Pricing Constant Maturity Swap Derivatives

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by

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# Contents

1 Introduction 3

2 Pricing Framework 5
   2.1 No-Arbitrage Theory .............................. 5
   2.2 Change of numéraire Theorem ...................... 7
   2.3 Examples of numéraires ............................ 8
      2.3.1 Value of a money market account as numéraire 8
      2.3.2 Zero coupon price as the numéraire .......... 9
      2.3.3 Annuity factor as numéraire ................. 10

3 Constant Maturity Swaps Derivatives 12
   3.1 CMS Swaps under the Hull’s Model ................ 13
   3.2 CMS Swaps under the Linear Swap Rate Model (LSM) 17
   3.3 CMS Caps and Floors .............................. 19

4 Average CMS options 22
   4.1 Levy Log-normal Approximation .................... 23

5 Pricing CMS Derivatives with Monte Carlo Simulation 26
   5.1 Hull-White Model ................................. 26
      5.1.1 Bond and Option pricing ...................... 28
      5.1.2 Simulation of the short interest rate .......... 30
      5.1.3 Model Calibration .............................. 31

6 Example of CMS Derivatives 34
   6.1 Example 1. CMS Swap .............................. 35
      6.1.1 Pricing CMS swaps under Hull’s and LSM Model 35
      6.1.2 Pricing CMS swaps using Monte Carlo simulation 38
   6.2 Example 2. Asian CMS Cap ......................... 40
      6.2.1 Pricing Asian CMS cap under Levy Approximation 41
      6.2.2 Pricing Asian CMS cap using Monte Carlo simulation 43

A Moments under Black’s model 44

B Detailed derivation of Black’s formula to price options 46
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Chapter 1

Introduction

During the last decades the interest rate derivatives market has expanded enormously and the instruments traded there tend to be more complicated every day. This has implied a need for sophisticated models to price and hedge these instruments.

Some of the many products that are traded there are Constant Maturity Swap (CMS) derivatives. These derivatives are primarily structured as swaps. CMS swaps differ from a regular fixed-to-float or float-to-float swap, because the floating leg does not reset periodically to LIBOR or other short term rate but resets to a long term rate like 10-year swap rate.

When pricing these instruments an adjustment to the swap rate is needed. This adjustment comes from the fact that the swap price is not a linear function of the interest rate. It is in fact a decreasing convex function of the rate.

To price derivatives we assumed forward prices equal the expected value of future prices, i.e., if we denote by $V(Y_T)$ the function that gives the price of a swap based on the future rate $Y_T$ at time $T$, and by $Y_0$ the current forward swap rate, we have the next relationship

$$V(Y_0) = E[V(Y_T)].$$

Since $V(Y)$ is a convex function, Jensen’s inequality gives

$$E[V(Y_T)] \geq V(E[Y_T]),$$

using the fact that the swap price is a Decreasing function of the rate, we also have

$$Y_0 \leq E[Y_T].$$

This means the expected value of the future swap rate will not be equal to the forward swap rate, as it is assumed in almost all the cases when pricing derivatives. To price CMS instruments, we will need to make a good estimation of the future swap rate, in order to change the previous inequality into an equality we will need to add another term to the right hand side. This term, denoted by CA, is called Convexity Adjustment.

$$Y_0 \approx E[Y_T] + CA.$$ 

In a mathematical framework this Convexity Adjustment is due to a wrong pricing under a measure that is not the natural martingale measure. In chapter
2 we will provide the martingale framework that is used to price CMS derivatives. In Chapter 3 we will describe two models to calculate the Convexity Adjustment. The models relies on a number of approximation that are widely used by market participants.

Other type of instruments that are also considered in this thesis are Asian CMS options. In Chapter 4 an approximation to price these instruments is provided. The challenge with these options is that the distribution of the average swap rate is not known when we make the usual assumption that the underlying swap rate follows a geometric Brownian motion. With a log-normal approximation we can get a Black-Scholes style formula.

One of the oldest approaches to price interest rate derivatives is based on modelling the evolution of the instantaneous short interest rate. In literature, there are many models trying to simulate the stochastic movements of this rate. In Chapter 5, an implementation of the no-arbitrage Hull-White short rate model is presented. The calibration procedure to find the parameters of this model is also included. In this calibration, we minimize the difference between Cap prices obtained from the Hull-White model and actual prevailing market prices.

Finally, in Chapter 6 some results obtained from the implemented models are compared. Two specific contracts are described and analysed.
Chapter 2

Pricing Framework

The aim of this chapter is to provide mathematical tools that will be used in the implementation of the interest rate models to price Constant Maturity Swap (CMS) derivatives. The theoretical framework that a pricing model should deal with will be introduced. The no arbitrage condition and the change of numéraire technique will also be presented. This chapter is mainly based on Pelsser [11] and Hull [8].

2.1 No-Arbitrage Theory

Consider a continuous trading market within a compact time interval $[0,T]$. The uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual conditions.

Consider also a financial market that consists of a number of assets. We denoted the price of an asset by $S(t)$. Each price process is driven by a Brownian motion $W_t$, and follows a stochastic differential equation (SDE) of the form:

$$dS(t) = \mu_t dt + \sigma_t dW_t,$$

where the processes $\mu_t$ and $\sigma_t$ are assumed to be $\mathcal{F}_t$-adapted and satisfying

$$\int_0^T |\mu_t| \, dt < \infty$$

$$\int_0^T \sigma_t^2 \, dt < \infty.$$

**Definition 1.** A trading strategy is a $n$-dimensional predictable process, $\delta(t) = (\delta_1(t), \delta_2(t), \ldots, \delta_n(t))$, where $\delta_j(t)$ denotes the number of units of asset $j$ held at time $t$.

The predictability condition means, in general terms, that the value $\delta_j(t)$ is known immediately before time $t$.

The value $V_t(\delta)$ at time $t$ of a trading strategy $\delta$ is given by

$$V_t(\delta) = \sum_{i=1}^n \delta_i(t)S_i(t).$$

---

1A filtration $\mathcal{F}$ satisfies the usual conditions if $\mathcal{F}$ is right continuous and $\mathcal{F}_t$ contains for all $t$, all the measure zero sets of $(\Omega, \mathcal{F}, P)$.  

2.1. No-Arbitrage Theory

A trading strategy is self-financing if
\[ dV_i(\delta) = \sum_{i=1}^{n} \delta_i(t) \, dS_i(t), \]
or equivalently
\[ V_i(\delta) = V_0(\delta) + \sum_{i=1}^{n} \int_{0}^{t} \delta_i(s) \, dS_i(s). \]

In other words, the price of a self-financing portfolio does not allow for infusion or withdrawal of capital. All changes in its value are due just to changes in the asset prices.

A fundamental assumption in pricing derivatives is the absence of arbitrage opportunities. This is equivalent to the absence of zero cost investment strategies that allow to make a profit without taking some risk of a loss.

**Definition 2.** An arbitrage opportunity is a self financing trading strategy \( \delta(t) \) such that
\[ P[V_T(\delta) > 0] > 0 \]
\[ P[V_T(\delta) \geq 0] = 1 \]
and
\[ V_0(\delta) = 0. \]

Another important concept in pricing theory is a numéraire. Geman et al. [6] introduced the next definition.

**Definition 3.** A numéraire is any asset with price process \( B(t) \) that pays no dividend, i.e.,
\[ B(t) > 0 \quad \forall \, t \in [0, T] \quad \text{a.s.} \]

Examples of numéraires are money market account, a zero-dividend stock, or the \( m \)-maturity zero-coupon bond.

The role of a numéraire \( B(t) \) is to discount other asset price processes, and express them as relative prices,
\[ S_j^B(t) = \frac{S_j(t)}{B(t)}. \]

Notice that the relative price of an asset is its price expressed in the units of the numéraire.

A probability measure \( Q^* \) on \( (\Omega, \mathcal{F}) \) is called an equivalent martingale measure for the above financial market with numéraire \( B(t) \), \( t \in [0, T] \), if it has the following properties

- \( Q^* \) is equivalent to \( P \), i.e. both measures have the same null-sets.\(^2\)
- The relative price processes \( S_j^B(t) \) are martingales under the measure \( Q^* \) for all \( j \), i.e., for \( t \leq s \) we have
\[ \mathbb{E}^*[S_j^B(s) | \mathcal{F}_t] = S_j^B(t). \]

\(^2\)\( Q^* \) is equivalent to \( P \) if \( Q^* \ll P \) and \( P \ll Q^* \) where \( \ll \) denotes absolutely continuity. And \( Q^* \ll P \) if \( Q^*(A) = 0 \) whenever \( P(A) = 0 \).
A contingent claim is defined as a $\mathcal{F}_T$-measurable random variable $H(T)$ such that $0 < \mathbb{E}^*[|H(T)|] < \infty$. This random variable can be interpreted as the uncertain payoff of a derivative.

**Definition 4.** A contingent claim is attainable if there exists a self-financing trading strategy $\delta$ such that $V_T(\delta) = H(T)$. The self-financing trading strategy is then called a replicating strategy.

**Definition 5.** A financial market is complete if all the contingent claims are attainable.

Suppose we can construct a new measure $\mathbb{Q}$ (the so-called risk neutral measure) such that all numéraire-expressed asset prices become martingales. This is the Unique Equivalent Martingale Measure Theorem, and in one of its versions it is loosely stated as follows:

**Theorem 2.1.1.** (Equivalent Martingale Measure Theorem). An arbitrage free market is complete if and only if there exists a unique equivalent martingale measure $\mathbb{Q}$.

An important consequence of this theorem is the arbitrage pricing law. It affirms that given a numéraire, there is a unique probability measure $\mathbb{Q}$ such that the relative price processes are martingales. So, given the numéraire $B(t)$ the value of any derivative $V(t)$ at time $t < T$ is uniquely determined by the $\mathbb{Q}$-expected value of its future value expressed in terms of the numéraire, i.e

$$\frac{V(t)}{B(t)} = \mathbb{E}^Q\left[\frac{V(T)}{B(T)} | \mathcal{F}_t\right]. \quad (2.1)$$

This value must be independent of the choice of numéraire, together with the measure defined by this numéraire. One should be able to change from one numéraire to another one, and the Radon Nikodym Theorem is the key to make this change.

### 2.2 Change of numéraire Theorem

**Theorem 2.2.1.** (Radon-Nikodym). Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$. If $\mathbb{P} \ll \mathbb{Q}$, there exists a unique non-negative $\mathcal{F}$-measurable function $f$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ such that

$$\mathbb{P}(A) = \int_A f \, d\mathbb{Q} \quad \forall \, A \in \mathcal{F}.$$ 

The measurable function $f$ is called the Radon-Nikodym derivative or density of $\mathbb{P}$ w.r.t. $\mathbb{Q}$ and is denoted by $\frac{d\mathbb{P}}{d\mathbb{Q}}$.

This theorem implies that for every random variable $X$ for which $\mathbb{E}^Q[X \frac{d\mathbb{P}}{d\mathbb{Q}}] < \infty$ we have

$$\mathbb{E}^\mathbb{P}[X] = \mathbb{E}^Q[X \frac{d\mathbb{P}}{d\mathbb{Q}}].$$

More generally, when dealing with conditional expectation, we have

$$\mathbb{E}^\mathbb{P}[X | \mathcal{F}_t] = \mathbb{E}^Q[S \frac{d\mathbb{P}}{d\mathbb{Q}} | \mathcal{F}_t] = \frac{\mathbb{E}^Q[X \frac{d\mathbb{P}}{d\mathbb{Q}} | \mathcal{F}_t]}{\mathbb{E}^Q[S \frac{d\mathbb{P}}{d\mathbb{Q}} | \mathcal{F}_t]}.$$
2.3. Examples of numé raires

A result proved by Geman et al. [6] affirms that if we have two numé raires \( N \) and \( M \) with numé raires measures \( Q^N \) and \( Q^M \) respectively, equivalent to the initial \( P \) measure, such that the price of any traded asset \( V \) relative to \( N \) or \( M \) is a martingale under \( Q^N \) or \( Q^M \) respectively, then Equation (2.1) implies

\[
N(t)\mathbb{E}^{Q^N}\left[\frac{V(T)}{N(T)}|\mathcal{F}_t\right] = M(t)\mathbb{E}^{Q^M}\left[\frac{V(T)}{M(T)}|\mathcal{F}_t\right].
\]

This can be written as

\[
\mathbb{E}^{Q^N}\left[C(T)|\mathcal{F}_t\right] = \mathbb{E}^{Q^M}\left[C(T)\frac{N(T)/N(t)}{M(T)/M(t)}|\mathcal{F}_t\right],
\]

where \( C(T) = V(T)/N(T) \).

The Radon-Nikodym derivative defining the measure \( Q^N \) on \( \mathcal{F} \) is then given by

\[
\frac{dQ^N}{dQ^M} = \frac{N(T)/N(t)}{M(T)/M(t)}.
\]

2.3 Examples of numé raires

In the valuation of interest rate options we encounter different numé raires, that are associated to different measures. Here some examples.

