and calculated it with (\cos60^\circ), but could not remember the sides of the right triangle.</td>
<td>stated that (\sin210^\circ = \sin(-30^\circ)); however, could not remember a correct configuration for the sides of a right triangle of 30°.</td>
</tr>
<tr>
<td>Distinguishing the graph of sine from the one of cosine.</td>
<td>Question was not asked.*</td>
<td>distinguished sine by stating it passes through the origin, also could explain the graph of cosine referring to the horizontal position of a moving point along the unit circle.</td>
<td>could distinguish and explain sine as the vertical position and cosine as horizontal position referring to the unit circle.</td>
<td>could distinguish, that sine starts from (0,0) and cosine from (0,1); explained reason for the latter as horizontal position is 1 at the beginning, referring to the moving point along unit circle.</td>
</tr>
</tbody>
</table>

* Some questions were not asked due to a shortage of time

Table 6.6: Summary of the student responses to the interview tasks
Drawing and Interpreting the Graph of Sine

Except for Student 17, all interviewees drew the graph of the sine function immediately, and then wrote down the coordinates. This shows that they memorized the sine curve, and drew it from memory. However, since they could explain their graphs sufficiently, it can be interpreted that their understanding of the graph was beyond simple memorization. Students 16 and 11 engaged in a similar strategy to explain why their graphs were the graph of the sine function by using travelled distances and the corresponding vertical positions of a moving point along the unit circle. Both could clearly explain the coordinates on the graph as arc lengths and vertical positions on the unit circle. The excerpt below shows how Student 16 explained the coordinates of her graph.

R: And what about this one where we have the highest value?
S16: [continuing...] this is – \( \frac{\pi}{2} \) [continuing with the others]
R: Can I ask you once more why this point is \( \frac{\pi}{2} \)? How do you know that?
S16: Because the coordinate on the x-axis is the travelled distance, and y-coordinate is the vertical position of your moving point. So, if you move this which is \( \frac{\pi}{2} \) [showing on the unit circle], the vertical position is 1. So, the point \( \frac{\pi}{2} \) is where the vertical position is 1.

Both Students 11 and 16 could also explain how they calculated the coordinates using the travelled distances. This can be considered as another indication that they possessed a good understanding regarding the connections to the unit circle. The excerpt below shows how Student 11 explained this.

R: Ok. Now. How do you know that when you move from here to here [showing on the unit circle from (1, 0) to (0, 1)] it corresponds to \( \frac{\pi}{2} \)?
S11: Because this is the half of the circle and it is \( \pi \). And half of it is \( \frac{\pi}{2} \).

Student 15, who also drew the graph very quickly, did not relate the graph to the arc lengths and the vertical position, unlike the others. Although he mentioned sine as the vertical position, he was basically considering a triangle in the first quadrant and related angles. This part can be seen in the excerpt below.

R: Well you drew here the unit circle [he used it to draw the graph]. What does sine mean on the unit circle?
S15: ... It is... the sine is the y... the vertical position. You can know that because this one [radius] is 1. Then you have here 60° and 30° and then you can calculate it... Because that one [radius] is 1.
R: Ok. Very good. If you consider the points on the graph, for instance \( \frac{\pi}{2} \), what does it mean on the unit circle?
S15: \( \frac{\pi}{2} \) is 90°. Hmm the distance \( \pi \) is 180°, \( \frac{2\pi}{2} \) is 270° and then the whole...
R: Ok. Then you are considering, if I am not mistaken, sine with angles in degrees.
S15: Yeah.

It was a very good point that he could explain why sine was defined as the vertical position by using the triangle he drew and using the ratio definition of sine by connecting it to the fact that the radius, or the hypotenuse of his triangle, was 1. This can be considered as a good level of conceptual understanding because he was combining different conceptual underpinnings. However, he was stuck...
on angles in degrees to describe sine although he mentioned at some point arc lengths as distances and that sine is the vertical position. When asked if he could define sine in another way considering the moving point on the unit circle, he did not explain travelled length and vertical position clearly.

At some point in the interview, he came to a very interesting point by conjecturing about the case of the radius was 2 instead of 1. Hence, the researcher asked him to elaborate more on it as it can be seen in the excerpt below although this was not intended in the interviews. (Line numbers were added for referencing purposes.)

Although he could not state clearly that the sine would not change, and he was a bit confused, it was good that he was considering arc lengths as well as angles. The graph he drew for the case that the radius was doubled is in Figure 6.13. However, his speech was far from clear, perhaps because of English, as can be derived from his contradictory arguments in lines 5, 13, 19, 21. For instance, in line 5 he stated that the graph would not change, but in line 13 he claimed that the sine value would still be the vertical position. Because of such unclear arguments from him, it is difficult to conclude that he was really aware of what would happen if the
radius was 2. However, his argument in line 19 might be a clue that his graph in Figure 6.12 shows the vertical position, which would be twice the corresponding sine value, not the graph of sine itself. In addition, he could not connect it to the definition of radian which would have been another way to realize the unchanging sine value.

On the other hand, Student 17 followed a different strategy to draw the graph because she could not draw it at once, and needed the help of the interviewer. She first determined the coordinates and then drew the graph unlike the others. After she was asked what sine was, she mentioned that it was the vertical position. Then she started drawing the unit circle and stated that there was travelled distance and the vertical position, and she started moving from the point (1, 0).

R: Ok. If you move from here to here [from (1, 0) to (0, 1)], what is the travelled distance?
S17: 1. Right? ... Ohh 1 is the vertical position and ...
R: Ok and what is the length you travelled?
S17: 1.
R: Are you sure? Why do you think it is 1.
S17: No idea.
R: Ok. If you moved along the whole circle, what is the distance you’d travel?
S17: $2\pi$
R: Yeah. And if you move not the whole but this part [researcher showing the quarter of the unit circle] ?
S17: Yeah, $\frac{\pi}{2}$
R: Yeah. Then here you travelled a distance of $\frac{\pi}{2}$ and you told me already what the vertical position is.
S17: 1.
R: So, if you determine the point $\frac{\pi}{2}$ it will give the vertical position 1.
S17: Yeah.

As seen in the excerpt above, she could not calculate the length of the arc between the points (1, 0) and (0, 1). However, as a positive point, she could manage to calculate it by relating it to the whole circle after the researcher asked her about the travelled distance along the whole circle. After that, she could also answer the same question for the arc between (1, 0) and (-1, 0), but when asked about sine corresponding to this arc, she mentioned that she had 180°. However, after she was reminded that sine was the vertical position, she could give the related sine value as 0. It is noteworthy here that despite her difficulties to draw the graph, she could still connect the information between the graph and the unit circle, and she could also explain negative values as the clockwise direction later on.

Three interviewees were asked about the concave down shape of the sine function by focusing only the part of the sine graph between (0, 0) and (\(\frac{\pi}{2}\), 1) to elucidate more about their interpretation skills of the sine graph. Only Student 17 was not asked about it due to a shortage of time. Students 16 and 11 provided again similar responses for this part, and Student 15 provided a better response revealing a good level of understanding. Both Student 16 and 11 mentioned that the points were connected by such a curve, but not by a line, and that if they had had a unit square instead of the unit circle, they
would have had a line connecting the two points. In addition, Student 16 commented on the reason why the curve is not concave up as it can be seen in the excerpt below.

...  
R: You said if we have the square, we will have lines. Which part of our lessons helped you see this transition?  
S16: Yeah we had this applet. We just increased the number of sides. If you have all the points, and if you do not know the logic, you will just draw it. If you just measure very point you will get this [curve].  
R: Ok. One more question about this. Ok now we are sure that we have such a curve but not a line. Why do we have this curve but not this [a concave up curve]. Maybe you can consider we are moving from here to here on the unit circle.  
S16: [explaining further...] I do not know how to explain it, but it is just like that. You just take points and you draw them in your graph. But it may also have something to do with how the function is. But I am not sure about that.

It was clear that she understood the reason why the two points were not connected by a line. Her reasoning why the graph was not concave up was interesting in that she just mentioned the effect of all the points which are the coordinates on the graph. Unlike her, Student 15, on the other hand, could provide a surprisingly good response for this issue. The following dialogue, which can be considered as another indication of his good level of understanding, took place after he stated clearly that they were not connected with a line because he was using the unit circle, not the square.

R: Ok. Now you explained to me it should not be a line but a curve. But then how do we know that the curve should be like this [concave down] but not like this [drawing concave up]? Do you have any idea?  
S15: Otherwise this [unit circle] should be like this [arc indicated by 1 in Figure 6.13].  
R: Ok. Why do you think so?  
S15: Hmm. Because here [on the unit circle] the first... the sine is not going very fast to the up and then it is going faster. And this one is also faster, faster.

As seen in the excerpt above, he was asked why they were connected with a concave down curve but not with a concave up one after he stated the reason why we did not have a line. Then he stated that we needed a shape like the one (arc indicated by 1) in Figure 6.14 for such a curve. To see if he really had an understanding about it, or if he just made this claim just due to the similar shapes of the figure [arc indicated by 1] and the concave up curve, he was asked for the reason. Surprisingly he could explain the reason with an idea of speed up and slow down motion, indicating to the rate of change in the vertical position with respect to the travelled arc length. It is also noteworthy to mention here that he was considering sine through arcs rather than only angles in these discussions unlike the first part of the interview.
Three interviewees were also asked to explain how they could distinguish the sine graph from the one of the cosine towards the end of each interview. Student 16 was not asked about this task because there was not enough time. All three interviewees could distinguish the graphs and explain the reason. Student 11 and Student 17 explained that cosine is defined as the horizontal position corresponding to an arc length, and exemplified their arguments with $x=\frac{\pi}{2}$. Student 15 explained that the horizontal position of a moving point along the unit circle at the starting point is 1.

**Function Concept and Relates Properties**

After the students drew the graph of the sine function, they were all asked what characteristics they could derive from the graph. Although such a question was discussed in the lessons, except Student 15 all the others could not interpret the question. Thus, they were first asked why the sine graph shows a function. Student 17 emphasized only an input-output mechanism while the others provided a formal definition. Student 16, for example, stated that “Because for every $x$, there is one $y$, not two or more.” Student 11, on the other hand, provided a response combining the formal definition with an input-output mechanism. The excerpt below shows that she was able to explain the single-valuedness although she could not retrieve the correct words, but she could explain it through the graph which is not a function as given in Figure 6.15.

**R:** Why do you think it is a function?

**S11:** Because of input and output. Every time you take a point, it just has one coordinate.

**R:** Can you try to explain again, maybe with some other words?

**S11:** If you have a graph like this [Figure 6.15], if you take this point here, there are two points

**R:** Ok. So, for every $x$ there should be only one $y$. And in the graph you drew, for an $x$ there are two $y$.

**S11:** Yes. And it is not a function.

![Figure 6.14: The figure for the graph to be concave up in the domain $[0, \frac{\pi}{2}]$](image1)

![Figure 6.15: Non-function example of the second interviewee](image2)
Periodicity was among the important properties related to the function concept which was asked students. Student 16 mentioned that she could not remember what it was, and because of the time limitation in the interview, it was decided not to pursue further her conception of periodicity. Similarly, Student 11 also showed uncertainty when she was asked about periodicity. Nonetheless, when she was reminded of the definition of the periodicity as the interval where a graph repeats itself, she could immediately connect it to the circumference of the unit circle, and stated that the period of the sine function is $2\pi$. It was good in terms of her understanding that she could elaborate further on periodicity and she could transit between the two representations, namely the graph and the unit circle, in a connected manner.

Student 17 showed a better understanding compared to the first two interviewees. Although she could not explain everything clearly with proper words, mostly because of her difficulties with English, she was able to reveal that she knew what periodicity was, and understood its meaning both on the graph and the unit circle. Considering that she was a below average student, it was remarkable that she possessed a better understanding than the first two interviewees. The following excerpt shows how she explained periodicity.

\[R: \text{ And what is the periodicity of this function, the sine function?}\]
\[S17: \text{Hmm. It is right from here and you are starting over again [showing } 2\pi \text{ on the graph] and on the same thing, } \ldots [\text{pointing to the second interval}]\]
\[R: \text{Yeah the interval it repeats itself...}\]
\[S17: \text{Yes.}\]
\[R: \text{Let me give you a better version of the graph, maybe it will be easier for you to read it.}\]
\[S17: \text{It is here. It is } 2\pi.\]
\[R: \text{Yeah. So you say the periodicity is } 2\pi. \text{ And if you consider the unit circle and the moving point again, what does it mean that the periodicity is } 2\pi.\]
\[S17: \text{You are going on one circle and then the periodicity...}\]

Student 15 also revealed a very good level of understanding of periodicity by explaining it with his own words about a copy and paste idea to explain the graph repeating itself. The excerpt below illustrates how he explained this.

\[R: \text{What is the period then?}\]
\[S15: 2\pi [\text{also showing it on the graph}] \text{ and then it starts over. You can copy this part [showing the part between (0, 0) and (2}\pi, 0)] \text{ and put it there [the second interval]}\]

He could also connect it to the circumference of the unit circle as the other did.

About the domain and range of the sine function, all of the interviewees provided good responses indicating that they understood them well. They all could explain them both on the graph and the unit circle by relating them to travelled distances and vertical positions as a good point regarding their understanding. Although Student 16 mixed up the terms domain and range as she had also mixed up

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20 She also mixed these two terms in the test two days after the interview.
vertical and horizontal positions, she could state and explain their meaning after she was corrected by the researcher. The excerpt below shows how she explained them.

...  
R: Ok. We have a function of travelled length showing the vertical position. In this case [-1, 1] should be about...
S16: ... The vertical position. So travelled distance can just go to infinity. And vertical position can never go higher than 1 and lower than -1.

Students 11 and 15 also explained the meaning of the range in the same way as Student 16 did after they stated the domain and range of the sine function correctly. Student 17, on the other hand, needed a follow-up question of the researcher to reveal that she engaged in a similar way of thinking as the others.

R: Ok. You showed me with your hand that domain is on the x-axis, every real number, and the range is [-1, 1]. Ok. What does it mean for the sine function to have the range of [-1, 1]?
S17: No idea.
R: Why are the points 2 and -2 not in the range?
S17: Because of the circle. The circle is going on 1 and -1 and not further.

To elucidate the students' understanding more regarding sine as a function of real numbers, they were shown the graph with integers instead of multiple values of $\pi$ with the designed scenario, except Student 16 because of time limitations in the interview with her. As a very good point regarding their understanding all of them mentioned that there was no difference between the graph with integers and the ones they had drawn using the arc lengths. They mentioned that $\pi$ is also a number, and could mark the points $\pi$ or $2\pi$ correctly on the graph given with integers. This is a good finding because it shows that they considered sine as a function of real numbers through arc lengths as aimed for in the lesson sequence, and the angle definition of sine was not confusing for them.

**Radians**

As a continuation to the discussions about the difference between the graph of sine with integers and with multiples of $\pi$, the students were asked about radians to find out their understanding of the transition to the function of real numbers from the sine of angles. In the first interview there was not such a follow up question because Student 16 was not given this task. There is evidence that all of the interviewees understood the role of radians in the transition to the real numbers because all of them defined the sine function with arc lengths, and knew the fact that the measure of an angle in radians is equal to the corresponding arc length. The following excerpt shows how Student 16 connected the radians to the arc lengths although she revealed once again uncertainty.

S16: And radian is like the travelled distance. And every radian we learnt like 45°, 60°, and 30°. And multiples of these. So, if you have like 90°, you have the radian of $\frac{\pi}{2}$. That is how you can always measure. If you have a travelled distance half pi, the angle is 90° because of that.
R: Ok. You said the angle corresponding to $\frac{\pi}{2}$ is 90°. And how many radians is that again?
S16: Oh I do not remember.
Student 17, on the other hand, was asked why the radian measure is equal to the corresponding arc length after she explained the relationship between them. However, she could not explain the role of the radius of the unit circle.

Student 15 gave the clearest connection from angles to real numbers as the domain of the sine function. He first explained what the real number 3.14 represents for the sine function considering the unit circle, and then he explained the relationship between an angle in radians and the corresponding arc length as the travelled distance. The following excerpt shows these, respectively.

\[ R: \text{ And here your vertical position is 1. But here you are explaining it by angles. But how can I explain it by as a real number } \frac{\pi}{2}? \]
\[ S15: \text{ Ohh. You can do that by pi is 3.14. So, } \frac{\pi}{2} \text{ is...} \]
\[ R: \text{ So, this real number } [3.14/2] \text{... What does it mean on the unit circle?... It is not an angle anymore, right?} \]
\[ S15: \text{ It means... The travelled distance.} \]
\[ \ldots \]
\[ R: \text{ Very good. Thank you... And do you remember what radian is?} \]
\[ S15: \text{ Radian is as... like this [showing an angle corresponding to an arc on the unit circle]. So, if you have a circle, this is a radian.} \]

As can be seen above, he could explain the role of radians through the arc lengths as travelled distances in the transition to real numbers although he could not give a formal definition of radians like the other interviewees.

**Calculating \( \sin \left(-\frac{\pi}{2}\right), \sin \left(\frac{3\pi}{2}\right) \text{ and } \sin(210^\circ) \)**

Student 16 and Student 17 showed uncertainties about the task \( \sin(-\frac{\pi}{2}) \). Although Student 16 showed the correct position on the unit circle, she could not give the answer. After she was asked she could also use the graph, she could give the result correctly. Perhaps, she could not remember that she needed to consider the vertical position on the unit circle. For \( \sin\left(\frac{3\pi}{2}\right) \), she could immediately provide the correct answer using the unit circle. Apparently, the previous task reminded her how she could find a sine value on the unit circle. When she was asked to explain the equality of the two sine values, she mentioned that they correspond to the same position on the unit circle.

Student 17, on the other hand, stated at first that she did not understand the question. After she was asked what the value of the sine function at \( -\frac{\pi}{2} \) is, she could interestingly give the correct answer. However, she got confused when she was asked \( \sin\left(\frac{3\pi}{2}\right) \). The researcher wrote it down as \((1\frac{1}{2})\pi\), and with the help of the researcher, she could find the correct answer. Student 11 revealed another confusion by considering \( -\pi \) as \( -\frac{\pi}{2} \) by mistake. After she was asked to check her answer on the graph of the sine function, she realized her mistake, and corrected herself by providing the correct answer. It
can be considered as a positive point of her understanding that she could correct herself as a result of connecting once again the information between the graph and the unit circle. Another indication of her good level of understanding was that she could explain the equality of the two sine values both by stating that they had the same position on the unit circle and connecting it to the periodicity. This can be seen in the following excerpt.

R: ... So, sin (-\(\frac{3\pi}{2}\)) is the same as sin (-\(\frac{\pi}{2}\)). Can you explain to me this equality using the periodicity?
S11: Minus goes this way and the other goes the other way, and then they come to the same point.
R: Ok. And can you connect this to the periodicity? And remember you said the periodicity of sine function is \(2\pi\).
S11: Yeah.
R: Can you see that connection here?
S11: One round is one period.
R: If you move like this [the direction related to sin (-\(\frac{\pi}{2}\))] and this [the direction related to sin (\(\frac{2\pi}{2}\)] , basically you move along the whole circle and that circumference is \(2\pi\).

Although the uncertainties or the mistakes of the students might have been due to carelessness at some point in the interview, they were surprising because the students were quite good on explaining the graph using the arc lengths and vertical positions on the unit circle. Student 15 was the only interviewee who could give the correct answers for the both questions at once using the correct positions on the unit circle. He could also explain very well the equality of the sine values with periodicity. The following excerpt shows how he connected it to periodicity, again by his own idea of copy and paste.

R: Could you connect their equality to the periodicity?... Could you explain their equality by the periodicity of the function?
S15: Yes.
R: How?
S15: Because the period is [showing the point \(2\pi\) on the x-axis.]... This one... And here [\(\frac{2\pi}{2}\), -1)] you have -1. If you copy this [(the part of the graph between (-\(\frac{\pi}{2}\), -1) and (0, 0)] there [from the point (\(2\pi\), 0 backward)], it is the same point.
R: So, you are showing me that the distance between these two points is \(2\pi\).
S15: Yes.

Although he preferred to use his own words to explain this phenomenon related to the periodicity instead of mathematical words, it can be considered as a sign of good his understanding. This is mainly because explaining a mathematical relationship in one’s own words using a different kind of reasoning requires a good understanding of the underlying mathematical ideas.

The students generally performed well on the task of calculating \(\sin 210^\circ\) although none of them could give the final response because they could not remember the correct side lengths of a right angled triangle of \(30^\circ\), \(60^\circ\), and \(90^\circ\). However, they could give the correct answer after they were given the side lengths by the researcher. Student 16 mentioned that she could calculate the result by sin \(30^\circ\) after indicating the correct rotation on the unit circle and the correct triangle. Students 11 and 17, on the
other hand, calculated it by \( \cos 60^\circ \) to calculate the required coordinate on the unit circle. Although Student 17 needed again the help and feedback of the researcher, it was a very good point of these two students’ understanding regarding integrating triangle trigonometry to the unit circle because they figured out that it could be calculated by \( \cos 60^\circ \) although such a task was done with \( \sin 30^\circ \) in the lessons. Figure 6.16 illustrates the work of Student 11.

![Figure 6.16: The work of the second interviewee for \( \sin (210^\circ) \)](image)

Student 15, on the other hand, followed a different strategy that was another sign of his good level of understanding. After determining the correct position on the unit circle he explained that \( \sin (210^\circ) = \sin (-30^\circ) \), and then he found the result. This can be considered as an indicator that a student with a good level of understanding can encounter even a standard kind of task from a different point of view by coming up with other valid mathematical arguments.

**The Students’ Ideas Regarding the Lessons**

The students were asked for their ideas about the lessons at the end of each interview. Student 16 mentioned that she liked the exercises in the lessons, but she complained that they did not get their homework papers back. Similarly, Student 17 criticized the same issue about the homework papers. Students 17 and 11 also emphasized their difficulties with English. In addition, Student 11 stated that the lessons were difficult for her because they were too fast for someone with dyslexia.

More importantly, the last three interviewees mentioned the different kinds of tasks in the lessons. Student 11 confirmed that what they were asked to do in the lessons was quite different from what they were used to as she put it:

\[ R: \quad \text{So you are not really used to that kind of questions.} \]

\[ S11: \quad \text{Not really. If we have a subject, we have one page with how it works, and then we have a lot of exercises to make, and then you understand how to make it and never explain why you do that like that.} \]
She also mentioned that she was good at calculations, but not at explanations. Hence, she gave the advice of including more exercises in the lessons and less explanations. Student 17 also gave the feedback that there should have been handouts with explanations to improve the lessons. Similarly, Student 15 stated that he did not know where to look for the answers of the questions, and added as an advice that “If you know at the beginning where it is going to be, then it is simple I think.”. Lastly, Students 17 and 15 said that at the beginning it was more difficult, but at the end everything was clear.

All these comments confirmed that the lessons were quite different for the students, and hence, some were not very satisfied with how the lessons were implemented. Apparently, it was also a factor which made the tasks difficult for them in the lesson because the tasks were not given in a traditional method based on giving the rules first and then exercises to be solved according to the rules, or because they did not know where to look for the answers in the words of Student 15. On the other hand, it was the main aim of the lesson sequence to facilitate the students’ learning through their mathematical investigations. In this sense, it is a positive sign that Students 17 and 15 mentioned that everything was clear at the end. Such issues are discussed thoroughly in the conclusion chapter.

**The Exceptional Situation of Student 15**

As was highlighted before in this section, Student 15 said at the beginning of the interview that he had gotten confused on the test with the part about drawing the graph of cosine. Although he had drawn the graph correctly at first, he changed it to the graph showing the horizontal position versus the travelled distance of a moving point along the unit square instead of the unit circle. He did this because he became confused after seeing the last part of the first question, which was marking the point corresponding to \(\cos (3)\). When he was asked what he was thinking about at the beginning of the interview, he stated that he had needed to draw the graph using the unit circle.

At the end of the interview, he was asked again about his mistake with the graph of cosine referring to the Trigonometry Test. He corrected himself by stating that he needed to draw the graph by using the unit circle, not the unit square. Afterwards, he was asked to mark the point corresponding to \(\cos(3)\) on the graph, and he could easily point to the correct position on the graph. This instance may also exemplify the certain kind of limitations of using a test to assess students’ understanding as such issues were mentioned in section 6.3.2.

**6.4.4 Summary Remarks**

When the four interviewees are compared, it can be claimed that Student 15 was the best followed by Student 11, Student 16 and Student 17, respectively, according to their performances in the
interviews. Especially the performance of Student 15 was extraordinary with his rich understanding regarding the connections among different concepts. He differed from the others by being able to explain why sine is the vertical position, explain periodicity in his own words, and relate the concave down curve to the movement along the unit circle. On the other hand, the performance of every interviewee was surprisingly good for the researcher given his low expectations. Even the understanding of Student 17, which was more limited compared to the others, was not bad at all considering that she could combine different concepts and explain them during the interview despite her uncertainties during the interview. She even possessed a better understanding of periodicity compared to the first two interviewees. It is likely that all these students gained quite good conceptual understanding as a result of the lesson design which asked for discussion and explanation, unlike in traditional lessons.

Connections between the graphs of the trigonometric functions and the unit circle were among the main issues in both lesson sequence and the interviews. All of the interviewees were found to be very successful at explaining these connections. This can be seen from the student responses to explaining the graph of the sine and cosine functions using the corresponding arcs and vertical positions on the unit circle, the meaning of the range and periodicity, and to calculating \( \sin \left( -\frac{\pi}{2} \right) \) and \( \sin \left( \frac{3\pi}{2} \right) \) both on the graph and the unit circle.

Another noteworthy issue was the transition from sine of angles to the sine as a function of real numbers. It was clear that the students conceptualized sine as a function of real numbers. All three interviewees, who were asked the difference of the sine graph when it is given with integer x values instead of with \( \pi \) values, could clearly explain that they were not different, and they could combine both versions by marking certain x values with \( \pi \) on the graph with integers. Furthermore, all interviewees were aware of the relationship between angles in radians and corresponding arc lengths. Hence, there is strong evidence that they conceptualized the transition to real numbers through arc lengths. However, when it came to explaining this transition with proper mathematical language, it was difficult, especially for Student 17. Student 15 was the only one who could explain this transition clearly through the arc lengths corresponding to real numbers. On the other hand, a common problem was the concept of radians for all interviewees. Although they were all aware of the mathematical fact that the radians are equal to the corresponding arc lengths for the unit circle, none of them could explain radians in a formal way.

As a result of the four interviews, it can generally be claimed that all of the interviewees developed a good level of understanding of the sine and cosine functions in varying levels despite some mistakes and uncertainties shown in the interviews. Since they were good at explaining the relationships and connections among different trigonometry contexts and concepts, it can be claimed that they
possessed a conceptual understanding after the lesson sequence. The following are among these connections for which the interviewees showed capability:

- Connecting the information between the sine graph and the unit circle,
- Explaining what the range of the sine function corresponds to on the unit circle,
- Explaining periodicity both on the graph and the unit circle,
- Stating the relationship between arc and corresponding central angles,
- Showing certain trigonometric equalities on the unit circle as well as explaining it by using periodicity,
- Integrating triangle trigonometry into unit circle trigonometry.