2.3.1 Value of a money market account as numé raire

The money market account numé raire is simply a deposit of \( \mathcal{E}1 \) that earns the instantaneous risk free rate \( r \). This interest rate may be stochastic. If we set \( B(t) \) equal to the money market account value at time \( t \), its value is given by the differential equation

\[
 dB(t) = r_t B(t) \, dt
\]

i.e.

\[
 B(t) = \exp\left( \int_0^t r(s) \, ds \right).
\]

By defining \( D(t,T) \) as the stochastic discount factor between two time instants \( t \) and \( T \), \( D(t,T) \) is given by

\[
 D(t,T) = \frac{B(t)}{B(T)} = \exp\left( -\int_t^T r(s) \, ds \right).
\]

The Equivalent Martingale Measure Theorem was first proved using the money market account as the numé raire, so the main result from (2.1) was

\[
 V(t) = \mathbb{E}^{Q}[V(T)\, D(t,T)|\mathcal{F}_t].
\]

We define \( P(t,T) \) as the value at time \( t \) of a zero-coupon bond that pays off \( \mathcal{E}1 \) at time \( T \). Letting \( V(t) = P(t,T) \) and noting that \( P(T,T) = 1 \), we have

\[
 P(t,T) = \mathbb{E}^{Q}[D(t,T)|\mathcal{F}_t].
\]

This is an important relationship between the discount factor and the bond price when the interest rate is not deterministic.
2.3. Examples of numéraires

2.3.2 Zero coupon price as the numéraire

For the fixed time \(T\), the \(T\)-forward measure is the one associated with the numéraire \(P(t,T)\). This measure is useful when we want to price derivatives with the same maturity \(T\). The related expectation is denoted by \(E^T\).

From Equation (2.3), changing from measure \(Q\) to the \(T\)-forward measure, we have

\[
V(t) = E^Q[V(T) D(t,T) | \mathcal{F}_t]
= E^T[V(T) D(t,T) \frac{D(T,T) P(t,T)}{D(t,T) P(T,T)} | \mathcal{F}_t]
= P(t,T) E^T[V(T) | \mathcal{F}_t].
\] (2.4)

Using the \(T\)-forward measure we simplify things. The discount factor is now outside the expectation operator. So what has been done, rather than assuming deterministic discount factors, is a change of measure. We have factored out the stochastic discount factor and replaced it with the related bond price, but in order to do so we had to change the probability measure under which the expectation is taken.

If we set \(V(t) = P(t,S)\), the last equations imply that

\[
P(t,S) = P(t,T) E^T [P(T,S) | \mathcal{F}_t]
\] (2.5)

for \(0 \leq t \leq T \leq S\).

Forward rates are interest rates for a future period of time implied by rates prevailing in the market today. These rates are martingales under the Forward measure (hence its name). Define \(R(t,S,T)\) as the simply compounded forward rate as seen at time \(t\) for the time interval \([S,T]\), we have

\[
E^T[R(t,S,T)|\mathcal{F}_u] = E^T \left[ \frac{1}{(T-S)} \left( \frac{P(t,S)}{P(t,T)} - 1 \right) \right] | \mathcal{F}_u
= \frac{1}{(T-S)} E^T \left[ \frac{P(t,S) - P(t,T)}{P(t,T)} \right] | \mathcal{F}_u
\] (2.6)

where \(P(t,S) - P(t,T)\) is a portfolio of two zero coupon bonds divided by the numéraire \(P(t,T)\), so it is a martingale under the \(T\)-forward measure. Then Equation (2.6) can be written as

\[
E^T[R(t,S,T)|\mathcal{F}_u] = \frac{1}{(T-S)} \left( \frac{P(u,S) - P(u,T)}{P(u,T)} \right)
= R(u,S,T)
\] (2.7)

for \(0 \leq u \leq t \leq S \leq T\).

We can conclude that the forward interest rate equals the expected future interest rate when considering the \(T\)-forward measure as the risk neutral measure.
2.3. Examples of numéraires

2.3.3 Annuity factor as numéraire

A swap is a contract that exchanges interest rate payments on some predetermined notional principal\(^3\) between two differently indexed legs. Consider a forward starting swap with start date \(t_0\) and payment dates \(t_1, t_2, \ldots, t_n\).

\[\text{Figure 2.1: Swap cash flow.}\]

Figure 2.1 shows the cash flows of a vanilla swap, which has a set of payment dates \(t_1, t_2, \ldots, t_n\). The time interval \([t_0, t_n]\) is known as the swap tenor.

Assume that the predetermined principal is \(\mathcal{E}1\) and that the rate in the fixed leg is given by \(K\). This leg pays out an amount of

\[\delta_i K\]

at every payment date; \(\delta_i\) represents the year fraction from \(t_{i-1}\) and \(t_i\). The total value \(V_{fix}(t)\) of this leg at time \(t\), with \(t \leq t_0 \leq t_n\), is then given by

\[V_{fix}(t) = KA(t)\] (2.8)

where

\[A(t) = \sum_{i=1}^{n} \delta_i P(t, t_i).\] (2.9)

Notice that \(A(t)\) is a linear combination of bond prices that is always positive, so it can be taken as a numéraire and indeed it is called the Annuity factor numéraire denoted by \(A\). The measure associated with this numéraire is the swap measure and is essential for pricing Constant Maturity Swaps.

The floating leg of this swap pays out an amount corresponding to the interest rate \(R(t_i)\) observed at time \(t_i\). This rate is reset at dates \(t_0, t_1, \ldots, t_{n-1}\) and is paid at dates \(t_1, t_2, \ldots, t_n\). This way of payment is referred as the natural time lag. The value of this floating leg can be calculated as the expected value of the discounted payment under the martingale measure, in this case, the \(t_i\)-forward measure, then we have

\[V_{float}(t) = \sum_{i=1}^{n} P(t, t_i) \delta_i \mathbb{E}_{t_i}[R(t_{i-1}, t_{i-1}, t_i)]\]

\[= \sum_{i=1}^{n} P(t, t_i) \delta_i \frac{1}{\delta_i} \left( \frac{P(t, t_{i-1})}{P(t, t_i)} - 1 \right)\]

\[= \sum_{i=1}^{n} (P(t, t_{i-1}) - P(t, t_i))\]

\[= P(t, t_0) - P(t, t_n).\] (2.10)

---

\(^3\)Notional principal of a derivative contract is a hypothetical underlying quantity upon which interest rate or other payment obligations are computed.
2.3. Examples of numéraires

The forward swap rate $SR(t, t_0, t_n)$ is the rate that makes the value of this contract zero at inception. It is given by

$$SR(t, t_0, t_n) = \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} \delta_i P(t, t_i)},$$

(2.11)

Its corresponding spot swap rate determined in the future is

$$SR(t_0, t_0, t_n) = \frac{P(t_0, t_0) - P(t_0, t_n)}{\sum_{i=1}^{n} \delta_i P(t_0, t_i)} = \frac{1 - P(t_0, t_n)}{\sum_{i=1}^{n} \delta_i P(t_0, t_i)}.$$  

(2.12)

The quantity $P(t, t_0) - P(t, t_n)$ can be seen as a tradable asset that, expressed in $A(t)$ units, coincides with the forward swap rate. So according to the Equivalent Martingale Measure Theorem, the expected value of the future swap rate is the current swap rate, i.e.

$$SR(t, t_0, t_n) = \mathbb{E}^A[SR(t_0, t_0, t_n)|\mathcal{F}_t]$$

where $\mathbb{E}^A$ denotes the expectation under the swap measure.

In the next chapter we will simplify notation by replacing $SR(t, t_0, t_n)$ by $SR(t, t_0)$, just keeping in mind that $t_n$ is the maturity of the swap. If $t = 0$, then we simply write $SR(0, t_0)$. At the same time the spot swap rate will be replaced by $SR(t_0, t_0)$. 

11
A constant maturity swap (CMS) is a contract that exchanges a swap rate\(^1\) with a certain time to maturity against a fixed rate or floating rate, that can be for example a LIBOR rate, on a given notional principal. The amount of the notional, however, is never exchanged. In a standard swap, we exchange a floating short term rate, like a LIBOR rate, against a fixed rate. In a CMS swap, the floating rate is no longer a short term rate, but a swap rate with a certain time to maturity. For example, we pay every 6 months the 5-year swap rate and receive a fixed rate payment.

It is because of the mix of short-term resetting on long-term rates that the CMS is a useful instrument. It gives investors the ability to place bets on the shape of the yield curve over time and enables positions or views on the longer part of the curve but with a shorter life.

When pricing this special swap, we can follow the same approach as with a basis or fixed/floating swap described in section 2.3.3. The fair value will be the value of the fixed rate leg less the value of the floating leg (where one is paying and the other receiving). The only hard part is coming up with the rates on the floating side. In this part we will use an adjustment that can be attributed to two factors:

- Convexity adjustment. This adjustment is the difference between the expected swap rate and the forward swap rate. Since the relationship between the swap price and the swap rate is non-linear, it is not correct to say that the expected swap rate is equal to the forward swap rate.

- Timing adjustment. It is necessary to make this adjustment when an interest rate derivative is structured so that it does not incorporate the natural time lag implied by the interest rate. This the case of the CMS rate where the swap rate instead of being paid during the whole life of the swap is paid only once.

In a mathematical framework the previous adjustments are due to a valuation done under the "wrong" martingale measure. We will analyse this in the following section.

---

\(^1\)A swap rate is the fixed rate in an interest rate swap that makes the swap to have a value of zero.
It is market practice to price European interest rate derivative using Black’s model. This classical approach is a variation of the Black-Scholes option model adapted to options where the underlying is an interest rate rather than a stock.

Black’s model involves calculating the expected payoff under the \( \mathbb{Q} \) risk-neutral measure, that makes the process driftless, i.e., if \( SR(t, t) \) is the underlying interest rate observed at time \( t \), this rate follows a geometric Brownian motion given by equation

\[
dSR(t, t) = \sigma_t SR(t, t) dW_t
\]

where \( W_t \) is a standard Brownian motion under the \( \mathbb{Q} \) measure and \( \sigma_t \) is the volatility. We will work under this assumption to price CMS derivatives.

### 3.1 CMS Swaps under the Hull’s Model

First, consider the case of a CMS swap where the rate is observed and paid on the same day. Usually underlying rates are set in advance, i.e., they are set at the start of each payment period and paid at the end of that period.

To get the price of a CMS payer swap\(^2\) we need to price the two different legs, the leg based on the swap rate (CMS leg) and the leg that can depend on a floating (LIBOR, Euribor) rate or on a fixed rate. This later leg will be simply called fix leg. The value of the CMS payer swap at time 0 is given by

\[
V_{\text{swap}}(0) = V_{\text{fix}}(0) - V_{\text{CMS}}(0).
\]

Formula (2.8) or (2.10) in the previous chapter give the value of the fix leg when the rate is fix or floating respectively. So we will focus on the pricing of the CMS leg.

Let \( t_0, t_1, \ldots, t_{m-1} \) be the reset dates of the CMS leg and let \( \delta_i = t_i - t_{i-1} \) for \( i = 1, \ldots, m \). The observed CMS rate is based on a swap that may have different payment frequency. Let \( q \) be the underlying swap’s payment frequency and \( n \) the number of payments. Define \( s_j = j \left( \frac{12}{q} \right) \), for \( j = 0, 1, \ldots, n \) as the time of the \( i^{th} \) payment (in months), then at every reset date \( t_i, i = 0, \ldots, m-1 \) the underlying swap has payment dates \( t_i + s_1, \ldots, t_i + s_n \), where \( t_i + s_j \) is a date \( j \left( \frac{12}{q} \right) \) months after \( t_i \).

From Equation (2.12) in the previous chapter, the observed swap rate at time \( t_i \) is defined as

\[
SR(t_i, t_i) = \frac{P(t_i, t_i) - P(t_i, t_i + s_n)}{\sum_{j=1}^{n} \alpha_j P(t_i, t_i + s_j)} \quad \text{(3.3)}
\]

\[
= \frac{1 - P(t_i, t_i + s_n)}{\sum_{j=1}^{n} \alpha_j P(t_i, t_i + s_j)}, \quad \text{(3.4)}
\]

where \( \alpha_j \) is the time between \( t_i + s_j \) and \( t_i + s_{j-1} \).

---

\(^2\)A payer CMS swap gives the owner the right to enter into a swap where he/she pays the fixed leg and receives the floating (CMS) leg.
3.1. CMS Swaps under the Hull’s Model

Consider a single CMS swap leg payment for the period \([t_{i-1}, t_i]\). Since the payment is made at time \(t_{i-1}\), according to equation (2.4), to factor out the discount term, we need to use the \(t_{i-1}\)-forward measure, then we have

\[
V_{CMS}(0) = \mathbb{E}^{t_{i-1}}[SR(t_{i-1}, t_{i-1}) \delta_i D(0, t_{i-1})]
\]

\[
= P(0, t_{i-1}) \delta_i \mathbb{E}^{t_{i-1}}[SR(t_{i-1}, t_{i-1})].
\]  

(3.5)

However \(SR(t_{i-1}, t_{i-1})\) is not a martingale under the \(t_{i-1}\)-forward measure, so its expected value is not simply the value of the forward swap rate at time zero.

To get the last expected value we will use a bond that has the same life and frequency as the underlying swap, with coupons equal to the forward swap rate \(SR(0, t_{i-1})\) and face value equal to \(\mathbb{E}\). Remember that the price of a bond is given by the present value of the stream of cash flows, so we have that the price of this bond at time \(t_{i-1}\) is given by

\[
B(t_{i-1}) = SR(0, t_{i-1}) \sum_{j=1}^{n} \alpha_j P(t_{i-1}, t_{i-1} + s_j) + P(t_{i-1}, t_{i-1} + s_n).
\]

Let \(y_{i-1}\) be the yield to maturity (ytm) or par yield\(^3\) of the bond with price \(B(t_{i-1})\). Assume \(G\) is the function that gives the price of a bond given the ytm, i.e.

\[
G(t_{i-1}, y_{i-1}) = SR(0, t_{i-1}) \sum_{j=1}^{n} \alpha_j \frac{1}{(1 + \frac{y_{i-1}}{2})^{2((t_{i-1}+s_j) - t_{i-1})}} \\
+ \frac{1}{(1 + \frac{y_{i-1}}{2})^{2((t_{i-1}+s_n) - t_{i-1})}},
\]

considering that the underlying swap rate is paid in a semi-annual basis.