As a general finding, the students’ responses and discourse during task solving in the interviews might be an indication that the designed lesson sequence helped the students to grasp the intended conceptual connections despite some missing points. These will be discussed more deeply in the findings and conclusion chapter.
The aim of the present study was to examine students’ concept development within and understanding just after the implementation of a new learning trajectory of trigonometric functions. Five lessons were implemented at a Dutch pre-university secondary school with twenty-four students. Data were collected through worksheets and audio recordings of the group discussions to answer the first research question about students’ concept development, and through a trigonometry test and four semi-structured interviews after the implementation to answer the second research question about students’ understanding.

In this chapter, research findings are presented and discussed separately according to the research questions in the first two sections. Afterwards, the effectiveness of the lesson design is discussed. This is followed by a discussion of the study. The chapter ends with suggestions for further research.

7.1 Conclusion and Discussion for Research Question 1

<table>
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<tr>
<th>Research Question 1</th>
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<td>What task-related difficulties do students face in their concept development within the designed lesson sequence based on a hypothesized learning trajectory of trigonometric functions?</td>
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The lesson design which aimed at integrated understanding of trigonometry was based on a hypothesized learning trajectory (Chapter 5). Students’ worksheet responses were analyzed together with the audio recordings of their group discussions in order to locate their difficulties with the worksheet tasks.

The analysis of student responses to worksheet tasks revealed that in general the students were successful on most of the tasks, but faced difficulties in four tasks regarding their concept development:

i. Drawing the graph showing the vertical position of a moving point along the unit square against the distance it travels \([1a]\),

ii. Deriving the formula \(s(x)=\sin\left(\frac{180^\circ x}{\pi}\right)\) of the graph showing the vertical position versus arc length on the unit circle \([3d]\),

iii. Converting \(\frac{180^\circ}{\pi}\) to radians and using it in the formula to transit to real numbers \([4d]\),

iv. Calculating \(\sin 210^\circ\) \([5b]\).

Students’ difficulties with the tasks i and ii indicate a more general difficulty for them: making mathematical generalizations from the concrete to the abstract. Although they did not have problems...
with individual components of the two tasks, mathematization requiring generalization based on these individual components was the main problem. Tasks iii and iv, on the other hand, indicate a difficulty in applying known relationships in different contexts.

**Task i:** What made it difficult for the students to draw the graph of the vertical position versus the travelled distance of a moving point along the unit square? It was clear that this task was very uncommon to the students. They were familiar with drawing the graphs of functions whose formulae were given but this graph did not have one, but once they understood the task, it was easy for them to plot the points to draw the graph. However, it was also difficult for them to decide if they needed to connect the points with lines or curves. It was found that students had not understood the idea of a constant rate of change in their group work. Their problem with the linear structure of the graph may have also been because they were in general used to smooth curves as function graphs. A piecewise function and its graph in this task were new to them as confirmed in the Diagnostic Test (Section 6.1).

**Task ii:** The most difficult task for the students was deriving the formula $s(x) = \sin\left(\frac{180\pi x}{\pi}\right)$ for the graph showing the vertical position against arc lengths on the unit circle. Before asking for the formula, the students were given preparatory tasks on which they needed both to explain the relationship between angles in degrees and arcs subtending these angles, and to use these arcs and angles in the equation $s(x) = \sin(\alpha)$ where $x$ and $\alpha$ denoted arcs and angles, respectively. To obtain the formula, the students needed to do a proportional calculation based on the relationship of angles and arcs. However, this was not easy for them. They did quite well in terms of the relationship between angles and subtending arcs, but they could not proceed with the proportional calculation. It was still unclear to them after the researcher’s explanations in the whole class discussion. The same difficulty was also observed when they attempted to obtain a similar formula for the cosine function. Because they could not overcome this difficulty in the entire instructional sequence, it was probably not only due to a difficulty to do the proportional calculations but also about the reasoning behind them. Although they were very successful in working with the relationship between specific angles and corresponding arcs, the generalization based on this relationship to proceed with proportions that included the variables $x$ and $\alpha$ was very troublesome for most students. This is the most plausible explanation regarding the reason of this difficulty. Again this was a very different kind of task for them, and also in the trigonometry test students had problems with proving equalities that included variables, rather than specific values.

**Task iii:** In their group discussions, very few students could convert $\frac{180^\circ}{\pi}$ to a radian, and use it in the formula to obtain $s(x) = \sin(x \text{ radians})$. The reason for this was probably that the radian concept was very new at this phase, and it was too challenging for most students although they were able to
convert degrees and radians to each other beforehand. The unusual form \( \frac{180^\circ}{\pi} \) of an angle measure might also have confused some students.

**Task iv:** The last difficulty for students was calculating the sine and cosine of 210°. Many students appeared unable to figure out how to use the ratio definitions of sine and cosine to calculate the required coordinates. Although most students could draw the correct rotation and determine the required vertical or horizontal positions, they could not come up with the final answers. The basic reason of this for some students was that they did not know or remember the side lengths of a right triangle of 30° and 60° by which they could have calculated the trigonometric values of these angles. However, many students could not even make the links between triangle trigonometry and the unit circle that were not explicit in the task. It was hard for them to make these links on their own.

Note that these findings are about the students’ difficulties during task solving in groups for their concept development of the sine and cosine functions. The more important issue was if students could overcome these difficulties in subsequent whole class discussions. It was observed that drawing the graph, converting \( \frac{180^\circ}{\pi} \) to a radian and the transition based on it to real numbers, and integrating triangle trigonometry to the unit circle for calculations such as sin 210° (tasks i, iii, and iv) were rather easy for students afterwards. However, they could not overcome the problem of deriving the formula of the function (task ii). As discussed above, the most probable reason underlying this difficulty was moving from specific values to a general relationship, in other words moving from the concrete to the abstract.

In general, students need more time when they are asked to think in a different way and from a new perspective. Therefore, spending more time on some mathematical activities based on generalizations may, for example, help students to understand how to derive a formula. Another alternative could be skipping the step of writing the formula if it is considered too abstract for a particular group of students. Nevertheless, it should be noted that this formula provides an important reason to use the unit of radians, that is, radians help to define trigonometric functions in the domain of real numbers, and thank to this concept the coefficient \( \frac{180^\circ}{\pi} \) can be neglected in more advanced calculations later on in calculus.

Clements and Sarama (2004) emphasized that a designed learning trajectory is hypothetical until it is implemented. In any way of teaching a mathematical topic, it is very likely that there will be some students having certain difficulties. These difficulties can even be opportunities to improve their learning. In this regard, although students faced some difficulties in the hypothetical learning
trajectory within the present study, it can be concluded that the trajectory mostly fits well with the students’ learning of trigonometric functions.

7.2 Conclusion and Discussion for Research Question 2

<table>
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<th>Research Question 2</th>
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<td>What characteristics relating to students’ understanding of sine and cosine can be found in the data resulting from the intervention based on a model of trigonometric understanding?</td>
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The main aim of the present study was to promote an integrated understanding of trigonometry based on three trigonometry contexts: triangle, unit circle and trigonometric function graphs. This was explained within a trigonometric understanding model in Chapter 4, which drove the analysis of the Trigonometry Test, and interviews to assess students’ understanding of trigonometric functions. A slightly simplified version of the model is given in Figure 7.1.

![Figure 7.1: A Model of Trigonometric Understanding](image)

The findings regarding the second research question are presented and discussed separately in this part according to these contexts and their connections. The numbered line segments represent the access of students’ understanding to the various aspects of these contexts and their connections. These aspects are given once again in Table 7.1. The results are reported below with respect to the code of each aspect.
Table 7.1: Aspects of the Trigonometric Understanding Model

It may be useful for the reader to keep in mind that the findings reported here were derived from the analysis of trigonometry test and the four interviews, and that they reflect general conclusions rather than details of individuals.

Note that Triangle Context (aspect 1A) was considered as students’ prior knowledge in this study upon which they could develop an integrated understanding. Hence the presentation of conclusions starts with the context of unit circle.
The Unit Circle Context

It can be concluded from the results of the data analysis that in general students developed a good level of understanding regarding the aspects in the context of unit circle, but it was found that when they were asked to use the coordinate definitions in different kinds of tasks which were not familiar to them, they had problems. The following are the characteristics of students’ understanding regarding the aspects within the unit circle context:

2A. Students knew the coordinate definitions of sine and cosine, and used them well to point to the correct position with the correct direction due to the negative and positive angles or real numbers.

2B. They were able to evaluate trigonometric functions of real numbers which is a multiple of π by associating the number to an arc on the unit circle. The use of function notation such as f(x)=sin x was confusing for some students and prevented them from being able to give the final answer in calculating trigonometric values, although they could indicate the correct positions on the unit circle.

2C. Students were not able to prove a trigonometric equality given with a variable. Most preferred using the unit circle rather than the graph, and all who succeeded to show the equality did it for a specific value of the variable. On the other hand, students were more successful to show equalities with specific angle values.

2D. Students understood the radian concept well enough for the construction of knowledge of trigonometric functions based on the relationship that the measure of an angle in radians is equal to the subtended arc’s length.

2E. Students were able to find the measure of an angle in degrees subtended by a given arc, and vice versa, by using the proportional relationship between radians and degrees.

Although the students learnt well the coordinate definitions of sine and cosine, they could not use them in some different kinds of tasks. Not everyone could provide the final result for the question sin (-\(\frac{\pi}{2}\)) in the test, but this was not the case in the interviews. Yet in both cases, students could show the correct position corresponding to the required trigonometric value. It also includes the correct direction on the unit circle due to the negative sign. Students did not have problems with negative angles, unlike what Martinez-Sierra (2008) reported as a problematic concept for students. It is likely that the function notation as f(x)=sin x was confusing for some students. This is because in the trigonometry test, the value of sin (-\(\frac{\pi}{2}\)) was asked in the form: “If f(x)=sin x, f (-\(\frac{\pi}{2}\))=?”. Although they could point to the correct position on the unit circle, and stated f (-\(\frac{\pi}{2}\))=sin (-\(\frac{\pi}{2}\)), some students did not provide the final result of -1. Trigonometric functions are not generally represented by such notation in teaching, so it might be useful to emphasize this notation for trigonometric functions in mathematics lessons, especially if students are aimed to conceptualize sine and cosine as functions.

Although many students (14 out of 24) were able to show a trigonometric relationship for specific angles, an overwhelming number of students (18 out of 24) were not able to prove a trigonometric
relationship with a variable. This is most probably related to students’ capabilities in doing mathematical proofs.

Students showed a good but somehow incomplete understanding of the radian concept. They were able to use radians to explain the relationship between arcs and subtended angles, and to explain the role of radians in transition to real numbers as the domain of trigonometric functions. An overwhelming number of students could find the measure of the angle subtended by an arc of \( \frac{\pi}{3} \), but fewer found the length of the arc subtending an angle of 60°. This finding is contrary to that of Orhun (2001) that students cannot relate angles to arcs, and that more students can find the length of an arc corresponding to 60° compared to the reverse. Orhun’s work was no doubt based on a different learning trajectory in the Turkish education system, where she did her research. As another positive finding, students generally could apply proportional reasoning in such calculations instead of replacing \( \pi \) directly by 180°. Additionally, some other difficulties and misconceptions reported in the literature about radians such as not being able to see radians as real numbers (Orhun, 2001; Topçu et al., 2006; Akkoç & Akbaş Gül, 2010), and seeing \( \pi \) as 180 instead of a number close to 3.14 (Akkoç & Akbaş Gül, 2010) were not observed among the students in this research. The emphasis on arcs in the first phase, and then referring to the angles in a connected manner are considered to have helped students in the present study to avoid or overcome such difficulties. Students’ understanding, however, was incomplete because they could not learn the formal definition of the radian concept; instead they conceptualized it based on the fact that the radian measure of an angle is equal to the length of the subtending arc.

**The Graph Context**

Note that drawing and explaining the graphs of sine and cosine functions were not considered under this context within the understanding framework of trigonometry as seen in Table 7.1, but under connections between unit circle and graph contexts because it requires connections to the unit circle context.

For the context of trigonometric function graphs, it can be concluded that students conceptualized sine and cosine as functions of real numbers, and they grasped how to interpret the graphs in terms of domain, range, and periodicity. The following are the characteristics of students’ understanding within the graph context of trigonometric understanding:

3A. Students were successful in explaining why the sine and cosine graphs represent functions by using either the formal definition “There is only one y for every x”, or a process definition based on an input-output mechanism, or a combined conception of the first two like “There is only one output for every input.”
3B. Students were successful in interpreting the trigonometric graphs to state the domain and range of the trigonometric functions.

3C. Students demonstrated a good understanding of periodicity by both explaining what periodicity is, and by stating the periods of the sine and cosine functions.

3D. Students did not prefer to use the graph of the sine function to show a trigonometric relationship with a variable, but they were not successful with the unit circle either, as mentioned in the unit circle context.

It seems a remarkable finding that almost every student could explain why sine and cosine graphs represent functions. Considering that this issue was not researched well in the literature, and that some researchers found that students are not able to explain why trigonometric concepts are functions (e.g., Weber, 2005; Challenger, 2009), it can be claimed that the present study added to the body of knowledge that students in this research study could conceptualize sine and cosine as functions based on either a formal or a process view. This ability seems to be largely related to the way trigonometry is taught. In this regard, the designed lesson sequence was effective. A second issue which seems to be ignored in the literature is periodicity. Students revealed a good understanding of periodicity in the context of trigonometric functions as well as of domain and range within this study. It should be noted that all these issues are also about the connections between the graph and unit circle context, through which a deeper understanding may develop.

**Connections between Triangle and Unit Circle Contexts**

From the analysis of the trigonometry test and interviews, it was concluded that students showed a good understanding regarding how to integrate right triangle definitions with the unit circle in order to calculate the trigonometric values of angles larger than 90°. However, most students could not calculate trigonometric values of well-known angles like 30°, 60°, 45° because they did not know or did not remember the side lengths of a right triangle with these angles. The following are the characteristics of students’ understanding regarding the connections between these two contexts based on different components of the aspect 4A:

- Students performed well in drawing the corresponding rotations on the unit circle, and in determining the required vertical or horizontal position.
- Students could determine a required reference triangle, and a proper trigonometric ratio to calculate the required trigonometric value, e.g. sin 45° for sin 225°.
- Some students showed a flexible use of using either sine or cosine in such tasks. For instance, they could calculate sin 210° with either sin 30° or cos 60°.
- Students were successful in stating the sign of trigonometric values.
- Students could not calculate the trigonometric values of certain acute angles because they did not remember the lengths of the sides of a right triangle with those angles. Hence they could not come up with the final answer in such a task as calculating sin 225°.