Expanding \(G(t_{i-1}, y_{i-1})\) in a second order Taylor series expansion with respect to \(y_{i-1}\) around \(SR(0, t_{i-1})\) we get

\[
G(t_{i-1}, y_{i-1}) \approx G(t_{i-1}, SR(0, t_{i-1})) + G'(t_{i-1}, SR(0, t_{i-1}))(y_{i-1} - SR(0, t_{i-1})) \\
+ \frac{1}{2} G''(t_{i-1}, SR(0, t_{i-1}))(y_{i-1} - SR(0, t_{i-1}))^2.
\]

The assumption is that \(y_{i-1}\) is close to the current forward swap rate. Notice that \(G(t_{i-1}, SR(0, t_{i-1})) = 1\) since a bond with coupons equal to the ytm trades at par, i.e. its price equals its face value. Taking conditional expectation under the \(t_{i-1}\)-forward measure, we obtain

\[
\mathbb{E}^{t_{i-1}}[G(t_{i-1}, y_{i-1})] \approx 1 + G'(t_{i-1}, SR(0, t_{i-1}))\mathbb{E}^{t_{i-1}}[y_{i-1} - SR(0, t_{i-1})] \\
+ \frac{1}{2} G''(t_{i-1}, SR(0, t_{i-1}))\mathbb{E}^{t_{i-1}}[(y_{i-1} - SR(0, t_{i-1}))^2].
\]

(3.6)

\(^3\)The par yield for a certain bond is the coupon rate that causes the bond price to equal its face value. It is merely a convenient way of expressing the price of a coupon-bearing bond in terms of a single interest rate.
3.1. CMS Swaps under the Hull’s Model

Using (2.5) we also get
\[ E^{t_i-1}[G(t_i-1, y_i-1)] = \mathbb{E}^{t_i-1}[B(t_i-1)] \]
\[ = SR(0, t_i-1) \frac{\sum_{j=1}^{n} \alpha_j P(0, t_i-1 + s_j)}{P(0, t_i-1)} + \frac{P(0, t_i-1 + s_n)}{P(0, t_i-1)} \]
\[ = \left( \frac{P(0, t_i-1) - P(0, t_i-1 + s_n)}{\sum_{j=1}^{n} \alpha_j P(0, t_i-1 + s_j)} \right) \sum_{j=1}^{n} \alpha_j P(0, t_i-1 + s_j) \frac{P(0, t_i-1)}{P(0, t_i-1)} + \frac{P(0, t_i-1 + s_n)}{P(0, t_i-1)} \]
\[ = 1. \]

From (3.6), we obtain
\[ E^{t_i-1}[(y_i-1 - SR(0, t_i-1))] \approx -\frac{1}{2} \frac{G''(t_i-1, SR(0, t_i-1))E^{t_i-1}[(y_i-1 - SR(0, t_i-1))^2]}{G'(t_i-1, SR(0, t_i-1))}. \]

It follows that
\[ E^{t_i-1}[y_i-1] \approx SR(0, t_i-1) - \frac{1}{2} \frac{G''(t_i-1, SR(0, t_i-1))E^{t_i-1}[(y_i-1 - SR(0, t_i-1))^2]}{G'(t_i-1, SR(0, t_i-1))}. \] (3.7)

To estimate \( E^{t_i-1}[(y_i-1 - SR(0, t_i-1))^2] \), this model uses the assumption that the change of rates between today and time \( t_i-1 \) is slight, i.e., that we can approximate \( SR(t_i-1, t_i-1) \) by \( y_i-1 \). Under this assumption we have
\[ E^{t_i-1}[(y_i-1 - SR(0, t_i-1))^2] \approx E^{t_i-1}[(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2] \] (3.8)

But the \( t_i-1 \)-forward measure is not the natural martingale measure for the swap rate, then to compute the last expectation we would need to make a change of measure. Using the Radon Nikodym Theorem we would get
\[ E^{t_i-1}[(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2] = \]
\[ \mathbb{E}^A [(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2 \left( \frac{P(t_i-1, t_i-1)/P(0, t_i-1)}{A(t_i-1)/A(0)} \right)] = \]
\[ \mathbb{E}^A [(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2] + \frac{A(0)}{P(0, t_i-1)} \text{Cov}^A [(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2, 1/A(t_i-1)]. \]

The value of the covariance term is unknown, and it is not always negligible. Then, to compute the expected value given by Equation (3.8), this model replaces the true \( Q^{t_i-1} \)-dynamics of the swap rate by its \( Q^A \)-dynamics. This approximation has been tested in [4], and it works well when the maturity of the swap is not too long. Using the fact that the swap rate follows Black’s model, Appendix A, Equation (A.5), gives the following approximation
\[ E^{t_i-1}[(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2] \approx \mathbb{E}^A [(SR(t_i-1, t_i-1) - SR(0, t_i-1))^2] \]
\[ = SR(0, t_i-1)^2 (e^{\sigma_{SR}^2 t_i-1} - 1) \]
\[ \approx SR(0, t_i-1)^2 \sigma_{SR}^2 t_i-1 \] (3.9)
where \( \sigma_{SR_{t_{i-1}}}^2 \) is the volatility of the forward swap rate. This volatility can be obtained from market swaption prices, since prices of European-style swaptions are also obtained under the Black’s model.

From (3.7) we can finally get
\[
\mathbb{E}^{t_i} \{ [SR(t_{i-1}, t_{i-1})] \} = SR(0, t_{i-1})
- \left( \frac{1}{2} G'(t_{i-1}, SR(0, t_{i-1})) \right) \left( SR(0, t_{i-1})^2 \sigma_{SR_{t_{i-1}}}^2 \right).
\]

Now, if we consider a CMS swap where the payments are not made at time \( t_{i-1} \) but at time \( t_i \), we would also need a “timing adjustment”, and instead of looking for \( \mathbb{E}^{t_i} \{ [SR(t_{i-1}, t_{i-1})] \} \) we would be interested in \( \mathbb{E}^{t_i} \{ [SR(t_{i-1}, t_{i-1})] \} \).

Using the Radon Nikodym Theorem we get
\[
\mathbb{E}^{t_i} \{ [SR(t_{i-1}, t_{i-1})] \} = \mathbb{E}^{t_i} \left[ SR(t_{i-1}, t_{i-1}) \left( \frac{P(t_{i-1}, t_{i-1})}{P(t_{i-1}, t_i)} \right) \right]
= \mathbb{E}^{t_i} \left[ \frac{1 + \delta R(t_{i-1}, t_{i-1}, t_i)}{1 + \delta R(0, t_{i-1}, t_i)} \right]
= \mathbb{E}^{t_i} \left[ [SR(t_{i-1}, t_{i-1})] \mathbb{E}^{t_i} \left[ \frac{1 + \delta R(t_{i-1}, t_{i-1}, t_i)}{1 + \delta R(0, t_{i-1}, t_i)} \right] \right]
+ \text{Cov}^{t_i} \left[ \frac{1 + \delta R(t_{i-1}, t_{i-1}, t_i)}{1 + \delta R(0, t_{i-1}, t_i)} \right]
= \mathbb{E}^{t_i} \left[ [SR(t_{i-1}, t_{i-1})]\right]
+ \frac{\delta_i}{1 + \delta R(0, t_{i-1}, t_i)} \text{Cov}^{t_i} \left[ [SR(t_{i-1}, t_{i-1}), R(t_{i-1}, t_{i-1}, t_i)] \right]
= \mathbb{E}^{t_i} \left[ [SR(t_{i-1}, t_{i-1})]\right]
+ \frac{\delta_i \rho_{SR,R} \text{Var}^{t_i}([SR(t_{i-1}, t_{i-1})]^{1/2} \text{Var}^{t_i}(R(t_{i-1}, t_{i-1}, t_i))^{1/2}}}{1 + \delta R(0, t_{i-1}, t_i)}
\]
(3.10)

where \( \rho_{SR,R} \) is the correlation between the CMS rate and the LIBOR rate.

To estimate the variance of \( R(t_{i-1}, t_{i-1}, t_i) \), notice that this variable is a martingale under the \( t_i \)-forward measure, Equation (2.7). Under the assumption that this rate follows Black’s model, using appendix A, we get
\[
\text{Var}^{t_i} (R(t_{i-1}, t_{i-1}, t_i)) = \mathbb{E}^{t_i} \left[ (R(t_{i-1}, t_{i-1}, t_i) - R(0, t_{i-1}, t_i))^2 \right]
= R(0, t_{i-1}, t_i)^2 (e^{\sigma_{SR_{t_{i-1}}}^2} - 1)
= R(0, t_{i-1}, t_i)^2 e^{\sigma_{SR_{t_{i-1}}}^2} - 1
\]

where \( \sigma_{SR_{t_{i-1}}}^2 \) is the volatility of the forward rate.

To compute \( \text{Var}^{t_i} (SR(t_{i-1}, t_{i-1})) \), from Equation (3.9) we have
\[
\text{Var}^{t_i} (SR(t_{i-1}, t_{i-1})) = \mathbb{E}^{t_i} \left[ (SR(t_{i-1}, t_{i-1}) - SR(0, t_{i-1}))^2 \right]
\approx SR(0, t_{i-1})^2 \sigma_{SR_{t_{i-1}}}^2 t_{i-1}
\]

16
where $\sigma^2_{R_{t_{i-1}}}$ is the volatility of the forward swap rate.

Substituting these in (3.10), we get

$$E^{t_{i-1}}[SR(t_{i-1}, t_{i-1})] = E^{t_{i}}[SR(t_{i-1}, t_{i-1})] + \frac{\delta_i \rho_{SR, R} R(0, t_{i-1}, t_i) \sigma_{SR_{t_{i-1}}} \sqrt{t_{i-1}} SR(0, t_{i-1}) \sigma_{SR_{t_{i-1}}} \sqrt{t_{i-1}}}{1 + \delta_i R(0, t_{i-1}, t_i)}.$$

We can finally get the expected value of the CMS rate

$$E^{t_{i}}[SR(t_{i-1}, t_{i-1})] = SR(0, t_{i-1}) - \frac{1}{2} G''(t_{i-1}, SR(0, t_{i-1})) \frac{SR(0, t_{i-1})^2 \sigma^2_{SR_{t_{i-1}}}}{1 + \delta_i R(0, t_{i-1}, t_i)} = SR(0, t_{i-1}) - CA - TA.$$  \tag{3.11}

We see that the expected value of this rate is equal to the forward swap rate plus two adjustments. The first term is the convexity adjustment and the second the timing adjustment.

As a result, the CMS rate depends on the following three components:

1. The yield curve via the swap rate and the annuity factor.
2. The volatility of the forward rate and the forward swap rate.
3. The correlation between the forward rate and the forward swap rate.

Once we have computed the CMS rate we can calculate the payments of this leg given by Equation (3.5). Adding all this payments we will get the value of the CMS swap leg. Finally, with the value of both legs, using Equation (3.2) we can get the price of the CMS swap.

Some problems of the previous convexity adjustment approximation are stressed in [13]. First, being just a second-order approximation, this model does not capture the "Skew" of the swap rate. The second problem is the assumption that we can replace the second moment under the $t_i$-forward measure by the second moment under the swap measure. For swap with large maturities this difference can be significant.

### 3.2 CMS Swaps under the Linear Swap Rate Model (LSM)

The Linear Swap Rate Model (LSM) is due to Hunt and Kennedy [9]. Let $SR(t, t)$ be the forward swap rate as described in section 2.3.3. From the previous section, we know that to calculate the payment at time $t_i$ of the floating leg of the CMS swap based in the $SR(t, t)$ rate, with $t < t_i$, we need to compute

$$P(0, t_i) \delta_i E^{t_i}[SR(t, t)]$$

where $\delta_i$ is the time between $t$ and $t_i$. 

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17
The main difficulty lies in computing the expected value of the swap rate $SR(t, t)$ under the $t_i$-forward measure, since it is not a martingale under this measure. Applying the Radon Nikodym Theorem to change measures we get

$$E^{t_i}[SR(t, t)] = E^{A} \left[ SR(t, t) \frac{P(t, t_i)}{P(0, t_i)} \frac{A(0)}{A(t)} \right]$$

$$= \frac{A(0)}{P(0, t_i)} E^{A} \left[ SR(t, t) \frac{P(t, t_i)}{A(t)} \right]$$

(3.12)

where $A(t) = \sum_{i=1}^{n} \alpha_i P(t, t_i)$ denotes the swap annuity factor.