Despite most students’ inability to calculate the trigonometric values of acute angles procedurally, they understood the connections between triangle and unit circle contexts. This means that their
understanding, limited to acute angles in the triangle context at the beginning, was extended through the unit circle to any angles. In the literature, there are some studies suggesting that the choice of which context to teach first, unit circle or triangle, results in that (first) choice being more effective (e.g., Kendal & Stacey, 1997; Weber, 2005). In this study, it was assumed and the findings show that an integrated approach is essential for students’ understanding. In addition, this finding is consistent with Brown (2005) who found that students who developed an integrated understanding of ratio and coordinate definitions of trigonometric concepts were more successful in performing calculations such as sin 210°.

**Connections between Unit Circle and Graph Contexts**

It can be concluded that students had a deep understanding of connections between the unit circle and graph contexts. The most remarkable finding is that their understanding of such connections is based on arcs. This can be seen in the following characteristics of their understanding regarding the connections of these two contexts.

5A. Regarding drawing and interpreting the graphs of trigonometric functions

5A.1. Although many students could not draw the graph of cosine on the trigonometry test, mostly because of unsuccessful attempts to memorize the graph, interviews showed that students could indeed draw and explain the sine graph. Students conceptualized trigonometric graphs through the arc lengths on the unit circle and corresponding vertical or horizontal positions.

5A.2. They explained the coordinates of the graphs with a journey metaphor based on arcs as travelled distances and corresponding vertical or horizontal positions, and they related the direction of the movement to the sign of angles or real numbers.

5A.3. They could explain why the graphs do not have a linear structure by comparing the cases with the unit square and the unit circle. Some could even explain the concave down shape of the sine graph.

5A.4. Students could distinguish the sine and cosine graphs by reasoning with arcs and corresponding horizontal and vertical positions.

5A.5. Students could mark the point on the graph corresponding to a given position on the unit circle.

5B. Regarding the function concept and related properties

5B.1. Students could explain the trigonometric functions through arc lengths in the domain of real numbers. Travelled arc lengths as the input and the vertical or horizontal position as the output of these functions constituted their process views of functions in the case of trigonometry.

5B.2. Although students could conceptualize trigonometric functions in the domain of real numbers well, trigonometry test results indicate that even though they could explain π as a real number by referring to the arc length, it was difficult for some of them to state or

22 For example, Student 17, who was one of the interviewees could draw and explain the sine graph coordinate by coordinate. However, she could not draw the correct graph in the trigonometry test the next day. This probably explains the situation for the others who could not draw it in the test because of their attempt to memorize the curves. In addition, the factors related to the formal testing situation such as time restriction, concern to earn a good grade, and stress might well have affected the performance of some students.
use this explicitly when it is asked as “what is x in \( y = \sin x \)” or “Mark the point on the graph corresponding to \( \cos(3) \).

5B.3. Students explained the role of radians in the transition from angles to real numbers by referring to the relationship that the angles in radians are equivalent to the lengths of the subtending arcs. This means that the transition \( \sin (x \text{ rad}) = \sin x \) was meaningful to them.

5B.4. Students could explain the range of trigonometric functions by referring to the unit circle where the vertical and horizontal positions cannot be higher than 1 and lower than -1.

5B.5. Students could connect the period of trigonometric functions to the circumference of the unit circle. They considered \( \pi \) as a real number referring to the arc lengths.

It can be concluded that the reason why many students could not draw the cosine graph in the test was due to their attempts to memorize the curve. In the interviews, one of the students could not draw the graph immediately, but she managed to draw it by considering some individual coordinates. However, in the test this was not the case for most students including the same interviewee because they tended to try to draw it from memory. It is a very positive finding, on the other hand, that students could explain the meaning of the coordinates through the connections to the unit circle which indicates a process view of functions.

A most compelling finding of this study was that students developed a connected understanding regarding the function concept. Their understanding of trigonometric functions based on travelled distances as inputs and the vertical or horizontal position as the output shows that they developed a process view of trigonometric functions. This is very crucial in trigonometry because trigonometric concepts cannot be considered as functions through their algebraic expressions, and they cannot be computed through specific arithmetic operations. As Breidenbach et al. (1992) explained, to attain a process view of functions students should not require recipes such as formulas. This rather confirms the students’ process view of trigonometric functions in the present study. It is also an indication of a deeper understanding through which they could combine different algebraic, geometric, and graphical reasoning to conceptualize trigonometric functions.

Another point worth mentioning here is students’ understanding of the domain of trigonometric functions. It is clear in the data analysis that they conceptualized sine and cosine as functions of real numbers based on the arc lengths, and they also grasped the role of radians in connecting this to angles. For example, when asked what \( \pi \) stands for in \( \sin \left( -\frac{\pi}{2} \right) \), many students referred to the length of the corresponding arc as a real number, and some students mentioned radians by additionally indicating the corresponding arc length, while only a few referred to degrees. However, relatively few students mentioned that \( x \) represents a real number in \( y = \sin x \), and some chose the option radians while a few chose degrees. The students giving arcs in their responses with radians, together with those who referred to radians and degrees simultaneously, indicate that these students might not have been able to distinguish among these in their statements in terms of the domain concept regarding...
trigonometric functions. These findings are contrary to Kang (2003) in that the students in the present study revealed a better understanding of trigonometric functions of real numbers. It is reassuring that some students, at least, could transit among degrees, radians, and real numbers, thereby showing a flexible use of trigonometric contexts.

The majority of students could not mark the point corresponding to \( \cos(3) \) on the cosine graph. This can be because students had difficulty in drawing the graph in the test. Another possibility is that the numerical form of trigonometric functions like \( \cos(3) \) might be confusing for students as it is not used very often. The finding that students could not point to \( \cos(3) \) on the graph is consistent with Orhun’s (2001) finding that students have difficulties with analytic trigonometry. Nevertheless, the finding of the present study that students could clearly explain that there is no difference between the trigonometric graphs given with integers and those with multiples of \( \pi \) because of the domain of real numbers is a contrasting issue. This may be because some students’ angle conceptions remained dominant in a way that they could not differentiate angle definitions from functions of real numbers, although they demonstrated knowledge of the transition from angles to real numbers. The evidence for this is clear in the responses of the interviewees.

The finding that an overwhelming majority of students could indicate the correct point corresponding to a given position on the unit circle reveals their good understanding. Brown (2005) stated this task to be one indicating a deep understanding. This finding in the present study is in contrast to her finding of students’ low performance on this task.

### 7.3 Effectiveness of the Instructional Design

Effectiveness of any instructional approach depends on the initial expectations regarding the kind of knowledge and understanding which students are aimed to attain. Do we consider it important for students to be able to explain why trigonometric functions have the properties they have? Or should they only state what properties they have? Is it enough, for instance, to state the period of sine as \( 2\pi \), which is quite easy to memorize? Or do we also expect students to explain why it is \( 2\pi \) by connecting the mathematics in the graph to the unit circle? Is it important to explain why it is a function? Do we find it at a sufficient mathematical level if students can draw trigonometric graphs by heart, or should they also explain why it is the graph of sine and then interpret it? The importance here is to consider what we expect our students to learn before we implement an instructional design. In a mechanical way of teaching mathematics, acquiring mathematical knowledge by memorizing, following cookbook procedures, and drilling may be considered sufficient. However, if it is essential to learn mathematics with understanding, then such questions should be answered in ways which address
students’ need to grasp conceptual connections and to learn to use them to explain mathematical processes.

The instructional sequence in this study was effective in that students developed in general a conceptual understanding of trigonometric functions as modeled in Figure 7.1. Most of the connections among the three contexts of trigonometry: triangle, unit circle and function graphs, are probably beyond what students would gain from traditional methods of teaching trigonometry. It is worth mentioning that the most beneficial aspect of the lesson sequence was its emphasis on arcs at an earlier stage than angles. It is clear that students understood the important connections through the arc lengths. Students were not found to have developed difficulties cited in the literature. The main reason for this is that their concept development was built up from arcs. As one of the main aims of the design, students could conceptualize sine and cosine as functions of real numbers within a process view based on their actions done with arcs. GeoGebra was very beneficial as a cognitive tool in this regard by also allowing the students to perform some actions beyond what they could have been able to demonstrate using only paper, pencil and a basic hand-held calculator. In addition, the indications are that performing relatively easy actions on the unit square before moving on to the unit circle, helped students to construct meaning.

In the lesson design, students transited to angles from arcs, and this helped them to develop a complete and connected understanding. They also developed a good understanding of the radian concept. Hence, the transition from arcs to angles in the lessons were found to be effective. Nevertheless, at the end of the five lessons students were not yet capable of the formal definition of radians. If students are aimed to gain the formal definition as well, another approach like the one suggested by Moore (2010) may be more effective. This would necessitate at least one more lesson in the sequence. In the present study, the radian concept was defined as the ratio of an arc to the radius of the circle, whereas Moore suggested teaching one radian as the measure of an angle corresponding to the \(\frac{1}{2\pi}\)th of the circumference of any circle. In any case it might be useful to lead students to investigate the radian concept for more circles, not only for the unit circle, then it could be easier for them to grasp the formal definition. Such a result was found in the study of Akkoç & Gül (2010).

As it can be seen from the data analysis and conclusions, students had some procedural problems, and there were some difficulties when they faced standard types of tasks in different forms. To overcome such issues, more practice could be added to the lesson design.
7.4 Discussion of the Study

The main positive point of the present study was the effectiveness of the instructional design in promoting students’ integrated understanding of trigonometric functions in a way that they did not develop difficulties and misconceptions, at least those reported in the literature. It is worth noting once more that the literature on trigonometry appears to be very limited, especially in terms of students’ understanding of trigonometric functions in the domain of real numbers. The present study included a broader approach by connecting important aspects of trigonometric understanding in an integrated manner. Trigonometric functions based on both formal and a process view of functions, their domain and range, periodicity, drawing and interpreting their graphs were not found in any research study in the literature. Some researchers only mentioned students’ difficulties to consider and explain trigonometric values as functions of real numbers (e.g., Kang, 2003; Orhun, 2001). This study shows that the designed learning trajectory helped students to develop a good understanding of trigonometric functions, their properties, and graphs based on connections between the unit circle and their graphs. The early emphasis on arc lengths was the main factor promoting such a good level of understanding. Arcs served really as the glue among different concepts and ideas for students’ understanding, as was assumed in the formulation of the understanding model (see Section 4.3).

The radian concept was reported by many researchers to be problematic for students (Orhun, 2001; Topçu et al., 2006; Akkoç & Gül, 2010). Martinez-Sierra (2008) made a very significant point by explaining the radian concept as the major break in students’ construction of trigonometric functions. In contrast, and given the limitations inherent in a small study, students in the present study developed a good understanding of radians. The transition from trigonometry of angles to trigonometric functions of real numbers was done through radians. The way this concept was covered – based on making connections between angles and arcs through radians specific to the unit circle – seemed to be helpful for the students. They received what they needed for their concept development of trigonometric functions in terms of the radian concept. This may have well been the reason why, at this point, they could not learn the formal definition, which addresses a broader aspect. They constructed their knowledge by conceptualizing radians based on the fact that the radian measure of an angle is equivalent to the length of the arc subtending this angle.

Such positive aspects of the study were in spite of the existence of some difficulties encountered in the implementation and data analysis.

The most important difficulty was about the type of tasks which affected students’ performances in and contributions to the lessons. All of the tasks were prepared in a way to lead students to mathematical investigation, through which they could develop understanding by interacting with peers and the researcher. However, this type of activity was not familiar to the students, and
occasionally some were not willing to participate or proceed without receiving clear and short rules, definitions, or answers from the researcher. As can be read in the interview analysis (Section 6.4), students mentioned that they were expecting to have rules at the beginning and then to do some exercises. A student stated very clearly the kind of tasks they were accustomed to engage in:

“If we have a subject, we have one page with how it works, and then we have a lot of exercises to make, and then you understand how to make it and never explain why you do that like that.”

[Student 11]

As another example, a student resisted to proceed on one of the worksheet tasks using the given definition of radians. They were expected to interpret radians for the unit circle by using its definition. She told the researcher that they had never learnt about radians before although a definition was given on the worksheet. So, she was not accustomed to learning a mathematics subject requiring thinking, investigation, understanding, and explanation.

Such instances due to the unfamiliar task types may have well influenced the motivation and performance of the students to contribute to the group and class discussions. This can be connected to the “didactic contract” of Brousseau (1997). A didactic contract determines the interactions and experiences in which students engage in their learning. The researcher was a stranger to the students, and he was trying to do something completely different from what the students had been practicing in that classroom as determined by their didactic contract with their teacher. It is important to consider that such a contract is usually established at the beginning of the academic year and holds for the entire course with that teacher. It was quite difficult for the researcher to break the original contract, and create a new one only in five lessons in the middle of the course.

A second difficulty in the lessons concerned the language of instruction. The lessons were taught and the data were collected in English, but neither the researcher nor the students were native speakers of English. It was more problematic for the students, although they were initially expected to be more fluent in English. Luckily, they could understand most of the spoken English in the lessons, but the mathematical tasks written in English were difficult for them to interpret, then think and discuss in Dutch, and finally present or write again in English. This also meant that they needed much more time for the tasks than initially planned. This caused timing problems in the implementation, together with the researcher’s rather unrealistic original allocation of time to each task. The problem of language was also confirmed by the cooperating teacher and another teacher who willingly implemented the same lesson materials. They stated that the students would have participated more actively and performed better if the instruction had been completely in Dutch.