To compute the right hand side of Equation (3.12), we will use the LSM. This model approximates the value of the numéraire ratio $\frac{P(t, t_i)}{A(0)}$ by a linear function of $SR(t, t)$, i.e.

$$\frac{P(t, t_i)}{A(t)} \approx B + C_{t_i} SR(t, t).$$

The values for $B$ and $C_{t_i}$ are defined consistently with the martingale property of $\frac{P(t, t_i)}{A(t)}$ under the swap measure.

$$\frac{P(0, t_i)}{A(0)} = E^{A} \left[ \frac{P(t, t_i)}{A(t)} \right]$$

$$\approx E^{A} \left[ B + C_{t_i} SR(t, t) \right]$$

$$= B + C_{t_i} SR(0, t)$$

and thus

$$C_{t_i} = \frac{\frac{P(0, t_i)}{A(0)} - B}{SR(0, t)}.$$ 

To determine $B$ notice that

$$1 = \sum_{i=1}^{n} \alpha_i \frac{P(t, t_i)}{A(t)}$$

$$= \sum_{i=1}^{n} \alpha_i B + \sum_{i=1}^{n} \alpha_i C_{t_i} SR(t, t)$$

$$= \sum_{i=1}^{n} \alpha_i B + \left( 1 - \sum_{i=1}^{n} \alpha_i B \right) \frac{SR(t, t)}{SR(0, t)}$$

which makes

$$B = \frac{1}{\sum_{i=1}^{n} \alpha_i}.$$ 

From (3.12), assuming that the swap rate follows Black’s model, we have
\[
\mathbb{E}^t_i[SR(t, t)] \approx \frac{1}{B + C_t, SR(0, t)} \mathbb{E}^h_i[SR(t, t) (B + C_t, SR(t, t))] \\
= \frac{1}{B + C_t, SR(0, t)} \mathbb{E}^h_i[B \cdot SR(t, t) + C_t, SR(t, t)^2] \\
= \frac{1}{B + C_t, SR(0, t)} (B \cdot SR(0, t) + C_t, \mathbb{E}^h_i[SR(t, t)^2]) \\
= \frac{1}{B + C_t, SR(0, t)} (B \cdot SR(0, t) + C \cdot SR(0, t)^2 e^{\sigma^2 t}) \\
= SR(0, t) \left( \frac{B + C_t, SR(0, t) e^{\sigma^2 t}}{B + C_t, SR(0, t)} \right) 
\]

where \( \sigma^2 \) is the volatility of the swap rate at time \( t \). Since we are assuming that the swap rate is log-normally distributed under the swap measure, this volatility can be implied from at-the-money vanilla swaptions market quotes that are calculated under the same assumption.

Notice that for large maturities, the swap rate volatility is higher. It means the convexity adjustment has a bigger impact on the forward swap rate. It is stated in [11] that for large maturities the term structure almost moves in parallel, and this movement is well described by the swap rate. So the the ratio \( P_i(t, t) \mathcal{A}(t) \) is closely approximated by a linear function of the swap rate.

### 3.3 CMS Caps and Floors

Some other products that are commonly traded in the market are options based on a CMS rate, i.e., options where the underlying is a swap rate. The key reason for the success of these products is that the swap rate is a long term rate index rather than a typical LIBOR 3-month or 6-month rate for other caps and floors, so it can be used to hedge instruments with long maturities.

A CMS cap is a derivative in which the buyer receives payments at the end of each period \( i \) when the swap rate \( SR(t_{i-1}, t_{i-1}) \) exceeds the agreed strike rate \( K \). These payments are based on a certain amount of money called notional. A CMS cap can be seen as a series of caplets that are single call options on a reference swap rate. The price of the cap will be sum of the present value of the value of these caplets. A CMS payer cap will for instance be a cap on the 10-year swap rate with a strike at 5%. This is an agreement to receive a payment for each period the 10 year swap rate exceeds 5% fixed rate.

A CMS floor is defined in a similar way. It is a series of European put options or floorlets on a specified swap rate. The buyer of the floor receives money, if on the maturity \( t_i \) of any of the floorlets the reference rate \( SR(t_{i-1}, t_{i-1}) \) is below the agreed strike price \( K \).

CMS based products have been widely used by insurance companies to solve their solvency problem (protect themselves against the rise of long dated interest rates). CMS options (caps and floors) provide suitable hedge requirements for insurance products like guaranteed investment contracts (GIC) and other negotiated term deposits. GIC are contracts that guarantee repayment of principal and a fixed or floating interest rate for a predetermined period of time.
3.3. CMS Caps and Floors

Guaranteed investment contracts are typically issued by life insurance companies and often bought for retirement plans.

CMS based products provide a suitable hedge to liabilities arising from GIC. A CMS Floor for instance, provides a hedge to GIC’s when rates are falling and the insurance company has to make guaranteed fixed interest payments. Similarly, CMS caps provide hedge in a rising environment.

The payoff of each CMS caplet/floorlet at time $t_i$ is given by

- $L \delta_i \max [SR(t_{i-1}, t_{i-1}) - K, 0]$
- $L \delta_i \max [K - SR(t_{i-1}, t_{i-1}), 0]$

where $L$ is the principal, $\delta_i$ is the time between time $t_i$ and $t_{i-1}$.

Practitioners focus on the computation of the forward CMS rate as they use this adjusted rate to price simple options on CMS (cap, floor, swaption). Once that the CMS rate has been adjusted they use it in the Black’s model to price options.

To compute the value at time zero of a caplet, according to the option pricing theory, we need to compute the expected value of the discounted payoff under the martingale measure. Since the payoff is made at time $t_i$ we will use the $t_i$-forward measure. Then we have

$$L \delta_i P(0, t_i) \mathbb{E}^{t_i} \left[ \max [SR(t_{i-1}, t_{i-1}) - K, 0] \right].$$

From the log-normal assumption, and using appendix B, we have that the value of the expected payoff at time 0 is given by

$$L \delta_i P(0, t_i) \left[ \mathbb{E}^{t_i} [SR(t_{i-1}, t_{i-1})] N(d_1) - K N(d_2) \right].$$

where $N(x)$ denotes the standard normal cumulative distribution function.

We know the value of $\mathbb{E}^{t_i} [SR(t_{i-1}, t_{i-1})]$, this is the adjusted swap rate from previous chapters, Equation (3.11) and Equation (3.13). This adjusted rate will be denoted by $SR(0, t_{i-1})'$.

The values for $d_1$ and $d_2$ are given by

$$d_1 = \frac{\ln \left( \frac{SR(0, t_{i-1})'}{K} \right) + \frac{\sigma^2 t_{i-1}}{2}}{\sigma \sqrt{t_{i-1}}}$$

$$d_2 = d_1 - \sigma \sqrt{t_{i-1}}.$$

Notice that the volatility $\sigma$ is multiplied by $\sqrt{t_{i-1}}$ because the swap rate $SR(t_{i-1}, t_{i-1})$ is observed at time $t_{i-1}$. The discount factor $P(0, t_i)$ reflects the fact that the payoff is at time $t_i$.

The value of the corresponding floorlet is given by

$$L \delta_i P(0, t_i) \left[ K N(-d_2) - \mathbb{E}^{t_i} [SR(t_{i-1}, t_{i-1})] N(-d_1) \right].$$

It has been emphasized in [7] that the main problem with the Black’s model is that it does not consider the smile or skew observed in the market volatility. And some recommendations to diminish this problem are made there. For a
CMS swap the volatility of at-the-money swaptions should be used. Out-of-the-money caplets and floorlets can be priced using the volatility for the strike $K$, and finally for in-the-money caplets or floorlets the call-put parity should be used, i.e. the price of an in-the-money caplet will be the price of a CMS swap plus the price of an out-of-the-money floorlet.
Chapter 4

Average CMS options

Average (Asian) options are path-dependant securities which payoff depends on the average of the underlying asset price over certain time interval. Stock prices, stock indices, commodities, exchange rates, and interest rates are example of underlying assets. Since no general analytical solution for the price of the Asian option is known, a variety of techniques have been developed to analyse the price of these financial instruments.

Asian options have a lower volatility, that makes them cheaper relative to their European counterparts. They were introduced partly to avoid a common problem for European options, where the speculators could drive up the gains from the option by manipulating the price of the underlying asset near to the maturity. The name ”Asian option” probably originates from the Tokyo office of bankers Trust, where it was first offered.

In this chapter we present a method to approximate the value of average CMS options (caps and floors). As explained in section 3.3, a cap or floor can be seen as a series of caplets or floorlets respectively, that are single options on a reference rate. For the case of an Asian CMS cap, the price of every single caplet will be calculated, to later add them up and have the price of the cap.

Let $SR(t_j, t_j)$ be the swap rate observed at time $t_j$. We suppose that the average is determined over the time interval $[t_1, t_N]$ and at points on this interval $t_j = t_1 + jh$ for $j = 0, 2, \ldots, N - 1$ where $h = (t_N - t_1)/N$.

Let $SR(t_j, t_j)$ be the swap rate observed at time $t_j$. We suppose that the average is determined over the time interval $[t_1, t_N]$ and at points on this interval $t_j = t_1 + jh$ for $j = 0, 2, \ldots, N - 1$ where $h = (t_N - t_1)/N$.

The running average $H(t_N)$ is defined as

$$H(t_N) = \frac{1}{N} \sum_{j=1}^{N} S(t_j, t_j).$$
4.1 Levy Log-normal Approximation

The average CMS option is characterized by the payoff function at time $t_N$ given by

$$L \delta t_N \max [H(t_N) - K, 0]$$

(4.1)

for a caplet option, and

$$L \delta t_N \max [K - H(t_N), 0]$$

(4.2)

for a floorlet option. In both cases $L$ is the notional, $\delta t_N$ is the time between each caplet/floorlet payment, and $K$ is the strike price of the option.

4.1 Levy Log-normal Approximation

To price the average CMS option we will need to compute the distribution of the arithmetic average. It is well known that no analytical solution exists for the price of European calls or puts written on the arithmetic average when the underlying index follows a log-normal process described in Section 3.1, equation (3.1).

Hence, the pricing of such options becomes an interesting field in finance. We will use an approximation by Levy [10], that consists in approximate the average $H(t_N)$ with a log-normal distribution. This approach relies on the fact that all the moments of the arithmetic average can be easily calculated even though the distribution of the average is unknown.

Based on the risk neutral pricing method, the value $C$ at time 0 of a caplet with payoff given by equation (4.1) is

$$C = P(0,t_N) \delta t_N \mathbb{E}^{t_N} \left[ \max [H(t_N) - K, 0] \right]$$

where $\mathbb{E}^{t_N}$ denotes the expectation under the $t_N$-forward measure.

Under the log-normal distribution assumption, the value of the last expectation will be a Black-Scholes style formula. Denoting by $f(x)$ the log-normal probability density function, we have

$$\mathbb{E}^{t_N} \left[ \max [H(t_N) - K, 0] \right] = \int_K^\infty x f(x) \, dx - K \int_K^\infty f(x) \, dx.$$

To solve the first integral

$$\int_K^\infty x f(x) \, dx = \int_K^\infty x \frac{1}{\lambda x \sqrt{2\pi}} \exp \left( - \frac{(\ln x - \nu)^2}{2\lambda^2} \right) \, dx$$

we will use a change of variable

$$y = \frac{\nu + \lambda^2 - \ln x}{\lambda}, \quad dy = -\frac{1}{\lambda x} \, dx.$$

It follows

$$\int_K^\infty x f(x) \, dx = \int_{y(K)}^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}(y - \lambda)^2 \right) dy$$

$$= \exp(\nu + \frac{1}{2} \lambda^2) \int_{y(K)}^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}y^2 \right) dy$$

$$= \exp(\nu + \frac{1}{2} \lambda^2) N \left( \frac{\nu + \lambda^2 - \ln K}{\lambda} \right)$$

(4.3)
where $N(x)$ denotes the standard normal cumulative distribution.

To compute the second integral we go along the same lines, using the change of variable $z = \frac{\nu - \ln x}{\lambda} \quad dz = -\frac{1}{\lambda x} \, dx$

we get

$$\int_{K}^{\infty} x f(x) \, dx = N\left(\frac{\nu - \ln K}{\lambda}\right).$$

(4.4)

Using (4.3) and (4.4) the price of a CMS caplet option is

$$C = P(0, t_N) \delta_{t_N} \left( \exp(\nu + \frac{1}{2} \lambda^2) N(d_1) - K N(d_2) \right)$$

(4.5)

where

$$d_1 = \frac{\nu + \lambda^2 - \ln K}{\lambda}$$

(4.6)

$$d_2 = d_1 - \lambda.$$  

(4.7)

The parameters $\nu$ and $\lambda$ can be estimated from the following expression for a log-normal distribution

$$\mathbb{E}[H(t_N)^t] = e^{\sigma^2 + \frac{1}{2} \sigma^2 \lambda^2}.$$  

(4.8)

Estimating the first and second moments, we get

$$\nu = 2 \ln \mathbb{E}[H(t_N)] - \frac{1}{2} \ln \mathbb{E}[H(t_N)^2]$$

$$\lambda^2 = \ln \mathbb{E}[H(t_N)^2] - 2 \ln \mathbb{E}[H(t_N)].$$

We still have to compute the moments of the average variable $H(t_N)$ under the $t_N$ forward measure. The first moment is given by

$$\mathbb{E}^{t_N}[H(t_N)] = \mathbb{E}^{t_N}\left[\frac{1}{N} \sum_{j=1}^{N} S(t_j, t_j)\right]$$

$$= \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}^{t_N}[S(t_j, t_j)]$$

$$= \frac{1}{N} \sum_{j=1}^{N} S(0, t_j)'$$

where $S(0, t_j)'$ denotes the convexity adjusted rate that we have extensively worked out in Chapter 3, equations (3.11) and (3.13). We need this convexity adjusted rate because for the valuation of the caplet we work under the $t_N$-forward measure rather than the martingale measure for the swap rate. Another difference in these payments is that they are not made one period after the rate is observed, but at the end of the average period. So for each swap rate observed at time $t_i$ and considered in the average payment made at time $t_N$, the timing adjustment will be made for these associated two forward measures.
4.1. Levy Log-normal Approximation

To compute the second moment of the variable $H(t_N)$, we assume that all
the swap rates $SR(t_i, t_j)$ have the same constant volatility $\sigma$, that can be im-
plied from swaption prices. We will use the same approximation of the second
moment under the $t_N$-forward measure by the second moment under the swap
measure $A$ that was used in Section 3.1.

Using Equations (A.4) and (A.6) from appendix A, we get

$$E^{t_N}[H(t_N)^2] = E^{t_N}\left(\frac{1}{N} \sum_{j=1}^{N} S(t_j, t_j)\right)^2$$

$$= \frac{1}{N^2} \left( \sum_{j=1}^{N} E^{t_N}[S(t_j, t_j)]^2 + \sum_{i \neq j} E^{t_N}[S(t_i, t_i)S(t_j, t_j)] \right)$$

$$\approx \frac{1}{N^2} \left( \sum_{j=1}^{N} E^{A}[S(t_j, t_j)]^2 + \sum_{i \neq j} E^{A}[S(t_i, t_i)S(t_j, t_j)] \right)$$

$$= \frac{1}{N^2} \left( \sum_{j=1}^{N} S(0, t_j)^2 e^{\sigma^2 t_j} + \sum_{i \neq j} S(0, t_i)S(0, t_j)e^{\sigma^2 \min\{t_i, t_j\}} \right).$$

Finally, from Equations (4.5) and (4.8) the price of a caplet is given by

$$C = P(0, t_N) \delta_{t_N} \left( E[H(t_N)]N(d_1) - KN(d_2) \right).$$

where

$$d_1 = \frac{\nu + \lambda^2 - \ln K}{\lambda}$$

$$d_2 = d_1 - \lambda.$$

The price of an average CMS floorlet can be computed similarly

$$P = P(0, t_N) \delta_{t_N} \left( KN(-d_2) - E[H(t_N)]N(-d_1) \right)$$

where $d_1$ and $d_2$ are given by equations (4.6) and (4.7) respectively.

This is the value of a single caplet or floorlet option, for the total CMS
option value it is necessary to calculate every single payment during the life of
the option and add them up.
Chapter 5

Pricing CMS Derivatives with Monte Carlo Simulation

Derivatives valuation involves the computation of expected values of complex functional of random paths. The Monte Carlo method can be applied to compute approximated values for these quantities. This method plays an important role in the valuation of some interest rate derivatives, that usually require the simulation of the entire yield curve. It simulates the component that includes uncertainty. We will use a Monte Carlo approximation to implement a short rate model.

The short rate, $r$, is the rate that applies to an infinitesimally short period of time. Derivatives prices depend on the process followed by $r$ in a risk neutral world. The most common short rate models are single-factor Markovian models, where there is only one source of uncertainty and the evolution of the short rate does not depend on previous interest rate movements. Some examples of these models are the Vasicek, Ho-Lee or Hull-White model. These models are widely used since they are easily implemented and under them some derivatives have a price expressed in a closed formula.

These models are described by a Stochastic Differential Equation (SDE), that under the risk neutral measure $\mathbb{Q}$ has the form

$$
    dr_t = \mu(r_t)dt + \sigma(r_t)dW_t
$$

(5.1)

where $\mu$ is the drift function, $\sigma$ is the diffusion function and $W$ is a $\mathbb{Q}$-Brownian motion. The short rate models differ from each other in how the functions $\mu$ and $\sigma$ are defined.

5.1 Hull-White Model

The Hull-White model (1994) is one of the no-arbitrage models that is designed to be consistent with the actual term structure of interest rate. In this model, also called extended Vasicek model, the function $\mu(r)$ is defined as

$$
    \mu(r) = \theta - ar
$$
5.1. Hull-White Model

where $a$ is a constant that represents the mean reversion parameter and $\theta$ is a function determined uniquely by the term structure.

The SDE that defines the Hull-White model is then given by

$$dr_t = [\theta_t - ar_t]dt + \sigma dW_t$$ (5.2)

the constants $a$ and $\sigma$ will be determined by a calibration process. In practice, the Hull-White model is calibrated by choosing the mean reversion rate and volatility in such a way so that they are consistent with option prices observed in the market.

The model implies that the short-term rate is normally distributed and subject to mean reversion. The mean reversion parameter $a$ ensures consistency with the empirical observation that long rates are less volatile than short rates.

**Lemma 5.1.1.** The process $r_t$ is normally distributed with mean

$$E[r_t] = e^{-at}r_0 + \int_0^t e^{-a(t-u)}\theta_u du$$ (5.3)

and variance

$$Var[r_t] = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$ (5.4)

**Proof.** Let $X_t = e^{at}r_t$, by Itô’s lemma we get

$$dX_t = \frac{\partial X}{\partial r_t}dr_t + \frac{\partial X}{\partial t}dt + \frac{1}{2} \frac{\partial^2 X}{\partial r_t^2} [dX_t]^2$$

$$= e^{at}dr_t + e^{at}ar_t dt.$$

Substituting the dynamics of $dr_t$, Equation (5.2), and expressing $dX_t$ in integral form, we get

$$X_t = X_0 + \int_0^t e^{au}(\theta_u - ar_u)du + \int_0^t e^{au}\sigma dW_u + \int_0^t e^{au}ar_u du$$

$$= X_0 + \int_0^t e^{au}\theta_u du + \int_0^t e^{au}\sigma dW_u.$$

Then

$$r_t = e^{-at}X_t$$

$$= e^{-at}r_0 + \int_0^t e^{-a(t-u)}\theta_u du + \int_0^t e^{-a(t-u)}\sigma dW_u.$$ (5.5)

We can now compute the moments of this random variable. First, notice that

$$E[\int_0^t e^{-a(t-u)}\sigma dW_u] = 0$$

then we have

$$E[r_t] = e^{-at}r_0 + \int_0^t e^{-a(t-u)}\theta_u du$$
5.1. Hull-White Model

and the variance is given by

\[
Var[r_t] = \mathbb{E} [r_t - \mathbb{E}[r_t]]^2 = \mathbb{E} \left[ \int_0^t e^{-a(t-u)} \sigma dW_u \right]^2 = \mathbb{E} \left[ \sigma^2 \int_0^t e^{-2a(t-u)} du \right] = \frac{\sigma^2}{2a} (1 - e^{-2at}).
\]

5.1.1 Bond and Option pricing

We denote by \( P(t, T) \) the price at time \( t \) of a zero coupon bond maturing at time \( T \). It’s value is given by

\[
P(t, T) = \mathbb{E}_Q \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right].
\]

where \( Q \) denotes the risk neutral measure. In order to compute this expectation we need to know the distribution of \( \int_t^T r_s ds \). Define

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}.
\]

From Equation (5.5), integrating \( r_s \) from \( t \) to \( T \), and interchanging integrals we get

\[
\int_t^T r_s ds = \int_t^T e^{-as} r_0 ds + \int_t^T \int_s^T e^{-a(s-u)} \theta_u du ds + \int_t^T \int_0^s e^{-a(s-u)} \sigma dW_u ds
\]

\[
= \int_t^T e^{-as} r_0 ds + \int_0^t \int_t^T e^{-a(s-u)} \theta_u ds du + \int_t^T \int_0^T e^{-a(s-u)} \theta_u ds du + \int_0^T \int_t^T e^{-a(s-u)} \sigma ds dW_u + \int_t^T \int_u^T e^{-a(s-u)} \sigma ds dW_u.
\]

Using (5.6) we rewrite this as

\[
\int_t^T r_s ds = r_t B(t, T) + e^{-at} B(t, T) \int_0^t e^{au} \theta_u du + \int_0^T B(u, T) \theta_u du
\]

\[
+ e^{-at} B(t, T) \int_0^t e^{au} dW_u + \sigma \int_t^T B(u, T) dW_u
\]

\[
= r_t B(t, T) + \int_t^T B(u, T) \theta_u du + \sigma \int_t^T B(u, T) dW_u.
\]

It follows that \( \int_t^T r_s ds \) is conditionally normally distributed with

\[
\mathbb{E}[\int_t^T r_s ds \mid \mathcal{F}_t] = r_t B(t, T) + \int_t^T B(u, T) \theta_u du
\]

and variance

\[
Var[\int_t^T r_s ds \mid \mathcal{F}_t] = \sigma^2 \int_t^T B(u, T)^2 du.
\]
5.1. Hull-White Model

Thus, the bond price can be calculated as the expected value of a log-normal distributed variable.

\[ P(t, T) = \mathbb{E}^Q[e^{-\int_t^T r_s ds} | F_t] \]
\[ = e^{-\int_t^T (A(t, T) - B(t, T)r_s + \frac{1}{2} \sigma^2 \int_t^T B(u, T)^2 du) du} \]
\[ = e^{A(t, T) - B(t, T)r_t} \]

where

\[ A(t, T) = \int_t^T \left( \frac{1}{2} \sigma^2 B(u, T)^2 - B(u, T)\theta_u \right) du. \] (5.7)

We have not given an explicit formula for the function \( \theta_T \), but we mentioned that it can be calculated from the initial term structure of the instantaneous forward rate. Denoting the theoretical instantaneous forward rate observed at time zero with maturity \( T \) by \( f(0, T) \), we have that its value is given by

\[ f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T} = -\frac{\partial (A(0, T) - B(0, T)r_0)}{\partial T}. \]

From Equation (5.7) we have

\[ \frac{\partial A(0, T)}{\partial T} = \frac{1}{2} \sigma^2 B(T, T)^2 + \sigma^2 \int_0^T B(u, T) \frac{\partial B(u, T)}{\partial T} du \]
\[ - B(T, T)\theta_T - \int_0^T \frac{\partial B(u, T)}{\partial T} \theta_u du \]

and from Equation (5.6) we get

\[ \frac{\partial B(t, T)}{\partial T} = e^{-a(T-t)}. \]

Putting this together we get

\[ f(0, T) = -\frac{\partial A(0, T)}{\partial T} + \frac{\partial B(0, T)}{\partial T}r_0 \]
\[ = \int_0^T e^{-a(T-u)}\theta_u du + \frac{\sigma^2}{a} \left[ \frac{1}{2} \left( 1 - e^{-2aT} \right) - \left( 1 - e^{-aT} \right) \right] + e^{-aT}r_0. \] (5.8)

Taking the derivative with respect to \( T \) to get the value of \( \theta_T \), we have

\[ \frac{\partial f(0, T)}{\partial T} = \theta_T - a \int_t^T e^{-a(T-u)}\theta_u du + \frac{\sigma^2}{a} \left( e^{-2aT} - e^{-aT} \right) - a e^{-aT}r_0. \]

Using Equation (5.8) last equation can be written as

\[ \frac{\partial f(0, T)}{\partial T} = \theta_T - af(0, T) + \frac{\sigma^2}{a} \left[ \frac{1}{2} \left( 1 - e^{-2aT} \right) - \left( 1 - e^{-aT} \right) \right] \]
\[ + ae^{-aT}r_0 + \frac{\sigma^2}{a} \left( e^{-2aT} - e^{-aT} \right) - ae^{-aT}r_0 \]
\[ = \theta_T - af(0, T) - \frac{\sigma^2}{2a} \left( 1 - e^{-2aT} \right). \]
If we denote by \( f^M(0, T) \) the market instantaneous forward rate, we would like to match the theoretical rate \( f(0, T) \) with the observed rate, which means

\[
\theta_T = \frac{\partial f^M(0, T)}{\partial T} + af^M(0, T) + \frac{\sigma^2}{2a}(1 - e^{-2aT}). \quad (5.9)
\]

This procedure is called fitting the model to the term structure of interest rate.

We can finally get the bond price substituting (5.9) into Equation (5.7). This yields

\[
P(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp \left( B(t, T)f^M(0, t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t, T)^2 - B(t, T)r_t \right),
\]

where \( P^M(0, T) \) denotes the market bond price seeing at time zero with maturity \( T \).

### 5.1.2 Simulation of the short interest rate

A more convenient representation of \( r_t \), uses a deterministic function \( \alpha_t \) that reflects the term structure at time 0, and a random process \( x_t \), that is independent of market data.

From Equation (5.5) we have that the short rate \( r_t \) can be expressed as

\[
r_t = e^{-at}r_0 + \int_0^t e^{-a(t-u)}\theta_u du + \int_0^t e^{-a(t-u)}\sigma dW_u
\]

\[
= \alpha_t + x_t
\]

(5.11)

where

\[
\alpha_t = e^{-at}r_0 + \int_0^t e^{-a(t-u)}\theta_u du
\]

and

\[
x_t = \int_0^t e^{-a(t-u)}\sigma dW_u.
\]

(5.12)

Substituting Equation (5.9) into the expression for \( \alpha_t \), we get

\[
\begin{align*}
\alpha_t &= e^{-at}r_0 + \int_0^t e^{-a(t-u)} \left[ \frac{\partial f^M(0, u)}{\partial u} + af^M(0, u) + \frac{\sigma^2}{2a}(1 - e^{-2au}) \right] du \\
&= e^{-at}r_0 + e^{-at} \left[ \int_0^t e^{au} \frac{\partial f^M(0, u)}{\partial u} du + \int_0^t e^{au} af^M(0, u) du \\
&\quad + \int_0^t e^{au} \frac{\sigma^2}{2a}(1 - e^{-2au}) du \right] \\
&= f^M(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.
\end{align*}
\]

To simulate the evolution of \( r_t \), we will use an Euler scheme, the simplest and most common discretization scheme. We have that for a small time step \( \Delta \), Equation (5.1) can be approximated by

\[
r_{\Delta(t+1)} - r_{\Delta t} = \mu(r_{\Delta t}, \theta_{\Delta t}) + \sigma(r_{\Delta t}, \theta_{\Delta t})(W_{\Delta(t+1)} - W_{\Delta t}).
\]

(5.13)
The increments \( W_{\Delta(t+1)} - W_{\Delta t} \) are independent normally distributed variables with mean zero and variance \( \Delta \).