Another difficulty related to the language issue was encountered while analyzing the audio recordings of the group discussions. Although they were asked to discuss in English as much as they could, the students naturally preferred to use Dutch in their group work. Because the researcher did not
understand Dutch, he could not analyze the data directly by himself. He received help from a friend, who was unconnected to the research. It was not possible to listen and transcribe everything. In order to reduce and make feasible the translation and transcription of group data, the researcher determined probable main points from the worksheet responses about which the discussions were likely to be interesting and useful. Then the corresponding parts of the audio were transcribed and translated to English.

The final difficulty was about assessing students’ understanding of trigonometric functions. Understanding can be a vague term to research unless the meaning of the term is clearly described. Unfortunately, the literature on teaching and learning trigonometry is far from being adequate in providing a useful framework in terms of trigonometric understanding. Only a few studies have stated clearly how understanding was conceived (e.g., Weber, 2005; Brown, 2005), but they could not cover all of the aspects considered within the present study. For instance, in none of these studies were the domain of real numbers or connections among the three contexts considered. Therefore, the researcher provided his own model of understanding to describe what is meant by students’ understanding of trigonometric functions within the present study. Since the model has not been tested by anyone else, there may well be as yet undiscovered limitations and weak points in it.

### 7.5 Suggestions for Future Research

The present study includes obvious limitations such as revealing findings from only a single mathematics classroom, and being based on an untested framework of understanding. Further research should address such limitations regarding this study. It is worth mentioning that the study reported in this thesis is easily repeatable, and the teacher guide presented in Appendix A may help others to implement the same or an alternative design.

It should be noted that the tested teaching approach within the study reported in this thesis was a new approach to teaching trigonometric functions. In this regard, it addresses the absence in the literature of different tested methods in terms of their effects on students’ understanding. There is considerable need for further research to contribute to the literature on this aspect and to the improvement of teaching trigonometry.

This study was conducted at a pre-university level (VWO) Dutch mathematics classroom. It might be interesting to try the new teaching approach in some other levels, for instance at a HAVO class in the Netherlands, or at different levels in any country. It will be invaluable to have more research findings on the effects of the provided learning trajectory in this thesis based on the presented understanding framework to see if it fits well with different groups of students.
Some parts of the new teaching approach can be at a very high level for some groups. For example, finding the formula of a graph showing the vertical position of a moving point along the unit circle versus the distance it travels was found to be tough for most students in this study. The steps of the learning trajectory basing the new model (Chapter 5) can be altered according to another researcher’s learning aims for students and to students’ general mathematical levels. It might be quite interesting to try some altered version of the learning trajectory used in this study in some future research. For instance, the step of coming up with the formula $s(x) = \sin\left(\frac{180x}{\pi}\right)$ could be skipped for lower level mathematics students, the emphasis of arcs should remain, and the meaning of radian concept should be compensated by the addition of another way. Trying such alternatives for different levels of students would enable a thorough exploration of the new teaching approach. A second factor to consider in altering the learning trajectory can be curricular aspects. Steps of the trajectory can be shaped with respect to certain curricular requirements or limitations, and researched to explain subsequent effects on students’ understanding.

Finally, although the teaching approach reported here may be rejected by some teachers because it is time consuming, it can also be very effective and efficient when students’ future studies of trigonometry are considered. If students gain a strong basic understanding as a result of such a teaching approach at the introductory level, it may help them learn and understand more advanced trigonometric topics more easily and deeply. Furthermore, some more advanced topics and properties of trigonometric functions, for example $\lim_{x \to 0} \frac{\sin x}{x} = 1$, are also valid for the function $s$ defined on the unit square before on the unit circle in the learning trajectory within this study. Introducing students to those through the function $s$ prior to sine could provide better results. Such issues are worth researching. So although it may be more time consuming, certainly when first used, in the end and for later courses students might well use the understand-explain strategies as a matter of course not only in this particular topic, but also in some others. Some future research should shed light on this issue by investigating if such an approach can make students’ future learning of trigonometry easier and better. This calls for a long term research study lasting more than a single year.
References


References


References


APPENDICES

APPENDIX A: TEACHER GUIDE WITH DESIGNED WORKSHEETS

APPENDIX B: WORKSHEETS USED FOR DATA COLLECTION

APPENDIX C: DIAGNOSTIC TEST

APPENDIX D: TRIGONOMETRY TEST

APPENDIX E: INTERVIEW TASKS
Appendices

Appendix A: Teacher Guide with Designed Worksheets

TEACHER GUIDE

This document was prepared to help you for the implementation of the lesson sequence on trigonometric functions. The teaching approach is different from traditional methods. First of all, the presented lesson materials here ignores the early introduction of angles to define sine and cosine functions. Instead, the main focus is arc lengths at the beginning. In addition, the sequence considers the graph of sine and cosine at the very beginning together with the connections to the unit circle. By this method, it is expected that students will be able to understand sine and cosine as functions of real numbers more easily. The lesson sequence has the focus of the function concept where students will be guided to learn them in an integrated manner by connecting different representations of the same trigonometric concepts: right triangle trigonometry, unit circle trigonometry, and the graphs of trigonometric functions.

The lesson sequence was designed in a way that students will learn by using their own inputs as much as possible, and by the social interactions in the class. Students will do their tasks in pairs, and after pair-work, a class discussion will take place every time. The following aspects are important to consider:

• Students should work with the same partner in every lesson.
• The teacher should guide students during their pair-work by walking around and giving feedback to the students.
• During the lessons, two worksheets will be given to each pair of students. One of them will be collected before the class discussion so that the researcher can use it as a data collection instrument.
• Students should be told that they should do their homework on their own, and that they need to deliver every homework sheet in the following lesson.

This document includes all lesson materials for the purpose of teacher use only. The students should not have the access to this document. For every lesson, a short lesson plan is included with lesson objectives, prerequisite knowledge and skills for students, lesson materials, and step-by-step lesson flow. Then copies of worksheets and homework sheets are included with answers and some important remarks in red to give some clues regarding the aims of the tasks and the issues to pay attention in the lessons.
LESSON 1

A. Objectives:

At the end of the lesson, students will be able to
1. Draw and interpret the graph $s$ of the travelled distance versus the vertical position of a point which moves along the unit square in both counter-clockwise and clockwise direction.
2. Define periodicity, and find the period of the function corresponding to the drawn graph $s$.
3. Find the points on the unit square corresponding to the given positions on the graph $s$.
4. Explain the function whose graph is $s$ with its input and output.

B. Prerequisite Knowledge and Skills:

Prior to the lesson, student should be able to
1. Draw the graph of a given linear relationship on the Cartesian coordinate system using coordinates.
2. Define what a function is as an input and output mechanism, and explain why a given relationship or a graph represents a function.

C. Materials:

1. Worksheets 1a, 1b, 2, and Homework Sheet 1
2. GeoGebra Applet 1
3. One computer for each pair of students
4. A computer and projector for the teacher
5. Whiteboard

D. Lesson-Flow

1. Introduction: 5 minutes
   • Discuss what a function is. Exemplify two relations one of which is not a function with their graphs below. Mention the single-valuedness

   ![Graphs](image-url)
2. **Lesson Body: 50 minutes**
   - Hand out *Worksheet 1a* (two per pair) and describe the task for the students. Explain the movement of the point P.
   - Give 10 minutes to the pairs to draw the graph illustrating the vertical position versus the travelled distance of a point in the counter-clockwise direction along the unit square. Collect one worksheet from each pair.
   - Ask a pair to draw the graph on the whiteboard. Discuss with the class about the graph for 5 minutes. To make sure that every pair has the correct graph at the end, give the pairs the attached graph.
   - Handout *Worksheet 1b* which aims to lead students to comprehending the graph and making it more meaningful.
   - Give 10 minutes to the pairs to complete the Worksheet 1b. Collect worksheets.
   - Spend 5 minutes to discuss the questions with the students.
   - In the discussion, remember to give the definition of periodicity and make sure that the students have realized that the period of s is equal to the perimeter of the unit square!
   - Hand out *Worksheet 2*. Explain how students can start the GeoGebra applet 1.
   - Give 10 minutes to the pairs to complete the worksheet.
   - Discuss the tasks of the Worksheet 2 in the rest of the lesson. The discussion may be completed in the next lesson if the time is not enough. Remember to emphasize that we are working on a function called s and that its input is the path-length travelled and the output is the corresponding vertical position. It is very basic to transit to the sine function later on.

3. **Closure: 5 minutes.**
   - Hand out Homework Sheet 1. Mention that it will be collected in the next lesson.
Worksheet 1a - Walking around the Unit Square

The unit square is the square whose center is placed at the origin of the Cartesian coordinate system with a side length of 2 units. An example of the unit square is shown on the coordinate system on the attached graph paper.

We consider a point P on the plane that moves along the unit square starting at (1, 0) in the counterclockwise direction.

Plot the vertical position of the moving point P against the distance travelled around the square. The x-axis represents the distance along the path that P has traveled and the y-axis represents the vertical position (y-coordinate) of P as it moves around the square.

Draw the graph on the attached graph paper!
Worksheet 1b - Walking around the Unit Square

Let us name the graph you have drawn $s$.

Discuss the following questions with your partner and make notes.

1. Consider the points A(2, 1), B(6, -1) & C(13, -1) on the graph of $s$. To which positions of P on the unit square do they each correspond?

   Remark: This question aims for students to connect the information of the graph to the unit square.

   Answer: A $\rightarrow$ (0, 1), B $\rightarrow$ (0, -1), C $\rightarrow$ (-1, -1)

2. Remark: Question 2 will be connected to periodicity.

   a. What is the distance along the path that P has moved when it returns to the starting position for the first time?

      8

   b. What is it for the second time and for the third time?

      16 & 24

   c. To what points on the graph $s$ do they correspond? What do you observe regarding the coordinates of these points?

      First time $\rightarrow$ (8, 0)
      Second time $\rightarrow$ (16, 0)
      Third time $\rightarrow$ (24, 0)

      Observation: x-coordinates are the positive multiples of 8 which is the perimeter of the unit square, and y – coordinates are always 0.

3. Write as many characteristics of the graph as you can think of.

   Open question. Possible answers: “It is bounded.”, “It repeats itself.”. Suggestion: Ask “What are the minimum and maximum values of the graph? What do they represent?”
Worksheet 2 - Walking along the Unit Square

Open Applet 1.

The applet shows the graph $s$ which you have just drawn. Dragging the point Q along the positive x-axis, observe the changes in the points P and S. Pay attention to how the motion of Q is related to the motion of P by considering what they represent.

Discuss the following with your partner and make notes:

1. What does the point S represent? What is its relation to P and Q?
   Remark: Students need to realize and explain the relationship between S, Q and P. This is a comprehension of the graph they have drawn and also observed on GeoGebra. S is the point in the graph which connects the information revealed by P and Q. x-coordinate of S is the x-coordinate of Q, and y-coordinate of S is the y-coordinate of P.

2. Position the point Q at (8, 0). Move to the left while keeping an eye on P and S. Now suppose that you can move Q to the left of the y-axis. How would you extend the graph $s$? Sketch your idea below.

3. How would you interpret the point (-5, 1) on the extended part of $s$? What does the extended part mean?
   Remark: The extended part shows the movement in the reverse direction. This is an important step to introduce negative angles later on. When the point P travels 5 units in the reverse direction (indicated by the minus sign), its vertical position is 1. The extended part shows the graph when the point P moves in the clockwise direction along the unit square.
4. Is the graph of a function?

   Remark: Since the emphasis of the lesson-sequence is the trigonometric functions, it is important to focus on the notion of function in this early phase. Make sure that students understand the input and output of the function because this is the basic to transit to the sine function through arc lengths.

   YES.

5. If your answer to Question 4 is yes, why is it a function? What are its input and output?

   Because of the single-valuedness, i.e. for every x value there is only one corresponding y value.
   Input of the function is the travelled distance, and the output is the vertical position.
   So, s is a function indicating the vertical position of P corresponding to any distance it has travelled.

6. If your answer to the Question 5 is no, why not?

   Not Applicable.
HOMEWORK – LESSON 1

1. Consider the graph $s$ below (travelled distance versus vertical position) we have drawn in the class. Answer the following questions:

   ![Graph](image_url)

   a. $s(8) = ?$ $s(10) = ?$

   $s(8) = 0$ $s(10) = 1$

   b. With a sentence or so explain what the equalities you have written in part a mean.

   $s(8) = 0$ means that when point P moves 8 units in the counter-clockwise direction starting from $(1,0)$, its vertical position is 0.

   Similarly for the second one, when it moves 10 units, its vertical position is 1.

   $s$ is a function of travelled distance, showing the vertical position.

   c. What is the period of the graph? How can you relate it to the unit square?

   The period is 8. It is the perimeter of the unit square.
LESSON 2

A. Objectives:

At the end of the lesson, students will be able to

1. Define the unit circle.
2. Explain that the period of the movement along the unit circle is $2\pi$.
3. Calculate the coordinates of the points corresponding to certain arc lengths.
4. Calculate the measures of the angles in degrees corresponding to certain arc lengths.
5. Find the formula of the function $s$ for the unit circle.

B. Prerequisite Knowledge and Skills:

Prior to the lesson, student should be able to

1. Define periodicity.
2. Calculate the perimeter of a circle.
3. Calculate the length of certain arcs corresponding to different central angles using proportions.
4. Use Pythagoras Theorem to find the side lengths.
5. Calculate the sine value of certain angles using right triangles.

C. Materials:

1. Worksheets 3, 4, 5, and Homework Sheet 2
2. GeoGebra Applet 2 and 3
3. One computer for each pair of students
4. A computer and projector for the teacher
5. Whiteboard

D. Lesson-Flow

1. Introduction: 5 minutes
   - Collect Homework 1. Discuss it shortly, and revise what periodicity means putting an emphasis its relation to the unit square.