For the Hull-White model, we will apply this approximation to the process \( x_t \) defined by Equation (5.12). Then we have

\[
e^{a\Delta(t+1)}x_{\Delta(t+1)} - e^{a\Delta t}x_{\Delta t} = \sigma \int_{\Delta t}^{\Delta(t+1)} e^{au} dW_u.
\]

Notice that these variables are also normally distributed with mean zero and variance \( \sigma^2 \int_{\Delta t}^{\Delta(t+1)} e^{2au} du = \frac{\sigma^2 e^{2a\Delta} - 1}{2a} \).

Hence

\[
e^{a\Delta(t+1)}x_{\Delta(t+1)} - e^{a\Delta t}x_{\Delta t} \sim e^{a\Delta t} \sqrt{\frac{1}{2\sigma^2} \left( e^{2a\Delta} - 1 \right)} Z_k, \tag{5.14}
\]

where \( Z_k \) is a standard Gaussian random variable generated in the \( k^{th} \) iteration.

We can finally write

\[
x_{\Delta(t+1)} - e^{a\Delta t}x_{\Delta t} \sim \sqrt{\frac{1}{2\sigma^2}} B(0, 2\Delta) Z_k.
\]

Once we have the short rate term structure \( r_t = \alpha_t + x_t \), we are able to compute the bond prices \( P(t, T) \), using formula (5.10), required to get the value of the swap rate \( SR(t_i, t_i) \) at time \( t_i \). This swap rate represents the pay off of the CMS leg. From Chapter 2, formula (2.12), we have

\[
SR(t_i, t_i) = \frac{1 - P(t_i, t_i + s_n)}{\sum_{j=1}^{n} \alpha_j P(t_i, t_i + s_j)}.
\]

The discount factors \( P(0, T) = \mathbb{E}^Q[e^{-\int_0^T r_s ds}] \) will be calculated using an approximation for the integral \( e^{-\int_0^T r_s ds} \approx e^{-\Delta \sum e_{r_e}} \).

### 5.1.3 Model Calibration

A term structure model has to be calibrated to the market before it can be used for valuation purposes. After fitting the term structure with the parameter \( \theta \), there are still two constants to be determined, \( a \) and \( \sigma \). The procedure of choosing the constants \( a \) and \( \sigma \) that match market prices for certain instruments with the prices we get with the model is called calibration.

To calibrate the model to market prices we need to find the best fit for \( a \) and \( \sigma \) simultaneously. In this case we will use Euro-Caplets derivatives as instruments to calibrate the model, since they are quite liquid in the market and since we have an analytical formula under the Hull-White model to price them, see Equation (5.16).

So, if we have \( M \) European caplet prices, we should minimize the function

\[
\sum_{i=1}^{M} \left( \frac{\text{model}(a, \sigma)_i - \text{market}_i}{\text{market}_i} \right)^2 \tag{5.15}
\]
5.1. Hull-White Model

where model \((a, \sigma)_i\) and market \(i\) stand for model and market prices respectively.

We will use the Euro-caplets volatility quotes from the market to calculate the market prices. According to the Black’s model, mentioned in section 3.3, the volatilities uniquely determined the caplet prices. The model prices will be computed using the analytical formulas under the Hull-White model provided in next subsection.

Finally, to minimize the previous function we will use a Levenberg-Marquardt algorithm, that is a popular alternative to the Gauss-Newton method of finding the minimum of a function that is a sum of squares of nonlinear functions.

**Caplet Prices under the Hull-White Model**

To price European options with maturity at time \(T\) under the Hull-White model, we need to compute the expected value of the payoff under the \(T\)-forward measure. We will need to express the dynamics of the short rate \(r_t\), given by equation (5.11), under this measure. An application of Girsanov’s Theorem is used to change the dynamics of the process \(x_t\), used in the decomposition of \(r_t = \alpha_t + x_t\), from the \(Q\) risk-neutral measure to the \(T\)-forward measure, see Brigo [4]. Then we have

\[
\begin{align*}
 dx(t) &= \left[-B(t,T)\sigma^2 - ax(t)\right] dt + \sigma dW^T(t) \\
 dW^T(t) &= dW(t) + \sigma B(t,T) dt
\end{align*}
\]

where \(B(t, T)\) is given by Equation (5.6), and \(W^T\) is a Brownian motion under the \(T\)-forward measure.

Then for \(s \leq t \leq T\)

\[
x_t = x_s e^{-a(t-s)} - M^T(s, t) + \sigma \int_s^t e^{-a(t-u)} dW^T(u)
\]

with

\[
M^T(s, t) = \frac{\sigma^2}{a^2} \left[1 - e^{-a(t-s)}\right] - \frac{\sigma^2}{2a^2} \left[e^{-a(T-t)} - e^{-a(T+t-2s)}\right].
\]

The short rate \(r_t\) conditional on \(\mathcal{F}_s\) and under the \(T\)-forward measure, is normally distributed with mean and variance given by

\[
\begin{align*}
\mathbb{E}[r_t | \mathcal{F}_s] &= x_s e^{-a(t-s)} - M^T(s, t) + \alpha(t) \\
\text{Var}[r_t | \mathcal{F}_s] &= \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)}).
\end{align*}
\]

Under this distribution, we can calculate the price at time \(t\) of a European caplet on a rate \(L(T, S)\), that can be a Euribor rate for the period \([T, S]\). This rate is reset at time \(T\) and paid at time \(S\). If the cap rate is equal to \(K\) and the notional value is \(X\) then its price is given by

\[
Cap(t, T, S) = X[P(t, T)N(-h_2) - (1 + K\tau)P(t, S)N(-h_1)]
\]

\(1\)The following formulas can be derived through the same methodology given in Appendix B.
where

\[ h_1 = \frac{1}{\sigma_p} \ln \frac{P(t, S)(1 + K\tau)}{P(t, T)} + \frac{\sigma_p^2}{2} \]

\[ h_2 = h_1 - \sigma_p \]

and

\[ \sigma_p = \sigma \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, S). \]
Chapter 6

Example of CMS Derivatives

In this chapter we compare the three models described in Chapters 3, 4 and 5 to price CMS based instruments like CMS swaps and Asian CMS swaps. The models have been implemented in MATLAB routines. A friendly interface has been developed to help the user to insert the characteristics of a CMS derivative and to choose a model to price it. The next figure shows the screen generated by the interface.

![CMS screen Interface](image)

Figure 6.1: CMS screen Interface
Once the screen has been filled, the instrument can be priced and an Excel file with the results is generated. In the next two examples, we will see this Excel file.

### 6.1 Example 1. CMS Swap

Consider a five-year CMS swap starting on 10th June 2010. They payment frequency is semi-annual in both legs. One leg will pay a 5% fix rate, and the other leg will pay the five-year semi-annually swap rate. This contract is over a notional of €10,000,000. The day-count convention is Actual/360. The Euribor-swap zero curve from 8th June 2010 will be used to value this contract; usually the reset dates are two days in advanced.

The zero swap term structure on this day was:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Zero Rate (Percentage)</th>
<th>Maturity</th>
<th>Zero Rate (Percentage)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MO</td>
<td>0.434</td>
<td>8 YR</td>
<td>2.69399</td>
</tr>
<tr>
<td>2 MO</td>
<td>0.54</td>
<td>9 YR</td>
<td>2.84076</td>
</tr>
<tr>
<td>3 MO</td>
<td>0.713</td>
<td>10 YR</td>
<td>2.96595</td>
</tr>
<tr>
<td>4 MO</td>
<td>0.801</td>
<td>11 YR</td>
<td>3.08075</td>
</tr>
<tr>
<td>5 MO</td>
<td>0.895</td>
<td>12 YR</td>
<td>3.18222</td>
</tr>
<tr>
<td>6 MO</td>
<td>0.999</td>
<td>15 YR</td>
<td>3.37877</td>
</tr>
<tr>
<td>1 YR</td>
<td>1.07641</td>
<td>20 YR</td>
<td>3.45766</td>
</tr>
<tr>
<td>2 YR</td>
<td>1.25712</td>
<td>25 YR</td>
<td>3.5702</td>
</tr>
<tr>
<td>3 YR</td>
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<tr>
<td>6 YR</td>
<td>2.31933</td>
<td>45 YR</td>
<td>2.97416</td>
</tr>
<tr>
<td>7 YR</td>
<td>2.52316</td>
<td>50 YR</td>
<td>2.91187</td>
</tr>
</tbody>
</table>

The inputs to calculate the price of the described CMS swap can be summarized in the next table.

<table>
<thead>
<tr>
<th>Valuation date</th>
<th>10-Jun-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>10-Jun-15</td>
</tr>
<tr>
<td>CMS payment frequency</td>
<td>2</td>
</tr>
<tr>
<td>Underlying swap payment frequency</td>
<td>2</td>
</tr>
<tr>
<td>Underlying swap tenor</td>
<td>5</td>
</tr>
</tbody>
</table>

As mentioned before, the interface helps to generates an Excel file that shows the price of the CMS based instrument. Some important data like cash flows, swap rates, convexity adjustments are also included in the Excel report that we will analyse later.

### 6.1.1 Pricing CMS swaps under Hull’s and LSM Model

To price the previously described CMS swap using the Hull’s convexity adjustment (section 3.1) the user should provide three very important inputs that can be obtained from the market.
6.1. Example 1. CMS Swap

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Swaption volatility on a five-year swap</td>
<td>0.22</td>
</tr>
<tr>
<td>Six-month caplet volatility</td>
<td>0.15</td>
</tr>
<tr>
<td>Correlation between swap and forward rate</td>
<td>0.7</td>
</tr>
</tbody>
</table>

An assumption used in the implementation of these models, is that the volatilities are constant through all period of time where the contract is in existence. However in the literature, there are some model where implied volatility is a function (possibly random) of time. It is also a common practise to use volatility surfaces, i.e., a matrix of (strike vs. forward start date), when pricing these derivatives. This allows the smile effect to be incorporated.

The LSM model does not need all the inputs from the previous table. From formula (3.13) we see that we only need the volatility of swaption, so this model is highly dependent on this parameter.

The Excel files that are generated for the LSM and the Hull’s model are very similar, and they are shown in table 6.1 and table 6.2 respectively.

After analyzing some results, it can be concluded that the size of the convexity adjustment is directly proportional to the following characteristics of the underlying swap rate

- Time to reset
- Volatility
- Maturity

We can also price CMS caps or CMS floors, we only need to select this instrument in the interface and to specify the respective cap or floor rate. As in the case of a swap instrument, the three models give similar values for a specific contract.
### Table 6.1: Hull’s convexity adjustment results.

<table>
<thead>
<tr>
<th>Payment date</th>
<th>Convexity adjustment</th>
<th>Timing adjustment</th>
<th>Total Adjustment</th>
<th>CMS rate</th>
<th>Fix leg CMS swap price</th>
<th>Floating leg CMS swap price</th>
<th>Discount Factor</th>
<th>CMS swap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/Dec/10</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.02048</td>
<td>254,166.67</td>
<td>104,094.13</td>
<td>0.99496</td>
<td>967,408.65</td>
</tr>
<tr>
<td>10/Jun/11</td>
<td>-0.00003</td>
<td>0.00000</td>
<td>-0.00003</td>
<td>0.02285</td>
<td>252,777.78</td>
<td>115,514.64</td>
<td>0.98920</td>
<td></td>
</tr>
<tr>
<td>12/Dec/11</td>
<td>-0.00007</td>
<td>0.00000</td>
<td>-0.00006</td>
<td>0.02532</td>
<td>256,944.44</td>
<td>130,130.85</td>
<td>0.98242</td>
<td></td>
</tr>
<tr>
<td>11/Dec/12</td>
<td>-0.00012</td>
<td>0.00001</td>
<td>-0.00012</td>
<td>0.02768</td>
<td>252,777.78</td>
<td>139,957.56</td>
<td>0.97490</td>
<td></td>
</tr>
<tr>
<td>10/Dec/12</td>
<td>-0.00019</td>
<td>0.00001</td>
<td>-0.00018</td>
<td>0.03005</td>
<td>252,777.78</td>
<td>151,923.84</td>
<td>0.96565</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>-0.00036</td>
<td>0.00003</td>
<td>-0.00033</td>
<td>0.03383</td>
<td>254,166.67</td>
<td>172,220.13</td>
<td>0.94301</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>-0.00046</td>
<td>0.00004</td>
<td>-0.00041</td>
<td>0.03540</td>
<td>252,777.78</td>
<td>178,948.09</td>
<td>0.92986</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>-0.00056</td>
<td>0.00006</td>
<td>-0.00050</td>
<td>0.03676</td>
<td>254,166.67</td>
<td>186,884.13</td>
<td>0.91579</td>
<td></td>
</tr>
<tr>
<td>10/Dec/14</td>
<td>-0.00067</td>
<td>0.00007</td>
<td>-0.00059</td>
<td>0.03786</td>
<td>252,777.78</td>
<td>191,399.46</td>
<td>0.90079</td>
<td></td>
</tr>
</tbody>
</table>