2. Lesson Body: 50 minutes
   - Hand out Worksheet 3 which will be a transition to the unit circle, and give students 10 minutes to complete the worksheet.
   - Discuss with the class the worksheet for 5 minutes. Define the unit circle (the circle which is approached by the polygon as its number of sides increase), and name travelled distances as arc lengths for the circle. Collect Worksheet 3.
   - Hand out the Worksheet 4 which Give students 10 minutes for their pair-work.
   - Discuss the questions with the class for 5 minutes. Then collect Worksheet 4.
   - Hand out Worksheet 5 with which the students will connect arc lengths and central angles, and come up with the formula of the function $s$. 
• Give students 15 minutes to complete the worksheet.
• Discuss the tasks in the rest of the lesson. Pay attention to emphasize the connections between concepts used to get the formula.

3. **Closure: 5 minutes.**

• Hand out Homework Sheet 2. Mention that it will be collected in the next lesson.
Worksheet 3

Open applet 2. In this applet you can see the graph $s$ for polygons with $n$ sides, not only for the unit square for the movement of $P$ in both clockwise and counter clockwise direction. Observe the changes in the graph as you modify $n$.

Remark: This applet demonstrates the increase of the number of the sides of the polygons starting with the unit square.

Discuss the followings with your partner and make notes.

1. For $n=8$, what is the period of the graph $s$? What about for $n=12$?

   $n = 8 \rightarrow$ period is $\approx 6.6$
   $n = 12 \rightarrow$ period is $\approx 6.4$

2. What happens to the polygon and the graph $s$ as the values of $n$ become larger?

   The polygon approaches a circle (later: the unit circle), and the graph becomes a smooth curve (later: the graph of sine).

3. **CLAIM:** For very large values of $n$, the period of the movement approaches to $2\pi$? Is this true or false? Verify your answer by using the graph and corresponding polygon?

   Remark: This part aims at a smooth introduction of $\pi$, and connecting periodicity, graph and polygon.
   This claim is true. The period is equal to the perimeter of the respective polygon. Since the polygon approaches the circle with radius 1, the period approaches its circumference, in other words $2\pi$.

4. For very large values of $n$, is the graph $s$ still representing a function? Why or why not?

   Yes because single-valuedness is not ruined. There is still only one $y$ value for each $x$. 
Worksheet 4 – Unit Circle

Let P be a point moving on the unit circle starting from (1, 0) in the counter-clockwise direction.

Remark: This worksheet aims for students to connect coordinates and arc lengths.

1. What are the coordinates of P when it has moved an arc length of $\frac{\pi}{2}$ along the unit circle?

   $(0, 1)$

2. What are the coordinates of P when it has moved an arc length of $\pi$ along the unit circle?

   $(1, 0)$

3. What are the coordinates of P when it has moved an arc length of $3\pi$ along the unit circle?

   $(1, 0)$

4. Consider now that P has moved an arc length of $\frac{\pi}{4}$ along the unit circle. Make a drawing and answer the following.

   Remark: Students need to use their prior knowledge of Pythagoras Theorem and trigonometric ratios in right triangles.
   a. How many degrees is the central angle subtending this distance?

   $45^\circ$

   b. What are the coordinates of P?

   $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
Worksheet 5 – Rotation of a line segment by a given angle

In the last activity, we have realized that an arc length of \( \frac{\pi}{4} \) corresponds to a central angle of 45°.

This is illustrated in the diagram below.

Using the corresponding central angles, the movement of the point P along the unit circle can also be described as a rotation of \(|OP|\) around the origin.

1. Consider the point P has moved \( \frac{3\pi}{2} \) along the unit circle in the counter-clockwise direction from (1, 0).
   a. What is the rotation angle corresponding to that movement? 270°
   b. What are the coordinates of P after this movement? (0, -1)
   c. To which point on the graph \( s \) (plotting travelled distance against the vertical position on the unit circle) does this position correspond? Make a sketch here.

   d. Open applet 3. Verify your answer to b and c. Consider once again what P & Q represent.
2. Find the following central angles corresponding to the given arc lengths.

\[ \begin{align*}
\pi & \rightarrow 180^\circ \\
\frac{\pi}{2} & \rightarrow 90^\circ \\
2\pi & \rightarrow 360^\circ \\
\frac{3\pi}{2} & \rightarrow 270^\circ 
\end{align*} \]

3. For the function \( s \), if \( s(x) = -1 \) what is the smallest positive value of \( x \)? What does \( x \) represent?

In other words what is the input of the function \( s \)?

\[ x = \frac{3\pi}{2} \]

\( x \) represents the travelled arc length or the arc length subtended by the rotation angle corresponding to this arc length.

4. CLAIM: The \( y \)-coordinate of \( S \) can be written as \( \sin (\alpha) \), for any position of \( P \). In other words for any \( x \), we can find an \( \alpha \) such that \( s(x) = \sin (\alpha) \). What could be meant by \( \alpha \)? Verify the claim for \( x = \frac{\pi}{4} \).

\( x \) refers to the arc length, and \( \alpha \) represents the rotation angle corresponding to that angle.

For \( x = \frac{\pi}{4} \), \( s\left(\frac{\pi}{4}\right) = \sin (45^\circ) = \frac{\sqrt{2}}{2} \) using a right triangle and the graph \( s \).

Fill in the blanks considering the relationship.

\[ \begin{align*}
s(\pi) &= \sin (\ldots180^\circ\ldots) = 0 \\
s\left(\frac{\pi}{2}\right) &= \sin (\ldots90^\circ\ldots) = 1 \\
s\left(\frac{3\pi}{2}\right) &= \sin (\ldots270^\circ\ldots) = -1 \\
s(2\pi) &= \sin (\ldots360^\circ\ldots) = 0
\end{align*} \]

5. Give the formula of the function \( s \).

(Hint: Use the relationship in 4 between arcs and angles.)

\[ s(x) = \ldots \sin\left(\frac{180^\circ x}{\pi}\right) \]

by using the proportional relationship of

\[ \begin{align*}
\text{arc} & \rightarrow \text{angle} \\
\pi & \rightarrow 180^\circ \\
x & \rightarrow \alpha
\end{align*} \]
1. In the lesson we found the formula $s(x) = \sin \left( \frac{180^\circ x}{\pi} \right)$ for our function $s$. Briefly explain here how we derived the formula. Pay attention to the corresponding arc lengths and central angles.

Remark: this part is a kind of reproducing the work which has been done in the class. It aims students to think about the steps of driving the formula on their own.

2. Using the formula compute $s(3)$. You need a calculator!

$$s(3) = \sin \left( \frac{180^\circ 3}{\pi} \right) \approx 0,14$$

Remark: This task will be connected to the sine function of real numbers after introducing radians in Lesson 3.
LESSON 3

A. Objectives:

At the end of the lesson, students will be able to
1. Define radian, and interpret it for the unit circle.
2. Convert degrees and radians to each other.
3. Explain sine as a function of real numbers, its properties.
4. Calculate the sine values of given angles or real numbers.

B. Prerequisite Knowledge and Skills:

Prior to the lesson, student should be able to
1. Find the domain and range of a function.
2. Use Pythagoras Theorem to find the side lengths.
3. Calculate the sine value of certain angles using right triangles.

C. Materials:

1. Worksheets 6 and 7, and Homework Sheet 3
2. GeoGebra Applet 4
3. One computer for each pair of students
4. A computer and projector for the teacher
5. Whiteboard

D. Lesson-Flow

1. Introduction: 10 minutes
   • Collect Homework 2. Summarize how to derive the formula of s in terms of degrees.

2. Lesson Body: 50 minutes
   • Hand out Worksheet 6 (one per pair) which will be a transition to the unit circle, and give students 15 minutes to complete the worksheet.
   • Discuss with the class the worksheet for 10 minutes. In the discussion, remember to emphasize the connections used to define sine as a function on real numbers. In other words, the relationship that an arc length is equal to the measure of the corresponding angle in radians for the unit circle. Hence, we can define sine as a function of real numbers instead of angles.
   • Collect Worksheet 6.
   • Hand out the Worksheet 7 and give students 15 minutes for their pair-work.
   • Discuss the questions with the class for 10 minutes. Then collect Worksheet 7.

3. Closure: 5 minutes.
   • Let the students use the applet 4 to visualize different angles units and arc lengths relations.
   • Hand out Homework Sheet 3. Mention that it will be collected in the next lesson.
Worksheet 6

Angles can be measured in degrees and radians.

The measure of an angle in radians is defined as the ratio of the arc length subtended by that angle to the radius of the circle.

The diagram below illustrates how the measure in radian is defined.

1. What does this mean for the unit circle? What is the measure of the angle in radians subtending an arc of \( \frac{\pi}{4} \)? What can you say about the relationship between the arc length and the angle?

Remark: This question is expected to help students why radian measure is equal to the corresponding arc length in the unit circle, which is an elaboration on the definition of radians, and why radian measure is expressed with \( \pi \).

The definition indicates that the measure of an angle in radians is equal to the arc length subtended by that angle. The angle subtending an arc of \( \frac{\pi}{4} \) is therefore \( \frac{\pi}{4} \) radians.

2. To what angle does the arc of \( 2\pi \) on the unit circle correspond?
   a. in radians \( \Rightarrow 2\pi \)
   b. in degrees \( \Rightarrow 360^\circ \)
3. Explain how to convert radians to degrees by a formula or by words.

   Using the relationship that \(2\pi\) radians is equal to \(360^\circ\), conversions can be carried out by the proportion, or by the following formula:

\[
\frac{\text{Degree}}{360^\circ} = \frac{\text{Radian}}{2\pi}
\]

4. Back to our formula \(s(x) = \sin\left(\frac{180^\circ x}{\pi}\right)\).

Remark: This task will make it meaningful for students why we need the unit of radians by connecting it to how sine can be defined as a function of real numbers.

   a. What is \(\frac{180^\circ}{\pi}\) in radians? \(\Rightarrow 1 \text{ radian}\)

   b. Using radians what does the formula become? \(\Rightarrow s(x) = \sin \left( x \text{ rad} \right)\)

   c. In the last version of our formula what can \(x\) denote?

      Rotation angle in radians, equivalently arc lengths (because they are equal for the unit circle), and hence can be any real number. Hence, radian unit can be omitted, and \(s(x)=\sin x\) for any real number \(x\).

5. Answer the following questions regarding some properties of our function \(s\):

   a. What is the domain of the sine function? Real Numbers

   b. What is the range of the sine function? \([-1, 1]\)

   c. What is its period? \(2\pi \approx 6.28\)
Worksheet 7

1. Consider an arc of length approximately 3,14. How can you find \( \sin(3,14) \)?

Remark: This question is not of the standard form because the students are asked to find the sine of a value given without \( \pi \). This may help them to comprehend it further to see sine as a function of real numbers.
(Refer here to the homework question about \( s(3) \). Mention \( s(3) = \sin (3 \text{ rad}) = \sin (3) \).) It is approximately \( \pi \). Hence, \( \sin \pi = 0 \).

2. How can you find \( \sin(-\frac{\pi}{2}) \)? What does a negative angle mean?

Remark: For the movement around the unit circle, we have discussed the meaning of the negative inputs. Now, students need to learn that negative angles refer to the rotations in the clockwise-direction.
Negative angles mean rotations in the clockwise direction. Hence it is -1.

3. In which quadrant(s), are the sine values of angles are negative? Why?

Sine values are negative for the angles in the 3\(^{rd}\) and 4\(^{th}\) quadrants because sine values stand on the y-axis.

4. Given the unit circle on the next page. Discuss the following:

Remark: This task aims at integrating different conceptions of trigonometry, in other words, triangle trigonometry and unit circle trigonometry.

a. What is the sine value of the angle \( \alpha \)? Why?

0,8 because it is the y-coordinate of the point B.

b. How can you connect this to the \( \sin \alpha \) in a right triangle?

See the triangle ODB on the attached unit circle.

\[
\sin \alpha = \frac{|DB|}{|OB|} = \frac{0,8}{1} = 0,8
\]

Because the radius of the unit circle is 1 unit, the sine of an angle in the first quadrant is equal to the y-coordinate.

5. Try to relate \( \sin 120^{\circ} \) (using the second attached unit circle) to sine as a ratio in a right triangle.

See the triangle ODB on the second attached unit circle. The sine of 120\(^{\circ}\) is equal to the sine of its supplementary angle (60\(^{\circ}\)) as seen in the triangle in the second quadrant.
6. How does the unit circle trigonometry extend your knowledge compared to right triangle trigonometry?

With right triangles
- it is not possible to find the sine values of angles greater than or equal to $90^\circ$.
- It is not possible to find the sine values of negative angles.
- It is not possible to define sine as a function of real numbers.
HOMEWORK - Lesson 3

1. Attached is the unit square on a graph paper. Consider again that the point P is moving along the unit square in the counter clockwise direction starting from the point (1, 0). On the graph paper, draw the graph of travelled distance versus the horizontal position (this time instead of vertical position).

   Remark: This task will be discussed to introduce cosine function in lesson 4. See attached graph paper.