### Table 6.2: Linear Swap Rate Model results.

<table>
<thead>
<tr>
<th>Payment date</th>
<th>CMS rate</th>
<th>Fix leg CMS swap price</th>
<th>Floating leg CMS swap price</th>
<th>Discount Factor</th>
<th>CMS swap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/Dec/10</td>
<td>0.02048</td>
<td>254,166.67</td>
<td>104,094.13</td>
<td>0.99496</td>
<td>967,408.65</td>
</tr>
<tr>
<td>10/Jun/11</td>
<td>0.02285</td>
<td>252,777.78</td>
<td>115,514.64</td>
<td>0.98920</td>
<td></td>
</tr>
<tr>
<td>12/Dec/11</td>
<td>0.02532</td>
<td>256,944.44</td>
<td>130,130.85</td>
<td>0.98242</td>
<td></td>
</tr>
<tr>
<td>11/Dec/12</td>
<td>0.02768</td>
<td>252,777.78</td>
<td>139,957.56</td>
<td>0.97490</td>
<td></td>
</tr>
<tr>
<td>10/Dec/12</td>
<td>0.03005</td>
<td>252,777.78</td>
<td>151,923.84</td>
<td>0.96565</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>0.03383</td>
<td>254,166.67</td>
<td>172,220.13</td>
<td>0.94301</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>0.03540</td>
<td>252,777.78</td>
<td>178,948.09</td>
<td>0.92986</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>0.03676</td>
<td>254,166.67</td>
<td>186,884.13</td>
<td>0.91579</td>
<td></td>
</tr>
<tr>
<td>10/Dec/14</td>
<td>0.03786</td>
<td>252,777.78</td>
<td>191,399.46</td>
<td>0.90079</td>
<td></td>
</tr>
<tr>
<td>10/Jun/14</td>
<td>0.03850</td>
<td>254,166.67</td>
<td>172,100.48</td>
<td>0.96565</td>
<td></td>
</tr>
<tr>
<td>10/Jun/14</td>
<td>0.03996</td>
<td>252,777.78</td>
<td>186,812.72</td>
<td>0.91579</td>
<td></td>
</tr>
<tr>
<td>10/Jun/15</td>
<td>0.04146</td>
<td>254,166.67</td>
<td>191,367.87</td>
<td>0.90079</td>
<td></td>
</tr>
</tbody>
</table>
6.1.2 Pricing CMS swaps using Monte Carlo simulation

A Monte Carlo simulation was also implemented in a MATLAB routine. The inputs are quite similar, but for this model the swap rate volatility, forward rate volatility and the correlation are not required. As explained in Chapter 5, to generate the short rate $r_t$, the process $x_t$ of Equation (5.14) needs to be simulated.

The main difficulty for this simulation is to estimate the constants $a$ and $\sigma$ that are required for the Hull-White model. Unlike the other parameters, $a$ and $\sigma$ are not directly provided by the market. The MATLAB code has an optimization routine, that calculates these two parameters. To calibrate the model we need to provide market data from plain vanilla caplets. These data include caplets maturities, implied volatilities and strikes that will be used to calculate caplet prices using the Black’s model (1976)\(^1\).

We calibrate on 15 caplets, each maturing in 1, 2, 3, , 15 years, and written on one year-Euribor rate. To determine the mean reversion $a$ and volatility $\sigma$, such that the model fits the prices of actively traded instruments for caps as closely as possible, see equation (5.15), consider the following at-the-money Euro-cap market data:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Strike</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.11</td>
<td>69.77</td>
</tr>
<tr>
<td>2</td>
<td>1.31</td>
<td>69.39</td>
</tr>
<tr>
<td>3</td>
<td>1.59</td>
<td>52.31</td>
</tr>
<tr>
<td>4</td>
<td>1.88</td>
<td>45.3</td>
</tr>
<tr>
<td>5</td>
<td>2.14</td>
<td>39.65</td>
</tr>
<tr>
<td>6</td>
<td>2.36</td>
<td>35.25</td>
</tr>
<tr>
<td>7</td>
<td>2.55</td>
<td>31.88</td>
</tr>
<tr>
<td>8</td>
<td>2.7</td>
<td>29.35</td>
</tr>
<tr>
<td>9</td>
<td>2.83</td>
<td>27.39</td>
</tr>
<tr>
<td>10</td>
<td>2.93</td>
<td>25.85</td>
</tr>
<tr>
<td>12</td>
<td>3.11</td>
<td>23.58</td>
</tr>
<tr>
<td>15</td>
<td>3.27</td>
<td>21.61</td>
</tr>
</tbody>
</table>

The optimization routine needs a guess of the parameters $a$ and $\sigma$. For example we might have a guess that $0 < a < 1$ and $0 < \sigma < 0.5$. In this example we used the values of 0.1 and 0.2 respectively. But with different starting values we will have the same final $a$ and $\sigma$ parameters. The optimization routine may take more time to converge, but $a$ and $\sigma$ will be the same since we are finding the values that minimize a square function.

Using the *lsqnonlin* MATLAB function, a minimization method based on the Levenberg-Marquardt algorithm, we got the values

$$a = 0.0399$$
$$\sigma = 0.0101$$

The results from the approximation of the caplets prices are also shown in the next table:

---

\(^1\)This model is described in Section 3.3 for pricing CMS caps and floors, and derived in appendix B. Notice that this caplets are based on a Euribor rate and not on a swap rate as in 3.3.
6.1. Example 1. CMS Swap

Table 6.3: Caplets prices

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Market Price</th>
<th>Model Price</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.391</td>
<td>4.565</td>
<td>3.96%</td>
</tr>
<tr>
<td>2</td>
<td>7.5563</td>
<td>7.7661</td>
<td>2.78%</td>
</tr>
<tr>
<td>3</td>
<td>10.791</td>
<td>10.5541</td>
<td>2.20%</td>
</tr>
<tr>
<td>4</td>
<td>13.0382</td>
<td>12.5619</td>
<td>3.65%</td>
</tr>
<tr>
<td>5</td>
<td>14.0092</td>
<td>13.5066</td>
<td>3.59%</td>
</tr>
<tr>
<td>6</td>
<td>14.2263</td>
<td>13.7855</td>
<td>3.10%</td>
</tr>
<tr>
<td>7</td>
<td>13.9917</td>
<td>13.662</td>
<td>2.36%</td>
</tr>
<tr>
<td>8</td>
<td>13.6782</td>
<td>13.4374</td>
<td>1.76%</td>
</tr>
<tr>
<td>9</td>
<td>13.2715</td>
<td>13.1234</td>
<td>1.12%</td>
</tr>
<tr>
<td>10</td>
<td>12.4878</td>
<td>12.554</td>
<td>0.53%</td>
</tr>
<tr>
<td>11</td>
<td>12.7095</td>
<td>12.6069</td>
<td>0.81%</td>
</tr>
<tr>
<td>12</td>
<td>10.3296</td>
<td>10.7116</td>
<td>3.70%</td>
</tr>
<tr>
<td>13</td>
<td>10.472</td>
<td>10.6959</td>
<td>2.14%</td>
</tr>
<tr>
<td>14</td>
<td>10.8366</td>
<td>10.862</td>
<td>0.23%</td>
</tr>
<tr>
<td>15</td>
<td>10.9891</td>
<td>11.0231</td>
<td>0.31%</td>
</tr>
</tbody>
</table>

Once the calibration is done and we get \( a \) and \( \sigma \) values, we can then simulated the process \( x_t \) given in equation (5.14), and finally compute bond prices. These bond prices are the key factor to compute any kind of rate.

Some attention should be given to the quality of the calibration, since it can be affected by the specific market conditions one is trying to reproduce. Keep in mind that we are using data for at-the-money caps, so what we have done is an at-the-money calibration. We should only use this model to price instruments with this payoff condition.

The main drawback of Hull-White model is that there exist some positive probability of \( r_t \) being negative, however in practice this probability is small. Under this model \( r_t \) is normally distributed with parameters

\[
\mu = f_M(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2
\]

\[
\lambda^2 = \frac{\sigma^2}{2a}(1 - e^{-2at})
\]

then the probability of negative values can be then determined by

\[
P(r_t < 0) = P\left(\frac{r_t - \mu}{\lambda} < -\frac{\mu}{\lambda}\right) = N\left(-\frac{\mu}{\lambda}\right)
\]

with \( N(x) \) denoting the standard normal cumulative distribution function.

The next two graphs show two hundred discount curves and swap rates that were obtained using formula (5.10) and formula (3.4) respectively. We can observe that some bond prices are greater than one, which happens when the short rate \( r_t \) is below zero. As illustrated by figure 6.2, in practice, the probability of this rate being below zero is indeed small.
6.2 Example 2. Asian CMS Cap

Finally, with the swap rate, the price of the CMS swap can be calculated. The next results were obtained after 1,000,000 simulations.

<table>
<thead>
<tr>
<th>Payment date</th>
<th>CMS rate</th>
<th>Fix leg</th>
<th>Floating leg</th>
<th>Discount Factor</th>
<th>CMS swap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/Dec/10</td>
<td>0.02048</td>
<td>254,166.67</td>
<td>104,094.13</td>
<td>0.9945</td>
<td></td>
</tr>
<tr>
<td>10/Jun/11</td>
<td>0.02265</td>
<td>252,777.78</td>
<td>114,481.97</td>
<td>0.9883</td>
<td></td>
</tr>
<tr>
<td>12/Dec/11</td>
<td>0.02508</td>
<td>256,944.44</td>
<td>128,859.93</td>
<td>0.9810</td>
<td></td>
</tr>
<tr>
<td>11/Jun/12</td>
<td>0.02722</td>
<td>252,777.78</td>
<td>137,602.58</td>
<td>0.9722</td>
<td></td>
</tr>
<tr>
<td>10/Dec/12</td>
<td>0.02962</td>
<td>252,777.78</td>
<td>149,749.74</td>
<td>0.9621</td>
<td></td>
</tr>
<tr>
<td>10/Jun/13</td>
<td>0.03145</td>
<td>252,777.78</td>
<td>158,992.27</td>
<td>0.9506</td>
<td></td>
</tr>
<tr>
<td>10/Dec/13</td>
<td>0.03330</td>
<td>254,166.67</td>
<td>169,259.42</td>
<td>0.9378</td>
<td></td>
</tr>
<tr>
<td>10/Jun/14</td>
<td>0.03480</td>
<td>252,777.78</td>
<td>175,948.31</td>
<td>0.9242</td>
<td></td>
</tr>
<tr>
<td>10/Dec/14</td>
<td>0.03606</td>
<td>254,166.67</td>
<td>183,280.36</td>
<td>0.9094</td>
<td></td>
</tr>
<tr>
<td>10/Jun/15</td>
<td>0.03728</td>
<td>252,777.78</td>
<td>188,456.03</td>
<td>0.8946</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.4: Monte Carlo Simulation results.

Comparing the price of the CMS swap under the three models, we see that the convexity adjustment approximation given by Hull’s model and by the LSM give reliable prices. These methods are more efficient that a Monte Carlo simulation that takes more time to price a derivative.

6.2 Example 2. Asian CMS Cap

In this section we will price an Asian cap instrument under the model described in Chapter 4. The results will also be compared with a Monte Carlo simulation.

Consider an Asian CMS swap where the fix leg pays out an Euribor 6-month rate plus a spread\(^2\) of 0.545% on a semiannual basis. The floating leg has an annual payoff given by

\[
\max[H(T), 4%]
\]

where \(H(T)\) is the arithmetic mean of 12-year swap rates paid at time \(T\). The arithmetic mean consists of 12 swap rates observed every month.

\(^2\)Spread is the difference between the fixed and the floating rate.
6.2. Example 2. Asian CMS Cap

This contract is over a notional of €41,000,000. The 6-year-contract started on December 31\textsuperscript{st}, 2007, and the valuation date was July 13\textsuperscript{th}, 2010.

6.2.1 Pricing Asian CMS cap under Levy Approximation

For the Levy approximation we assumed that the distribution of the average swap rate is log-normal. Then, we can get a Black style formula to price options on average rates.

This model has also been implemented in a MATLAB routine and can be chosen in the screen in figure 6.1. We have the option to calculate the convexity adjustment for the swap rate using either the Hull’s adjustment, section 3.1; or the LSM adjustment, section 3.2.

We will use the two convexity adjustment models described in Example 1, then we will need similar inputs. Consider the next table with market data:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Swaption volatility on a twelve-year swap</td>
<td>0.2</td>
</tr>
<tr>
<td>Six-month caplet volatility</td>
<td>0.15</td>
</tr>
<tr>
<td>Correlation between swap and forward rate</td>
<td>0.7</td>
</tr>
</tbody>
</table>

In this contract, the two legs of the swap have different payment frequency. The frequency of the leg that pays out the Euribor rate is semi-annual, while the frequency of the CMS leg is annual. The Excel file that is generated, differentiates the payments of each leg.

The price of the Euribor leg has been calculated using formula (1.17). We do not need convexity correction for this leg, so we will get the same result under the Hull’s model and the LSM. Next table illustrates the Excel file generated for this leg. It contains detailed information such as expiry date of each payment, forward rates, discount factors and the total price of this leg.
6.2. Example 2. Asian CMS Cap

<table>
<thead>
<tr>
<th>Payment date</th>
<th>Euribor Forward rate</th>
<th>Payment</th>
<th>Discount Factor</th>
<th>Euribor leg price</th>
</tr>
</thead>
<tbody>
<tr>
<td>31/Dec/10</td>
<td>0.0107</td>
<td>330,409.85</td>
<td>0.9951</td>
<td></td>
</tr>
<tr>
<td>30/Jun/11</td>
<td>0.0136</td>
<td>390,408.94</td>
<td>0.9884</td>
<td></td>
</tr>
<tr>
<td>31/Dec/11</td>
<td>0.0143</td>
<td>404,103.16</td>
<td>0.9814</td>
<td>3,116,683.47</td>
</tr>
<tr>
<td>30/Jun/12</td>
<td>0.0161</td>
<td>442,142.55</td>
<td>0.9736</td>
<td></td>
</tr>
<tr>
<td>31/Dec/12</td>
<td>0.0191</td>
<td>503,954.77</td>
<td>0.9644</td>
<td></td>
</tr>
<tr>
<td>30/Jun/13</td>
<td>0.0215</td>
<td>551,693.48</td>
<td>0.9543</td>
<td></td>
</tr>
<tr>
<td>31/Dec/13</td>
<td>0.0236</td>
<td>596,432.39</td>
<td>0.9431</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.5: Euribor leg.

The most interesting part of the MATLAB routine is to price the CMS leg. The payoff of this leg is different from the one given by equation (4.1), but once we have an explicit value for the later we can easily get the value of the leg we are interested in. Since the expected value is a linear operator, it follows that

$$E\left[\max[H(T), K]\right] = E\left[\max[H(T) - K, 0]\right] + K$$

So to price the payoff of our Asian contract, we will only need to compute the pay off of equation (4.1) and add the strike value.

Next table displays the value of the Asian CMS cap with payoff given by equation (6.1).

<table>
<thead>
<tr>
<th>Payment date</th>
<th>CMS rate</th>
<th>Payment floating leg</th>
<th>Discount Factor</th>
<th>CMS cap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>31/Dec/10</td>
<td>0.0305</td>
<td>1,640,000.00</td>
<td>0.9951</td>
<td>6,632,197.51</td>
</tr>
<tr>
<td>2/Jan/12</td>
<td>0.0328</td>
<td>1,665,554.68</td>
<td>0.9813</td>
<td></td>
</tr>
<tr>
<td>31/Dec/12</td>
<td>0.0354</td>
<td>1,721,375.84</td>
<td>0.9644</td>
<td></td>
</tr>
<tr>
<td>31/Dec/13</td>
<td>0.0378</td>
<td>1,808,635.06</td>
<td>0.9431</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.6: Price of the Asian CMS cap using Hull’s convexity adjustment.

We can compare it with the output file generated when the LSM is used to adjust the swap rate.

<table>
<thead>
<tr>
<th>Payment date</th>
<th>CMS rate</th>
<th>Payment floating leg</th>
<th>Discount Factor</th>
<th>CMS cap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>31/Dec/10</td>
<td>0.0305</td>
<td>1,631,899.08</td>
<td>0.9951</td>
<td>6,630,039.85</td>
</tr>
<tr>
<td>2/Jan/12</td>
<td>0.0327</td>
<td>1,634,294.77</td>
<td>0.9813</td>
<td></td>
</tr>
<tr>
<td>31/Dec/12</td>
<td>0.0354</td>
<td>1,659,258.28</td>
<td>0.9644</td>
<td></td>
</tr>
<tr>
<td>31/Dec/13</td>
<td>0.0377</td>
<td>1,704,587.72</td>
<td>0.9431</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.7: Price of the Asian CMS cap using LSM convexity adjustment.

We can see that the choice of the model to calculate the convexity adjustment does not produce a big difference in the Asian CMS cap price. The total price of this contract will be given by the difference between the prices of the leg that pays out the Euribor rate and the CMS cap leg.
6.2.2 Pricing Asian CMS cap using Monte Carlo simulation

A Monte Carlo simulation can also be used to price the described Asian CMS cap. We will assume that the short rate follows the Hull-White model described in section 5.1. For the model calibration, we will use the same data that was used in Example 1. The values for the parameters $a$ and $\sigma$ we obtained with that calibration were

$$a = 0.0399$$
$$\sigma = 0.0101$$

Once we have simulated the short rate, we can calculate the swap rate and estimate the payoff given by equation (6.1) for every simulation. We will discount it with its respective simulated discount factor, and finally compute the mean of all the payoffs. Next table shows the file generated for this model.

<table>
<thead>
<tr>
<th>Payment date</th>
<th>CMS rate</th>
<th>Payment floating leg</th>
<th>Discount Factor</th>
<th>CMS cap price</th>
</tr>
</thead>
<tbody>
<tr>
<td>31/Dec/10</td>
<td>0.0315</td>
<td>1,630,605.54</td>
<td>0.9943</td>
<td></td>
</tr>
<tr>
<td>2/Jan/12</td>
<td>0.0335</td>
<td>1,643,452.46</td>
<td>0.9805</td>
<td></td>
</tr>
<tr>
<td>31/Dec/12</td>
<td>0.0364</td>
<td>1,677,197.26</td>
<td>0.9629</td>
<td>6,672,725.82</td>
</tr>
<tr>
<td>31/Dec/13</td>
<td>0.0390</td>
<td>1,721,470.54</td>
<td>0.9411</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.8: Price of the Asian CMS cap using Monte Carlo Simulation.

If we compare the results of the Levy approximation with the results of the Monte Carlo simulation, we see that the difference between the prices is quite small and can be consider sufficiently accurate for applications. The Levy approximation is computationally more efficient, and can be used for a faster valuation.
Appendix A

Moments under Black’s model

The Black’s model can be generalized into a class of models known as log-normal forward models, the main assumption is that the modelled rate at maturity is log-normally distributed. Let $S_i$ be the modelled rate. Under the risk neutral measure $Q$, $S_i$ is a martingale. The process followed by this rate is given by

$$dS_i(t) = \sigma_i(t) S_i(t) dW(t)$$  \hspace{1cm} (A.1)

where $\sigma_i(t)$ is the instantaneous volatility, and $W_i(t)$ is a standard Brownian motion under the $Q$ measure. Using Itô’s formula the solution of last equation is

$$S_i(T) = S_i(0) e^{\int_0^T \sigma_i(t) dW(t)} e^{-\frac{1}{2} \int_0^T \sigma_i^2(t) dt}.$$  \hspace{1cm} (A.2)

The stochastic integral $\int_0^T \sigma_i(t) dW(t)$ is well defined since $\sigma_i(t)$ is a deterministic function. It is normally distributed with mean zero. If $\int_0^T \sigma_i(t)^2 dt < \infty$ then the variance is given by

$$\text{Var}\left[ \int_0^T \sigma_i(t) dW(t) \right] = \mathbb{E}\left[ \left( \int_0^T \sigma_i(t) dW(t) \right)^2 \right]$$

$$= \int_0^T \sigma_i(t)^2 dt$$

where we used Itô’s isometry in the last step.

It is worth noting that the moment generating function of a random variable $X$ that is $\mathcal{N}(\mu, \sigma^2)$ distributed is given by

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2} \sigma^2 t^2}.$$  \hspace{1cm} (A.3)

We will often need to compute the second moment of the log-normally distributed rate $S_i$, that can be written as

$$\mathbb{E}^Q[S_i(T)^2] = \mathbb{E}^Q\left[ \int_0^T \sigma_i(t)^2 dW(t) - \int_0^T \sigma_i^2(t) dt \right]$$

$$= S_i(0)^2 e^{\int_0^T \sigma_i^2(t) dt} \mathbb{E}^Q\left[ \int_0^T e^{2 \int_0^T \sigma_i(t) dW(t)} \right]$$

$$= S_i(0)^2 e^{\int_0^T \sigma_i^2(t) dt}.$$

44
The last equality is an application of (A.3).

If \( \sigma \) is constant, then the expected value becomes

\[
E^Q[S_i(T)^2] = S_i(0)^2 e^{\sigma^2_i T}.
\] (A.4)

Another quantity that is frequently used is

\[
E^Q[(S_i(T) - S_i(0))^2] = E^Q\left[S_i(0)^2 \left( e^{\int_0^T \sigma_i(t) dW(t)} - \frac{1}{2} \int_0^T \sigma_i^2(t) dt \right)^2 \right]
\]

\[
= S_i(0)^2 \left[ e^{2 \int_0^T \sigma_i(t) dW(t) - \int_0^T \sigma_i^2(t) dt} \right.
\]

\[
+ E^Q\left[ -2 e^{\int_0^T \sigma_i(t) dW(t) - \int_0^T \sigma_i^2(t) dt} + 1 \right]
\]

\[
= S_i(0)^2 \left( e^{\int_0^T \sigma_i^2(t) dt} - 1 \right)
\]

\[
= S_i(0)^2 \left( e^{\sigma^2_i T} - 1 \right).
\] (A.5)

And finally, the expectation of the product of two rates is given by

\[
E^Q[S_i(T_i)S_j(T_j)] = E^Q\left[S_i(0)e^{\int_0^{T_i} \sigma_i(t) dW(t) - \frac{1}{2} \int_0^{T_i} \sigma_i^2(t) dt}
\]

\[
* S_j(0)e^{\int_0^{T_j} \sigma_j(t) dW(t) - \frac{1}{2} \int_0^{T_j} \sigma_j^2(t) dt}
\]

\[
= S_i(0)S_j(0)e^{-\frac{1}{2} \left( \sigma_i^2 T_i + \sigma_j^2 T_j \right)}
\]

\[
* E^Q\left[ e^{\int_0^{T_i} \sigma_i(t) dW(t) + \int_0^{T_j} \sigma_j(t) dW(t)} \right].
\]

To compute the last expectation we will use (A.3). Notice that the covariance between the variables \( \int_0^{T_i} \sigma_i(t) dW(t) \) and \( \int_0^{T_j} \sigma_j(t) dW(t) \) is given by \( \sigma_i \sigma_j \min\{T_i, T_j\} \), when the volatility \( \sigma_i \) and \( \sigma_j \) are constants.

Then we have

\[
E^Q[S_i(T_i)S_j(T_j)] = S_i(0)S_j(0)e^{-\frac{1}{2} \left( \sigma_i^2 T_i + \sigma_j^2 T_j \right)}
\]

\[
* e^{\sigma_i \sigma_j \min\{T_i, T_j\}}
\]

\[
= S_i(0)S_j(0)e^{\sigma_i \sigma_j \min\{T_i, T_j\}}
\] (A.6)
Appendix B

Detailed derivation of Black’s formula to price options

To price a caplet with underlying rate $S_t(T)$ observed at time $T$, that follows the Black’s model described in previous appendix, equation (A.1), we need to compute the expected value of its payoff

$$E_Q[\max[S_t(T) - K, 0]]$$

where $Q$ denotes the equivalent martingale measure.

From Equation (A.2), we have that

$$S_t(T) = S_t(0)e^{\int_0^T \sigma_s(t) dW(t) - \frac{1}{2} \int_0^T \sigma_s^2(t) dt} = S_t(0)e^{m + V y}$$

where $m$ is given by

$$m = -\frac{1}{2} \int_0^T \sigma_s^2(t) dt$$

and $V^2$ has the form

$$V^2 = \int_0^T \sigma_s^2(t) d(t)$$

the random variable $y$ is standard normally distributed.

Then the expected value can be written as

$$E_Q[\max[S_t(T) - K, 0]] = E_Q[\max[S_t(0)e^{m+V y} - K, 0]]$$

$$= \int_{-\infty}^{\infty} \max[S_t(0)e^{m+V y} - K, 0] f(y) dy$$

where $f(x)$ is the normal probability density function.

Note that $S_t(0)e^{m+V y} - K > 0$ if and only if

$$y > \frac{-\ln \left( \frac{S_t(0)}{K} \right) - m}{V} =: y'$$
then we have

$$E^Q \left[ \max \{S_i(T) - K, 0\} \right] = \int_{y'}^{\infty} (S_i(0)e^{m+V} - K)f(y) \, dy$$

$$= S_i(0) \int_{y'}^{\infty} e^{m+V} f(y) \, dy - K \int_{y'}^{\infty} f(y) \, dy$$

$$= S_i(0) \frac{1}{\sqrt{2\pi}} \int_{y'}^{\infty} e^{-\frac{1}{2}y'^2 + m+V} \, dy - K \left(1 - \mathcal{N}(y')\right)$$

$$= S_i(0) e^{m+\frac{1}{2}V^2} \frac{1}{\sqrt{2\pi}} \int_{y'}^{\infty} e^{-\frac{1}{2}(y-V)^2} \, dy - K \left(1 - \mathcal{N}(y')\right)$$

$$= S_i(0) \frac{1}{\sqrt{2\pi}} \int_{y'\infty}^{\infty} e^{-\frac{1}{2}z^2} \, dz - K \left(1 - \mathcal{N}(y')\right)$$

$$= S_i(0) (1 - \mathcal{N}(y' - V)) - K (1 - \mathcal{N}(y'))$$

$$= S_i(0) \mathcal{N}(-y' + V) - KN(-y')$$

$$= S_i(0) \mathcal{N}(d_1) - KN(d_2)$$

where

$$d_1 = \ln \frac{S_i(0)}{K} + \frac{1}{2} \int_0^T \sigma_i(t)^2 dt$$

$$\sqrt{\int_0^T \sigma_i(t)^2 dt}$$

and

$$d_2 = d_1 - \sigma_i \sqrt{T}.$$
Bibliography