2. Explain with a sentence or so how to find the sine of some given angles in the unit circle.

   Find the sine values of the given angles and show your work with a drawing for each. No calculators.

   a. \(30^\circ\) \(\rightarrow\) Using a right triangle, it is \(\frac{1}{2}\).

   b. \(\frac{3\pi}{2}\) rad \(\rightarrow\) Using the unit circle, it is -1.

   c. \(150^\circ\) \(\rightarrow\) Using the unit circle and the right triangle of \(30^\circ\), it is \(\frac{1}{2}\).
3. This part is a kind of summary of our lessons so far. For each bullet, write a short explanation or fill in the blanks. Make sure you understand all. This is a good review for you.

a. The movement of a point along the unit circle can be considered in two ways:
   - With arc lengths
   - …Rotations……

b. Arc lengths and corresponding central angles in …radian.. unit are equal for the unit circle because …the radius of the unit circle is 1………………

c. Using radians we have a function of rotation angles, and if we consider the relation of …rotation angles in radian.. and …..arc lengths …. in the unit circle, we can omit the angle unit and define the sine function on the domain of …real numbers…

d. Using x for arc lengths and α for the corresponding central angle the formula
   \[ s(x) = \sin \left( \frac{180^\circ x}{\pi} \right) \] implies to \( s(x) = \sin x \) because \( \frac{180^\circ}{\pi} = 1 \) radian.

e. Negative angles represent …a rotation in the clockwise-direction…………………………………………………………

f. We can find the sine values of angles on the …y….. axis.

g. Sine values in the …….third and fourth………… quadrant(s) are negative.

h. Consider the following two function:
   - \( f(x) = x + 5 \)
   - \( g(x) = \sin x \)

   Evaluate the functions for \( x = \pi \).
   \[ f(\pi) = \pi + 5 \approx 8.14 \]
   \[ g(\pi) = \sin \pi = 0 \]

   Please comment on the differences between these functions?

   We cannot compute sine function with algebraic operations unlike the function \( f \).

i. Find \( \sin 210^\circ \). Explain your work drawing a unit circle. How can you integrate your knowledge of triangle trigonometry to unit circle trigonometry for this question?

   Using the right triangle of \( 30^\circ \) in the third quadrant, \( \sin 210^\circ = -\frac{1}{2} \)

j. Consider three kinds of trigonometry that we have dealt for sine: right triangle trigonometry, unit circle trigonometry and graph of the sine function. Comment on the relationships among them. You can summarize your ideas in words, drawings, or both.

OPEN-ENDED
LESSON 4

A. Objectives:
At the end of the lesson, students will be able to
1. Find the formula of the function $c$ indicating the change in the horizontal position of a point moving along the unit circle with respect to the arc length it travels.
2. Explain cosine as a function of real numbers, its properties.
3. Calculate the cosine values of given angles or real numbers.

B. Prerequisite Knowledge and Skills:
Prior to the lesson, student should be able to
1. Find the domain and range of a function.
2. Use the relationship between angles in radians and arc lengths in the unit circle
3. Convert angle units to each other.
4. Use Pythagoras Theorem to find the side lengths.
5. Calculate the cosine value of certain angles using right triangles.

C. Materials:
1. Worksheets 8 and 9 and Homework Sheet 4
2. A computer and projector for the teacher
3. Whiteboard

D. Lesson-Flow
1. Introduction: 10 minutes
   • Collect Homework 3. To summarize the main points of the previous lessons, a classroom discussion will take place using the last part of the Homework 3. Teacher can show the bullets of this homework task in ppt slides which is attached in the folder “Teacher’s Materials”.

2. Lesson Body: 50 minutes
   • Recall the task in the third homework on drawing the graph of horizontal position of a moving point along the unit square versus the distance it travels. Ask students how they would expect if the number of the sides of the polygon increases.
   • After a short discussion, show students the graph on GeoGebra. Then hand out Worksheet 8 (one per pair) about the cosine function, and give students 15 minutes to complete the worksheet.
   • Discuss with the class the worksheet for 10 minutes. In the discussion, remember to emphasize the connections which are used to define cosine as a function on real numbers. In other words, the relationship that an arc length is equal to the measure of the corresponding angle in radians for the unit circle.
   • Collect Worksheet 8.
   • Hand out the Worksheet 9 and give students 15 minutes for their pair-work.
   • Discuss the questions with the class for 10 minutes. In the discussion, remember to pay attention to how the students use different representations of sine and cosine regarding the given relationship in the task, and at how students decide which graph belongs to the sine function and which to the cosine.
3. **Closure: 5 minutes.**
   - Ask students if they have questions.
   - Hand out Homework Sheet 4. Mention that it will be collected in the next lesson.
Worksheet 8

Given below is the graph c of the travelled distance against the horizontal position of a moving point moving along the unit circle in the counter-clockwise direction starting from (1, 0).

1. Write down the coordinates of the point A, B, C, D, E and G.

   Remark: The students are aimed to connect the information gathered from the unit circle and the given graph.

   \[
   A\left(-\frac{3\pi}{2}, 0\right), \quad B\left(\frac{\pi}{2}, 0\right), \quad C\left(\frac{\pi}{2}, 0\right), \quad D\left(\frac{3\pi}{2}, 0\right), \quad E\left(\frac{5\pi}{2}, 0\right), \quad G(2\pi, 1)
   \]

2. What does the part of the graph on the negative x-axis side mean?

   Rotation in the clockwise direction.

3. Consider the relationship between corresponding arc lengths and central angles for the unit circle. Given the claim that for every x (travelled distance) there is an \(\alpha\) such that \(c(x) = \cos(\alpha)\). Reproduce the work we have done for sine to find a formula for \(c(x)\).

   Remark: This is a kind of re-producing the work which was done for the sine function. Considering time limitations, the task was shortened to this version.

   By using the proportional relationship of

   \[
   \begin{align*}
   \frac{\text{arc}}{\pi} & \Rightarrow \frac{\text{angle}}{180^\circ} \\
   x & \Rightarrow \alpha
   \end{align*}
   \]

   \[
   \alpha = \frac{180^\circ x}{\pi}
   \]

4. If you use radians, what can you say about the formula you have found in part 3?

   Remark: Remember to emphasize clearly how to define cosine as a function of real numbers in the discussion phase.

   Using radians, the formula becomes \(c(x) = \cos(x \text{ rad})\), and using the relation that angles in radians are equivalent to the corresponding arc-lengths the function \(c\) is the same as cosine function.
5. How can you find the cosine value of a given angle using the unit circle?

Cosine value of a given angle can be found on the x-axis bounded by the unit circle.

6. For the cosine function, answer:

   a. What is the domain? \( \mathbb{R} \)

   b. What is the range? \([-1, 1]\)

   c. What is the periodicity? Why? \(2\pi\) because it repeats itself in every \(2\pi\) because of the circumference of the unit circle.

   d. \(\cos\left(-\frac{\pi}{2}\right) = 0\) (unit circle) \(\cos(30^\circ) = \frac{\sqrt{3}}{2}\) (right triangle)

      \(\cos(-3.14) = -1\) (approximately)
Worksheet 9

\[\sin x = \cos \left( \frac{x}{2} - x \right) \text{ for } 0 \leq x \leq \frac{x}{2}.\]

Remark: The tasks in this worksheet aims at helping students to see how different representations of sine and cosine concepts can be used, i.e. right angle trigonometry, unit circle trigonometry, and their graphs.

1. Show that the equality above holds by using a right triangle.

![Right Triangle Diagram]

Using the angles in degrees, the equality can be shown in a right triangle.

In the figure, \( \sin x = \frac{b}{c} \),

And obviously, \( \cos(90^\circ - x) = \frac{b}{c} \)

Hence the sine of an angle is equal to the cosine of its complementary angle.

2. Show that the equality holds by using the unit circle.

The given interval is the first quadrant.

![Unit Circle Diagram]

In the figure \( \sin x = G \), and using the triangle OAG, \( \cos \frac{\pi}{2} - x = G \).
3. Below is a part of the graphs of the sine and cosine functions. Show the equality using the graphs.

Hint: First decide which graph is sine and which is cosine, then use the given $x$ on the figure to show the equality.

Remark: Pay attention how students distinguish between the graphs of sine and cosine!

As seen in the figure, $\sin x = \cos \left( \frac{\pi}{2} - x \right) = A$
1. Fill in the table for the sine and cosine values of the given real numbers by using the unit circle. Pay attention to what these values corresponds to in the unit circle!

<table>
<thead>
<tr>
<th>x</th>
<th>sin(x)</th>
<th>cos(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>- $\pi$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$-\frac{3\pi}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>- $2\pi$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{5\pi}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$3\pi$</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
2. Sketch the graph of the sine function here by using the values from the table in part 1.

3. Sketch the graph of the cosine function by using the values from the table in part 2.
LESSON 5

A. Objectives:
At the end of the lesson, students will be able to
1. Explain how to solve the equations of the form \( \sin x = -1, 0, 1 \) or \( \cos x = -1, 0, 1 \).
2. Explain how to solve the equations of the form \( \sin A = \sin A \) and \( \cos A = \cos B \).
3. Solve such equations in given intervals.

B. Prerequisite Knowledge and Skills:
Prior to the lesson, student should be able to
1. Calculate sine and cosine values of given angles or real numbers using the unit circle.
2. State the periodicity of sine and cosine functions, and explain their meaning.

C. Materials:
1. Worksheets 10 and Homework Sheet 5
2. Whiteboard

D. Lesson-Flow
1. Introduction: 10 minutes
   • Collect Homework 4. As a review, illustrate how to draw the graphs of sine and cosine functions by using the multiples of \( \pi \) as inputs. And then show students the graphs on GeoGebra using integers on the x-axis by inserting \( y = \sin(x) \) to the input box. (Also attached in the folder “Teacher’s Materials). Ask students what the difference is. The aim here is to reinforce that \( \pi \) is a real number not an angle corresponding to 180°. Using \( \pi \) values the intersection points are clear.

2. Lesson Body: 40 minutes
   • Hand out Worksheet 10. Give students 20 minutes to complete it.
   • Discuss the tasks with the class for about 20 minutes. In the discussion, summarize how to solve the equations of the given forms. The aim is to help students figure out how they can solve them in a meaningful way using their knowledge of unit circle trigonometry, and also integrating it to their knowledge about the graphs, instead of memorizing the rules.
   • Collect the Worksheets.

3. Closure: 5 minutes.
   • Write on the whiteboard the equation \( \sin (2x - \frac{\pi}{3}) = \sin (x + \frac{\pi}{4}) \). Let them solve it on their own and then let one of them to solve it on the whiteboard.
   • Handout Homework 5.
Worksheet 10 – Some Trigonometric Equations

1. Given the equations $\sin x = 0$, $\sin x = 1$ & $\sin x = -1$.

Remark: Students need to elaborate further on the unit circle definitions of sine. After filling in the table, they may come up with the general solution using the idea of periodicity.

a. Fill in the table for any four values of $x$ satisfying the equations.

<table>
<thead>
<tr>
<th></th>
<th>$\sin x = 0$</th>
<th>$\sin x = 1$</th>
<th>$\sin x = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{3\pi}{2}$</td>
</tr>
<tr>
<td>$x$</td>
<td>$\pi$</td>
<td>$\frac{5\pi}{2}$</td>
<td>$\frac{7\pi}{2}$</td>
</tr>
<tr>
<td>$x$</td>
<td>$2\pi$</td>
<td>$\frac{9\pi}{2}$</td>
<td>$\frac{11\pi}{2}$</td>
</tr>
<tr>
<td>$x$</td>
<td>$3\pi$</td>
<td>$\frac{13\pi}{2}$</td>
<td>$\frac{15\pi}{2}$</td>
</tr>
</tbody>
</table>

b. How many solutions are there for each of these equations. How would you explain that there are more than one solution to the equations?

There are infinitely many solutions for such equations. This is because of the periodicity.

c. All solutions of $\sin x = 0$ can be expressed as $x = \ldots \ldots \pi \cdot k \ldots$ where $k \in \mathbb{Z} \ldots \ldots$

All solutions of $\sin x = 1$ can be expressed as $x = \ldots \frac{\pi}{2} + \pi \cdot k \ldots$ where $k \in \mathbb{Z}$

All solutions of $\sin x = -1$ can be expressed as $x = \ldots -\frac{\pi}{2} + \pi \cdot k \ldots$ where $k \in \mathbb{Z}$

d. For the equation $\sin x = 1$, how would you explain your answer using the graph of the sine function?

Remark: Students are expected to integrate their knowledge about the graph of the sine with this task.
Showing that the points differing by the multiples of $2\pi$ on the graph of sine have the same values.
2. Given the equation of the form $\sin A = \sin B$.

$A = B$ is the trivial solution of this equation, but it is not the only one. Discuss with your partner how to solve the equations of this form. Write all possible solutions considering the possible relationships between $A$ and $B$.

Using unit circle,
Possible solutions:
- $A = B$
- $A = B + k \cdot 2\pi$
- $A = \pi - B$
- $A = \pi - B + k \cdot 2\pi$

Hence the solution of the equation $\sin A = \sin B$ can be expressed as:

$$A = B + k \cdot 2\pi \quad \text{or} \quad A = \pi - B + k \cdot 2\pi$$
where $k$ is an integer.

3. Do the same as in part 2 for the equations of the form $\cos A = \cos B$

Using unit circle,
Possible solutions:
- $A = B$
- $A = B + 2k \cdot \pi$
- $A = -B$
- $A = -B + k \cdot 2\pi$

Hence the solution of the equation $\cos A = \cos B$ can be expressed as:

$$A = B + k \cdot 2\pi \quad \text{or} \quad A = -B + k \cdot 2\pi$$
where $k$ is an integer.
HOMEWORK – Lesson 5

A. Explain with a sentence or so why trigonometric equations have more than one solutions?
This is because they reveal a periodic behavior. Hence we include the terms of $2k\pi$ in the general solutions.

B. Solve the following trigonometric equation for the given intervals or give a general solution if no interval is mentioned.

1. $\sin(4x - \frac{\pi}{3}) = 1$
   
   $4x - \frac{\pi}{3} = \frac{\pi}{2} + k \cdot 2\pi$ where $k \in \mathbb{Z}$
   
   $4x = \frac{5\pi}{6} + k \cdot 2\pi$ where $k \in \mathbb{Z}$
   
   $x = \frac{5\pi}{24} + \frac{k\pi}{2}$ where $k \in \mathbb{Z}$

2. $\cos(3x - \pi) = 0$
   
   $3x - \pi = \frac{\pi}{2} + k \cdot \pi$ where $k \in \mathbb{Z}$
   
   $3x = \frac{3\pi}{2} + k \cdot \pi$ …… where $k \in \mathbb{Z}$ ……..
   
   $x = \frac{\pi}{2} + \frac{k\pi}{3}$ where $k \in \mathbb{Z}$ ……..

3. $\sin x \cdot \cos(\frac{x}{2} - \frac{\pi}{6}) = 0$
   
   $\sin x = 0$ or $\cos(\frac{x}{2} - \frac{\pi}{6}) = 0$
   
   $x = k \cdot \pi$ where $k \in \mathbb{Z}$ or $\frac{x}{2} - \frac{\pi}{6} = \frac{\pi}{2} + k \cdot \pi$ where $k \in \mathbb{Z}$
   
   $x = \frac{4\pi}{3} + k \cdot 2\pi$ where $k \in \mathbb{Z}$
   
   Hence $x = k \cdot \pi$ or $x = \frac{4\pi}{3} + k \cdot 2\pi$ where $k \in \mathbb{Z}$

4. $\sin(2x - \frac{\pi}{3}) = \sin (x + \frac{\pi}{4})$ in the interval $[0, 2\pi]$
   
   $2x - \frac{\pi}{3} = x + \frac{\pi}{4} + k \cdot 2\pi$ or $2x - \frac{\pi}{3} = \pi - (x + \frac{\pi}{4}) + k \cdot 2\pi$
   
   $x = \frac{7\pi}{12} + k \cdot 2\pi$ or $x = \frac{13\pi}{36} + k \cdot 2\pi$
   
   The equation has solutions in the given interval for $k = 0$
   
   Hence solutions are $x = \frac{7\pi}{12}$ or $x = \frac{13\pi}{36}$
5. \( \cos(x - \frac{\pi}{3}) = \cos(2x) \)

\[
x - \frac{\pi}{3} = 2x + k \cdot 2\pi \quad \text{or} \quad x - \frac{\pi}{3} = -(2x) + k \cdot 2\pi
\]

\[
x = \frac{\pi}{3} - k \cdot 2\pi \quad \text{or} \quad x = \frac{\pi}{9} + \frac{k \cdot 2\pi}{3}
\]
Appendix B: Worksheets Used for Data Collection

WORKSHEET 1a - Walking around the Unit Square

The unit square is the square whose center is placed at the origin of the Cartesian coordinate system with a side length of 2 units. An example of the unit square is shown on the coordinate system on the attached graph paper.

We consider a point P on the plane that moves along the unit square starting at (1, 0) in the counterclockwise direction.

Plot the vertical position of the moving point P against the distance travelled around the square. The x-axis represents the distance along the path that P has traveled and the y-axis represents the vertical position (y-coordinate) of P as it moves around the square.

Draw the graph on the attached graph paper!
Worksheet 1 b - Walking around the Unit Square

Let us name the graph you have drawn $s$.

Discuss the following questions with your partner and make notes.

1. Consider the points $A(2, 1)$, $B(6, -1)$ & $C(13, -1)$ on the graph of $s$. To which positions of $P$ on the unit square do they each correspond?

2.
   a. What is the distance along the path that $P$ has moved when it returns to the starting position for the first time?

   b. What is it for the second time and for the third time?

   c. To what points on the graph $s$ do they correspond? What do you observe regarding the coordinates of these points?

3. Write as many characteristics of the graph as you can think of.
Worksheet 2- Walking along the Unit Square

Open Applet 1.
The applet shows the graph s which you have just drawn. Dragging the point Q along the positive x-axis, observe the changes in the points P and S. Pay attention to how the motion of Q is related to the motion of P by considering what they represent.

Discuss the following with your partner and make notes:

1. What does the point S represent? What is its relation to P and Q?

2. Position the point Q at (8, 0). Move to the left while keeping an eye on P and S. Now suppose that you can move Q to the left of the y-axis. How would you extend the graph s? Sketch your idea below.

3. How would you interpret the point (-5, 1) on the extended part of s? What does the extended part mean?
4. Is the graph of a function?

5. If your answer to Question 4 is yes, why is it a function? What are its input and output?

6. If your answer to the Question 5 is no, why not?
Worksheet 3

Open applet 2.

In this applet you can see the graph $s$ for polygons with $n$ sides, not only for the unit square for the movement of $P$ in both clockwise and counter clockwise direction. Observe the changes in the graph as you modify $n$.

Discuss the followings with your partner and make notes.

1. For $n = 8$, what is the period of the graph $s$? What about for $n = 12$?

2. What happens to the polygon and the graph $s$ as the values of $n$ become larger?

3. CLAIM: For very large values of $n$, the period of the movement approaches to $2\pi$? Is this true or false? Verify your answer by using the graph and corresponding polygon.

4. For very large values of $n$, is the graph $s$ still representing a function? Why or why not?
Worksheet 5 – Rotation of a line segment by a given angle

In the last activity, we have realized that an arc length of \( \frac{\pi}{4} \) corresponds to a central angle of 45°.

This is illustrated in the diagram below.

Using the corresponding central angles, the movement of the point P along the unit circle can also be described as a rotation of |OP| around the origin.

1. Consider the point P has moved \( \frac{3\pi}{2} \) along the unit circle in the counter-clockwise direction from (1, 0).
   a. What is the rotation angle corresponding to that movement?
   b. What are the coordinates of P after this movement?
   c. To which point on the graph s (plotting travelled distance against the vertical position on the unit circle) does this position correspond? Mark it on the graph below.
   d. Open applet 3. Verify your answer to b and c. Consider once again what P & Q represent.
2. Find the following central angles corresponding to the given arc lengths.

\[
\begin{align*}
\pi & \rightarrow \\
\frac{\pi}{2} & \rightarrow \\
2\pi & \rightarrow \\
\frac{3\pi}{2} & \rightarrow
\end{align*}
\]

3. For the function \( s(x) = -1 \) what is the smallest positive value of \( x \)? What does \( x \) represent? In other words, what is the input of the function \( s \)?

4. **CLAIM:** The y-coordinate of \( S \) can be written as \( \sin (\alpha) \), for any position of \( P \). In other words for any \( x \), we can find an \( \alpha \) such that \( s(x) = \sin (\alpha) \). What could be meant by \( \alpha \)? Verify the claim for \( x = \frac{\pi}{4} \) using the graph \( s \).

Fill in the blanks considering the relationship.

\[
\begin{align*}
s(\pi) &= \sin (\ldots\ldots) = \\
s\left(\frac{\pi}{2}\right) &= \sin (\ldots\ldots) = \\
s\left(\frac{3\pi}{2}\right) &= \sin (\ldots\ldots) = \\
s(2\pi) &= \sin (\ldots\ldots) =
\end{align*}
\]
5. Give the formula of the function $s$.

(Hint: Use the relationship in 4 between arcs and angles.)

$$s(x) = \ldots$$
Angles can be measured in degrees and radians.

The measure of an angle in radians is defined as the ratio of the arc length subtended by that angle to the radius of the circle.

The diagram below illustrates how the measure in radian is defined.

1. What does this mean for the unit circle? What is the measure of the angle in radians subtending an arc of $\frac{\pi}{4}$? What can you say about the relationship between the arc length and the angle?

2. To what angle does the arc of $2\pi$ on the unit circle correspond?
   a. in radians $\Rightarrow$
   b. in degrees $\Rightarrow$

3. Explain how to convert radians to degrees by a formula or by words.
4. Back to our formula \( s(x) = \sin \left( \frac{180^\circ x}{\pi} \right) \).
   
   a. What is \( \frac{180^\circ}{\pi} \) in radians?

   b. Using radians what does the formula become?

   c. In the last version of our formula what can \( x \) denote?

5. Answer the following questions regarding some properties of our function \( s \):
   
   a. What is the domain of the sine function?

   b. What is the range of the sine function?

   c. What is its period?
Worksheet 7

1. What is the sine value of the angle $\alpha$? How can you connect this to the $\sin \alpha$ in a right triangle?
2. Try to relate \( \sin 120^\circ \) to sine as a ratio in a right triangle.
Worksheet 8

Part 1

Given below is the graph $c$ of the travelled distance against the horizontal position of a moving point moving along the unit circle starting from $(1, 0)$.

1. Write down the coordinates of the point A, B, C, D, E and G.
Part 2

2. For the cosine function, answer:

a. What is the domain?

b. What is the range?

c. What is the period? Why?

d. \( \cos \left( -\frac{\pi}{2} \right) = ? \)

\( \cos (-3.14) = ? \)

\( \cos (30^\circ) = ? \)
Worksheet 9

\[ \sin x = \cos \left( \frac{x}{2} - x \right) \text{ for } 0 \leq x \leq \frac{\pi}{2}. \]

1. Show that the equality above holds by using a right triangle.

2. Show that the equality holds by using the unit circle.
3. Below is a part of the graphs of the sine and cosine functions. Show the equality using the graphs.

Hint: First decide which graph is sine and which is cosine, then use the given x on the figure to show the equality.
Worksheet 10

1. Show that \( \cos x = \cos (\text{-}x) \).

2. Calculate \( \cos 210^\circ \) and \( \sin 210^\circ \).
Appendix C: Diagnostic Test

The purpose of this test is to find out how much you know before our lessons on trigonometry. The results will be kept confidential. It will not affect your mathematics mark. The use of this test is only for research purposes.

The test includes 7 questions. You have 30 minutes to complete the test. Write your answers clearly in the parts allocated for each question. Try to be as much explanatory as possible. No calculators are allowed. Your answers must be in English. If you think you have difficulties with some English words, you can ask your teacher.

Thank you for participating in the research.

Good Luck!

Özcan Demir
1. Given $\sin \alpha = 3/5$, compute $\cos \alpha$. Show your work on a drawing.

2. Explain in a sentence or so what a function is. If you can give a definition of a function, then please do so.
3. **a.** Write in the box YES if the given graph represents a function, write NO if it does not.

   a.
   
   b.

   c.

   d.

   e.

   b. Briefly explain your answer for the last graph (e)?
4. Given the function $f(x) = x^2 + 5$, answer the following.

   a. What is the domain (domein) of the function?

   b. What is the range (bereik) of the function?

   c. $f(4) =$ ?

   d. If $f(z) = 30$, find $z$. 
5.

Given is the graph of \( y = (x-2)^3 + 2 \). Find the y coordinate of the point A.
6. Given is the graph $s$.

a. Can $s$ represent a function? Why or why not?

b. $s(1) =$?

c. If $s(x) = 0$, what are the possible values of $x$?

d. If the point $(-1, y)$ is on the graph of $s$, $y =$?
7.

A circle with center $O$ and radius $r = 2$ is given in the figure. An ant is walking along the circle.

a. When the ant returns to its starting point $A$, what is the distance he has walked? Explain your work.

b. What is the distance it has walked from $A$ to $D$. Explain your work.

c. What is the distance when it has walked from $D$ to $C$. Explain your work.
Appendix D: Trigonometry Test

NAME:
SURNAME:

TEST

TRIGONOMETRIC FUNCTIONS

9 MARCH 2012

Time: 55 minutes

NO CALCULATORS !!!

GOOD LUCK

Özcan Demir
1. 
   a. Draw the graph of $y = \cos x$ on the figure below. Write down the coordinates of some special points you use on the graph.

   ![Graph of $y = \cos x$](image)

   b. Write down the characteristics that you can derive from the graph.

   c. Mark the point corresponding to $\cos (3)$ on the graph.
2. Which of the choices below does $x$ in the function $y = \sin x$ represent? Explain your choice.

   iv. An angle in degrees  
   v. An angle in radians  
   vi. A real number

3. 
   a. Why is $\text{sine}$ a function?  
   
   b. What is the period of the $\text{sine}$ function? What does periodicity actually mean?  
   
   c. What is the domain and the range of the $\text{sine}$ function?
4. For the function \( f(x) = \sin x \),
   
a. Calculate \( f(-\frac{\pi}{2}) \). Explain your answer.

b. What does \( \pi \) stand for in part a?

5. What is the value of \( \sin (225^\circ) \)? Explain (Show) how you got to your answer.
6. Are \( \cos 340^\circ \) and \( \cos 200^\circ \) equal, opposite, or neither? Use the unit circle below to show your answer.

7. Show that \( \sin(-x) = -\sin x \) on the unit circle or on the graph of sine function.
8.

a. What is the arc length corresponding to a central angle 60° in the unit circle?

b. How many degrees is the central angle that corresponds to an arc of length $\frac{\pi}{3}$?
9. A, B, C and D are points on the unit circle as shown in the figure below. They correspond to some rotations about the origin in the counter-clockwise direction starting from (1, 0).

a. Arrange in increasing order the sine values of the angles corresponding to these rotations: \( \sin A, \sin B, \sin C \) and \( \sin D \).

b. Mark the point that corresponds to B on the graph of the sine function below.
Appendix E: Interview Tasks

1. Draw the graph of \( y = \sin x \).
   
   a. How would you explain to me that this is the graph of sine?
   
   b. Why does it look like this? Why do you connect the points with such a curve but not with a line?
   
   c. What characteristics can you derive from the graph? What can you say about the graph? (function, domain, range, periodicity [definitions]).
   
   d. The following is a conversation between a mathematics teacher and a student. How would you respond if you were the teacher?

   Teacher: "The following is the graph of the sine function:"

   ![Graph of sine function]

   Student: "There is a mistake in this graph because the x values of the function are not angles, but numbers. We defined sine of rotation angles as the y coordinate of the terminal point of the rotation angle on the unit circle."

2. \( \sin \left( -\frac{\pi}{2} \right) = ? \) What does negative mean? Why? \( \sin \left( \frac{3\pi}{2} \right) = ? \) How can you explain that they are equal? How can you connect it to the periodicity?

3. \( \sin \left( 210^\circ \right) = ? \)
4. Which graph is of sine/cosine? How can you distinguish them?

![Graph of sine and cosine functions]

5. What do you think of our lessons? What things did you find hard/easy? Advice for improvement?